The inspection paradox is one of my favorite results in probability theory. It says that for a renewal process with a finite average waiting time, an observer will spend more time waiting for the next arrival than the average waiting time itself. For example, if a bus arrives at a bus stop every 5 minutes on average, someone waiting to get on the bus will wait longer than 5 minutes on average. This is certainly paradoxical, but nonetheless feels true from experience. I prove this fact below.

Proof. To start, we define some variables. Let X_t be the value of the renewal process at time t; let J_{X_t} be the time of the X_t'th arrival, and S_{X_t} the holding time for the X_t'th arrival. Now, consider time t in the interval $[J_{X_t}, J_{X_t+1}]$, i.e. we have observed X_t but not $X_t + 1$. Then, we have

$$\begin{aligned}
P(S_{X_t+1} > x) &= \int_0^\infty P(S_{X_t+1} > x | J_{X_t} = s) f_{J_{X_t}}(s)\ ds \\
&= \int_0^\infty P(S_{X_t+1} > x | S_{X_t+1} > t - s) f_{J_{X_t}}(s)\ ds \\
&= \int_0^\infty \frac{P(S_{X_t+1} > x, S_{X_t+1} > t - s)}{P(S_{X_t+1} > t - s)} f_{J_{X_t}}(s)\ ds \\
&= \int_0^\infty \frac{1 - F(\max\{x, t - s\})}{1 - F(t - s)} f_{J_{X_t}}(s)\ ds \\
&= \int_0^\infty \min\{\frac{1 - F(x)}{1 - F(t - s)}, \frac{1 - F(t - s)}{1 - F(t - s)}\} f_{J_{X_t}}(s)\ ds \\
&= \int_0^\infty \min\{\frac{1 - F(x)}{1 - F(t - s)}, 1\} f_{J_{X_t}}(s)\ ds \\
&\geq \int_0^\infty (1 - F(x)) f_{J_{X_t}}(s)\ ds \\
&= 1 - F(x) \\
&= P(S_1 > x)
\end{aligned}$$

□