The Borel-Cantelli lemma is one of my favorite results in measure and probability theory. And the proof is particularly elegant to me.

Claim. For finite measure space (X, Σ, μ), if $\{E_n\}_{n=1}^{\infty} \subset \Sigma$ and $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then $\mu(\limsup_{n\to\infty} E_n) = 0$.

Proof. By definition, we have

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

To tidy things up, we define

$$A_n := \bigcup_{k=n}^{\infty} E_k,$$

so, substituting we get

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} A_n.$$

Of course, since we are only removing elements from $\{A_n\}$ as n increases, we have $A_1 \supset A_2 \supset A_3$.... Thus, by continuity from above–a basic property of measure–we can write

$$\begin{aligned}
\mu(\limsup_{n\to\infty} E_n) &= \mu(\bigcap_{n=1}^{\infty} A_n) \\
&= \lim_{n\to\infty} \mu(A_n) && \text{(by continuity from above)} \\
&= \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) && \text{(substituting back)} \\
&\leq \lim_{n\to\infty} (\sum_{k=n}^{\infty} \mu(E_k)) && \text{(by countable subadditivity property of measure)} \\
&= 0.
\end{aligned}$$

Note that the last step follows from the fact that $\sum_{n=1}^{\infty} \mu(E_n)$ converges. Meaning, for any $\epsilon > 0$, there exists some N such that for each $k \geq N$, we have $\sum_{k=N}^{\infty} \mu(E_k) < \epsilon$. Hence, we can make the tail end of the sum arbitrarily small. Therefore, the limit in the second to last line converges to 0. And lastly, by non-negativity of measure, we have the limit superior bounded above and below by 0. Thus, $\mu(\limsup_{n\to\infty} E_n) = 0$. $\square$