Bolzano-Weierstrass Theorem and its consequences

Below we prove the Bolzano-Weierstrass Theorem and use it to prove some general theorems that are helpful in determining whether sequences and series converge.

Bolzano-Weierstrass Theorem. *If (x_n) is bounded, then it contains a convergent subsequence (x_{n_k}).*

Proof. Let (x_n) be bounded. Then there exists $M \in \mathbb{N}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. We will construct a series of nested, closed intervals I_k for $k \in \mathbb{N}$ such that the length of I_k tends to 0, and each I_k contains infinitely many terms. We show that the point contained in I_k for all $k \in \mathbb{N}$ (which exists by the nested interval property) is the limit of our subsequence.

Consider the interval $[-M, M]$ which the sequence (x_n) lives in. Now bisect it into two intervals $[-M, 0]$ and $[M, 0]$. One of these intervals is guaranteed to contain infinitely many terms. Choose that interval and call it I_1. Choose n_1 such that $x_{n_1} \in I_1$. Now bisect I_1 into two intervals, and again choose the interval with infinitely many terms to be I_2, and choose $n_2 > n_1$ such that x_{n_2} is in I_2. Continue this process so that $n_1 < n_2 < ...$ and $I_1 \supseteq I_2 \supseteq ....$ There exists $x \in I_k$ for all $k \in \mathbb{N}$ by the nested interval property. Note that $\lim_{k\to\infty} |I_k| = \lim_{k\to\infty} M \left(\frac{1}{2}\right)^{k-1} = 0$. Hence, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have $|I_k| < \epsilon$. Thus, because $x_{n_k} \in I_N$ for all $k \geq N$, and $x \in I_N$, we know $|x_{n_k} - x| < \epsilon$ for all $k \geq N$. So $(x_{n_k}) \longrightarrow x$.

□

Now will use the Bolzano-Weierstrass theorem in proving the equivalence of being convergent and Cauchy. The Cauchy criterion is a handy tool we have for assessing convergence without identifying a specific limit. We need the following lemma to be able to use BW.

Lemma. *If (x_n) is Cauchy, then it is bounded.*

Proof. Let (x_n) be Cauchy. Take $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < 1$. This implies that $|x_n| < |x_N| + 1$ for all $n \geq N$. Taking $M = \max\{|x_1|, |x_2|, ..., |x_N| + 1\}$, it is clear that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Thus (x_n) is bounded. □

Now we can prove the equivalence of convergence and being Cauchy.

Theorem. A sequence (x_n) is convergent if and only if it is Cauchy.

Proof. We start with the backwards direction. Let (x_n) be Cauchy. Then there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$ we have $|x_n - x_m| < \frac{\epsilon}{2}$. Moreover, because (x_n) is Cauchy, our lemma tells us that it is bounded. By Bolzano-Weierstrass, it must contain a convergent subsequence (x_{n_k}). Let $x_{n_k} \longrightarrow x$. Then there exists $N_2 \in \mathbb{N}$ such that for all $n_k \geq N_2$, we have $|x_{n_k} - x| < \frac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$. Then for $n, n_k \geq N$

$$\begin{aligned} |x_n - x| &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So $x_n \longrightarrow x$.
Now for the forwards direction. Let (x_n) converge. Let $\epsilon > 0$. Then there exists N such that for all $n \geq N$, we have $|x_n - x| < \frac{\epsilon}{2}$. For $n, m \geq N$,

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x| + |x_m - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So (x_n) is Cauchy. We are done. □