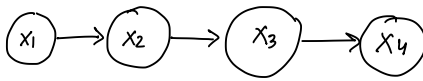


Q1



Joint distribution

$$(a) \quad P(x_1, x_2, x_3, x_4) = p(x_1) p(x_2 | x_1) p(x_3 | x_1, x_2) p(x_4 | x_1, x_2, x_3)$$

$$(b) \quad x_1 \sim \mathcal{N}(0, \sigma^2) \quad \text{Random walk model}$$

$$x_t | x_{t-1} \sim \mathcal{N}(x_{t-1}, \sigma^2) \quad t=2, 3, 4$$

We have,

$$x_1 \sim \mathcal{N}(0, \sigma^2)$$

$$x_2 | x_1 \sim \mathcal{N}(x_1, \sigma^2)$$

$$x_3 | x_2 \sim \mathcal{N}(x_2, \sigma^2)$$

$$x_4 | x_3 \sim \mathcal{N}(x_3, \sigma^2)$$

$$x_t = x_{t-1} + \underbrace{n_t}_{\sim \mathcal{N}(0, \sigma^2)}$$

$$\left. \begin{array}{l} \textcircled{1} \quad x_2 = x_1 + n_1 \\ \textcircled{2} \quad x_3 = x_2 + n_2 = x_1 + n_1 + n_2 \\ \textcircled{3} \quad x_4 = x_3 + n_3 = x_1 + n_1 + n_2 + n_3 \end{array} \right\} n_1, n_2, n_3 \sim \mathcal{N}(0, \sigma^2)$$

$$x_1 = n_0 \sim \mathcal{N}(0, \sigma^2)$$

$$\mathbb{E}[x_1] = 0$$

$$\textcircled{1} \quad \mathbb{E}[x_2] = \mathbb{E}[x_1 + n_1] = \mathbb{E}[x_1] + \mathbb{E}[n_1] = 0$$

$$\textcircled{2} \quad \mathbb{E}[x_3] = \mathbb{E}[x_2 + n_2] = 0$$

$$\textcircled{3} \quad \mathbb{E}[x_4] = 0$$

$$\Sigma = \left[\begin{array}{cccc} \text{Var}(n_1) & \text{Cov}(n_1, n_2) & \text{Cov}(n_1, n_3) & \text{Cov}(n_1, n_4) \\ \text{Cov}(n_2, n_1) & \text{Var}(n_2) & \text{Cov}(n_2, n_3) & \text{Cov}(n_2, n_4) \\ \text{Cov}(n_3, n_1) & \text{Cov}(n_3, n_2) & \text{Var}(n_3) & \text{Cov}(n_3, n_4) \\ \text{Cov}(n_4, n_1) & \text{Cov}(n_4, n_2) & \text{Cov}(n_4, n_3) & \text{Var}(n_4) \end{array} \right] \begin{array}{l} \text{row 1} \\ \text{row 2} \\ \text{row 3} \\ \text{row 4} \end{array}$$

row 1

$$\text{Var}(n_1) = \mathbb{E}[n_1^2] = \sigma^2$$

$$\begin{aligned}\text{Cov}(n_1, n_2) &= \mathbb{E}[n_1, n_2] - \mathbb{E}[n_1] \mathbb{E}[n_2] \\ &= \mathbb{E}[n_1, n_2] = \mathbb{E}[n_1 (n_1 + n_1)] \\ &= \mathbb{E}[n_1^2] + \mathbb{E}[n_1] \mathbb{E}[n_1] \\ &= \sigma^2\end{aligned}$$

$$\begin{aligned}\text{Cov}(n_1, n_3) &= \mathbb{E}[n_1, n_3] - \mathbb{E}[n_1] \mathbb{E}[n_3] \\ \mathbb{E}[n_1, n_3] &= \mathbb{E}[n_1 (n_1 + n_1 + n_2)] \\ &= \mathbb{E}[n_1^2 + n_1 n_1 + n_1 n_2] \\ &= \sigma^2\end{aligned}$$

Similarly,

$$\begin{aligned}\text{Cov}(n_1, n_4) &= \mathbb{E}[n_1, (n_1 + n_1 + n_2 + n_3)] - \mathbb{E}[n_1] \mathbb{E}[n_4] \\ &= \sigma^2\end{aligned}$$

row 2

$$\begin{aligned}\text{Cov}(n_2, n_1) &= \mathbb{E}[n_2, n_1] - \mathbb{E}[n_2] \mathbb{E}[n_1] \\ &= \mathbb{E}[(n_1 + n_1)(n_1)] = \sigma^2\end{aligned}$$

$$\begin{aligned}\text{Cov}(n_2, n_3) &= \mathbb{E}[n_2, n_3] - \mathbb{E}[n_2] \mathbb{E}[n_3] \\ &= \mathbb{E}[(n_1 + n_1)(n_1 + n_1 + n_2)] \\ &= \mathbb{E}[n_1^2] + \mathbb{E}[n_1^2] + 0 \\ &= 2\sigma^2\end{aligned}$$

$$\begin{aligned}\text{Cov}(n_2, n_4) &= \mathbb{E}[n_2, n_4] - \mathbb{E}[n_2] \mathbb{E}[n_4] \\ &= \mathbb{E}[(n_1 + n_1)(n_1 + n_1 + n_2 + n_3)] \\ &= \mathbb{E}[n_1^2 + n_1 n_1 + n_1 n_2 + n_1 n_3 + n_1^2 + n_1 n_2 + n_1 n_3] \\ &= 2\sigma^2\end{aligned}$$

$$\begin{aligned}\text{Var}(n_2) &= \mathbb{E}[n_2^2] \\ &= \mathbb{E}[(n_1 + n_1)(n_1 + n_1)] = \mathbb{E}[n_1^2 + n_1^2 + 2n_1 n_1] = 2\sigma^2\end{aligned}$$

row 3

$$\text{Var}(n_3) = \mathbb{E}[n_3^2] = \mathbb{E}[(n_1 + n_1 + n_2)^2] = 3\sigma^2$$

$$\text{Cov}(n_3, n_1) = \mathbb{E}[(n_1 + n_1 + n_2)(n_1)] = \mathbb{E}[n_1^2 + n_1 n_1 + n_2 n_1] = \sigma^2$$

$$\text{Cov}(n_3, n_2) = \mathbb{E}[(n_1 + n_1 + n_2)(n_1 + n_1)] = 2\sigma^2$$

$$\text{Cov}(n_3, n_4) = \mathbb{E}[(n_1 + n_1 + n_2)(n_1 + n_1 + n_2 + n_3)] = 3\sigma^2$$

row 4

$$\text{Cov}(n_4, n_1) = \mathbb{E}[n_4 n_1] - \mathbb{E}(n_4) \mathbb{E}(n_1) = \sigma^2$$

$$\text{Cov}(n_4, n_2) = \mathbb{E}[(n_1 + n_1 + n_2 + n_3)(n_1 + n_1)] = 2\sigma^2$$

$$\text{Cov}(n_4, n_3) = 3\sigma^2$$

$$\text{Var}(n_4) = \mathbb{E}[n_4^2] = 4\sigma^2$$

(b) Covariance matrix $\Sigma =$

$$\Sigma = \text{Cov}(n_1, n_2, n_3, n_4) = \begin{bmatrix} \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 & 2\sigma^2 & 2\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 3\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 4\sigma^2 \end{bmatrix}$$

(c) Inverse of $\Sigma \rightarrow$ Precision matrix

$$\Sigma^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

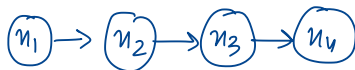
zeros \swarrow

[Inversion done in MATLAB]

(d) We see that this matrix Σ^{-1} is banded diagonal

$$\Sigma_{1,3}^{-1}, \Sigma_{1,4}^{-1}, \Sigma_{2,4}^{-1}, \Sigma_{3,1}^{-1}, \Sigma_{4,1}^{-1}, \Sigma_{4,2}^{-1} = 0$$

Given graph structure



the zeros imply that n_1, n_3 don't have a direct link
 n_1, n_4
 n_2, n_4

that is n_1 does not have n_3 or n_4 as parent
 n_2 does not have n_4 as parent
 n_3 does not have n_2 as parent
 n_4 does not have n_1 or n_2 as parent