Q1 $\quad X_1 \to X_2 \to X_3 \to X_4$

Joint distribution

(a) $p(x_1, x_2, x_3, x_4) = p(x_1)\, p(x_2 | x_1)\, p(x_3 | x_1, x_2)\, p(x_4 | x_1, x_2, x_3)$

(b) $$x_1 \sim \mathcal{N}(0, \sigma^2)$$ Random walk model

$$x_t | x_{t-1} \sim \mathcal{N}(x_{t-1}, \sigma^2) \qquad t = 2, 3, 4$$

We have,

$$x_1 \sim \mathcal{N}(0, \sigma^2)$$
$$x_1 | x_2 \sim \mathcal{N}(x_1, \sigma^2)$$
$$x_3 | x_2 \sim \mathcal{N}(x_2, \sigma^2)$$
$$x_4 | x_3 \sim \mathcal{N}(x_3, \sigma^2)$$

$$x_t = x_{t-1} + \underbrace{n_t}_{\mathcal{N}(0, \sigma^2)}$$

$$\left.\begin{aligned}
&① \quad x_2 = x_1 + n_1 \\
&② \quad x_3 = x_2 + n_2 = x_1 + n_1 + n_2 \\
&③ \quad x_4 = x_3 + n_3 = x_1 + n_1 + n_2 + n_3
\end{aligned}\right\} n_1, n_2, n_3 \sim \mathcal{N}(0, \sigma^2)$$

$$x_1 = n_0 \sim \mathcal{N}(0, \sigma^2)$$

$$\mathbb{E}[x_1] = 0$$

① $\mathbb{E}[x_2] = \mathbb{E}[x_1 + n_1] = \mathbb{E}[x_1] + \mathbb{E}[n_1] = 0$

② $\mathbb{E}[x_3] = \mathbb{E}[x_2 + n_2] = 0$

③ $\mathbb{E}[x_4] = 0$

$$\Sigma = \begin{array}{cc}
\begin{bmatrix}
\mathrm{Var}(x_1) & \mathrm{Cov}(x_1, x_2) & \mathrm{Cov}(x_1, x_3) & \mathrm{Cov}(x_1, x_4) \\
\mathrm{Cov}(x_2, x_1) & \mathrm{Var}(x_2) & \mathrm{Cov}(x_2, x_3) & \mathrm{Cov}(x_2, x_4) \\
\mathrm{Cov}(x_3, x_1) & \mathrm{Cov}(x_3, x_2) & \mathrm{Var}(x_3) & \mathrm{Cov}(x_3, x_4) \\
\mathrm{Cov}(x_4, x_1) & \mathrm{Cov}(x_4, x_2) & \mathrm{Cov}(x_4, x_3) & \mathrm{Var}(x_4)
\end{bmatrix} &
\begin{array}{l}
\text{row 1} \\ \text{row 2} \\ \text{row 3} \\ \text{row 4}
\end{array}
\end{array}$$

row 1

$$\mathrm{Var}(x_1) = \mathbb{E}[x_1^2] = \sigma^2$$

$$\mathrm{Cov}(x_1, x_2) = \mathbb{E}[x_1, x_2] - \overbrace{\mathbb{E}[x_1]\ \mathbb{E}[x_2]}^{0}$$

$$= \mathbb{E}[x_1, x_2) \quad = \mathbb{E}[\,x_1\ (x_1 + n_1)\,]$$

$$= \mathbb{E}[x_1^2] + \mathbb{E}(x_1)\ \mathbb{E}(n_1)$$

$$= \sigma^2$$

$$\mathrm{Cov}(x_1, x_3) = \mathbb{E}[x_1, x_3] - \mathbb{E}[x_1]\ \mathbb{E}[x_3]$$

$$\mathbb{E}[x_1, x_3] = \mathbb{E}[x_1\ (x_1 + n_1 + n_2)]$$

$$= \mathbb{E}[x_1^2 + x_1 n_1 + x_1 n_2]$$

$$= \sigma^2$$

Similarly, $\mathrm{Cov}(x_1, x_4) = \mathbb{E}[x_1\ ,\ (x_1 + n_1 + n_2 + n_3)] - \mathbb{E}[x_1]\ \mathbb{E}[x_4]$

$$= \sigma^2$$

row 2

$$\mathrm{Cov}(x_2, x_1) = \mathbb{E}[x_2, x_1] - \mathbb{E}(x_2)\ \mathbb{E}[x_1)$$

$$= \mathbb{E}[(x_1 + n_1)(x_1)] = \sigma^2$$

$$\mathrm{Cov}(x_2, x_3) = \mathbb{E}[x_2, x_3) - \mathbb{E}(x_2)\ \mathbb{E}(x_3)$$

$$= \mathbb{E}[(x_1 + n_1)\ (x_1 + n_1 + n_2)$$

$$= \mathbb{E}[x_1^2] + \mathbb{E}[n_1^2] + 0$$

$$= 2\sigma^2$$

$$\mathrm{Cov}(x_2, x_4) = \mathbb{E}(x_2\ ,\ x_4] - \mathbb{E}(x_2)\ \mathbb{E}(x_4)$$

$$= \mathbb{E}((x_1 + n_1)\ (x_1 + n_1 + n_2 + n_3))$$

$$= \mathbb{E}[x_1^2 + x_1 n_1 + x_1 n_2 + x_1 n_3 + n_1^2 + n_1 n_2 + n_1 n_3]$$

$$= 2\sigma^2$$

$$\mathrm{Var}(x_2) = \mathbb{E}(x_2^2)$$

$$= \mathbb{E}[(x_1 + n_1)(x_1 + n_1)) = \mathbb{E}[x_1^2 + n_1^2 + 2x_1 n_1] = 2\sigma^2$$

row3

$$\mathrm{Var}(x_3) = \mathbb{E}[x_3^2] = \mathbb{E}[(x_1+n_1+n_2)^2] = 3\sigma^2$$

$$\mathrm{Cov}(x_3, x_1) = \mathbb{E}[(x_1+n_1+n_2)(x_1)] = \mathbb{E}[x_1^2 + n_1 x_1 + n_2 x_1] = \sigma^2$$

$$\mathrm{Cov}(x_3, x_2) = \mathbb{E}\,((x_1+n_1+n_2)(x_1+n_1)] = 2\sigma^2$$

$$\mathrm{Cov}(x_3, x_4) = \mathbb{E}[(x_1+n_1+n_2)(x_1+n_1+n_2+n_3)] = 3\sigma^2$$

row4

$$\mathrm{Cov}(x_4, x_1) = \mathbb{E}[x_4, x_1) - \mathbb{E}(x_4)\mathbb{E}(x_1) = \sigma^2$$

$$\mathrm{Cov}(x_4, x_2) = \mathbb{E}[(x_1+n_1+n_2+n_3)(x_1+n_1)] = 2\sigma^2$$

$$\mathrm{Cov}(x_4, x_3) = 3\sigma^2$$

$$\mathrm{Var}(x_4) = \mathbb{E}[x_4^2] = 4\sigma^2$$

(b) Covariance matrix $\Sigma$ =

$$\Sigma = \mathrm{Cov}(x_1, x_2, x_3, x_4) = \begin{bmatrix} \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 & 2\sigma^2 & 2\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 3\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 4\sigma^2 \end{bmatrix}$$

(c) Inverse of $\Sigma$ → Precision matrix

$$\Sigma^{-1} = \frac{1}{\sigma^2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

[Inversion done in MATLAB]

zeros

(d) We see that this matrix $\Sigma^{-1}$ is banded diagonal

$$\Sigma^{-1}_{1,3},\ \Sigma^{-1}_{1,4},\ \Sigma^{-1}_{2,4},\ \Sigma^{-1}_{3,1},\ \Sigma^{-1}_{4,1},\ \Sigma^{-1}_{4,2} = 0$$

Given graph structure $x_1 \to x_2 \to x_3 \to x_4$

the zeros imply that
$x_1, x_3$
$x_1, x_4$
$x_2, x_4$
don't have a direct link

that is $x_1$ does not have $x_3$ or $x_4$ as parent
$x_2$ does not have $x_4$ as parent
$x_3$ does not have $x_2$ as parent
$x_4$ does not have $x_1$ or $x_2$ as parent