Intermediate Value Theorem

Continuous functions have many nice properties. The intermediate value property is one of them. Its proof is a great exercise in learning how to control continuous functions to derive other related properties.

Intermediate Value Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$, and let there exist α such that $f(a) < \alpha < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = \alpha$.*

Proof. Define $K := \{x \in [a, b] : f(x) \leq \alpha\}$ and $c := \sup K$. We claim that $f(c) = \alpha$ for $c \in (a, b)$. First we show that $c \in (a, b)$. We know $c = \sup K$ exists because $a \in K$ and b is an upper bound for K. Now we show $c > a$. Because f is continuous at a, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Take $\epsilon = \alpha - f(a)$. Then for x such that $|x - a| < \delta$, $|f(x) - f(a)| < \alpha - f(a)$, implying that $f(x) < \alpha$. Let $I_1 := [a, \min\{a + \delta, b\})$. Then $I_1 \subset K$, so $c := \sup K \geq \sup I_1 = \min\{a + \delta, b\} > a$.

We follow a similar argument to show $c := \sup K < b$. Because f is continuous at b, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - b| < \delta$, then $|f(x) - f(b)| < \epsilon$. Take $\epsilon = f(b) - \alpha$. Then for x such that $|x - b| < \delta$, we have $|f(x) - f(b)| < f(b) - \alpha$, implying that $f(x) > \alpha$. Let $I_2 := (\max\{a, b - \delta\}, b]$. Then for $x \in I_2$, we know $f(x) > \alpha$. Hence x is an upper bound for K. Thus, $c := \sup K < b$. We have shown that $c \in (a, b)$.

Now we prove $f(c) = \alpha$ by showing $\alpha - \epsilon < f(c) < \alpha + \epsilon$ for every $\epsilon > 0$. Since f is continuous at c, for every $\epsilon > 0$, there exists $\delta_1 > 0$ such that if $|x - c| < \delta_1$, then $|f(x) - f(c)| < \epsilon$. And since (a, b) is open, there exists $\delta_2 > 0$ such that $(c - \delta_2, c + \delta_2) \subset (a, b)$. Take $\delta = \min\{\delta_1, \delta_2\}$. For $x \in (c - \delta, c + \delta)$, we have $f(x) - \epsilon < f(c) < f(x) + \epsilon$. Since $c := \sup K$, we know there exists $x^* \in (c - \delta, c]$ such that $f(x^*) \leq \alpha$. So $f(c) < f(x^*) + \epsilon \leq \alpha + \epsilon$. And there exits $x^{**} \in [c, c + \delta)$ such that $f(c) > f(x^{**}) - \epsilon \geq \alpha - \epsilon$. We have shown $f(c) \in (\alpha - \epsilon, \alpha + \epsilon)$ for every $\epsilon > 0$. Thus it must be true $f(c) = \alpha$, and we are done. $\square$