Finding Optimal Portfolio on Asset with Nonlinear Optimization

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1. Abstract

The field of finance places great emphasis on portfolio optimization, which involves determining the optimal distribution of assets in a portfolio to achieve maximum returns while minimizing risk. Nonlinear optimization is frequently utilized to solve complex problems that involve numerous variables and constraints. In this article, we will explore how nonlinear optimization can be applied to portfolio optimization by focusing on the example of a stock portfolio. We will examine the Minimum Variance Portfolio (MVP), which seeks to minimize the risk associated with individual stocks, and propose an alternative method that aims to achieve a higher rate of return.

2. Introduction

There are a multitude of investment strategies available to investors, ranging from low-risk options that offer steady, but modest returns to high-risk options that aim to achieve significant returns. Regardless of the strategy, the primary goal is to minimize volatility and achieve high returns. Portfolio models and stock price prediction models have been developed to aid investors in this pursuit, with most portfolio models focusing on maximizing returns through efficient allocation of investments in stocks.

One such model is the Markowitz Model, developed by American economist Harry Markowitz, who is considered the father of modern portfolio theory. In 1990, Markowitz was awarded the Nobel Prize for his contributions to modern portfolio theory, which has since been extensively studied and analyzed.

This article will provide an overview of the Markowitz model, as well as explore the Minimum Variance Portfolio (MVP) model that was inspired by it. Additionally, this paper will delve into how nonlinear optimization can be used to formulate the MVP model. Furthermore, the paper will introduce a Modified Minimum Variance Portfolio (MMVP) model that has the potential to outperform the MVP model in terms of returns. Finally, we will test the performance of both models on four randomly selected groups of three stocks each and review the results.

3. Several Models of Asset Allocation

3.1 Markowitz's model

The Markowitz model is an investment technique designed to maximize portfolio returns while minimizing risk. The model suggests that risk can be reduced by selecting stocks in each sector that are less correlated, thereby diversifying the portfolio to reduce the similarity between stocks. This approach is in line with the adage "Don't put all your eggs in one basket." The model employs mathematical calculations to enable a systematic analysis of risk.

For instance, an investor with a portfolio of 30 technology stocks will likely experience high correlation. However, if the investor holds a portfolio of 30 stocks spread across different investment sectors, the risk will be lower.

One of the limitations of the Markowitz model is its reliance on historical data, as well as assumptions that may not be relevant to the stock market. Additionally, the model uses mean variance instead of potential risk, which may not accurately reflect real-world scenarios. Nevertheless, the Markowitz model remains significant in the world of investment as it laid the foundation for modern investment theory.

3.2 Minimum Variance Portfolio

The Minimum Variance Portfolio (MVP) is a specific portfolio that is constructed using Markowitz Portfolio Theory. It is not the same thing as the theory of Markowitz Portfolio Theory. Markowitz Portfolio Theory provides a framework for constructing portfolios that optimize risk and return, while the Minimum Variance Portfolio is one possible outcome of that framework.

The MVP minimizes investment risks. In this method, investors determine the variance to reduce the portfolio’s volatility. Investors combine stock holdings in such a way that the price volatility of the entire portfolio is lower. The least variance method considers the investment weight and the variance of each investment. Therefore, to minimize diversification, investors diversify their investments.

The key to this model is the use of variance as a measure of risk. Stocks with high volatility are likely to have high variance. Conversely, stocks with low volatility are likely to have small variance. A portfolio with high variance (Risk) increases the likelihood of facing significant losses in the near future. Therefore, variance is used as a measure for judging a portfolio.

The formula used in MVP is as follows:

V: The expected risk of the portfolio
N: The number of stocks included in the portfolio
w_i: The Asset Allocation Weight (i=1,2,…,N)
σ_i: Standard deviation of the rate of return of item i (i=1,2,…,N)
σ_{ij}: Covariance of returns of items i and j (i,j=1,2,…,N)

$$Minimize \quad V = \sum_{i=1}^{N} {w_i}^2 \sigma^2 + \sum_{i=1}^{N} \sum_{j \neq j}^{N} w_i w_j \sigma_{ij}$$

$$Subject\ to \quad \sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0\ for\ i = 1,2, \ldots, N$$

<Fig 1: MVP formula>

V represents the diversification of the portfolio, and we want to find the proportion that provides the minimum variance. And the sum of each weight is 1, which means that all resources are distributed. A condition in which 'w_i is 0 or greater' means that a negative weight is not allowed. That means, 'Short selling is not allowed'. As indicated in the objective function, since a quadratic function is used, nonlinear optimization must be used to find the optimal value.

Next, we will look at the process of deriving the expression. To make things simpler, rather than deriving n terms, let's try deriving for two stocks. In the end, what MVP is looking for is the combined variance result, i.e. $Var(aX + bY)$. We want to derive the equation below.

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

Linearity of Expectations

$$E(aX) = aE(X)$$
$$E(X + Y) = E(X) + E(Y)$$
$$E(aX + bY) = aE(X) + bE(Y)$$

Properties of variance

$$\begin{aligned} Var(aX) &= E[\left(aX - E(aX)\right)^2] \\ &= E[a\{X - E(X)\})^2] \\ &= E[a^2\{X - E(X)\}^2] \\ &= E[a^2]\, E[\{X - E(X)\}^2] \\ &= a^2 Var(X) \end{aligned}$$
$$Var(X + Y) = E[((X + Y) - E(X + Y))^2]$$

$$
\begin{aligned}
&= E[((X - E(X)) + (Y - E(Y)))^2] \\
&= E[\left(X - E(X)\right)^2 + \left(Y - E(Y)\right)^2 + 2\left(X - E(X)\right)\left(Y - E(Y)\right)] \\
&= E[\left(X - E(X)\right)^2] + E[\left(Y - E(Y)\right)^2] \\
&\quad + 2E[\left(X - E(X)\right)\left(Y - E(Y)\right)] \\
&= Var(X) + Var(Y) + 2Cov(X, Y)
\end{aligned}
$$

If we combine above equations about variance, we can get the result below.

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$$

If we apply earlier equations to 3 variables, we can get the result below.

$$
\begin{aligned}
Var(aX + bY + cZ) = {} & a^2Var(X) + b^2Var(Y) + c^2Var(Z) \\
& + 2abCov(X, Y) + 2acCov(X, X) + 2bcCov(Y, Z)
\end{aligned}
$$

3.2.1 Advantages and disadvantages of MVP

The advantages are as follows:

1. The portfolio becomes resistant to a risk
2. Diversification helps investors understand different sectors.
3. A variety of financial instruments fit this investment strategy.

The disadvantages are as follows:

1. It is more inclined towards variance and tends to overlook potential risks.
2. It does not guarantee good returns and is only based on historical data.
3. The model does not account for associated costs like broker commissions, taxes, and other charges.

3.3 Modified MVP Model (MMVP)

Inspired by Markowitz's theory, MVP is a pioneering framework for minimizing risk and generating returns, but it has limitations. When variance is minimized, the risk is stable, but the best returns are difficult to obtain. Therefore, this paper proposes a model that can achieve a higher rate of return by including the rate of return in the objective function while minimizing the variance among portfolio components.

The purpose is to: Instead of increasing the risk a little bit, it is to find weights of asset that can achieve a better rate of return than MVP.

In other words, we modified the MVP to maximize return by including 'rate of return' in the objective function. And to unify the returns, we divided them by the standard deviation. The risk may be slightly higher, but the risk is still included in the denominator, so the risk is manageable. In addition, if stocks are completely excluded, large profits may be missed, so we allocate a weight of at least 15% per stock. Finally, to avoid biasing the asset allocation, we put an upper limit on each asset to no more than two-thirds.

Unlike the MVP model, which only focuses on portfolio risk reduction, this is an improved method that seeks to both reduce risk and improve returns. The objective function and constraints are shown in the equation below.

The formula used in Modified MVP is as follows:

V: The expected return of the portfolio
N: The number of stocks included in the portfolio
w_i: The Asset Allocation Weight (i=1,2,…,N)
r_i: The return ratio of the stock (i=1,2,…,N)
σ_i: Standard deviation of the rate of return of item i (i=1,2,…,N)
σ_{ij}: Covariance of returns of items i and j (i,j=1,2,…,N)

$$Maximize \quad V = \frac{\sum_{i=1}^{N} \frac{w_i r_i}{\sigma_i}}{\sum_{i=1}^{N} {w_i}^2 \sigma^2 + \sum_{i=1}^{N} \sum_{j \neq j}^{N} w_i w_j \sigma_{ij}}$$

$$Subject\ to \quad \sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0.15\ for\ i = 1,2, \dots, N$$

$$w_i \geq 0.67\ for\ i = 1,2, \dots, N$$

<Fig 2: Modified MVP formula>

4. Implementation on MVP and MMVP

4.1 Implementation overview

Two programs for MVP and MMVP are developed to get an optimal solutions using AMPL, a mathematical programming language for solving optimization problems. The programs were written using AMPL syntax.

4.2 Implementation on MVP

$$Minimize \quad V = \sum_{i=1}^{N} {w_i}^2 \sigma^2 + \sum_{i=1}^{N} \sum_{j \neq j}^{N} w_i w_j \sigma_{ij}$$

$$Subject\ to \quad \sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0\ for\ i = 1,2, \dots, N$$

The core source code is below.

```
# Target variables
var w1;
var w2;
var w3;

var riskVar = w1^2 * stdev1^2 + w2^2 * stdev2^2 + w3^2 * stdev3^2 + 2 * w1 * w2 * cov12 + 2 * w1
* w3 * cov13 + 2 * w2 * w3 * cov23;
```

```
minimize MVP: riskVar; # MVP
subject to c1: w1 + w2 + w3 = 1;
subject to c2: w1 >= 0;
subject to c3: w2 >= 0;
subject to c4: w3 >= 0;
```

<Fig 3: The core implementation of MVP>

4.3 Implementation of MMVP

$$Maximize \qquad V = \frac{\sum_{i=1}^{N} \frac{w_i r_i}{\sigma_i}}{\sum_{i=1}^{N} w_i{}^2 \sigma^2 + \sum_{i=1}^{N} \sum_{j \neq j}^{N} w_i w_j \sigma_{ij}}$$

$$Subject\ to \qquad \sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0.15\ for\ i = 1,2, \dots, N$$

$$w_i \geq 0.67\ for\ i = 1,2, \dots, N$$

The core source code is below.

```
# Target variables
var w1;
var w2;
var w3;

var expRet = w1*ret1/stdev1 + w2*ret2/stdev2 + w3*ret3/stdev3;
var riskVar = w1^2 * stdev1^2 + w2^2 * stdev2^2 + w3^2 * stdev3^2 + 2 * w1 * w2 * cov12 + 2 * w1
* w3 * cov13 + 2 * w2 * w3 * cov23;
var totalRet = expRet / riskVar;

maximize MMVP: totalRet; # MMVP
subject to c1: w1 + w2 + w3 = 1;
subject to c2: w1 >= 0.15;
subject to c3: w2 >= 0.15;
subject to c4: w3 >= 0.15;
subject to c5: w1 <= 0.67;
subject to c6: w2 <= 0.67;
subject to c7: w3 <= 0.67;
```

<Fig 4: The core implementation of Modified MVP>

5 Experimental results

5.1 The purpose of the experiment

1. Apply nonlinear optimization to the MVP theory to find the optimal weight value that minimizes risk.
2. Apply nonlinear optimization to the MMVP theory to find the optimal weighting value that can achieve a better rate of return than MVP even at the expense of a little risk.

5.2 Experiment setup overview

To implement and validate each model, the entire experiment is divided into three phases.

1. Collect price data for each stock item (Yahoo Finance website)
2. Calculate the constant value of each event (Excel)
3. Execute a non-linear optimization program to derive the result (AMPL implementation)

The prices of stocks were extracted for three years from the Yahoo Finance site. In addition, the constant values (standard deviation and covariance) required for the nonlinear optimization formula were obtained using Excel. Full values can be found in appendix at the end of the document. Finally, a nonlinear optimization program that accepts the constant value as a factor was implemented as AMPL. For test brevity, the AMPL implementation supports three stocks.

5.3 The test data and scenario

The experiment consisted of:
1. 3 stocks were randomly selected so that the sectors do not overlap.
2. Indicators were obtained with monthly price data for the past 3 years (2020, 2021, 2022).
3. After investing in December 2022 with the calculated ratio, compare the rate of return and variance when held until recently (4/10/2023).

The above test was repeatedly applied to the 4 groups and the final conclusion was drawn.

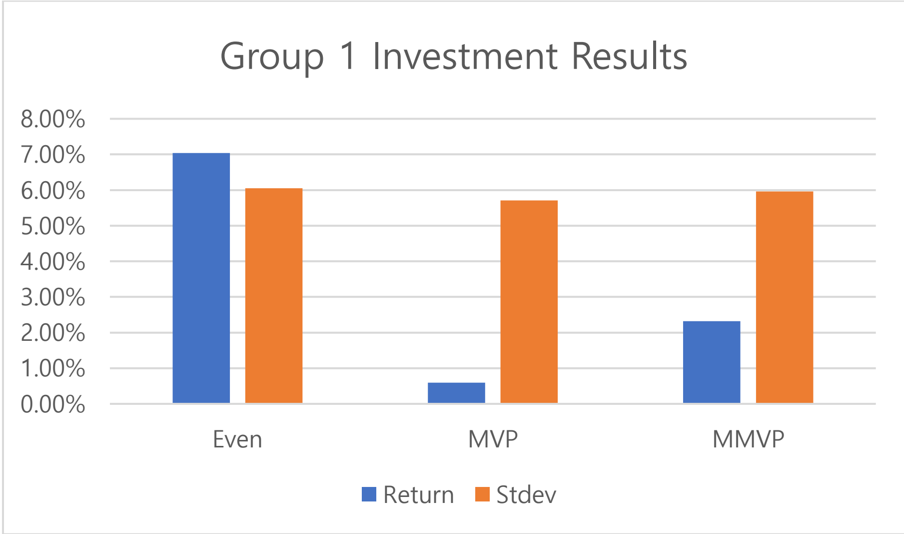
Stock sector: Technology, Healthcare and Consumption	
Group 1	Apple, United health care, Coca-Cola
Group 2	Microsoft, J&J, Walmart
Group 3	Netflix, Merck, Pepsi
Group 4	Google, AbbVie, P&G

<Table 1: Test group for the test>

5.4 The test result

The results of the investment for holding 3 stocks in each of the 4 groups, are presented below. The test is divided into three cases: equal distribution, MVP-based asset allocation, and MMVP-based asset allocation.

Upon analyzing the graphs and tables, it can be observed that MMVP outperformed MVP in all 4 groups. While MVP aims to minimize risk (variance), MMVP achieves this by taking into account the return on MVP.



<Fig 5: The test result of Group 1>

Group 1	Apple	United H	Coca-Cola	Return	Variance	Stdev
Even	33.00%	33.00%	33.00%	7.04%	0.37%	6.05%
MVP	8.30%	40.40%	51.20%	0.60%	0.33%	5.71%
MMVP	15.00%	67.00%	18.00%	2.32%	0.35%	5.96%

<Table 2: The test result of Group 1>

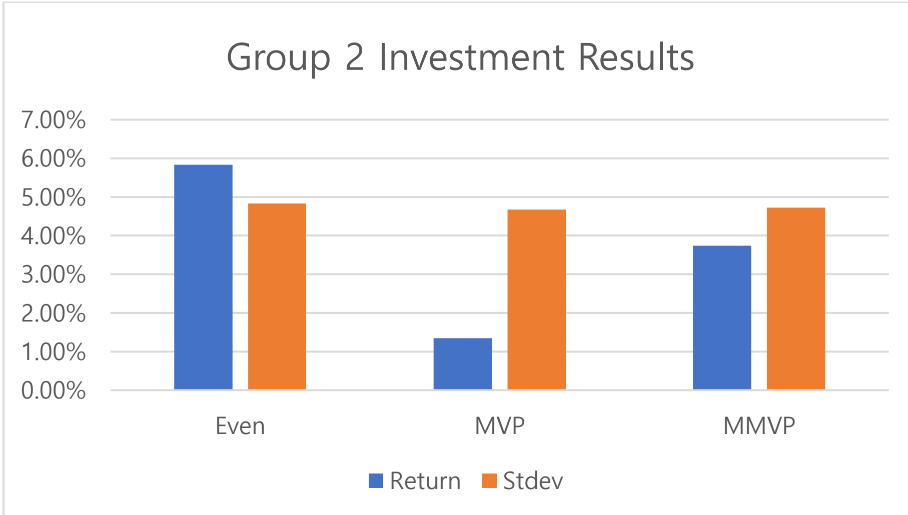
Let's look at the table above as the result of Group 1. 'Even' is the result when all assets are invested equally, MVP is for when MVP is applied and MMVP is for when MMVP is applied. MVP determined that risk is minimal when it allocated 8.3% to Apple, 40.4% to United Health care, and 51.2% to Coca-Cola. Unlike MVP, MMVP judged that risk is a bit higher when 15%, 67%, and 18% are invested, but higher returns can be achieved.

In other words, MVP found an optimal ratio that could only minimize risk while MMVP found an optimal ratio for a higher rate of return at a slightly higher risk than MVP. Similar results were observed in Group 2, 3, and 4.

In the end, as reviewed in the previous equation, it can be seen that MMVP guarantees a better rate of return (a slight increase in risk) than MVP by including rate of return and risk in the objective function at the same time.

It is also noticeably that the highest rate of return was achieved in the case of equal distribution. This is primarily due to the recent significant rise in the IT sector, while the price of other sectors remain almost the same.

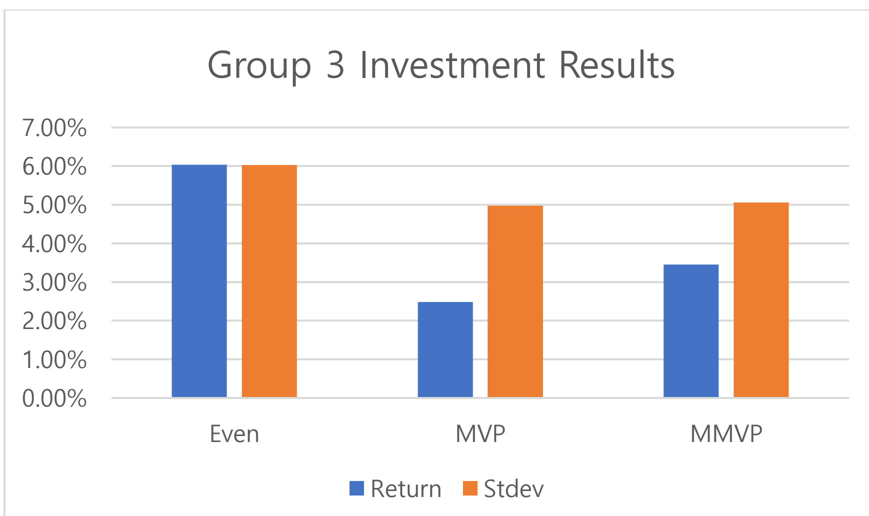
Below is the result of Group 2, 3, and 4.



<Fig 6: The test result of Group 2>

Group 2	MS	J&J	Walmart	Return	Variance	Stdev
Even	33.30%	33.30%	33.30%	5.83%	0.23%	4.83%
MVP	18.90%	54.10%	26.80%	1.34%	0.22%	4.67%
MMVP	29.70%	45.90%	24.24%	3.74%	0.22%	4.73%

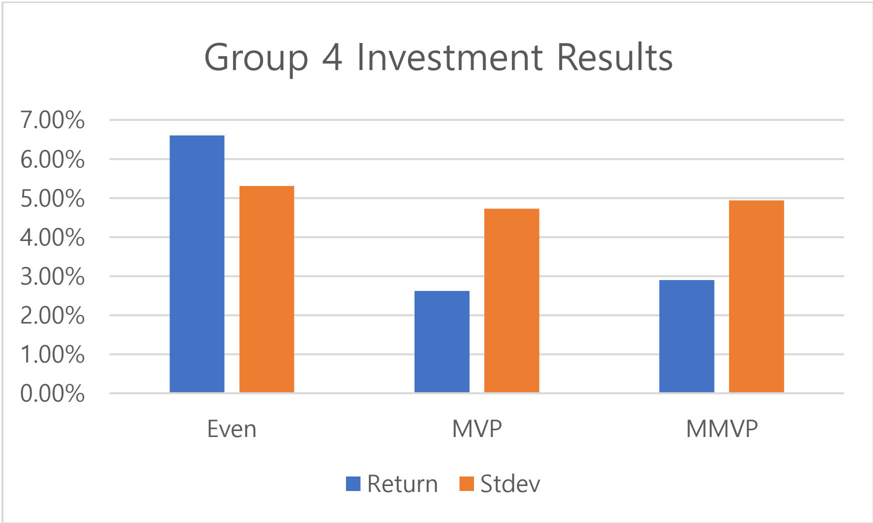
<Table 3: The test result of Group 2>



<Fig 7: The test result of Group 3>

Group 3	Netflix	Merck	Pepsi	Return	Variance	Stdev
Even	33.30%	33.30%	33.30%	6.04%	0.36%	6.03%
MVP	8.10%	17.30%	74.50%	2.48%	0.25%	4.97%
MMVP	15.00%	18.80%	66.10%	3.45%	0.26%	5.06%

<Table 4: The test result of Group 3>



<Fig 8: The test result of Group 4>

Group 4	Google	Abbvie	P&G	Return	Variance	Stdev
Even	33.30%	33.30%	33.30%	6.60%	0.28%	5.31%
MVP	13.80%	20.60%	65.40%	2.62%	0.22%	4.72%
MMVP	15.00%	37.60%	47.60%	2.90%	0.24%	4.94%

<Table 5: The test result of Group 4>

6. Conclusion

It is clear that nonlinear optimization has a wide range of applications in the real world and portfolio optimization is only one example. Taking MVP (Minimum Variance Portfolio) and MMVP (Modified MVP) as examples, since the objective function uses a two-dimensional equation, it is obvious that they are good examples for nonlinear optimization.

The main goal of modern portfolio theory is to maximize returns while minimizing risk. And by writing a nonlinear optimizer, we were able to find the optimal solution for portfolio optimization. In addition, as a way to compensate for the shortcomings of MVP, Modified MVP was introduced and the results were compared through tests, and it was found that it actually showed better results. This paper demonstrated the importance of nonlinear optimization in solving real-world problems and confirmed the effectiveness of Modified MVP as a tool for portfolio optimization.

7. Future work

7.1 Adding Stock Functionality

The aim of this document was to determine the optimal asset allocation ratio for three specific assets. However, the number of assets can vary depending on the investor's preference. Therefore, the addition of a stock input feature that allows users to adjust the asset input according to their individual preference is expected to enhance the utility of this program.

7.2 Streamlining Execution Steps

To obtain the optimal weight for investing money, this paper employed Excel, AMPL, and a website for referencing stock prices. However, the multi-step process involved in this approach can be time-consuming. Extracting the relevant information from the stock price website and automating the necessary calculations can significantly reduce the time taken to perform tests and increase the applicability of the results presented in this document.

8. Contributions of team members

I did this research on my own. Byeongchan Gwak: 100%

9. References

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Appendix A

Group 1's indicators

	Apple	United H	Coca-Cola
Return rate	0.019472	0.021127	0.004438
Variance	0.009357	0.004053	0.003864
STDEV	0.096731	0.063663	0.062161

<Table A.1: Indicators of Group 1>

Covariance	Apple	United H	Coca-Cola
Apple	0.009357		
United H	0.003054	0.004053	
Coca-Cola	0.002442	0.002684	0.003864

<Table A.2: Covariance of Group 1>

Stock Price ($)	Apple	United H	Coca-Cola
2022-12-01	129.93	530.18	63.61
2023-04-10	161.73	521.69	62.63
return	24.47%	-1.60%	-1.54%

<Table A.3: Stock price of Group 1>

Group 2's indicators

	MS	J&J	Walmart
Return rate	0.012171	0.006277	0.007998
Variance	0.004761	0.002791	0.003694
STDEV	0.068998	0.052827	0.060779

<Table A.4: Indicators of Group 2>

Covariance	MS	J&J	Walmart
MS	0.004761		
J&J	0.001418	0.002791	
Walmart	0.001943	0.001535	0.003694

<Table A.5: Covariance of Group 2>

Stock Price ($)	MS	J&J	Walmart
2022-12-01	239.82	176.65	141.79
2023-04-10	283.92	164.35	150.43
return	18.39%	-6.96%	6.09%

<Table A.6: Stock price of Group 2>

Group 3's indicators

	Netflix	Merck	Pepsi
Return rate	0.005822	0.011119	0.008223
Variance	0.017009	0.004612	0.002680
STDEV	0.130419	0.067913	0.051765

<Table A.7: Indicators of Group 3>

Covariance	Netflix	Merck	Pepsi
Netflix	0.017009		
Merck	0.000771	0.004612	
Pepsi	0.001296	0.002167	0.002680

<Table A.8: Covariance of Group 3>

Stock Price ($)	Netflix	Merck	Pepsi
2022-12-01	294.88	110.95	180.66
2023-04-10	340.15	112.56	183.06
return	15.35%	1.45%	1.33%

<Table A.9: Stock price of Group 3>

Group 4's indicators

	Google	Abbvie	P&G
Return rate	0.009674	0.022585	0.007023
Variance	0.007070	0.005512	0.002876
STDEV	0.084082	0.074240	0.053625

<Table A.10: Indicators of Group 4>

Covariance	Google	Abbvie	P&G
Google	0.007070		
Abbvie	0.002931	0.005512	
P&G	0.000998	0.001059	0.002876

<Table A.11: Covariance of Group 4>

Stock Price ($)	Google	Abbvie	P&G
2022-12-01	88.73	161.61	151.56
2023-04-10	106.52	161.59	151.23
return	20.05%	-0.01%	-0.22%

<Table A.12: Indicators of Group 4>