MATH 502 - Real Analysis: A Brief Summary

Bilge Köksal

Contents

1 Locally Convex Spaces

Definition 1.1. Given a vector space E and τ a topology on E, (E, τ) is a **Topological Vector Space** (TVS) if the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ where $x, y \in E$ and $\lambda \in \mathbb{K}$ are continuous.

Definition 1.2. A TVS E is a **Locally Convex Space** (LCS) if $0 \in E$ has a fundamental system of absolutely convex neighborhoods denoted by $\mathscr{F}(0)$, that is every neighborhood of 0 in E necessarily contains some $U \in \mathscr{F}(0)$.

Theorem 1.3 (Gauge Function)**.** *Let E be a TVS and let V be an absolutely convex neighborhood of* 0*. Then there is a unique seminorm $p_V : E \mapsto \mathbb{K}$ called the gauge function of V such that*

$$V = \{x \in E : p_V(x) < 1\}$$

In light of Theorem 1.3 a TVS (E, τ) is locally convex if and only if there is a *unique* family of seminorms $(p_i : i \in I)$ that induces the topology τ, which correspond to the gauge functions of the sets in $\mathscr{F}(0)$ that defines the topology on E in the sense that the finite intersections of the collection $\{B_{J,\epsilon} : J \subset I, |J| < \infty, \epsilon > 0\}$ where

$$B_{J,\epsilon} := \{x \in E : p_j(x) < \epsilon \quad \forall j \in J\}$$

is a neighborhood base of $0 \in E$.

Theorem 1.4. *A LCS is normable iff it contains an open convex set.*

2 Hahn Banach Theorem: Implications and Applications

Theorem 2.1 (Hahn Banach Theorem)**.** *Let X vector space and p be a semi-norm on X. If λ is a linear form defined on $Y \subset X$, dominated by p i.e.,*

$$\lambda(y) \leq p(y), \quad \forall y \in Y$$

then there is Λ, a linear form on X that is an extension of λ and is again dominated by p on X.

Theorem 2.2 (Hahn Banach Theorem Geometric Version)**.** *Let E be TVS, $A \in E$ an open convex set and M a vector subspace of X s.t. $A \cap M = \varnothing$. Then there is a hyperplane H in E s.t. $H \supset M$ and $H \cap A = \varnothing$.*

Theorem 2.3 (Hahn-Banach Separation Theorem)). *Let A and B be convex subsets of a LCS E. Then there is a linear form f on E such that f separates A and B in the sense that*

$$f(A) \cap f(B) = \varnothing$$

Intuitively, the **main idea** of the Hahn Banach theorem is that there are enough continuous linear forms on E to separate the elements of E. This fact is extremely useful in applications. We will outline some applications throughout this summary.

Definition 2.4. Let E be a TVS. $K \subset E$ is **total** in E if for $f : E \to \mathbb{R}$ that is continuous and linear we have

$$f\big|_K \equiv 0 \Rightarrow f \equiv 0$$

One can prove that K is total in $E \Rightarrow$ the span of K is dense in E by taking an element $x \in \overline{spanK}^c$ and applying Hahn Banach Theorem (geometric version) in an attempt to separate x and $\overline{spanK}$ which will lead to a contradiction. Thus we have a new tool with which we can **recognize dense subspaces** of a TVS. Some examples of what one can show through this type of argument are:

- $\{x \in l^p : \sum x_n = 0\}$ is dense in l^p
- $\{x \in L^p(X, \mu) : \int_X f d\mu = 0\}$ is dense in $L^p(X, \mu)$

Another example of an application is that we can prove that $L^1(\mathbb{R})$ is not reflexive: define a linear form λ on $C_b(\mathbb{R}) \subset L^\infty(\mathbb{R})$ such that $\lambda : f \mapsto f(0)$. Then there is a Hahn-Banach extension Λ on $L^\infty(\mathbb{R})$. We can show that there is no element of $L^1(\mathbb{R})$ representing Λ and hence

$$(L^\infty(\mathbb{R}))' \neq L^1(\mathbb{R})$$

When working in the Hilbert space case, one can show from the analytic form of Hahn Banach Theorem that there is in fact an explicitly computable **unique norm preserving extension**, that is $\|\Lambda\| = \|\lambda\|$. This is a consequence of the Riesz representation theorem so it cannot be generalized to an arbitrary TVS. For a general normed spaces, the Hahn Banach extension is either unique, or there are infinitely many such extensions Λ: if Λ_1 and Λ_2 are two Hahn Banach Extensions, then so is $t\Lambda_1 + (1-t)\Lambda_2$ for any $t \in [0, 1]$.

The **existence of a fundamental solution** in the space of distributions for a constant coefficient differential operator $P(D)$ relies on the analytic form of Hahn-Banach theorem. A fundamental solution of $P(D)$ on say $\mathbb{R}^n$, is a distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying

$$P(D)E = \delta$$

The significance of finding the fundamental solution is that, the solution u to the PDE

$$P(D)u = f$$

is precisely $u = E * f$ which follows from the commutation of D and convolution. We will explore this example which is the Malgrange-Ehrenpreis Theorem further after defining adjoint operators.

Further applications of the Hahn Banach Theorem will be highlighted as we introduce new concepts.

3 Duality

Definition 3.1. (E, E') is called a **dual pair** if E separates the points and E' and E' separates the points of E, i.e. $\forall x \in E$, $\exists x' \in E'$ s.t. $\langle x, x' \rangle \neq 0$ and vice versa.

Definition 3.2. The **weak topology** on E is the weakest topology for which the maps $x \mapsto \langle x, x' \rangle$ are continuous. It is denoted by $\sigma(E, E')$.

The space of continuous linear forms on $(E, \sigma(E, E'))$ is precisely E' and we have $\sigma(E, E') \subseteq \tau_{\|\cdot\|}$ since the maps $x \mapsto \langle x, x' \rangle$ are already continuous w.r.t. $\tau_{\|\cdot\|}$.

Remark 3.3. If E is a finite dimensional normed space then $\sigma(E, E') = \tau_{\|\cdot\|}$ but if E is infinite dimensional then the weak topology is *strictly* coarser than the strong topology. As an example, the unit ball in E is never open in $\sigma(E, E')$. Consequently, $\sigma(E, E')$ **is not metrizable**.

In light of the above remark, it is easy to see that weakly closed subsets of E are also strongly closed (norm-closed), but strongly closed $\not\Rightarrow$ weakly closed, however, for any convex set $C \subset E$, through an argument involving the <u>Hahn-Banach (separation) theorem</u>

$$C \text{ is weakly closed} \Leftrightarrow C \text{ is strongly closed}$$

Idea of Proof. assuming C is strongly closed take $x \notin C$, then there is a linear form f separating x and C. Using f we show that C^c is weakly open.

Theorem 3.4. $(E, \sigma(E, E'))$ ***is Hausdorff.***

The proof again relies on Hahn-Banach Theorem, the main idea is that we separate distinct points of $(E, \sigma(E, E'))$ by a linear form.

Theorem 3.5 (Banach-Alaoglu)**.** *The closed unit ball of the dual space is always weakly compact.*

The reason we are interested in weak topologies is that coarser topologies contain more compact sets since there are less open covers of any set. The Banach Alaoglu theorem is an example of this fact since the closed unit ball in an infinite dimensional Banach space is never compact.

Theorem 3.6. *If E is a separable normed space, the closed unit ball B'_1 of the dual space $(E', \sigma(E', E))$ is metrizable.*

Take any dense sequence $(x_n)_{n\in\mathbb{N}}$ in E, then for $x', y' \in B'_1$ the metric

$$d(x', y') = \sum_{n=1}^{\infty} \frac{|\langle x_n, x' - y' \rangle|}{2\|x_n\|^2 + 1}$$

induces the weak topology on B'_1.

An Unusual Example (Schur's Theorem). l^1 is an infinite dimensional space with the property that strong convergence is equivalent to weak convergence. Note that this does not contradict Remark 3.3 since two different topologies can have the same convergent sequences, afterall $(l^1, \sigma(l^1, l^\infty))$ is not metrizable. *Idea of Proof.* Since l^1 is separable, the closed unit ball B'_1 in $(l^\infty, \sigma(l^\infty, l^1))$ is metrizable and it is compact by Banach Alaoglu Theorem. Thus we can write B'_1 as the union of some convenient subsets and apply Baire Category Theorem.

In general, weak convergence $\not\Rightarrow$ strong convergence but by **Mazur's Lemma**, $x_n \to x$ weakly $\Rightarrow$ there is a sequence of convex combinations of x_n's converging to x strongly.

Theorem 3.7. *Let X be a normed space. Then $f : X \to \mathbb{K}$ is continuous iff it is weakly continuous.*

Idea of Proof. The direct implication is clear whereas the proof of the converse implication relies on a purely algebraic lemma: for $\{f_i\}_{i=1}^n \subset X^*$,

$$\bigcap_{i=1}^{n} ker(f_i) \subset ker(f) \Rightarrow f \in span\{f_1, ..., f_n\}$$

this can be proved by induction on n. Assuming f is weakly continuous, the inverse image of the unit ball under f contains some absolutely convex element of the neighborhood base, determined by say $x_1', ..., x_n' \in X'$. We proceed to show that $ker(f)$ contains $\bigcap_{i=1}^n x_i'$ so that by the lemma, f is a linear combination of continuous functionals and hence continuous.

3.1 Polars

Definition 3.8. Let (E, E') be a dual pair. Then for $A \subset E$ we define the **polar** $A^{\text{o}} \subset E'$ of A as

$$A^{\text{o}} \coloneqq \{x' \in E' : \sup_{x \in A} |\langle x, x' \rangle| \leq 1\}$$

Theorem 3.9. *Let E be a metrizable LCHS, then E' equipped with the strong topology is metrizable then $\Rightarrow$ E' is normable.*

Idea of Proof. Since metric spaces are first-countable we can write E' as

$$E' = \bigcup_{n=1}^{\infty} U_n^{\text{o}}$$

where $\{U_n\} = \mathscr{F}_E(0)$. Then assuming E' is complete, by the Baire Category Theorem there is a U^{o} with $\varnothing$ interior. Since U^{o} is bounded in E', E' has a bounded neighborhood $\Rightarrow$ E' is normable.

3.2 Adjoints of Linear Maps and Applications to PDEs

Definition 3.10. Let (E, E'), (F, F') be dual pairs and $t \in L(E, F)$. There is a unique $t' \in L(F', E')$ called the **adjoint** of t such that $\forall y' \in F'$ and $x \in E$

$$\langle x, t'(y') \rangle = \langle t(x), y' \rangle$$

Theorem 3.11. *Let (E, E'), (F, F') be dual pairs and $t \in L(E, F)$.*

$$t'(F') \subset E' \Leftrightarrow t \text{ is } (\sigma(E, E'), \sigma(F, F'))\text{-continuous}$$

This concept is extremely important in the modern study of analysis of PDEs. The adjoint D' of the differentiation operator D is used to define **weak derivatives** (or derivatives *in the sense of distributions*) for functions that are not differentiable or even continuous. D' can be meaningful even if D is not.

Let $\mathscr{D}$ denote the space of C^∞ functions of compact support on some domain. The main idea behind the proof of the existence of fundamental solutions to a constant coefficient differential operator $P(D)$ is that we can define a linear form F on $P(D)\mathscr{D}$ with

$$F : \phi \mapsto \phi(0) \in \mathbb{K}$$

If we can show that F is continuous in the topology of $\mathscr{D}$, since $P(D)\mathscr{D}$ is embedded in $\mathscr{D}'$, by Hahn-Banach Theorem F has an extension E on $\mathscr{D}'$. Then E is a fundamental solution since

$$\langle \phi, P(D)E \rangle = \langle P'(D)\phi, E \rangle = \langle P'(D)\phi, F \rangle = \phi(0) = \langle \delta, \phi \rangle$$

Moreover Theorem 3.11 can also be applied to PDEs: take t to be the Fourier transform on the Schwartz space and t' to be the Fourier transform of tempered distributions.

4 Filters and Precompactness

Definition 4.1. For any set X, $\mathscr{F} \subset \mathscr{P}(X)$ is a **filter** if

- $\varnothing \notin \mathscr{F}$
- $A, B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$
- $A \in \mathscr{F}$ and $A \supset C \Rightarrow C \in \mathscr{F}$

The **motivation behind using filters** is that, we want to talk about notions of convergence, Cauchy sequences, completeness, precompactness etc. in arbitrary topological spaces. In a topological space X, for any $a \in X$, the collection of neighborhoods $\mathscr{U}_a$ of a is a filter, in fact it is the smallest filter that *converges* to a.

$\mathscr{F}$ is a **Cauchy filter** on the LCS E if

$$U \in \mathscr{U}_E(0) \Rightarrow \exists F \in \mathscr{F} : F - F \subset U$$

This is the generalization of the notion of Cauchy sequences to a LCS. Thus using this notion we can define completeness: $A \subset E$ is **complete** if any Cauchy filter containing A, converges to some $a \in A$, i.e. $\mathscr{U}_E(a) \subset \mathscr{F}$.

Definition 4.2. In a LCS E, a subset A is **precompact** if for all $U \in \mathscr{F}_E(0)$, A is *small of order* U, i.e. $A - A \subset U$. A more practical definition is that there is a finite set $K \subset A$ with $A \subset K + U$.

Any precompact set in a LCS is bounded and hence if a HLCS contains a precompact neighborhood, it is finite dimensional.

5 Polar Topologies and Topologies of the Dual Pair

Now for a collection $\mathscr{A} \subset \mathscr{P}(E)$ we define a **polar topology** ξ' on E' by the neighborhood base $\mathscr{F}_{E'}(0) = \mathscr{A}^{\circ} = \{A^{\circ} : A \in \mathscr{A}\}$. ξ' is the topology of uniform convergence on the sets of $\mathscr{A}$, it is also called the topology of $\mathscr{A}$-convergence.

Theorem 5.1. *E is reflexive if and only if the unit ball of E is compact under the topology of uniform convergence on finite subsets of E'.*

So for example, since any Hilbert space or the space $L^p(dt)$ for $p > 1$ are reflexive, the unit ball in all of these spaces are weakly compact.

The strong topology on E, denoted by β is the topology of uniform convergence on $\sigma(E', E)$-bounded subsets of E' whereas the weak topology $\sigma(E, E')$ is the topology of uniform convergence on finite subsets of E'. Note that both of them are polar topologies, but β need not be a topology of dual pair.

A natural question that one might ask is **which topologies are topologies of the dual pair?** The Mackey Arens Theorem gives such a characterization of topologies that are compatible with duality.

Theorem 5.2 (Mackey Arens Theorem)**.** *Let (E,E') be a dual pair and ξ a topology on E. Then ξ is a topology of dual pair if and only if ξ is the (polar) topology of uniform convergence on a set of absolutely convex $\sigma(E', E)$-compact subsets of E'.*

This implies that the topology of uniform convergence on *all* $\sigma(E', E)$-compact subsets of E', denoted by τ, is the finest topology of dual pair. $\tau = \tau(E, E')$ is called the **Mackey topology**. On a Mackey space $(E, \tau(E, E'))$, $t : E \to F$ is weakly continuous $\Rightarrow$ t is continuous. This characterization of continuity in Mackey spaces is very useful in applications. From the definition of the strong topology β one can easily see that we have $\beta \supseteq \tau$. We will see that $\beta = \tau = \sigma$ when E is barrelled.

6 Barrelled Spaces

Definition 6.1. A set in a LCS is a **barrel** if it is absolutely convex, closed and absorbent.

Definition 6.2. A LCTVS E is **barrelled** if it every barrel in E is a neighborhood in E.

One of the most important properties of a barrel is that it absorbs every complete and compact set. Barrels in E can be characterized as the polars of weakly bounded subsets of the dual E'. Whether or not a set is a barrel does not depend on the topology on E'. Being a barrel is a property of dual pair, it is more of a geometric property.

If E is a barrelled space, given any weakly bounded $A' \subset E'$, $B = A'^{\circ}$ is a barrel in E. By the definition of the polar of a set,

$$B = \bigcap_{x' \in A'} (x')^{-1}([-1, 1])$$

and since E is barrelled, B is a neighborhood so that A' is equicontinuous.

E is barrelled $\Leftrightarrow$ every $\sigma(E', E)$-bounded $A' \subset E'$ is equicontinuous and hence the topology on E is the strong topology β (this follows trivially from the definition). Then since the Mackey topology τ is the finest topology of the dual pair and $\beta \supset \tau(E, E')$, the topology on E is also the Mackey topology. Then $A' \subset E'$ is $\sigma(E', E)$-compact $\Rightarrow \overline{co}(A')$ is $\sigma(E', E)$-compact.

Theorem 6.3. *In a barreled space E, the same sets are bounded in every topology of the dual pair.*

So we have outlined some very nice properties of barreled spaces, but we also need to know which spaces are barreled so that we can apply this theory. As an example we have the following theorem:

Theorem 6.4. *Every Fréchet space F is barreled.*

Idea of Proof. Let B be a barrel in F, then $\cup_{n=1}^{\infty} nB = E$ since B is absorbent. Then since F is complete we can apply Baire Category Theorem to conclude that B is a neighborhood.

7 Topologies on $L(E, F)$

Let $\mathscr{A}$ be a collection of weakly bounded subsets of E and $\mathscr{V}$ be an absolutely convex neighborhood base of 0 in F. For $A \in \mathscr{A}$ and $V \in \mathscr{V}$ we define the set

$$W_{A,V} := \{t \in L(E, F) : t(A) \subset V\}$$

One can show that the set $\{W_{A,V} : A \in \mathscr{A}, V \in \mathscr{V}\}$ defines a locally convex topology on $L(E, F)$ called the topology of $\mathscr{A}$-convergence.

Let $\mathscr{A}_f$ be the collection of all finite subsets of E, the topology of $\mathscr{A}_f$ convergence is the coarsest topology on $L(E, F)$ whereas $\mathscr{A}_\beta$-convergence topology is finest one where $\mathscr{A}_\beta$ is the collection of *all* bounded subsets of E

Theorem 7.1. *If $T \subset L(E, F)$ is equicontinuous, then T is bounded in any topology of $\mathscr{A}$-convergence.*

Idea of Proof. If V is an absolutely convex neighborhood in F, then by the definition of equicontinuity $\bigcap_{t\in T} t^{-1}(V)$ is a neighborhood in E, so it absorbs any bounded set $A \subset E$. Then we can show that $\bigcup_{t\in T} t(A)$ is absorbed by V and hence T is absorbed by $W_{A,V}$.

The converse of this theorem requires an additional assumption that E is barreled:

Theorem 7.2 (Banach-Steinhaus)**.** *Let E be a barreled space and F be a HLCS. Then any pointwise bounded subset of $L(E, F)$ is equicontinuous.*

Idea of Proof. Let $T \subset L(E, F)$ and V be a closed and absolutely convex neighborhood in F. Define the set

$$B = \bigcap_{t\in T} t^{-1}(V) \neq \varnothing$$

B is clearly closed. We can show that B is absorbent and hence a barrel in E. Then since E is barreled, B is neighborhood and so T is equicontinuous.

Thus on $L(E, F)$ we have **strongly bounded = weakly bounded**. Keeping in mind that (by Baire's Theorem) any Banach space is barrelled, a famous application of the Banach-Steinhaus theorem on Banach spaces is the Uniform Boundedness Principle:

Corollary 7.3 (Uniform Boundedness Principle)**.** *: Let E be a Banach space and F be a normed space. Given any $T \subset L(E,F)$*

$$\forall x \in E \ \sup_{t\in T} \|t(x)\|_F < \infty \Rightarrow \sup_{t\in T} \|t\| < \infty$$

Remark 7.4. In the Banach-Steinhaus Theorem the assumption that E is barelled is essential. We can observe this by considering the special case $F = \mathbb{K}$ in which the statement becomes: *$T \subset E'$ is $\sigma(E', E)$-bounded $\Rightarrow T$ is equicontinuous.* In Section 6 we pointed out that this statement holds iff E is barreled.

8 Inductive and Projective Limit Topologies

We will only introduce the main idea and give an important example. Let $\mathscr{U}$ denote the neighborhood base of the LCS E and M be a (necessarily) closed subspace of E. Let π denote the canonical projection

$$\pi : E \to {}^{E}\!/_{M}$$

Then $\pi(\mathscr{U})$ is a neighborhood base in ${}^{E}\!/_{M}$ defining the **quotient topology** which is the finest topology in the final space in which π is continuous. So the topology we can "put" on the final space has a an upper limit this is called the principle of **inductive limit**, whereas the topology on the initial space E has a lower limit (coarsest topology) corresponding to the **projective limit**. Then we just apply this idea to a family of initial or target spaces.
An Important Example. A very important example of a inductive limit topology is the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decaying smooth functions. Its topology is defined by the norms $(p_{\alpha,k} : \alpha, k \in \mathbb{N}^n)$ such that

$$p_{\alpha,k}(f) = \sup_{x\in\mathbb{R}^n} |(1 + \|x\|^{\alpha}) D^k f(x)|$$

In fact $\mathscr{S}(\mathbb{R}^n)$ is a Montel and hence barrelled. It is not normable since otherwise it would be finite dimensional because the unit ball would be compact. Although the topology is metrizable, for $f, g \in \mathscr{S}(\mathbb{R}^n)$ let

$$d(f,g) := \sum_{\alpha,k\in\mathbb{N}^n} 2^{-n} \frac{p_{\alpha,k}(f-g)}{1 + p_{\alpha,k}(f-g)}$$

$(\mathscr{S}(\mathbb{R}^n), d)$ is a Fréchet space with no bounded neighborhoods since it is not normable. Moreover, we have

$$\mathscr{S} = \bigcap_{\alpha,k\in\mathbb{N}^n} {}^{\mathscr{S}}\!/_{p^{-1}_{\alpha,k}(0)}$$

References

[1] A.S. Üstünel Math 502 Spring 2020-2021 Lecture Notes.

[2] A.P. Robertson, W. Robertson *Topological Vector Spaces* Cambridge University Press, 1973.

[3] H.H. Schaeffer, M.P. Wolff *Topological Vector Spaces* Springer Verlag, 1999.