Hahn Banach Theorem

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Zorn's Lemma

Let $\mathcal{S}$ be a non-empty partially ordered set and suppose that every chain in $\mathcal{S}$ has an upper bound. Then $\mathcal{S}$ contains at least one maximal element.

Def: Sublinear Functional

Given a vector space X and a real valued functional f on X is sublinear if;

- f is subadditive: $\forall x,\ y \in X,\ f(x+y) \leq f(x) + f(y)$
- f is positive homogeneous: $\forall x \in X$ and $\forall \alpha \in \mathbb{R}$ s.t. $\alpha \geq 0,\ f(\alpha x) = \alpha f(x)$

E.g.: A norm on a vector space is a sublinear functional.

Hahn Banach Theorem (Extension of Linear Functionals)

Theorem

Let X be a real vector space and b a sublinear functional on X. Let f be a linear functional defined on $Z \subset X$ s.t.

$$f(x) \leq b(x), \ \forall x \in Z.$$

Then f has a linear extension $\widetilde{f}$ s.t.

$$\widetilde{f}(x) \leq b(x), \ \forall x \in X.$$

i.e. $\widetilde{f}$ is linear and $\widetilde{f}(x) = f(x), \ \forall x \in Z$.

Proof

Step 1: We will show that the set of all linear extensions of f, bounded by b has a maximal element.

Let E be the set of all linear extensions g of f s.t. $g(x) \leq b(x),\ \forall x \in \mathcal{D}(g)$. We have that $f \in E$, so $E \neq \emptyset$. Define a partial order on E s.t. given $g,\ h \in E$, $g \preceq h$ means h is a linear extension of g, that is, $\mathcal{D}(g) \subseteq \mathcal{D}(h)$ and $\forall x \in \mathcal{D}(g)$ we have $g(x) = h(x)$.

Given C, a chain in E, we will define an upper bound $u(x)$ as follows: $\forall g \in C$ let $u(x) = g(x)\ \ \forall x \in \mathcal{D}(g)$. Then u is a linear functional with domain

$$\mathcal{D}(u) = \bigcup_{g \in C} \mathcal{D}(g)$$

Clearly $g \preceq u\ \forall g \in C$. Hence u is an upper bound of C. Thus by Zorn's Lemma, $\exists \widetilde{f} \in E$ s.t. $\widetilde{f}$ is maximal.

Proof

Step 2: We will show that $\mathcal{D}(\widetilde{f}) = X$.

Suppose not. Then $\exists y_1 \in X - \mathcal{D}(\widetilde{f})$. Now define $Y_1 := span(\mathcal{D}(\widetilde{f}) \cup y_1)$. Then given $x \in Y_1$, x has a unique representation $x = y + \alpha y_1$ where $y \in \mathcal{D}(\widetilde{f})$ and $\alpha \in \mathbb{R}$. Now define a functional h on Y_1 s.t.

$$h(x) = h(y + \alpha y_1) = \widetilde{f}(y) + \alpha c$$

where $c \in \mathbb{R}$. h is clearly a linear functional and $\forall y \in \mathcal{D}(\widetilde{f})$, we have $h(y) = \widetilde{f}(y)$. Since $y_1 \in \mathcal{D}(h),\ h$ is a proper extension of $\widetilde{f}$.

Now we have to show that $h \in E$, i.e. $h(x) \leq b(x),\ \forall x \in \mathcal{D}(h)$, so that we can obtain a contradiction.

Proof

Step 3: We must show that with a suitable $c \in \mathbb{R}$ we have, $\forall y \in \mathcal{D}(\widetilde{f})$ and $\alpha \in \mathbb{R}$

$$h(y + \alpha y_1) = \widetilde{f}(y) + \alpha c \leq b(y + \alpha y_1)$$

Due to the linearity of $\widetilde{f}$ and sublinearity of b, given $y,\ z \in \mathcal{D}(\widetilde{f})$ we have that

$$\widetilde{f}(y) - \widetilde{f}(z) = \widetilde{f}(y - z) \leq b(y - z) = b(y + y_1 - y_1 - z) \leq b(y + y_1) + b(-y_1 - z)$$

$$\Rightarrow -b(-y_1 - z) - \widetilde{f}(z) \leq b(y + y_1) - \widetilde{f}(y)$$

Since the RHS depends only on y and the LHS depends only on z, define

$$k_0 := \sup_{z \in \mathcal{D}(\widetilde{f})} \left\{ -b(-y_1 - z) - \widetilde{f}(z) \right\}$$

$$k_1 := \inf_{y \in \mathcal{D}(\widetilde{f})} \left\{ b(y_1 + y) - \widetilde{f}(y) \right\}$$

Hence we have $k_0 \leq k_1$ and $\exists c \in \mathbb{R}$ s.t. $k_0 \leq c \leq k_1$. Then $\forall y, z \in \mathcal{D}(\widetilde{f})$,

$$-b(-y_1 - z) - \widetilde{f}(z) \leq c \tag{1}$$

$$c \leq b(y_1 + y) - \widetilde{f}(y) \tag{2}$$

Proof

Given $x \in Y_1,\ x = y + \alpha y_1$ for some $y \in \mathcal{D}(\widetilde{f})$ and $\alpha \in \mathbb{R}$

<u>**Case 1:** $\alpha < 0$</u>

We will use (1). Let $z := \dfrac{y}{\alpha}$, then we have

$$-b(-y_1 - \frac{y}{\alpha}) - \widetilde{f}(\frac{y}{\alpha}) \leq c \ \Rightarrow\ \alpha b(-y_1 - \frac{y}{\alpha}) + \alpha \widetilde{f}(\frac{y}{\alpha}) \leq -\alpha c$$

$$\Rightarrow h(y + \alpha y_1) = \widetilde{f}(y) + \alpha c \leq -\alpha b(-y_1 - \frac{y}{\alpha}) = b(\alpha y_1 + y) = b(x)$$

$$\Rightarrow h(x) \leq b(x)$$

<u>**Case 2:** $\alpha = 0$</u>

$x = y \in \mathcal{D}(\widetilde{f})$, then $h(x) = \widetilde{f}(x) \leq b(x)$

Proof

<u>**Case 3:** $\alpha > 0$</u>

We will use (2). Let $y := \frac{y}{\alpha}$, then we have

$$c \leq b(y_1 + \frac{y}{\alpha}) - \widetilde{f}(\frac{y}{\alpha}) \;\Rightarrow\; \alpha c \leq \alpha b(y_1 + \frac{y}{\alpha}) - \alpha \widetilde{f}(\frac{y}{\alpha})$$

$$\Rightarrow h(y + \alpha y_1) = \widetilde{f}(y) + \alpha c \leq \alpha b(y_1 + \frac{y}{\alpha}) = b(\alpha y_1 + y) = b(x)$$

$$\Rightarrow h(x) \leq b(x)$$

Hence $h \in E$ and $\widetilde{f} \preceq h$, which is a contradiction since $\widetilde{f}$ was assumed to be maximal. Thus we have $\mathcal{D}(\widetilde{f}) = X$. $\square$

Remark.

Let X be a vector space over $\mathbb{R}$, x and y are distinct elements of X. Then there is a continuous linear functional f on X s.t. $f(x) \neq f(y)$

Consider the subspace $span(x - y)$ of X. Then define the functional f on $span(x - y)$ s.t.

$$f(x - y) = ||x - y||$$

Then

$$f(x) - f(y) = f(x - y) = ||x - y|| \neq 0$$
$$\Rightarrow f(x) \neq f(y)$$

Hahn Banach Theorem(Generalized)

Theorem.

Let X be a vector space and b a subadditive, real-valued functional on X s.t. for every scalar α and $x \in X$,

$$b(\alpha x) = |\alpha| b(x)$$

Let f be a linear functional defined on $Z \subset X$ such that

$$|f(x)| \leq b(x), \ \forall x \in Z$$

Then f has a linear extension $\widetilde{f}$ from Z to X such that

$$|\widetilde{f}(x)| \leq b(x), \ \forall x \in X$$

Proof

Case 1: X is a real vector space.

$$|f(x)| \leq b(x) \Rightarrow f(x) \leq b(x),\ \forall x \in Z$$

Hence by the previous theorem, there exists a linear extension $\widetilde{f}$ of f such that $\widetilde{f}(x) \leq b(x)\ \forall x \in X$. Then we have,

$$-\widetilde{f}(x) = \widetilde{f}(-x) \leq b(-x) = |-1|b(x) = b(x),$$

so $\widetilde{f}(x) \geq -b(x)$. Hence we have that $|\widetilde{f}(x)| \leq b(x),\ \forall x \in X$

Proof

Case 2: X is a complex vector space.
Then Z is also a complex vector space, hence we can write

$$f(x) = f_1(x) + if_2(x) \forall x \in Z$$

where f_1 and f_2 are real-valued functionals. Regard X and Z as real vector spaces X_r and Z_r. Then f_1 and f_2 are linear functionals of Z_r. Also $f_1(x) \leq |f(x)|$, thus we have $f_1(x) \leq b(x)$, $\forall x \in Z_r$. So by the previous Hahn Banach theorem, there is a linear extension $\widetilde{f_1}$ of f_1 s.t.

$$\widetilde{f_1}(x) \leq b(x), \ \forall x \in X_r$$

Proof

Going back to Z $\forall x \in Z$ we have,

$$i(f_1(x) + if_2(x)) = if(x) = f(ix) = f_1(ix) + if_2(ix).$$

Equating the real parts, we obtain $f_2(x) = -f_1(ix)$, $\forall x \in Z$. Now set

$$\widetilde{f}(x) = \widetilde{f}_1(x) - i\widetilde{f}_1(ix), \ \forall x \in X.$$

Then if $x \in Z$, we have $\widetilde{f}(x) = f_1(x) - if_1(ix) = f_1(x) + if_2(x) = f(x)$. Hence $\widetilde{f}$ is an extension of f. Now we have to show that $\widetilde{f}$ is linear and $|\widetilde{f}(x)| \leq b(x)$, $\forall x \in X$. $\widetilde{f}$ is linear since given $x \in X$ and $a + ib$ any complex scalar s.t. a and b are real, we have,

$$\begin{aligned} \widetilde{f}((a+ib)x) &= \widetilde{f}_1(ax + ibx) - i\widetilde{f}_1(iax - bx) \\ &= a\widetilde{f}_1(x) + b\widetilde{f}_1(ix) - i(a\widetilde{f}_1(ix) - b\widetilde{f}_1(x)) \qquad (3) \\ &= (a+ib)(\widetilde{f}_1(x) - i\widetilde{f}_1(ix)) = (a+ib)\widetilde{f}(x) \end{aligned}$$

Proof

Now we prove that $|\widetilde{f}(x)| \leq b(x),\ \forall x \in X$. Since $b(x) \geq 0\ \forall x \in X$, it holds for x s.t. $\widetilde{f}(x) = 0$. So let x be s.t. $\widetilde{f}(x) \neq 0$. Then since $\widetilde{f}(x) = |\widetilde{f}(x)|e^{i\theta}$, we have $|\widetilde{f}(x)| = \widetilde{f}(x)e^{-i\theta} = \widetilde{f}(e^{-i\theta}x)$ which is in $\mathbb{R}$, and hence equal to its real part, which is $\widetilde{f}_1(e^{-i\theta}x)$. Hence we have,

$$|\widetilde{f}(x)| = \widetilde{f}(e^{-i\theta}x) = \widetilde{f}_1(e^{-i\theta}x) \leq b(e^{-i\theta}x) = |e^{-i\theta}|b(x) = b(x).$$

$\square$

Quotient Normed Spaces

Let $(X, \|\cdot\|_X)$ be a normed space and M, a subspace of X. Define the set

$$X/M := \{x + M | x \in X\}$$

and addition and multiplication on X/M s.t. for $x + M, y + M \in X/M$,

$$(x + M) + (y + M) = (x + y) + M$$

and for $\alpha \in \mathbb{R}$ we have,

$$\alpha(x + M) = (\alpha x) + M$$

Proposition. Let M be a closed subspace of $(X, \|\cdot\|_X)$. Define $\|\cdot\| : X/M \to [0, +\infty)$ s.t.

$$\|x + M\| = \inf_{m \in M} \|x + m\|_X$$

Then $\|\cdot\|$ defines a norm on X/M.

Proof.

- Homogeneity: Let $\alpha \in \mathbb{R}$ and $x + M \in X/M$. If $\alpha = 0$ then
$$\|\alpha(x+M)\| = \|(\alpha x) + M\| = \|0 + M\| = 0$$
since $0 \in M$. If $\alpha \neq 0$ then $\alpha^{-1} m \in M$ so we have,
$$\begin{aligned} \|\alpha(x+M)\| = \|(\alpha x) + M\| = \inf_{m\in M} \|\alpha x + m\|_X &= \inf_{m\in M} |\alpha| \|x + \frac{m}{\alpha}\|_X \\ &= |\alpha| \inf_{m\in M} \|x + m\|_X = |\alpha|\|x + M\| \end{aligned} \tag{4}$$

- Triangle Inequality: Let $x + M, y + M \in X/M$, then we have
$$\begin{aligned} \|(x+M) + (y+M)\| = \|(x+y) + M\| &= \inf_{m\in M} \|x + y + m\|_X \\ &= \inf_{m\in M} \|x + y + 2m\|_X \\ &\leq \inf_{m\in M} (\|x + m\|_X + \|y + m\|_X) \\ &\leq \inf_{m\in M} \|x + m\|_X + \inf_{m\in M} \|y + m\|_X \\ &= \|x + M\| + \|y + M\| \end{aligned} \tag{5}$$

- Definiteness: Suppose $\|(x+M)\| = 0$ Then $\inf\limits_{m\in M} \|x+m\|_X =$ $\text{dist}(x, M) = 0$. Since M is closed, this means $x \in M$, so $x + M = M$

  Conversely if $x + M = M$, then $x \in M \Rightarrow \|x + M\| = \text{dist}(x, M) = 0$

Definition. Let X be a normed space and M, a subspace of X. Then $\mathcal{Q} : X \to X/M$ s.t. $\mathcal{Q}(x) := x + M \ \forall x \in X$ is the Quotient Map.

Remark. If M is closed, then $\|\mathcal{Q}(x)\| \leq \|x\|_X \ \forall x \in X$

$$\|\mathcal{Q}(x)\| = \|x+M\| = \inf_{m\in M} \|x+m\|_X \leq \|x\|_X$$

Theorem (4.3-3).

Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then there exists a bounded linear functional $\widetilde{f}$ on X such that

$$||\widetilde{f}|| = 1, \quad \widetilde{f}(x_0) = ||x_0||$$

Corollary($*$). Let X be a normed space, M a closed subset of X, $x_0 \in X - M$ and $d =$dist(x_0, M). Then there is a linear functional g on X s.t. $g(x_0) = 1$, $g(x) = 0 \ \forall x \in M$ and $\|g\| = d^{-1}$.

Proof. $\mathcal{Q} : X \to X/M$ be the quotient map. Since $x_0 + M \neq 0$, by Theorem 4.3-3 $\exists f \in (X/M)'$ s.t. $\|f\| = 1$ and $f(x_0 + M) = d$. Let $g := d^{-1} f \circ Q \in X'$. Then $g(x_0) = 1$ and $g(x) = 0$ for $x \in \mathcal{M}$. Also

$$|g(x)| = d^{-1}|g(\mathcal{Q}(x))| \leq d^{-1}\|\mathcal{Q}(x)\| \leq \|x\|_X \Rightarrow \|g\| \leq d^{-1}$$

Since $\|f\| = 1, \exists (x_n)_n$ s.t. $|f(x_n + M)| \to_n 1$ and $\|x_n + M\| < 1 \ \forall n$. Let $y_n \in M$ s.t. $\|x_n + y_n\| < 1 \ \forall n$, then $|g(x_n + y_n)| = d^{-1}|f(x_n + M)| \to d^{-1}$. Hence $\|g\| = d^{-1}$

Banach "Limit"

Limit Functional.

Define a functional L on c s.t. for $x = (\xi_j)_{j\in\mathbb{N}} \in c,\ L(x) := \lim_{n\to\infty} \xi_n$. Then L is a linear functional on c s.t. $||L|| = \sup_{||x||_\infty=1} |L(x)| = 1$. We also have, for $x \in c$, if $x' = (\xi_2, \xi_3, ...)$, then $L(x) = L(x')$ and if $\xi_j \geq 0\ \forall j \in \mathbb{N}$, then $L(x) \geq 0$.

Theorem.

There is a linear functional $L : l^\infty \to \mathbb{F}$ satisfying the following conditions:

1. $||L|| = 1$
2. $\forall x = (\xi_j)_{j\in\mathbb{N}} \in c,\ L(x) = \lim_{n\to\infty} \xi_n$
3. if $x = (\xi_j)_{j\in\mathbb{N}} \in l^\infty$ and $\xi_j \geq 0\ \forall j \in \mathbb{N}$, then $L(x) \geq 0$
4. if $x = (\xi_j)_{j\in\mathbb{N}} \in l^\infty$ and $x' = (\xi_2, \xi_3, ...)$, then $L(x) = L(x')$

Proof

First assume $\mathbb{F} = \mathbb{R}$. For $x = (\xi_j)_{j \in \mathbb{N}} \in l^\infty$ let x' denote $(\xi_2, \xi_3, ...)$. Define $\mathcal{M} := \{x - x' | x \in l^\infty\}$ and let 1 denote the sequence $(1, 1, 1, ...) \in l^\infty$.

Claim 1: $\text{dist}(1, \mathcal{M}) = 1$

Proof: Since $0 \in \mathcal{M}$, $\text{dist}(1, \mathcal{M}) \leq 1$. Let $x = (\xi_j)_{j \in \mathbb{N}} \in l^\infty$, if $\exists j \in \mathbb{N}$ s.t. $\xi_j - \xi_{j+1} \leq 0$ then $||1 - (x - x')||_\infty \geq |1 - (\xi_j - \xi_{j+1})| \geq 1$. Suppose $\xi_j - \xi_{j+1} > 0 \; \forall j \in \mathbb{N}$, then $\xi_{j+1} \leq \xi_j \; \forall j$. Since $x = \in l^\infty$, $\lim_{n \to \infty} \xi_n$ exists. Hence $\lim_{n \to \infty} \xi_n - \xi_{n+1} = 0$ and thus $||1 - (x - x')||_\infty = 1$. In either case we have that $\text{dist}(1, \mathcal{M}) \geq 1$. Hence $\text{dist}(1, \mathcal{M}) = 1$

By Corollary($*$) there exists $L \in (l^\infty)'$ s.t. $\|L\| = 1$, $L(1) = 1$ and $L(\mathcal{M}) = 0$. Hence L satisfies (1) and since $L(x - x') = 0 \Rightarrow L(x) = L(x')$, we have (4).

Proof

Claim 2: $c_0 \subseteq \ker L$

Proof: Given $x = (\xi_j)_{j\in\mathbb{N}} \in l^\infty$ let $x^{(n)} := (\xi_{n+1}, \xi_{n+2}, ...)$. Then notice that $x^{(n+1)} - x = (x^{(n+1)} - x^{(n)}) + ... + (x' - x) \in \mathcal{M}$. Hence $L(x) = L(x^{(n)})\ \forall n \geq 1$.

Now, given $\epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|\xi_n| \leq \epsilon$ for $n \geq N$. Hence we have that

$$|L(x)| = |L(x^{(N)})| \leq ||x^{(N)}||_\infty = \sup\{\xi_n | n > N\} < \epsilon.$$

Thus $x \in \ker L$. So $c_0 \subseteq \ker L$. Hence L satisfies (2).

To show (3), let $y = (\lambda_j)_{j\in\mathbb{N}} \in l^\infty$ s.t. $\lambda_j \geq 0\ \forall j \in \mathbb{N}$ and $L(y) < 0$. Replace y with $y/\|y\|_\infty$, then we still have $L(y) < 0$ and $1 \geq \lambda_j \geq 0$. But then $\|1 - y\|_\infty \leq 1$ and $L(1-y) = 1 - L(y) > 1$ which is a contradiction since $L(1) = 1$.

Proof

Now assume $\mathbb{F} = \mathbb{C}$. Let L_1 be the linear functional obtained in the previous part. Given $x \in l^{\infty}_{\mathbb{C}}$, $x = x_1 + ix_2$ where $x_1, x_2 \in l^{\infty}_{\mathbb{R}}$. Define L on l^{∞} s.t. $L(x) = L_1(x_1) + iL_1(x_2)$. L is $\mathbb{C}$-linear and satisfies (2), (3) and (4) since L_1 satisfies them, so it remains to show that $\|L\| = 1$.

Let $E_1, E_2, ..., E_m$ be pairwise disjoint subsets of $\mathbb{N}$ and let $\alpha_1, ...\alpha_m \in \mathbb{C}$ with $|\alpha_k| \leq 1$. Put

$$x = \sum_{k=1}^{m} \alpha_k \chi_{Ek}$$

So $x \in l^{\infty}$ and $\|x\|_{\infty} \leq 1$. Then

$$L(x) = \sum_{k=1}^{m} \alpha_k L(\chi_{Ek}) = \sum_{k=1}^{m} \alpha_k L_1(\chi_{Ek})$$

But $L_1(\chi_{Ek}) \geq 0$ and $\sum_{k=1}^{m} L_1(\chi_{Ek}) = L_1(\chi_E)$ where $E = \bigcup\limits_{k=1}^{m} E_k$. Hence $\sum_{k=1}^{m} L_1(\chi_{Ek}) \leq 1$. Since $|\alpha_k| \leq 1 \;\; \forall k = 1, ..., m$, we have $|L(x)| \leq 1$

Proof

Now assume $x = (\lambda_j)_{j\in\mathbb{N}}$ is an arbitrary element of $l^\infty_{\mathbb{R}}$ where $\|x\|_\infty \leq 1$. Then w.l.o.g. assume $\lambda_j \geq 0$ for all $j \in \mathbb{N}$. Then define the sequence $x_n = (\xi_j^{(n)})_{j\in\mathbb{N}}$ s.t.

$$\xi_j^{(n)} = \frac{\lfloor n\lambda_j \rfloor}{n}$$

Then $\|x - x_n\| \leq 1/n$ and x_n takes at most $n+1$ different values.

If $x \in l^\infty_{\mathbb{C}}$, we have $x = a + ib$ where $a, b \in l^\infty_{\mathbb{R}}$ so the sequence $(a_n + ib_n)_n$ as defined above, converges to x, each element takes only finite many values and $\|x_n\|_\infty \leq 1$. Hence each x_n can be written as

$$x_n = \sum\nolimits_{k=1}^{m_n} \alpha_{kn} \chi_{Ekn}$$

Hence $L(x_n) \leq 1$. Since $L(x_n) \to L(x)$, $L(x) \leq 1$. Hence $\|L\| \leq 1$ and since $L(1) = 1$, we conclude that $\|L\| = 1$ $\square$