Claim. For finite measure space (X, Σ, μ), if $\{E_n\}_{n=1}^{\infty} \subset \Sigma$ and $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then $\mu(\limsup_{n\to\infty} E_n) = 0$.

Proof. By definition, we have

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

To tidy things up, we define the inner union as

$$A_n := \bigcup_{k=n}^{\infty} E_k$$

. Substituting our newly defined A_n into the equation, we can write.

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} A_n.$$

Now we note the descending structure of A_n. Since we only remove elements from $\{A_n\}$ as n increases, we have $A_1 \supset A_2 \supset A_3$.... Thus, by continuity from above (a property of measure) we can write

$$\begin{aligned}
\mu(\limsup_{n\to\infty} E_n) &= \mu(\bigcap_{n=1}^{\infty} A_n) \\
&= \lim_{n\to\infty} \mu(A_n) && \text{(by continuity from above)} \\
&= \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) && \text{(substituting back)} \\
&\leq \lim_{n\to\infty} (\sum_{k=n}^{\infty} \mu(E_k)) && \text{(by countable subadditivity property of measure)} \\
&= 0.
\end{aligned}$$

The last line above follows intuitively. We are given that $\sum_{n=1}^{\infty} \mu(E_n)$ converges, so the tail end of the sum must tend to 0. More formally, but briefly: by definition, the corresponding partial sums $S_1, S_2, S_3, ...$ must converge, say, to L. Thus, for any $\epsilon > 0$, there exists some N such that for $k - 1 = N$, we have $|S_{k-1} - L| < \epsilon$. Decomposing L into the k-1th partial sum and the remaining tail, we subtract the two identical partial sums. We are left with the tail $\sum_{k=N}^{\infty} \mu(E_k) < \epsilon$. This is the limit in the second to last line, and it converges to 0. To complete the proof, by non-negativity of measure, we have the limit superior bounded below by 0, too. Thus, $\mu(\limsup_{n\to\infty} E_n) = 0$. $\square$