

The Poincare Recurrence Theorem implies that almost every element in  $B$  returns to  $B$  infinitely often.

**Claim.** There exists infinitely many integers  $n_1 < n_2 < n_3 \dots$  such that for  $x \in B$  we have  $T^{n_i}x \in B$  almost everywhere.

*Proof.* Let

$$D = \{x \in B : T^k x \in B \text{ for finitely many } k \geq 1\}.$$

Meaning, to be in  $D$  you are an element of the set  $B$  such that after a finite number of transformations you never return to  $B$ . We can rewrite the condition of  $D$  in terms of our familiar set  $F$  which is comprised of the elements of  $B$  that never return to  $B$ ,

$$D = \{x \in B : T^k x \in F \text{ for some } k \geq 0\}.$$

So  $D$  is a subset of the union of the successive preimages of  $F$  under  $T$

$$D \subseteq \bigcup_{k=0}^{\infty} T^{-k}F.$$

Thus, by monotonicity of measure, we have

$$\mu(D) \leq \mu\left(\bigcup_{k=0}^{\infty} T^{-k}F\right) = 0$$

(where we proved the union of the preimages of repeated transformations had measure 0 in the proof of Poincare's Recurrence Theorem). Hence, almost all  $x \in B$  return to  $B$  infinitely often. □