Computer Vision

Lecture 7 – Learning in Graphical Models

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Robotics, Computer Vision, Deep Learning, Machine Learning, System Software

Agenda

7.1
Conditional Random Fields

Inference vs. Learning

Markov Random Field:

$$p(x_1, \dots, x_{100}) = \frac{1}{Z} \exp \left\{ \sum_i \psi_i(x_i) + \lambda \sum_{i \sim j} \psi_{ij}(x_i, x_j) \right\}$$

- So far: **Inference**
  - Marginal distributions: $p(x_i) = \sum_{x \setminus x_i} p(x_1, \dots x_{100})$
  - MAP solution: $x_1^*, \dots, x_{100}^* = \operatorname{argmax}_{x_1, \dots, x_{100}} p(x_1, \dots x_{100})$
- Now: **Learning**
  - Estimate parameters (here regularization strength λ) from dataset
- Remark: In the literature, potentials are sometimes defined as the negative log factors, but here we will consider them as generic features and omit the sign

Conditional Random Fields

Markov Random Field:

$$p(\mathcal{X}) = \frac{1}{Z} \exp \left\{ \sum_{i} \psi_i(x_i) + \lambda \sum_{i \sim j} \psi_{ij}(x_i, x_j) \right\}$$

- Reason about output variables $\mathcal{X} \in \mathbb{X}$ given one particular model instantiation

Structured Output Learning:

$$f_{\mathbf{w}} : \mathbb{X} \rightarrow \mathbb{Y}$$

- Inputs $\mathcal{X} \in \mathbb{X}$ can be any kind of objects
- Outputs $\mathcal{Y} \in \mathbb{Y}$ are complex (structured) objects
  - images, text, parse trees, folds of a protein, computer programs, . . .

Conditional Random Fields

Markov Random Field:

$$p(\mathcal{X}) = \frac{1}{Z} \exp \left\{ \sum_i \psi_i(x_i) + \lambda \sum_{i \sim j} \psi_{ij}(x_i, x_j) \right\}$$

- Reason about output variables $\mathcal{X} \in \mathbb{X}$ given one particular model instantiation

Conditional Random Field:

$$p(\mathcal{Y}|\mathcal{X}, \mathbf{w}) = \frac{1}{Z} \exp \left\{ \sum_i \psi_i(\mathcal{X}, y_i) + \lambda \sum_{i \sim j} \psi_{ij}(\mathcal{X}, y_i, y_j) \right\}$$

- Make conditioning of output $\mathcal{Y}$ on input $\mathcal{X}$ and parameters $\mathbf{w}$ explicit (here $\mathbf{w} = \lambda$)
- MRF notation: outputs $\mathcal{X} \in \mathbb{X} \Rightarrow$ CRF notation: inputs $\mathcal{X} \in \mathbb{X}$, outputs $\mathcal{Y} \in \mathbb{Y}$
- Learning: Estimate $\mathbf{w}$ from dataset $\mathcal{D} = \{(\mathcal{X}^1, \mathcal{Y}^1), \ldots, (\mathcal{X}^N, \mathcal{Y}^N)\}$

Conditional Random Fields

Conditional Random Field – General Form:

$$p(\mathcal{Y}|\mathcal{X},\mathbf{w}) = \frac{1}{Z(\mathcal{X},\mathbf{w})} \exp\{\langle \mathbf{w}, \psi(\mathcal{X},\mathcal{Y})\rangle\}$$

- **Feature function:** $\psi(\mathcal{X},\mathcal{Y}) : \mathbb{X} \times \mathbb{R}^M \rightarrow \mathbb{R}^D$ (concatenates potentials/features)
  Graphical model specifies decomposition of ψ into potentials (=log factors) ψ_k:

$$\psi(\mathcal{X},\mathcal{Y}) = \left(\psi_1(\mathcal{X},\mathcal{Y}_1), \ldots, \psi_K(\mathcal{X},\mathcal{Y}_K)\right)$$

- **Parameter vector:** $\mathbf{w} \in \mathbb{R}^D$ (M: num. output nodes, D: dim. of feature space)
  Note that this model is much more flexible than a model with a single λ
- **Partition function:** $Z(\mathcal{X},\mathbf{w}) = \sum_{\mathcal{Y}} \exp\{\langle \mathbf{w}, \psi(\mathcal{X},\mathcal{Y})\rangle\}$
- **Learning:** Estimate $\mathbf{w}$ from dataset $\mathcal{D} = \{(\mathcal{X}^1,\mathcal{Y}^1), \ldots, (\mathcal{X}^N,\mathcal{Y}^N)\}$

7.2
Parameter Estimation

Parameter Estimation

Goal: Maximize likelihood of outputs $\mathcal{Y}$ conditioned on inputs $\mathcal{X}$ wrt. $\mathbf{w}$, assuming independent and identically distributed (IID) data (likelihood factorizes):

$$\hat{\mathbf{w}}_{ML} = \underset{\mathbf{w} \in \mathbb{R}^D}{\operatorname{argmax}} \; p(\mathcal{Y}|\mathcal{X}, \mathbf{w}) \quad \text{with} \quad p(\mathcal{Y}|\mathcal{X}, \mathbf{w}) = \prod_{n=1}^{N} p(\mathcal{Y}^n|\mathcal{X}^n, \mathbf{w})$$

In other words, find parameter vector $\hat{\mathbf{w}}_{ML}$ such that $p_{model}(\mathcal{Y}|\mathcal{X}, \hat{\mathbf{w}}_{ML}) \approx p_{data}(\mathcal{Y}|\mathcal{X})$.

This is equivalent to minimizing the **negative conditional log-likelihood:**

$$\hat{\mathbf{w}}_{ML} = \underset{\mathbf{w} \in \mathbb{R}^D}{\operatorname{argmin}} \; \mathcal{L}(\mathbf{w}) \quad \text{with} \quad \mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N} \log p(\mathcal{Y}^n|\mathcal{X}^n, \mathbf{w})$$

Parameter Estimation

Goal: Minimize negative conditional log-likelihood $\mathcal{L}(\mathbf{w})$

$$\hat{\mathbf{w}}_{ML} = \underset{\mathbf{w}\in\mathbb{R}^D}{\operatorname{argmin}} \; \mathcal{L}(\mathbf{w})$$

$$\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N} \log p(\mathcal{Y}^n|\mathcal{X}^n, \mathbf{w}) \\
&= -\sum_{n=1}^{N} \left[\log \frac{1}{Z(\mathcal{X}^n, \mathbf{w})} \exp\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle\}\right] \\
&= -\sum_{n=1}^{N} [-\log Z(\mathcal{X}^n, \mathbf{w}) + \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle] \\
&= -\sum_{n=1}^{N} \left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}\in\mathbb{Y}} \exp\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\}\right]
\end{aligned}$$

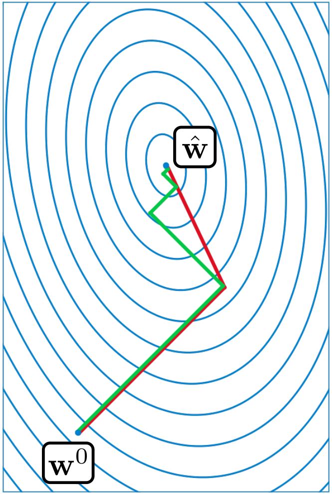
Optimization

Gradient Descent:

- Pick step size η and tolerance ϵ
- Initialize $\mathbf{w}^0$
- Repeat until $\|\mathbf{v}\| < \epsilon$
  - $\mathbf{v} = \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}) = \sum_{i=1}^{N} \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w})$
  - $\mathbf{w}^{t+1} = \mathbf{w}^{t} - \eta \mathbf{v}$

Variants:

- Line search (green)
- Conjugate gradients (red)
- All require gradients, some (e.g., line search) require function evaluation



Gradient of Negative Conditional Log-Likelihood

$$
\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N} \left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n) \rangle - \log \sum_{\mathcal{Y}} \exp \{ \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}) \rangle \} \right] \\
\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N} \left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \frac{\sum_{\mathcal{Y}} \exp \{ \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}) \rangle \} \psi(\mathcal{X}^n, \mathcal{Y})}{\sum_{\mathcal{Y}} \exp \{ \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}) \rangle \}} \right] \\
&= -\sum_{n=1}^{N} \left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}} \frac{\exp \{ \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}) \rangle \}}{\sum_{\mathcal{Y}'} \exp \{ \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}') \rangle \}} \psi(\mathcal{X}^n, \mathcal{Y}) \right] \\
&= -\sum_{n=1}^{N} \left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}} p(\mathcal{Y} | \mathcal{X}^n, \mathbf{w}) \psi(\mathcal{X}^n, \mathcal{Y}) \right] \\
&= -\sum_{n=1}^{N} \left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \mathbb{E}_{\mathcal{Y} \sim p(\mathcal{Y} | \mathcal{X}^n, \mathbf{w})} \psi(\mathcal{X}^n, \mathcal{Y}) \right]
\end{aligned}
$$

Gradient of Negative Conditional Log-Likelihood

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \mathbb{E}_{\mathcal{Y}\sim p(\mathcal{Y}|\mathcal{X}^n,\mathbf{w})}\psi(\mathcal{X}^n, \mathcal{Y})\right]$$

When is $\mathcal{L}(\mathbf{w})$ minimal?

$$\mathbb{E}_{y\sim p(\mathcal{Y}|\mathcal{X}^n,\mathbf{w})}\psi(\mathcal{X}^n, \mathcal{Y}) = \psi(\mathcal{X}^n, \mathcal{Y}^n) \Rightarrow \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) = 0$$

- Interpretation: we aim at **expectation matching**: $\mathbb{E}_{\mathcal{Y}\sim p(\cdot)}\psi(\mathcal{X}, \mathcal{Y}) = \psi(\mathcal{X}, \mathcal{Y}^{\text{obs}})$, but discriminatively: only for $\mathcal{X} \in \{\mathcal{X}^1, \ldots, \mathcal{X}^N\}$

Note:

- $\mathcal{L}(\mathbf{w})$ convex (Hessian positive semi-definite) $\Rightarrow \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) = 0 \Rightarrow$ **global optimum**
- Only true as $p(\mathcal{Y}|\mathcal{X}, \mathbf{w})$ is log-linear in $\mathbf{w} \in \mathbb{R}^D$ (we will also see non-linear models)

Computational Complexity

Task: For gradient descent with line search we must evaluate $\mathcal{L}(\mathbf{w})$ and $\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w})$:

$$\begin{aligned}\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}\in\mathbb{Y}} \exp\left\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\right\}\right] \\ \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}\in\mathbb{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})\psi(\mathcal{X}^n, \mathcal{Y})\right]\end{aligned}$$

Problem: $\mathbb{Y}$ is typically very (exponentially) large!

- Binary image segmentation: $|\mathbb{Y}| = 2^{640\times 480} \approx 10^{92475}$
- We must use the structure in $\mathbb{Y}$, or we are lost!

Computational Complexity

$$
\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}\in\mathbb{Y}} \exp\left\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\right\}\right] \\
\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}\in\mathbb{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})\psi(\mathcal{X}^n, \mathcal{Y})\right]
\end{aligned}
$$

Computational complexity: $O(NC^M D)$

- N: number of samples in dataset ($\approx$ 100 to 1,000,000)
- M: number of output nodes ($\approx$ 100 to 1,000,000)
- C: maximal number of labels per output node ($\approx$ 2 to 100)
- D: dimensionality of feature space ψ

Computational Complexity

$$
\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}\in\mathbb{Y}} \exp\left\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\right\}\right] \\
\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}\in\mathbb{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})\psi(\mathcal{X}^n, \mathcal{Y})\right]
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Probabilistic Inference to the Rescue

Remember: in a graphical model, **features and weights decompose** as follows

$$\psi(\mathcal{X},\mathcal{Y}) = (\psi_1(\mathcal{X},\mathcal{Y}_1),\ldots,\psi_K(\mathcal{X},\mathcal{Y}_K)) \qquad \mathbf{w} = (\mathbf{w}_1,\ldots,\mathbf{w}_K)$$

Thus, the **partition function simplifies** as:

$$\begin{aligned}
\sum_{\mathcal{Y}} \exp\{\langle \mathbf{w}, \psi(\mathcal{X}^n,\mathcal{Y})\rangle\} &= \sum_{\mathcal{Y}} \exp\left\{\sum_k \langle \mathbf{w}_k, \psi_k(\mathcal{X}^n,\mathcal{Y}_k)\rangle\right\} \\
&= \sum_{\mathcal{Y}} \prod_k \underbrace{\exp\{\langle \mathbf{w}_k, \psi_k(\mathcal{X}^n,\mathcal{Y}_k)\rangle\}}_{\text{k'th factor}}
\end{aligned}$$

- Can be efficiently calculated/approximated using **message passing**
  (run sum-product belief propagation, sum over any of the unnorm. marginals)

Probabilistic Inference to the Rescue

Similarly, the **feature expectation simplifies** as:

$$
\begin{aligned}
\sum_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})\psi(\mathcal{X}^n, \mathcal{Y}) &= \mathbb{E}_{\mathcal{Y}\sim p(\mathcal{Y}|\mathcal{X}^n,\mathbf{w})}\psi(\mathcal{X}^n, \mathcal{Y}) \\
&= \left(\mathbb{E}_{\mathcal{Y}\sim p(\mathcal{Y}|\mathcal{X}^n,\mathbf{w})}\psi_k(\mathcal{X}^n, \mathcal{Y}_k) \right)_{k\in 1,\dots,K} \\
&= \left(\mathbb{E}_{\mathcal{Y}_k\sim p(\mathcal{Y}_k|\mathcal{X}^n,\mathbf{w})}\psi_k(\mathcal{X}^n, \mathcal{Y}_k) \right)_{k\in 1,\dots,K} \\
&= \left(\sum_{\mathcal{Y}_k} p(\mathcal{Y}_k|\mathcal{X}^n, \mathbf{w})\psi_k(\mathcal{X}^n, \mathcal{Y}_k) \right)_{k\in 1,\dots,K}
\end{aligned}
$$

- Now only C^F terms in sum over $\mathcal{Y}_k$ (C: max. number of labels, F: largest order)
- Marginals $p(\mathcal{Y}_k|\mathcal{X}^n, \mathbf{w})$ can be calculated efficiently (e.g., with BP)

Computational Complexity

$$
\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}} \exp\left\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\right\}\right] \\
\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})\psi(\mathcal{X}^n, \mathcal{Y})\right]
\end{aligned}
$$

Computational complexity: $\cancel{O(NC^M D)} \quad \rightarrow \quad O(NKC^F D)$

- N: number of samples in dataset (≈ 100 to 1,000,000)
- M: number of output nodes (≈ 100 to 1,000,000)
- C: maximal number of labels per output node (≈ 2 to 100)
- D: dim. of feature space, K: number of factors, F: order of largest factor (≈ 2-3)

Computational Complexity

$$\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N} \left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n) \rangle - \log \sum_{\mathcal{Y}} \exp\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}) \rangle\} \right]$$

$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N} \left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w}) \psi(\mathcal{X}^n, \mathcal{Y}) \right]$$

Computational complexity: $O(NKC^F D)$

- N: number of samples in dataset ($\approx$ 100 to 1,000,000)
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Computational Complexity

Learning on large datasets:

- Processing all N training samples for one gradient update is slow
- Furthermore, often not all data fits into memory (as in deep learning)

How can we estimate parameters in this setting?

- Simplify model to make gradient updates faster $\Rightarrow$ results get worse
- Train model on subsampled dataset $\Rightarrow$ ignores information
- Parallelize across CPUs/GPUs $\Rightarrow$ bottlenecks, doesn't save computation
- Stochastic gradient descent

Stochastic Gradient Descent (SGD)

Stochastic Gradient Descent:

- In each gradient step:
  - Create random subset $\mathcal{D}' \subset \mathcal{D}$ (typically $\mathcal{D}' \leq 256$)
  - Follow approximate gradient:

$$\nabla_{\mathbf{w}} \approx - \sum_{\color{red}(\mathcal{X}^n, \mathcal{Y}^n) \in \mathcal{D}'} \left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \mathbb{E}_{\mathcal{Y} \sim p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})} \psi(\mathcal{X}^n, \mathcal{Y}) \right]$$

Comments:

- Line search no longer possible $\Rightarrow$ extra step-size hyper-parameter η
- SGD converges to $\operatorname{argmin}_{\mathbf{w}} \mathcal{L}(\mathbf{w})$! (if η chosen right)
- SGD needs more iterations, but each one is faster
- See also: Bottou & Bousquet: The Tradeoffs of Large Scale Learning, NIPS 2007

Computational Complexity

$$\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N}\left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}} \exp\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\}\right]$$

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n, \mathcal{Y}^n) - \sum_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}^n, \mathbf{w})\psi(\mathcal{X}^n, \mathcal{Y})\right]$$

Computational complexity: $O(NKC^F D)$

- N: number of samples in dataset ($\approx$ 100 to 1,000,000)
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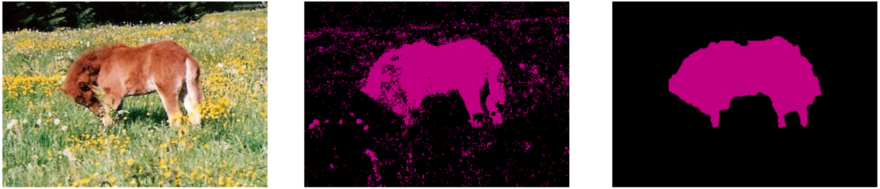
Applications / Feature Functions

Semantic Segmentation:

- $\psi_i(\mathcal{X}, y_i) \in \mathbb{R}^{\approx 1000}$: local image features (e.g., bag of words, deep features)
  $\rightarrow \langle \mathbf{w}_i, \psi_i(\mathcal{X}, y_i) \rangle$: local classifier (like logistic regression)
- $\psi_{ij}(y_i, y_j) = [y_i = y_j] \in \mathbb{R}^1$: test for same label
  $\rightarrow \langle w_{ij}, \psi_{ij}(y_i, y_j) \rangle$: penalizer for label changes (if $w_{ij} > 0$)
- combined: $\operatorname{argmax}_{\mathcal{Y}}\, p(\mathcal{Y}|\mathcal{X}, \mathbf{w})$ is smoothed version of local cues



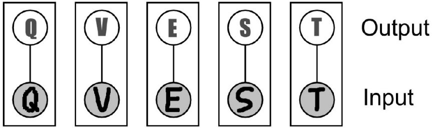
original local classification local + smoothness

Shotton, Winn, Rother and Criminisi: TextonBoost: Joint Appearance, Shape and Context Modeling for Multi-class Object Recognition and Segmentation. ECCV, 2006.

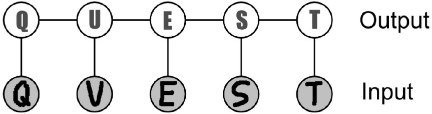
Applications / Feature Functions

Handwriting Recognition:

- $\psi_i(\mathcal{X}, y_i) \in \mathbb{R}^{\approx 1000}$: image representation (e.g., pixels, gradients)
  $\rightarrow \langle \mathbf{w}_i, \psi_i(\mathcal{X}, y_i) \rangle$: local classifier for letters
- $\psi_{ij}(y_i, y_j) = \mathbf{e}_{y_i} \mathbf{e}_{y_j}^\top \in \mathbb{R}^{26\times 26}$: letter/letter indicator (matrix with one element = 1)
  $\rightarrow \langle \mathbf{w}_{ij}, \psi_{ij}(y_i, y_j) \rangle$: encourage/suppress letter combinations
- Combined: $\operatorname{argmax}_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}, \mathbf{w})$ is “corrected” version of local cues



local classification



local + ”correction”

Applications / Feature Functions

Pose Estimation:

- $\psi_i(\mathcal{X}, y_i) \in \mathbb{R}^{\approx 1000}$: image representation (e.g., HoG, deep features)
  $\rightarrow \langle \mathbf{w}_i, \psi_i(\mathcal{X}, y_i) \rangle$: local confidence map
- $\psi_{ij}(y_i, y_j) = \text{fit}(y_i, y_j) \in \mathbb{R}^1$: test for geometric fit / pose prior
  $\rightarrow \langle w_{ij}, \psi_{ij}(y_i, y_j) \rangle$: penalizer for unrealistic poses
- Combined: $\operatorname{argmax}_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}, \mathbf{w})$ is sanitized version of local cues

original | local classification | local + geometry

Ferrari, Marin-Jimenez and Zisserman: Progressive Search Space Reduction for Human Pose Estimation. CVPR, 2008

Applications / Feature Functions

Typical feature functions for CRFs in computer vision:

- Unary terms $\psi_i(\mathcal{X}, y_i)$: local representation, high-dimensional
  $\rightarrow \langle \mathbf{w}_i, \psi_i(\mathcal{X}, y_i) \rangle$: local classifier
- Pairwise terms $\psi_{ij}(y_i, y_j)$: prior knowledge, typically low-dimensional
  $\rightarrow \langle w_{ij}, \psi_{ij}(y_i, y_j) \rangle$: penalize inconsistencies
- Pairwise terms sometimes also depend on $\mathcal{X}$: $\psi_{ij}(\mathcal{X}, y_i, y_j)$

Learning adjusts parameters:

- Unary weights $\mathbf{w}_i$: learn local linear classifiers
- Pairwise weights w_{ij}: learn importance of smoothing/penalization
- $\operatorname{argmax}_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}, \mathbf{w})$ is cleaned up version of local prediction

Piece-wise Training

Sometimes, training the entire model at once is not easy:

- If terms actually depend on parameters in non-linear fashion
- If features are high-dimensional, learning can be very slow

Alternative: **Piece-wise Training**

- Pre-train classifiers $p(y_i|\mathcal{X})$; set $\psi_i(\mathcal{X}, y_i) = \log p(y_i|\mathcal{X}) \in \mathbb{R}$
- Learn one-dimensional weight per classifier: $\langle w_i, \psi_i(\mathcal{X}, y_i)\rangle$

Advantage:

- Lower dimensional feature vector during training/inference $\rightarrow$ faster
- $\log p(y_i|\mathcal{X})$ can be stronger classifiers, e.g., non-linear SVMs, CNNs, ..

Disadvantage

- If local classifiers are bad, CRF training cannot fix this

Summary

Given:

- Training set $\mathcal{D} = \{(\mathcal{X}^1, \mathcal{Y}^1), \ldots, (\mathcal{X}^N, \mathcal{Y}^N)\}$ with $(\mathcal{X}^n, \mathcal{Y}^n) \overset{\text{i.i.d.}}{\sim} p_{data}(\mathcal{X}, \mathcal{Y})$
- Feature function: $\psi(\mathcal{X}, \mathcal{Y}) : \mathbb{X} \times \mathbb{R}^M \to \mathbb{R}^D$

Task:

- Find parameter vector $\hat{\mathbf{w}}_{ML}$ such that
$p_{model}(\mathcal{Y}|\mathcal{X}, \hat{\mathbf{w}}_{ML}) = \frac{1}{Z(\mathcal{X}, \hat{\mathbf{w}}_{ML})} \exp\{\langle \hat{\mathbf{w}}_{ML}, \psi(\mathcal{X}, \mathcal{Y})\rangle\} \approx p_{data}(\mathcal{Y}|\mathcal{X})$

Minimize negative conditional log-likelihood:

$$\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N} \left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n)\rangle - \log \sum_{\mathcal{Y}} \exp\{\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y})\rangle\} \right]$$

- Convex optimization problem $\to$ gradient descent leads to global optimum
- Training needs repeated runs of probabilistic inference $\Rightarrow$ must be fast

Summary

Gradient of negative conditional log-likelihood:

$$\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N} \left[\langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}^n) \rangle - \log \sum_{\mathcal{Y} \in \mathbb{Y}} \exp \{ \langle \mathbf{w}, \psi(\mathcal{X}^n, \mathcal{Y}) \rangle \} \right]$$

Problem	**Solution**	**Method**
$\lvert\mathbb{Y}\rvert$ too large	exploit structure	belief propagation
N too large	mini-batches	stochastic gradient descent
D too large	trained ψ	piece-wise training

7.3
Deep Structured Models

Motivation

Log-Linear Models:

$$p(\mathcal{Y}|\mathcal{X},\mathbf{w}) = \frac{1}{Z(\mathcal{X},\mathbf{w})}\exp\{\langle\mathbf{w},\psi(\mathcal{X},\mathcal{Y})\rangle\}$$

- Log-linear in the parameters $\mathbf{w} \Rightarrow$ features must do all the heavy lifting
- Only linear combination of features is learned

Deep Structured Models:

$$p(\mathcal{Y}|\mathcal{X},\mathbf{w}) = \frac{1}{Z(\mathcal{X},\mathbf{w})}\exp\{\psi(\mathcal{X},\mathcal{Y},\mathbf{w})\}$$

- Potential functions directly parametrized via $\mathbf{w}$
- Results in a much more flexible model (ψ can represent, e.g., a neural network)

Deep Structured Models

Negative Log-Likelihood and its Gradient:

$$\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n,\mathcal{Y}^n,\mathbf{w}) - \log\sum_{\mathcal{Y}}\exp\{\psi(\mathcal{X}^n,\mathcal{Y},\mathbf{w})\}\right]$$

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) = -\sum_{n=1}^{N}\left[\nabla_{\mathbf{w}}\psi(\mathcal{X}^n,\mathcal{Y}^n,\mathbf{w}) - \sum_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}^n,\mathbf{w})\nabla_{\mathbf{w}}\psi(\mathcal{X}^n,\mathcal{Y},\mathbf{w})\right]$$

- Similar form as for log-linear models
- Differences to log-linear model highlighted in red

Deep Structured Models

Negative Log-Likelihood and its Gradient:

$$
\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\psi(\mathcal{X}^n,\mathcal{Y}^n,\mathbf{w}) - \log\sum_{\mathcal{Y}}\exp\{\psi(\mathcal{X}^n,\mathcal{Y},\mathbf{w})\}\right] \\
\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) &= -\sum_{n=1}^{N}\left[\nabla_{\mathbf{w}}\psi(\mathcal{X}^n,\mathcal{Y}^n,\mathbf{w}) - \sum_{\mathcal{Y}} p(\mathcal{Y}|\mathcal{X}^n,\mathbf{w})\nabla_{\mathbf{w}}\psi(\mathcal{X}^n,\mathcal{Y},\mathbf{w})\right]
\end{aligned}
$$

- Again, sums can be efficiently computed as features decompose

$$
\psi(\mathcal{X},\mathcal{Y},\mathbf{w}) = \big(\psi_1(\mathcal{X},\mathcal{Y}_1,\mathbf{w}),\ldots,\psi_K(\mathcal{X},\mathcal{Y}_K,\mathbf{w})\big)
$$

Deep Structured Models

Algorithm:

- Forward pass to compute $\psi_k(\mathcal{X}, \mathcal{Y}_k, \mathbf{w})$
- Backward pass to obtain gradients $\nabla_{\mathbf{w}} \psi(\mathcal{X}^n, \mathcal{Y}, \mathbf{w})$
- Compute marginals using message passing
- Update parameters $\mathbf{w}$

What is the problem with this approach?

- Very slow as forward and backward pass are required to calculate features and gradients for GM inference in every gradient update step

Alternatives:

- Interleave learning and inference [Chen et al., ICML 2015], but still slow
- Unrolled inference (simple, but we loose probabilistic interpretation)

Chen, Schwing, Yuille and Urtasun: Learning Deep Structured Models. ICML, 2015.

Inference Unrolling

Inference Unrolling

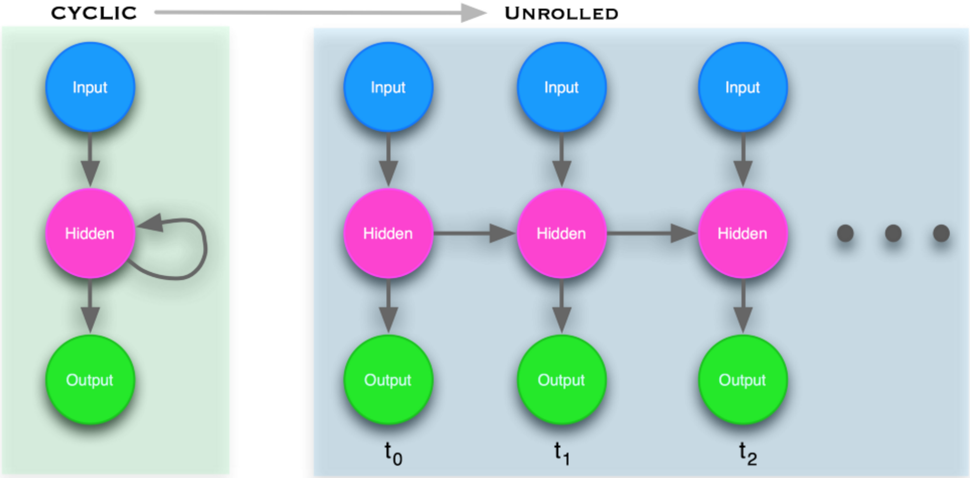
Idea:

- Consider inference as sequence of small computations
- "Unroll" a **fixed** number of inference iterations similar to RNN
- Compute gradients using automatic differentiation

Remarks:

- Now: empirical risk minimization
- Thus purely deterministic approach, giving up probabilistic viewpoint
- But often fast enough for efficient training in deep models
- Effectively integrates structure of the problem into architecture of the network
- Can be thought of as a form of regularization (hard constraint)

Inference Unrolling



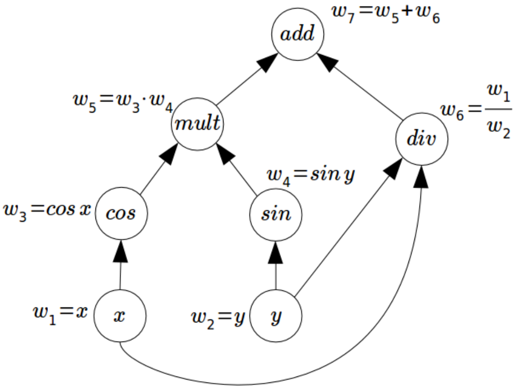
Automatic Differentiation

Automatic Differentiation:

- Rewrite complicated function as **composition** of simple functions: $f = f_0 \circ f_1 \circ \cdots \circ f_n$
- Each simple function f_k has a simple derivative
- Use chain rule: $\frac{\partial f_0}{\partial f_1} \frac{\partial f_1}{\partial f_2} \cdots \frac{\partial f_n}{\partial x}$
- **Example:**

$$f(x, y) = \cos(x)\sin(y) + \frac{x}{y}$$

Computation Graph:

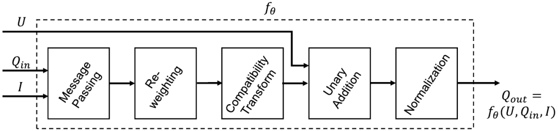


Examples

Conditional Random Fields as Recurrent Neural Networks

$$E(\mathbf{x}) = \sum_i \psi_u(x_i) + \sum_{i<j} \psi_p(x_i, x_j), \qquad (1)$$

$$\psi_p(x_i, x_j) = \mu(x_i, x_j) \sum_{m=1}^{M} w^{(m)} k_G^{(m)}(\mathbf{f}_i, \mathbf{f}_j), \qquad (2)$$



Algorithm 1 Mean-field in dense CRFs [29], broken down to common CNN operations.

$Q_i(l) \leftarrow \frac{1}{Z_i} \exp\left(U_i(l)\right)$ for all i $\triangleright$ Initialization

while not converged **do**

  $\tilde{Q}_i^{(m)}(l) \leftarrow \sum_{j \neq i} k^{(m)}(\mathbf{f}_i, \mathbf{f}_j) Q_j(l)$ for all m $\triangleright$ Message Passing

  $\check{Q}_i(l) \leftarrow \sum_m w^{(m)} \tilde{Q}_i^{(m)}(l)$ $\triangleright$ Weighting Filter Outputs

  $\hat{Q}_i(l) \leftarrow \sum_{l' \in \mathcal{L}} \mu(l, l') \check{Q}_i(l')$ $\triangleright$ Compatibility Transform

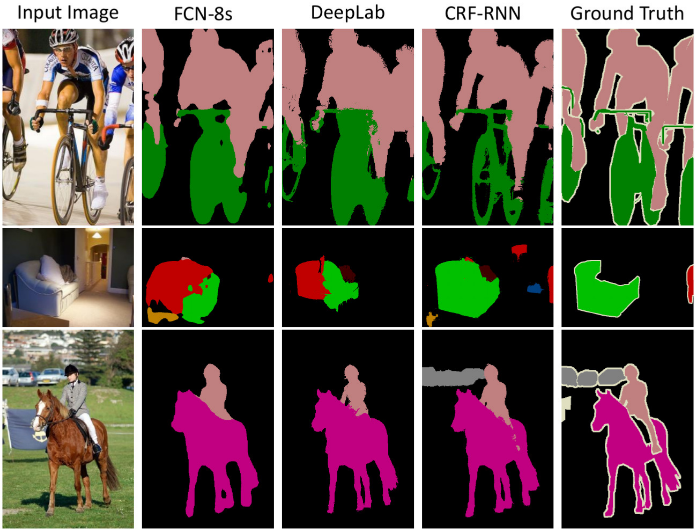
  $\breve{Q}_i(l) \leftarrow U_i(l) - \hat{Q}_i(l)$ $\triangleright$ Adding Unary Potentials

  $Q_i \leftarrow \frac{1}{Z_i} \exp\left(\breve{Q}_i(l)\right)$ $\triangleright$ Normalizing

end while

Zheng et al.: Conditional Random Fields as Recurrent Neural Networks. ICCV, 2015.

Conditional Random Fields as Recurrent Neural Networks



Zheng et al.: Conditional Random Fields as Recurrent Neural Networks. ICCV, 2015.

RayNet: Learning Volumetric 3D Reconstruction

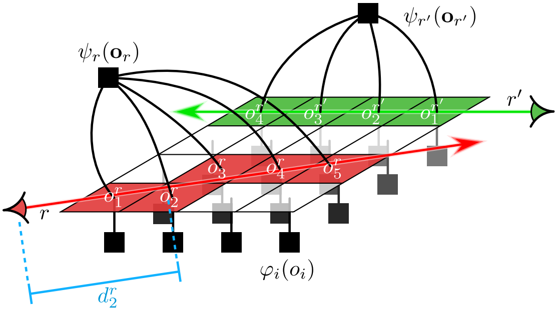
Distribution over voxel occupancies:

$$p(\mathbf{o}) = \frac{1}{Z} \prod_{i \in \mathcal{X}} \underbrace{\varphi_i(o_i)}_{\text{unary}} \prod_{r \in \mathcal{R}} \underbrace{\psi_r(\mathbf{o}_r)}_{\text{ray}}$$

$$\varphi_i(o_i) = \gamma^{o_i}(1-\gamma)^{1-o_i}$$

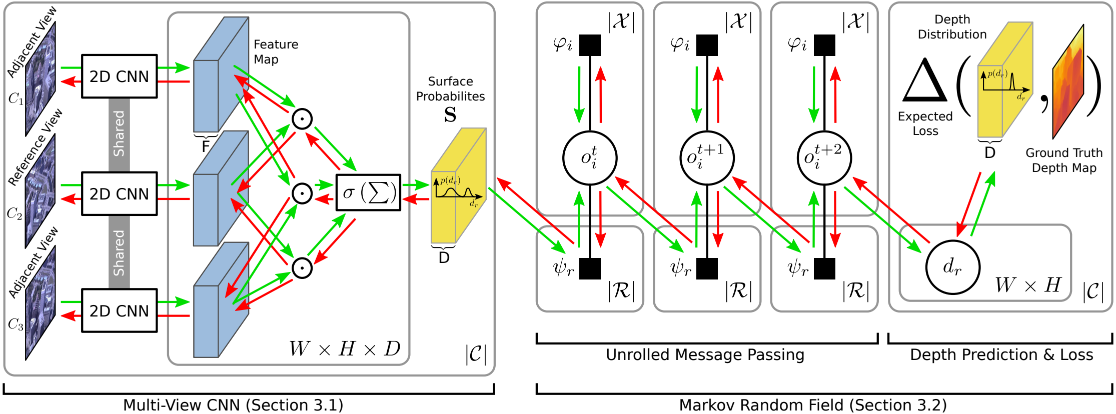
$$\psi_r(\mathbf{o_r}) = \sum_{i=1}^{N_r} o_i^r \prod_{j<i} (1-o_j^r) s_i^r$$

Corresponding factor graph:



Paschalidou, Ulusoy, Schmitt, van Gool and Geiger: RayNet: Learning Volumetric 3D Reconstruction with Ray Potentials. CVPR, 2018.

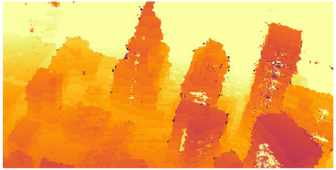
RayNet: Learning Volumetric 3D Reconstruction



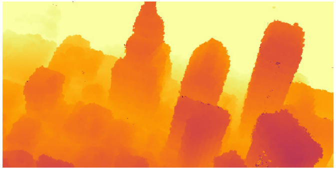
Paschalidou, Ulusoy, Schmitt, van Gool and Geiger: RayNet: Learning Volumetric 3D Reconstruction with Ray Potentials. CVPR, 2018.

RayNet: Learning Volumetric 3D Reconstruction

(a) Image

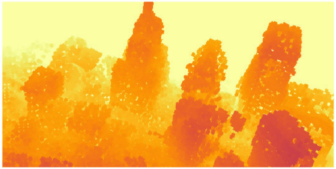


(b) Ours (CNN)

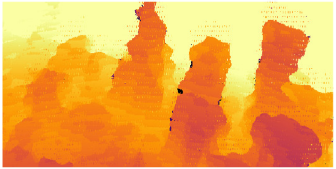


(c) Ours (CNN+MRF)

(d) ZNCC



(e) Ulusoy et al. [35]



(f) Hartmann et al. [14]

Paschalidou, Ulusoy, Schmitt, van Gool and Geiger: RayNet: Learning Volumetric 3D Reconstruction with Ray Potentials. CVPR, 2018.