Double sums

Analysis is often concerned with how certain mathematical operations respect limits. In this case, we are interested in how addition respects limits. More specifically, given a doubly indexed array $\{a_{ij} : i, j \in \mathbb{N}\}$, when does

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

i.e. when is addition commutative in infinite settings? Proving the following theorem is good practice with the triangle inequality, and more generally, learning how to control convergent sequences to produce another convergent sequence.

Theorem. *Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array. If $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$ converges, then*

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Proof. First define

$$t_{mn} := \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|.$$

The sequence (t_{nn}) is bounded by assumption, and is monotonically increasing. Hence it converges by the monotone convergence theorem. Equivalently, (t_{nn}) is Cauchy. Now we show (s_{nn}) is also Cauchy. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|t_{nn} - t_{mm}| < \epsilon$. Without loss of generality, let $m > n$. We have Then

$$\begin{aligned}
|s_{mm} - s_{nn}| &= \left| \sum_{i=n+1}^{m}\sum_{j=1}^{m} a_{ij} + \sum_{i=1}^{n}\sum_{j=n+1}^{m} a_{ij} \right| \\
&\leq \left| \sum_{i=n+1}^{m}\sum_{j=1}^{m} |a_{ij}| + \sum_{i=1}^{n}\sum_{j=n+1}^{m} |a_{ij}| \right| \\
&= |t_{mm} - t_{nn}| \\
&< \epsilon.
\end{aligned}$$

for all $m, n \geq N$. Hence (s_{nn}) is Cauchy, and equivalently, converges. Now we know the limit of s_{nn} exists, and we can set

$$S = \lim_{n\to\infty} s_{nn}.$$

Now let $S = \lim_{n\to\infty} s_{nn}$. We want to show that $S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$. Define $B := \sup\{t_{mn} : m, n \in \mathbb{N}\}$. The definition of the supremum guarantees that there exists $N \in \mathbb{N}$ such that

$$B - \frac{\epsilon}{2} < t_{nn} \leq B.$$

And—without loss of generality—since $t_{mn} \geq t_{nn}$ for $m \geq n$, it is also true that

$$B - \frac{\epsilon}{2} < t_{mn} \leq B$$

for all $m, n \geq N$. Thus for all $m, n \geq N$, we know $|t_{mn} - t_{nn}| < \frac{\epsilon}{2}$.

Now we show that there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$|s_{mn} - S| < \epsilon.$$

Assuming $m > n$, we choose N_1 such that for all $m, n \geq N_1$ we have $|t_{mn} - t_{nn}| < \frac{\epsilon}{2}$, and choose N_2 such that for all $n \geq N_2$ we have $|s_{nn} - S| < \frac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$. Then we get

$$\begin{aligned}
|s_{mn} - S| &= |s_{mn} + s_{nn} - s_{nn} + S| \\
&\leq |s_{mn} - s_{nn}| + |s_{nn} - S| \\
&= \left| \sum_{i=n+1}^{m} \sum_{j=1}^{n} a_{ij} \right| + |s_{nn} - S| \\
&\leq \left| \sum_{i=n+1}^{m} \sum_{j=1}^{n} |a_{ij}| \right| + |s_{nn} - S| \\
&= |t_{mn} - t_{nn}| + |s_{nn} - S| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, for all $m, n \geq N$ we have $|s_{mn} - S| < \epsilon$.

So far, we have shown that we can get infinitely close to the limit of the double sum S with finite "rectangles" of summands. The next step is showing that when we extend the width (columns) of the rectangle infinitely, we can still get infinitely close to the limiting sum S by choosing how many rows to add up. Meaning, we need to show that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m \geq N$, we have

$$\left| \sum_{i=1}^{m} r_i - S \right| < \epsilon$$

where $r_i = \sum_{j=1}^{\infty} a_{ij}$ i.e. the infinite sum of the i'th row.

First note that for each $i \in \mathbb{N}$, there exists K_i such that for all $n \geq K_i$ we have

$$\left| r_i - \sum_{j=1}^{n} a_{ij} \right| < \frac{\epsilon}{2m}.$$

Take $K = \max\{K_1, ..., K_m\}$. Also note that from above we can choose N' such that for all $m, n \geq N'$ we have $|s_{mn} - S| < \frac{\epsilon}{2}$. Take $N = \max\{K, N'\}$. Then, for $m, n \geq N$

$$\begin{aligned}
\left| \sum_{i=1}^{m} r_i - S \right| &= \left| \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - S \right| \\
&\leq \left| \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right| + \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - S \right| \\
&= \left| \sum_{i=1}^{m} \left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| + |s_{mn} - S| \\
&\leq \sum_{i=1}^{m} \left| r_i - \sum_{j=1}^{n} a_{ij} \right| + |s_{mn} - S| \\
&< \sum_{i=1}^{m} \frac{\epsilon}{2m} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = S$. We can use a similar approach to show that the double sum where we sum the rows first converges to the same limit once we establish that each column sum converges to a finite quantity. This can be shown using the comparison test. Intuitively it makes sense as well—if the entire double sum does not escape to infinity, then no sum over a single column should escape to infinity either.

□