Claim. The limit

$$q_{ij} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)},$$

exists and is well defined for each $i, j \in \{0, 1, ..., N-1\}$ where the quantity $p_{ij}^{(k)}$ is the k-step transition probability from state i to state j.

Proof. First we note that the left-shift transformation T is measure preserving. By the ergodic theorem, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_k=j\}}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_0=j\}}(T^k x) = f^*(x),$$

such that f^* is integrable. Using the above equality, and the fact that $\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_0=j\}} \leq 1$ for all n, we can use the dominated convergence theorem to rearrange the formula for q_{ij}.

$$\begin{aligned}
q_{ij} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)} \\
&= \frac{1}{\pi_i} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\{x \in X : x_0 = i, x_k = j\}) \\
&= \frac{1}{\pi_i} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_X \mathbb{1}_{\{x:x_0=i,x_k=j\}}\, d\mu(x) \\
&= \frac{1}{\pi_i} \int_X \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{x:x_0=i,x_k=j\}}\, d\mu(x) && \text{(by DCT)} \\
&= \frac{1}{\pi_i} \int_X f^*(x) \mathbb{1}_{\{x:x_0=i\}}\, d\mu(x) \\
&= \frac{1}{\pi_i} \int_{\{x:x_0=i\}} f^*(x)\, d\mu(x).
\end{aligned}$$

Since f^* is integrable, we know q_{ij} exists and is well-defined. □