Supplementary material on simulation study

1 Parametrization

We often find it convenient to start with a simplified version of the Hyperbolic distribution containing only the shape parameters. To this end, we consider the following expression:

$$f_X(x') = \frac{\gamma'}{2\alpha' K_1(\gamma')} e^{-\alpha'\sqrt{1+x'^2}+\beta' x'}. \tag{1}$$

Here, $\gamma' = \sqrt{\alpha'^2 - \beta'}$. This probability density leads to the following mean $\mu'_{\alpha',\beta'}$ and variance $\sigma'^2_{\alpha',\beta'}$:

$$\mu'_{\alpha',\beta'} = \frac{\beta' K_2(\gamma')}{\gamma' K_1(\gamma')} \tag{2}$$

$$\sigma'^2_{\alpha',\beta'} = \frac{K_2(\gamma')}{\gamma' K_1(\gamma')} + \frac{\beta'^2}{\gamma'^2}\left(\frac{K_3(\gamma')}{K_1(\gamma')} - \frac{K_2(\gamma')^2}{K_1(\gamma')^2}\right). \tag{3}$$

For PCs, we require a zero mean and a variance equal to λ_j (following from the eigenvalues of the correlation matrix). We consider therefore the following transformations to eliminate the mean $\mu'_{\alpha',\beta'}$ and to replace the variance $\sigma'^2_{\alpha',\beta'}$ by λ_j:

$$\begin{aligned}
x &= \sqrt{\lambda_j}(x' - \mu'_{\alpha',\beta'})/\sigma'_{\alpha',\beta'} \\
\delta &= \sqrt{\lambda_j}/\sigma'_{\alpha',\beta'} \\
\alpha &= \alpha'/\delta \\
\beta &= \beta'/\delta \\
\mu &= -\mu'_{\alpha',\beta'}\delta
\end{aligned} \tag{4}$$

By using the expressions in of Eq. (4) in Eq. (1), we can eliminate the primed variables to equivalently obtain the parametrization:

$$f_{\mathrm{HB}}(x) = \frac{\gamma}{2\delta\alpha K_1(\delta\gamma)} e^{-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)}. \tag{5}$$

The parametrizations of Eq. (4) ensures that the mean equals zero and the variance equals λ_j when varying the shape parameters α' and β'. These parametrizations are therefore encountered in the simulation scripts.

Using $\delta^2 = \chi$ and $\gamma^2 = \psi$, we obtain the form:

$$f_{\mathrm{HB}}(x) = \frac{\sqrt{\psi/\chi}}{2\alpha K_1(\sqrt{\chi\psi})} e^{-\alpha\sqrt{\chi+(x-\mu)^2}+\beta(x-\mu)}. \tag{6}$$

This is the parametrization used in the paper by Gubbels et al. (2025) on Principal Component Copulas for Capital Modelling and Systemic Risk.