CHAPTER 14

SORTING AND SEARCHING

CHAPTER GOALS

To study several sorting and searching algorithms

To appreciate that algorithms for the same task can differ widely in performance

To understand the big-Oh notation

To estimate and compare the performance of algorithms

To write code to measure the running time of a program

CHAPTER CONTENTS

One of the most common tasks in data processing is sorting. For example, an array of employees often needs to be displayed in alphabetical order or sorted by salary. In this chapter, you will learn several sorting methods as well as techniques for comparing their performance. These techniques are useful not just for sorting algorithms, but also for analyzing other algorithms.

Once an array of elements is sorted, one can rapidly locate individual elements. You will study the *binary search* algorithm that carries out this fast lookup.

14.1 Selection Sort

In this section, we show you the first of several sorting algorithms. A *sorting algorithm* rearranges the elements of a collection so that they are stored in sorted order. To keep the examples simple, we will discuss how to sort an array of integers before going on to sorting strings or more complex data. Consider the following array `a`:

[0]	[1]	[2]	[3]	[4]
11	9	17	5	12

The selection sort algorithm sorts an array by repeatedly finding the smallest element of the unsorted tail region and moving it to the front.

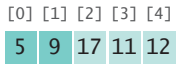
An obvious first step is to find the smallest element. In this case the smallest element is 5, stored in `a[3]`. We should move the 5 to the beginning of the array. Of course, there is already an element stored in `a[0]`, namely 11. Therefore we cannot simply move `a[3]` into `a[0]` without moving the 11 somewhere else. We don't yet know where the 11 should end up, but we know for certain that it should not be in `a[0]`. We simply get it out of the way by *swapping* it with `a[3]`:

[0]	[1]	[2]	[3]	[4]
5	9	17	11	12

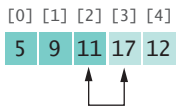
Now the first element is in the correct place. The darker color in the figure indicates the portion of the array that is already sorted.

In selection sort, pick the smallest element and swap it with the first one. Pick the smallest element of the remaining ones and swap it with the next one, and so on.

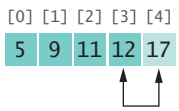
Next we take the minimum of the remaining entries `a[1] . . . a[4]`. That minimum value, 9, is already in the correct place. We don't need to do anything in this case and can simply extend the sorted area by one to the right:



Repeat the process. The minimum value of the unsorted region is 11, which needs to be swapped with the first value of the unsorted region, 17:



Now the unsorted region is only two elements long, but we keep to the same successful strategy. The minimum value is 12, and we swap it with the first value, 17:



That leaves us with an unprocessed region of length 1, but of course a region of length 1 is always sorted. We are done.

Let's program this algorithm, called **selection sort**. For this program, as well as the other programs in this chapter, we will use a utility method to generate an array with random entries. We place it into a class `ArrayUtil` so that we don't have to repeat the code in every example. To show the array, we call the static `toString` method of the `Arrays` class in the Java library and print the resulting string (see Section 7.3.4). We also add a method for swapping elements to the `ArrayUtil` class. (See Section 7.3.8 for details about swapping array elements.)

This algorithm will sort any array of integers. If speed were not an issue, or if there were no better sorting method available, we could stop the discussion of sorting right here. As the next section shows, however, this algorithm, while entirely correct, shows disappointing performance when run on a large data set.

Special Topic 14.2 discusses insertion sort, another simple sorting algorithm.

section_1/SelectionSorter.java

```
1  /**
2     The sort method of this class sorts an array, using the selection
3     sort algorithm.
4  */
5  public class SelectionSorter
6  {
7     /**
8        Sorts an array, using selection sort.
9        @param a the array to sort
10    */
11    public static void sort(int[] a)
12    {
13       for (int i = 0; i < a.length - 1; i++)
14       {
15          int minPos = minimumPosition(a, i);
16          ArrayUtil.swap(a, minPos, i);
17       }
18    }
```

```
19
20    /**
21       Finds the smallest element in a tail range of the array.
22       @param a the array to sort
23       @param from the first position in a to compare
24       @return the position of the smallest element in the
25       range a[from] . . . a[a.length - 1]
26    */
27    private static int minimumPosition(int[] a, int from)
28    {
29       int minPos = from;
30       for (int i = from + 1; i < a.length; i++)
31       {
32          if (a[i] < a[minPos]) { minPos = i; }
33       }
34       return minPos;
35    }
36 }
```

section_1/SelectionSortDemo.java

```
1  import java.util.Arrays;
2
3  /**
4     This program demonstrates the selection sort algorithm by
5     sorting an array that is filled with random numbers.
6  */
7  public class SelectionSortDemo
8  {
9     public static void main(String[] args)
10    {
11       int[] a = ArrayUtil.randomIntArray(20, 100);
12       System.out.println(Arrays.toString(a));
13
14       SelectionSorter.sort(a);
15
16       System.out.println(Arrays.toString(a));
17    }
18 }
```

section_1/ArrayUtil.java

```
1  import java.util.Random;
2
3  /**
4     This class contains utility methods for array manipulation.
5  */
6  public class ArrayUtil
7  {
8     private static Random generator = new Random();
9
10    /**
11       Creates an array filled with random values.
12       @param length the length of the array
13       @param n the number of possible random values
14       @return an array filled with length numbers between
15       0 and n - 1
16    */
17    public static int[] randomIntArray(int length, int n)
18    {
```

```
19          int[] a = new int[length];
20          for (int i = 0; i < a.length; i++)
21          {
22             a[i] = generator.nextInt(n);
23          }
24
25          return a;
26       }
27
28       /**
29          Swaps two entries of an array.
30          @param a the array
31          @param i the first position to swap
32          @param j the second position to swap
33       */
34       public static void swap(int[] a, int i, int j)
35       {
36          int temp = a[i];
37          a[i] = a[j];
38          a[j] = temp;
39       }
40    }
```

Program Run

```
[65, 46, 14, 52, 38, 2, 96, 39, 14, 33, 13, 4, 24, 99, 89, 77, 73, 87, 36, 81]
[2, 4, 13, 14, 14, 24, 33, 36, 38, 39, 46, 52, 65, 73, 77, 81, 87, 89, 96, 99]
```

1. Why do we need the `temp` variable in the `swap` method? What would happen if you simply assigned `a[i]` to `a[j]` and `a[j]` to `a[i]`?
2. What steps does the selection sort algorithm go through to sort the sequence 6 5 4 3 2 1?
3. How can you change the selection sort algorithm so that it sorts the elements in descending order (that is, with the largest element at the beginning of the array)?
4. Suppose we modified the selection sort algorithm to start at the end of the array, working toward the beginning. In each step, the current position is swapped with the minimum. What is the result of this modification?

Practice It Now you can try these exercises at the end of the chapter: R14.2, R14.10, E14.1, E14.2.

14.2 Profiling the Selection Sort Algorithm

To measure the performance of a program, you could simply run it and use a stopwatch to measure how long it takes. However, most of our programs run very quickly, and it is not easy to time them accurately in this way. Furthermore, when a program takes a noticeable time to run, a certain amount of that time may simply be used for loading the program from disk into memory and displaying the result (for which we should not penalize it).

In order to measure the running time of an algorithm more accurately, we will create a `StopWatch` class. This class works like a real stopwatch. You can start it, stop

it, and read out the elapsed time. The class uses the `System.currentTimeMillis` method, which returns the milliseconds that have elapsed since midnight at the start of January 1, 1970. Of course, you don't care about the absolute number of seconds since this historical moment, but the *difference* of two such counts gives us the number of milliseconds in a given time interval.

Here is the code for the `StopWatch` class:

section_2/StopWatch.java

```
/**
   A stopwatch accumulates time when it is running. You can
   repeatedly start and stop the stopwatch. You can use a
   stopwatch to measure the running time of a program.
*/
public class StopWatch
{
   private long elapsedTime;
   private long startTime;
   private boolean isRunning;

   /**
      Constructs a stopwatch that is in the stopped state
      and has no time accumulated.
   */
   public StopWatch()
   {
      reset();
   }

   /**
      Starts the stopwatch. Time starts accumulating now.
   */
   public void start()
   {
      if (isRunning) { return; }
      isRunning = true;
      startTime = System.currentTimeMillis();
   }

   /**
      Stops the stopwatch. Time stops accumulating and is
      is added to the elapsed time.
   */
   public void stop()
   {
      if (!isRunning) { return; }
      isRunning = false;
      long endTime = System.currentTimeMillis();
      elapsedTime = elapsedTime + endTime - startTime;
   }

   /**
      Returns the total elapsed time.
      @return the total elapsed time
   */
   public long getElapsedTime()
   {
      if (isRunning)
      {
```

```
51          long endTime = System.currentTimeMillis();
52          return elapsedTime + endTime - startTime;
53       }
54       else
55       {
56          return elapsedTime;
57       }
58    }
59
60    /**
61       Stops the watch and resets the elapsed time to 0.
62    */
63    public void reset()
64    {
65       elapsedTime = 0;
66       isRunning = false;
67    }
68 }
```

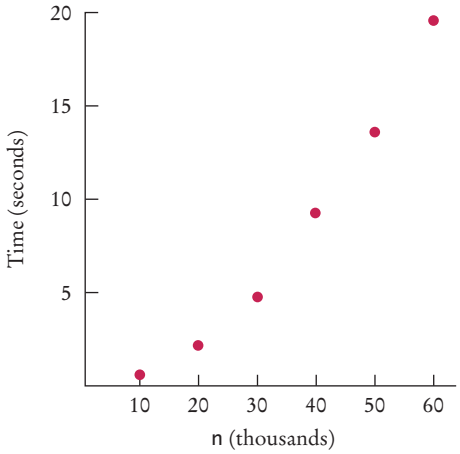
Here is how to use the stopwatch to measure the sorting algorithm's performance:

section_2/SelectionSortTimer.java

```
1  import java.util.Scanner;
2
3  /**
4     This program measures how long it takes to sort an
5     array of a user-specified size with the selection
6     sort algorithm.
7  */
8  public class SelectionSortTimer
9  {
10    public static void main(String[] args)
11    {
12       Scanner in = new Scanner(System.in);
13       System.out.print("Enter array size: ");
14       int n = in.nextInt();
15
16       // Construct random array
17
18       int[] a = ArrayUtil.randomIntArray(n, 100);
19
20       // Use stopwatch to time selection sort
21
22       StopWatch timer = new StopWatch();
23
24       timer.start();
25       SelectionSorter.sort(a);
26       timer.stop();
27
28       System.out.println("Elapsed time: "
29             + timer.getElapsedTime() + " milliseconds");
30    }
31 }
```

Program Run

```
Enter array size: 50000
Elapsed time: 13321 milliseconds
```



n	Milliseconds
10,000	786
20,000	2,148
30,000	4,796
40,000	9,192
50,000	13,321
60,000	19,299

Figure 1 Time Taken by Selection Sort

To measure the running time of a method, get the current time immediately before and after the method call.

By starting to measure the time just before sorting, and stopping the stopwatch just after, you get the time required for the sorting process, without counting the time for input and output.

The table in Figure 1 shows the results of some sample runs. These measurements were obtained with an Intel processor with a clock speed of 2 GHz, running Java 6 on the Linux operating system. On another computer the actual numbers will look different, but the relationship between the numbers will be the same.

The graph in Figure 1 shows a plot of the measurements. As you can see, when you double the size of the data set, it takes about four times as long to sort it.

5. Approximately how many seconds would it take to sort a data set of 80,000 values?
6. Look at the graph in Figure 1. What mathematical shape does it resemble?

Practice It Now you can try these exercises at the end of the chapter: E14.3, E14.6.

14.3 Analyzing the Performance of the Selection Sort Algorithm

Let us count the number of operations that the program must carry out to sort an array with the selection sort algorithm. We don't actually know how many machine operations are generated for each Java instruction, or which of those instructions are more time-consuming than others, but we can make a simplification. We will simply count how often an array element is *visited*. Each visit requires about the same amount of work by other operations, such as incrementing subscripts and comparing values.

Let n be the size of the array. First, we must find the smallest of n numbers. To achieve that, we must visit n array elements. Then we swap the elements, which takes

two visits. (You may argue that there is a certain probability that we don't need to swap the values. That is true, and one can refine the computation to reflect that observation. As we will soon see, doing so would not affect the overall conclusion.) In the next step, we need to visit only $n - 1$ elements to find the minimum. In the following step, $n - 2$ elements are visited to find the minimum. The last step visits two elements to find the minimum. Each step requires two visits to swap the elements. Therefore, the total number of visits is

$$\begin{aligned} n + 2 + (n - 1) + 2 + \cdots + 2 + 2 &= n + (n - 1) + \cdots + 2 + (n - 1) \cdot 2 \\ &= 2 + \cdots + (n - 1) + n + (n - 1) \cdot 2 \\ &= \frac{n(n + 1)}{2} - 1 + (n - 1) \cdot 2 \end{aligned}$$

because

$$1 + 2 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$$

After multiplying out and collecting terms of n, we find that the number of visits is

$$\frac{1}{2}n^2 + \frac{5}{2}n - 3$$

We obtain a quadratic equation in n. That explains why the graph of Figure 1 looks approximately like a parabola.

Now simplify the analysis further. When you plug in a large value for n (for example, 1,000 or 2,000), then $\frac{1}{2}n^2$ is 500,000 or 2,000,000. The lower term, $\frac{5}{2}n - 3$, doesn't contribute much at all; it is only 2,497 or 4,997, a drop in the bucket compared to the hundreds of thousands or even millions of comparisons specified by the $\frac{1}{2}n^2$ term. We will just ignore these lower-level terms. Next, we will ignore the constant factor $\frac{1}{2}$. We are not interested in the actual count of visits for a single n. We want to compare the ratios of counts for different values of n. For example, we can say that sorting an array of 2,000 numbers requires four times as many visits as sorting an array of 1,000 numbers:

$$\frac{\left(\frac{1}{2} \cdot 2000^2\right)}{\left(\frac{1}{2} \cdot 1000^2\right)} = 4$$

The factor $\frac{1}{2}$ cancels out in comparisons of this kind. We will simply say, "The number of visits is of order n^2." That way, we can easily see that the number of comparisons increases fourfold when the size of the array doubles: $(2n)^2 = 4n^2$.

Computer scientists use the big-Oh notation to describe the growth rate of a function.

To indicate that the number of visits is of order n^2, computer scientists often use **big-Oh notation**: The number of visits is $O(n^2)$. This is a convenient shorthand. (See Special Topic 14.1 for a formal definition.)

To turn a polynomial expression such as

$$\frac{1}{2}n^2 + \frac{5}{2}n - 3$$

into big-Oh notation, simply locate the fastest-growing term, n^2, and ignore its constant coefficient, no matter how large or small it may be.

We observed before that the actual number of machine operations, and the actual amount of time that the computer spends on them, is approximately proportional to the number of element visits. Maybe there are about 10 machine operations

(increments, comparisons, memory loads, and stores) for every element visit. The number of machine operations is then approximately $10 \times \frac{1}{2}n^2$. As before, we aren't interested in the coefficient, so we can say that the number of machine operations, and hence the time spent on the sorting, is of the order n^2 or $O(n^2)$.

Selection sort is an $O(n^2)$ algorithm. Doubling the data set means a fourfold increase in processing time.

The sad fact remains that doubling the size of the array causes a fourfold increase in the time required for sorting it with selection sort. When the size of the array increases by a factor of 100, the sorting time increases by a factor of 10,000. To sort an array of a million entries (for example, to create a telephone directory), takes 10,000 times as long as sorting 10,000 entries. If 10,000 entries can be sorted in about 3/4 of a second (as in our example), then sorting one million entries requires well over two hours. We will see in the next section how one can dramatically improve the performance of the sorting process by choosing a more sophisticated algorithm.

7. If you increase the size of a data set tenfold, how much longer does it take to sort it with the selection sort algorithm?
8. How large does n need to be so that $\frac{1}{2}n^2$ is bigger than $\frac{5}{2}n - 3$?
9. Section 7.3.6 has two algorithms for removing an element from an array of length n. How many array visits does each algorithm require on average?
10. Describe the number of array visits in Self Check 9 using the big-Oh notation.
11. What is the big-Oh running time of checking whether an array is already sorted?
12. Consider this algorithm for sorting an array. Set k to the length of the array. Find the maximum of the first k elements. Remove it, using the second algorithm of Section 7.3.6. Decrement k and place the removed element into the kth position. Stop if k is 1. What is the algorithm's running time in big-Oh notation?

Practice It Now you can try these exercises at the end of the chapter: R14.4, R14.6, R14.8.

Special Topic 14.1

Oh, Omega, and Theta

We have used the big-Oh notation somewhat casually in this chapter to describe the growth behavior of a function. Here is the formal definition of the big-Oh notation: Suppose we have a function $T(n)$. Usually, it represents the processing time of an algorithm for a given input of size n. But it could be any function. Also, suppose that we have another function $f(n)$. It is usually chosen to be a simple function, such as $f(n) = n^k$ or $f(n) = \log(n)$, but it too can be any function. We write

$$T(n) = O(f(n))$$

if $T(n)$ grows at a rate that is bounded by $f(n)$. More formally, we require that for all n larger than some threshold, the ratio $T(n)/f(n) \leq C$ for some constant value C.

If $T(n)$ is a polynomial of degree k in n, then one can show that $T(n) = O(n^k)$. Later in this chapter, we will encounter functions that are $O(\log(n))$ or $O(n \log(n))$. Some algorithms take much more time. For example, one way of sorting a sequence is to compute all of its permutations, until you find one that is in increasing order. Such an algorithm takes $O(n!)$ time, which is very bad indeed.

Table 1 shows common big-Oh expressions, sorted by increasing growth.

Strictly speaking, $T(n) = O(f(n))$ means that T grows no faster than f. But it is permissible for T to grow much more slowly. Thus, it is technically correct to state that $T(n) = n^2 + 5n - 3$ is $O(n^3)$ or even $O(n^{10})$.

Table 1 Common Big-Oh Growth Rates

Big-Oh Expression	Name
$O(1)$	Constant
$O(\log(n))$	Logarithmic
$O(n)$	Linear
$O(n \log(n))$	Log-linear
$O(n^2)$	Quadratic
$O(n^3)$	Cubic
$O(2^n)$	Exponential
$O(n!)$	Factorial

Computer scientists have invented additional notation to describe the growth behavior of functions more accurately. The expression

$$T(n) = \Omega(f(n))$$

means that T grows at least as fast as f, or, formally, that for all n larger than some threshold, the ratio $T(n)/f(n) \geq C$ for some constant value C. (The Ω symbol is the capital Greek letter omega.) For example, $T(n) = n^2 + 5n - 3$ is $\Omega(n^2)$ or even $\Omega(n)$.

The expression

$$T(n) = \Theta(f(n))$$

means that T and f grow at the same rate—that is, both $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$ hold. (The Θ symbol is the capital Greek letter theta.)

The Θ notation gives the most precise description of growth behavior. For example, $T(n) = n^2 + 5n - 3$ is $\Theta(n^2)$ but not $\Theta(n)$ or $\Theta(n^3)$.

The notations are very important for the precise analysis of algorithms. However, in casual conversation it is common to stick with big-Oh, while still giving an estimate as good as one can make.

Special Topic 14.2

Insertion Sort

Insertion sort is another simple sorting algorithm. In this algorithm, we assume that the initial sequence

```
a[0] a[1] . . . a[k]
```

of an array is already sorted. (When the algorithm starts, we set `k` to `0`.) We enlarge the initial sequence by inserting the next array element, `a[k + 1]`, at the proper location. When we reach the end of the array, the sorting process is complete.

For example, suppose we start with the array

11	9	16	5	7

Of course, the initial sequence of length 1 is already sorted. We now add `a[1]`, which has the value 9. The element needs to be inserted before the element 11. The result is

9	11	16	5	7

Next, we add a[2], which has the value 16. This element does not have to be moved.

We repeat the process, inserting a[3] or 5 at the very beginning of the initial sequence.

Finally, a[4] or 7 is inserted in its correct position, and the sorting is completed.

The following class implements the insertion sort algorithm:

```
public class InsertionSorter
{
   /**
      Sorts an array, using insertion sort.
      @param a the array to sort
   */
   public static void sort(int[] a)
   {
      for (int i = 1; i < a.length; i++)
      {
         int next = a[i];
         // Move all larger elements up
         int j = i;
         while (j > 0 && a[j - 1] > next)
         {
            a[j] = a[j - 1];
            j--;
         }
         // Insert the element
         a[j] = next;
      }
   }
}
```

How efficient is this algorithm? Let n denote the size of the array. We carry out $n - 1$ iterations. In the kth iteration, we have a sequence of k elements that is already sorted, and we need to insert a new element into the sequence. For each insertion, we need to visit the elements of the initial sequence until we have found the location in which the new element can be inserted. Then we need to move up the remaining elements of the sequence. Thus, $k + 1$ array elements are visited. Therefore, the total number of visits is

$$2 + 3 + \cdots + n = \frac{n(n+1)}{2} - 1$$

FULL CODE EXAMPLE

Go to wiley.com/go/javacode to download a program that illustrates sorting with insertion sort.

We conclude that insertion sort is an $O(n^2)$ algorithm, on the same order of efficiency as selection sort.

Insertion sort is an $O(n^2)$ algorithm.

Insertion sort has a desirable property: Its performance is $O(n)$ if the array is already sorted—see Exercise R14.17. This is a useful property in practical applications, in which data sets are often partially sorted.

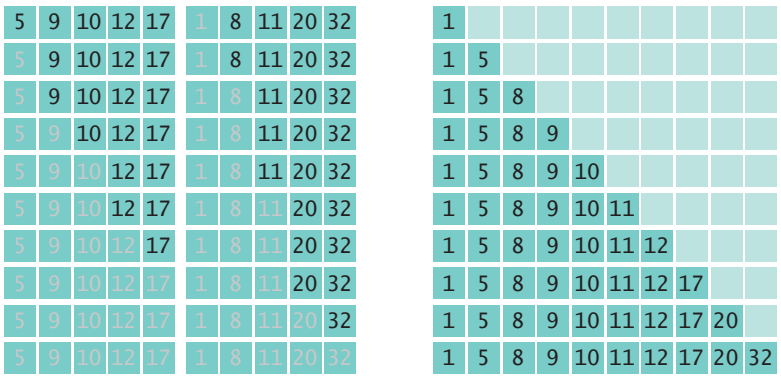
Insertion sort is the method that many people use to sort playing cards. Pick up one card at a time and insert it so that the cards stay sorted.

14.4 Merge Sort

In this section, you will learn about the **merge sort** algorithm, a much more efficient algorithm than selection sort. The basic idea behind merge sort is very simple.

Suppose we have an array of 10 integers. Let us engage in a bit of wishful thinking and hope that the first half of the array is already perfectly sorted, and the second half is too, like this:

Now it is simple to *merge* the two sorted arrays into one sorted array, by taking a new element from either the first or the second subarray, and choosing the smaller of the elements each time:



In fact, you may have performed this merging before if you and a friend had to sort a pile of papers. You and the friend split the pile in half, each of you sorted your half, and then you merged the results together.

The merge sort algorithm sorts an array by cutting the array in half, recursively sorting each half, and then merging the sorted halves.

That is all well and good, but it doesn't seem to solve the problem for the computer. It still must sort the first and second halves of the array, because it can't very well ask a few buddies to pitch in. As it turns out, though, if the computer keeps dividing the array into smaller and smaller subarrays, sorting each half and merging them back together, it carries out dramatically fewer steps than the selection sort requires.

In merge sort, one sorts each half, then merges the sorted halves.

Let's write a `MergeSorter` class that implements this idea. When the `MergeSorter` sorts an array, it makes two arrays, each half the size of the original, and sorts them recursively. Then it merges the two sorted arrays together:

```
public static void sort(int[] a)
{
   if (a.length <= 1) { return; }
   int[] first = new int[a.length / 2];
   int[] second = new int[a.length - first.length];
   // Copy the first half of a into first, the second half into second
   . . .
   sort(first);
   sort(second);
   merge(first, second, a);
}
```

The `merge` method is tedious but quite straightforward. You will find it in the code that follows.

section_4/MergeSorter.java

```
/**
   The sort method of this class sorts an array, using the merge
   sort algorithm.
*/
public class MergeSorter
{
   /**
      Sorts an array, using merge sort.
      @param a the array to sort
   */
   public static void sort(int[] a)
   {
      if (a.length <= 1) { return; }
      int[] first = new int[a.length / 2];
      int[] second = new int[a.length - first.length];
      // Copy the first half of a into first, the second half into second
      for (int i = 0; i < first.length; i++)
      {
         first[i] = a[i];
      }
      for (int i = 0; i < second.length; i++)
      {
         second[i] = a[first.length + i];
      }
      sort(first);
      sort(second);
      merge(first, second, a);
   }

   /**
      Merges two sorted arrays into an array.
      @param first the first sorted array
      @param second the second sorted array
      @param a the array into which to merge first and second
   */
   private static void merge(int[] first, int[] second, int[] a)
   {
      int iFirst = 0;  // Next element to consider in the first array
      int iSecond = 0;  // Next element to consider in the second array
      int j = 0;  // Next open position in a

      // As long as neither iFirst nor iSecond past the end, move
      // the smaller element into a
      while (iFirst < first.length && iSecond < second.length)
      {
         if (first[iFirst] < second[iSecond])
         {
            a[j] = first[iFirst];
            iFirst++;
         }
         else
         {
            a[j] = second[iSecond];
            iSecond++;
```

```
55            }
56            j++;
57         }
58
59         // Note that only one of the two loops below copies entries
60         // Copy any remaining entries of the first array
61         while (iFirst < first.length)
62         {
63            a[j] = first[iFirst];
64            iFirst++; j++;
65         }
66         // Copy any remaining entries of the second half
67         while (iSecond < second.length)
68         {
69            a[j] = second[iSecond];
70            iSecond++; j++;
71         }
72      }
73   }
```

section_4/MergeSortDemo.java

```
1  import java.util.Arrays;
2
3  /**
4     This program demonstrates the merge sort algorithm by
5     sorting an array that is filled with random numbers.
6  */
7  public class MergeSortDemo
8  {
9     public static void main(String[] args)
10    {
11       int[] a = ArrayUtil.randomIntArray(20, 100);
12       System.out.println(Arrays.toString(a));
13
14       MergeSorter.sort(a);
15
16       System.out.println(Arrays.toString(a));
17    }
18 }
```

Program Run

```
[8, 81, 48, 53, 46, 70, 98, 42, 27, 76, 33, 24, 2, 76, 62, 89, 90, 5, 13, 21]
[2, 5, 8, 13, 21, 24, 27, 33, 42, 46, 48, 53, 62, 70, 76, 76, 81, 89, 90, 98]
```

13. Why does only one of the two `while` loops at the end of the `merge` method do any work?
14. Manually run the merge sort algorithm on the array 8 7 6 5 4 3 2 1.
15. The merge sort algorithm processes an array by recursively processing two halves. Describe a similar recursive algorithm for computing the sum of all elements in an array.

Practice It Now you can try these exercises at the end of the chapter: R14.11, E14.4, E14.11.

14.5 Analyzing the Merge Sort Algorithm

The merge sort algorithm looks a lot more complicated than the selection sort algorithm, and it appears that it may well take much longer to carry out these repeated subdivisions. However, the timing results for merge sort look much better than those for selection sort.

Figure 2 shows a table and a graph comparing both sets of perfor0mance data. As you can see, merge sort is a tremendous improvement. To understand why, let us estimate the number of array element visits that are required to sort an array with the merge sort algorithm. First, let us tackle the merge process that happens after the first and second halves have been sorted.

Each step in the merge process adds one more element to `a`. That element may come from `first` or `second`, and in most cases the elements from the two halves must be compared to see which one to take. We'll count that as 3 visits (one for a and one each for `first` and `second`) per element, or $3n$ visits total, where n denotes the length of `a`. Moreover, at the beginning, we had to copy from `a` to `first` and `second`, yielding another $2n$ visits, for a total of $5n$.

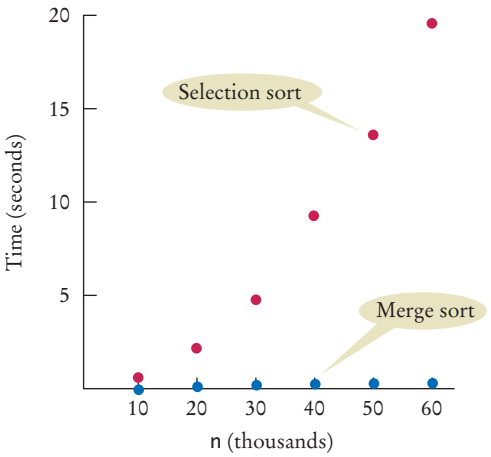
If we let $T(n)$ denote the number of visits required to sort a range of n elements through the merge sort process, then we obtain

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + 5n$$

because sorting each half takes $T(n/2)$ visits. Actually, if n is not even, then we have one subarray of size $(n - 1)/2$ and one of size $(n + 1)/2$. Although it turns out that this detail does not affect the outcome of the computation, we will nevertheless assume for now that n is a power of 2, say $n = 2^m$. That way, all subarrays can be evenly divided into two parts.

Unfortunately, the formula

$$T(n) = 2T\left(\frac{n}{2}\right) + 5n$$



n	Merge Sort (milliseconds)	Selection Sort (milliseconds)
10,000	40	786
20,000	73	2,148
30,000	134	4,796
40,000	170	9,192
50,000	192	13,321
60,000	205	19,299

Figure 2 Time Taken by Selection Sort

does not clearly tell us the relationship between n and $T(n)$. To understand the relationship, let us evaluate $T(n/2)$, using the same formula:

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + 5\frac{n}{2}$$

Therefore

$$T(n) = 2 \times 2T\left(\frac{n}{4}\right) + 5n + 5n$$

Let us do that again:

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + 5\frac{n}{4}$$

hence

$$T(n) = 2 \times 2 \times 2T\left(\frac{n}{8}\right) + 5n + 5n + 5n$$

This generalizes from 2, 4, 8, to arbitrary powers of 2:

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 5nk$$

Recall that we assume that $n = 2^m$; hence, for $k = m$,

$$\begin{aligned} T(n) &= 2^m T\left(\frac{n}{2^m}\right) + 5nm \\ &= nT(1) + 5nm \\ &= n + 5n\log_2(n) \end{aligned}$$

Because $n = 2^m$, we have $m = \log_2(n)$.

To establish the growth order, we drop the lower-order term n and are left with $5n\log_2(n)$. We drop the constant factor 5. It is also customary to drop the base of the logarithm, because all logarithms are related by a constant factor. For example,

$$\log_2(x) = \log_{10}(x)/\log_{10}(2) \approx \log_{10}(x) \times 3.32193$$

Hence we say that merge sort is an $O(n\log(n))$ algorithm.

Merge sort is an $O(n\log(n))$ algorithm. The $n\log(n)$ function grows much more slowly than n^2.

Is the $O(n\log(n))$ merge sort algorithm better than the $O(n^2)$ selection sort algorithm? You bet it is. Recall that it took $100^2 = 10{,}000$ times as long to sort a million records as it took to sort 10,000 records with the $O(n^2)$ algorithm. With the $O(n\log(n))$ algorithm, the ratio is

$$\frac{1{,}000{,}000\log(1{,}000{,}000)}{10{,}000\log(10{,}000)} = 100\left(\frac{6}{4}\right) = 150$$

Suppose for the moment that merge sort takes the same time as selection sort to sort an array of 10,000 integers, that is, 3/4 of a second on the test machine. (Actually, it is much faster than that.) Then it would take about 0.75×150 seconds, or under two minutes, to sort a million integers. Contrast that with selection sort, which would take over two hours for the same task. As you can see, even if it takes you several hours to learn about a better algorithm, that can be time well spent.

FULL CODE EXAMPLE

Go to `wiley.com/go/javacode` to download a program for timing the merge sort algorithm.

In this chapter we have barely begun to scratch the surface of this interesting topic. There are many sorting algorithms, some with even better performance than merge sort, and the analysis of these algorithms can be quite challenging. These important issues are often revisited in later computer science courses.

SELF CHECK

16. Given the timing data for the merge sort algorithm in the table at the beginning of this section, how long would it take to sort an array of 100,000 values?

17. If you double the size of an array, how much longer will the merge sort algorithm take to sort the new array?

Practice It Now you can try these exercises at the end of the chapter: R14.7, R14.14, R14.16.

Special Topic 14.3

The Quicksort Algorithm

Quicksort is a commonly used algorithm that has the advantage over merge sort that no temporary arrays are required to sort and merge the partial results.

The quicksort algorithm, like merge sort, is based on the strategy of divide and conquer. To sort a range `a[from] . . . a[to]` of the array `a`, first rearrange the elements in the range so that no element in the range `a[from] . . . a[p]` is larger than any element in the range `a[p + 1] . . . a[to]`. This step is called *partitioning* the range.

For example, suppose we start with a range

5	3	2	6	4	1	3	7

Here is a partitioning of the range. Note that the partitions aren't yet sorted.

3	3	2	1	4	6	5	7

You'll see later how to obtain such a partition. In the next step, sort each partition, by recursively applying the same algorithm to the two partitions. That sorts the entire range, because the largest element in the first partition is at most as large as the smallest element in the second partition.

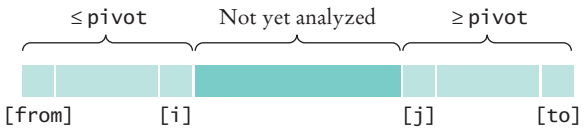
1	2	3	3	4	5	6	7

Quicksort is implemented recursively as follows:

```
public static void sort(int[] a, int from, int to)
{
   if (from >= to) { return; }
   int p = partition(a, from, to);
   sort(a, from, p);
   sort(a, p + 1, to);
}
```

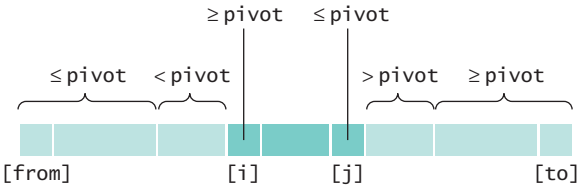
Let us return to the problem of partitioning a range. Pick an element from the range and call it the *pivot*. There are several variations of the quicksort algorithm. In the simplest one, we'll pick the first element of the range, `a[from]`, as the pivot.

Now form two regions `a[from] . . . a[i]`, consisting of values at most as large as the pivot and `a[j] . . . a[to]`, consisting of values at least as large as the pivot. The region `a[i + 1] . . . a[j - 1]` consists of values that haven't been analyzed yet. (See the figure below.) At the beginning, both the left and right areas are empty; that is, `i = from - 1` and `j = to + 1`.



Partitioning a Range

Then keep incrementing `i` while `a[i] < pivot` and keep decrementing `j` while `a[j] > pivot`. The figure below shows `i` and `j` when that process stops.



Extending the Partitions

Now swap the values in positions `i` and `j`, increasing both areas once more. Keep going while `i < j`. Here is the code for the `partition` method:

```
private static int partition(int[] a, int from, int to)
{
   int pivot = a[from];
   int i = from - 1;
   int j = to + 1;
   while (i < j)
   {
      i++; while (a[i] < pivot) { i++; }
      j--; while (a[j] > pivot) { j--; }
      if (i < j) { ArrayUtil.swap(a, i, j); }
   }
   return j;
}
```

On average, the quicksort algorithm is an $O(n \log(n))$ algorithm. There is just one unfortunate aspect to the quicksort algorithm. Its *worst-case* run-time behavior is $O(n^2)$. Moreover, if the pivot element is chosen as the first element of the region, that worst-case behavior occurs when the input set is already sorted—a common situation in practice. By selecting the pivot element more cleverly, we can make it extremely unlikely for the worst-case behavior to occur. Such "tuned" quicksort algorithms are commonly used because their performance is generally excellent. For example, the `sort` method in the `Arrays` class uses a quicksort algorithm.

Another improvement that is commonly made in practice is to switch to insertion sort when the array is short, because the total number of operations using insertion sort is lower for short arrays. The Java library makes that switch if the array length is less than seven.

FULL CODE EXAMPLE

Go to `wiley.com/go/javacode` to download a program that demonstrates the quicksort algorithm.

In quicksort, one partitions the elements into two groups, holding the smaller and larger elements. Then one sorts each group.

14.6 Searching

Searching for an element in an array is an extremely common task. As with sorting, the right choice of algorithms can make a big difference.

14.6.1 Linear Search

Suppose you need to find your friend's telephone number. You look up the friend's name in the telephone book, and naturally you can find it quickly, because the telephone book is sorted alphabetically. Now suppose you have a telephone number and you must know to what party it belongs. You could of course call that number, but suppose nobody picks up on the other end. You could look through the telephone book, a number at a time, until you find the number. That would obviously be a tremendous amount of work, and you would have to be desperate to attempt it.

This thought experiment shows the difference between a search through an unsorted data set and a search through a sorted data set. The following two sections will analyze the difference formally.

A linear search examines all values in an array until it finds a match or reaches the end.

If you want to find a number in a sequence of values that occur in arbitrary order, there is nothing you can do to speed up the search. You must simply look through all elements until you have found a match or until you reach the end. This is called a **linear** or **sequential search**.

A linear search locates a value in an array in $O(n)$ steps.

How long does a linear search take? If we assume that the element `v` is present in the array `a`, then the average search visits $n/2$ elements, where n is the length of the array. If it is not present, then all n elements must be inspected to verify the absence. Either way, a linear search is an $O(n)$ algorithm.

Here is a class that performs linear searches through an array `a` of integers. When searching for a value, the `search` method returns the first index of the match, or `-1` if the value does not occur in `a`.

section_6_1/LinearSearcher.java

```
1  /**
2     A class for executing linear searches in an array.
3  */
4  public class LinearSearcher
5  {
6     /**
7        Finds a value in an array, using the linear search
8        algorithm.
9        @param a the array to search
10       @param value the value to find
11       @return the index at which the value occurs, or -1
12       if it does not occur in the array
13    */
14    public static int search(int[] a, int value)
15    {
16       for (int i = 0; i < a.length; i++)
17       {
18          if (a[i] == value) { return i; }
19       }
20       return -1;
```

```
21    }
22 }
```

section_6_1/LinearSearchDemo.java

```
1  import java.util.Arrays;
2  import java.util.Scanner;
3
4  /**
5     This program demonstrates the linear search algorithm.
6  */
7  public class LinearSearchDemo
8  {
9     public static void main(String[] args)
10    {
11       int[] a = ArrayUtil.randomIntArray(20, 100);
12       System.out.println(Arrays.toString(a));
13       Scanner in = new Scanner(System.in);
14
15       boolean done = false;
16       while (!done)
17       {
18          System.out.print("Enter number to search for, -1 to quit: ");
19          int n = in.nextInt();
20          if (n == -1)
21          {
22             done = true;
23          }
24          else
25          {
26             int pos = LinearSearcher.search(a, n);
27             System.out.println("Found in position " + pos);
28          }
29       }
30    }
31 }
```

Program Run

```
[46, 99, 45, 57, 64, 95, 81, 69, 11, 97, 6, 85, 61, 88, 29, 65, 83, 88, 45, 88]
Enter number to search for, -1 to quit: 12
Found in position -1
Enter number to search for, -1 to quit: -1
```

14.6.2 Binary Search

Now let us search for an item in a data sequence that has been previously sorted. Of course, we could still do a linear search, but it turns out we can do much better than that.

Consider the following sorted array a. The data set is:

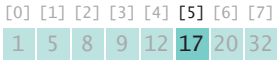
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
1	5	8	9	12	17	20	32

We would like to see whether the value 15 is in the data set. Let's narrow our search by finding whether the value is in the first or second half of the array. The last value

in the first half of the data set, `a[3]`, is 9, which is smaller than the value we are looking for. Hence, we should look in the second half of the array for a match, that is, in the sequence:

Now the last value of the first half of this sequence is 17; hence, the value must be located in the sequence:

The last value of the first half of this very short sequence is 12, which is smaller than the value that we are searching, so we must look in the second half:



A binary search locates a value in a sorted array by determining whether the value occurs in the first or second half, then repeating the search in one of the halves.

It is trivial to see that we don't have a match, because 15 ≠ 17. If we wanted to insert 15 into the sequence, we would need to insert it just before `a[5]`.

This search process is called a **binary search**, because we cut the size of the search in half in each step. That cutting in half works only because we know that the sequence of values is sorted.

The following class implements binary searches in a sorted array of integers. The `search` method returns the position of the match if the search succeeds, or `-1` if the value is not found in `a`. Here, we show a recursive version of the binary search algorithm.

section_6_2/BinarySearcher.java

```
1  /**
2     A class for executing binary searches in an array.
3  */
4  public class BinarySearcher
5  {
6     /**
7        Finds a value in a range of a sorted array, using the binary
8        search algorithm.
9        @param a the array in which to search
10       @param low the low index of the range
11       @param high the high index of the range
12       @param value the value to find
13       @return the index at which the value occurs, or -1
14       if it does not occur in the array
15    */
16    public int search(int[] a, int low, int high, int value)
17    {
18       if (low <= high)
19       {
20          int mid = (low + high) / 2;
21
22          if (a[mid] == value)
23          {
24             return mid;
25          }
26          else if (a[mid] < value )
27          {
```

```
28             return search(a, mid + 1, high, value);
29          }
30          else
31          {
32             return search(a, low, mid - 1, value);
33          }
34       }
35       else
36       {
37          return -1;
38       }
39    }
40 }
```

Now let's determine the number of visits to array elements required to carry out a binary search. We can use the same technique as in the analysis of merge sort. Because we look at the middle element, which counts as one visit, and then search either the left or the right subarray, we have

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

Using the same equation,

$$T\left(\frac{n}{2}\right) = T\left(\frac{n}{4}\right) + 1$$

By plugging this result into the original equation, we get

$$T(n) = T\left(\frac{n}{4}\right) + 2$$

That generalizes to

$$T(n) = T\left(\frac{n}{2^k}\right) + k$$

As in the analysis of merge sort, we make the simplifying assumption that n is a power of 2, $n = 2^m$, where $m = \log_2(n)$. Then we obtain

$$T(n) = 1 + \log_2(n)$$

Therefore, binary search is an $O(\log(n))$ algorithm.

A binary search locates a value in a sorted array in $O(\log(n))$ steps.

That result makes intuitive sense. Suppose that n is 100. Then after each search, the size of the search range is cut in half, to 50, 25, 12, 6, 3, and 1. After seven comparisons we are done. This agrees with our formula, because $\log_2(100) \approx 6.64386$, and indeed the next larger power of 2 is $2^7 = 128$.

Because a binary search is so much faster than a linear search, is it worthwhile to sort an array first and then use a binary search? It depends. If you search the array only once, then it is more efficient to pay for an $O(n)$ linear search than for an $O(n \log(n))$ sort and an $O(\log(n))$ binary search. But if you will be making many searches in the same array, then sorting it is definitely worthwhile.

18. Suppose you need to look through 1,000,000 records to find a telephone number. How many records do you expect to search before finding the number?

19. Why can't you use a "for each" loop `for (int element : a)` in the `search` method?

20. Suppose you need to look through a sorted array with 1,000,000 elements to find a value. Using the binary search algorithm, how many records do you expect to search before finding the value?

Practice It Now you can try these exercises at the end of the chapter: R14.12, E14.10, E14.12.

Computing & Society 14.1 The First Programmer

Before pocket calculators and personal computers existed, navigators and engineers used mechanical adding machines, slide rules, and tables of logarithms and trigonometric functions to speed up computations. Unfortunately, the tables—for which values had to be computed by hand—were notoriously inaccurate. The mathematician Charles Babbage (1791–1871) had the insight that if a machine could be constructed that produced printed tables automatically, both calculation and typesetting errors could be avoided. Babbage set out to develop a machine for this purpose, which he called a *Difference Engine* because it used successive differences to compute polynomials. For example, consider the function $f(x) = x^3$. Write down the values for $f(1)$, $f(2)$, $f(3)$, and so on. Then take the *differences* between successive values:

```
1
    7
8
    19
27
    37
64
    61
125
    91
216
```

Repeat the process, taking the difference of successive values in the second column, and then repeat once again:

```
1
    7
8       12
    19      6
27      18
    37      6
64      24
    61      6
125     30
    91
216
```

Now the differences are all the same. You can retrieve the function values by a pattern of additions—you need to know the values at the fringe of the pattern and the constant difference. You can try it out yourself: Write the highlighted numbers on a sheet of paper and fill in the others by adding the numbers that are in the north and northwest positions.

Replica of Babbage's Difference Engine

This method was very attractive, because mechanical addition machines had been known for some time. They consisted of cog wheels, with 10 cogs per wheel, to represent digits, and mechanisms to handle the carry from one digit to the next. Mechanical multiplication machines, on the other hand, were fragile and unreliable. Babbage built a successful prototype of the Difference Engine and, with his own money and government grants, proceeded to build the table-printing machine. However, because of funding problems and the difficulty of building the machine to the required precision, it was never completed.

While working on the Difference Engine, Babbage conceived of a much grander vision that he called the *Analytical Engine*. The Difference Engine was designed to carry out a limited set of computations—it was no smarter than a pocket calculator is today. But Babbage realized that such a machine could be made *programmable* by storing programs as well as data. The internal storage of the Analytical Engine was to consist of 1,000 registers of 50 decimal digits each. Programs and constants were to be stored on punched cards—a technique that was, at that time, commonly used on looms for weaving patterned fabrics.

Ada Augusta, Countess of Lovelace (1815–1852), the only child of Lord Byron, was a friend and sponsor of Charles Babbage. Ada Lovelace was one of the first people to realize the potential of such a machine, not just for computing mathematical tables but for processing data that were not numbers. She is considered by many to be the world's first programmer.

14.7 Problem Solving: Estimating the Running Time of an Algorithm

In this chapter, you have learned how to estimate the running time of sorting algorithms. As you have seen, being able to differentiate between $O(n \log(n))$ and $O(n^2)$ running times has great practical implications. Being able to estimate the running times of other algorithms is an important skill. In this section, we will practice estimating the running time of array algorithms.

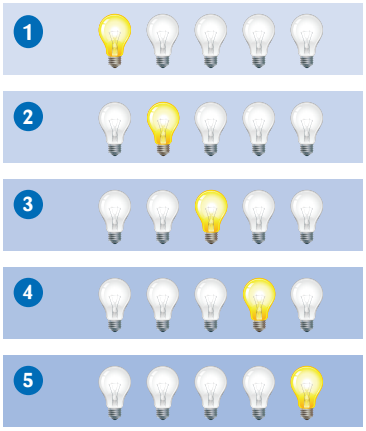
14.7.1 Linear Time

Let us start with a simple example, an algorithm that counts how many elements have a particular value:

```
int count = 0;
for (int i = 0; i < a.length; i++)
{
   if (a[i] == value) { count++; }
}
```

What is the running time in terms of n, the length of the array?

Start with looking at the pattern of array element visits. Here, we visit each element once. It helps to visualize this pattern. Imagine the array as a sequence of light bulbs. As the ith element gets visited, imagine the ith bulb lighting up.



Now look at the work per visit. Does each visit involve a fixed number of actions, independent of n? In this case, it does. There are just a few actions—read the element, compare it, maybe increment a counter.

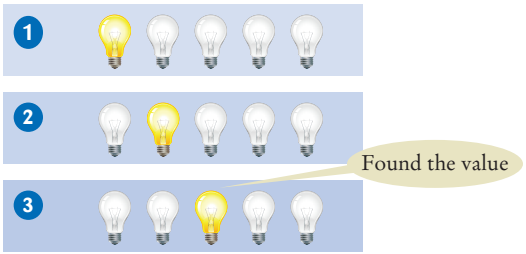
Therefore, the running time is n times a constant, or $O(n)$.

A loop with n iterations has $O(n)$ running time if each step consists of a fixed number of actions.

What if we don't always run to the end of the array? For example, suppose we want to check whether the value occurs in the array, without counting it:

```
boolean found = false;
for (int i = 0; !found && i < a.length; i++)
{
   if (a[i] == value) { found = true; }
}
```

Then the loop can stop in the middle:



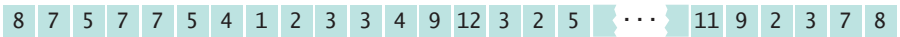
Is this still $O(n)$? It is, because in some cases the match may be at the very end of the array. Also, if there is no match, one must traverse the entire array.

14.7.2 Quadratic Time

Now let's turn to a more interesting case. What if we do a lot of work with each visit? Here is an example: We want to find the most frequent element in an array.

Suppose the array is

It's obvious by looking at the values that 7 is the most frequent one. But now imagine an array with a few thousand values.



We can count how often the value 8 occurs, then move on to count how often 7 occurs, and so on. For example, in the first array, 8 occurs once, and 7 occurs three times. Where do we put the counts? Let's put them into a second array of the same length.

Then we take the maximum of the counts. It is 3. We look up where the 3 occurs in the counts, and find the corresponding value. Thus, the most common value is 7.

Let us first estimate how long it takes to compute the counts.

```
for (int i = 0; i < a.length; i++)
{
   counts[i] = Count how often a[i] occurs in a
}
```

A loop with n iterations has $O(n^2)$ running time if each step takes $O(n)$ time.

We still visit each array element once, but now the work per visit is much larger. As you have seen in the previous section, each counting action is $O(n)$. When we do $O(n)$ work in each step, the total running time is $O(n^2)$.

This algorithm has three phases:

1. Compute all counts.
2. Compute the maximum.
3. Find the maximum in the counts.

We have just seen that the first phase is $O(n^2)$. Computing the maximum is $O(n)$—look at the algorithm in Section 7.3.3 and note that each step involves a fixed amount of work. Finally, we just saw that finding a value is $O(n)$.

The big-Oh running time for doing several steps in a row is the largest of the big-Oh times for each step.

How can we estimate the total running time from the estimates of each phase? Of course, the total time is the sum of the individual times, but for big-Oh estimates, we take the *maximum* of the estimates. To see why, imagine that we had actual equations for each of the times:

$$T_1(n) = an^2 + bn + c$$
$$T_2(n) = dn + e$$
$$T_3(n) = fn + g$$

Then the sum is

$$T(n) = T_1(n) + T_2(n) + T_3(n) = an^2 + (b + d + f)n + c + e + g$$

But only the largest term matters, so $T(n)$ is $O(n^2)$.

Thus, we have found that our algorithm for finding the most frequent element is $O(n^2)$.

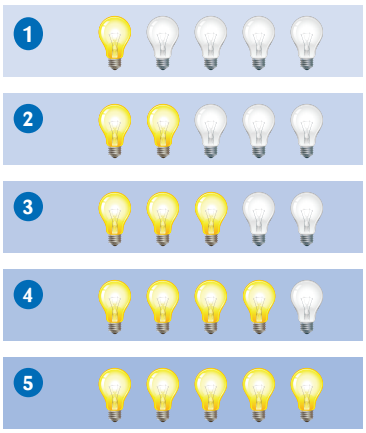
14.7.3 The Triangle Pattern

Let us see if we can speed up the algorithm from the preceding section. It seems wasteful to count elements again if we have already counted them.

Can we save time by eliminating repeated counting of the same element? That is, before counting `a[i]`, should we first check that it didn't occur in `a[0] ... a[i - 1]`?

Let us estimate the cost of these additional checks. In the ith step, the amount of work is proportional to i. That's not quite the same as in the preceding section, where you saw that a loop with n iterations, each of which takes $O(n)$ time, is $O(n^2)$. Now each step just takes $O(i)$ time.

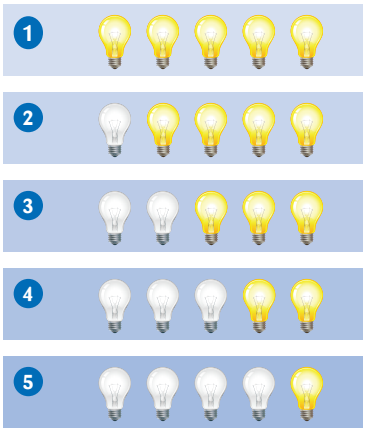
To get an intuitive feel for this situation, look at the light bulbs again. In the second iteration, we visit `a[0]` again. In the third iteration, we visit `a[0]` and `a[1]` again, and so on. The light bulb pattern is



A loop with n iterations has $O(n^2)$ running time if the ith step takes $O(i)$ time.

If there are n light bulbs, about half of the square above, or $n^2/2$ of them, light up. That's unfortunately still $O(n^2)$.

Here is another idea for time saving. When we count `a[i]`, there is no need to do the counting in `a[0] ... a[i - 1]`. If `a[i]` never occurred before, we get an accurate count by just looking at `a[i] ... a[n - 1]`. And if it did, we already have an accurate count. Does that help us? Not really—it's the triangle pattern again, but this time in the other direction.



That doesn't mean that these improvements aren't worthwhile. If an $O(n^2)$ algorithm is the best one can do for a particular problem, you still want to make it as fast as possible. However, we will not pursue this plan further because it turns out that we can do much better.

14.7.4 Logarithmic Time

Logarithmic time estimates arise from algorithms that cut work in half in each step. You have seen this in the algorithms for binary search and merge sort, and you will see it again in Chapter 17.

An algorithm that cuts the size of work in half in each step runs in $O(\log(n))$ time.

In particular, when you use sorting or binary search in a phase of an algorithm, you will encounter logarithmic time in the big-Oh estimates.

Consider this idea for improving our algorithm for finding the most frequent element. Suppose we first *sort* the array:

That cost us $O(n \log(n))$ time. If we can complete the algorithm in $O(n)$ time, we will have found a better algorithm than the $O(n^2)$ algorithm of the preceding sections.

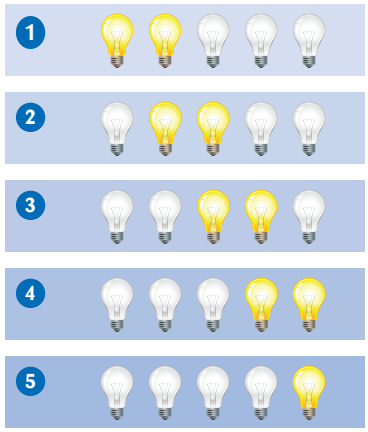
To see why this is possible, imagine traversing the sorted array. As long as you find a value that was equal to its predecessor, you increment a counter. When you find a different value, save the counter and start counting anew:

Or in code,

```
int count = 0;
for (int i = 0; i < a.length; i++)
{
```

```
      count++;
      if (i == a.length - 1 || a[i] != a[i + 1])
      {
         counts[i] = count;
         count = 0;
      }
   }
```

That's a constant amount of work per iteration, even though it visits two elements:



FULL CODE EXAMPLE

Go to wiley.com/go/javacode to download a program for comparing the speed of algorithms that find the most frequent element.

$2n$ is still $O(n)$. Thus, we can compute the counts in $O(n)$ time from a sorted array. The entire algorithm is now $O(n \log(n))$.

Note that we don't actually need to keep all counts, only the highest one that we encountered so far (see Exercise E14.8). That is a worthwhile improvement, but it does not change the big-Oh estimate of the running time.

21. What is the "light bulb pattern" of visits in the following algorithm to check whether an array is a palindrome?

```
for (int i = 0; i < a.length / 2; i++)
{
   if (a[i] != a[a.length - 1 - i]) { return false; }
}
return true;
```

22. What is the big-Oh running time of the following algorithm to check whether the first element is duplicated in an array?

```
for (int i = 1; i < a.length; i++)
{
   if (a[0] == a[i]) { return true; }
}
return false;
```

23. What is the big-Oh running time of the following algorithm to check whether an array has a duplicate value?

```
for (int i = 0; i < a.length; i++)
{
   for (j = i + 1; j < a.length; j++)
   {
      if (a[i] == a[j]) { return true; }
```

```
      }
   }
   return false;
```

24. Describe an $O(n \log(n))$ algorithm for checking whether an array has duplicates.

25. What is the big-Oh running time of the following algorithm to find an element in an $n \times n$ array?

```
for (int i = 0; i < n; i++)
{
   for (j = 0; j < n; j++)
   {
      if (a[i][j] == value) { return true; }
   }
}
return false;
```

26. If you apply the algorithm of Section 14.7.4 to an $n \times n$ array, what is the big-Oh efficiency of finding the most frequent element in terms of n?

Practice It Now you can try these exercises at the end of the chapter: R14.9, R14.13, R14.19, E14.8.

14.8 Sorting and Searching in the Java Library

When you write Java programs, you don't have to implement your own sorting algorithms. The `Arrays` and `Collections` classes provide sorting and searching methods that we will introduce in the following sections.

14.8.1 Sorting

The `Arrays` class implements a sorting method that you should use for your Java programs.

The `Arrays` class contains static `sort` methods to sort arrays of integers and floating-point numbers. For example, you can sort an array of integers simply as

```
int[] a = . . .;
Arrays.sort(a);
```

That `sort` method uses the quicksort algorithm—see Special Topic 14.3 for more information about that algorithm.

The `Collections` class contains a `sort` method that can sort array lists.

If your data are contained in an `ArrayList`, use the `Collections.sort` method instead; it uses the merge sort algorithm:

```
ArrayList<String> names = . . .;
Collections.sort(names);
```

14.8.2 Binary Search

The `Arrays` and `Collections` classes contain static `binarySearch` methods that implement the binary search algorithm, but with a useful enhancement. If a value is not found in the array, then the returned value is not –1, but $-k - 1$, where k is the position before which the element should be inserted. For example,

```
int[] a = { 1, 4, 9 };
int v = 7;
int pos = Arrays.binarySearch(a, v);
// Returns -3; v should be inserted before position 2
```

14.8.3 Comparing Objects

The sort method of the Arrays class sorts objects of classes that implement the Comparable interface.

In application programs, you often need to sort or search through collections of objects. Therefore, the Arrays and Collections classes also supply sort and binarySearch methods for objects. However, these methods cannot know how to compare arbitrary objects. Suppose, for example, that you have an array of Country objects. It is not obvious how the countries should be sorted. Should they be sorted by their names or by their areas? The sort and binarySearch methods cannot make that decision for you. Instead, they require that the objects belong to classes that implement the Comparable interface type that was introduced in Section 10.3. That interface has a single method:

```
public interface Comparable
{
   int compareTo(Object otherObject);
}
```

The call

```
a.compareTo(b)
```

must return a negative number if a should come before b, 0 if a and b are the same, and a positive number otherwise.

Several classes in the standard Java library, such as the String and Date classes, implement the Comparable interface.

You can implement the Comparable interface for your own classes as well. For example, to sort a collection of countries, the Country class would need to implement this interface and provide a compareTo method:

```
public class Country implements Comparable
{
   public int compareTo(Object otherObject)
   {
      Country other = (Country) otherObject;
      if (area < other.area) { return -1; }
      else if (area == other.area) { return 0; }
      else { return 1; }
   }
}
```

This method compares countries by their area. Now you can pass an array of countries to the Arrays.sort method:

```
Country[] countries = new Country[n];
// Add countries
Arrays.sort(countries); // Sorts by increasing area
```

FULL CODE EXAMPLE

Go to wiley.com/go/javacode to download a program that demonstrates the Java library methods for sorting and searching.

Whenever you need to carry out sorting or searching, use the methods in the Arrays and Collections classes and not those that you write yourself. The library algorithms have been fully debugged and optimized. Thus, the primary purpose of this chapter was not to teach you how to implement practical sorting and searching algorithms. Instead, you have learned something more important, namely that different algorithms can vary widely in performance, and that it is worthwhile to learn more about the design and analysis of algorithms.

27. Why can't the `Arrays.sort` method sort an array of `Rectangle` objects?
28. What steps would you need to take to sort an array of `BankAccount` objects by increasing balance?
29. Why is it useful that the `Arrays.binarySearch` method indicates the position where a missing element should be inserted?
30. Why does `Arrays.binarySearch` return $-k - 1$ and not $-k$ to indicate that a value is not present and should be inserted before position k?

Practice It Now you can try these exercises at the end of the chapter: E14.9, E14.13, E14.14.

Common Error 14.1

The `compareTo` Method Can Return Any Integer, Not Just –1, 0, and 1

The call `a.compareTo(b)` is allowed to return *any* negative integer to denote that a should come before `b`, not necessarily the value –1. That is, the test

```
if (a.compareTo(b) == -1) // ERROR!
```

is generally wrong. Instead, you should test

```
if (a.compareTo(b) < 0) // OK
```

Why would a `compareTo` method ever want to return a number other than –1, 0, or 1? Sometimes, it is convenient to just return the difference of two integers. For example, the `compareTo` method of the `String` class compares characters in matching positions:

```
char c1 = charAt(i);
char c2 = other.charAt(i);
```

If the characters are different, then the method simply returns their difference:

```
if (c1 != c2) { return c1 - c2; }
```

This difference is a negative number if `c1` is less than `c2`, but it is not necessarily the number –1.

Special Topic 14.4

The Parameterized `Comparable` Interface

As of Java version 5, the `Comparable` interface is a parameterized type, similar to the `ArrayList` type:

```
public interface Comparable<T>
{
   int compareTo(T other)
}
```

The type parameter specifies the type of the objects that this class is willing to accept for comparison. Usually, this type is the same as the class type itself. For example, the `Country` class would implement `Comparable<Country>`, like this:

```
public class Country implements Comparable<Country>
{
   . . .
   public int compareTo(Country other)
   {
      if (area < other.area) { return -1; }
      else if (area == other.area) { return 0; }
      else { return 1; }
   }
```

```
   . . .
}
```

The type parameter has a significant advantage: You need not use a cast to convert an `Object` parameter variable into the desired type.

Special Topic 14.5

The `Comparator` Interface

Sometimes you want to sort an array or array list of objects, but the objects don't belong to a class that implements the `Comparable` interface. Or, perhaps, you want to sort the array in a different order. For example, you may want to sort countries by name rather than by value.

You wouldn't want to change the implementation of a class simply to call `Arrays.sort`. Fortunately, there is an alternative. One version of the `Arrays.sort` method does not require that the objects belong to classes that implement the `Comparable` interface. Instead, you can supply arbitrary objects. However, you must also provide a *comparator* object whose job is to compare objects. The comparator object must belong to a class that implements the `Comparator` interface. That interface has a single method, `compare`, which compares two objects.

As of Java version 5, the `Comparator` interface is a parameterized type. The type parameter specifies the type of the `compare` parameter variables. For example, `Comparator<Country>` looks like this:

```
public interface Comparator<Country>
{
   int compare(Country a, Country b);
}
```

The call

```
comp.compare(a, b)
```

must return a negative number if `a` should come before `b`, `0` if `a` and `b` are the same, and a positive number otherwise. (Here, `comp` is an object of a class that implements `Comparator<Country>`.)

For example, here is a `Comparator` class for country:

```
public class CountryComparator implements Comparator<Country>
{
   public int compare(Country a, Country b)
   {
      if (a.area < b.area) { return -1; }
      else if (a.area == b.area) { return 0; }
      else { return 1; }
   }
}
```

To sort an array of countries by area, call

```
Arrays.sort(countries, new CountryComparator());
```

WORKED EXAMPLE 14.1 **Enhancing the Insertion Sort Algorithm**

Learn how to implement an improvement of the insertion sort algorithm shown in Special Topic 14.2. The enhanced algorithm is called *Shell sort* after its inventor, Donald Shell. Go to `wiley.com/go/javaexamples` and download Worked Example 14.1.

CHAPTER SUMMARY

Describe the selection sort algorithm.

- The selection sort algorithm sorts an array by repeatedly finding the smallest element of the unsorted tail region and moving it to the front.

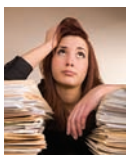
Measure the running time of a method.

- To measure the running time of a method, get the current time immediately before and after the method call.

Use the big-Oh notation to describe the running time of an algorithm.

- Computer scientists use the big-Oh notation to describe the growth rate of a function.
- Selection sort is an $O(n^2)$ algorithm. Doubling the data set means a fourfold increase in processing time.
- Insertion sort is an $O(n^2)$ algorithm.

Describe the merge sort algorithm.



- The merge sort algorithm sorts an array by cutting the array in half, recursively sorting each half, and then merging the sorted halves.

Contrast the running times of the merge sort and selection sort algorithms.

- Merge sort is an $O(n \log(n))$ algorithm. The $n \log(n)$ function grows much more slowly than n^2.

Describe the running times of the linear search algorithm and the binary search algorithm.

- A linear search examines all values in an array until it finds a match or reaches the end.
- A linear search locates a value in an array in $O(n)$ steps.
- A binary search locates a value in a sorted array by determining whether the value occurs in the first or second half, then repeating the search in one of the halves.
- A binary search locates a value in a sorted array in $O(\log(n))$ steps.

Practice developing big-Oh estimates of algorithms.

- A loop with n iterations has $O(n)$ running time if each step consists of a fixed number of actions.
- A loop with n iterations has $O(n^2)$ running time if each step takes $O(n)$ time.
- The big-Oh running time for doing several steps in a row is the largest of the big-Oh times for each step.

- A loop with n iterations has $O(n^2)$ running time if the ith step takes $O(i)$ time.
- An algorithm that cuts the size of work in half in each step runs in $O(\log(n))$ time.

Use the Java library methods for sorting and searching data.

- The `Arrays` class implements a sorting method that you should use for your Java programs.
- The `Collections` class contains a `sort` method that can sort array lists.
- The `sort` method of the `Arrays` class sorts objects of classes that implement the `Comparable` interface.

STANDARD LIBRARY ITEMS INTRODUCED IN THIS CHAPTER

```
java.lang.Comparable<T>
   compareTo
java.lang.System
   currentTimeMillis
java.util.Arrays
   binarySearch
   sort
java.util.Collections
   binarySearch
   sort
java.util.Comparator<T>
   compare
```

REVIEW QUESTIONS

■ **R14.1** What is the difference between searching and sorting?

■■ **R14.2** *Checking against off-by-one errors.* When writing the selection sort algorithm of Section 14.1, a programmer must make the usual choices of `<` versus `<=`, `a.length` versus `a.length - 1`, and `from` versus `from + 1`. This is fertile ground for off-by-one errors. Conduct code walkthroughs of the algorithm with arrays of length 0, 1, 2, and 3 and check carefully that all index values are correct.

■■ **R14.3** For the following expressions, what is the order of the growth of each?

a. $n^2 + 2n + 1$

b. $n^{10} + 9n^9 + 20n^8 + 145n^7$

c. $(n + 1)^4$

d. $(n^2 + n)^2$

e. $n + 0.001n^3$

f. $n^3 - 1000n^2 + 10^9$

g. $n + \log(n)$

h. $n^2 + n \log(n)$

i. $2^n + n^2$

j. $\dfrac{n^3 + 2n}{n^2 + 0.75}$

■ **R14.4** We determined that the actual number of visits in the selection sort algorithm is

$$T(n) = \frac{1}{2}n^2 + \frac{5}{2}n - 3$$

We characterized this method as having $O(n^2)$ growth. Compute the actual ratios

$$T(2{,}000)/T(1{,}000)$$
$$T(4{,}000)/T(1{,}000)$$
$$T(10{,}000)/T(1{,}000)$$

and compare them with

$$f(2{,}000)/f(1{,}000)$$
$$f(4{,}000)/f(1{,}000)$$
$$f(10{,}000)/f(1{,}000)$$

where $f(n) = n^2$.

■ **R14.5** Suppose algorithm A takes five seconds to handle a data set of 1,000 records. If the algorithm A is an $O(n)$ algorithm, approximately how long will it take to handle a data set of 2,000 records? Of 10,000 records?

■■ **R14.6** Suppose an algorithm takes five seconds to handle a data set of 1,000 records. Fill in the following table, which shows the approximate growth of the execution times depending on the complexity of the algorithm.

	$O(n)$	$O(n^2)$	$O(n^3)$	$O(n \log(n))$	$O(2^n)$
1,000	5	5	5	5	5
2,000					
3,000		45			
10,000					

For example, because $3{,}000^2/1{,}000^2 = 9$, the algorithm would take nine times as long, or 45 seconds, to handle a data set of 3,000 records.

■■ **R14.7** Sort the following growth rates from slowest to fastest growth.

$O(n)$	$O(n \log(n))$
$O(n^3)$	$O(2^n)$
$O(n^n)$	$O(\sqrt{n})$
$O(\log(n))$	$O(n\sqrt{n})$
$O(n^2 \log(n))$	$O(n^{\log(n)})$

■ **R14.8** What is the growth rate of the standard algorithm to find the minimum value of an array? Of finding both the minimum and the maximum?

■ **R14.9** What is the big-Oh time estimate of the following method in terms of n, the length of `a`? Use the "light bulb pattern" method of Section 14.7 to visualize your result.

```
public static void swap(int[] a)
{
   int i = 0;
   int j = a.length - 1;
   while (i < j)
   {
      int temp = a[i];
      a[i] = a[j];
      a[j] = temp;
      i++;
      j--;
   }
}
```

■ **R14.10** Trace a walkthrough of selection sort with these sets:

a. 4 7 11 4 9 5 11 7 3 5

b. –7 6 8 7 5 9 0 11 10 5 8

■ **R14.11** Trace a walkthrough of merge sort with these sets:

a. 5 11 7 3 5 4 7 11 4 9

b. 9 0 11 10 5 8 –7 6 8 7 5

■ **R14.12** Trace a walkthrough of:

a. Linear search for 7 in –7 1 3 3 4 7 11 13

b. Binary search for 8 in –7 2 2 3 4 7 8 11 13

c. Binary search for 8 in –7 1 2 3 5 7 10 13

■■ **R14.13** Your task is to remove all duplicates from an array. For example, if the array has the values

4 7 11 4 9 5 11 7 3 5

then the array should be changed to

4 7 11 9 5 3

Here is a simple algorithm: Look at `a[i]`. Count how many times it occurs in `a`. If the count is larger than 1, remove it. What is the growth rate of the time required for this algorithm?

■■■ **R14.14** Modify the merge sort algorithm to remove duplicates in the merging step to obtain an algorithm that removes duplicates from an array. Note that the resulting array does not have the same ordering as the original one. What is the efficiency of this algorithm?

■■ **R14.15** Consider the following algorithm to remove all duplicates from an array: Sort the array. For each element in the array, look at its next neighbor to decide whether it is present more than once. If so, remove it. Is this a faster algorithm than the one in Exercise R14.13?

■■■ **R14.16** Develop an $O(n \log(n))$ algorithm for removing duplicates from an array if the resulting array must have the same ordering as the original array. When a value occurs multiple times, all but its first occurrence should be removed.

■■■ **R14.17** Why does insertion sort perform significantly better than selection sort if an array is already sorted?

■■■ **R14.18** Consider the following speedup of the insertion sort algorithm of Special Topic 14.2. For each element, use the enhanced binary search algorithm that yields the insertion position for missing elements. Does this speedup have a significant impact on the efficiency of the algorithm?

■■ **R14.19** Consider the following algorithm known as *bubble sort*:

```
While the array is not sorted
   For each adjacent pair of elements
      If the pair is not sorted
         Swap its elements.
```

What is the big-Oh efficiency of this algorithm?

■■ **R14.20** The *radix sort* algorithm sorts an array of n integers with d digits, using ten auxiliary arrays. First place each value v into the auxiliary array whose index corresponds to the last digit of v. Then move all values back into the original array, preserving their order. Repeat the process, now using the next-to-last (tens) digit, then the hundreds digit, and so on. What is the big-Oh time of this algorithm in terms of n and d? When is this algorithm preferable to merge sort?

■■ **R14.21** A *stable sort* does not change the order of elements with the same value. This is a desirable feature in many applications. Consider a sequence of e-mail messages. If you sort by date and then by sender, you'd like the second sort to preserve the relative order of the first, so that you can see all messages from the same sender in date order. Is selection sort stable? Insertion sort? Why or why not?

■■ **R14.22** Give an $O(n)$ algorithm to sort an array of n bytes (numbers between –128 and 127). *Hint:* Use an array of counters.

■■ **R14.23** You are given a sequence of arrays of words, representing the pages of a book. Your task is to build an index (a sorted array of words), each element of which has an array of sorted numbers representing the pages on which the word appears. Describe an algorithm for building the index and give its big-Oh running time in terms of the total number of words.

■■ **R14.24** Given two arrays of n integers each, describe an $O(n \log(n))$ algorithm for determining whether they have an element in common.

■■■ **R14.25** Given an array of n integers and a value v, describe an $O(n \log(n))$ algorithm to find whether there are two values x and y in the array with sum v.

■■ **R14.26** Given two arrays of n integers each, describe an $O(n \log(n))$ algorithm for finding all elements that they have in common.

■■ **R14.27** Suppose we modify the quicksort algorithm from Special Topic 14.3, selecting the middle element instead of the first one as pivot. What is the running time on an array that is already sorted?

■■ **R14.28** Suppose we modify the quicksort algorithm from Special Topic 14.3, selecting the middle element instead of the first one as pivot. Find a sequence of values for which this algorithm has an $O(n^2)$ running time.

PRACTICE EXERCISES

■ **E14.1** Modify the selection sort algorithm to sort an array of integers in descending order.

■ **E14.2** Modify the selection sort algorithm to sort an array of coins by their value.

■■ **E14.3** Write a program that automatically generates the table of sample run times for the selection sort algorithm. The program should ask for the smallest and largest value of `n` and the number of measurements and then make all sample runs.

■ **E14.4** Modify the merge sort algorithm to sort an array of strings in lexicographic order.

■■■ **E14.5** Write a telephone lookup program. Read a data set of 1,000 names and telephone numbers from a file that contains the numbers in random order. Handle lookups by name and also reverse lookups by phone number. Use a binary search for both lookups.

■■ **E14.6** Implement a program that measures the performance of the insertion sort algorithm described in Special Topic 14.2.

■ **E14.7** Implement the bubble sort algorithm described in Exercise R14.19.

■■ **E14.8** Implement the algorithm described in Section 14.7.4, but only remember the value with the highest frequency so far:

```
int mostFrequent = 0;
int highestFrequency = -1;
for (int i = 0; i < a.length; i++)
   Count how often a[i] occurs in a[i + 1]...a[a.length - 1]
   If it occurs more often than highestFrequency
      highestFrequency = that count
      mostFrequent = a[i]
```

■■■ **E14.9** Write a program that sorts an `ArrayList<Country>` in decreasing order so that the largest country is at the beginning of the array. Use a `Comparator`.

■■ **E14.10** Consider the binary search algorithm in Section 14.6. If no match is found, the `search` method returns –1. Modify the method so that if `a` is not found, the method returns $-k - 1$, where k is the position before which the element should be inserted. (This is the same behavior as `Arrays.binarySearch`.)

■■ **E14.11** Implement the `sort` method of the merge sort algorithm without recursion, where the length of the array is a power of 2. First merge adjacent regions of size 1, then adjacent regions of size 2, then adjacent regions of size 4, and so on.

■■■ **E14.12** Use insertion sort and the binary search from Exercise E14.10 to sort an array as described in Exercise R14.18. Implement this algorithm and measure its performance.

■ **E14.13** Supply a class `Person` that implements the `Comparable` interface. Compare persons by their names. Ask the user to input ten names and generate ten `Person` objects. Using the `compareTo` method, determine and the first and last person among them and print them.

■■ **E14.14** Sort an array list of strings by increasing *length*. *Hint:* Supply a `Comparator`.

■■■ **E14.15** Sort an array list of strings by increasing length, and so that strings of the same length are sorted lexicographically. *Hint:* Supply a `Comparator`.

PROGRAMMING PROJECTS

■■ **P14.1** Implement the following modification of the quicksort algorithm, due to Bentley and McIlroy. Instead of using the first element as the pivot, use an approximation of the median. (Partitioning at the actual median would yield an $O(n \log(n))$ algorithm, but we don't know how to compute it quickly enough.)

If $n \leq 7$, use the middle element. If $n \leq 40$, use the median of the first, middle, and last element. Otherwise compute the "pseudomedian" of the nine elements `a[i * (n - 1) / 8]`, where `i` ranges from 0 to 8. The pseudomedian of nine values is $\text{med}(\text{med}(v_0, v_1, v_2), \text{med}(v_3, v_4, v_5), \text{med}(v_6, v_7, v_8))$.

Compare the running time of this modification with that of the original algorithm on sequences that are nearly sorted or reverse sorted, and on sequences with many identical elements. What do you observe?

■■■ **P14.2** Bentley and McIlroy suggest the following modification to the quicksort algorithm when dealing with data sets that contain many repeated elements.

Instead of partitioning as

(where ≤ denotes the elements that are ≤ the pivot), it is better to partition as

However, that is tedious to achieve directly. They recommend to partition as

and then swap the two = regions into the middle. Implement this modification and check whether it improves performance on data sets with many repeated elements.

■ **P14.3** Implement the radix sort algorithm described in Exercise R14.20 to sort arrays of numbers between 0 and 999.

■ **P14.4** Implement the radix sort algorithm described in Exercise R14.20 to sort arrays of numbers between 0 and 999. However, use a single auxiliary array, not ten.

■■ **P14.5** Implement the radix sort algorithm described in Exercise R14.20 to sort arbitrary `int` values (positive or negative).

■■■ **P14.6** Implement the `sort` method of the merge sort algorithm without recursion, where the length of the array is an arbitrary number. Keep merging adjacent regions whose size is a power of 2, and pay special attention to the last area whose size is less.

ANSWERS TO SELF-CHECK QUESTIONS

1. Dropping the `temp` variable would not work. Then `a[i]` and `a[j]` would end up being the same value.
2. 1 | 5 4 3 2 6
   1 2 | 4 3 5 6
   1 2 3 4 5 6
3. In each step, find the *maximum* of the remaining elements and swap it with the current element (or see Self Check 4).
4. The modified algorithm sorts the array in descending order.
5. Four times as long as 40,000 values, or about 37 seconds.
6. A parabola.
7. It takes about 100 times longer.
8. If n is 4, then $\frac{1}{2}n^2$ is 8 and $\frac{5}{2}n - 3$ is 7.
9. The first algorithm requires one visit, to store the new element. The second algorithm requires $T(p) = 2 \times (n - p - 1)$ visits, where p is the location at which the element is removed. We don't know where that element is, but if elements are removed at random locations, on average, half of the removals will be above the middle and half below, so we can assume an average p of $n / 2$ and $T(n) = 2 \times (n - n / 2 - 1) = n - 2$.
10. The first algorithm is $O(1)$, the second $O(n)$.
11. We need to check that `a[0]` ≤ `a[1]`, `a[1]` ≤ `a[2]`, and so on, visiting $2n - 2$ elements. Therefore, the running time is $O(n)$.
12. Let n be the length of the array. In the kth step, we need k visits to find the minimum. To remove it, we need an average of $k - 2$ visits (see Self Check 9). One additional visit is required to add it to the end. Thus, the kth step requires $2k - 1$ visits. Because k goes from n to 2, the total number of visits is

$$2n - 1 + 2(n - 1) - 1 + \ldots + 2 \cdot 3 - 1 + 2 \cdot 2 - 1 =$$
$$2(n + (n - 1) + \ldots + 3 + 2 + 1 - 1) - (n - 1) =$$
$$n(n + 1) - 2 - n + 1 = n^2 - 3$$

   (because $1 + 2 + 3 + \ldots + (n - 1) + n = n(n + 1)/2$)
   Therefore, the total number of visits is $O(n^2)$.
13. When the preceding `while` loop ends, the loop condition must be false, that is, `iFirst >= first.length` or `iSecond >= second.length` (De Morgan's Law).
14. First sort 8 7 6 5. Recursively, first sort 8 7. Recursively, first sort 8. It's sorted. Sort 7. It's sorted. Merge them: 7 8. Do the same with 6 5 to get 5 6. Merge them to 5 6 7 8. Do the same with 4 3 2 1: Sort 4 3 by sorting 4 and 3 and merging them to 3 4. Sort 2 1 by sorting 2 and 1 and merging them to 1 2. Merge 3 4 and 1 2 to 1 2 3 4. Finally, merge 5 6 7 8 and 1 2 3 4 to 1 2 3 4 5 6 7 8.
15. If the array size is 1, return its only element as the sum. Otherwise, recursively compute the sum of the first and second subarray and return the sum of these two values.
16. Approximately $(100{,}000 \cdot \log(100{,}000)) / (50{,}000 \cdot \log(50{,}000)) = 2 \cdot 5 / 4.7 = 2.13$ times the time required for 50,000 values. That's $2.13 \cdot 192$ milliseconds or approximately 409 milliseconds.
17. $$\frac{2n\log(2n)}{n\log(n)} = 2\frac{(1 + \log(2))}{\log(n)}.$$
    For $n > 2$, that is a value < 3.
18. On average, you'd make 500,000 comparisons.
19. The `search` method returns the index at which the match occurs, not the data stored at that location.
20. You would search about 20. (The binary log of 1,024 is 10.)
21. 

22. It is an $O(n)$ algorithm.
23. It is an $O(n^2)$ algorithm—the number of visits follows a triangle pattern.
24. Sort the array, then make a linear scan to check for adjacent duplicates.
25. It is an $O(n^2)$ algorithm—the outer and inner loops each have n iterations.

26. Because an $n \times n$ array has $m = n^2$ elements, and the algorithm in Section 14.7.4, when applied to an array with m elements, is $O(m \log(m))$, we have an $O(n^2 \log(n))$ algorithm. Recall that $\log(n^2) = 2 \log(n)$, and the factor of 2 is irrelevant in the big-Oh notation.

27. The `Rectangle` class does not implement the `Comparable` interface.

28. The `BankAccount` class would need to implement the `Comparable` interface. Its `compareTo` method must compare the bank balances.

29. Then you know where to insert it so that the array stays sorted, and you can keep using binary search.

30. Otherwise, you would not know whether a value is present when the method returns `0`.