What is a limit

Remark 1. ***Objectives***

- *What is a limit*
- *What is a one-sided limit*
- *When does a limit not exist*
- *What does it mean for a function to be continuous*
- *If we know a function is continuous, how can this help us find a limit*

Computational goals

- *Calculate limits from a graph (or state that the limit does not exist)*
- *Estimate limits using nearby values*
- *Find interval of continuity*
- *Use continuity to evaluate limits*

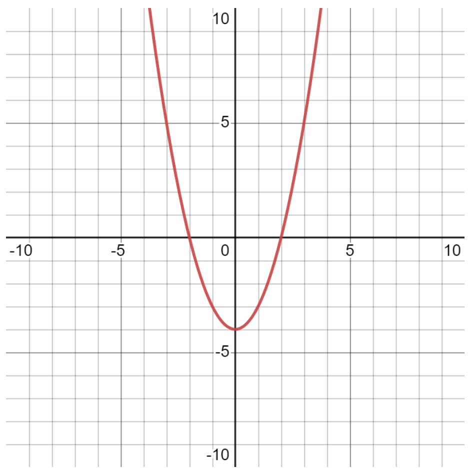
Finding Limits Graphically

Lesson What is a limit

YouTube link: https://www.youtube.com/watch?v=U1PxqDjVfBQ

Learning outcomes:
Author(s):

Quiz



Question 1 $\lim_{xto3} f(x) = \boxed{5}$

Explanation. *That's right! You selected the correct response. The limit as x approached 3 is 5. This is the y-value the graph is getting close to when the x-value is near 3.*

Finding Limits Numerically

Lesson Evaluating a rational function

YouTube link: `https://www.youtube.com/watch?v=YTKoob7m3DM&t=36s`

The computation we just did plugged 3 into $f(x)$ and tells us $f(3)$. It does not necessarily tell us $lim_{xto3} fracx^2 - 1x - 1$.

Since the limit is supposed to be telling us what the y-value is near when x is near 3, we can estimate this by plugging in numbers near 3, such as

$$2.9, 2.99, 2.999, \ldots, 3.001, 3.01, 3.1$$

(eventually the following should link the correct subpages)

- I need a lot of help Watch a video about estimating limits by checking nearby numbers.

See I need a lot of help at `whatisalimit`

- I need a little help Watch a shortened video.
- I think I understand Skip to the quiz.

Estimating limits video

YouTube link: https://www.youtube.com/watch?v=sM4lzNqAdiA

Estimating limits video (short)

YouTube link: https://www.youtube.com/watch?v=sM4lzNqAdiA&t=185s

Quiz

Question 2 *Which of the following statements is true regarding the relationship between the limit as x goes to a of $f(x)$ and $f(a)$ That is, the relationship between the limit and the function value at the point $x = a$. (You should have 2 attempts; not sure if there's a way to do this in Ximera.)*

Multiple Choice:

(a) *The limit of $f(x)$ as x goes to a and the function value $f(a)$ always give the same value because the function must be approaching its function value.*

(b) *The limit of $f(x)$ as x goes to a and the function value $f(a)$ always give different values since they are by definition different quantities.*

(c) *The limit of $f(x)$ as x goes to a and the function value $f(a)$ are by definition different quantities. They may or may not have the same numerical value, depending on the behavior of $f(x)$ near $x = a$.* ✓

Explanation. *That's right! The limit of $f(x)$ as $x \to a$ and the function value $f(a)$ are by definition different quantities. They may or may not have the same numerical value, depending on the behavior of $f(x)$ near $x = a$.*

See I need a little help at whatisalimit
See I think I understand at whatisalimit

Lesson Estimating limits from nearby numbers

YouTube link: https://www.youtube.com/watch?v=YTKoob7m3DM&t=150s

What would you like to do next (eventually the following should link the correct subpages; maybe put them in expandable environments)

- See how to find this limit graphically.
- See another example of limits estimated with a table.
- Continue the lesson and learn about one-sided limits.

Example Finding a limit graphically

YouTube link: https://www.youtube.com/watch?v=why41NH7U4k

Example Estimating a limit using nearby values

YouTube link: https://www.youtube.com/watch?v=C-xihdPs9s

One-Sided Limits

We can extend these same concepts and definitions to the idea of a one-sided limit...

$\lim_{xtoa^+} f(x)$ means "What value does $f(x)$ approach as x approaches a only from the right side"

$\lim_{xtoa^-} f(x)$ means "What value does $f(x)$ approach as x approaches a only from the left side"

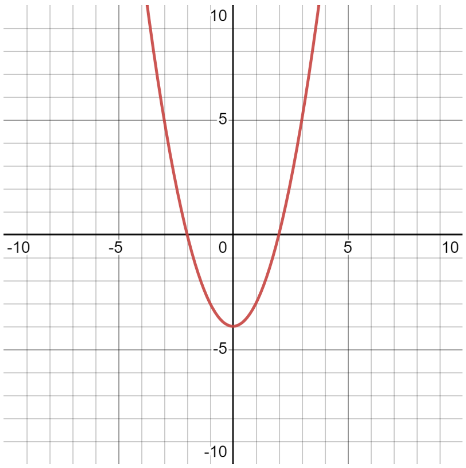
Lesson one-sided limits

YouTube link: https://www.youtube.com/watch?v=KI5tjq2yrcI

See See how to find this limit graphically. at whatisalimit
See See another example of limits estimated with a table. at whatisalimit
See Continue the lesson and learn about one-sided limits. at whatisalimit

Practice problem

Take a few minutes to practice by evaluating the following (this is not graded; click to reveal answers)



- $f(5)$ Value 4
- $\lim_{xto5^+} f(x)$ Value 6
- $\lim_{xto5^-} f(x)$ Value 2

Limits which "Do Not Exist"

Sometimes, a limit may not exist.

Explanation. *A limit does not exist if $f(x)$ does not approach a single value as x approaches a.*

We use the notation DNE (which stands for "Does Not Exist") when a limit does not exist.

Use the links below to explore three scenarios in which the limit does not exist (all three should be required; not sure how to do this in Ximera). After you have explored all three, click to complete the assessment.

Oscillations

YouTube link: `https://www.youtube.com/watch?v=EoT5FZdt6qs`

One-sided limits do not match

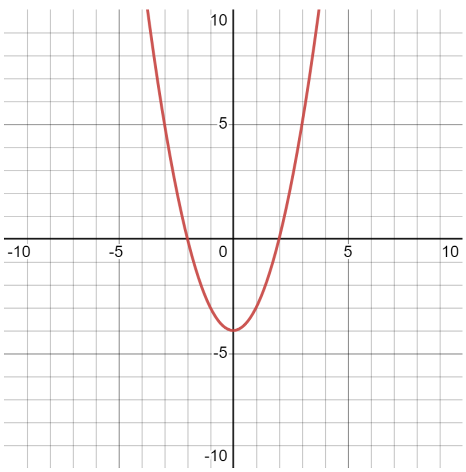
YouTube link: https://www.youtube.com/watch?v=KI5tjq2yrcI&t=131s

Infinite limits

YouTube link: https://www.youtube.com/watch?v=gCuh1DmvvQE

Assessment

Question 3 *What is the limit of the pictured function as x approaches 5 (You have 2 attempts.)*



Multiple Choice:

(a) *4*

(b) *DNE*

(c) *2* ✓

(d) *6*

Explanation. *That's right! The correct response is DNE. While the one-sided limits do exist, the limit as $x \to 5$ of this function does not exist because the one-sided limits are not equal.*

Continuity

Lesson Continuity at a point

YouTube link: https://www.youtube.com/watch?v=ReDZpc5jhCw

Requirements for continuity

In order for the function $f(x)$ to be continuous at a point $x = a$, the following conditions must be met

(a) $f(a)$ is defined

(b) $\lim_{xtoa} f(x)$ exists

(c) $\lim_{xtoa} f(x) = f(a)$ (the function value and limit are equal at a).

Explanation. ***Intuitive vs. Formal Definitions.*** *Intuitively, the graph of f is continuous at the point a if the graph near a can be traced "without lifting your pencil". While this way of thinking about continuity can be helpful to us in understanding the concept, it is not the mathematical definition. If you are asked to show a function is continuous at a point, you must use the mathematical definition given above!*

Quiz criteria for continuity

Question 4 *Which of the following is NOT a criteria for continuity at $x = a$ (You have one attempt.)*

Multiple Choice:

(a) *The function is defined at the point $x = a$*

(b) *The limit as x approaches a exists*

(c) *The limit as x approaches a is equal to the function value at a*

(d) *The function has no sharp turns or oscillations near $x = a$.*

Explanation. *That's right! The function has no sharp turns or oscillations near $x = a$ is not required for continuity.*

Continuity on an Interval

Now that we've studied continuity at a point, let's expand our knowledge and learn what it means for a function to be continuous on an interval $[a, b]$

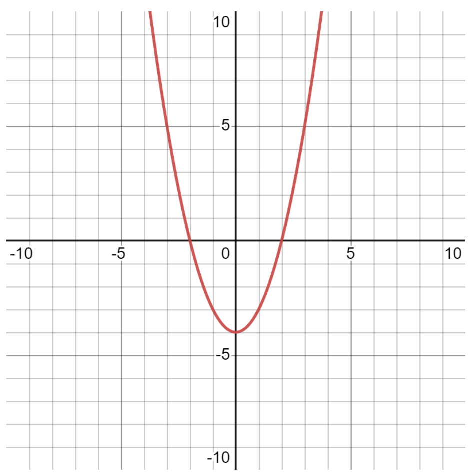
Lesson

YouTube link: `https://www.youtube.com/watch?v=ReDZpc5jhCw&t=213s`

To summarize...

Explanation.

- *The function $f(x)$ is continuous from the left at point a if* $\lim_{xtoa^-} f(x) = f(a)$
- *The function $f(x)$ is continuous from the right at point a if* $\lim_{xtoa^+} f(x) = f(a)$

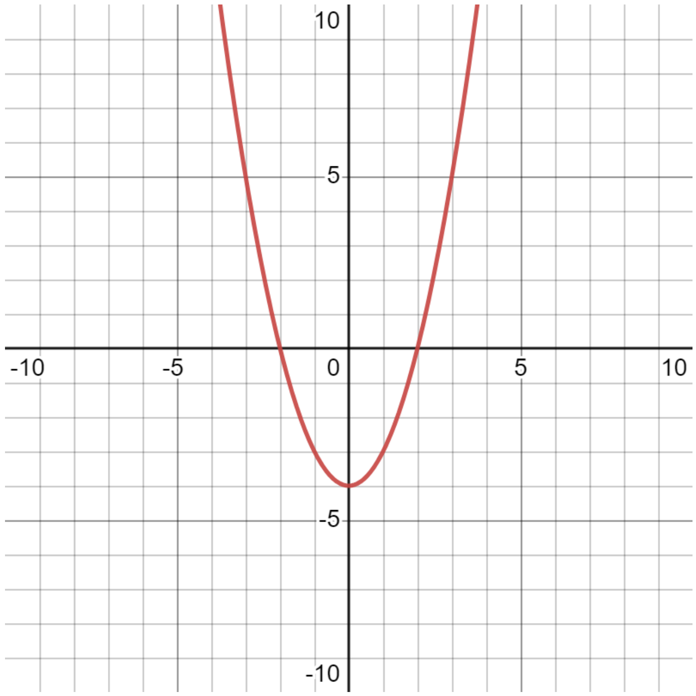


The function $f(x)$ is continuous on the open interval (a, b) if it is continuous at all points of the interval

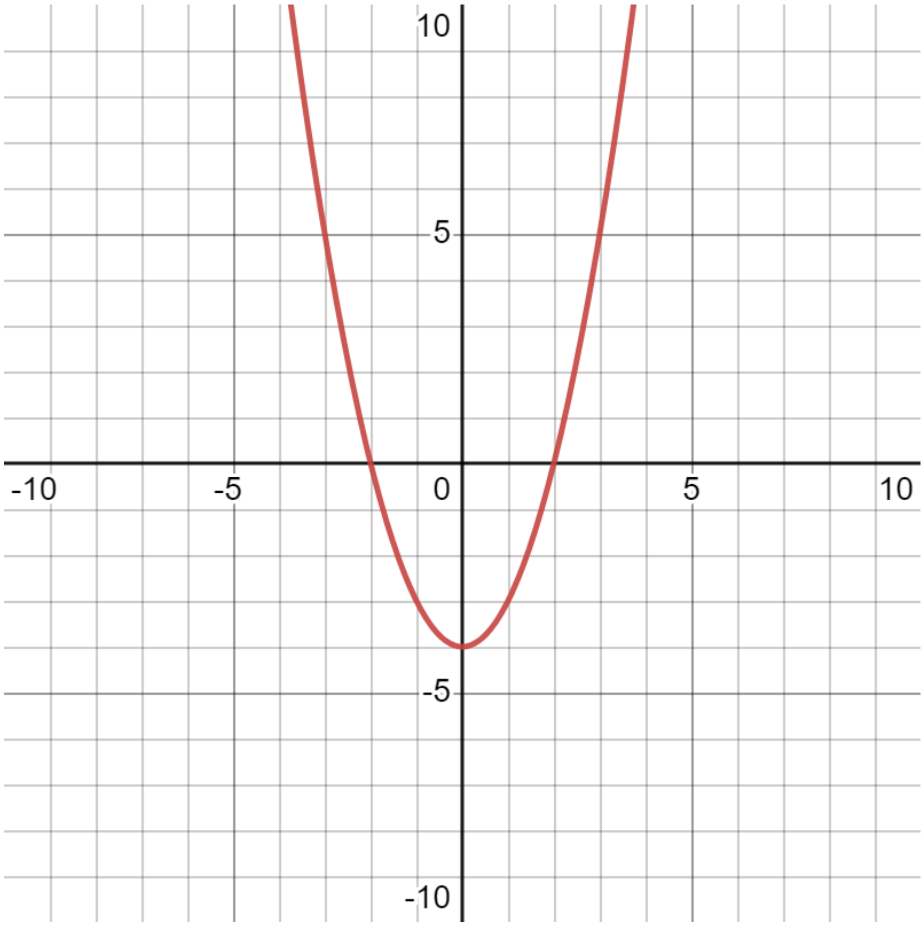
The function $f(x)$ is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and if $lim_{xtoa^+} f(x) = f(a)$ and $lim_{xtob^-} f(x) = f(b)$. In other words, $f(x)$ is continuous on the closed interval $[a, b]$ if it is continuous at all points of the interior (a, b), left continuous at $x = b$, and right continuous at $x = a$.

Examples

Pictured are the graphs of two functions which are continuous for all real numbers. The first is a polynomial and the second is the cosine function.



Below is an example of a discontinuous graph.



Famous functions

Theorem 1. *The following functions are continuous on their natural domains, for k a real number and b a positive real number.*

- *Constant function*
$$f(x) = k$$
- *Identity function*
$$f(x) = x$$
- *Power function*
$$f(x) = x^b$$
- *Exponential function*
$$f(x) = b^x$$

- *Logarithmic function*
$$f(x) = log_b(x)$$
- *Sine and cosine*
$$f(x) = sin(x), \qquad f(x) = cos(x)$$

We can now take limits of these functions without having to guess by evaluating them at nearby numbers. We know that for all values of a in the domain of these functions, we can find the limit as x approaches a just by plugging a into the function!

$$\lim_{xtoa} f(x) = f(a)$$

Quiz

Question 5 *Use continuity to find this limit*

$$\lim_{x \to \pi} sin(x)$$

Multiple Choice:

(a) *0, because* $sin(\pi) = 0$. ✓

(b) *0 because if we plug numbers near* pi *like* 3.1, 3.14, *and* 3.15 *into* $sin(x)$, *we get out very small values.*

(c) *We don't know how to find the exact value of this limit yet.*

Explanation. *That's right! Since we know that* $sin(x)$ *is continuous at* $x = \pi$, *we know that the limit as* $x \to \pi$ *is equal to what we get when we plug* π *into* $sin(x)$.