Real Spectral Theorem

We used the Real Spectral Theorem to bound mixing times for Markov chains in Fall 2021. The following are useful lemmas in the proof of the Real Spectral Theorem. They are proved in the appendix.

Lemma 1.1: *Let T be a self-adjoint operator, and let $b, c \in \mathbb{R}$ such that $b^2 < 4c$. Then the operator $T^2 + bT + cI$ is invertible.*

Lemma 1.2: *If T is a self-adjoint operator on a non-zero real vector space V, then there exist $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that its minimal polynomial factors completely into the form*

$$p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_m).$$

Lemma 1.3: *If the minimial polynomial of an operator T on a finite dimensional vector space V factors completely into the form*

$$p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_m)$$

for some $\lambda_1, ..., \lambda_m \in \mathbb{R}$ or $\mathbb{C}$, then there exits some basis of V such that the matrix of T is upper-triangular with respect to that basis.

Real Spectral Theorem: *Let T be an operator on a real, finite dimensional inner product space V. Then the following are equivalent.*

(i) *T is self-adjoint.*

(ii) *There exists an orthonormal basis of V such that the matrix of T is diagonal with respect to that basis.*

(iii) *There exists an orthonormal basis of V consisting of eigenvectors of T.*

Proof. If $\dim V = 0$, the theorem is trivial. Let $\dim V > 0$.

(i) $\implies$ (ii). Let T be self-adjoint. Let p be the minimal polynomial of T. From Lemma 1.2, there exist $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that for each $x \in \mathbb{R}$

$$p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_n).$$

Then there exists a basis of V for which the matrix of T is upper triangular by Lemma 1.3. And since T is self-adjoint, $T = T^*$, and the matrices of T and T^* are the same (the adjoint of a real valued matrix is its transpose). So the matrix of T denoted $M(T)$ must be diagonal.

(ii) $\implies$ (i). Assume there exists an orthonormal basis of V for which the matrix of T is diagonal with respect to that basis. Denote this matrix $M(T)$. Since $M(T)$ is diagonal, $M(T) = M(T)^*$. Thus $T = T^*$, and T is self-adjoint.

(ii) $\iff$ (iii). Let $e_1, ..., e_n$ be an orthonormal basis of V. Then for $k = 1, ..., n$, we have

$$T(e_k) = \lambda_k e_k$$

if and only if $\lambda_1, ..., \lambda_n$ are on the diagonal of the matrix of T. This is equivalent to λ_k being an eigenvalue for eigenvector e_k. Thus, there exists an orthonormal basis of V for which the matrix of T is diagonal if and only if there exists an orthonormal basis of V consisting of eigenvectors of T.
We have shown all three statements are equivalent. □

Appendix

Proof of Lemma 1.1

Let T be a self-adjoint operator, and let $b, c \in \mathbb{R}$ such that $b^2 < 4c$. Then the operator $T^2 + bT + cI$ is invertible.

Proof. Assume $b, c \in \mathbb{R}$ and $b^2 < 4c$. Let $v \in V$ and $v \neq 0$. We show that the operator $T^2 + bT + cI$ is injective i.e. $\text{null}(T^2 + bT + cI) = \{0\}$ by showing the following inner product is never equal to 0. We have

$$\begin{aligned}
\langle (T^2 + bT + cI)(v), v\rangle &= \langle (T^2)(v), v\rangle + b\langle T(v), v\rangle + c\langle v, v\rangle \\
&= \langle T(v), T(v)\rangle + b\langle T(v), v\rangle + c\langle v, v\rangle && \text{(since T is self-adjoint)} \\
&\geq \|T(v)\|^2 - |b||\langle T(v), v\rangle| + c\,\|v\|^2 \\
&\geq \|T(v)\|^2 - |b|\,\|T(v)\|\,\|v\| + c\,\|v\|^2 && \text{(by Cauchy-Schwarz)} \\
&= \|T(v)\|^2 - |b|\,\|T(v)\|\,\|v\| - \frac{b^2}{4}\,\|v\|^2 + \frac{b^2}{4}\,\|v\|^2 + c\,\|v\|^2 && \text{(completing the square)} \\
&= (\|T(v)\| - \frac{|b|}{2}\,\|v\|)^2 + \|v\|^2\,(c - \frac{b^2}{4}) \\
&> 0
\end{aligned}$$

where the last line follows because the first term is squared and therefore non-negative, and the second term is positive by assumption. Hence, the inner product between the operator $T^2 + bT + cI$ applied to v and v itself is always positive. Therefore, the operator $T^2 + bT + cI \neq 0$ and $\text{null}(T^2 + bT + cI) = \{0\}$, and so it is injective, and equivalently invertible. □

Proof of Lemma 1.2

If T is a self-adjoint operator on a non-zero real vector space V, then there exists $\lambda_1, ..., \lambda_m$ such that its minimal polynomial factors completely into the form $p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_m)$.

Proof. Let $\lambda_1, ..., \lambda_m$ be the eigenvalues of T. They are real-valued because T is self-adjoint, and are the zeros of the minimal polynomial p of T. Assume for sake of contradiction that

$$p(x) \neq (x - \lambda_1) \cdot ... \cdot (x - \lambda_m).$$

If p is constant, then V is the zero vector space, but we assumed V is non-zero, so p can't be constant. Then, (by another lemma not included) it must be the case that

$$p(x) = q(x)(x^2 + bx + c)$$

where $b, c \in \mathbb{R}$ and $b^2 < 4c$ and $\deg q < \deg p$. Since p is the minimal polynomial of T, we have

$$p(T) = q(T)(T^2 + bT + cI) = 0.$$

And the operator $T^2 + bT + cI$ is invertible (i.e. $\text{null}(T^2 + bT + cI) = \{0\}$) by Lemma 1.1, so $q(T) = 0$. But we assumed $\deg q < \deg p$, so we have a contradiction. Thus, $p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_m)$ for real-valued $\lambda_1, ..., \lambda_m$. □

Proof of Lemma 1.3

Lemma 1.3: *If the minimial polynomial of an operator T on a finite-dimensional vector space V factors completely into the form*

$$p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_m)$$

for some $\lambda_1, ..., \lambda_m \in \mathbb{R}$ or $\mathbb{C}$, then there exists some basis of V such that the matrix of T is upper-triangular with respect to that basis.

Proof. Let $p(x) = (x - \lambda_1) \cdot ... \cdot (x - \lambda_m)$ be the minimal polynomial of the operator T. We use induction on m. Let $m = 1$. Then $p(T) = (T - \lambda_1 I) = 0$ by assumption, which implies $T = \lambda_1 I$ which is of course upper-triangular with respect to any basis of V. Now let $m > 1$, and let the desired result be true for all positive integers less than m. Define $\mathcal{U} := \text{range}(T - \lambda_m I)$. We know $\mathcal{U}$ is invariant under T (null spaces and ranges of polynomials applied to operators are invariant under that operator). So $T|_{\mathcal{U}}$ is an operator on $\mathcal{U}$. If $u \in \mathcal{U}$, then $u = (T - \lambda_m I)(v)$ for some $v \in V$. Now define q to be the minimal polynomial of $T|_{\mathcal{U}}$, and define $r(T) = (T - \lambda_1 I) \cdot ... \cdot (T - \lambda_{m-1} I)$. Then

$$r(T)(u) = (T - \lambda_1 I) \cdot ... \cdot (T - \lambda_m I)(v) = p(T)(v) = 0$$

by our assumption about the form of the minimal polynomial p of T. Hence, $r(T)$ is a polynomial multiple of $q(T)$. Therefore, there exist $\alpha_1, ..., \alpha_n \in \{\lambda_1, ..., \lambda_{m-1}\}$ for $n \leq m - 1$ such that $q(T) = (T - \alpha_1 I) \cdot ... \cdot (T - \alpha_n I) = 0$. By our induction hypothesis, there exits a basis $u_1, ..., u_M$ of $\mathcal{U}$ such that $T|_{\mathcal{U}}$ is upper-triangular with respect to that basis. Extend this basis of $\mathcal{U}$ to a basis of V so that $u_1, ..., u_M, v_1, ..., v_N$ is a basis of V. Now we show an equivalent condition for upper-triangulability, which is that $T(a_k) \in \text{span}(a_1, ..., a_k)$ for each $k = 1, ..., \dim V$ where $a_1, ..., a_k$ is a basis for V. In our case, if we take any u_k for $k = 1, ..., M$ we have

$$T(u_k) = T|_{\mathcal{U}}(u_k) \in \text{span}(u_1, ..., u_k)$$

since $T|_{\mathcal{U}}$ is upper-triangular. Now take any v_k for $k = M + 1, ..., N$. We make a basic manipulation to the expression $T(v_k)$ so that we have

$$T(v_k) = (T - \lambda_m I)(v_k) + \lambda_m v_k.$$

We see that $(T - \lambda_m I)(v_k) \in \mathcal{U}$ by definition, and therefore $(T - \lambda_m I)(v_k) \in \text{span}(u_1, ..., u_M)$. Consequently,

$$(T - \lambda_m I)(v_k) + \lambda_m v_k \in \text{span}(u_1, ..., u_M, v_k) \subset \text{span}(u_1, ..., u_M, v_1, ..., v_k).$$

This is equivalent to T being upper-triangular with respect to the extended basis. We are done. $\square$