

The Poincare Recurrence Theorem implies that almost every element in B returns to B infinitely often.

Claim. There exists infinitely many integers $n_1 < n_2 < n_3 \dots$ such that for $x \in B$ we have $T^{n_i}x \in B$ almost everywhere.

Proof. Let

$$D = \{x \in B : T^k x \in B \text{ for finitely many } k \geq 1\}.$$

Meaning, to be in D you are an element of the set B such that after enough transformations you never return to B . We can rewrite the condition of D in terms of our familiar set F comprised of the elements of B that never return to B ,

$$D = \{x \in B : T^k x \in F \text{ for some } k \geq 0\}.$$

In words, if you are an element of D , you are an element of B with the extra condition that you never return to B after some finite amount of transformations. Therefore

$$D \subseteq \bigcup_{k=0}^{\infty} T^{-k} F.$$

Thus, by monotonicity of measure, we have

$$\mu(D) \leq \mu\left(\bigcup_{k=0}^{\infty} T^{-k} F\right) = 0$$

(where we proved the union of the preimages of repeated transformations had measure 0 in the proof of Poincare's Recurrence Theorem). Hence, almost all $x \in B$ return to B infinitely often. □