Limiting proportion of time spent in state

We're often interested in the proportion of time a Markov chain would spend in a given state if the chain ran forever. We can use this lemma to prove equivalencies of ergodicity in Markov chains and irreducibility later on.

Lemma: The limit

$$q_{ij} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)},$$

exists and is well defined for each $i, j \in \{0, 1, ..., N-1\}$ in the state space, where the quantity $p_{ij}^{(k)}$ is the k-step transition probability from state i to state j.

Proof. Let T be the left-shift transformation. It's measure preserving. Let x be a sequence of random variables generated by a Markov chain. Then by the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_k=j\}}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_0=j\}}(T^k x) = f^*(x),$$

such that f^* is integrable. Using the above equality, and the fact that $\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_0=j\}}(T^k x) \leq 1$ for all n, we can use the dominated convergence theorem to rearrange the equality for q_{ij}. We have

$$\begin{aligned}
q_{ij} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)} \\
&= \frac{1}{\pi_i} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\{x \in X : x_0 = i, x_k = j\}) \\
&= \frac{1}{\pi_i} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_X \mathbb{1}_{\{x:x_0=i,x_k=j\}}\, d\mu \\
&= \frac{1}{\pi_i} \int_X \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_0=i,x_k=j\}}\, d\mu && \text{(by DCT)} \\
&= \frac{1}{\pi_i} \int_X f^*(x) \mathbb{1}_{\{x:x_0=i\}}\, d\mu \\
&= \frac{1}{\pi_i} \int_{\{x:x_0=i\}} f^*(x)\, d\mu.
\end{aligned}$$

Since f^* is integrable, we know q_{ij} exists and is well-defined. $\square$