Fatou's Lemma

If (f_n) is a sequence of non-negative, measurable functions, then

$$\liminf_{n\to\infty}\left(\int f_n\right) \geq \int \liminf_{n\to\infty} f_n.$$

Proof. We apply the Monotone Convergence Theorem to the non-negative, increasing sequence of functions $(\inf_{k\geq n} f_k)_{n\geq 1}$. First note that by definition, we have

$$\liminf_{n\to\infty}\int f_k = \lim_{n\to\infty}\left(\inf_{k\geq n}\int f_k\right) \tag{1}$$

Because $\inf_{k\geq n} f_k \leq f_k$ for all $k \geq n$, by monotonicity of the integral we have that

$$\int \inf_{k\geq n} f_k \leq \inf_{k\geq n}\int f_k \tag{2}$$

Combining (1) and (2) we get

$$\begin{aligned}
\liminf_{n\to\infty}\int f_k &= \lim_{n\to\infty}\left(\inf_{k\geq n}\int f_k\right)\\
&\geq \lim_{n\to\infty}\int \inf_{k\geq n} f_k\\
&= \int \lim_{n\to\infty}\inf_{k\geq n} f_k && \text{(by MCT)}\\
&= \int \liminf_{n\to\infty} f_k.
\end{aligned}$$

The desired inequality has been shown. □

Dominated Convergence Theorem

Let (f_n) be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all x and some function f. And suppose $|f_n| \leq g$ for some integrable function g. Then

$$\lim_{n\to\infty}\int f_n\; d\mu = \int f\; d\mu.$$

Proof. To prove the desired statement, we show that

$$\limsup_{n\to\infty}(\int f_n\; d\mu) \leq \int f\; d\mu \leq \liminf_{n\to\infty}(\int f_n\; d\mu)$$

which implies

$$\lim_{n\to\infty}\int f_n\; d\mu = \int f d\mu.$$

Note that $g - f_n \geq 0$ and is integrable so we can apply Fatou's Lemma to it. We have

$$\begin{aligned}
\int g\; - \int f = \int (g-f) &&& \text{(by linearity of integrals)}\\
&= \int \liminf_{n\to\infty}(g-f_n) && \text{(f_n converges to f pointwise)}\\
&\leq \liminf_{n\to\infty}\int (g-f_n) && \text{(by Fatou's Lemma)}\\
&= \int g\; - \limsup_{n\to\infty}\int f_n && \text{(by linearity and $\liminf = -\limsup$).}
\end{aligned}$$

It follows that $\limsup_{n\to\infty} \int f_n \leq \int f$. We apply Fatou's Lemma to $g + f_n$ (which is possible because $|g + f_n| \leq 2g$ and $2g$ is integrable).

$$\begin{aligned}
\int g + \int f &= \int (g + f) && \text{(by linearity of integrals)} \\
&= \int \liminf_{n\to\infty} (g + f_n) && \text{(f_n converges to f pointwise)} \\
&\leq \liminf_{n\to\infty} \int (g + f_n) && \text{(by Fatou's Lemma)} \\
&= \int g + \liminf_{n\to\infty} \int f && \text{(by linearity).}
\end{aligned}$$

It follows that $\liminf_{n\to\infty} \int f \geq \int f$. Thus, we've shown the desired inequality, and it must be the case that $\lim_{n\to\infty} \int f_n = \int f$. $\square$