DEVISING THE OPTIMAL HANDOFF STRATEGY FOR THE 4 $\times$ 100 METER RELAY

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Abstract. For competitive track and field athletes, there are often small differences in athletic ability. This necessitates the implementation of optimal racing strategy, especially for team events like the 4 $\times$ 100 meter relay. Perhaps the most important decision coaches must make for this event is how close each receiving runner should allow each corresponding inbound runner to get before accelerating in preparation for the baton exchange. In this paper, I formally model a coach's problem of determining the optimal set of cue distances at which each receiving runner should begin to accelerate. Allowing for some variation in each athlete's ability parameters, I find that the average cue distance is roughly 7m per exchange, and that my model is not particularly sensitive to small measurement errors.

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Introduction

At elite levels of competition, differences in ability between athletes are often small, necessitating the implementation of effective strategy. This is especially the case for sprinting events in track and field, and in particular for the 4 × 100 relay, a team event in which teams of 4 sprinters run separate portions of a 400 meter race. Arguably the most important features of this race are the baton exchanges that occur between runners – properly executed handoffs can ensure a seamless transition of the baton from one runner to another, while poor handoffs result in inefficient uses of energy. The key determinant of successful handoffs is the distance between incoming runners (the runners preparing to pass the baton) and receiving runners at the time that the receiving runner begins to accelerate. If the receiving runner takes off too late, she forces the incoming runner to slow down in order to hand off the baton. If the receiving runner takes off too early, however, she risks outrunning the incoming runner, resulting in a disqualification.

In preparing for 4 × 100 meter relay, it is often the job of a coach to determine when each receiving runner should begin their acceleration. The goal of this paper is to present a mathematical model in which a coach aims to minimize his team's finishing time by selecting a set of "cue distances" that determine how close each incoming runner should be to her receiving runner when the receiving runner begins her acceleration. Taking individual athletic parameters into account, such a model could aid a coach's decisions in preparing for races against comparably talented competition.

There is a well-established literature that focuses on building mathematical models to predict optimal racing strategies. Perhaps the most influential work is the analysis done by [2], who developed a formal strategy model for races between 60 and 10,000 meters. Analyses like the one conducted in [4] have examined the maximum time elite sprinters can spend accelerating, while studies like [6] and [3] have examined factors that influence 4 × 100 performance, as well as the optimal ordering of runners for the relay. There have been a relative paucity of studies that examine how to optimize race performance, given an order of athletes, by optimizing baton exchanges. My paper attempts to address this void in the literature.

Assumptions

- There are two rounds of modeling. In the first round, runner speed is not affected by turning radius. This assumption is relaxed in the second round.
- Runners are not affected by factors like wind and temperature.
- Each runner applies their maximal level of possible propulsive force for their portion of the relay. This is not affected by the team's relative position in the race.
- Each runner starts at the beginning of the acceleration zone, ten meters away from the exchange zone.
- The order of runners for a given relay is fixed.

4 × 100m Rules

The 4 × 100 relay is a 400 meter race in which each competing team has four runners, each of whom runs a portion of the race while carrying a baton that they must successfully exchange with the next runner upon the completion of their leg of the race. The race is run around a 400 meter racing track, and each team's final time for the race corresponds to the time at which their final runner crosses the finish line.

Important to the success of each team is efficient baton exchanges. As there are four runners, there are three exchanges that occur during the race, with three exchange zones that the exchanges must occur in. Each exchange zone is 20 meters long, with the midpoint of the first exchange zone being 100 meters from the starting line. Each subsequent exchange zone's midpoint is 100 meters from the midpoint of the previous exchange zone. There is a 10 meter acceleration zone leading up to each exchange zone which receiving runners start in. Figure 1 displays each exchange and acceleration zone for a regulation 400 meter track.

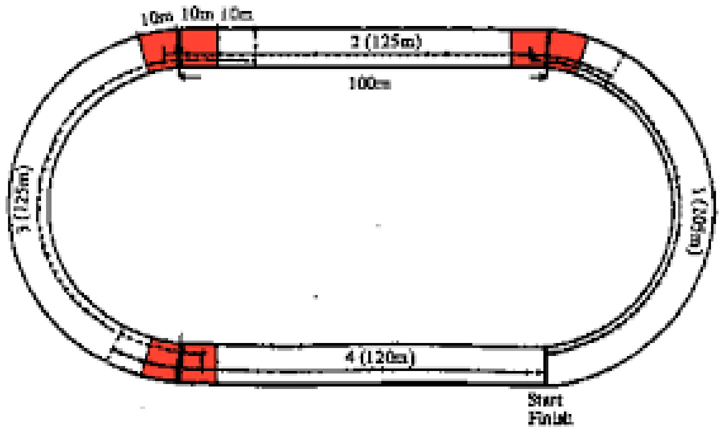

FIGURE 1. Standard exchange and acceleration zones for the 4×100 meter relay. Image comes from BrianMac.

The Model

In this section, I develop a model that defines an optimization problem for a coach attempting to optimize performance in the 4×100 meter relay, given a set of runners with a particular order already specified. It is the coach's goal to determine how close each receiving runner should let the inbound runner approach before beginning to accelerate as a means of optimizing the team's overall time. My model is an extension of the model developed in [2]. Following suggestions outlined in [5], I adapt this paper's equation of motion so that for sprinting races, runners begin to decelerate over time rather than maintain a constant velocity.

The key component of every relay is the exchanges between runners, which depend on the relative positions, $x(t)$, of each runner, as well as the relative speeds $v(t)$. Exchanges between runners i and j can be modeled by the following decoupled system of differential equations, which determine each runner's position and velocity:

$$\frac{dx_i}{dt} = v_i$$
$$\frac{dv_i}{dt} = F_i - \frac{v_i}{\tau} - \alpha_i t_i$$
$$\frac{dx_j}{dt} = v_j$$
$$\frac{dv_j}{dt} = F_j - \frac{v_j}{\tau} - \alpha_j t_j$$

As in [2], we have that

$$x_i(t) = x_i(t_i) + \int_{t_i}^{t} v_i(t)dt \tag{1}$$

where t_i is the time at which runner i began running.

Parameters vary across runners – here, F_i denotes the maximum propulsive force for runner i, τ_i a damping constant for runner i's velocity, and α_i a constant of decay for runner i (see Table 5 in the appendix for descriptions of variables and their units). For simplicity, it is assumed that exchanges occur when $x_i = x_j$.

Assume that runner i is handing off to runner j. Note that when $x_i = x_j$, it cannot be the case that $v_i < v_j$. If it were, then as each runner's velocity function is continuous, runner j would have had to have been behind runner i and caught up to to him/her at the time of exchange, while we assume that runner j starts ahead of runner i. If $v_i > v_j$ then runner i would run into and surpass runner j. Thus, we must have that $v_i = v_j$ for an optimal exchange.

Note that t records the total time since the start of the race, yet each runner only runs for a portion of the race. Each runner i has an associated start time t_i at which they begin to accelerate.[1] Thus, *each non-leadoff runner's* position is given by

$$x_i(t) = \begin{cases} 80 + (i-2)100 & t \leq t_i \\ x_i(t_i) + \int_{t_i}^{t} v_i(t-t_i)dt & t > t_i \end{cases} \tag{2}$$

To simplify notation, I write $v_i(t_i)$ rather than $v_i(t-t_i)$ henceforth. Each runner's initial acceleration, however, is not cued by the total accumulated time in the race but rather the interpersonal distance between runners i and j. For a given distance s_i at which a coach mandates for the receiving athlete to begin accelerating, the associated time will satisfy $x_i(t_i) = x_j(t_i) - s_i$. Thus, the time of exchanges between runners are a function of each runner's start time, which is a function of each s_i. Define l_i to be the amount of time that each runner holds the baton. Note that this implies

$$l_i = \begin{cases} t_* - e_{i-1} & i = 4 \\ e_i - e_{i-1} & i \in \{2,3\} \\ e_i & i = 1 \end{cases} \tag{3}$$

where e_i denotes the time at which runner i handed off the baton to runner $i+1$, and t_* denotes the time at which the team's final runner crossed the finish line. Each exchange time is a function of the start cues so that each l_i is a function of each start cues. Note that t_*, given by

$$t_* = \sum_{i=i}^{4} l_i(s_i) \tag{4}$$

is then also a function of the starting distance cues. It is then the coach's problem to choose a vector of starting distance cues S to optimize the following problem:

$$t_* = \min_S \sum_{i=1}^{4} l_i(S) \tag{5}$$

subject to the constraints

$$x_i(e_i) = x_j(e_i) \tag{6}$$

$$v_i(e_i) = v_j(e_i) \tag{7}$$

$$10 \leq x_j(e_i) - x_j(s_j) \leq 30 \tag{8}$$

where $(i = 1,2,3 \quad j = 1,2,3,4)$. Interestingly, the satisfaction of constraints 6 - 8 is a sufficient condition for the optimization of t_*: This is outlined by the following Proposition (see Appendix for proof):

Proposition 1. *Consider the optimization problem outlined in equation (5). If a set of starting cues S satisfies constraints (6) - (8), then it is the unique solution to (5).*

Solving the Model. Consider a handoff between arbitrary runners i and j. As each constraint must be satisfied, we must have that

$$x_i(e_i(s_j)) = x_j(e_i(s_j))$$
$$v_i(e_i(s_j)) = v_j(e_i(s_j))$$

where we write $e_i(s_i)$ to emphasize that the time of the exchange is a function of the determined cue distance. The optimal s_j will then satisfy

$$x_i(e_i(s_j)) - x_j(e_i(s_j)) = 0 \tag{9}$$

$$v_i(e_i(s_j)) - v_j(e_i(s_j)) = 0 \tag{10}$$

Each x_i and v_i correspond to the solution of the decoupled system of differential equations outlined earlier, and (as described earlier) the solution to this system depends on the starting distance cues by shifting the time at which each non-leadoff runner begins her acceleration.

[1] For the first runner, $s_1 = 0$ trivially assuming no delayed reaction to the starting gun.

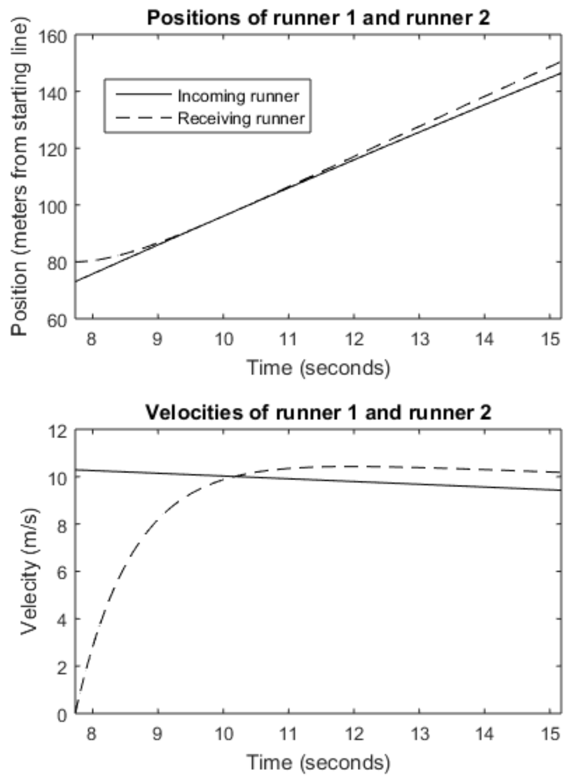

FIGURE 2. Position and velocities for two runners given a starting cue of roughly 7m. Note that the conditions of $v_i(e_i) = v_j(e_i)$ and $x_i(e_i) = x_j(e_i)$ are satisfied here.

It is useful to consider an example of calculating distance cues for a specific exchange. Starting with runners 1 and 2, a coach can determine s_2. This is done by noting that t_2 is the value that satisfies $x_2(t_2) = x_1(t_2) + s$, where s is a variable for the possible distance cues. As this varies by s, we may write $t_2 = f(s)$, where f is a function that maps each s to the appropriate t_2. Thus,

$$x_1(t) - x_2(f(s)) = 0$$
$$v_1(t) - v_2(f(s)) = 0$$

The t and s that satisfy these two equations are the coach's e_1 and s_2, respectively. Given s_2, the coach can determine s_3 to satisfy (9) and (10) between runners 2 and 3, and finally choose s_4 in the same manner.

Figure 2 displays an example of constraints (9) and (10) being satisfied between runners 1 and 2 of a relay team.

Example: Determining Exchanges for a Relay

For the highest levels of competition where differences in athletic capability across teams are small, it is useful to implement an optimal handoff strategy to maximize the chances of success. In this section, I examine the solution to the coach's problem outlined above for a particular set of parameters corresponding to Olympic-caliber athletes. Using data on world record sprinting times, [2] estimated values of $F = 12.2$ and $\tau = 0.892$. Keeping in line with these estimations, I assign similar parameter values to each of four runners comprising a relay team in the model, allowing for some variation. These values are displayed in Table 1.

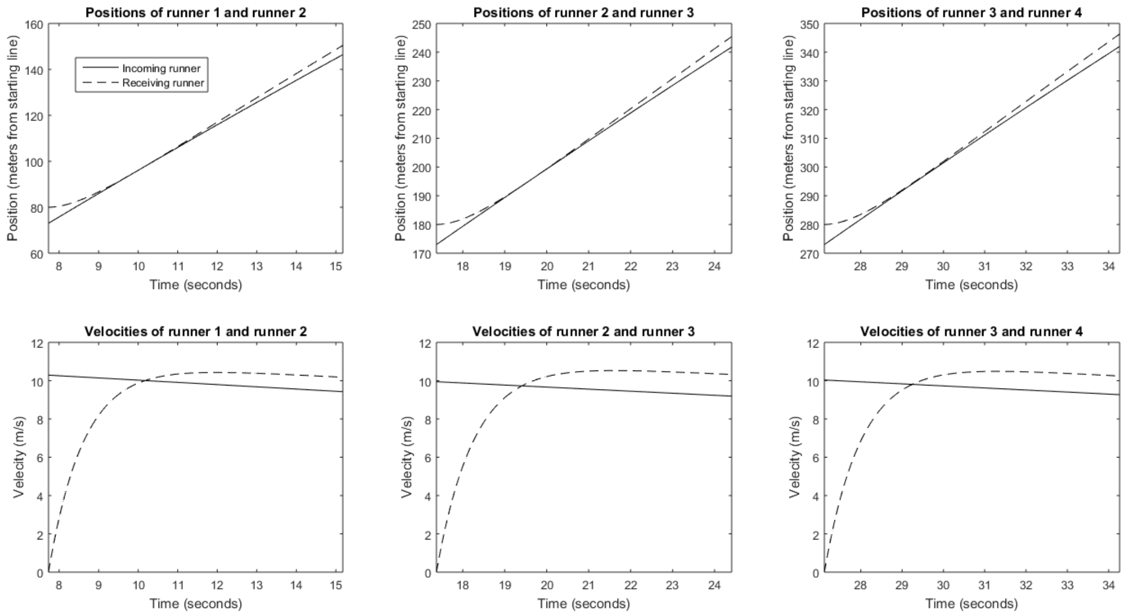

FIGURE 3. Distances and velocities between exchanging runners. Note that the major constraints are satisfied.

TABLE 1. Parameter values, as well as corresponding handoff/leg times and cue distances, for each runner R_i.

	R_1	R_2	R_3	R_4
F_i	12.400	12.200	12.300	12.300
τ_i	0.89400	0.89200	0.89300	0.89200
α_i	0.13000	0.12000	0.12000	0.13000
l_i	10.175	9.2321	9.8571	10.905
s_i	0.00000	**6.9553**	**6.9880**	**6.7716**
e_i	10.175	19.408	29.265	40.169
Total time		40.169		

While each runner has different parameter values, it is intuitive that the first and last legs of the relay (the amount of time that the baton spends in the hands of the first and last runners) are the longest. The first runner begins with the baton from a static start, accelerating from a velocity of zero, and the last runner actually runs for the longest distance with the baton as she/he does not hand off to another runner. As emphasized earlier, each handoff corresponds with the satisfaction of constraints 6 and 7. Figure 3 displays the satisfaction of these constraints for each handoff.

As data on relay times and cue distances is sparse, it is difficult to assess the validity of my model empirically. While one could analyze videos of existing races among elite athletes, this would require the implementation of advanced visual analysis software that I do not possess. Rudimentary observation of numerous elite track races suggest that my cue distances are within range of the ones implemented at professional levels of competition, though a more sophisticated analysis would be necessary to affirm this assertion.

Relaxing the no-turning assumption. To this point, my model has ignored the impact that track curvature can have on the above optimization problem, as well as the consequences curvature can have for finishing times. Work from [1] suggests that typical track dimensions can slow finishing finishing times by 4 to 5 percent. To account for this affect, I give each runner a "turning" parameter

TABLE 2. Parameter values, leg/exchange times and cue distances for each runner.

	R_1	R_2	R_3	R_4
F_i	12.400	12.200	12.300	12.300
τ_i	0.89400	0.89200	0.89300	0.89200
τ_i'	0.85824	0.85632	0.85728	0.85632
α_i	0.13000	0.12000	0.12000	0.13000
l_i	10.197	9.6266	10.153	10.698
s_i	0.00000	**6.7650**	**6.4700**	**6.1150**
e_i	10.197	19.823	29.977	40.674

$\tau' = \delta\tau$, where I use values of δ ranging from 0.95 to 0.97 – each runner's top speed is increasing in τ, so decreasing τ accounts for the fact that runners cannot run as fast while rounding turns. When a runner is rounding turns, the value τ' replaces τ in the runner's distance and velocity expressions.

From Figure 1, we see that runner 1 runs their entire leg on a curve, while runner 2 begins on a curve and ends on the straightaway, runner 3 begins on the straightaway and ends on the turn, and runner 4 begins on a curve and ends on a straightaway. If $\tilde{v}(t)$ denotes a runners turning velocity, this implies that each runners velocity is a piece-wise function, depending on whether they are turning or not. For simplicity, I assume that switching between turning and running straight at the time of each baton exchange. For example, for runner 2 we have that

$$\frac{dx_2}{dt} = \begin{cases} v_1(t-t_2), & t > e_1 \\ \tilde{v}_1(t-t_2), & t \leq e_1 \end{cases}$$

Subsequently, we have that

$$x_2(t) = \begin{cases} 80, & t \leq t_2 \\ 80 + \int_{t_2}^{t} \tilde{v}_2(t-t_2)dt, & t \in [t_2, e_1] \\ x_2(e_2) + \int_{e_1}^{t} v_2(t-t_1)dt, & t > e_1 \end{cases}$$

where $v_2(e_1 - t_2) = \tilde{v}_2(e_1)$.

The coach's problem is still to choose a vector S of starting distance cues to minimize the sum of leg times, only now each runner's parameters vary with their position on the track. With this framework in place, I am able to run a similar analysis where track curvature is able to affect top running speed. Table 2 displays the results of this analysis. For ease of comparison, I use the same parameter values as the ones used for my main analysis above. Predictably, each leg is slower than it previously was, as each leg involves some amount of turning. Note that the second runner has the fastest leg by a noticeable margin, as they run the majority of the straightaway and, unlike the final runner, are able to run with the baton for a short distance before handing off to another runner.

Most significantly, notice that each cue distance is shorter. The longest cue distance corresponds to the exchange between runners 2 (who benefited from running their leg on the straightaway) and 3, which should be expected. For handoffs occurring on turns, the cue distance is smaller – this is a consequence of the fact that these handoffs are occurring at lower levels of velocity. Note that the final time of the relay is slightly slower than in the case where curvature is assumed to have no effect on speed, which is consistent with the finding of [1].

A Simple Order Analysis. While my model can be used to find the optimal strategy for a given order of runners, it does not have an elegant method of determining an optimal ordering of runners on a relay other than exhaustively comparing each possible permutation of runners. Despite this, using my model to determine the optimal order of runners with given parameters can provide insight into general strategies coaches could employ, and is particularly easy to do for particular extreme cases.

Suppose that a relay team is comprised of four runners: three of these runners are identical in their parameter values, while the fourth runner is the fastest of the group, with parameter values to

TABLE 3. Possible ordering of a relay with one type A runner and three type B runners, with subsequent finishing times when applying the model's handoff strategy.

	R_1	R_2	R_3	R_4	t_*
O_1	A	B	B	B	**40.26**
O_2	B	A	B	B	**40.65**
O_3	B	B	A	B	**40.47**
O_4	B	B	B	A	**40.40**

reflect this fact. Effectively, this reduces the total number of possible orders for a coach to consider from 4 factorial down to 4: she can place her fastest runner first, second, third or fourth in the order.

For the purpose of this example, suppose runners come in two types: type A and type B, and that a given team has three type B runners and one type A runner. Let the type B runners have parameters $F_B = 12.2$, $\tau_B = 0.892$, and the type A runner have parameters $F_A = 12.4$ and $\tau_A = 0.894$. For simplicity, assume that $\alpha = 0.12$ across all runners. Table 3 displays the four possible unique orders a coach could employ, given the ability parameters of each athlete, as well as the team's predicted finishing time when implementing the model's optimal handoff strategy.

From Table 3, we see that the fastest time corresponds to the case where the fastest runner leads off, and that the second-fastest time corresponds to the case where the faster runner runs the final leg of the relay. This makes intuitive sense: the leadoff runner is the only runner who begins running with the baton from a standstill, while each other runner receives the baton while already running at a positive velocity. Given that the type A runner is able to accelerate the fastest out of all of the athletes, the results from my model imply that the team benefits the most from placing leading off with their fastest runner, as this allows for the quickest acceleration of the baton at the beginning of the race. It is also intuitive that second-best order corresponds to placing runner A last – as explained earlier, this allows for the baton to spend a greater share of distance with runner A, the fastest of the group. These results suggest that when a team has one "superstar" runner with the other runners possessing similar talents, it is optimal to have the superstar runner lead the race off.

Sensitivity Analysis

In implementing my model, a coach would need to have each of her athletes perform tests in order to measure values of F, τ and α for each runner.[2] Due to variation in performances, it is possible that there will be small degrees of measurement error in estimating each athlete's parameters. A natural concern arises that if small deviations in parameter values produce large changes in the model's output, then inputting parameters with small measurement errors into the model will result in cue distance suggestions that are far from what would be optimal given the actual values of these parameters.

In this section I perform a sensitivity analysis to examine whether one should be significantly concerned regarding small levels of measurement error. To do this, I take each runner's initial parameter values outlined in Table 1, and scale their values by a factor $\lambda \in (0.97, 1.03)$ randomly. The resulting parameter values, as well as the model's subsequent output, are included in Table 4.

After assigning random "measurement error" shocks to each of the initial parameter values, note that each runner's measured parameters are either slightly higher or lower than before. For example, in this scenario 3 out of 4 runners are assigned values of F that are underestimates of their true parameters, while 3 out of 4 runners are assigned values of τ that are *overestimates* of their true parameters. The model predicts optimal cue distances that, on average, deviate from the prediction with the true parameter values in absolute value by 3.67 percent. Thus, given reasonable ranges of measurement error, the model provides suggestions of starting distance cues that are within a reasonable distance of the truly optimal cue distances. This implies that my model is not particularly sensitive to small measurement errors.

[2]Practically, this could be implemented by finding a runner's top level of acceleration F, and also recording a runner's distance throughout a time trial. From this data, the coach could fit each athlete's τ and α parameters.

Table 4. Model output when introducing measurement error to initial parameter values.

	R_1	R_2	R_3	R_4
F_i	12.303	12.341	12.126	12.296
τ_i	0.90364	0.90017	0.89506	0.88352
α_i	0.12992	0.12146	0.12040	0.13046
l_i	9.9616	9.7014	9.6274	11.073
s_i	0.00000	**7.2221**	**7.2403**	**6.5295**
e_i	9.9616	19.663	29.290	40.363
Total time:		40.363		

Generalizations

My modeling framework could be adapted to analyze other relay races in track and field. For example, my model would be well-suited for analyzing the 4 × 200 meter relay – a less commonly run, but nonetheless import race. As each runner would race for a distance lower than the critical distance suggested by [2], my current model could be used with minor alterations to optimize handoffs in this race.

My model could also be generalized to the 4 × 400 meter relay, an important event in championship track meets. Unlike the 4 × 200, in this event each runner races over a distance further than the critical distance suggested by [2]. Thus, one would need to account for further decay of energy systems in modeling the distances and velocities of each runner, though one could still impose the optimal handoff conditions for such a race to derive optimal cue distances.

Finally, my model could be adapted to characterize racing strategies in longer events. A common racing strategy is for runners to make a surge in pace when the distance between them and their opponent falls within a certain threshold. As my model determines optimal cue distances for relay runners, it could be altered to determine optimal cue distances at which runners should surge in response to the relative positions and speeds of their opponents.

Strengths and Weaknesses

As noted, my model is able to provide accurate predictions of cue distances among elite athletes. By allowing for each runner to have her own talent parameters, my model allows for appropriate adjustments of strategy based on relative talent levels of teammates. Further, my model can be adjusted to account for the observation that runners rounding turns tend to run slower than they otherwise would. By adapting the equation of motion developed in [2] (as suggested by [5]) I am able to capture the observation that runners tend to decelerate towards the end of the 100 meter race.

While there are many strong features of my model, it also has multiple shortcomings. For example, my model does not explicitly attempt to optimize the order of runners on a relay, but rather minimize total time for a given order of runners. While one could exhaustively compare the 4 factorial different possible orders and their subsequent minimized times, this is not an elegant method, and could become computationally intensive if there are more than 4 potential runners looking to fill the 4 relay slots. Fortunately, my model can easily be used to examine ordering in extreme cases, such as the case in which a team has one runner significantly faster than three similar runners. Along with not possessing an elegant manner by which to optimize ordering of runners, mymodel does not account for external forces, such as wind or inclement weather, nor does it account for psychological influences – it is possible that a team's position in a race relative to other teams could influence the top speed of its runners through psychological channels. Unfortunately, my model cannot address these potential factors.

References

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Appendix

Proposition 2. *Consider the optimization problem outlined in equation (5). If a set of starting cues S satisfies constraints (6) - (8), then it is the unique solution to (5).*

Proof. Suppose $S = (0, s_2, s_3, s_4)$ is chosen so that the desired constraints are satisfied. It suffices to show that this obtains a solution to (5) by showing that S is the unique solution to constraints (6) - (8).

Consider an arbitrary candidate solution vector $S' = (0, s_2', s_3', s_4')$ such that $s_2' > s_2$. This implies that runner 2 takes off earlier than initially suggested. Thus, we must have that $e_2 < e_2'$. But note that at e_2, $x_1(e_2) = x_2(e_2)$ and $v_1(e_2) = v_2(e_2)$. With the assumption that the incoming runner is decelerating and the receiving runner accelerating within the exchange region (a reasonable assumption) then we must have that for all $e > e_2$, $v_2(e_2) > v_1(e_2)$. But then runner 2 pulls away from runner 1 and the handoff never occurs.

If $s_2^{'} < s_2$, then we must have that $e_2' < e_2$. Since runner 1 is decelerating, $v_1(e_2') > v_1(e_2)$. But we also must have that $v_2(e_2') < v_2(e_2)$, which implies that $v_1(e_2') > v_2(e_2')$. But this implies that an inefficient exchange occurs.

Since s_2' cannot satisfy the constraints, let $S' = (0, s_2, s_3', s_4')$. Then, the same argument can be applied for the remaining 2 components of S', implying that, given an S that satisfies (6) and (7) exits, it is unique. □

Derivation of expressions for v and x. As the systems of differential equations for each exchange are decoupled, a modeler can use analytic expressions for each v_i and x_i in finding the solution of the model outlined in this paper. Of course, the solutions can also be found numerically – using a numerical method may be especially desirable if one anticipates the model's system of differential equations changing in its functional form at some point during their analysis. Nonetheless, using analytic expressions may be appropriate if one does not anticipate altering the functional forms of the original system of differential equations. I derive these expressions here.

For an arbitrary runner i, we have that

$$\begin{aligned}
\frac{dv_i}{dt} &= F_i - \alpha_i t - \frac{v_i}{\tau_i} \\
\frac{dv_i}{dt} + \frac{v_i}{\tau_i} &= F_i - \alpha_i t \\
e^{\frac{t}{\tau_i}} \left(\frac{dv_i}{dt} + \frac{v_i}{\tau_i} \right) &= e^{\frac{t}{\tau_i}} (F_i - \alpha_i t) \\
\frac{d}{dt} \left(v_i e^{\frac{t}{\tau}} \right) &= e^{\frac{t}{\tau_i}} (F_i - \alpha_i t) \\
v_i e^{\frac{t}{\tau}} &= \int \left(e^{\frac{t}{\tau_i}} F_i - e^{\frac{t}{\tau_i}} F_i \alpha_i t \right) dt
\end{aligned}$$

Using integration by parts for the second term on the righthand side, one will find

$$v_i e^{\frac{t}{\tau_i}} = \tau_i F_i e^{\frac{t}{\tau_i}} - \alpha_i \tau_i v_i e^{\frac{t}{\tau_i}} (t - \tau_i) + k \qquad \text{(where } k \in \mathbb{R}\text{)}$$

For the initial condition $v_i(0) = 0$, one will find that

$$v_i(t) = \tau_i F_i e^{\frac{t}{\tau_i}} - \alpha_i \tau_i (t - \tau_i) - (\tau_i F_i + \alpha_i \tau_i^2) e^{\frac{-t}{\tau_i}}$$

Note that $x_i(t) = \int v_i(t) dt$. We are ignoring the shift of initial position, as well as the translation of time $t_i - t$ here for simplicity. Simple integration of v_i, while imposing an initial condition of $x_i(0) = 0$ yields

$$x_i(t) = F_i \tau_i t - \frac{\alpha_i \tau_i t^2}{2} + \alpha_i \tau_i^2 t + F_i \tau_i^2 e^{-t/\tau_i} + \alpha_i \tau_i^3 e^{-t/\tau_i} - (F_i \tau_i^2 + \alpha_i \tau_i^3)$$

One can use these expressions in place of numerical solutions in determining the solution to the model outlined in this paper, for given parameter values.

Symbol	Meaning	Units
F	Maximum propulsive force	m/sec^2
τ	Damping constant for acceleration	sec
α	Decay constant of maximum propulsive force	m/sec^3
x	Position	meters
v	Velocity	m/sec
t	Time	seconds
l	Leg time	seconds
e	Time of exchange	seconds
s	Cue distance	meters

TABLE 5. Variable symbols with meanings and associated units.