Compactness and the Heine-Borel Theorem

Compactness is crucial to many important analytical arguments. For instance, we know continuous functions on compact sets are uniformly continuous, which is not always true on non-compact sets. The Heine-Borel theorem completely characterizes compactness. First we define compactness in the following way.

Definition (Compactness). A set $K \subset \mathbb{R}$ is said to be compact if every sequence in K has a convergent subsequence whose limit is also in K—meaning, if (x_n) is a sequence in K, then there exists a subsequence (x_{n_k}) such that $\lim_{n_k \to \infty} x_{n_k} = x$ and $x \in K$.

Heine-Borel Theorem. *Let K be a compact set in $\mathbb{R}$. The following three claims are equivalent:*

(i) K is compact

(ii) K is closed and bounded

(iii) Every open cover of K contains a finite subcover.

Proof. (i) $\implies$ (ii). Let K be compact. First we show K is closed. Let x be a limit point of K. By definition of limit point, there exists a sequence (x_n) in K and a subsequence (x_{n_k}) such that $\lim_{n_k \to \infty} x_{n_k} = x$. Because K is compact, the limit of (x_{n_k}) must also be in K i.e. $x \in K$. Hence K contains its limit points, and is therefore closed. Now we show that K is bounded. For sake of contradiction, assume K is not bounded. We construct an unbounded sequence in K. For every $n \in \mathbb{N}$, we can choose $|x_n| > n$. Then every subsequence (x_{n_k}) is also unbounded (we can pick $|x_{n_k}| > n$ by picking $n_k > n$). Therefore the subsequence (x_{n_k}) does not converge, contradicting our assumption that K is compact.

(ii) $\implies$ (i). Let K be closed and bounded, and let (x_n) be a sequence in K. By Bolzano-Weierstrass, there exists a convergent subsequence (x_{n_k}). Let $\lim_{n_k \to \infty} x_{n_k} = x$. So x is a limit point of K. Because K is closed, we know $x \in K$. Therefore, there exists a convergent subsequence whose limit is also in K. Therefore K is compact.

(iii) $\implies$ (ii). Let every open cover of K have a finite subcover. First we show that K is bounded. We construct our own open cover of K. Clearly the collection of neighborhoods $\{N_\epsilon(x) : x \in K\}$ for $\epsilon = 1$ is an open cover for K. By assumption, there exists a finite subcover $\{N_\epsilon(x_1), ..., N_\epsilon(x_n)\}$. Hence for every $x \in K$ it is true that $|x| < \max\{|x_1|, ..., |x_n|\} + 1$. So K is bounded.

Now we need to show that K is closed. For sake of contradiction, assume K is not closed. Then there exists a limit point y of K such that $y \notin K$. Because y is not in K, it is true that $|x - y| > 0$ for every $x \in K$. Now we construct an open cover for K and derive a contradiction. Consider the collection of neighborhoods $\{N_{\epsilon_x}(x) : x \in K\}$ where $\epsilon_x := \frac{|x-y|}{2}$. By assumption, there exists a finite subcover $\{N_{\epsilon_{x_1}}(x_1), ..., N_{\epsilon_{x_n}}(x_n)\}$. Let

$$\epsilon_0 := \min\left\{\frac{|x_i - y|}{2} : 1 \le i \le n\right\}.$$

We pick $N \in \mathbb{N}$ such that $|y - y_N| < \epsilon_0$. The contradiction arises when we realize if $|y - y_N| < \epsilon_0$, then $y_n \notin \bigcup_{i=1}^n N_{\epsilon_{x_i}(x_i)}$. Specifically, we can rearrange the triangle inequality to show that $|x_i - y_N| > \epsilon_0$ for $1 \le i \le n$. Hence K is closed.

(ii) $\implies$ (iii). Let K be closed and bounded, and therefore compact. For sake of contradiction, assume that every open cover of K does not admit a finite subcover. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of K. Let I_0 be a compact set containing K. Now bisect I_0 into two closed intervals. We are guaranteed that at least one of the two resulting intervals cannot be finitely subcovered with the given open cover—otherwise, K could be covered by the finite union of the two finite subcovers for each half. In other words, bisecting I_0 does not change the fact that K cannot be finitely subcovered. Choose one of the resulting intervals that cannot be finitely subcovered and call it I_1. Note that I_1 is compact (closed subset of K). Continue in this way so that

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

and each I_n is compact and cannot be finitely subcovered for $n \geq 0$. Moreover, $\lim_{n\to\infty} |I_n| = 0$. Thus for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $|I_n| < \epsilon$ for all $n \geq N$. By the Nested Compact Set property, there exists x such that $x \in I_n$ for every $n \geq 0$. Because $\{O_\lambda : \lambda \in \Lambda\}$ covers K, there exists $\lambda_0 \in \Lambda$ such that $x \in O_{\lambda_0}$. By the openness of O_{λ_0}, we can choose ϵ small enough such that $N_\epsilon(x) \subseteq O_{\lambda_0}$. We take N large enough so that $I_N \subset N_\epsilon(x)$ e.g. by taking N large enough such that $|I_N| < \frac{\epsilon}{2}$. This gives rise to the contradiction. We have shown that I_N can be finitely subcovered by O_{λ_0}, even though we constructed I_N to have no finite subcover. Thus every open cover of K admits a finite subcover.

□

That completes the proof of the Heine-Borel theorem. It completely characterizes compactness. An alternate proof of the last implication (ii) $\implies$ (iii) is also given below for extra practice.

Proof. (ii) $\implies$ (iii). Consider the special case where K is a closed and bounded (and therefore compact) interval $[a, b]$. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$. Define the set

$$S := \{x \in K : [a, x] \text{ has a finite subcover}\}.$$

We show that S is non-empty and bounded, hence $s := \sup S$ exists. It is clear that if we take $x = a$ then the interval $[a, a]$ has a finite subcover, so S is non-empty. It is clear that we can take $x = b$ as an upperbound for S as well. Now we show that $s = b$. Since we have established b as an upperbound for S, we know $s \leq b$. For sake of contradiction, assume $s < b$. Then there exists λ_0 such that $s \in O_{\lambda_0}$. Because O_{λ_0} is open, there exists $\epsilon > 0$ such that $N_\epsilon(s) \subseteq O_{\lambda_0}$ and $N_\epsilon(s) \subset [a, b]$. By definition of the supremum, there exists $x \in S$ such that $s - \epsilon < x \leq s < b$. Hence there exists a finite subcover $\{O_{\lambda_0}, ..., O_{\lambda_n}\}$ for $[a, x]$. This gives rise to the contradiction: there must exist $y \in [a, b]$ such that $y \in N_\epsilon(s) \subseteq O_{\lambda_0}$ and $y > s$. Hence $[a, y]$ has a finite subcover, specifically the same finite subcover for $[a, x]$. Therefore it cannot be the case that $\sup S = s$. We have shown $s = b$. It remains to be shown that there exists a finite subcover for arbitrary K. If K is closed and bounded, K must be the union of a finite number of closed and bounded intervals and isolated points. If $\{O_\lambda : \lambda \in \Lambda\}$ is an open cover for K, then each closed interval or isolated point has a finite subcover, and the union of these finite subcovers is also finite, so K also has a finite subcover.

□

We provide a quick proof of the nested compact set property which we used above.

Lemma (Nested Compact Set Property). *Let I_n be non-empty and compact for every $n \in \mathbb{N}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\bigcap_{n=1}^\infty I_n$ is also non-empty.*

Proof. We construct a sequence as follows: for every $n \in \mathbb{N}$, pick x_n to be in I_n. Because the sets are nested, $x_n \in I_1$ for every $n \in \mathbb{N}$. Because I_1 is compact, there exists a subsequence (x_{n_k}) such that $\lim_{n_k\to\infty} x_{n_k} = x$ where $x \in I_1$. It is also the case that $x \in I_n$ for every $n \geq 1$. Note that $x_{n_k} \in I_n$ when $n_k \geq n$. Fix $n_0 \in \mathbb{N}$. We can ignore the first n_0 terms of (x_{n_k}) to produce another subsequence entirely in I_{n_0}. Because each set is compact, and subsequences of convergent sequences converge to the same limit, it must be the case that the subsequence converges to x and $x \in I_{n_0}$. Since n_0 was arbitrary, $x \in I_n$ for all $n \in \mathbb{N}$, and equivalently $x \in \bigcap_{n=1}^\infty I_n$.

□