ASEN 3111 Computational Lab #5: Flow Over Finite Wings

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I. Introduction

The goal of this lab is to write a program that solves the fundamental equation of Prandtl Lifting Line Theory, evaluate the theory's accuracy with respect to Fourier coefficient expansion, and use it to assess the relationship between span efficiency factor and wing shape. Prandtl Lifting Line Theory inherits several assumptions[a] and this implementation is restricted to rectangular wings with linearly varying chord length, geometric twist, aerodynamic twist, and change in airfoil shape from root to tip.

The fundamental equation of Prandtl Lifting Line Theory for finite wings with thick airfoils is given by

$$\alpha(\theta) = \frac{4b}{a_0(\theta)c\theta}\sum_{n=1}^{\infty} A_n \sin(n\theta) + \alpha_{L=0}(\theta) + \sum_{n=1}^{\infty} nA_n \frac{\sin(n\theta)}{\sin(\theta)} \tag{1}$$

Although the equation is expressed as an infinite sum, the series converges to the solution and the equation can hence be approximated by truncating the series expansion for circulation using N odd terms (only odd terms are used because we are considering steady, level flight):

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_{2n-1}\sin((2n-1)\theta) \tag{2}$$

The angles we use to designate locations along the span for evaluation are defined by

$$\theta_n = \frac{n\pi}{2N} \qquad for\ n = 1, 2, ..., N \tag{3}$$

After solving for the N Fourier coefficients, we can find the span efficiency factor, e, using Equation 4. With e known, we can find c_L and c_{Di} using Equations 5 and 6 respectively.

$$e = \frac{1}{(1+\delta)} \qquad where\ \delta = \sum_{n=2}^{\infty} \frac{A_n}{A_1}^2 \tag{4}$$

$$c_L = A_1 \pi AR \tag{5}$$

$$c_{Di} = \frac{c_L^2}{\pi e AR} \tag{6}$$

And we can then use these to find the actual lift and drag with Equations 7 and 8 respectively.

$$L = c_L q_\infty S \tag{7}$$

$$D = c_{Di} q_\infty S \tag{8}$$

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[a] Prandtl Lifting Line Theory assumes incompressible, inviscid (thus irrotational), flow over non-swept finite wings with high AR.

I.A. Analysis of Error and Number of Fourier Coefficients

In analyzing the error associated with using a given number of Fourier coefficients, it is necessary to define an "actual" case against which to compare the results obtained using a specific number of coefficients. For the conditions specified by Problem #2 (see Results for details), plotting the results using different numbers of coefficients depicts an obvious convergence to an approximate solution somewhere between $N = 30$ and $N = 100$. Out of ease of computational intensity and good measure, N was taken to be 200 for the calculations of the "actual" case.

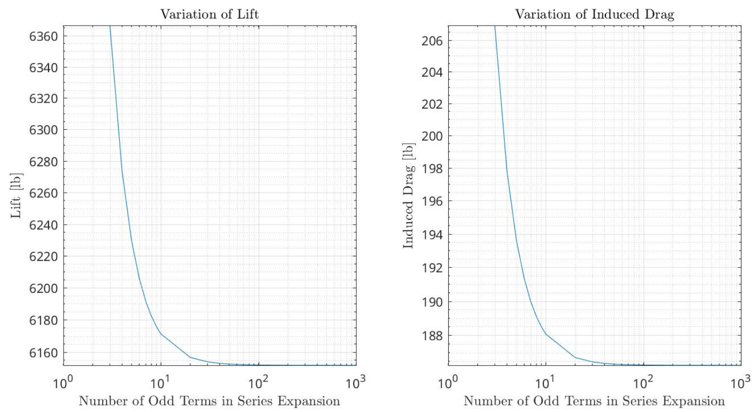

Figure 1

II. Results

For Problem #2, we considered a wing with a span of 40 ft, that varied linearly from a NACA 4412 with $c_r = 6\ ft$, 7°AOA, $\alpha_{L=0} = -4.2811°$[b], and lift slope of 6.8597 rad^{-1} at the root to a NACA 2412 with $c_t = 2ft$, 4°AOA, $\alpha_{L=0} = -2.148°$, and a lift slope of 6.8586 rad^{-1} at the tip. This wing was considered to be in a flow at sea level where $V_\infty = 130\ mi/hr$. The resulting lift and drag calculated for this wing using $N = 200$ are seen in Table 1 below.

Table 1: Results for Finite Wing (Problem #2)

Lift	**Drag**
6151.78 [lb]	186.27 [lb]

In the analysis of error, it is useful to find the number of Fourier coefficient needed to satisfy standard uncertainty thresholds. These are given in Table 2 below.

Table 2: Number of Odd Fourier Coefficients Needed for Given Uncertainties

	Lift: N Needed	**Drag: N Needed**
5% Error	3	5
1% Error	6	10
0.1% Error	18	31

A plot of span efficiency vs. taper ratio for a thin wing with no geometric or aerodynamic twist and be seen in Figure 2 below.

[b] $\alpha_{L=0}$ and lift slopes were computed using the vortex panel code from Computational Lab #4. It was necessary to sample coefficients of lift at different angles of attack and interpolate for the $\alpha_{L=0}$ and differentiate for the lift slopes.

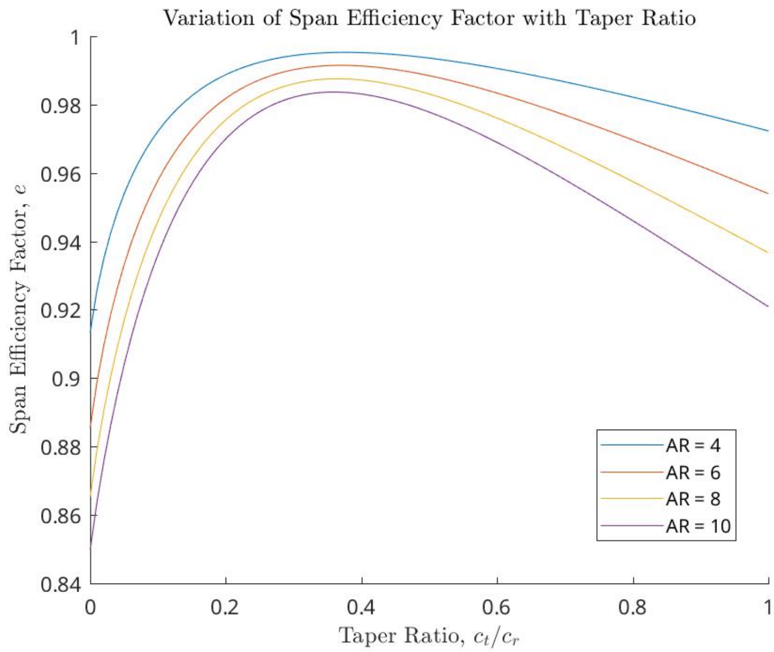

Figure 2

III. Discussion

The error associated with using a given number of odd Fourier coefficients is seen to decrease exponentially as N increases. This can be seen in Figure 3—plotting error results using log-scales produces a linear trend, hence the relationship is exponential. This result shows that a reasonable accuracy can be obtained using a relatively small number of Fourier coefficients.

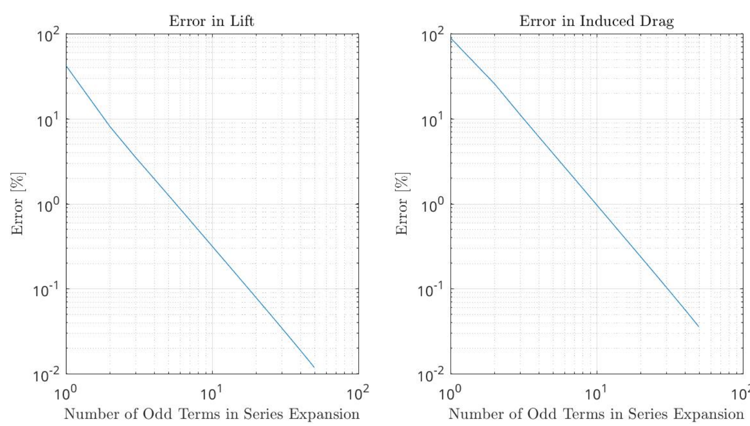

Figure 3

The values used in the plot of span efficiency factor vs. taper ratio (Figure 2) were found using a uniform geometric angle of attack of 1°. The values obtained do not depend on this AOA; however, using $\alpha = 0°$ did not produce any results with this Prandtl Lifting Line Theory implementation. This is likely because no lift or induced drag can be assessed for a thin airfoil in incompressible flow with no angle of attack, hence no span efficiency factor can be obtained. Also note that a lift-slope of 2π was used for these calculations since we are dealing with a thin airfoil.

In Figure 2, it is easily seen that an increase in aspect ratio overall decreases the span efficiency factor.

In addition, it can be seen that, for all aspect ratios, a maximum on these curves exist, hence there is an optimal taper ratio somewhere around 0.4 in this case. These results show that span efficiency factor can be maximized by minimizing AR and using a taper ratio of around 0.4.

Acknowledgments

Rachel Mamich was very helpful by pointing me in the right direction on varying c_t/c_r for problem #3. and no

References

[1] Anderson, Jr., John D., "*Fundamentals of Aerodynamics*", McGraw-Hill, 5th ed., 2011.

[2] Evans, John "*ASEN 3111 Computational Lab #5 : Flow Over Finite Wings*",*Desire2Learn*' University of Colorado Boulder.

[3] Kuethe and Chow "*Aerodynamic Characteristics of Airfoils*","*The Airfoil of Arbitrary Thickness and Camber*",*Desire2Learn*' University of Colorado Boulder.