Math 2000 Notes

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Contents

Throughout this text we will denote:

- $\mathbb{N}$ as the set of natural numbers, these are the non-negative whole numbers. Note, 0 is a natural number.
- $\mathbb{Z}$ as the set of integers.
- $\mathbb{Q}$ as the set of rational numbers.
- $\mathbb{R}$ as the set of real numbers.
- $\mathbb{C}$ as the set of complex numbers.

By $\mathbb{Z}^+$ and $\mathbb{Z}^-$, we mean the set of all positive and negative integers respectively. By $\mathbb{Z}^{\geq 0}$ and $\mathbb{Z}^{\leq 0}$, we mean the set of all non-negative and non-positive integers respectively. Similarly, we may replace the set of integers, $\mathbb{Z}$ in the previous with $\mathbb{Q}$ or $\mathbb{R}$ for the same meaning. Note that as positive and negative are not defined on the complex plane, we cannot do the same for $\mathbb{C}$.

1 Logical Forms And Equivalence

Definition 1.1. A *statement* (or *proposition*) is a sentence that is either true or false, but not both.

Definition 1.2. A *statement form* (or *proposition form*) is an expression made up of statement variables and logical connections (such as $\neg$, $\vee$, or $\wedge$) which when substituting statements for statement variables becomes a statement.

Note, statement forms are acting as *Platonic forms* of statements.

Definition 1.3. The *truth value* of a given statement is true if that sentence is itself true otherwise, the truth value of that statement is false.

Definition 1.4. Let p be a statement form. The *negation* of p, written $\neg p$, is the statement form with the opposite truth value of p.

Note, the symbol $\neg$ is not the only symbol used to denote negation. For instance, it is not uncommon to see the symbol $\sim$ used in other logic texts. Further, many c-like programming-languages will use the symbol ! for the same meaning. The Python programming-language deserves a special call-out on this note, it allows for the use of the symbol 'not' for its not symbol (which fits nicely with its use of 'and' and 'or' for its and and or logical connectives).

Definition 1.5. Let p and q be statements forms. The *disjunction* of p and q, written $p \vee q$, is the statement form that is true when either p or q is true and false precisely when p and q are both false.

Definition 1.6. Let p and q be statements forms The *conjunction* of p and q, written $p \wedge q$, is the statement form that is true precisely when both p and q are true and is otherwise false.

Just as in arithmetic, when more than one logical connective is used, we

- perform the operations from left to right,
- evaluate parenthetical terms first,
- treat $\neg$ similar to a minus sign.

Definition 1.7. A *truth table* for a statement form displays the truth values corresponding to every possible combination of truth values for its component statement variables.

Example 1.1. Write the truth tables for logical connectives: $\neg$ (not), $\vee$ (or), and $\wedge$ (and) :

p	$\neg p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.2. Truth table for the statement form $(p \vee q) \wedge \neg(p \wedge q)$:
Note, in this example we also include the truth table for the sub-statements forming the larger statement. This is not necessary, though this does reduce the risk for error at the cost of space and ink.

p	q	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Example 1.3. Write the truth table for the statement form $(p \wedge q) \vee \neg r$:
Note, in this example, we also include the truth table for the sub-statements forming the larger statement. This is not necessary, though this does reduce the risk for error at the cost of space and ink.

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Definition 1.8. Two statement forms are said to be *logically equivalent* if, and only if, they have identical truth values for each possible truth value assignment for their statement variables. Let P and Q be statement forms, we write $P \equiv Q$ to mean that P is logically equivalent to Q.

Definition 1.9. Two statements are said to be *logically equivalent* if, and only if, they have logically equivalent forms after replacing identical component statements with statement variables.

Proposition 1.1. (Double negation): *The statement form p is logically equivalent to $\neg\neg p$, i.e. $p \equiv \neg\neg p$.*

Proof. In the truth table below, we will write $\neg\neg p$ and $\neg(\neg p)$. This is to highlight that we are negating the statement form $\neg p$.

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

We note that the column for p is identical to the column for $\neg\neg p$. Thus, we can conclude that after substituting a statement for the statement variable p in the statement forms p and $\neg\neg p$ the statement forms will have identical truth values. Whence, the statement forms are identical. $\square$

The conclusion of the previous definition is certainly a mouthful, but worth repeating. Similar to the way statement forms provide a layer of abstraction to statements, examining the truth tables of two statement forms to determine equivalence abstracts away substituting actual statements for statement variables.

Example 1.4. The statement form $\neg(p \wedge q)$ is not equivalent to the statement form $\neg p \wedge \neg q$ ($\neg(p \wedge q) \not\equiv \neg p \wedge \neg q$).

p	q	$\neg(p \wedge q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

Note the middle two rows of the $\neg(p \wedge q)$ and the $\neg p \wedge \neg q$ do not have the same values. These rows are counterexamples to the two statement forms being logically equivalent.

Proposition 1.2. (DeMorgan's Law):

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

and

$$\neg(p \vee q) \equiv \neg p \wedge \neg q\,.$$

Proof. We'll show this by examining the truth tables for each statement form and noting that the relevant columns are identical.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	F	F
T	F	T	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

□

Exercise 1.1. Use DeMorgan's, Proposition 1.2, to write the negation of the following statements:

1. John is 6 feet tall or he weighs less than 200 pounds.

2. The bus was late or Tom's watch was slow.

3. $-1 \leq x \leq 4$.

Definition 1.10. A *tautology* is a statement form that is true independent of the truth value assignments of its truth value assignments. A statement whose statement form is a tautology is a *tautological statement.*

Definition 1.11. A *contradiction* is a statement form that is false independent of the truth value assignments of its truth value assignments. A statement whose statement form is a contradiction is a *contradictory statement.*

Theorem 1.3. *Let p, q, and r be statement variables, τ a tautology, and c a contradiction. Then the following hold:*

1. *Commutativity: $p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$.*

2. *Associativity: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ and $(p \vee q) \vee r \equiv p \vee (q \vee r)$.*

3. *Distribution:* $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ *and* $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
4. *Identity:* $p \wedge \tau \equiv p$ *and* $p \vee c \equiv p$.
5. *Negation:* $p \wedge \neg p \equiv c$ *and* $p \vee \neg p \equiv \tau$.
6. *Double negation:* $\neg\neg p \equiv p$.
7. *Idempotent:* $p \wedge p \equiv p$ *and* $p \vee p \equiv p$.
8. *Universal bound:* $p \vee \tau \equiv \tau$ *and* $p \wedge c \equiv c$.
9. *DeMorgan's Law:* $\neg(p \wedge q) \equiv \neg p \vee \neg q$ *and* $\neg(p \vee q) \equiv \neg p \wedge \neg q$.
10. *Absorption:* $p \vee (p \wedge q) \equiv p$ *and* $p \wedge (p \vee q) \equiv q$.
11. *Negation of* τ *and* c*:* $\neg\tau \equiv c$ *and* $\neg c \equiv \tau$.

Proof. Claims (6) and (9) have been proved in Proposition 1.1 and Proposition 1.2 respectively. The remaining claims are left as an exercise to the reader. □

Exercise 1.2. Use truth tables to show that claims of Theorem 1.3.

2 Conditional Statements

Definition 2.1. Let p and q be statements forms. The *conditional statement* "p implies q", written $p \to q$, is the statement form that is false precisely when p is true and q is false (that is, when the statement "If p, then q" is violated).

In the conditional $p \to q$, p is refered to as the *hypothesis* and q is called the *conclusion*.

Conditional statements, $p \to q$ can be expressed, in English, in many ways. For instance, the conditional may be presented as "If p, then q" or "q by p".

A conditional statement is said to be vacuously true when the hypothesis is false.

In expressions with other logical connectives, $\to$ is performed last.

Example 2.1. The following is the truth table for the conditional statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 2.2. Construct the truth table for the statement form $(p \vee \neg q) \to \neg p$.

p	q	$(p \vee \neg q) \to \neg p$
T	T	F
T	F	F
F	T	T
F	F	T

Proposition 2.1. $p \to q \equiv \neg p \vee q$

Proof. Left as an exercise to the reader. □

Exercise 2.1. Show, using a truth table, Proposition 2.1.

Example 2.3. Rewrite the following as an if-then statement (in English):

"Either you get your work in on time or you're fired.".

Let $\neg p$ be the statement "you get your work in on time". This means that p is equivalent to the statement " you do not get your work in on time" Let q the statement "you are fired". Then the original statement can be written as $\neg p \vee q$. Using Proposition 2.1, we have that $\neg p \vee q \equiv p \to q$. Rewriting this in English we get:

"If you do not get your work in on time, then you are fired.".

Example 2.4. From Proposition 2.1, we have that $p \to q \equiv \neg p \vee q$. Applying DeMorgan's Law, Proposition 1.2, and using double negation, Proposition 1.1, we have that

$$\begin{aligned} \neg(p \to q) &\equiv \neg(\neg p \vee q) && (1)\\ &\equiv \neg\neg p \wedge \neg q && (2)\\ &\equiv p \wedge \neg q\,. && (3) \end{aligned}$$

Whence, $\neg(p \to q) \equiv p \wedge \neg q$.

Example 2.5. Write the negation of:

"If Sara lives in Athens, then Sara lives in Greece.".

Let p be the statement "Sara lives in Athens" and q the statement "Sata lives in Greece". Then the given statement can be written as $p \to q$. As we saw in Example 2.4, $\neg(p \to q)$ is equivalent to $p \wedge \neg q$. Thus, the negation of the original statement is:

"Sara lives in Athens and Sara does not live in Greece.".

Definition 2.2. The *contrapositive* of a conditional statement form $p \to q$ is the statement form $\neg q \to \neg p$.

Proposition 2.2. $p \to q \equiv \neg q \to \neg q$

Proof. Left as an exercise to the reader. □

Exercise 2.2. Show, using a truth table, Proposition 2.2

Example 2.6. Write the contrapositive of:

"If today is Martin Luther King Jr. Day, then tomorrow is Tuesday.".

Similar to as in Exercise 2.3, let p be the statement "today is Martin Luther King Jr. Day" and q the statement "tomorrow is Tuesday". So, $\neg p$ is equivalent to the statement "today is not Martin Luther King Jr. Day" and $\neg q$ is the statement "tomorrow is not Tuesday".

Using Proposition 2.2, we have that $p \to q \equiv \neg q \to \neg p$. Rewriting this in English we get:

"If tomorrow is not Tuesday, then today is not Martin Luther King Jr. Day.".

Definition 2.3. The *converse* of a conditional statement form $p \to q$ is the statement form $q \to p$.

Definition 2.4. The *inverse* of a conditional statement form $p \to q$ is the statement form $\neg p \to \neg q$.

Example 2.7. Write (in English) the converse and inverse of:

"If today is Martin Luther King Jr. Day, then tomorrow is Tuesday.".

Just as in Example 2.6, let p be the statement "today is Martin Luther King Jr. Day" and q the statement "tomorrow is Tuesday". So, $\neg p$ is equivalent to the statement "today is not Martin Luther King Jr. Day" and $\neg q$ is the statement "tomorrow is not Tuesday"

Finally, the from Definition 2.3, the converse of the original statement is $q \to p$ or in English:

"If tomorrow is Tuesday, then today is Martin Luther King Jr. Day.".

And, using Definition 2.4 the inverse of the original statement is $\neg p \to \neg q$ or in English:

"If today is not Martin Luther King Jr. Day, then tomorrow is not Tuesday.".

Exercise 2.3. Show that the converse and inverse of a conditional statement $p \to q$ are equivalent.

Definition 2.5. Let p and q be statements forms. The *biconditional statement* "p if, and only if q", written $p \leftrightarrow q$, is the statement form that is true precisely when p and q have the same truth value.

We ofter abbreviate "if, and only if" by iff.

Example 2.8. The following is the truth table for the conditional statement $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercise 2.4. Show, using a truth table, that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

3 Predicates and Quantifiers

Note the sentence

“They are a college student.”

is not a statement as it may be true or false depending on the value of “They”. In this sentence “They” is a free variable. Similarly, “$x + y > 0$” is not a statement, it has more familiar variables x and y.

The predicate refers to the part of the sentence with some or all of its nouns removed. That is, in the sentence

“James is a student at the University of North Texas.”,

“James” is the subject and

“is a student at the University of North Texas.”

is the predicate.

In logic, predicates are formed in much the same way.

Definition 3.1. A *predicate* is a sentence which contains a finite number of variables which becomes a statement when a value is assigned to each of its variables. The *domain* of a predicate variable is the set of possible values which that variable may take.

Example 3.1. Let $P(x)$ denote

“x is a student at the University of North Texas.”

and $Q(x, y)$ denote

“x is a student at y.”.

x is the predicate variable for $P(x)$ while both x and y are predicate variables for $Q(x, y)$.

Note that when values are substituted for x and y in $P(x)$ and $Q(x, y)$, these sentences become statements. For instance if we substitute x for “Taylor” and y for“Boise State University”, $P(x)$ and $Q(x, y)$ become the statements

“Taylor is a student at the University of North Texas.”

and

"Taylor is a student at Boise State University."

respectively. Whence, $P(x)$ and $Q(x, y)$ are predicates

In Example 3.1, we referred to the example's predicates with their full names, $P(x)$ and $Q(x, y)$. When there is no room for confusion and the variables or other ornaments are of little importance to the statement being made, we may drop those decorations from the symbol. That is, the concluding sentence of Example 3.1 could be written "Whence, P and Q are predicates.".

Example 3.2. Let $P(x)$ denote "$x^2 > x$" with $\mathbb{R}$, the set of real numbers, as the domain of x.

$P(2)$ and $P(-\frac{1}{2})$ are true. This is because $P(2)$ and $P(-\frac{1}{2})$ denote the statement "$4 > 2$" and "$\frac{1}{4} > -\frac{1}{2}$" respectively. On the other hand, $P(\frac{1}{2})$, which is the statement "$\frac{1}{4} > \frac{1}{2}$" is false.

Definition 3.2. Let $P(x)$ be a predicate with variable x with domain D. The *truth set* of $P(x)$ is the set of elements in D so that $P(x)$ is true after substituting that element for x. That is, it is the set $\{\, x \in D \mid P(x) \,\}$.

Example 3.3. Let $P(x)$ be as in Example 3.2. It is not difficult to see that for any real number x with $x < 0$, that $x^2 > 0 > x$. So, for any real number x with $x < 0$ $P(x)$ holds.

Now, if x is a real number with $0 \leq x \leq 1$, then we can see that $0 \leq x^2 \leq x \leq 1$. So, if x is a real number with $0 \leq x \leq 1$, then $P(x)$ is false.

Finally, if x is a real number with $1 < x$, then we have that $x < x^2$. Thus, if x is a real number with $1 < x$, then $P(x)$ is true.

Puting this together, we have that the truth set of $P(x)$ are the real numbers x such that $(x < 0) \vee (1 < x)$.

Example 3.4. Let $Q(n)$ be "n is a factor of 8. Find the truth set of $Q(n)$ where:

1. the domain of n is the set of positive integers, $\mathbb{Z}^+$
$$\left\{\, n \in \mathbb{Z}^+ \mid Q(n) \,\right\} = \{\, 1, 2, 4, 8 \,\}\ .$$
2. the domain of n is the set of integers, $\mathbb{Z}$.
$$\{\, n \in \mathbb{Z} \mid Q(n) \,\} = \{\, -8, -4, -2, -1, 1, 2, 4, 8 \,\}\ .$$

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