Taylor series

Taylor series are one of the crown jewels of analysis. You can represent most well-behaved functions as polynomials of infinite order. For reasonably well-behaved functions, their Taylor series are continous and infinitely differentiable at the points for which they converge. Here we show why Taylor series have these nice properties.

We start with Abel's Lemma, then apply it to prove Abel's Theorem. Abel's Theorem guarantees uniform convergence of a power series on intervals for which the power series converges (other theorems require absolute convergence which is stricter). Uniform convergence is key in proving Taylor series are continuous and infinitely differentiable (when some other reasonable criteria are met).

Abel's Lemma

Let $b_1 \geq b_2, \cdots \geq 0$, and let $\sum_{k=1}^{\infty} a_k$ be a series for which the partial sums are bounded. Meaning, for all $n \in \mathbb{N}$, there exists some $A > 0$ such that

$$|a_1 + ... + a_n| \leq A.$$

Then, for all $n \in \mathbb{N}$,

$$|a_1 b_1 + ... + a_n b_n| \leq A b_1.$$

Proof. Let $s_n := a_1 + ... + a_n$. Then, using the summation by parts formula, we have

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k b_k\right| &\leq \left|s_n b_{n+1} + \sum_{k=1}^{n} s_k (b_k - b_{k+1})\right| \\
&\leq A b_{n+1} + A \sum_{k=1}^{n} (b_k - b_{k+1}) && \text{(absolute partials are bounded by } A\text{)} \\
&= A b_{n+1} + A(b_1 - b_{n+1}) && \text{(telescoping sum)} \\
&= A b_1.
\end{aligned}$$

□

Now we can apply Abel's Lemma to Abel's Theorem, which guarantees uniform convergence on intervals for which the power series converges.

Abel's Theorem

Let $g(x) := \sum_{n=0}^{\infty} a_n x^n$ converge at $x = R > 0$. Define $g_n(x) := a_n x^n$. Then $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on the closed interval $[0, R]$ i.e. the partial sums converge uniformly to g on $[0, R]$.

Proof. We use the Cauchy Criterion for uniform convergence of series to prove the statement. Let $\epsilon > 0$. We need to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$, and for all $x \in [0, R]$, we have

$$|g_{m+1}(x) + g_{m+2}(x) + ... + g_n(x)| < \epsilon.$$

First, we rewrite

$$g(x) := \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n \left(\frac{x}{R}\right)^n.$$

Note that by assumption $\sum a_n R^n$ converges, thus its tail sum can be made arbitrarily small. By the Cauchy Criterion for convergent series, we can choose N such that for every $n > m \geq N$, we have

$$\left|a_{m+1} R^{m+1} + a_{m+2} R^{m+2} + ... + a_n R^n\right| < \frac{\epsilon}{2}.$$

If we let $n \to \infty$, we have

$$\left|\sum_{j=1}^{\infty} a_{m+j} R^{m+j}\right| < \frac{\epsilon}{2}.$$

Note that the quantity $\left(\frac{x}{R}\right)^{m+j}$ is monotonically decreasing in $m+j$ given our choice of x, and it is also >0. So we can apply Abel's Lemma to the sum

$$\begin{aligned}|g_{m+1}(x)+g_{m+2}(x)+...+g_n(x)| &= \left|a_{m+1}R^{m+1}\left(\frac{x}{R}\right)^{m+1}+...+a_nR^n\left(\frac{x}{R}\right)^n\right| \\ &\leq \frac{\epsilon}{2}\left(\frac{x}{R}\right)^{m+1} \\ &< \epsilon.\end{aligned}$$

We have shown that $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on the closed interval $[0,R]$. Note that we chose the tail sum of the power series at R to be less than $\frac{\epsilon}{2}$ since Abel's Lemma does not yield a *strict* inequality. We need the truncated tail sum of the $g_k(x)$'s to be *strictly* less than ϵ. □

We now know power series are continuous at the points for which they converge (it is well known that the limit of a sequence of continuous functions is also continuous if the sequence of functions converges uniformly—in this case, each function in the sequence is continuous because the sum of continuous functions is also continuous). Now we show that you can differentiate the power series term-by-term to get the power series representation of the target function's derivative.

Differentiated series

If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R,R)$, then the differentiated series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges on $(-R,R)$ as well. Because the differentiated series is itself a power series, it converges uniformly on any compact set $K \subset (-R,R)$.

Proof. First we show that for $0<s<1$, the quantity ns^{n-1} is bounded. Note that

$$\lim_{n\to\infty}\frac{(n+1)s^n}{ns^{n-1}} = s < 1,$$

and thus by the Ratio Test, the sum $\sum_{n=1}^{\infty} ns^{n-1}$ converges, and is therefore bounded. Choose $M>0$ such that $ns^{n-1} \leq M$ for all $n \in \mathbb{N}$. Now for $x \in (-R,R)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} na_nx^{n-1} &= \sum_{n=1}^{\infty} nt^{n-1}a_n\left(\frac{x}{t}\right)^{n-1} \\ &\leq M\sum_{n=1}^{\infty} a_nt^{n-1} && \left(\left(\frac{x}{t}\right)^{n-1}<1\right)\end{aligned}$$

which converges by assumption because $t \in (-R,R)$. Hence, we can conclude that $\sum_{n=1}^{\infty} na_nx^{n-1}$ converges uniformly on $(-R,R)$. □

Thus, by the differentiable limit theorem, if $f(x) := \sum_{n=0}^{\infty} a_nx^n$ converges on $(-R,R)$, we can differentiate the series term-by-term i.e. $f'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$.

Now that we've proven that power series have nice properties where they converge, we will prove Lagrange's Remainder Theorem for Taylor series.

Lagrange's Remainder Theorem

Let a function f be differentiable $N+1$ times on $(-R,R)$. Define $a_n := \frac{f^{(n)}(0)}{n!}$, for $n=0,1,2,...,N$, and

$$S_N(x) := a_0 + a_1x + a_2x^2 + ... + a_nx^N,$$

and $E_N(x) := f(x) - S_N(x)$. Then for $x \in (-R,R)$, there exists some c such that $|c|<|x|$ and

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$

Meaning, the error in the Nth order Taylor approximation is given by the $(N+1)$th term in the full Taylor expansion evaluated at some $|c|<|x|$.

Proof. First note that $f^{(n)}(0) = S_N^{(n)}(0)$ for all $0 \leq n \leq N$ since we have chosen to approximate f with a polynomial of degree N that has the same derivatives as f at 0 up to and including the Nth derivative. Therefore, $E_N^{(n)}(0) = 0$ for $0 \leq n \leq N$, and $E_N^{(N+1)}(x) = f^{(N+1)}(x)$ since $S_N^{(N+1)}(x) = 0$. Now consider the generalized mean value theorem which says that for differentiable functions g and h on the interval $[a, b]$, there exists some $c \in [a, b]$ such that

$$\frac{g(b) - g(a)}{h(b) - h(a)} = \frac{g'(c)}{h'(c)}.$$

Let's apply the generalized mean value theorem for $x > 0$ to the functions $E_N(x)$ and x^{N+1} on the interval $[0, x]$. We have

$$\frac{E_N(x) - E_N(0)}{x^{N+1} - 0^{N+1}} = \frac{E_N(x)}{x_1^{N+1}} = \frac{E'_N(x_1)}{(N+1)x^N}$$

for some $x_1 \in (0, x)$. Note that we can write the last term in that sequence of equalities as

$$\frac{E'_N(x_1) - E'_N(0)}{(N+1)x_1^N - (N+1)0^N} = \frac{E''_N(x_2)}{(N+1)(N)x_2^{N-1}}$$

for some $x_2 \in (0, x_1)$. Continuing in this way, we get

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}.$$

It follows that

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}$$

for some $c \in (0, x)$. $\square$