Uniform convergence and differentiability

For a sequence of functions (f_n) such that $f_n \longrightarrow f$ pointwise, we know $f' = \lim_{n\to\infty} f'_n$ when the sequence (f'_n) converges uniformly. But we can say something stronger. Particularly, when (f'_n) converges uniformly, not only does $f' = \lim_{n\to\infty} f'_n$, but (f_n) converges uniformly too.

Exercise 6.3.7 in *Understanding Analysis*

Claim: *Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$ such that $f_n \to f$ pointwise. Assume $f'_n \to g$ uniformly on $[a,b]$. If there exists a point x_0 in $[a,b]$ such that $(f_n(x_0)) \to f(x_0)$, then $f_n \to f$ uniformly on $[a,b]$.*

Proof. Let $\epsilon > 0$. To show (f_n) converges uniformly, we use the Cauchy Criterion for uniform convergence i.e. we show there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ and for all $x \in [a,b]$, we have

$$|f_n(x) - f_m(x)| < \epsilon.$$

First, define $g_{n,m}(x) = f_n(x) - f_m(x)$., By the triangle inequality, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &= |g_{n,m}(x) - g_{n,m}(x_0)| + |f_n(x_0) - f_m(x_0)|. \end{aligned}$$

Since $(f_n(x_0))$ converges by assumption, by the Cauchy criterion, we choose N_1 such that for $n, m \geq N_1$, we have $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$. Each f_n is differentiable by assumption, and thus $g'_{n,m} = f'_n - f'_m$, so we can apply the Mean Value Theorem to $g_{n,m}$. Specifically, there exists $c \in (x, x_0)$ such that

$$g'_{n,m}(c) = \frac{g_{n,m}(x) - g_{n,m}(x_0)}{x - x_0}.$$

Thus, we have

$$\begin{aligned} |g_{n,m}(x) - g_{n,m}(x_0)| + |f_n(x_0) - f_m(x_0)| &= |x - x_0| \left| \frac{g_{n,m}(x) - g_{n,m}(x_0)}{x - x_0} \right| + |f_n(x_0) - f_m(x_0)| \\ &= |x - x_0||g'_{n,m}(c)| + |f_n(x_0) - f_m(x_0)| \\ &\leq |b - a||g'_{n,m}(c)| + |f_n(x_0) - f_m(x_0)|. \end{aligned}$$

By assumption, $(g'_{n,m})$ converges uniformly to 0 as both n and m tend to ∞. So we can pick $N_2 \in \mathbb{N}$ such that for $n, m \geq N_2$ we have $|g'_{n,m}(c)| < \frac{\epsilon}{2|b-a|}$. Let $N = \max\{N_1, N_2\}$. Then for $n, m \geq N$, we have

$$\begin{aligned} |b - a||g'_{n,m}(c)| + |f_n(x_0) - f_m(x_0)| &< |b - a| \frac{\epsilon}{2|b - a|} + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So $|f_n(x) - f_m(x)| < \epsilon$ for any $x \in [a,b]$, and thus (f_n) converges uniformly on $[a,b]$. .

$\square$