# Pseudodifferential Operators and the Atiyah-Singer Index Formula

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**Abstract**

This paper will delve into a heat equation approach to solving the Atiyah-Singer index theorem on a Riemann surface. The methods we shall use have their roots in the theory of pseudodifferential operators and their symbol calculus. The main goal of the paper is to provide an explicit computation of the index of a Dirac operator through heat trace asymptotics.

MSc Mathematics Project - 7CCMMS50
Candidate Number: Y00526

September 7, 2018

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# Notation

In order to succinctly capture higher order derivatives, we introduce the notation of multi-indices. An $n$-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ where each $\alpha_j$ is a non-negative integer is called an n-dimensional multi-index. We define its length $|\alpha|$ to be the sum of its entries $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and its factorial as $\alpha! := \alpha_1! \dots \alpha_n!$. Raising $x \in \mathbb{R}^n$ to a multi-index exponent $\alpha$ is simply the product $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Now let $X$ be an open subset of $\mathbb{R}^n$ and $f \in \mathcal{C}^k(X)$. So long as $|\alpha| \leqslant k$, we write the derivatives[1] of $f$ as $D_x^\alpha f := i^{-|\alpha|}\partial_x^\alpha f$, where

$$\partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

# 1 Introduction

## 1.1 Overview

The purpose of this essay is to prove the local Atiyah-Singer index theorem for the case of a Riemann surface. The index problem at its core is the verification that the topological and analytical indices of an elliptic (pseudo)differential operator coincide. The theorem itself has its roots in conjectures made by Gelfand who suggested that one could write the index of a Fredholm operator solely in terms of topological quantities. This conjecture arose from the fact that a Fredholm operator's index is invariant under homotopy. More precisely, if $\{D_t\}$ is a continuous family of differential operators, the index of $D_t$ is independent of $t$.

In the general case, one concerns oneself with a particular type of elliptic differential operator called a twisted Dirac operator $\not{\partial}^{\mathcal{V}}$. This operator maps between smooth sections of Hermitian vector bundles equipped with metric compatible connections. The prototypical manifold $\mathcal{M}$ over which these bundles are defined is of even dimension, Riemannian, closed (compact and without boundary) and oriented with a spin structure equipped to its tangent bundle. Thus, the index theorem reads as follows:

**Theorem 1** (Atiyah-Singer Index Formula)

$$\text{ind } \not{\partial}^{\mathcal{V}} = \left\langle \widehat{\mathcal{A}}(\mathcal{M})\text{Ch}(\mathcal{V}), [\mathcal{M}] \right\rangle = \frac{1}{(2\pi i)^{n/2}} \int_{\mathcal{M}} \widehat{\mathcal{A}}(\mathcal{M}, R)\text{Ch}(\mathcal{V}, F)$$

where $\widehat{\mathcal{A}}(\mathcal{M})$ and $\text{Ch}(\mathcal{V})$ are topological quantities depending on the manifold $\mathcal{M}$.

[1] The presence of $i^{-|\alpha|}$ in the definition of $D_x^\alpha$ is to simplify expressions later on.

We shall focus more specifically on a Dirac type operator $\overline{\partial}$ that maps between smooth sections of holomorphic vector bundles defined over a Riemann surface $\Sigma$ of genus $g_\Sigma$ with Hermitian metric $\mathfrak{h}$. Thus, our index formula shall read:

**Theorem 2** (Riemann-Roch-Hirzebruch Index Formula)

$$\mathrm{ind}\,\overline{\partial} = \frac{1}{2\pi i}\int_\Sigma \overline{\partial}\partial \log\det E + \frac{N}{4\pi i}\int_\Sigma \overline{\partial}\partial \log \mathfrak{h}$$

The approach that we shall take is via the heat equation. From a physical viewpoint, under the action of a so-called heat flow, a suitable map will propagate over a manifold $\mathcal{M}$. The analysis of such a flow is often sufficient in gleaning topological information about said manifold. Indeed, the short term and long term behaviour of objects called heat traces serve as a tool to link local and global properties of $\mathcal{M}$. The index of a Dirac-type operator is the quantity that Atiyah and Bott investigated and found is independent of time and whose course under the heat flow suggested that it depends solely on the topological information of $\mathcal{M}$.

## 1.2 Review of Differential Geometry

We'll begin by reviewing some key structures in differential geometry that shall play host to our incoming analysis.

**Definition** The tuple $(E, \pi, \mathcal{M}, V)$ comprising of topological spaces $E$, $\mathcal{M}$, $V$ and a surjective map $\pi\colon E \to \mathcal{M}$ is called a fibre bundle over a base space $\mathcal{M}$ with typical fibre $V$ if there exists an open cover $\{U_i\}_i$ of $\mathcal{M}$ and a family of homeomorphisms $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times V$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[d, "\pi"'] & U_i \times V \arrow[d, "\mathrm{proj}_1"] \\
U_i \arrow[r, "\mathrm{id}"'] & U_i
\end{tikzcd}$$

We call the fibre bundle $E$ and each $\varphi_i$ a local trivialisation of $E$ over $U_i$. The map $\mathrm{proj}_1$ denotes the usual projection $X \times Y \to X$ onto the first element defined by $(x, y) \mapsto x$. In essence, such a trivialisation helps us to locally regard our map $\pi$ as a projection like $\mathrm{proj}_1$.

**Definition** A section $s$ of a fibre bundle $E$ is a continuous map $s\colon \mathcal{M} \to E$ such that the following diagram commutes

$$\begin{tikzcd}
 & E \arrow[d, "\pi"] \\
\mathcal{M} \arrow[ur, "s"] & \mathcal{M} \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

i.e. a section is a continuous right inverse of $\pi$. The space of all smooth sections of $E$ is denoted $\Gamma(E) = \Gamma(\mathcal{M}, E)$.

There are some commonly used subclasses of vector bundles:

- In the case that our typical fibre $V$ is an $n$-dimensional complex vector space and the homeomorphisms $\varphi_i$ may be chosen so that for every $x \in U_i \cap U_j$, the transition map $\varphi_j \circ \varphi_i^{-1}\colon \{x\} \times V \to \{x\} \times V$ is a linear isomorphism, we say that $E$ is a complex vector bundle of rank $n$.
- If $E, \mathcal{M}$ are smooth manifolds, $\pi$ is a smooth map and the $\varphi_i$ can be taken as diffeomorphisms, the vector bundle $E$ is then called smooth (or differentiable).
- A Hermitian metric on a differentiable complex vector bundle $E$ is an assignment $\mathfrak{h}$ to each fibre $E_x$ of $E$, a Hermitian inner product $\langle \cdot, \cdot \rangle_x$. This assignment must be chosen such that for any open set $U \subseteq \mathcal{M}$ and any $\eta, \theta \in \Gamma(U; E)$, the function

$$\begin{aligned} \langle \eta, \theta \rangle \colon U &\longrightarrow \mathbb{C} \\ : x &\longmapsto \langle \eta(x), \theta(x) \rangle_x \end{aligned}$$

  is smooth on $U$. When equipped with a Hermitian metric, we call $E$ a Hermitian vector bundle.
- If $E \to \mathcal{M}$ is a complex vector bundle where both $E$ and $\mathcal{M}$ are complex manifolds, and $\pi$ is holomorphic, we call $E$ a holomorphic vector bundle. In the language of trivialisations, one requires that every induced map $\varphi_{ji}\colon U_j \cap U_i \to \mathrm{GL}(n, \mathbb{C})$ is holomorphic.

**Definition** A Riemannian manifold is a smooth manifold $\mathcal{M}$ equipped with a positive-definite, symmetric $(0, 2)$ tensor field $g$ i.e. a smooth section of the bundle $T^*\mathcal{M} \otimes T^*\mathcal{M}$ such that:

- for each $p \in \mathcal{M}$, $g$ determines an inner product on each $T_p\mathcal{M}$ via $\langle X, Y \rangle := g(X, Y)$ where $X, Y \in T_p\mathcal{M}$,
- $g(X, X) > 0$ for $X \neq 0$ and $g(X, X) = 0 \iff X = 0$.

We call this tensor field a Riemannian metric, $g$.

In local coordinates $(x^1, \dots, x^n)$, the Riemannian metric takes the form

$$g = \sum_{ij} g_{ij}(x) \mathrm{d}x^i \otimes \mathrm{d}x^j$$

where $(g_{ij}(x))_{ij} = (g_x(\partial_{x^i}, \partial_{x^j}))_{ij}$ is the matrix representation of $g$ at $x$ whose entries are smooth maps in $x$.

A vital ingredient required to integrate on a manifold is a volume form. It turns out that in local coordinates and with respect to a positively oriented basis of the tangent space $T_x\mathcal{M}$, we can associate a volume form with a Riemannian metric. We define it to be the differential form $\mathrm{dvol}_{\mathcal{M}}$ given by $\sqrt{\det g}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. This allows us to, for instance, calculate the "volume" of such a manifold.

# 2 Pseudodifferential Operators

## 2.1 The Fourier Transform

We begin by recalling a few facts about the Fourier transform which acts as a starting point for the discussion of pseudodifferential operators. The space of Schwartz class functions, $\mathcal{S}(\mathbb{R}^n)$, is the set of $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ that satisfy for any multi-indices $\alpha, \beta$

$$\|u\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha D_x^\beta u(x)| < \infty.$$

The condition above characterises the rapidly decreasing nature of these functions at infinity. Indeed, they decay faster than any monomial and thus, polynomial grows. We define the Fourier transform of $u \in \mathcal{S}(\mathbb{R}^n)$ as

$$\mathcal{F}(u)(\xi) = \widehat{u}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \mathrm{d}x \in \mathcal{S}(\mathbb{R}^n).$$

The semi-norms $\|u\|_{\alpha,\beta}$ defined above generate a topology with respect to which $\mathcal{S}(\mathbb{R}^n)$ is complete and thus a Fréchet space. Schwartz space is of particular importance as the Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ defines a topological isomorphism and its inverse is defined by

$$\mathcal{F}^{-1}(\hat{u})(x) = u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi) \mathrm{d}\xi.$$

Since $C_c^\infty(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ densely, and $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$, we conclude that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Thus, we can extend the Fourier transform to a unitary operator on $L^2(\mathbb{R}^n)$. The Fourier transform exhibits several important properties that will be useful in what follows. Namely, for $f, g \in \mathcal{S}(\mathbb{R}^n)$:

- $\langle f, g\rangle_{L^2} = \langle \widehat{f}, \widehat{g}\rangle_{L^2}$ (Parseval's formula)
- $\widehat{f \star g} = \widehat{f} \cdot \widehat{g}$
- $D_\xi^\alpha \widehat{f}(\xi) = (-1)^{|\alpha|} \widehat{x^\alpha f}(\xi)$
- $\xi^\alpha \widehat{f}(\xi) = \widehat{D_x^\alpha f}(\xi)$

where $\star$ denotes the convolution of two functions

$$(f \star g)(x) = \int f(x-y) g(y) \mathrm{d}y$$

and $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2$.

The Fourier transform also allows us to give an integral representation of the action of a differential operator. We'll shortly generalise the notion of a differential operator to a pseudodifferential operator via this identification. To this end, let $u \in \mathcal{S}(\mathbb{R}^n)$ and for each multi-index $\alpha$ with $|\alpha| \leqslant d$, suppose that $a_\alpha(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$. Consider the corresponding differential operator $P$ of order $d$, defined for $x \in \mathbb{R}^n$ by

$$P = P(x, D) = \sum_{|\alpha| \leqslant d} a_\alpha(x) D_x^\alpha.$$

Via Fourier inversion, we see that

$$\begin{aligned} P(x, D)u(x) &= P(x, D)\ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \sum_{|\alpha| \leqslant d} a_\alpha(x) D_x^\alpha \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sum_{|\alpha| \leqslant d} a_\alpha(x) \xi^\alpha \widehat{u}(\xi) \mathrm{d}\xi \\ &=: \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi) \mathrm{d}\xi. \end{aligned}$$

We call the polynomial expression $p(x, \xi) =: \sigma(P)$ the symbol of the differential operator $P$. The highest order part of the symbol of $P$ is often called the principal symbol of the operator and we write

$$\sigma_d(P) = \sum_{|\alpha| = d} a_\alpha(x) \xi^\alpha.$$

## 2.2 Pseudodifferential Operators on $\mathbb{R}^n$

Considering the above derivation of a symbol, we need not restrict ourselves to symbols of polynomial form. We owe the following definition to Hörmander's pioneering paper [4].

**Definition** Let $\Omega$ be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R} \cup \{-\infty\}$ and $\rho, \delta \in [0, 1]$. We say that $p$ is a symbol of order $m$ and type $(\rho, \delta)$ if:

- $p$ is $\mathcal{C}^\infty$ in $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
- $p$ has compact support in the $x$ variable,
- and for every compact $K \subseteq \Omega$ and all multi-indices $\alpha, \beta$, there exists a constant $C = C(\alpha, \beta, K)$ for which

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leqslant C\,(1 + \|\xi\|)^{m - \rho|\alpha| + \delta|\beta|}.$$

We denote the set of all such symbols by $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$. We shall only need the subclass $S^m_{1,0}$ and we'll refer to it by the shorthand $S^m$. If a symbol $p$ is in $S^m_{\rho,\delta}$ for every $m \in \mathbb{R} \cup \{-\infty\}$, one calls $p$ infinitely smoothing and we denote the space of infinitely smoothing symbols by $S^{-\infty}$.

**Definition** The associated pseudodifferential operator[2] $P\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ to a symbol $p \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ is therefore defined by

$$Pu(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{u}(\xi)đ\xi,$$

where we've denoted each $(2\pi)^{-1}\mathrm{d}\xi_i$ by $đ\xi_i$ and $(2\pi)^{-n}\mathrm{d}\xi$ by $đ\xi$.

Pseudodifferential operators enjoy a number of important properties.

(i) Differential operators $P$ are local in the sense that on an open set $U$, if $f \equiv 0$ then $Pf \equiv 0$. The wider class of pseudodifferential operators satisfies the weaker property of being pseudo-local i.e. on an open set $U$, $\operatorname{sing\,supp} Pf \subseteq \operatorname{sing\,supp} f$. To this end, we define the singular support of a distribution $u$ as the complement of the largest open set on which $u$ is smooth.

(ii) The action of an order d pseudodifferential operator $P\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to a continuous linear map on Sobolev space $H^s(\mathbb{R}^n)$, $P\colon H^s(\mathbb{R}^n) \to H^{s-d}(\mathbb{R}^n)$, for every $s \in \mathbb{R}$. This is proven by taking a symbol $p \in S^d$ and demonstrating the equivalent property of boundedness of its associated pseudodifferential operator $P$.

## 2.3 Symbol Calculus and the Algebra of Compactly Supported ΨDOs

A most important property of compactly supported pseudodifferential operators is that they form an algebra[3] under multiplication (i.e. composition) and the operation of taking adjoints. One defines the equivalence relation $\sim$ on symbols $p \sim q$ if their difference is an infinitely smoothing symbol. It follows from (ii) and the Sobolev Embedding Theorem that the ΨDO associated to $p \in S^{-\infty}$ is a smoothing operator i.e. its image is a subset of $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus, the induced equivalence relation on the associated space of ΨDOs quotients out smoothing operators. Now, if we let $P$ and $Q$ be pseudodifferential operators with associated symbols $p(x,\xi)$ and $q(x,\xi)$ respectively, it follows that $PQ$ and $P^*$ are also pseudodifferential with symbols, for a multi-index $\mu$:

$$\sigma(PQ) \sim \sum_\mu \frac{1}{\mu!}\partial^\mu_\xi p(x,\xi) D^\mu_x q(x,\xi) \qquad \sigma(P^*) \sim \sum_\mu \frac{1}{\mu!}\partial^\mu_\xi D^\mu_x p^*(x,\xi).$$

[2]We shall often interchangeably use the shorthand ΨDO.

[3]This fact follows partly from an expansion of the class of symbols one admits to associate with pseudodifferential operators. The proof is somewhat lengthy and can be found as Lemma 1.2.2. in [2].

## 2.4 Generalisations of Pseudodifferential Operators

Of interest to us will be pseudodifferential operators defined on more general domains than $\mathcal{S}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$. For example, we'll be considering operators that map between sections of vector bundles. Hence, let $\mathcal{M}$ denote a smooth and closed[4] Riemannian manifold of dimension $n$ whose atlas is denoted $\{(U_i, \varphi_i)\}_{i \in I}$.

Before we state the following two theorems, both of which have been taken from [8], we note that $H^s(E)$ will denote the generalisation of the familiar Sobolev space $H^s(\mathbb{R}^n)$ to a vector bundle $E$ over $\mathcal{M}$. A thorough exposition on this generalisation can be found in the third chapter of [1].

**Theorem 3** (Rellich's Theorem) The natural inclusion $\iota \colon H^s(E) \hookrightarrow H^t(E)$ for $t < s$ is a compact operator i.e. the image of a closed ball is compact.

**Theorem 4** (Sobolev Embedding Theorem) If $s > \lfloor n/2 \rfloor + k + 1$, then $H^s(E) \subseteq \Gamma^k(E)$.

We now have the appropriate terminology to reformulate our pseudodifferential operators on more general spaces.

**Definition** We call a map $P \colon \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ a pseudodifferential operator if for each chart $(U_i, \varphi_i)$ of $\mathcal{M}$, the induced map $\widetilde{P}_{\varphi_i} \colon \mathcal{C}^\infty(\varphi_i(U_i)) \to \mathcal{C}^\infty(\varphi_i(U_i))$ is a $\Psi$DO.

Their existence simply boils down to sewing together locally Euclidean $\Psi$DOs via the familiar partition of unity argument. Adding an extra level of abstraction, we can define a pseudodifferential operator between sections of Hermitian vector bundles $E$ and $F$ over $\mathcal{M}$.

**Definition** We call a map $P \colon \Gamma(E) \to \Gamma(F)$ a pseudodifferential operator of order $m$ if on a local level, $P$ takes on the form of a matrix whose components are each $\Psi$DOs of order $m$. More formally, one must take an open cover of $\mathcal{M}$ whose elements trivialise both vector bundles.

In a similar way to how one does so on $\mathbb{R}^n$, a $\Psi$DO $P \colon \Gamma(E) \to \Gamma(F)$ of order $m$ between sections of vector bundles extends to a bounded linear operator $P \colon H^s(E) \to H^{s-m}(F)$ for every $s \in \mathbb{R}$. We shall be interested in elliptic differential operators[5] $D$ between such sections of vector bundles. The characteristic feature of these operators is that they have symbols that are linear isomorphisms for every non-zero element of their domains. In particular, it follows that such an elliptic differential operator is Fredholm.

[4] A manifold is closed if it is both compact and its boundary is empty.

[5] These will be complex linear maps between sections of vector bundles that induce a partial differential operator on the local level for any choices of charts of the base manifold and local trivialisations of the bundles. Details of the standard theory and examples of such constructions can be found on page 114 of [8].

# 3 The Atiyah-Singer Index Formula

## 3.1 Index and Fredholm Operators

We begin with a formal definition of the index of a linear map.

**Definition** Let $D\colon V \to W$ be a linear map between two vector spaces $V$ and $W$. The index of $D$, denoted $\mathrm{ind}(D)$, is the difference $\dim\ker D - \dim\ \mathrm{coker} D$ where the cokernel of $D$ is the quotient of the codomain of $D$ by its image.

The index is always an integer and happens to be an example of a quantity that is invariant under homotopy.

**Example 1** The simplest example of an index calculation pertains to a linear map $T : V \to W$ between finite dimensional vector spaces. It follows from the rank-nullity theorem of linear algebra that

$$\begin{aligned}
\mathrm{ind}(T) &= \dim\ \ker(T) - \dim\ \mathrm{coker}(T) \\
&:= \dim\ \ker(T) - \dim(W/\mathrm{im}(T)) \\
&= \dim\ \ker(T) - (\dim(W) - \dim(\mathrm{im}(T))) \\
&= \dim\ \ker(T) + \dim\ \mathrm{im}(T) - \dim(W) \\
&= 0.
\end{aligned}$$

A class of operators whose elements always have a well-defined (i.e. finite) index is the space of Fredholm operators.

**Definition** We call a bounded linear operator $T\colon \mathcal{H}_1 \to \mathcal{H}_2$, between Hilbert spaces, Fredholm if both its kernel and cokernel are finite dimensional. Equivalently, $T$ is Fredholm if it's invertible modulo a compact operator i.e. if there exists a bounded linear map $S\colon \mathcal{H}_2 \to \mathcal{H}_1$ such that both $TS - \mathrm{id}_{\mathcal{H}_2}$ and $ST - \mathrm{id}_{\mathcal{H}_1}$ are compact. In this case, we call $S$ a parametrix for $T$.

**Example 2** In the case of pseudodifferential operators $A, B$ with respective symbols $a(x,\xi)$ and $b(x,\xi)$, we say that $B$ is a parametrix for $A$ if $AB = I$ modulo a smoothing operator. On the associated symbol level, $(a \circ b)(x,\xi) = \mathrm{id}$ modulo an infinitely smoothing symbol. This fact will be used without proof later on when appealing to the theory of asymptotic approximations.

### 3.1.1 The Operators $\Delta$ and $\widetilde{\Delta}$

From this point onwards, let $E^+$ and $E^-$ be Hermitian vector bundles over $\mathcal{M}$. Denote their sum[6] by $E = E^+ \oplus E^-$.

[6] This is indeed a vector bundle in its own right. We shall define these more formally later on.

The heat equation approach has its roots in the McKean-Singer index formula. In particular, it was the realisation that the index of a first order, elliptic differential operator $D\colon \Gamma(E^+) \to \Gamma(E^-)$ can be expressed in terms of traces of the so-called heat operators. With respect to the metrics on $\mathcal{M}$ and $E^\pm$, denote the adjoint of $D$ by $D^*\colon \Gamma(E^-) \to \Gamma(E^+)$. Now consider the compositions

$$\begin{aligned} \Delta &\coloneqq D^*D\colon \Gamma(E^+) \to \Gamma(E^+) \\ \widetilde{\Delta} &\coloneqq DD^*\colon \Gamma(E^-) \to \Gamma(E^-). \end{aligned}$$

It immediately follows that $\Delta$ is a positive-definite and self-adjoint differential operator of degree 2. Also, $\Delta^{-1}$ is a pseudodifferential operator of order $-2$. Thus, $\Delta^{-1}$ is a compact operator whose spectrum accumulates at 0. Consequently, $\operatorname{spec}(\Delta)$ is a discrete subset $\{\mu_i\}_{i\in\mathbb{N}}$ of $[0,\infty)$ that comprises of eigenvalues gathering at infinity i.e. $0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \to \infty$. An analogous statement can be made for $\operatorname{spec}(\widetilde{\Delta}) = \{\widetilde{\mu}_i\}_{i\in\mathbb{N}}$.

Before we state the McKean-Singer theorem itself, we shall take a slight detour via the heat equation in order to understand the forthcoming heat operators in the index formula expression.

### 3.1.2 The Heat Equation

The heat equation on a Riemannian manifold $\mathcal{M}$ is the elliptic partial differential equation

$$\begin{cases} \dfrac{\partial s}{\partial t} + \Delta s \\ s(0,x) = f(x) \in L^2(E). \end{cases}$$

The Laplacian on $\mathcal{M}$ is denoted $\Delta$, $s$ is a time dependent section of $E$ in the sense that $s = s(t,x)\colon (0,\infty) \times \mathcal{M} \to \operatorname{End}(E)$, and $f$ is the prescribed initial condition (or initial heat distribution). We call the continuous map that solves the heat equation the heat operator and suggestively denote it by $e^{-t\Delta}$ i.e. $s(t,x) = e^{-t\Delta} f(x)$.

An important property of the heat operator $e^{-t\Delta}$ is that it is of trace-class. This comes as a direct consequence of $e^{-t\Delta}$ being a smoothing operator. We shall prove the latter fact in due course.

**Definition** One says that a bounded operator $A$ on a separable (infinite-dimensional) Hilbert space $\mathcal{H}$ is of trace class if for any orthonormal basis $\{e_n\} \subseteq \mathcal{H}$, the quantity $\sum_n \langle Ae_n, e_n\rangle$ is finite.

Since the heat operator is of trace-class, we can invoke Lidskii's theorem and express its trace as a sum of its eigenvalues i.e.

$$\operatorname{tr}(e^{-t\Delta}) = \sum_{\mu_i \in \operatorname{spec}(\Delta)} e^{-t\mu_i}.$$

Note that one sums the rightmost summand over a multi-set representation of spec($\Delta$). For instance, if $\mu_1$ has multiplicity two in spec($\Delta$), our sum would take on the form

$$\sum_{\{\mu_1,\mu_1,\mu_2,\dots\}} e^{-t\mu_i} = e^{-t\mu_1} + e^{-t\mu_1} + e^{-t\mu_2} + \dots .$$

## 3.2 McKean-Singer Index Formula

We now state the awaited McKean-Singer index theorem.

**Theorem 5** (McKean-Singer) For $t > 0$, $\operatorname{ind}(D) = \operatorname{tr}(e^{-t\Delta}) - \operatorname{tr}(e^{-t\widetilde{\Delta}})$.

We begin with a comparison of the eigenspaces of our newly defined operators $\Delta$ and $\widetilde{\Delta}$. Suppose that $\mu \in \operatorname{spec}(\Delta) \setminus \{0\}$. Then for some $\varphi \in \Gamma(E^+)$ we have that $\Delta\varphi = \mu\varphi$. In other words, $(D^*D)\varphi = \mu\varphi$. Let $D\varphi =: \rho \in \Gamma(E^-)$. By associativity and linearity,

$$\begin{aligned}
\widetilde{\Delta}\rho &= (DD^*)\rho \\
&= (DD^*)(D\varphi) \\
&= D((D^*D)\varphi) \\
&= D(\mu\varphi) = \mu\rho \implies \mu \in \operatorname{spec}(\widetilde{\Delta}) \setminus \{0\}.
\end{aligned}$$

The above holds for a basis of eigensections[7] of $L^2(E)$ and by the symmetry between $\Delta$ and $\widetilde{\Delta}$, we conclude the reverse inclusion $\operatorname{spec}(\widetilde{\Delta}) \setminus \{0\} \subseteq \operatorname{spec}(\Delta) \setminus \{0\}$ holds.

Since the non-zero eigenspaces of $\Delta$ and $\widetilde{\Delta}$ are isomorphic and their associated operators $e^{-t\Delta}$ and $e^{-t\widetilde{\Delta}}$ are of trace class, we may re-write the right-hand side of the McKean-Singer index expression above to read

$$\begin{aligned}
\operatorname{tr}(e^{-t\Delta}) - \operatorname{tr}(e^{-t\widetilde{\Delta}}) &= \sum_{\mu \in \operatorname{spec}\Delta} e^{-t\mu} - \sum_{\widetilde{\mu} \in \operatorname{spec}\widetilde{\Delta}} e^{-t\widetilde{\mu}} \\
&= \sum_{\{\mu=0\}} e^{-t\mu} - \sum_{\{\widetilde{\mu}=0\}} e^{-t\widetilde{\mu}} \\
&= \dim \ker(\Delta) - \dim \ker(\widetilde{\Delta}).
\end{aligned}$$

The final equality follows from the aforementioned comment on summing over multi-sets in order to take into account eigenvalues with multiplicity greater than 1.

The final step in establishing the McKean-Singer identity is to translate our dimensional difference between the kernels of $\Delta$ and $\widetilde{\Delta}$ to the differential operators we began with, $D$ and $D^*$. To this end, we make use of the subsequent lemma.

[7] This is because $\Delta$ and $\widetilde{\Delta}$ being positive and self-adjoint implies that each of their respective eigenspaces are finite dimensional and thus form an orthogonal decomposition of $L^2(E^+)$ and $L^2(E^-)$ respectively. This is indeed the content of Theorem 3.25 from [1].

**Lemma 1** For $D\colon \Gamma(E^+) \to \Gamma(E^-)$, $\ker(D) = \ker(\Delta)$ and $\ker(D^*) = \ker(\widetilde{\Delta})$. Furthermore, these spaces are subsets of $\Gamma(E^+)$ and $\Gamma(E^-)$ respectively.

*Proof.* Suppose that $D\varphi = 0$. It follows that $0 = D^*(D\varphi) = \Delta\varphi$ so $\ker(D) \subseteq \ker(\Delta)$. For the reverse inclusion, we argue using the inner product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(E)}$, defined by $\langle \eta, \xi \rangle_{L^2(E)} = \langle \eta, \xi \rangle_{L^2(E^\pm)}$ if both $\eta, \xi$ belong to the same subset $L^2(E^\pm)$ and 0 otherwise.

Let $\varphi \in \ker(\Delta)$ i.e. $\Delta\varphi = D^*D\varphi = 0$. Thus, $0 = \langle \Delta\varphi, \varphi \rangle = \langle D\varphi, D\varphi \rangle = \|D\varphi\|^2$. Since norms separate points i.e. $\|\varphi\| = 0 \iff \varphi = 0$, we conclude that $D\varphi = 0$ i.e. $\varphi \in \ker(D)$. Thus, $\ker(D) = \ker(\Delta)$. The above working is symmetric in $D$ and $D^*$ so without any more work we have that $\ker(D^*) = \ker(\widetilde{\Delta})$. $\square$

Putting everything together from the spectral analysis above, we confirm the McKean-Singer index formula:

$$\begin{aligned} \mathrm{tr}(e^{-t\Delta}) - \mathrm{tr}(e^{-t\widetilde{\Delta}}) = \ldots &= \dim \ker(\Delta) - \dim \ker(\widetilde{\Delta}) \\ &= \dim \ker(D) - \dim \ker(D^*) \\ &=: \mathrm{ind}(D). \end{aligned}$$

### 3.2.1 An Alternate Analytic Viewpoint via Superspaces

Despite seeming out of place at first, the language of supergeometry shall enable us to make a few powerful analytical observations about the index of $D$. To start, we shall succinctly express $\mathrm{tr}(e^{-t\Delta}) - \mathrm{tr}(e^{-t\widetilde{\Delta}})$ as a “supertrace" or graded trace of a new heat operator $e^{-t\not{\Delta}}$ on $L^2(E)$.

**Definitions**

- A superspace $W$ is a $\mathbb{Z}_2$-graded vector space i.e. $W$ is subject to a decomposition of the form $W = W^+ \oplus W^-$. The grading operator $\varepsilon$ of such a superspace $W$ is an endomorphism on $W$ that can be realised as a block matrix

$$\begin{pmatrix} \mathrm{id}_{W^+} & 0 \\ 0 & -\mathrm{id}_{W^-} \end{pmatrix}$$

  with respect to the decomposition of $W$ into $W^+$ and $W^-$.

- The supertrace of an endomorphism $A$ on $W$ is defined as $\mathrm{str}(A) = \mathrm{tr}(\varepsilon A)$. Upon identifying $A \in \mathrm{End}(W)$ with a block matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\mathrm{str}(A) = \mathrm{tr}(a) - \mathrm{tr}(d)$.

- A superbundle $E = E^+ \oplus E^-$ over $\mathcal{M}$ is a vector bundle for which $E^+$ and $E^-$ are themselves vector bundles over $\mathcal{M}$ and the fibres of $E$ are superspaces.

Now if $\pi\colon E^+ \oplus E^- \to \mathcal{M}$ is a vector bundle, a smooth section $s$ of the vector bundle direct sum $E^+ \oplus E^-$ over $\mathcal{M}$ is a smooth map $s = (s_1, s_2)\colon \mathcal{M} \to E^+ \oplus E^-$ where $s_1 \in \Gamma(\mathcal{M}; E^+)$ and $s_2 \in \Gamma(\mathcal{M}; E^-)$.

With the machinery of superbundles and superspaces under our belt, we can now form new operators $\not{\partial}$ and $\not{\Delta}$ in terms of $D, D^*, \Delta$ and $\widetilde{\Delta}$. Let $\not{\partial}\colon \Gamma(\mathcal{M}; E^+ \oplus E^-) \to \Gamma(\mathcal{M}; E^+ \oplus E^-)$ be the operator whose action is defined by

$$\not{\partial} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} : \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \longmapsto \not{\partial} s = \begin{pmatrix} D^* s_2 \\ D s_1 \end{pmatrix},$$

where $s_1 \in \Gamma(E^+)$ and $s_2 \in \Gamma(E^-)$. We can now associate with $\not{\partial}$, its square:

$$\not{\Delta} := \not{\partial}^2 = \begin{pmatrix} D^* D & 0 \\ 0 & D D^* \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{pmatrix}.$$

Using these operators, it can be shown[8] that

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{str}(e^{-t\not{\Delta}}) = \frac{-1}{2} \operatorname{str}\left( \left[ \not{\partial}, \not{\partial} e^{-t\not{\Delta}} \right] \right).$$

As the supertrace is designed to vanish on supercommutators, we see that this derivative is equal to 0 for all $t \geqslant 0$. We'll be most interested in the equality between the long and short-term behaviours exhibited by $\operatorname{str}(e^{-t\not{\Delta}})$ i.e.

$$\lim_{t \to 0^+} \operatorname{str}(e^{-t\not{\Delta}}) = \lim_{t \to \infty} \operatorname{str}(e^{-t\not{\Delta}}).$$

We've already calculated an expression for the supertrace of $e^{-t\not{\Delta}}$ for $t > 0$ as the index of $D$ - that was the content of the McKean-Singer index formula. Thus, in what follows, we'll focus our attention on the left-hand side, assuming the limit indeed exists as $t \to 0^+$.

[8] As found in pages 120 and 121 from [7].

## 3.3 Asymptotic Expansion of the Heat Kernel

In order to determine whether an expression for the limit of the supertrace of $e^{-t\not{\Delta}}$ as $t \to 0^+$ exists, we tap into the theory of functional calculi and pseudodifferential operators. Specifically, the functional calculus for unbounded operators on Hilbert spaces $\mathcal{H}$ (linear maps with domain densely included in $\mathcal{H}$) will be useful in determining expressions for the traces of the heat kernels of $e^{-t\Delta}$ and $e^{-t\widetilde{\Delta}}$.

Recall that Cauchy's integral formula from complex analysis links, in a suitable neighbourhood, the value of a function at a point to a contour integral on an enclosing curve. Namely, if $f\colon U \to \mathbb{C}$ is holomorphic on a domain $U$ and $\gamma \subseteq U$ is a closed curve oriented anticlockwise about $z$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega.$$

In analogous fashion, one can define for an unbounded operator $T$ on $L^2(E^+)$,

$$f(T) = \frac{1}{2\pi i} \int_\Pi f(\nu)(\nu I - T)^{-1}\, \mathrm{d}\nu,$$

where $\Pi$ is a curve (infinite loop) oriented anti-clockwise about the spectrum[9] of $T$ and $f$ is holomorphic and bounded in a neighbourhood of $T$'s spectrum. In particular, letting $t > 0$, we shall confine our interest to the[10] heat operator $e^{-t\Delta}\colon L^2(E^+) \to L^2(E^+)$ as the contour integral defined for $\psi \in L^2(E^+)$ by

$$(e^{-t\Delta}\psi)(x) = \int_\Pi e^{-t\nu}((\Delta - \nu I)^{-1}\psi)(x)\,đ\nu,$$

where $đ\nu = \frac{i}{2\pi}\mathrm{d}\nu$ is responsible for the sign change of $(\Delta - \nu I)^{-1}$ in the integrand.

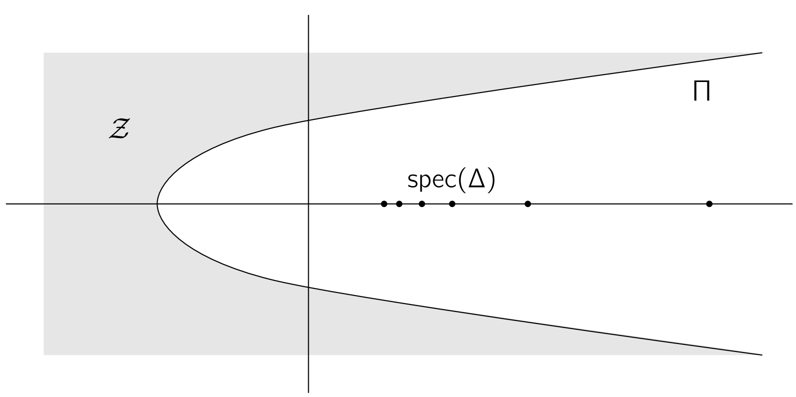


Figure 1: An illustration of a suitable infinite loop $\Pi$ enclosing $\mathrm{spec}(\Delta) \subseteq [0, \infty)$.

[9]In particular, $\nu \in \mathcal{Z}$ where $\mathcal{Z}$ is the shaded area that is equal to the union of $\Pi$ and the subset of the complement of $\Pi$ in $\mathbb{C}$ that doesn't contain $\mathrm{spec}(\Delta)$.

[10]Analogous statements will hold for $e^{-t\widetilde{\Delta}}$ as well.

We call the map $(\Delta - \nu I)^{-1} \colon H^s(E^+) \to H^{s+2}(E^+)$ the resolvent operator of $\Delta$. Since $\Delta$ is a 2nd order differential operator, we would expect that once applied to a section, the section's number of derivatives would reduce by 2. This is because Sobolev spaces, in a sense, count[11] the number of $L^2$ derivatives a map has. Accordingly, we'd expect the resolvent map to increase the subscript $s$ to $s+2$.

Our general strategy will be to express the resolvent map as an oscillatory integral whose (resolvent) kernel involves a symbol that depends on a complex parameter. After this, we will asymptotically expand the symbol of the resolvent kernel, thereby yielding the asymptotic expansion of the heat trace:

$$\begin{aligned}(e^{-t\Delta}\psi)(x) &= \int_\Pi e^{-t\nu}((\Delta - \nu I)^{-1}\psi)(x)đ\nu \\ &= \int_\Pi e^{-t\nu} \int_{\mathcal{M}} R_\Delta(x,y,\nu)\psi(y)\mathrm{dvol}_{\mathcal{M}}(y)đ\nu \\ &= \int_{\mathcal{M}} \underbrace{\left(\int_\Pi e^{-t\nu} R_\Delta(x,y,\nu)đ\nu\right)}_{H_t(x,y)} \psi(y)\mathrm{dvol}_{\mathcal{M}}(y).\end{aligned}$$

The final line's interchange of the order of integration $\int_\Pi \int_{\mathcal{M}} = \int_{\mathcal{M}} \int_\Pi$ is a consequence of applying Fubini's theorem. We're justified in applying Fubini's theorem as we're working on a compact manifold.

In order to work with the resolvent kernel, we introduce a generalisation of our formerly defined symbol (and its corresponding pseudodifferential operator).

**Definition** A symbol of order $m$ and type $(\rho, \delta) = (1, 0)$ depending on a complex parameter $\nu \in \mathcal{Z}$ is a map $r(x, \xi, \nu)$ that satisfies the following properties:

- $r$ is $\mathcal{C}^\infty$ in $x$ and $\xi$, and analytic in $\nu$,
- $r$ has compact support in the $x$ variable,
- and for every compact $K \subseteq \Omega$ and all multi-indices $\alpha, \beta$, there exists a constant $C = C(\alpha, \beta, K)$ for which[12]

$$|D_x^\beta D_\xi^\alpha r(x,\xi,\nu)| \leqslant C\left(1 + \|\xi\| + |\nu|^{1/2}\right)^{m-|\alpha|}.$$

One pressing issue confronts us. The resolvent operator $(\Delta - \nu I)^{-1}$ is not pseudodifferential for $\nu \in \mathcal{Z}$. Fortunately, we can approximate (to arbitrarily high accuracy) $(\Delta - \nu I)$ by a parametrix. We refer the reader to sections 1.6 and 1.7 of [2] for a detailed explanation as to why this can be done.

[11] This observation follows from considering equivalent norms on Sobolev space and using the Fourier transform to interchange multiplication and differentiation.

[12] The denominator of the exponent of $|\nu|$ corresponds to the order of the associated $\Psi$DO. In our case, we shall be considering $\Delta$ which is elliptic and of order 2.

For our purposes, we assume that this is true and by an abuse of notation, we write $(\Delta - \nu I)^{-1}$ for the parametrix itself. In that respect, we proceed to say that the resolvent kernel $R_\Delta$ of $(\Delta - \nu I)^{-1}$ takes the form[13]

$$R_\Delta(x, y, \nu) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} r(x, y, \xi, \nu) đ\xi$$

for some symbol amplitude $r(x, y, \xi, \nu)$ depending on a complex parameter and having compact support in $x$ and $y$. It follows from re-writing the exponential in terms of its own derivative and integration by parts, that the resolvent kernel is smooth off the diagonal.

**Lemma 2** The resolvent kernel $R_\Delta$ is smooth off the diagonal $\{x = y\}$.

*Proof.* As we're only considering symbols of type $(1, 0)$, the necessary[14] hypothesis that $\rho > 1$ for $R_\Delta$ being smooth off the diagonal is fulfilled. The following argument is an adaptation of Proposition 2.1 from [5]. First of all, note that for $j \in \mathbb{N}$, $e^{i(x-y)\cdot\xi} = (x-y)^{-1} D_{\xi_j} e^{i(x-y)\cdot\xi}$. Let $\beta \geqslant 0$ be a multi-index and consider

$$\begin{aligned}
(x-y)^{|\beta|} R_\Delta(x, y, \nu) &= \int_{\mathbb{R}^n} (x-y)^{|\beta|} e^{i(x-y)\cdot\xi} r(x, y, \xi, \nu) đ\xi \\
&= \int_{\mathbb{R}^n} \left( D_\xi^\beta e^{i(x-y)\cdot\xi} \right) r(x, y, \xi, \nu) đ\xi \\
&= \int_{\mathbb{R}^n} (-1)^{|\beta|} e^{i(x-y)\cdot\xi} D_\xi^\beta r(x, y, \xi, \nu) đ\xi
\end{aligned}$$

where the factor of $(-1)^{|\beta|}$ is introduced through integrating by parts $|\beta|$ times and the boundary terms all vanish because $r$ has compact support in $\xi$. Since $r$ is a symbol of order $m$ depending on a complex parameter $\nu$, we have the bound

$$|D_x^\beta D_\xi^\alpha r(x, y, \xi, \nu)| \leqslant C_{\alpha,\beta}(1 + \|\xi\| + |\nu|^{1/2})^{m-|\alpha|}.$$

So long as $m - |\beta| < -n$ our integral is absolutely convergent. We can choose $|\beta|$ to be as large as we desire so there's no issue and $(x-y)^\beta R_\Delta$ is continuous. Analogously, upon differentiating $(x-y)^\beta R_\Delta$ with respect to $\xi$, say $j$-times, it is also absolutely convergent by virtue of the inequality $m - |\alpha| - j < -n$. In other words, $(x-y)^\beta R_\Delta \in \mathcal{C}^j$ for any $j$. $\square$

**Lemma 3** Another important fact is that the resolvent kernel is singular along the diagonal $\{x = y\}$.

The proof of this fact follows by regularising the resolvent kernel, which is realised as an oscillatory integral, by introducing a new parameter. The process involves integrating by parts and using appropriate step functions to obtain an expression for the kernel.

[13] Despite $x$ and $y$ being elements of a manifold, we consider their difference. The reason why this suggestive notation poses no problem is somewhat down to an abuse of notation. We're in fact working in local coordinates and with appropriately chosen trivialisations.

[14] The proof can be found under Lemma 1.1 in [5].

Despite the fact that the resolvent kernel is singular along the diagonal, the heat (Schwartz) kernel of $e^{-t\Delta}$ is actually smooth on the very same diagonal.

The reason being is that one can formally integrate by parts[15] with respect to the complex parameter $\nu \in \mathcal{Z}$:

$$\begin{aligned}
\int_{\Pi} e^{-t\nu}(\Delta - \nu I)^{-1} đ\nu &= \int_{\Pi} \frac{-1}{t}\partial_\nu e^{-t\nu}(\Delta - \nu I)^{-1} đ\nu \\
&= \frac{-1}{t}\left( \cancelto{0}{(\Delta - \nu I)^{-1} e^{-t\nu}\Big|_{\partial\Pi}} - \int_{\Pi} e^{-t\nu}(\Delta - \nu I)^{-2} đ\nu \right) \\
&= \int_{\Pi} \frac{1}{t} e^{-t\nu}(\Delta - \nu I)^{-2} đ\nu \\
&\qquad \vdots \text{ (proceeding inductively)} \\
&= \int_{\Pi} \frac{1}{t^k} e^{-t\nu} k!(\Delta - \nu I)^{-k-1} đ\nu.
\end{aligned}$$

The boundary term in each successive integration by parts vanishes because $\Pi$ is a closed curve in $\mathbb{C}$ and thus $\partial\Pi = \emptyset$. Note that $(\Delta - \nu I)^{-k-1} \colon H^s(E^+) \to H^{s+2(k+1)}(E^+)$ by simply composing $(\Delta - \nu I)^{-1}$ with itself. As we can repeat this procedure as many times as we please, the resulting mapping is independent of $k$. Thus, the range of our arbitrarily iterated map $(\Delta - \nu I)^{-k-1}$ as $\mathbb{Z} \ni k \to \infty$ is a subset of

$$\bigcap_{t \in \mathbb{R}} H^t(E^+) \subseteq \Gamma(E^+).$$

The fact that we consider an intersection of these ranges follows from the natural inclusion $H^s(E^+) \hookrightarrow H^t(E^+)$ for $t < s$ and the Sobolev Embedding Theorem. As an operator whose image is contained in $\Gamma(E^+)$, the heat operator is smoothing.

**Theorem 6** Since the heat operator $e^{-t\Delta}$ is smoothing, it is consequently of trace class and

$$\mathrm{tr}(e^{-t\Delta}) = \int_{\mathcal{M}} H_t(x, x)\mathrm{dvol}_{\mathcal{M}}(x),$$

where $H_t(x, x)$ is the heat (Schwartz) kernel $H_t(x, y)\big|_{x=y}$.

*Proof.* The proof for the general case of a smoothing operator $A$ on $L^2(\mathcal{M})$ can be found as Theorem 6.10 in [1]. $\square$

[15] The process of integration by parts of a contour integral over a differentiable path follows immediately from the real case once we invoke a parameterisation.

### 3.3.1 An Expression for the Symbol of the Resolvent Operator

Computing the components of the symbol of the resolvent operator depends heavily on having an explicit local coordinate representation of the Laplacian on $\mathcal{M}$. These components, that we'll label by $r_{-2-j}$, will be computed recursively and turn out to be quasi-homogeneous functions. We'll aim for an asymptotic approximation of the form

$$R_\Delta(x,y,\nu) \sim \sum_{j\geqslant 0}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} r_{-2-j}(x,\xi,\nu)đ\xi.$$

In local coordinates, it is easily demonstrable that the Laplace-Beltrami operator on $\mathcal{M}$ is

$$\begin{aligned}\Delta &= \sum_{i,j} -\frac{1}{\sqrt{\det g}}\partial_j\left(g^{ij}\sqrt{\det g}\partial_i\right)\\ &= \sum_{i,j=1}^{n} -g^{ij}(x)\partial_j\partial_i + \sum_{k=1}^{n} b_k(x)\partial_j + c(x),\end{aligned}$$

where $(g^{ij})$ is the matrix inverse to $(g_{ij})$ and the second line is a strategic splitting into summands of decreasing order (namely, 2, 1 and 0). As mentioned in the first section on pseudodifferential operators, the symbol $\sigma(P)$ of a differential operator $P$ in local coordinates is obtained by replacing $\partial_j$ with $i\xi_j$ in its expression. Accordingly, the symbol of the Laplacian is given by

$$\begin{aligned}\sigma(\Delta)(x,\xi) &= \overbrace{\sum_{i,j=1}^{n} g^{ij}(x)\xi_j\xi_i}^{=:|\xi|_g^2} + \sum_{k=1}^{n}\widetilde{b_k}(x)\xi_k + c(x)\\ &:= a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi)\end{aligned}$$

where $\widetilde{b_k}(x) = ib_k(x)$ and each $a_i(x,\xi)$ is homogeneous of degree $j$ in $\xi$.

Referring back to example 2, we shall let $A = \Delta - \nu I$, $B$ be a parametrix for $A$ and denote by $a(x,\xi)$, $b(x,\xi)$ their respective symbols. As found in Lemma 1.3.1(a) of [2], by supposing that $b$ exists to begin with, one obtains via an inductive process, an asymptotic approximation for $b(x,\xi) := r(x,\xi,\nu) \sim \sum_{j\geqslant 0} r_{-2-j}(x,\xi,\nu)$. We state the resulting expression of this method in the following lemma.

**Lemma 4** Let $\mu = (\mu_1,\ldots,\mu_n)$ be a multi-index. The $r_{-2-j}(x,\xi,\nu)$ are given by the recursive formulae:

$$\begin{cases}\text{For } j=0, & r_{-2} = (a_2(x,\xi)-\nu)^{-1}\\ \text{Otherwise,} & r_{-2-j} = -r_{-2}\displaystyle\sum_{\substack{|\mu|+k+l=j\\ l<j}}\frac{1}{\mu!}\partial_\xi^\mu a_{2-k}D_x^\mu r_{-2-l}.\end{cases}$$

Upon substituting this asymptotic expansion for the symbol $r(x, y, \xi, \nu)$ back into the resolvent kernel, our heat kernel takes the form

$$\begin{aligned} H_t(x, y) &:= \int_\Pi e^{-t\nu} R_\Delta(x, y, \nu) đ\nu \\ &\sim \int_\Pi e^{-t\nu} \sum_{j=0}^\infty \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} r_{-2-j}(x, \xi, \nu) đ\xi đ\nu \\ &= \sum_{j=0}^\infty \int_{\mathbb{R}^n} \int_\Pi e^{-t\nu} e^{i(x-y)\cdot\xi} r_{-2-j}(x, \xi, \nu) đ\nu đ\xi \end{aligned}$$

where we've used Fubini's theorem and interchanged summations with integrals.

In order to manipulate our asymptotic expansion into a standard form found in the literature, we make a change of variables and note that each $r_{-2-j}$ is quasi-homogeneous of type $(0, 1, 2)$ and degree $-2-j$ i.e. that $r_{-2-j}(x, s\xi, s^2\nu) = s^{-2-j} r_{-2-j}(x, \xi, \nu)$. Let $\mu = t\nu$ so that $d\mu = t d\nu$ and $\xi = t^{-1/2}\eta$ which implies that $d\xi = t^{-n/2} d\eta$. Thus, we have that

$$\begin{aligned} H_t(x, y) &\sim \sum_{j=0}^\infty \int_{\mathbb{R}^n} \int_\Pi e^{-\mu} e^{i(x-y)\cdot\xi} r_{-2-j}\left(x, \frac{\eta}{t^{1/2}}, \frac{\mu}{t}\right) \frac{1}{t} đ\mu \frac{1}{t^{n/2}} đ\eta \\ &= \sum_{j=0}^\infty \left( \int_{\mathbb{R}^n} \int_\Pi e^{-\mu} e^{i(x-y)\cdot\xi} r_{-2-j}(x, \eta, \mu) đ\mu đ\eta \right) t^{\frac{-n+j}{2}} \\ &=: \sum_{j=0}^\infty h_{\frac{-n+j}{2}}(x, y) t^{\frac{-n+j}{2}}. \end{aligned}$$

Referring back to theorem 6, we can express the trace of our heat operator in terms of an asymptotic expansion. Importantly, note that the smoothing operator over which the equivalence $\sim$ is defined vanishes once we take the trace, thereby allowing us to have equality in the second line below:

$$\begin{aligned} \mathrm{tr}(e^{-t\Delta}) &= \int_{\mathcal{M}} H_t(x, x) \mathrm{dvol}_{\mathcal{M}}(x) \\ &= \int_{\mathcal{M}} \sum_{j=0}^\infty \left( \int_{\mathbb{R}^n} \int_\Pi e^{-\nu} r_{-2-j}(x, \xi, \nu) đ\nu đ\xi \right) t^{\frac{-n+j}{2}} \mathrm{dvol}_{\mathcal{M}}(x) \\ &=: \sum_{j=0}^\infty \underbrace{\left( \int_{\mathcal{M}} h_{\frac{-n+j}{2}}(x) \mathrm{dvol}_{\mathcal{M}}(x) \right)}_{=: h_{\frac{-n+j}{2}}} t^{\frac{-n+j}{2}}, \end{aligned}$$

where $h_{\frac{-n+j}{2}}(x) := h_{\frac{-n+j}{2}}(x, x)$.

An entirely analogous conclusion holds for $\widetilde{\Delta}$ and its associated heat trace:

$$\begin{aligned}
\operatorname{tr}(e^{-t\widetilde{\Delta}}) &= \int_{\mathcal{M}} \widetilde{H_t}(x,x)\mathrm{dvol}_{\mathcal{M}}(x) \\
&= \int_{\mathcal{M}} \sum_{j=0}^{\infty} \left( \int_{\mathbb{R}^n} \int_{\Pi} e^{-\nu} \widetilde{r_{-2-j}}(x,\xi,\nu) đ\nu đ\xi \right) t^{\frac{-n+j}{2}} \mathrm{dvol}_{\mathcal{M}}(x) \\
&=: \sum_{j=0}^{\infty} \underbrace{\left( \int_{\mathcal{M}} \widetilde{h_{\frac{-n+j}{2}}}(x)\mathrm{dvol}_{\mathcal{M}}(x) \right)}_{\widetilde{h_{\frac{-n+j}{2}}}} t^{\frac{-n+j}{2}}.
\end{aligned}$$

We shall now specialise our calculations to a Riemann surface $\Sigma$ of genus $g_\Sigma$.

### 3.4 Riemann Surfaces

**Definition** A Riemann surface is a smooth, connected complex manifold $\Sigma$ of complex dimension 1 (real dimension $n = 2$) whose atlas $\{(U_i, \varphi_i)\}$ contains homeomorphisms $\varphi_i$, each from an open subset $U_i \subseteq \Sigma$ to an open subset of the complex plane. Crucially, the associated transition maps $\varphi_i \circ \varphi_j^{-1}$ are holomorphic wherever defined.

We shall suppose that $\Sigma$ is endowed with a Hermitian metric $\mathfrak{h}$ given on a local coordinate patch[16] $U_i \subseteq \Sigma$ by $\mathfrak{h}(z,\overline{z})\mathrm{d}z \otimes \mathrm{d}\overline{z}$. Comparable to the case for a Riemannian manifold, the metric $\mathfrak{h}$ induces a volume form

$$\mathrm{dvol}_{\mathfrak{h}}(\Sigma) = \frac{i}{2}\mathfrak{h}(z,\overline{z})\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

### 3.5 The Riemann-Roch-Hirzebruch Theorem

The main goal of this subsection is to introduce the classical Riemann-Roch-Hirzebruch identity and explore its links to the $\overline{\partial}^{\mathcal{V}}$ operator and its related Laplacian-type operators, $\Delta$ and $\widetilde{\Delta}$. This subsection has been inspired by the more involved exposition of Kotake in [9].

To this end, we begin by considering a collection of transition maps $g_{ij}\colon U_i \cap U_j \to \mathrm{GL}(n,\mathbb{C})$ that define a holomorphic vector bundle $\mathcal{V} \to \Sigma$ of rank $n$. Suppose that relative to a covering $U_i$ of $\Sigma$, $\mathcal{V}$ is equipped with a Hermitian metric $E$ given by a system $E_j\colon U_j \to \mathrm{GL}(n,\mathbb{C})$ of Hermitian matrices. In order to define a global metric on $\mathcal{V}$, we require further that our $E_i$ satisfy the compatibility criterion $g_{ij}^* E_i g_{ij} = E_j$.

This metric induces a Hermitian structure on the determinant line bundle $\det E := \Lambda^N E$ where $\operatorname{rank}(E) = N$ and this new Hermitian structure satisfies the compatibility condition $\det(E_j) = |\det(g_{ij})|^2 \det(E_i)$ obtained by taking determinants of the prior criterion on the $E_i$.

[16] Equivalently, in complex local coordinates $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$.

Since the first Chern classes $c_1(E)$ and $c_1(\det E)$ coincide, and the transition functions $\det g_{ij}\colon U_i \cap U_j \to \mathbb{C}^\times$ are holomorphic, we can represent $C_1(E)$ by an exterior differential form of type $(1,1)$ that is closed i.e. $\overline{\partial}\partial \log \det E \in \Lambda^{(1,1)} T^*_{\mathbb{C}}\Sigma$. We call this form the curvature form in $\det E$.

The first Chern number of $\mathcal{V}$ is then the integer

$$c_1(\mathcal{V}) = \frac{1}{2\pi i} \int_\Sigma \overline{\partial}\partial \log \det E.$$

Similarly, the differential form $\overline{\partial}\partial \log \mathfrak{h} \in \Lambda^{(1,1)} T^*_{\mathbb{C}}\Sigma$ integrates to compute the familiar Euler characteristic of the surface $\Sigma$

$$c_1(\Sigma) = \frac{1}{2\pi i} \int_\Sigma \overline{\partial}\partial \log \mathfrak{h} = 2 - 2g_\Sigma = \chi(\Sigma).$$

In classical terminology, $c_1(\mathcal{V})$ and $c_1(\Sigma)$ are known as the degrees of $\mathcal{V}$ and the holomorphic tangent (or canonical) bundle, respectively.

**Theorem 7** (Classical Riemann-Roch-Hirzebruch Formula) If we write $H^0(\Sigma, \mathcal{V})$ for the $0^{\text{th}}$ degree Dolbeaut cohomology group[17] for the holomorphic vector bundle $\mathcal{V} \to \Sigma$ and let $\widetilde{\mathcal{V}}$ be the twisted holomorphic vector bundle[18] $\mathcal{V}^* \otimes T^{(1,0)}_{\mathbb{C}}\Sigma$, then

$$\dim H^0(\Sigma, \mathcal{V}) - \dim H^0(\Sigma, \widetilde{\mathcal{V}}) = \deg \mathcal{V} + \frac{\operatorname{rank}(\mathcal{V})\chi(\Sigma)}{2}.$$

The link between our heat trace asymptotic expansion and the above theorem relies crucially on the definition of a new operator $\overline{\partial} \coloneqq \overline{\partial}^{\mathcal{V}} \colon \Omega^0(\Sigma, \mathcal{V}) \to \Omega^{(0,1)}(\Sigma, \widetilde{\mathcal{V}})$. We call $\overline{\partial}$ a Dirac operator and its index is in fact equal to the left-hand side of the equality above. We summarise these observations in the following proposition without proof.

**Proposition 1** For the $\overline{\partial}$ operator, $\ker \overline{\partial} \cong H^0(\Sigma, \mathcal{V})$ and $\ker \overline{\partial}^* \cong H^0(\Sigma, \widetilde{\mathcal{V}})$. Thus, the index of $\overline{\partial}$ is equal to the left-hand side of the Classical Riemann-Roch-Hirzebruch Formula.

After carefully arguing via the general Riemann-Roch-Hirzebruch formula for a holomorphic vector bundle over a Kähler manifold $\mathcal{K}$, one arrives at an alternative formula for the index of $\overline{\partial}$:

$$\operatorname{ind}(\overline{\partial}) = \frac{1}{2\pi i} \int_\Sigma \overline{\partial}\partial \log \det E + \frac{N}{4\pi i} \int_\Sigma \overline{\partial}\partial \log \mathfrak{h}.$$

Our aim is to now match up our heat approach to this newly obtained formula for $\operatorname{ind}(\overline{\partial})$.

[17] This is a complex analogue of the de Rham cohomology group.

[18] We call $\mathcal{V}^*$ the dual bundle of $\mathcal{V} \to \Sigma$ and define it as the vector bundle $\pi^*\colon \mathcal{V}^* \to \Sigma$ whose fibres are dual to the fibres of $\mathcal{V}$.

## 3.6 Explicit Formulae for $\overline{\partial}$, $\Delta$ and $\widetilde{\Delta}$

In our local chart $U$, our Dirac operator $\overline{\partial}\colon \Omega^0(\Sigma, \mathcal{V}) \to \Omega^{(0,1)}(\Sigma, \widetilde{\mathcal{V}}) = \Gamma(\Sigma, \mathcal{V}^* \otimes T_{\mathbb{C}}^{(0,1)}\Sigma)$ is given by the formula $\overline{\partial}|_U = \partial_{\overline{z}} d\overline{z}$ and its adjoint takes the form

$$\overline{\partial}^*|_U = -\mathfrak{h}^{-1}\partial_z - (\mathfrak{h}E^t)^{-1}(\partial_z E^t).$$

The associated Laplacians shall be the maps

$$\begin{aligned}\Delta &:= \overline{\partial}^*\overline{\partial}\colon \Omega^0(\Sigma, \mathcal{V}) \to \Omega^0(\Sigma, \mathcal{V})\\ \widetilde{\Delta} &= \overline{\partial}\,\overline{\partial}^*\colon \Gamma(\Sigma, \mathcal{V}^* \otimes T^{1,0}\mathcal{V}) \to \Gamma(\Sigma, \mathcal{V}^* \otimes T^{1,0}\mathcal{V}).\end{aligned}$$

In complex local coordinates $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$ on the aforementioned chart $U$, we see that $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ form a basis of $T_{\mathbb{C}}\Sigma$. On this chart, our Laplacian operators take on the forms

$$\Delta = -\underbrace{\frac{\mathfrak{h}(z,\overline{z})^{-1}}{4}}_{:=\gamma(x_1,x_2)=\gamma}\left(\partial_{x_1}^2 + \partial_{x_2}^2\right) - \underbrace{\frac{1}{2i}(\mathfrak{h}E^t)^{-1}\frac{\partial E^t}{\partial z}}_{:=\alpha(x_1,x_2)=\alpha}(i\partial_{x_1} - \partial_{x_2})$$

$$\widetilde{\Delta} = -\frac{\mathfrak{h}(z,\overline{z})^{-1}}{4}\left(\partial_{x_1}^2 + \partial_{x_2}^2\right) - \underbrace{\frac{1}{2i}E\frac{\partial(\mathfrak{h}E)^{-1}}{\partial z}}_{:=\widetilde{\alpha}(x_1,x_2)=\widetilde{\alpha}}(i\partial_{x_1} - \partial_{x_2}).$$

Thus, $\Delta$ has local symbol

$$\begin{aligned}\sigma(\Delta) &= \gamma|\xi|^2 + \alpha(\xi_1 + i\xi_2) + 0\\ &= a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi).\end{aligned}$$

### 3.6.1 Calculating the $r_{-2-j}$ for $j \in \{0,1,2\}$

From lemma 4, we can begin to calculate the $r_{-2-j}(x,\xi,\nu)$ and from these terms, we can take on the $h_{\frac{-n+j}{2}}(x)$. The first term is simply $r_{-2} = (a_2(x,\xi) - \nu)^{-1} = (\gamma|\xi|^2 - \nu)^{-1}$.

Calculating $r_{-3}$:

$$r_{-3} = -r_{-2}\sum_{\substack{|\mu|+k+l=1\\ l<1}} \frac{1}{\mu!}\partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}$$

Since $l < 1$, it must be 0. This means that $|\mu| + k = 1$ i.e. $|\mu| = 0 \vee |\mu| = 1$. In either case, the length $|\mu| = 1$ so $\mu! = 1$. Our sum can be simplified further by splitting it in two over the indexing set $\{|\mu| = 0 \wedge k = 1\} \cup \{|\mu| = 1 \wedge k = 0\}$.

$$\begin{aligned}
r_{-3} &= -r_{-2} \sum_{|\mu|+k=1} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \\
&= -r_{-2} \sum_{\substack{|\mu|=1 \\ k=0}} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} - r_{-2} \sum_{\substack{|\mu|=0 \\ k=1}} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \\
&= -r_{-2} \left( \sum_{s=1}^{2} \partial_{\xi_s} a_2 D_{x_s} r_{-2} \right) - r_{-2}^2 a_1 \\
&= 2r_{-2}^3 \sum_{s=1}^{2} \gamma (D_{x_s}\gamma) \xi_s |\xi|^2 - r_{-2}^2 \alpha(\xi_1 + i\xi_2)
\end{aligned}$$

The final line follows from substituting in the expressions for $\partial_{\xi_s} a_2 = 2\gamma\xi_s$ and $D_{x_s} r_{-2}^m = -m r_{-2}^{m+1} |\xi|^2 (D_{x_s}\gamma)$ where $m = 1$. For $m \in \mathbb{N}$, this identity will soon prove useful.

Calculating $r_{-4}$:

$$r_{-4} = -r_{-2} \sum_{\substack{|\mu|+k+l=2 \\ l<2}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}$$

Similarly to $r_{-3}$, the restriction on $l \in \mathbb{N}_0$ means that it takes on either 0 or 1 as its value and we break the sum up accordingly:

$$r_{-4} = \underbrace{-r_{-2} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2}}_{(1)} \; \underbrace{-r_{-2} \sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-3}}_{(2)}$$

Before we once again partition the summations (1) and (2) above, we state a few expressions that shall simplify the resulting formulae in the form of a lemma. Their proofs are merely consequences of differentiation.

**Lemma 5** For $s, t, m \in \mathbb{N}_0$, the following equalities hold:

- $D_{x_s} D_{x_s} r_{-2}^m = m(m+1) r_{-2}^{m+2} (D_{x_s}\gamma)^2 |\xi|^4 - m r_{-2}^{m+1} (D_{x_s x_s}^2 \gamma) |\xi|^2$,
- $\partial_{\xi_t} \partial_{\xi_s} a_2(x, \xi) = \begin{cases} 0, & \text{if } s \neq t \\ 2\gamma, & \text{if } s = t. \end{cases}$ $\square$

Now we may deal with (1).

(i) $|\mu| = 0$ and $k = 2$

In this case, $\mu! = 1$ and our sum becomes $a_0 r_{-2} = 0$ as $a_0(x, \xi) = 0$.

(ii) $|\mu| = 1$ and $k = 1$

This condition implies that $\mu! = 1$ and $\mu$ is either $(1, 0)$ or $(0, 1)$. The sum is therefore

$$\begin{aligned}
&\partial_\xi^{(1,0)} a_1 D_x^{(1,0)} r_{-2} + \partial_\xi^{(0,1)} a_1 D_x^{(0,1)} r_{-2} \\
&= \partial_{\xi_1} \alpha(\xi_1 + i\xi_2) D_{x_1} r_{-2} + \partial_{\xi_2} \alpha(\xi_1 + i\xi_2) D_{x_2} r_{-2} \\
&= -\alpha r_{-2}^2 |\xi|^2 (D_{x_1}\gamma) - i\alpha r_{-2}^2 |\xi|^2 (D_{x_2}\gamma)
\end{aligned}$$

(iii) $|\mu| = 2$ and $k = 0$

Now $\mu$ takes on the following values $(0, 2)$, $(2, 0)$ and $(1, 1)$. In the first two cases, $\mu! = 2$ and the latter yields $\mu! = 1$. Upon substituting the equalities in the preceding lemma, the terms reduce to

$$2r_{-2}^3 \sum_{t=1}^{2} \gamma |\xi|^4 (D_{x_t}\gamma)^2 - r_{-2}^2 \sum_{t=1}^{2} \gamma |\xi|^2 (D_{x_t x_t}^2 \gamma).$$

Moving on to (2). The iterative nature of the $r_{-2-j}$ means that we must substitute $r_{-3}$ back into (2) to obtain:

$$\sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} \; D_x^\mu \left( 2r_{-2}^3 \sum_{s=1}^{2} \gamma (D_{x_s}\gamma) \xi_s |\xi|^2 - r_{-2}^2 \alpha(\xi_1 + i\xi_2) \right)$$

(i) $|\mu| = 0$ and $k = 1$

This case is simple as we need not differentiate $r_{-3}$ in the summand of (2), thereby leaving us with:

$$2r_{-2}^3 \sum_{s=1}^{2} \alpha\gamma (D_{x_s}\gamma) \xi_s |\xi|^2 (\xi_1 + i\xi_2) - \alpha r_{-2}^2 \alpha (\xi_1 + i\xi_2)^2.$$

(ii) $|\mu| = 1$ and $k = 0$

The expression for the derivative $D_x^\mu r_{-3}$ involves quite a few terms.

$$\begin{aligned}
D_x^\mu r_{-3} = \; &-6r_{-2}^4 |\xi|^2 (D_x^\mu \gamma) \sum_{s=1}^{2} \gamma (D_{x_s}\gamma) \xi_s |\xi|^2 \; + \; 2r_{-2}^3 \sum_{s=1}^{2} (D_x^\mu \gamma)(D_{x_s}\gamma) \xi_s |\xi|^2 \\
&+ 2r_{-2}^3 \sum_{s=1}^{2} \gamma (D_x^\mu D_{x_s}\gamma) \xi_s |\xi|^2 + 2r_{-2}^3 |\xi|^2 (D_x^\mu \gamma) \alpha (\xi_1 + i\xi_2) - r_{-2}^2 (D_x^\mu \alpha)(\xi_1 + i\xi_2)
\end{aligned}$$

Fortunately, after we take the outermost summation (over $\{|\mu| + k = 1\}$), the fact that $\partial_{\xi_t} a_2 = 2\gamma\xi_t$ enables us to collect like-terms and considerably reduce the clutter.

Thus, the final expression for $r_{-4}$ is the lengthy sum (in increasing powers of $|\xi|^2$):

$$\begin{aligned}
r_{-4} \;=\; &+2r_{-2}^3\sum_{t=1}^{2}\gamma(D_{x_t}\alpha)(\xi_1+i\xi_2)\xi_t \\
&+\alpha r_{-2}^3\alpha(\xi_1+i\xi_2)^2 \\
&+i\alpha r_{-2}^3(D_{x_2}\gamma)\ |\xi|^2+\alpha r_{-2}^3(D_{x_1}\gamma)\ |\xi|^2 \\
&+r_{-2}^3\sum_{t=1}^{2}\gamma(D^2_{x_t x_t}\gamma)\ |\xi|^2 \\
&-2r_{-2}^4\sum_{t=1}^{2}\alpha\gamma(D_{x_t}\gamma)\xi_t(\xi_1+i\xi_2)\ |\xi|^2 \\
&-4r_{-2}^4\sum_{t=1}^{2}\gamma(D_{x_t}\gamma)\alpha\xi_t(\xi_1+i\xi_2)\ |\xi|^2 \\
&-4r_{-2}^4\sum_{t=1}^{2}\sum_{s=1}^{2}\gamma^2(D^2_{x_t x_s}\gamma)\xi_t\xi_s\ |\xi|^2 \\
&-4r_{-2}^4\sum_{t=1}^{2}\sum_{s=1}^{2}\gamma(D_{x_t}\gamma)(D_{x_s}\gamma)\xi_t\xi_s\ |\xi|^2 \\
&-2r_{-2}^4\sum_{t=1}^{2}\gamma(D_{x_t}\gamma)^2\ |\xi|^4 \\
&+12r_{-2}^5\sum_{t=1}^{2}\sum_{s=1}^{2}\gamma(D_{x_t}\gamma)\gamma(D_{x_s}\gamma)\xi_s\xi_t\ |\xi|^4
\end{aligned}$$

The expression for $\widetilde{r_{-4}}$ is obtained similarly by replacing $\alpha$ with $\widetilde{\alpha}$ in the expression of $r_{-4}$.

### 3.6.2 Evaluating the $h_{\frac{-n+j}{2}}(x)$ terms

Now that we've calculated $r_{-2-j}$ for $j \in \{0,1,2\}$, we can turn our attention back to the coefficient functions in our asymptotic expansion for the heat trace associated with $\Delta$. We state here a few identities in a preparatory lemma to once again refer back upon in order to simplify the evaluation of the incoming integrals.

**Lemma 6** Let $k \in \mathbb{N}$ and $\Pi$ be the same curve as mentioned in the definition of $e^{-t\Delta}$. Then,

$$\int_{\Pi} e^{-\nu}(\beta|\xi|^2-\nu I)^{-k}\,đ\nu = \frac{e^{-\beta|\xi|^2}}{(k-1)!}. \qquad (\star)$$

Let $n, m \in \mathbb{N}_0$ and $\beta > 0$. Then

$$\int_{\mathbb{R}^2} \xi_1^{2n}\xi_2^{2m} e^{-\beta|\xi|^2}\mathrm{d}\xi = \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\beta^{n+\frac{1}{2}}\beta^{m+\frac{1}{2}}}. \tag{$\square$}$$

If either $n$ or $m$ is odd, we have that

$$\int_{\mathbb{R}^2} \xi_1^{n}\xi_2^{m} e^{-\beta|\xi|^2}\mathrm{d}\xi = 0. \tag{$\triangle$}$$

*Proof.* We shall use Cauchy's derivative formula without proof.

**Theorem 8** (Cauchy's Derivative Formula) Let $f : U \to \mathbb{C}$ be a holomorphic function on an open subset $U$ of the complex plane. Let $\gamma$ be a piecewise smooth, simple and closed curve in $U$ about some point $z$, oriented counter-clockwise. For $k \in \mathbb{N}_0$, we have that

$$f^k(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\omega)}{(\omega - z)^{k+1}}\ \mathrm{d}\omega.$$

$\square$

($\star$): Now let $f(\nu) = e^\nu$ which is entire on $\mathbb{C}$ and let $\gamma = \Pi$:

$$\begin{aligned}\int_\Pi e^{-\nu}(\beta|\xi|^2 - \nu I)^{-k}đ\nu &\overset{\nu\mapsto-\nu}{=} -\int_\Pi e^{\nu}(\nu I + \beta|\xi|^2)^{-k}đ\nu \\ &= \frac{1}{2\pi i}\int_\Pi e^\nu(\nu I - (-\beta|\xi|^2))^{-k}\mathrm{d}\nu \\ &= \frac{1}{(k-1)!}f^{(k)}(-\beta|\xi|^2) \quad \text{(by theorem 6)} \\ &= \frac{e^{-\beta|\xi|^2}}{(k-1)!}\end{aligned}$$

($\square$): The result follows from writing the double integral over $\mathbb{R}^2$ as a product of integrals over $\mathbb{R}$ and differentiating under the integral sign. Introduce a new parameter $\beta$ and consider the integral

$$I(\beta) = \int_{\mathbb{R}} e^{-\beta x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{\beta^{1/2}}.$$

Since the integrand of $I(\beta)$ and its partial derivative with respect to $\beta$ are continuous in both $x$ and $\beta$, we may differentiate under the integral sign i.e.

$$\begin{aligned}-\frac{\sqrt{\pi}}{2\beta^{3/2}} = \frac{\mathrm{d}}{\mathrm{d}\beta}I(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta}\int_{\mathbb{R}} e^{-\beta x^2}\mathrm{d}x &= \int_{\mathbb{R}}\left(\frac{\partial}{\partial\beta}e^{-\beta x^2}\right)\mathrm{d}x \\ &= -\int_{\mathbb{R}} x^2 e^{-\beta x^2}\mathrm{d}x \implies \frac{\sqrt{\pi}}{2\beta^{3/2}} = \underbrace{\int_{\mathbb{R}} x^2 e^{-\beta x^2}\mathrm{d}x}_{(\|)}.\end{aligned}$$

Note that the conditions for differentiating under the integral sign are once again satisfied by the rightmost integral ($\|$). It turns out that we can proceed inductively in this manner to obtain for $k \in \mathbb{N}$:

$$\int_{\mathbb{R}} x^{2k} e^{-\beta x^2} \mathrm{d}x = \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)\sqrt{\pi}}{2^k} \frac{1}{\beta^{k+\frac{1}{2}}} =: \frac{\Gamma(k+\frac{1}{2})}{\beta^{k+\frac{1}{2}}}.$$

The result follows immediately from this.

($\triangle$): This simply follows from the integral of an odd function on a symmetric domain being equal to 0. The criterion of the integrand being odd is exactly in the hypothesis that either $n$ or $m$ is odd.

Since the real dimension of our Riemann surface $\Sigma$ is 2, we are specifically calculating $h_{-1+\frac{j}{2}}(x)$ for $j \in \{0, 1, 2, \ldots\}$.

Calculating $h_{-1}(x)$ and $h_{-1}$:

$$\begin{aligned} h_{-1}(x) &= \int_{\mathbb{R}^2} \int_{\Pi} e^{-\nu} r_{-2}(x, \xi, \nu) \text{đ}\nu \text{đ}\xi \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\Pi} e^{-\nu} (\gamma|\xi|^2 - \nu)^{-1} \text{đ}\nu \mathrm{d}\xi \\ &\overset{(\star)}{=} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-\gamma|\xi|^2} \mathrm{d}\xi \\ &\overset{(\square)}{=} \frac{1}{4\pi^2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\gamma^{\frac{1}{2}}\gamma^{\frac{1}{2}}} \\ &= \frac{1}{4\pi\gamma}. \end{aligned}$$

Using that $(4\gamma)^{-1} = \mathfrak{h}$ and $\mathrm{dvol}_{\mathfrak{h}}(\Sigma) = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 = \mathrm{dvol}_{\Sigma}(x)$, we can evaluate $h_{-1}$:

$$\begin{aligned} h_{-1} = \int_{\Sigma} \mathrm{tr}(h_{-1}(x)) \mathrm{dvol}_{\Sigma}(x) &= \int_{\Sigma} \mathrm{tr}((4\pi\gamma)^{-1} I_N) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &= \int_{\Sigma} \frac{\mathrm{tr}(I_N)}{\pi} \mathfrak{h} \frac{i}{2} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\ &= \int_{\Sigma} \frac{\mathrm{tr}(I_N)}{\pi} \mathrm{dvol}_{\mathfrak{h}}(\Sigma) \\ &= \frac{N}{\pi} \mathrm{vol}_{\mathfrak{h}}(\Sigma) \end{aligned}$$

Calculating $h_{-\frac{1}{2}}(x)$ and $h_{-\frac{1}{2}}$:

$$\begin{aligned}
h_{-\frac{1}{2}}(x) &= \int_{\mathbb{R}^2}\int_{\Pi} e^{-\nu} r_{-3}(x,\xi,\nu)đ\nu đ\xi \\
&= 2\sum_{s=1}^{2}\int_{\mathbb{R}^2}\left(\int_{\Pi} e^{-\nu} r_{-2}^{3} đ\nu\right)\gamma(D_{x_s}\gamma)\xi_s|\xi|^2 đ\xi - \int_{\mathbb{R}^2}\left(\int_{\Pi} e^{-\nu} r_{-2}^{2} đ\nu\right)\alpha(\xi_1+i\xi_2)đ\xi \\
&\overset{(\star)}{=} \sum_{s=1}^{2}\gamma(D_{x_s}\gamma)\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\xi_s|\xi|^2 đ\xi - \alpha\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}(\xi_1+i\xi_2)đ\xi
\end{aligned}$$

All of these terms evaluate to 0 by ($\triangle$). The second term does so because both $\xi_1$ and $\xi_2$ have odd exponents in their respective integrands. Likewise, if we expand out the sum

$$\sum_{s=1}^{2}\xi_s|\xi|^2 = \xi_1^3 + \xi_1\xi_2^2 + \xi_1^2\xi_2 + \xi_2^3,$$

we see that each term contains one $\xi_s$ (for $s = 0, 1$) with an odd exponent. Thus, $h_{-\frac{1}{2}} = 0$.

**Proposition 2** Akin to the above calculation, for odd $j \in \mathbb{N}$, $h_{\frac{-n+j}{2}}(x) = 0$ and so $h_{\frac{-n+j}{2}} = 0$.

It follows that the asymptotic expansion of the heat trace associated with $\Delta$ is a linear combination comprising solely of integer powers of $t$. As mentioned in the superspace section, we shall consider the limit as $t \to 0^+$ of $\mathrm{str}(e^{-t\not{\Delta}})$. This means that all terms involving $t^a$ for $\mathbb{Z} \ni a > 0$ will vanish as $t \to 0^+$ and we will be left with $h_{-1}t^{-1} + h_{-\frac{1}{2}}t^{-\frac{1}{2}} + h_0$ and their $e^{-t\widetilde{\Delta}}$ associated counterparts. We shall label this observation by ($\star$).

Thus, the calculation of our desired index can be reduced to the following:

$$\begin{aligned}
\mathrm{ind}(\overline{\partial}) &= \lim_{t\to 0^+}\left(\mathrm{tr}(e^{-t\Delta}) - \mathrm{tr}(e^{-t\widetilde{\Delta}})\right) \\
&= \lim_{t\to 0^+}\left[\left(h_{-1}t^{-1} + h_{-\frac{1}{2}}t^{-\frac{1}{2}} + h_0 + h_{\frac{1}{2}}t^{\frac{1}{2}} + \dots\right) - \left(\widetilde{h_{-1}}t^{-1} + \widetilde{h_{-\frac{1}{2}}}t^{-\frac{1}{2}} + \widetilde{h_0} + \widetilde{h_{\frac{1}{2}}}t^{\frac{1}{2}} + \dots\right)\right] \\
&= \lim_{t\to 0^+}\left[\left(h_{-1} - \widetilde{h_{-1}}\right)t^{-1} + \left(h_{-\frac{1}{2}} - \widetilde{h_{-\frac{1}{2}}}\right)t^{-\frac{1}{2}} + \left(h_0 - \widetilde{h_0}\right)\right] \quad \text{(by } (\star) \text{ and Proposition 2)}
\end{aligned}$$

A valuable observation is that the corresponding terms $\widetilde{r_{\frac{-2+j}{2}}}$ differ from the original $r_{\frac{-2+j}{2}}$ only by replacing $\alpha$ with $\widetilde{\alpha}$. We summarise the consequences of this detail in the following lemma.

**Lemma 7** If $r_{\frac{-2+j}{2}}$ is independent of $\alpha$, then $r_{\frac{-2+j}{2}} = \widetilde{r_{\frac{-2+j}{2}}}$. In particular, this means that both $h_{-1} - \widetilde{h_{-1}}$ and $h_{-\frac{1}{2}} - \widetilde{h_{-\frac{1}{2}}}$ are equal to 0.

Therefore, our index calculation becomes $\operatorname{ind}(\overline{\partial}) = h_0 - \widetilde{h_0}$. We shall write $\mathfrak{H}_0(x)$ for the modified version of $h_0(x)$ where we've left out all terms independent of $\alpha$. Analogously, we shall write $\widetilde{\mathfrak{H}_0}(x)$ for the corresponding version of $\widetilde{h_0}(x)$ where we've left out terms not involving $\widetilde{\alpha}$.

Calculating $\mathfrak{H}_0(x)$:

The first term we shall consider is

$$\int_{\mathbb{R}^2}\int_{\sqcap} e^{-\nu} 2r_{-2}^3\gamma\sum_{t=1}^{2}(D_{x_t}\alpha)(\xi_1+i\xi_2)\xi_t đ\nu đ\xi$$
$$\overset{(\star)}{=} \int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\gamma\sum_{t=1}^{2}(D_{x_t}\alpha)(\xi_1+i\xi_2)\xi_t đ\xi$$

Since the summation in the integrand expands into

$$\sum_{t=1}^{2}(D_{x_t}\alpha)(\xi_1+i\xi_2)\xi_t = (D_{x_1}\alpha)(\xi_1^2+\underbrace{i\xi_2\xi_1}) + (D_{x_2}\alpha)(\underbrace{\xi_1\xi_2}+i\xi_2^2),$$

the terms with a brace underneath them vanish by ($\triangle$) and we shall use ($\square$) to evaluate the remaining terms to obtain

$$\frac{1}{4\pi^2}\gamma\left((D_{x_1}\alpha)+i(D_{x_2}\alpha)\right)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\gamma^2} = \frac{1}{4\pi}\gamma\left((D_{x_1}\alpha)+i(D_{x_2}\alpha)\right)\frac{1}{2\gamma^2}.$$

The second term is

$$\int_{\mathbb{R}^2}\int_{\sqcap} e^{-\nu}\alpha r_{-2}^3\alpha(\xi_1+i\xi_2)^2 đ\nu đ\xi$$
$$\overset{(\star)}{=} \frac{1}{2}\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha^2(\xi_1+i\xi_2)^2 đ\xi$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha^2(\xi_1^2+2i\xi_1\xi_2-\xi_2^2) đ\xi$$
$$\overset{(\triangle)}{=} \frac{1}{2}\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha^2(\xi_1^2-\xi_2^2) đ\xi$$
$$\overset{(\square)}{=} 0$$

Our third term takes the form

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\Pi} e^{-\nu}\alpha r_{-2}^{3}(D_{x_1}\gamma)|\xi|^2 đ\nu đ\xi \\
&\overset{(\star)}{=} \frac{1}{2}\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha(D_{x_1}\gamma)(\xi_1^2+\xi_2^2)đ\xi \\
&= \frac{1}{4\pi^2}\frac{1}{2}\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha(D_{x_1}\gamma)(\xi_1^2+\xi_2^2)\mathrm{d}\xi \\
&\overset{(\triangle)}{=} \frac{1}{4\pi^2}\alpha(D_{x_1}\gamma)\left(\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\gamma^2}\right) \\
&= \frac{1}{4\pi}\frac{1}{2}\alpha(D_{x_1}\gamma)\frac{1}{\gamma^2}.
\end{aligned}$$

Analogously, our fourth term evaluates to

$$\int_{\mathbb{R}^2}\int_{\Pi} e^{-\nu} i\alpha r_{-2}^{3}(D_{x_2}\gamma)|\xi|^2 đ\nu đ\xi = \frac{1}{4\pi}\frac{1}{2}i\alpha(D_{x_2}\gamma)\frac{1}{\gamma^2}.$$

Our fifth term of $h_0(x)$ is slightly more complicated. We skip a few straight-forward manipulations of terms vanishing via $(\triangle)$ as they follow largely from prior observations and state the final result:

$$\begin{aligned}
&-2\int_{\mathbb{R}^2}\int_{\Pi} e^{-\nu} r_{-2}^{4}\alpha\gamma\sum_{t=1}^{2}(D_{x_t}\gamma)\xi_t(\xi_1+i\xi_2)|\xi|^2 đ\nu đ\xi \\
&\overset{(\star)}{=} \frac{-2}{3!}\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha\gamma\sum_{t=1}^{2}(D_{x_t}\gamma)\xi_t(\xi_1+i\xi_2)|\xi|^2 đ\xi \\
&= \frac{-1}{3}\frac{1}{4\pi^2}\left(\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha\gamma(D_{x_1}\gamma)(\xi_1^4+\xi_1^2\xi_2^2)\mathrm{d}\xi + i\int_{\mathbb{R}^2} e^{-\gamma|\xi|^2}\alpha\gamma(D_{x_2}\gamma)(\xi_2^4+\xi_1^2\xi_2^2)\mathrm{d}\xi\right) \\
&= \frac{1}{4\pi}\frac{-1}{3}\alpha\gamma\left((D_{x_1}\gamma)+i(D_{x_2}\gamma)\right)\frac{1}{\gamma^3}
\end{aligned}$$

Similarly, our final term takes the form

$$\begin{aligned}
&-4\int_{\mathbb{R}^2}\int_{\Pi} e^{-\nu} r_{-2}^{4}\gamma\sum_{t=1}^{2}(D_{x_t}\gamma)\alpha\xi_t(\xi_1+i\xi_2)|\xi|^2 đ\nu đ\xi \\
&= \frac{1}{4\pi}\frac{-2}{3}\gamma\left((D_{x_1}\gamma)+i(D_{x_2}\gamma)\right)\alpha\frac{1}{\gamma^3}
\end{aligned}$$

Finally, our expression for $\mathfrak{H}_0(x)$ is as follows:

$$\begin{aligned}
4\pi\mathfrak{H}_0(x) = \gamma\left((D_{x_1}\alpha) + i(D_{x_2}\alpha)\right)\frac{1}{2\gamma^2} \\
&+ \frac{1}{2}\alpha(D_{x_1}\gamma)\frac{1}{\gamma^2} \\
&+ \frac{1}{2}i\alpha(D_{x_2}\gamma)\frac{1}{\gamma^2} \\
&- \frac{1}{3}\alpha\gamma\left((D_{x_1}\gamma) + i(D_{x_2}\gamma)\right)\frac{1}{\gamma^3} \\
&- \frac{2}{3}\gamma\left((D_{x_1}\gamma) + i(D_{x_2}\gamma)\right)\alpha\frac{1}{\gamma^3}.
\end{aligned}$$

In order to express $\mathfrak{H}_0(x)$ in a more manageable form for when we take traces and calculate $\mathfrak{H}_0$, we refer back to our complex local coordinate system $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$. In this coordinate system, it follows that $D_{x_1} = (-i)(\partial_z + \partial_{\overline{z}})$, $D_{x_2} = (\partial_z - \partial_{\overline{z}})$. Using these identities, our expression simplifies to

$$\mathfrak{H}_0(x) = -\frac{1}{2\pi}\partial_{\overline{z}}\left((E^t)^{-1}\frac{\partial E^t}{\partial z}\right).$$

This implies that

$$\begin{aligned}
\mathfrak{H}_0 &= \int_\Sigma \mathrm{tr}(\mathfrak{H}_0(x))\mathrm{dvol}_\Sigma(x) \\
&= -\frac{1}{2\pi}\int_\Sigma \mathrm{tr}\left(\partial_{\overline{z}}\left((E^t)^{-1}\frac{\partial E^t}{\partial z}\right)\right)\frac{i}{2}\mathrm{d}z\wedge\mathrm{d}\overline{z} \\
&= \frac{1}{4\pi i}\int_\Sigma \overline{\partial}\partial\log\det E^t \\
&= \frac{1}{4\pi i}\int_\Sigma \overline{\partial}\partial\log\det E,
\end{aligned}$$

where we have used the matrix identity $\partial\log\det E = \mathrm{tr}(E^{-1}\partial E)$.

Calculating $\widetilde{\mathfrak{H}_0}(x)$:

As for $\widetilde{\mathfrak{H}_0}(x)$, we use the same identities as above and replace every $\alpha$ with $\widetilde{\alpha}$ to obtain

$$\begin{aligned}
4\pi\widetilde{\mathfrak{H}_0}(x) &= -2\left(\partial_{\overline{z}}\log\mathfrak{h}\right)\left(\partial_z\log E\right) - 2\partial_{\overline{z}}\partial_z\log E^{-1} - 2\mathfrak{h}\left(\partial_{\overline{z}}\partial_z\mathfrak{h}^{-1}\right) \\
&\quad + 2\left(\partial_z\log\mathfrak{h}\right)\left(\partial_{\overline{z}}\log\mathfrak{h}\right) + 2\left(\partial_z\log E\right)\left(\partial_{\overline{z}}\log\mathfrak{h}\right) \\
&= 2\partial_{\overline{z}}\partial_z\log E - 2\mathfrak{h}\left(\partial_{\overline{z}}\partial_z\mathfrak{h}^{-1}\right) + 2\left(\partial_z\log\mathfrak{h}\right)\left(\partial_{\overline{z}}\log\mathfrak{h}\right) \\
&= 2\partial_{\overline{z}}\partial_z\log E + 2\partial_{\overline{z}}\partial_z\log\mathfrak{h}.
\end{aligned}$$

The transition from the penultimate line to the final line uses a manipulation of the identity

$$\partial_{\overline{z}}\partial_z \log \mathfrak{h} = \mathfrak{h}^{-1}\left(\partial_{\overline{z}}\partial_z \mathfrak{h}\right) - \mathfrak{h}^{-2}\left(\partial_{\overline{z}}\mathfrak{h}\right)\left(\partial_z \mathfrak{h}\right).$$

Namely, replacing $\mathfrak{h}$ with $\mathfrak{h}^{-1}$, multiplying by 2 and rearranging the above identity gives:

$$2\partial_{\overline{z}}\partial_z \log \mathfrak{h} = 2\mathfrak{h}^2\left(\partial_{\overline{z}}\mathfrak{h}^{-1}\right)\left(\partial_z \mathfrak{h}^{-1}\right) - 2\mathfrak{h}\left(\partial_{\overline{z}}\partial_z \mathfrak{h}^{-1}\right).$$

Thus, we have that $\widetilde{\mathfrak{H}_0}(x) = \dfrac{1}{2\pi}\partial_{\overline{z}}\partial_z \log E + \dfrac{1}{2\pi}\partial_{\overline{z}}\partial_z \log \mathfrak{h}$.

Consequently, we calculate $\widetilde{\mathfrak{H}_0}$ as

$$\begin{aligned}
\widetilde{\mathfrak{H}_0} &= \int_\Sigma \operatorname{tr}\left(\widetilde{\mathfrak{H}_0}(x)\right) \mathrm{dvol}_\Sigma(x) \\
&= \int_\Sigma \operatorname{tr}\left(\frac{1}{2\pi}\partial_{\overline{z}}\partial_z \log E + \frac{1}{2\pi} I_N \partial_{\overline{z}}\partial_z \log \mathfrak{h}\right) \mathrm{dvol}_{\mathfrak{h}}(\Sigma) \\
&= -\frac{1}{4\pi i}\int_\Sigma \overline{\partial}\partial \log\det E - \frac{N}{4\pi i}\int_\Sigma \overline{\partial}\partial \log \mathfrak{h}.
\end{aligned}$$

Finally, we end up with the Riemann-Roch-Hirzebruch index formula:

$$\begin{aligned}
\operatorname{ind}(\overline{\partial}) = h_0 - \widetilde{h_0} &= \mathfrak{H}_0 - \widetilde{\mathfrak{H}_0} \\
&= \frac{1}{2\pi i}\int_\Sigma \overline{\partial}\partial \log\det E + \frac{N}{4\pi i}\int_\Sigma \overline{\partial}\partial \log \mathfrak{h}.
\end{aligned}$$

## 3.7 Concluding Remarks

The Atiyah-Singer index theorem is a somewhat miraculous result which intertwines a variety of different fields in its several proofs. The theorem is also powerful in the sense that it generalises several well-known theorems including the Chern-Gauss-Bonnet theorem for surfaces. The heat equation approach that we've taken, in particular, demonstrates the commonly referred to 'remarkable cancellations' in the supertrace of $e^{-t\not{\Delta}}$. In the general case of the Atiyah-Singer index theorem, the study of Clifford bundles, Clifford algebras and spin geometry is of paramount importance in the general calculations.

## References


[1] Roe, John. *Elliptic operators, topology and asymptotic methods. (Second Edition)* Pitman Research Notes in Mathematics Series, **395**. Longman, Harlow (1988)

[2] Gilkey, P. B. *Invariance theory, the heat equation and the Atiyah-Singer index theorem.* Mathematics Lecture Series, **11**. Publish or Perish, Wilmington, DE (1984)

[3] Trèves, François. *Introduction to Pseudodifferential and Fourier Integral Operators. I.* Plenum Press, New York (1980)

[4] Hörmander, Lars. *Fourier Integral Operators. I.* Acta Mathematica, **127**. (1971)

[5] Taylor, Michael Eugene. *Partial Differential Equations II. (Second Edition)* Applied Mathematical Sciences, **116**. (1996)

[6] Shubin, Mikhail A. *Pseudodifferential Operators and Spectral Theory.* Springer, New York (1987)

[7] Getzler, E., Berline, N. and Vergne, M. *Heat Kernels and Dirac Operators.* Springer, New York (1992)

[8] Wells, Raymond O. Jr. *Differential Analysis on Complex Manifolds. (Third Edition)* Springer-Verlag, Graduate Texts in Mathematics, **65**. (2008)

[9] Kotake, Takeshi. *An analytic proof of the classical Riemann-Roch theorem.* Global Analysis, Proceedings of Symposia in Pure Mathematics, **16** pp. 137-146 (1970)