Analysis of Oscillatory Integrals and Applications to IBVPs

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Abstract

We demonstrate the applications of oscillatory integral theory on initial boundary value problems (IBVP) by considering the Dirichlet ibvp for the linear Schrödinger equation (LS) on the half-line with a uniform approach that can be extended to a large class of IBVPs. By using Fokas method we will obtain a formula for the weak solution of the LS involving oscillatory integrals. Using some basic tools from oscillatory integral theory we will prove sharp regularity results for the LS. The estimates obtained in the process can then be used to extend the proof of local well-posedness of the NLS on the half-line to low regularity solutions through a fixed point argument.

Contents

1 Introduction

1.1 Weak Solutions

An initial boundary value problem (IBVP) is of the form

$$\begin{aligned}&F(D^k u, ..., Du, u) = f\\&u(x, 0) = u_0(x)\\&u(0, t) = g_0(t)\end{aligned}$$

where $F(D^k u, ..., Du, u) = f$ is a partial differential equation (PDE) of order k, f is the interior source, u_0 is the initial data and g_0 is the boundary data.

An IBVP is well-posed if

(i) the solution exists

(ii) the solution is unique

(iii) the solution is continuously dependent on the initial and boundary data

A *classical solution* to an IBVP of order k is a k times differentiable function that satisfies the PDE and condition (iii). Not all PDEs can be solved in the classical sense because it might be too difficult to solve or a sufficiently smooth solution may not exist at all. In this case to make the problem well-posed one weakens the notion of a solution enough to guarantee its existence by considering a wider class of candidates. In general, if a classical solution exists, then it should also satisfy the definition of the weak solution.

So in general a weak solution of a PDE of order k is not k times differentiable in the classical sense, if it were then by the uniqueness of the classical solution we would have found the classical solution.

We can view an IBVP as an operator

$$\Psi : [u_0, g_0, f] \mapsto u$$

where u is the solution of the IBVP with the given data $[u_0, g_0, f]$. So conditions (i) and (ii) of well-posedness are equivalent to Ψ being well-defined as a function and condition (iii) is equivalent to Ψ being continuous on its domain. Ψ is called the *solution operator*. We will get back to this operator when defining sharp regularity.

1.2 The Schwartz Space and Tempered Distributions

Definition 1.1. The space of rapidly decreasing functions (Schwartz space) $\mathcal{S}(\mathbb{R}^n)$ is defined as

$$\mathcal{S}(\mathbb{R}^n) \coloneqq \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha f(x)| < \infty \;\forall \alpha, \beta \in \mathbb{N}^n\}$$

It is easy to check that for $N, k \in \mathbb{N}$, the functional

$$p_{N,k} : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}_+$$

defined by

$$p_{N,k}(f) := \max_{|\alpha| \le k} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |D^\alpha f(x)|$$

is a norm on $\mathcal{S}(\mathbb{R}^n)$. We equip $\mathcal{S}(\mathbb{R}^n)$ with the locally convex topology generated by the family of norms $(p_{N,k})_{N,k \in \mathbb{N}}$.

Let $\mathcal{S}'(\mathbb{R}^n)$ denote the topological dual of $\mathcal{S}(\mathbb{R}^n)$ with the weak topology generated by the norms $(p_{N,k})_{N,k \in \mathbb{N}}$. So $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ if and only if there exist $N, k \in \mathbb{N}$ and $C > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$|u(f)| \le C p_{N,k}(f)$$

The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions.*

1.3 Sobolev Spaces

Definition 1.2. For $\alpha \in \mathbb{N}$ and $u, w \in L^2(\Omega)$, where Ω is a domain in $\mathbb{R}$ we say that w is the **weak derivative** of u of order α if for all smooth test functions ϕ with compact support in Ω we have

$$\langle u, D^\alpha \phi \rangle = (-1)^{|\alpha|} \langle w, \phi \rangle$$

For $k, n \in \mathbb{N}$ and $1 \le p < \infty$ we defined the Sobolev space $W^{k,p}$ as

$$W^{k,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n) \ \forall \ |\alpha| \le k\}$$

The derivatives are in the weak sense.

We equip $W^{k,p}(\mathbb{R}^n)$ with the norm $\|u\|_{k,p} = \left(\sum_{|\alpha|<k} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}}$, making $W^{k,p}$ a Banach space. From now on, we let Ω be a Lipschitz domain. We define the space $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \ |\alpha| \le k\}$$

$$\|u\|_{W^{k,p}(\Omega)} = \inf\{\|U\|_{k,p} : U\big|_\Omega = u, \ U \in W^{k,p}(\mathbb{R}^n)\}$$

Notice that for $k \in \mathbb{N}$ we have $W^{k+1,p} \subset W^{k,p}$ and $W^{k+1,p}$ contains smoother functions compared to $W^{k,p}$ The motivation behind working in these spaces is that the norm measures both the size and the regularity of a function, so that the nested Sobolev Spaces form a scale of regularity: elements of $W^{k,p}$ become smoother as k increases.

We will mostly be working with L^2-based Sobolev spaces

$$H^k(\Omega) := W^{k,2}(\Omega)$$

So far, we have only defined Sobolev spaces of index in $\mathbb{N}$. Now in order to be able to talk about "sharper" results, we need a better scale which motivates us to define Sobolev spaces $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

Definition 1.3. For $s \in \mathbb{R}$, the Sobolev space H^s of order s on $\mathbb{R}^n$ is defined as

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \ (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}$$

where $\widehat{f}$ is the Fourier transform of f. We equip $H^s(\mathbb{R}^n)$ with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}\|_{L^2(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}}$$

For $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ the Sobolev space of order s on Ω is defined as

$$H^s(\Omega) := \{f : f = F\Big|_{\Omega}, F \in H^s(\mathbb{R}^n)\}$$

with the norm

$$||f||_{H^s(\Omega)} = \inf\{F \in H^s(\mathbb{R})^n : ||F||_{H^s(\mathbb{R}^n)}, F\big|_{\Omega} = f\}$$

making $H^s(\Omega)$ a Banach space.

For $k \in \mathbb{N}$ the norm on $\| \cdot \|_{H^k(\Omega)}$ is equivalent to the norm $\| \cdot \|_{k,2}$. This follows from Parseval's identity and the fact that $\widehat{Df}(\xi) = i\xi \widehat{f}(\xi)$.

Proposition 1.4. *For $s > \frac{1}{2}$ and $\Omega \subset \mathbb{R}$, $H^s(\Omega)$ is an algebra, i.e. for any $u, v \in H^s(\Omega)$ we have*

$$\|uv\|_{H^s(\Omega)} \leq \|u\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)}$$

This is a special case of the more general claim: for $\Omega \subset \mathbb{R}^n$, $H^s(\Omega)$ is an algebra if and only if $s > \frac{n}{2}$. See appendix for proof.

Theorem 1.5 (Sobolev Trace Theorem)**.** *[6] For $s > \frac{1}{2}$ and $\Omega \subset \mathbb{R}^n$ there is a unique operator γ called the trace operator such that*

$$\gamma : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

This is a result of the fact that $H^s(\Omega)$ is embedded in $C^0(\Omega)$ for $s > \frac{1}{2}$.

The functions in $H^s(\Omega)$ are said to be of high regularity for $s > \frac{1}{2}$, and of low regularity otherwise. Naturally, the algebra structure makes it easier to work in the high regularity setting. We will be able to observe the significance of this in the last section when reviewing the proof of the local well-posedness of the NLS on the half-line because additional estimates and admissibility assumptions will be required when working in the low regularity setting.

2 Linear Schrödinger Equation on the Half Line

The IBVP for the linear Schrödinger equation on the half-line with Dirichlet boundary conditions is stated as follows:

$$iu_t + u_{xx} = 0, \qquad x \in (0,\infty), t \in (0,T) \tag{2.1a}$$
$$u(x,0) = u_0(x) \qquad x \in (0,\infty] \tag{2.1b}$$
$$u(0,t) = g_0(t) \qquad t \in [0,T] \tag{2.1c}$$

From now on we will denote the solution operator of (2.1) with Ψ.

Question. How smooth must u_0 and g_0 be so that $\Psi[u_0, g_0, 0]$ belongs to the space $C([0,T]; H_x^s(\mathbb{R}_+))$?

The answer, as we will see in Section 4 is that (u_0, g_0) must belong to the space $H_x^s(\mathbb{R}_+)H_t^{\frac{2s+1}{4}}(0,T)$. In this case, for $s > \frac{1}{2}$ the IBVP (2.1a) should be supplemented with the compatibility condition

$$g_0(0) = u_0(0)$$

which follows from the Sobolev Trace Theorem since u has an extension $u(0,0) = g_0(0) = u_0(0)$. For $s > \frac{5}{2}$ we have the additional compatibility conditions on the data:

$$g_0^{(j)}(0) = i^j u_0^{(2j)}(0), \quad 1 \leq j < \frac{2s-1}{4}$$

The question "how smooth?" is answered by which Sobolev spaces u_0 and g_0 belong to. Notice that the answer to this question gives us some specific information about the smoothing properties of the PDE (2.1a). As an example, the solution $u(x,t)$ might be smoother than $u_0(x) = u(x,0)$ at time $t > 0$. This is called *smoothing effect*. In our case, (2.1a) is *conservative* in the sense that the smoothness of the solution does not change as time passes. First we reduce the problem to make the calculations easier.

2.1 "Decompose and Re-unify"

Given $u \in H^s(\mathbb{R}_+)$ we have by the definition of the Sobolev norm and the approximation property of the infimum, there exists $U \in H^s(\mathbb{R})$ such that

$$\|U\|_{H^s(\mathbb{R})} \leq 2\|u\|_{H^s(\mathbb{R}_+)}$$

and U is an extension of u to $\mathbb{R}$. Let $* : H^s(\mathbb{R}_+) \to H^s(\mathbb{R})$ denote a fixed extension operator with the property

$$\|u^*\|_{H^s(\mathbb{R})} \leq 2\|u\|_{H^s(\mathbb{R}_+)}$$

Consider the following Cauchy problem (2.2) and boundary value problem (2.3) with initial data $v(x,0) = 0$.

$$iy_t + y_{xx} = 0 \qquad x \in \mathbb{R}, t > 0 \tag{2.2a}$$
$$y(x,0) = u_0^*(x) \in H_x^s(\mathbb{R}) \qquad x \in \mathbb{R} \tag{2.2b}$$

$$iv_t + v_{xx} = 0 \qquad x, t > 0 \tag{2.3a}$$
$$v(x,0) = 0 \qquad x > 0 \tag{2.3b}$$
$$v(0,t) = g_0(t) - y(0,t) \qquad t > 0 \tag{2.3c}$$

Now observe that if y is a solution of (2.2) and v is a solution of (2.3) then since $u_0^*\big|_{\mathbb{R}_+} = u_0$ one can easily verify that

$$\Psi[u_0, g_0, 0] = u = y\big|_{\mathbb{R}_+} + v \tag{2.4}$$

This decomposition reduces the problem into an easier one since Cauchy problems have been studied in the past, we have all the information we need about the problem (2.2).

So in order to establish well-posedness for the LS on the half line we will need certain estimates for the Cauchy Problem. We first take the Fourier transform of the PDE. Since $\widehat{y_x}(\xi,t) = i\xi\widehat{y}(\xi,t)$, it follows that $\widehat{y_{xx}}(\xi) = -\xi^2\widehat{y}(\xi)$. Hence we obtain the first order ODE

$$\widehat{y}_t + i\xi^2\widehat{y} = 0$$

which has the solution

$$\widehat{y}(\xi,t) = c(\xi)e^{-i\xi^2 t}$$

where c is some function of ξ. Taking $t = 0$ we have $c(\xi) = \widehat{u_0^*}(\xi)$. So we have $\widehat{y}(\xi,t) = \widehat{u}_0^*(\xi)e^{-i\xi^2 t}$ which implies that

$$y(x,t) = \int_{\mathbb{R}} e^{i\xi x - i\xi^2 t}\widehat{u_0^*(\xi)}d\xi \tag{2.5}$$

Now, we will use some previously obtained results on the Cauchy problem.

2.2 Previous Results on the LS Cauchy Problem on $\mathbb{R}$

Theorem 2.1 (LS ivp with Sobolev data [2])**.** *The solution given by* (2.5) *satisfies the following estimates:*

1. ***Space Estimates:*** *For $s \in \mathbb{R}$, $y(\cdot,t) \in C([0,T]; H_x^s(\mathbb{R}))$ with*

$$\|y(\cdot,t)\|_{H_x^s(\mathbb{R})} = \|u_0^*\|_{H_x^s(\mathbb{R})} \tag{2.6}$$

2. ***Time Estimates:*** *For $s \geq -\frac{1}{2}$, $y(x,\cdot) \in C(\mathbb{R}; H_t^{\frac{2s+1}{4}}(0,T))$ with*

$$\sup_{x\geq 0} \|y(x,\cdot)\|_{H_t^{\frac{2s+1}{4}}(0,T)} \leq c_s(1+\sqrt{T})\|u_0^*\|_{H_x^s(\mathbb{R})} \tag{2.7}$$

Remark 2.2. Notice that the Sobolev Trace Theorem 1.5 guarantees the existence of the extension $y(0,t)$ of $y\big|_{\mathbb{R}_+}$ only when $y\big|_{\mathbb{R}_+} \in H^s(\mathbb{R}_+)$ for $s > \frac{1}{2}$. However the time estimates in Theorem 2.1 imply that under the additional assumption that $y\big|_{\mathbb{R}_+}$ solves the ibvp (2.2) the trace $y(0,t)$ not only exists, but also belongs to the same space as $y\big|_{\mathbb{R}_+}$ for $s \geq 0$. This is called hidden boundary regularity.

The space estimates of 2.1 follow trivially from (2.5). The proof of the time estimates are somewhat technical and will be omitted for now.

Proposition 2.3 (mixed norm estimates [11])**.** *For $0 \leq s \leq 1$ and $r < \infty$, the solution given by* (2.5) *satisfies the following mixed norm estimate:*

$$\|y\|_{L_t^\lambda(0,T;W_x^{s,r}(\mathbb{R}))} \lesssim \|u_0^*\|_{H_x^s(\mathbb{R})} \tag{2.8}$$

Outline of Proof.
One can first prove the estimate in the case $s = 0$ which is

$$\|y\|_{L_t^\lambda(0,T;L_x^r(\mathbb{R}))} \lesssim \|u_0^*\|_{L_x^2(\mathbb{R})} \tag{2.9}$$

Then one can verify that if y solves (2.2) then y_x solves (2.2) with initial data u_x^*, which implies that (2.9) holds for y_x and u_x^*. Now by the definition of the norm on $W^{1,r}$ and (2.9), we have

$$\|y\|_{L_t^\lambda(0,T;W_x^{1,r}(\mathbb{R}))} \lesssim \|u_0^*\|_{H_x^1(\mathbb{R})} \tag{2.10}$$

In other words, the solution operator $\Phi : u_0^* \to y$ belongs to the intersection

$$\mathcal{L}(L_t^\lambda(0,T;W_x^{0,r}(\mathbb{R})), H_x^0(\mathbb{R})) \cap \mathcal{L}(L_t^\lambda(0,T;W_x^{1,r}(\mathbb{R})), H_x^1(\mathbb{R}))$$

Then the result follows after an interpolation argument using the fact that

$$[W^{0,r}(\mathbb{R}), W^{1,r}(\mathbb{R})]_\theta = W^{\theta,r}(\mathbb{R})$$

The details for this fact can be found in [9], Chapter 7.

Remark 2.4. The mixed norm ($L_t^\lambda W_x^{s,r}$) estimates are redundant for proving well-posedness and sharp regularity results for solutions in the space $C_t H_x^s(\mathbb{R})$ for the LS Cauchy problem (2.2) and for solutions in the space $C_t H_x^s(\mathbb{R}_+)$ for the LS IBVP (2.1). The motivation behind Proposition 4.2 will be apparent when dealing with the NLS.

2.3 Reduction of the Problem and Sharp Regularity

Note that $y\big|_{\mathbb{R}_+}$ is a solution of the IBVP

$$\begin{aligned}
&iu_t + u_{xx} = 0, && x \in (0,\infty), t \in (0,T) \\
&u(x,0) = u_0(x) \in H_x^s(\mathbb{R}_+) && \\
&u(0,t) = y(0,t) \in H_t^{\frac{2s+1}{4}}(0,T)) && t \in [0,T]
\end{aligned}$$

By the definition of $*$ and the $H^s(\Omega)$ norm it follows that

$$\|y\big|_{\mathbb{R}_+}(\cdot,t)\|_{H^s_x(\mathbb{R}_+)} \leq \|y(\cdot,t)\|_{H^s_x(\mathbb{R})} = \|u_0^*\|_{H^s_x(\mathbb{R})} \leq 2\|u_0\|_{H^s_x(\mathbb{R}_+)}$$

$$\Rightarrow y\big|_{\mathbb{R}_+} \in C([0,T];H^s_x(\mathbb{R}_+))$$

Remark 2.5. What we know so far is that there are functions in $H^s_x(\mathbb{R}_+) \times H_t^{\frac{2s+1}{4}}(0,T)$ that are mapped into $C([0,T];H^s_x(\mathbb{R}_+))$ by Ψ. So we have an upper bound for the regularity of the IBVP in the sense that the answer to our question can not be a smaller space containing smoother functions. In order to reach the exact answer one has to prove that $H^s_x(\mathbb{R}_+) \times H_t^{\frac{2s+1}{4}}(0,T)$ is in fact maximal with respect to this property i.e. all functions at this regularity level are mapped into $C([0,T];H^s_x(\mathbb{R}_+))$ by Ψ. This is called a *sharp regularity result*, it is *sharp* in the sense that one can not replace s with a larger number. We will come back to this idea in Section 4.

Also observe that we have reduced the original problem (2.1) since we can fix $u_0 = 0$ in (2.1) because of the decomposition (2.4). Unlike the LS Cauchy Problem, we do not have a lot of past results on the LS IBVP. So we move on to formulating a weak solution through the Fokas Method which is also known as the Uniform Transform Method since it can be applied to a wide class of IBVPs.

2.4 Obtaining Weak Solutions to the Linear Schrödinger Equation on the Half Line via Fokas Method

Step 1: Assume that u and g_0 are smooth, have sufficient decay at infinity and we have

$$u = \Psi[0, g_0, 0]$$

We will take the Fourier transform of (2.1a) on the half-Line to obtain the Global Relation (GR) and an expression for u involving the half-line Fourier transform of the boundary data g_0 and an unknown function.

$$\begin{aligned}
\hat{u}_t(\lambda,t) &= \int_0^\infty e^{-i\lambda x}u_t(x,t)dx = i\int_0^\infty e^{-i\lambda x}u_{xx}(x,t)dx \\
&= ie^{i\lambda x}u_x(x,y)\Big|_{x=0}^{\infty} - i\int_0^\infty(-i\lambda)e^{-i\lambda x}u_x(x,t)dx \\
&= ie^{i\lambda x}u_x(x,y)\Big|_{x=0}^{\infty} - \lambda e^{i\lambda x}u(x,t)\Big|_{x=0}^{\infty} + \lambda\int_0^\infty(-i\lambda)e^{-i\lambda x}u(x,t)dx \\
&= ie^{i\lambda x}u_x(x,y)\Big|_{x=0}^{\infty} - \lambda e^{i\lambda x}u(x,t)\Big|_{x=0}^{\infty} - i\lambda^2\hat{u}(\lambda,t)
\end{aligned}$$

Notice that the boundary values of the anti-derivatives are bounded if and only if $\lambda_I \leq 0$. Thus, on the lower half complex λ-plane we have the Global Relation

$$\hat{u}_t(\lambda,t) = -ig_1(t) + \lambda g_0(t) - i\lambda^2\hat{u}(\lambda,t), \qquad \lambda_{\mathbb{I}} \leq 0 \tag{GR}$$

where $g_1(t) := u_x(0,t)$ is an unknown function. Now we can obtain an expression for $\hat{u}$ as follows:

$$\begin{aligned}
\widehat{u}_t(\lambda,t) + i\lambda^2\widehat{u}(\lambda,t) &= \lambda g_0(t) - ig_1(t) && \lambda_{\mathbb{I}} \le 0\\
\Rightarrow \left(e^{i\lambda^2 t}\widehat{u}(\lambda,t)\right)_t &= e^{i\lambda^2 t}(\lambda g_0(t) - ig_1(t)) && \lambda_{\mathbb{I}} \le 0\\
\Rightarrow e^{i\lambda^2 t}\widehat{u}(\lambda,t) &= \int_0^t e^{i\lambda^2\tau}(\lambda g_0(\tau) - ig_1(\tau))d\tau && \lambda_{\mathbb{I}} \le 0
\end{aligned}$$

Now define the functions $\widetilde{g_j}$ for $j = 0,1$ such that,

$$\widetilde{g_j}(\lambda,t) = \int_0^t e^{i\lambda\tau} g_j(\tau)d\tau$$

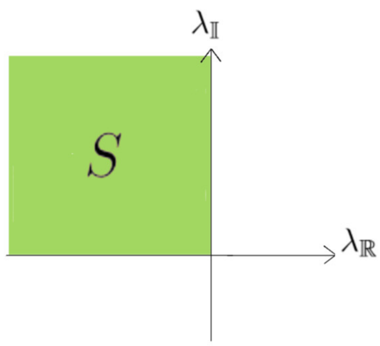
Then the expression for $\hat{u}$ becomes,

$$\hat{u}(\lambda,t) = e^{-i\lambda^2 t}[\lambda\widetilde{g_0}(\lambda^2,t) - i\widetilde{g_1}(\lambda^2,t)] \tag{2.11}$$

Now we can obtain an expression for $u(x,t)$ by taking the inverse Fourier transform,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^2 t}[\lambda\widetilde{g_0}(\lambda^2,t) - i\widetilde{g_1}(\lambda^2,t)]d\lambda \tag{2.12}$$

Step 2: We will deform the integral in (8) to the boundary of a region $D_+ \subset \mathbb{C}$.
Notice that the integral on the right hand side is convergent whenever the exponentials are bounded, i.e. $(i\lambda)_{\mathbb{R}} \le 0$ and $(-i\lambda^2)_{\mathbb{R}} \le 0$. This region is $S = \{\lambda \in \mathbb{C} : \lambda_{\mathbb{I}} \le 0, \lambda_{\mathbb{R}} \ge 0\}$.



Let

$$\phi_0(\lambda) = \lambda e^{-i\lambda^2 t}\widetilde{g_0}(\lambda^2,t) \qquad \lambda \in S \tag{2.13}$$
$$\phi_1(\lambda) = e^{-i\lambda^2 t}\widetilde{g_1}(\lambda^2,t) \qquad \lambda \in S \tag{2.14}$$

Integration by parts yields, for $j = 0,1$

$$e^{-i\lambda^2 t}\widetilde{g_j}(\lambda^2,t) = e^{-i\lambda^2 t}\int_0^t e^{i\lambda^2\tau} g_j(\tau)\,d\tau = \frac{g_j(t)}{i\lambda^2} - e^{-i\lambda^2 t}\frac{g_j(0)}{i\lambda^2}$$

So ϕ_1 is $O(\lambda^{-2})$ as $|\lambda| \to \infty$ in S.

Consider the contour $K_R = C_R \cup \gamma_1 \cup \gamma_2 \subset S$ where

$$\begin{aligned}
&C_R : \lambda = Re^{it}, && t \in [\frac{\pi}{2}, \pi] \\
&\alpha_R : \lambda = -t, && t \in [0, R] \\
&\beta_R : \lambda = it, && t \in [0, R]
\end{aligned}$$

Since $e^{i\lambda x}\phi_0$ and $e^{i\lambda x}\phi_1$ are analytic on S, by the Cauchy Theorem we have, $\int_{K_R} e^{i\lambda x}\phi_0 = 0 = \int_{K_R} e^{i\lambda x}\phi_1$. Since ϕ_1 is $O(\lambda^{-2})$, for some constant C we have

$$\int_{C_R} e^{i\lambda x}\phi_0 \leq \max_{|\lambda|=R} \frac{\pi R}{2} |\, e^{i\lambda x}\phi_0(\lambda)| \leq CR^{-2} \to 0$$

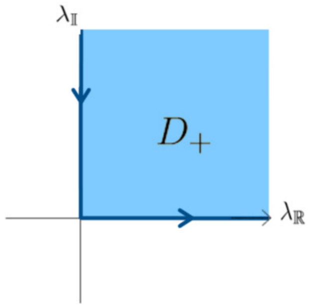
$$\begin{aligned}
\Rightarrow \int_{-\infty}^{0} e^{i\lambda x}\phi_1 \, d\lambda &= \lim_{R\to\infty} \int_{\gamma_1} e^{i\lambda x}\phi_1 \\
&= \lim_{R\to\infty} \big(- \int_{\gamma_2} e^{i\lambda x}\phi_1 - \int_{C_R} e^{i\lambda x}\phi_1 \big) \\
&= - \int_{i\mathbb{R}_+} e^{i\lambda}\phi_1
\end{aligned}$$

Since ϕ_1 is $O(\lambda^{-2})$, ϕ_0 is $O(\lambda^{-1})$ and since $x > 0$ by Jordan's Lemma we have that $\int_{C_R} \lambda e^{i\lambda x}\phi_0 \to 0$ as $R \to \infty$ and hence by the same argument

$$\int_{-\infty}^{0} e^{i\lambda x}\phi_0 \, d\lambda = - \int_{i\mathbb{R}_+} e^{i\lambda}\phi_0$$

$$\begin{aligned}
\Rightarrow u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x}[\phi_0(\lambda) - i\phi_1(\lambda)] \, d\lambda \\
&= \frac{1}{2\pi} \int_{i\mathbb{R}_+} e^{i\lambda x}[i\phi_1(\lambda) - \phi_0(\lambda)] \, d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} e^{i\lambda x}[\phi_0(\lambda) - i\phi_1(\lambda)] \, d\lambda \\
&= \frac{1}{2\pi} \int_{\partial D_+} e^{i\lambda x - i\lambda^2 t}[\lambda \widetilde{g_0}(\lambda^2, t) - i\widetilde{g_1}(\lambda^2, t)] \, d\lambda \qquad (2.15)
\end{aligned}$$

where D_+ is the region $\{\lambda \in \mathbb{C} : \lambda_{\mathbb{I}}, \lambda_{\mathbb{R}} \geq 0\}$ with the orientation shown in the figure.



Step 3: We will use the Global Relation in order to eliminate the term $\widetilde{g_1}$ in (2.15) which is an unknown function.
Notice that since the GR holds for $\lambda \in \mathbb{C}$ such that $\lambda_{\mathbb{I}} \leq 0$, by the transformation $\lambda \mapsto -\lambda$ we have a an equation that holds on $\lambda_{\mathbb{I}} \geq 0$.

$$\hat{u}(-\lambda, t) = e^{-i\lambda^2 t}[-\lambda\widetilde{g_0}(\lambda^2, t) - i\widetilde{g_1}(\lambda^2, t)], \qquad \lambda_{\mathbb{I}} \geq 0 \tag{2.16}$$

In particular, it holds on ∂D_+ so we can use (11) to rewrite u as

$$u(x,t) = \frac{1}{2\pi}\int_{\partial D_+} 2\lambda e^{ix-i\lambda^2 t}\widetilde{g_0}(\lambda^2, t) + e^{i\lambda x}\widehat{u}(-\lambda, t)\, d\lambda$$

The functions $e^{i\lambda x}$ and $\widehat{u}(-\lambda, t)$ are both analytic and bounded on the upper half complex λ-plane. Now by integration by parts, as $\lambda \to \infty$ we have

$$\widehat{u}(-\lambda, t) = \int_0^\infty e^{i\lambda x} u(x,t)\, dx \sim -\frac{u(0,t)}{i\lambda}$$

So $\widehat{u}(-\lambda, t)$ is $O(\lambda^{-1})$ thus by Cauchy Theorem and Jordan's Lemma it follows that

$$\int_{\partial D_+} e^{i\lambda x}\widehat{u}(-\lambda, t)\, d\lambda = \lim_{R\to\infty}\int_{-\bar{C_R}} e^{i\lambda x}\widehat{u}(-\lambda, t)\, d\lambda = 0$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_{\partial D_+} \lambda e^{i\lambda x - i\lambda^2 t}\widetilde{g_0}(\lambda^2, t)\, d\lambda \tag{2.17}$$

Note that although this solution was obtained by first assuming that g_0 is smooth and has sufficient decay, (2.17) makes sense under weaker regularity conditions on g_0. So (2.17) in particular defines a weak solution for (2.3).

3 Oscillatory Integrals

Lemma 3.1 (Van der Corput Lemma [5])**.** *Let $\phi : [a,b] \to \mathbb{R}$ be a smooth function such that $|\phi^{(k)}(x)| \geq 1\ \forall x \in (a,b)$. Then we have*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-\frac{1}{k}}$$

if either (i) $k \geq 2$ or (ii) $k = 1$ and ϕ' is monotonic. c_k is independent of ϕ and λ.

Proof. (ii)Assume $k = 1$ and ϕ' is monotonic. Integration by parts yields:

$$\int_a^b e^{i\lambda\phi} dx = e^{i\lambda\phi}(i\lambda\phi')^{-1}\Big|_a^b - \int_a^b e^{i\lambda\phi} \frac{d}{dx}(i\lambda\phi')^{-1} dx$$

Notice the integral on the RHS is bounded by λ^{-1}:

$$\begin{aligned}
\left| \int_a^b e^{i\lambda\phi} \frac{d}{dx}(i\lambda\phi')^{-1} dx \right| &\leq \int_a^b \left| e^{i\lambda\phi}(i\lambda)^{-1} \frac{d}{dx}(\frac{1}{\phi'}) dx \right| \\
&\leq \lambda^{-1} \int_a^b \left| \frac{d}{dx}(\frac{1}{\phi'}) dx \right| \\
&= \lambda^{-1} \left| \int_a^b \frac{d}{dx}(\frac{1}{\phi'}) dx \right| \\
&\leq \lambda^{-1} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \\
&\leq \lambda^{-1}
\end{aligned}$$

where the last two inequalities follow from the assumptions we made on ϕ'. Also note that for any $x \in [a,b]$, $\left| \frac{e^{i\lambda\phi(x)}}{i\lambda\phi'(x)} \right| = \left| \frac{1}{\lambda\phi'(x)} \right| \leq \frac{1}{\lambda}$. So we obtain a bound for the integral

$$\left| \int_a^b e^{i\lambda\phi} dx \right| \leq \left| \frac{e^{i\lambda\phi(b)}}{i\lambda\phi'(b)} \right| + \left| \frac{e^{i\lambda\phi(a)}}{i\lambda\phi'(a)} \right| + \left| \int_a^b e^{i\lambda\phi} \frac{d}{dx}(i\lambda\phi')^{-1} dx \right| \leq \frac{1}{\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{3}{\lambda}$$

Hence by taking $c_1 = 3$ we have $\left| \int_a^b e^{i\lambda\phi} dx \right| \leq c_1 \lambda^{-1}$

(i) Now we will work with the case $k \geq 2$ by induction on k. We already have the base case. So assume the lemma holds for some $k \in \mathbb{N}$ and that $|\phi^{(k+1)}(x)| \geq 1$ on (a,b). W.l.o.g. let $\phi^{(k+1)}(x) \geq 1$. Then notice that $\phi^{(k)}$ is monotone increasing so $|\phi^{(k)}|$ can have at most one zero in the interval (a,b).

<u>Case 1</u>: $\exists c \in (a,b)$ such that $\phi^{(k)}(c) = 0$. Then given any $\delta > 0$, we have $|\phi^{(k)}(x)| \geq \delta$ outside of $(c - \delta, c + \delta)$ because $\phi^{(k+1)}(x) \geq 1$. For δ small enough we have

$$\int_a^b e^{i\lambda\phi}dx = \int_a^{c-\delta} e^{i\lambda\phi}dx + \int_{c-\delta}^{c+\delta} e^{i\lambda\phi}dx + \int_{c+\delta}^{b} e^{i\lambda\phi}dx$$

On $(a, c-\delta)$ we have $\frac{|\phi^{(k)}(x)|}{\delta} \geq 1$ so by the induction hypothesis,

$$\left| \int_a^{c-\delta} e^{i\lambda\phi}dx \right| = \left| \int_a^{c-\delta} e^{i(\lambda\delta)\frac{\phi}{\delta}}dx \right| \leq c_k(\lambda\delta)^{-\frac{1}{k}}$$

By the same argument $\left| \int_{c+\delta}^{b} e^{i\lambda\phi}dx \right| \leq c_k(\lambda\delta)^{-\frac{1}{k}}$. Also note that $\left| \int_{c-\delta}^{c+\delta} e^{i\lambda\phi}dx \right| \leq 2\delta$. Hence we arrive at a bound for the integral:

$$\left| \int_a^b e^{i\lambda\phi}dx \right| \leq 2\delta + 2c_k(\lambda\delta)^{-\frac{1}{k}}$$

Case 2: $\phi^{(k)}$ has no zeros in the interval (a,b). Then since $\phi^{(k)}$ is monotone increasing, the minimum value is obtained at $x = a$. So we have for any $\delta \in (0, b-a)$, $|\phi^{(k)}(x)| \geq \delta$ in the interval $(a+\delta, b)$. So by the same argument as in Case 1 we have:

$$\left| \int_a^b e^{i\lambda\phi}dx \right| \leq \left| \int_a^{a+\delta} e^{i\lambda\phi}dx \right| + \left| \int_{a+\delta}^{b} e^{i\lambda\phi}dx \right| \leq \delta + c_k(\lambda\delta)^{-\frac{1}{k}} \leq 2\delta + 2c_k(\lambda\delta)^{-\frac{1}{k}}$$

So in either case we have the inequality $\left| \int_a^b e^{i\lambda\phi}dx \right| \leq 2\delta + 2c_k(\lambda\delta)^{-\frac{1}{k}}$. Then take $\delta = \lambda^{-\frac{1}{k+1}}$

$$\left| \int_a^b e^{i\lambda\phi}dx \right| \leq 2\lambda^{-\frac{1}{k+1}} + 2c_k\lambda^{-\frac{1}{k+1}} = (2+2c_k)\lambda^{-\frac{1}{k+1}}$$

Letting $c_{k+1} = 2 + 2c_k$, we reach the desired bound for the integral. Then using $c_1 = 3$ to solve the recurrence relation we obtain the result:

$$\left| \int_a^b e^{i\lambda\phi}dx \right| \leq (5 \cdot 2^{k-1} - 2) \cdot \lambda^{-\frac{1}{k}}$$

□

Corollary 3.2 ([5]). *Under the same assumptions on ϕ we have the following bound for $I(\lambda)$*

$$\left| \int_a^b e^{i\lambda\phi(x)}\psi(x)dx \right| \leq c_k\lambda^{-\frac{1}{k}} \left[|\psi(b)| + \int_a^b |\psi'(x)|dx \right]$$

Proof. Define the function $F : [a,b] \to \mathbb{C}$ s.t.

$$F(x) := \int_a^x e^{i\lambda\phi(t)}dt$$

Then by using FTOC, integration by parts and the fact that $F(a) = 0$ we obtain

$$\begin{aligned}
|I(\lambda)| &= \left| \int_a^b F'(x)\psi(x)dx \right| \\
&= \left| F(x)\psi(x)\Big|_a^b - \int_a^b F(x)\psi'(x)dx \right| \\
&\leq |F(b)\psi(b)| + \int_a^b |F(x)\psi'(x)|dx \\
&\leq ||F||_\infty |\psi(b)| + ||F||_\infty \int_a^b |\psi'(x)|dx \\
&\leq c_k \lambda^{-\frac{1}{k}} \left[|\psi(b)| + \int_a^b |\psi'(x)|dx \right]
\end{aligned}$$

where the last inequality follows from the Van der Corput Lemma. Hence we arrive at a bound for $I(\lambda)$ that is independent of the phase ϕ. $\square$

4 Sharp Regularity Results and Mixed Norm Estimates for LS on the Half Line

In this section we will use the results we obtained in the previous section to obtain mixed norm estimates and sharp regularity results for the solution of the Dirichlet problem $\Psi[0, g_0, 0]$ of the Schrödinger equation on the half-line. Two approaches will be outlined.

Theorem 4.1 ([2]). *Suppose that $s > \frac{1}{2}$ and $\frac{2s+1}{4} \notin \mathbb{N} + \frac{1}{2}$., then the UTM formula (2.17) defines a solution u to the IBVP (2.1) such that $u(\cdot, t) \in C([0, T]; H^s_x(\mathbb{R}_+))$ with the following space estimate,*

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{H^s_x(\mathbb{R})} \lesssim \|u_0\|_{H^s_x(\mathbb{R}_+)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} \tag{4.1}$$

Theorem 4.2 ([12]). *Let $s \geq 0$, $g_0 \in H_t^{\frac{2s+1}{4}}(\mathbb{R})$ with $suppg_0 \subset [0, T)$ and (λ, r) be Schrödinger admissible that is $2 \leq \lambda, r \leq \infty$ and*

$$\frac{1}{\lambda} + \frac{1}{2r} = \frac{1}{4}$$

*Then the solution $u = \Psi[0, g_0, 0]$ obtained in (**??**) belongs to the space $C([0, T]; H^s_x(\mathbb{R}_+))$ and satisfies the estimate*

$$\|u\|_{L^\lambda_t([0,T]; W^{s,r}_x(\mathbb{R}_+))} \lesssim \|g_0\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \tag{4.2}$$

where the constant of inequality depends on s.

Remark 4.3. Theorem 4.1 settles the discussion in Remark 2.5, it is a sharp regularity result for the LS IBVP. Notice the assumption $s > \frac{1}{2}$, in the next section this Theorem will be uused when proving the local well-posedness of *high regularity solutions* of the NLS. Theorem 4.2 is a more general result, extending the $L^\infty_t H^s_x$ norm estimates in Theorem 4.1 to $L^\lambda_t W^{s,r}_x$ mixed norm estimates for any admissible pair (λ, r) and is valid for $s \geq 0$. In particular it is also a sharp regularity result and it shows that the Fokas method is able to define weak solutions to the LS below the Banach algebra threshold $C_t H^{\frac{1}{2}x}$. One motivation behind Theorem 4.2 is that it implies the local well-posedness of some low regularity solutions of the NLS, this idea will be revisited in the next section.

Proof of Theorem 4.2. First we split the solution u into two parts.

$$\begin{aligned} u(x,t) &= \frac{1}{\pi} \int_{\partial D_+} \lambda e^{ix - i\lambda^2 t} \widetilde{g_0}(\lambda^2, T)\, d\lambda && (4.3) \\ &= \frac{1}{\pi} \int_0^\infty e^{-\lambda x + i\lambda^2 t} \lambda \widehat{g_0}(\lambda^2) d\lambda + \frac{1}{\pi} \int_0^\infty e^{i\lambda x - i\lambda^2 t} \lambda \widehat{g_0}(-\lambda^2) d\lambda && (4.4) \end{aligned}$$

Note that we assume $suppg_0 \subset [0, T)$ in order to have $\widetilde{g_0}(\lambda^2, T) = \widehat{g_0}(-\lambda^2)$ which follows from the support condition. We will estimate the integrals separately.

4.1 Real axis

$$u_2(x,t) := \frac{1}{\pi}\int_0^\infty e^{i\lambda x - i\lambda^2 t}\lambda\widehat{g_0}(-\lambda^2)d\lambda$$

Define the function G_2 as the inverse Fourier transform of

$$\widehat{G_2}(\lambda) := \begin{cases} 2\lambda\widehat{g_0}(-\lambda^2) & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}$$

Then we can rewrite u_2 in terms of G_2 as

$$u_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\lambda x - i\lambda^2 t}\widehat{G_2}(\lambda)d\lambda \tag{4.5}$$

Notice that (4.5) makes sense for all $x \in \mathbb{R}$ and when considered as a function defined on $\mathbb{R}$, it is the solution of the Cauchy problem (2.2) with initial data G_2. Hence by the previous results for the Cauchy problem, Proposition 2.3,

$$\|u_2\|_{L_t^\lambda(0,T;W_x^{s,r}(\mathbb{R}))} \lesssim \|G_2\|_{H^s(\mathbb{R})} \tag{4.6}$$

Note also that we have

$$\begin{aligned}\|G_2\|^2_{H^s(\mathbb{R})} &= \|\langle\cdot\rangle^s\widehat{G_2}\|_{L^2(\mathbb{R})} = \int_0^\infty (1+\xi^2)^s 4\xi^2|\widehat{g_0}(-\xi^2)|^2\,d\xi \\ &= \frac{1}{2}\int_0^\infty (1+\tau)^s\sqrt{\tau}|\widehat{g_0}(-\tau)|^2\,d\tau \\ &\lesssim \int_{-\infty}^\infty (1+\tau^2)^{\frac{2s+1}{4}}|\widehat{g_0}(\tau)|^2\,d\tau = \|g_0\|^2_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}\end{aligned} \tag{4.7}$$

where $\langle\cdot\rangle = 1 + |\cdot|^2$. So it follows that

$$\|u_2\|_{L_t^\lambda(0,T;W_x^{s,r}(\mathbb{R}_+))} \leq \|u_2\|_{L_t^\lambda(0,T;W_x^{s,r}(\mathbb{R}))} \lesssim \|g_0\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \tag{4.8}$$

4.2 Imaginary Axis

$$u_1(x,t) := \frac{1}{2\pi}\int_0^\infty e^{-\lambda x + i\lambda^2 t}2\lambda\widehat{g_0}(\lambda^2)d\lambda$$

The estimation of the integral on the imaginary axis will be much more complicated. This is the part where oscillatory integral theory is used.

Define the function G_1 as the inverse Fourier transform of

$$\widehat{G_1}(\lambda) := \begin{cases} \dfrac{\lambda\widehat{g_0}(\lambda^2)}{2\pi} & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}$$

Then we can rewrite u_1 as

$$\begin{aligned}u_1(x,t) &= \int_{-\infty}^\infty e^{-\lambda x + i\lambda^2 t}\widehat{G_1}(\lambda)\,d\lambda = \lim_{b\to\infty}\int_0^b e^{-\lambda x + i\lambda^2 t}\int_{-\infty}^\infty e^{i\lambda\tau}G_1(\tau)\,d\tau\,d\lambda \\ &= \lim_{b\to\infty}\int_{-\infty}^\infty K(\tau;x,t,b)G_1(\tau)\,d\tau\end{aligned} \tag{4.9}$$

where $K(\tau; x, t, b) = \int_0^b e^{i\lambda^2 t - i\lambda\tau} e^{-\lambda\tau}\, d\lambda$

Proposition 4.4. *Let $I(\tau, t, k) := \int_0^k e^{its^2 - s\tau}\, ds$ then there is $c \in \mathbb{R}$ that is independent of $t, \tau \in \mathbb{R}$ and satisfies*

$$|I(\tau, t, k)| \leq c|t|^{-\frac{1}{2}}$$

Proof. Define the function $\phi_{t,\tau}(s) = s^2 - \frac{s\tau}{t}$ Then for fixed $t \in \mathbb{R} - \{0\}$ and $\tau \in \mathbb{R}$ we have $|\phi''_{t,\tau}(s)| = 2 > 1$ on $[0, k]$. Thus by Van der Corput Lemma it follows that for some $c \in \mathbb{R}$

$$|I(\tau, t, k)| = \left| \int_0^k e^{it\phi_{t,\tau}(s)}\, ds \right| \leq c|t|^{-\frac{1}{2}}$$

□

Proposition 4.5. $|K(\tau; x, t, b)| = \left| \int_0^b e^{i\lambda^2 t - i\lambda\tau} e^{-\lambda\tau}\, d\lambda \right| \leq c|t|^{-\frac{1}{2}}$ *for $t \neq 0$ uniformly in $x, b \in \mathbb{R}_+, \tau \in \mathbb{R}$*

Proof. Let $F(\lambda; \tau, t) := \int_0^\lambda e^{its^2 - s\tau}\, ds$. So we have that

$$K(\tau; x, t, b) = \int_0^b \frac{d}{d\lambda} F(\lambda; \tau, t) e^{-\lambda x}\, d\lambda$$

$$\Rightarrow |K(\tau; x, t, b)| \leq |F(b; \tau, t)e^{-bx}| + \int_0^b |F(\lambda; \tau, t)| \left| \frac{e^{-\lambda x}}{x} \right| d\lambda$$

Now notice that $\forall b, x \in \mathbb{R}_+$ we have $|e^{-bx}| \leq 1$ and

$$\int_0^b \left| \frac{e^{-\lambda x}}{x} \right| d\lambda = x \int_0^b |e^{-\lambda x}|\, d\lambda = (1 - e^{-bx}) \leq 1$$

$\Rightarrow |K(\tau; x, t, b)| \leq c|t|^{-\frac{1}{2}}$ for a constant c, independent of x, b and τ. □

Now recalling the expression (4.9) of u_1, Proposition 4.5 implies that

$$\|u_1(\cdot, t)\|_{L_x^\infty(\mathbb{R}_+)} \lesssim \frac{1}{\sqrt{t}} \|G_1\|_{L_t^1(\mathbb{R})} \tag{4.10}$$

By the boundedness of the Laplace transform (see Appendix) it follows that

$$\|u_1(\cdot, t)\|_{L_x^2(\mathbb{R}_+)} \lesssim ||G_1||_{L_t^2(\mathbb{R})} \tag{4.11}$$

Thus by the Riesz-Thorin Interpolation Theorem (see Appendix), we can interpolate between (4.10) and (4.11) to obtain that if $2 \leq r \leq \infty$ and r' is the conjugate of r and λ and r are Schrödinger admissable then,

$$\|u_1(\cdot, t)\|_{L_x^r(\mathbb{R}_+)} \lesssim t^{-(\frac{1}{2} - \frac{1}{r})} ||G_1||_{L_t^{r'}(\mathbb{R})} = t^{-\frac{2}{\lambda}} ||G_1||_{L_t^{r'}(\mathbb{R})} \tag{4.12}$$

$$\Rightarrow \Big\| \int_0^T u_1(\cdot, t-s)\, ds \Big\|_{L^r_x(\mathbb{R}_+)} \le \int_0^T \|u_1(\cdot, t-s)\|_{L^r_x(\mathbb{R}_+)}\, ds \tag{4.13}$$

$$\le \int_0^T |t-s|^{-\frac{2}{\lambda}} \|G_1\|_{L^{r'}_t(\mathbb{R})}\, ds \tag{4.14}$$

Now observing that (4.15) is a *Riesz Potential*, by the Hardy-Littlewood-Sobolev fractional integration theorem (see Appendix) it follows that

$$\Big\| \int_0^T u_1(\cdot, t-s)\, ds \Big\|_{L^\lambda_t(0,T;L^r_x(\mathbb{R}_+))} \lesssim \|G_1\|_{L^\lambda_t(0,T;L^r_x(\mathbb{R}))} \tag{4.15}$$

Now let $\psi \in C_c([0,T);\mathcal{D}(\mathbb{R}_+))$ where C_c denotes compactly supported continuous functions and $\mathcal{D}$ denotes compactly supported smooth functions. Then we have,

$$\begin{aligned}
&\Big| \int_{-\infty}^{\infty} \langle u_1(\cdot, t), \psi(t)\rangle_{L^2(\mathbb{R}_+)}\, dt \Big| \\
&= \Big| \int_{-\infty}^{\infty} \int_0^{\infty} \Big(\int_{-\infty}^{\infty} e^{-\lambda x + i\lambda^2 t} \widehat{G_1}(\lambda)\, d\lambda \Big) \overline{\psi}(x,t) dx dt \Big| \\
&= \Big| \int_0^T \int_0^{\infty} \Big(\int_{-\infty}^{\infty} e^{-\lambda x + i\lambda^2 t} \widehat{G_1}(\lambda)\, d\lambda \Big) \overline{\psi}(x,t) dx dt \Big| \\
&= \lim_{b\to\infty} \Big| \int_0^b \widehat{G_1}(\lambda) \overline{\int_0^T \int_0^\infty e^{-\lambda x + i\lambda^2 t} \psi(x,t) dx dt d\lambda} \Big| \\
&= \lim_{b\to\infty} \Big| \int_{-\infty}^{\infty} G_1(\tau) \overline{\int_0^T \int_0^\infty \overline{K}(\tau; x, t, b) \psi(x,t) dx dt d\tau} \Big|
\end{aligned} \tag{4.16}$$

Lemma 4.6. *If $\psi \in C_c([0,T);\mathcal{D}(\mathbb{R}_+))$ and $I(\tau;b) = \int_0^T \int_0^\infty \overline{K}(\tau;x,t,b)\psi(x,t)dxdt$ then we have*

$$\|I(\tau,b)\|_{L^2_\tau(\mathbb{R})} \lesssim \|\psi\|_{L^{\lambda'}(0,T;L^{r'}(\mathbb{R}_+))}$$

Proof.

$$\begin{aligned}
\|I(\tau,b)\|^2_{L^2_\tau(\mathbb{R})} &= \int_{-\infty}^{\infty} \Big(\int_0^T \int_0^\infty \overline{K}\psi(x,t) dx dt \Big) \Big(\overline{\int_0^T \int_0^\infty \overline{K}\psi(x,t) dx dt} \Big) d\tau \\
&= \int_0^T \int_0^\infty \psi(x,t) \int_0^T \int_0^\infty \overline{\psi}(y,s) L(x,y,t,s,b) dy ds dx dt
\end{aligned} \tag{4.17}$$

where $L(x,y,t,s,b) = \int_{-\infty}^{\infty} \overline{K}(\tau;x,t,b)K(\tau;y,s,b)d\tau$. Now recalling the definition of K and by the finite line Fourier transform and its inverse we have

$$\begin{aligned}
L(x,y,t,s,b) &= \int_0^b \int_{-\infty}^{\infty} \int_0^b e^{-\lambda x - i\lambda^2 t + i\lambda\tau} e^{-\widetilde{\lambda} y + i\widetilde{\lambda}^2 s - i\widetilde{\lambda}\tau} d\lambda d\widetilde{\lambda} d\tau \\
&= 2\pi \int_0^b e^{-\lambda(x+y) - i\lambda^2(t-s)} d\lambda
\end{aligned}$$

Thus by Proposition 4.4 we have that for $t \neq s$,

$$|L(x, y, t, s; b)| \lesssim |t - s|^{-\frac{1}{2}} \tag{4.18}$$

uniformly in $x, y, b \in \mathbb{R}_+$. Now we present very similar argument to the one we used to prove (4.12), that is, letting $A = \int_0^\infty \overline{\psi}(y, s)L(x, y, t, s, b)dy$ for simplicity, the estimate (4.18) implies

$$\|A\|_{L^\infty_x(\mathbb{R}_+)} \lesssim |t - s|^{-\frac{1}{2}} \|\psi\|_{L^1(\mathbb{R}_+)} \tag{4.19}$$

Then, writing A in terms of a Bochner integral of the Laplace transform of $\overline{\psi}$, by the boundedness of the Laplace transform on $L^2(\mathbb{R}_+)$ we obtain

$$\|A\|_{L^2_x(\mathbb{R}_+)} \lesssim \|\psi\|_{L^2(\mathbb{R}_+)} \tag{4.20}$$

Then again we can interpolate between (4.19) and (4.20) and arrive at the estimate

$$\|A\|_{L^r_x(\mathbb{R}_+)} \lesssim |t - s|^{-(\frac{1}{2}-\frac{1}{r})} \|\psi\|_{L^{r'}(\mathbb{R}_+)} \tag{4.21}$$

where r' is the conjugate of r. Now let (λ, r) be an admissible pair. Then using the properties of the Bochner integral and (4.21) it follows that

$$\Big\| \int_0^T Ads \Big\|_{L^r_x(\mathbb{R}_+)} \leq \int_0^T \|A\|_{L^r_x(\mathbb{R}_+)} ds \lesssim \int_0^T |t - s|^{-\frac{2}{\lambda}} \|\psi\|_{L^{r'}(\mathbb{R}_+)} ds$$

Now letting $\alpha = 1 + \frac{2}{\lambda}$ we have

$$\frac{1}{\lambda'} - \alpha = (1 - \frac{1}{\lambda}) - (1 - \frac{2}{\lambda}) = \frac{1}{\lambda}$$

we can apply Hardy-Littlewood-Sobolev fractional integration theorem to see that the L^λ_t norm of the LHS is dominated by the $L^{\lambda'}_t$ norm of the RHS. Thus

$$\Big\| \int_0^T Ads \Big\|_{L^\lambda_t(0,T;L^r_x(\mathbb{R}_+))} \lesssim \|\psi\|_{L^{\lambda'}_t(0,T;L^{r'}_x(\mathbb{R}_+))} \tag{4.22}$$

Now we can apply Hölder's inequality twice to (4.17) and use (4.22) to obtain

$$\begin{aligned} \|I(\tau, b)\|^2_{L^2_\tau(\mathbb{R})} &\leq \int_0^T \|\psi(\cdot, t)\|_{L^{r'}(\mathbb{R}_+)} \|A\|_{L^r(\mathbb{R}_+)} dt \\ &\leq \|\psi\|_{L^{\lambda'}(0,T;L^{r'}(\mathbb{R}_+))} \|A\|_{L^\lambda(0,T;L^r(\mathbb{R}_+))} \\ &\lesssim \|\psi\|^2_{L^{\lambda'}(0,T;L^{r'}(\mathbb{R}_+))} \end{aligned}$$

□

Combining (4.16) and Lemma 4.6 by duality we have, $\forall \psi \in C_c([0, T); \mathcal{D}(\mathbb{R}_+))$

$$\Big| \int_{-\infty}^\infty \langle u_1(\cdot, t), \psi(t)\rangle_{L^2(\mathbb{R}_+)} \, dt \Big| \lesssim \|G_1\|_{L^2(\mathbb{R})} \|\psi\|_{L^{\lambda'}(0,T;L^{r'}(\mathbb{R}_+))}$$

$$\Rightarrow \|u_1\|_{L^\lambda(0,T;L^r(\mathbb{R}_+))} \lesssim \|G_1\|_{L^2(\mathbb{R}_+)} \tag{4.23}$$

Now, recalling the definition of u_1, differentiating with respect to x brings a scalar multiple of into the integrand, so by the identity $\widehat{DG_1(\lambda)} = i\lambda G_1(\lambda)$ we have the

$$D_x u_1(x,t) = \int_{-\infty}^{\infty} e^{-\lambda x + i\lambda^2 t} \lambda \widehat{G_1}(\lambda)\, d\lambda = -i \int_{-\infty}^{\infty} e^{-\lambda x + i\lambda^2 t} \widehat{D(G_1)}(\lambda)\, d\lambda$$

So we can apply the same arguments to obtain

$$\|D_x u_1\|_{L^\lambda(0,T;L^r(\mathbb{R}_+))} \lesssim \|DG_1\|_{L^2(\mathbb{R})} \tag{4.24}$$

Thus recalling the definition of the Sobolev norm, (4.2) and (4.24) together imply

$$\|u_1\|_{L^\lambda(0,T;W^{1,r}(\mathbb{R}_+))} \lesssim \|G_1\|_{H^1(\mathbb{R})} \tag{4.25}$$

Now notice that since $W^{0,r}(\mathbb{R}_+) = L^r(\mathbb{R}_+)$ we have the desired inequality (4.1) for $s = 0$ in (4.2) and $s = 1$ in (4.25) so we can interpolate between the two to obtain

$$\|u_1\|_{L^\lambda(0,T;W^{s,r}(\mathbb{R}_+))} \lesssim \|G_1\|_{H^s(\mathbb{R})} \tag{4.26}$$

for $0 \leq s \leq 1$. Then we can repeat the derivative argument and obtain the same estimate for all $s \geq 0$. Now with a very similar argument as in (4.7) we have $\|G_1\|_{H^s(\mathbb{R})} \lesssim \|g_0\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}$ and the conclusion follows $\square$

5 Local Well-posedness of the cubic NLS on the Half Line

Definition 5.1. An IBVP is *locally well-posed* if there is sufficiently small $T > 0$ such that the IBVP is well-posed for $t \in [0, T]$

In this section we will review the proof of the local well-posedness of high regularity solutions to the Dirichlet NLS on the half-line:

$$iu_t + u_{xx} = k|u|^p u \qquad x \in (0, \infty), t \in (0, T), p > 0 \tag{5.1a}$$
$$u(x, 0) = u_0(x) \in H_x^s(0, \infty) \qquad x \in (0, \infty] \tag{5.1b}$$
$$u(0, t) = g_0(t) \in H_t^{\frac{2s+1}{4}}(0, T) \qquad t \in [0, T] \tag{5.1c}$$

We will work only with the case $p = 2$ which is referred to as the cubic NLS. The argument for other values of p is the same, but requires some more detail.

Theorem 5.2 (Local well-posedness of NLS with Sobolev data [2])**.** *Let $s > \frac{1}{2}$ and $p = 2$. Then there is $T^* \in (0, T]$ such that there is a unique solution $u \in C([0, T^*], H_x^s(\mathbb{R}_+))$ of* (5.1) *satisfying the space estimate*

$$\|u(t)\|_{H^s(\mathbb{R}_+)} \lesssim \|u_0\|_{H^s(\mathbb{R}_+)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} \qquad t \in [0, T^*] \tag{5.2}$$

Proof. Again, we decompose the solution into three parts. Consider the Cauchy Problems (5.3) and (5.4) together with the IBVP (5.5):

$$iy_t + y_{xx} = 0 \qquad x \in \mathbb{R}, t \in (0, T) \tag{5.3a}$$
$$y(x, 0) = u_0^*(x) \qquad x \in \mathbb{R} \tag{5.3b}$$

$$iv_t + v_{xx} = f^* \qquad x \in \mathbb{R}, t \in (0, T) \tag{5.4a}$$
$$v(x, 0) = 0 \qquad x \in \mathbb{R} \tag{5.4b}$$

$$iz_t + z_{xx} = 0, \qquad x \in \mathbb{R}_+ t \in (0, T) \tag{5.5a}$$
$$z(x, 0) = 0 \qquad x \in \mathbb{R}_+ \tag{5.5b}$$
$$z(0, t) = g_0(t) - y(0, t) - v(0, t) \qquad t \in [0, T] \tag{5.5c}$$

This is the same situation as in section 2.1. So letting y, v, z be solutions to (5.3),(5.4),(5.5) respectively, observe that

$$\Psi[u_0, g_0, f] = y\big|_{\mathbb{R}_+} + v\big|_{\mathbb{R}_+} + z \tag{5.6}$$

Fix $u_0 \in H_x^s(\mathbb{R}_+)$ and $g_0 \in H_t^{\frac{2s+1}{4}}(0, T)$. Let Θ denote the mapping

$$\Theta : u \mapsto \Psi[u_0, g_0, k|u|^2 u]$$

Now notice that if we can show that Θ has a fixed point, then the fixed point will solve the NLS (5.1).
We will use Theorem 2.1 so we attempt to prove that Θ has a fixed point in $\mathcal{X}_T = C([0,T]; H_x^s(\mathbb{R}_+))$ for sufficiently small T. But first, we need to prove some additional estimates.

Claim 5.3. $\sup_{t\in[0,1]} \|\Psi[0,0,f](t)\|_{H_x^s(\mathbb{R}_+)} \lesssim T \sup_{t\in[0,T]} \|f(t)\|_{H_x^s(\mathbb{R}_+)}$

Proof. Let $u = \Psi[0,0,f]$. Then by taking the Half-line Fourier Transform of $iu_t + u_{xx} = f$, it is easy to see that

$$\widehat{u_t}(\lambda,t) = -i\lambda^2\widehat{(u)}(\lambda,t) - i\widehat{f}(\lambda,t)$$

$$\Rightarrow e^{i\lambda^2 t}\widehat{u}(\lambda,t) = -i\int_0^t \widehat{(f)}(\lambda,\tau)d\tau$$

$$\begin{aligned}\Rightarrow u(x,t) &= -i\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\int_0^t f(\lambda,\tau)d\tau d\lambda\\ &= -i\int_0^t \Psi[f,0,0](x,\tau)\,d\tau\end{aligned}$$

It follows from the space estimates for the Cauchy Problem with initial data f that $\|\Psi[0f,0,0](t)\|_{H_x^s(\mathbb{R}_+)} \lesssim \|f\|_{H_x^s(\mathbb{R}_+)}$, so we have

$$\begin{aligned}\sup_{t\in[0,1]} \|\Psi[0,0,f](t)\|_{H_x^s(\mathbb{R}_+)} &= \sup_{t\in[0,1]} \|\int_0^t \Psi[f,0,0](x,\tau)\,d\tau\|_{H_x^s(\mathbb{R}_+)}\\ &\le \sup_{t\in[0,1]}\int_0^t \|\Psi[f,0,0](x,\tau)\|_{H_x^s(\mathbb{R}_+)}\,d\tau\\ &\lesssim \sup_{t\in[0,1]}\int_0^t \|f(\tau)\|_{H_x^s(\mathbb{R}_+)}\,d\tau\\ &\le T\sup_{t\in[0,1]} \|f(t)\|_{H_x^s(\mathbb{R}_+)}\end{aligned}$$

Claim 5.4. *Let B_R^T denote the closed unit ball of radius R in $\mathcal{X}_T$. Then there is $R, T > 0$ such that*

1. *B_R^T is invariant under Θ and*

2. *$\Theta : B_R^T \to B_R^T$ is contractive.*

Proof.
Given any $u \in B_R^T$ we have

$$\Theta u = \Psi[u_0, g_0, k|u|^2u] = y\big|_{\mathbb{R}_+} + v\big|_{\mathbb{R}_+} + z$$

where y, v and z are the solutions of (5.3),(5.4) and (5.5) respectively. Note that v and z are dependent on $f = k|u|^2u$. Then we have

$$
\begin{aligned}
\|\Theta u\|_{\mathcal{X}_T} &\leq \|y\big|_{\mathbb{R}_+}\|_{\mathcal{X}_T} + \|v\big|_{\mathbb{R}_+}\|_{\mathcal{X}_T} + \|z\|_{\mathcal{X}_T} \\
&\lesssim \|u_0\|_{H^s_x(\mathbb{R}_+)} + \|u\|^3_{\mathcal{X}_T} + \|z(0,t)\|_{H_t^{\frac{2s+1}{4}}(0,T)}
\end{aligned} \tag{5.8}
$$

where the estimates for $\|y\big|_{\mathbb{R}_+}\|_{\mathcal{X}_T}$ and $\|z\|_{\mathcal{X}_T}$ follow from Theorem 2.1 and we use the algebra property to arrive at

$$
\begin{aligned}
\|v\big|_{\mathbb{R}_+}\|_{\mathcal{X}_T} &= \sup_{t\in[0,T]} \|k|u|^2u\| \\
&\leq k \sup_{t\in[0,T]} \|u\|^3 \\
&= k\|u\|^3_{\mathcal{X}_T}
\end{aligned} \tag{5.9}
$$

Now recalling the definitions of y, v, z and using Claim 5.3 together with (5.9) we have

$$
\begin{aligned}
\|z(0,t)\|_{H_t^{\frac{2s+1}{4}}(0,T)} &\leq \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} + \|y(0,t)\|_{H_t^{\frac{2s+1}{4}}(0,T)} + \|v(0,t)\|_{H_t^{\frac{2s+1}{4}}(0,T)} \\
&\lesssim \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} + \|u_0\|_{H^s_x(\mathbb{R}_+)} + T\|u\|^3_{\mathcal{X}_T}
\end{aligned} \tag{5.10}
$$

Hence (5.10) and (5.8) together imply that for some constant $c > 0$ we have

$$
\|\Theta u\|_{\mathcal{X}_T} \leq c(\|u_0\|_{H^s_x(\mathbb{R}_+)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} + T\|u\|^3_{\mathcal{X}_T}) \tag{5.11}
$$

Then taking $R = 2c\|u_0\|_{H^s_x(\mathbb{R}_+)}$ we have

$$
\|\Theta u\|_{\mathcal{X}_T} \leq \frac{R}{2} + c(\|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} + T\|u\|^3_{\mathcal{X}_T}) \tag{5.12}
$$

Notice that we can take T small enough so that we have

$$
c(\|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} + T\|u\|^3_{\mathcal{X}_T}) \leq \frac{R}{2}
$$

So $\|\Theta u\|_{\mathcal{X}_T} \leq R$ and hence there is $R, T' > 0$ such that $B_R^{T'}$ is Θ invariant and clearly so is B_R^t for $t \in [0, T']$

Now fixing such R, for the second part of our claim we will prove that there is $T^* \in [0, T]$ such that $\Theta : B_R^{T*} \to B_R^{T*}$ is a contraction.

Given $u_1, u_2 \in B_R^T$ we have,

$$
\begin{aligned}
\Theta u_1 - \Theta u_2 &= (\Psi[u_0, g_0, 0] + \Psi[0, 0, k|u_1|^2u_1]) - (\Psi[u_0, g_0, 0] + \Psi[0, 0, k|u_2|^2u_2]) \\
&= (\Psi[0, 0, k(|u_1|^2u_1] - |u_2|^2u_2)])
\end{aligned}
$$

Hence, by Claim 5.3 we have that for some constant $m > 0$

$$\|\Theta u_1 - \Theta u_2\|_{\mathcal{X}_T} \leq mT\|k(|u_1(t)|^2 u_1(t)] - |u_2(t)|^2 u_2(t)]\|_{\mathcal{X}_T} \tag{5.13}$$

Now observe that

$$|u_1|^2 u_1 - |u_2|^2 u_2 = (|u_1|^2 + |u_2|^2)(u_1 - u_2) + u_1 u_2 \overline{(u_1 - u_2)} \tag{5.14}$$

Note that this step (5.14) is much more complicated when we take arbitrary $p > 0$, (see [3]). Then again, using the algebra property we arrive at

$$\begin{aligned}\|\Theta u_1 - \Theta u_2\|_{\mathcal{X}_T} &\leq |k|mT(\|u_1\|_{\mathcal{X}_T} + \|u_2\|_{\mathcal{X}_T})^2 \|u_1 - u_2\|_{\mathcal{X}_T} \\ &\leq 4|k|mTR^2 \|u_1 - u_2\|_{\mathcal{X}_T}\end{aligned}$$

Take $T^* < \min\{\frac{1}{4kmR^2}, T'\}$. Then Θ is a contraction, as desired. Since $T^* < T'$, B_R^{T*} is also Θ-invariant. Now we can use Banach Fixed Point Theorem and the completeness of $\mathcal{X}_T$ to deduce that Θ has a fixed point in $\mathcal{X}_T$. □

Remark 5.5. For $s \geq 0$, one can define $\mathcal{X}_T$ to be the space $L_t^\lambda(0, T; L^r(\mathbb{R}_+))$ and use a similar argument, this time involving the Strichartz estimates in Theorem (4.2). After proving the existence of a fixed point, one can show that in fact the fixed point belongs to the space $C([0, T]; H_x^s(\mathbb{R}_+))$.

A Appendix

Proposition A.1. *For $s \geq \frac{1}{2}$, the space $H^s(\Omega)$ is an algebra.*

Proof. We will prove for $\Omega = \mathbb{R}$, the proposition then follows by considering the definition of $\|\cdot\|_{H^s(\Omega)}$. First note that for $s > 0$ given $\xi, \omega \in \mathbb{R}$ we have,

$$\begin{aligned}
(1 + |\xi|^2)^s &\leq (1 + |\xi - \omega|^2 + |\xi - \omega||\omega| + |\omega|^2)^s \\
&\leq (1 + 2|\xi - \omega|^2 + 2|\omega|^2)^s \\
&\lesssim (1 + |\xi - \omega|^2 + 1 + |\omega|^2)^s \\
&\lesssim (1 + |\xi - \omega|^2)^s + (1 + |\omega|^2)^s \qquad (1.1)
\end{aligned}$$

We will denote $\langle \xi \rangle \coloneqq \sqrt{1 + |\xi|^2}$ for simplicity. Now given $u, v \in H^s(\Omega)$ we have

$$\begin{aligned}
\langle \xi \rangle^s |\widehat{uv}(\xi)| &= \langle \xi \rangle^s |(\widehat{u} * \widehat{v})(\xi)| \\
&\leq \int_{-\infty}^{\infty} \langle \xi \rangle^s |\widehat{u}(\xi - \omega)\widehat{v}(\omega)| \, d\omega \\
&\lesssim \int_{-\infty}^{\infty} \langle \xi - \omega \rangle^s |\widehat{u}(\xi - \omega)\widehat{v}(\omega)| + \langle \omega \rangle^s |\widehat{u}(\xi - \omega)\widehat{v}(\omega)| \, d\omega \qquad (1.2) \\
&\leq |\langle \cdot \rangle^s \widehat{u}| * |\widehat{v}| + |\langle \cdot \rangle^s \widehat{v}| * |\widehat{u}|
\end{aligned}$$

where (1.2) follows from (1.1). Now by Young's inequality for convolution we have $\|f * g\|_2 \leq \|f\|_2 \|g\|_1$ for all$f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Hence

$$\begin{aligned}
\|uv\|_{H^s(\mathbb{R})} &= \|\langle \xi \rangle^s |\widehat{uv}(\xi)\|_{L^2(\mathbb{R})} \\
&\lesssim \|\langle \cdot \rangle^s \widehat{u}\|_{L^2(\mathbb{R})} \|v\|_{L^1(\mathbb{R})} + \|\langle \cdot \rangle^s \widehat{v}\|_{L^2(\mathbb{R})} \|u\|_{L^1(\mathbb{R})} \\
&= \|u\|_{H^s(\mathbb{R})} \|\widehat{v}\|_{L^1(\mathbb{R})} + \|v\|_{H^s(\mathbb{R})} \|\widehat{u}\|_{L^1(\mathbb{R})} \qquad (1.3)
\end{aligned}$$

Finally notice that for $s > \frac{1}{2}$, $\|\widehat{u}\|_{L^1(\mathbb{R})} \lesssim \|u\|_{H^s(\mathbb{R})}$ since by Cauchy-Schwartz inequality we have

$$\|\widehat{u}\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} \langle \xi \rangle^{-s} \langle \xi \rangle^s |\widehat{u}(\xi)| \, d\xi \leq \|\langle \cdot \rangle^{-2s}\|_{L^1(\mathbb{R})} \|u\|_{H^s(\mathbb{R})}$$

Now, $\|\langle \cdot \rangle^{-2s}\|_{L^1(\mathbb{R})} < \infty$ for $-2s < 1$ hence we have $\|\widehat{u}\|_{L^1(\mathbb{R})} \lesssim \|u\|_{H^s(\mathbb{R})}$ and so it follows from (1.3) that $\|uv\|_{H^s(\mathbb{R})} \lesssim \|u\|_{H^s(\mathbb{R})} \|v\|_{H^s(\mathbb{R})}$ as desired. □

Lemma A.2 ([2])**.** *The Laplace Transform $\mathcal{L}$ is bounded on $L^2(0, \infty)$.*

Proof. Given any $f \in L^2(0, \infty)$ we have,

$$\begin{aligned}
\Big|\mathcal{L}(f)(x)\Big|^2 &= \Big|\int_0^\infty e^{-kx} f(x) \, dk\Big|^2 \leq \int_0^\infty e^{-kx} k^{-\frac{1}{2}} \, dk + \int_0^\infty e^{-kx} k^{\frac{1}{2}} |f(x)|^2 \, dk \\
&\leq \sqrt{\pi} x^{\frac{1}{2}} + \int_0^\infty e^{-kx} k^{\frac{1}{2}} |f(x)|^2 \, dk
\end{aligned}$$

$$\begin{aligned}\|\mathcal{L}(f)\|_{L^2}^2 &\leq \sqrt{\pi}\int_0^\infty \int_0^\infty x^{-\frac{1}{2}}e^{-kx}k^{\frac{1}{2}}|f(k)|^2\,dk\,dx \\ &= \sqrt{\pi}\int_0^\infty \Big(\int_0^\infty x^{-\frac{1}{2}}e^{-kx}k^{\frac{1}{2}}\,dx\Big)|f(k)|^2\,dk \\ &= \sqrt{\pi}\int_0^\infty \sqrt{\pi}|f(k)|^2\,dk = \pi\|f\|_{L^2}^2 \qquad \square\end{aligned}$$

Theorem A.3 (Riesz-Thorin Interpolation Theorem [8]). *Given* $1 \leq p_0, p_1 \leq \infty$ *and* $\theta \in (0,1)$ *define*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$f \in L^{p_0} \cap L^{p_1} \Rightarrow \|f\|_{p_\theta} \leq \|f\|_{p_0}^{1-\theta}\|f\|_{p_1}^{\theta} \tag{1.4}$$

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and* $\theta \in (0,1)$. *Let* T *be a linear map with*

$$\|T\|_{L^{p_0}\to L^{q_0}} = N_0 < \infty$$

$$\|T\|_{L^{p_1}\to L^{q_1}} = N_1 < \infty$$

then we have,

$$f \in L^{p_0} \cap L^{p_1} \Rightarrow \|Tf\|_{q_\theta} \leq N_0^{1-\theta}N_1^{\theta}\|f\|_{p_\theta}$$

Hence T *extends uniquely as a continuous map from* L^{p_θ} *to* L^{q_θ}.

Theorem A.4 (Hardy-Littlewood-Sobolev Fractional Integration Theorem[4]). *Let* $1 < q < p < \infty$ *and* $0 < \alpha < 1$ *with*

$$\frac{1}{q} = \frac{1}{p} - \alpha$$

Then we have the inequality

$$\Big\|\int_{-\infty}^\infty |x-y|^{\alpha-1}f(y)dy\Big\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}$$

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