Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes, I have read and understood the honor code.

Concept: Projection

2. (2 points) Consider a matrix A and a vector $\mathbf{b}$ which does not lie in the column space of A. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on to the column space of A. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \\ 11 \\ 8 \end{bmatrix}$, find $\mathbf{b}$.

Solution: Given $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \\ 11 \\ 8 \end{bmatrix}$.

We know that the projection matrix P corresponding to A is given by :
$P = A(A^\top A)^{-1}A^\top$.

$$A^\top = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix},\ A^\top A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 9 & 10 \end{bmatrix}$$

$$=> (A^\top A)^{-1} = (\tfrac{1}{19}) \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} \left(\tfrac{1}{19}\right) \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \left(\tfrac{1}{19}\right) \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 11 & -8 & 2 \\ 1 & -8 & 11 & 2 \end{bmatrix}.$$

$$=> P = \left(\tfrac{1}{19}\right) \begin{bmatrix} 2 & 3 & 3 & 4 \\ 3 & 14 & -5 & 6 \\ 3 & -5 & 14 & 6 \\ 4 & 6 & 6 & 8 \end{bmatrix}.$$

Since $\mathbf{p}$ is the projection of $\mathbf{b}$ on the column space of A, we have $P\ \mathbf{b} = \mathbf{p}$.

To solve for $\mathbf{b}$ in the above system of linear equations we apply Gaussian Elimination :

So we have, $\left(\tfrac{1}{19}\right) \begin{bmatrix} 2 & 3 & 3 & 4 \\ 3 & 14 & -5 & 6 \\ 3 & -5 & 14 & 6 \\ 4 & 6 & 6 & 8 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 11 \\ 8 \end{bmatrix}.$

Perform row operations : $(R_2 : R_2 + \frac{-3}{2}R_1)$, $(R_3 : R_3 + \frac{-3}{2}R_1)$, $(R_4 : R_4 + (-2)R_1)$.

Then we have, $\left(\tfrac{1}{19}\right) \begin{bmatrix} 2 & 3 & 3 & 4 \\ 0 & \frac{19}{2} & \frac{-19}{2} & 0 \\ 0 & \frac{-19}{2} & \frac{19}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 4 \\ -5 \\ 5 \\ 0 \end{bmatrix}.$

Perform row operations : $(R_3 : R_3 + R_2)$ and let $\mathbf{b} = [b_1, b_2, b_3, b_4]$.

Then we have, $\left(\tfrac{1}{19}\right) \begin{bmatrix} 2 & 3 & 3 & 4 \\ 0 & \frac{19}{2} & \frac{-19}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 0 \\ 0 \end{bmatrix}.$

We observe that, third and fourth columns are free columns so the corresponding free variables are b_3 and b_4.

Now we have,

$$\begin{aligned} 2b_1 + 3b_2 + 3b_3 + 4b_4 &= 76 \\ b_2 - b_3 &= -10 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Let $b_3 = t$ and $b_4 = k$ then $b_2 = t - 10$ and $b_1 = 53 - 3t - 2k \ \forall k, t \in \mathbb{R}$.

We have infinite solutions for $P\ \mathbf{b} = \mathbf{p}$ and hence $\mathbf{b} = \{ \begin{bmatrix} 53 - 3t - 2k \\ t - 10 \\ t \\ k \end{bmatrix}, \forall k, t \in \mathbb{R}\}$.

3. (2 points) Consider the following statement: Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

(a) Give one example where the above statement is True.

Solution: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{b_1} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{b_2} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$.

We know that the projection matrix P corresponding to A is given by :
$P = A(A^\top A)^{-1}A^\top$.

$$A^\top A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Then, $(A^\top A)^{-1} = I^{-1} = I$. So, $P = A(A^\top A)^{-1}A^\top = AIA^\top = AA^\top = I$.

Therefore, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Let the projection of $\mathbf{b_1}$ on the column space of A be $\mathbf{p_1}$ and projection of $\mathbf{b_2}$ on the column space of A be $\mathbf{p_2}$.

Then, $\mathbf{p_1} = P\ \mathbf{b_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Then, $\mathbf{p_2} = P\ \mathbf{b_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$.

Here, the projections of $\mathbf{b_1}$, $\mathbf{b_2}$ on the column space of A i.e. $\mathbf{p_1}$, $\mathbf{p_2}$ are not equal.

(b) Give one example where the above statement is False.

Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{b_1} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{b_2} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$.

We know that the projection matrix P corresponding to A is given by :
$P = A(A^\top A)^{-1}A^\top$.

$$A^\top A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Then, $(A^\top A)^{-1} = I^{-1} = I$. So, $P = A(A^\top A)^{-1}A^\top = AIA^\top = AA^\top$.

Therefore, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Let the projection of $\mathbf{b_1}$ on the column space of A be $\mathbf{p_1}$ and projection of $\mathbf{b_2}$ on the column space of A be $\mathbf{p_2}$.

Then, $\mathbf{p_1} = P\,\mathbf{b_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

Then, $\mathbf{p_2} = P\,\mathbf{b_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

Here, the projections of $\mathbf{b_1}$, $\mathbf{b_2}$ on the column space of A i.e. $\mathbf{p_1}$, $\mathbf{p_2}$ are equal.

(c) Based on the above examples, state the generic condition under which the above statement will be True or False.

Solution:
Given condition : Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

The given condition is True except when $(\mathbf{b_1} - \mathbf{b_2})$ is orthogonal to the column space of A.

Reason : Let A be a $m \times n$ matrix and let $\mathbf{b_1}$, $\mathbf{b_2}$ be two $m \times 1$ vectors such that $(\mathbf{b_1} - \mathbf{b_2})$ is orthogonal to the column space of A.

That is $A_i^\top(\mathbf{b}_1 - \mathbf{b}_2) = 0 \; \forall i \in \{1, 2, 3, \ldots, n\}$ where A_i denotes i^{th} column of A.

Equivalently, $A^\top(\mathbf{b}_1 - \mathbf{b}_2) = \mathbf{0} => (A(AA^\top)^{-1})A^\top(\mathbf{b}_1 - \mathbf{b}_2) = \mathbf{0}$.

$=> P(\mathbf{b}_1 - \mathbf{b}_2) = \mathbf{0}$ where $P = A(AA^\top)^{-1}A^\top$ is the projection matrix corresponding to projecting onto the column space of A.

$=> P\mathbf{b}_1 = P\mathbf{b}_2 = \mathbf{p}$ (say) i.e. the projection of $\mathbf{b}_1$ and the projection of $\mathbf{b}_2$ onto the column space of A are equal.

Therefore, if two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are such that ($\mathbf{b_1}$ - $\mathbf{b_2}$) is orthogonal to the column space of A then they have the same projection $\mathbf{p}$ on the column space of A .

4. (2 points) (a) Find the projection matrix P_1 that projects onto the line through $\mathbf{a} = \begin{bmatrix}1\\3\end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to $\mathbf{a}$.

Solution: Given $\mathbf{a} = \begin{bmatrix}1\\3\end{bmatrix}$. Also P_1 is the projection matrix that projects onto the line through $\mathbf{a}$.

We know that $P_1 = \frac{aa^\top}{a^\top a} => P_1 = \frac{\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix}}{\begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}} = \frac{\begin{bmatrix}1 & 3\\3 & 1\end{bmatrix}}{10} => P_1 = \frac{1}{10}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}$.

For any non-zero vector $\mathbf{b}$, projection of $\mathbf{b}$ on line passing through $\mathbf{a}$ i.e. $\mathbf{p} = \frac{\mathbf{a}\mathbf{a}^\top b}{\mathbf{a}^\top\mathbf{a}}$.

Consider ($\mathbf{b}$ - $\mathbf{p}$), then $\mathbf{a}^\top$($\mathbf{b}$ - $\mathbf{p}$) $= \mathbf{a}^\top\mathbf{b} - \frac{\mathbf{a}^\top\mathbf{a}\mathbf{a}^\top\mathbf{b}}{\mathbf{a}^\top\mathbf{a}} = \mathbf{a}^\top\mathbf{b} - \mathbf{a}^\top\mathbf{b} = \mathbf{0}$.

Let ($\mathbf{b}$ - $\mathbf{p}$) $= \mathbf{e}$ then we have $\mathbf{e}$ is perpendicular to $\mathbf{a}$ => $\mathbf{e}$ is projection of $\mathbf{b}$ on a line perpendicular to the line passing through the line passing through $\mathbf{a}$.

Then $\mathbf{e} = P_2\ \mathbf{b}$———(**eqn1**).

Consider $\mathbf{e} = (\mathbf{b} - \mathbf{p}) = (\mathbf{b}I - \frac{\mathbf{a}\mathbf{a}^\top\mathbf{b}}{\mathbf{a}^\top\mathbf{a}}) = (I - \frac{\mathbf{a}\mathbf{a}^\top}{\mathbf{a}^\top\mathbf{a}})\ \mathbf{b} = (I - P_1)\ \mathbf{b}$———(**eqn2**).

From (**eqn1**), (**eqn2**) we have,
$P_2\mathbf{b} = (I - P_1)\mathbf{b} => P_2\mathbf{b} - (I - P_1)\mathbf{b} = \mathbf{0} => (P_2 - I + P_1)\mathbf{b} = \mathbf{0}$.

Since $\mathbf{b}$ is non-zero => $(P_2 - I + P_1) = \mathbf{0}$ => $P_2 + P_1 = I$.———(**eqn3**).

Hence, we have $P_2 = I$ - $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ - $\frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$.

Therefore, $P_1 = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ and $P_2 = \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$.

(b) Compute $P_1 + P_2$ and P_1P_2 and explain the result.

Solution: We have $P_1 + P_2 = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Explanation :
Consider line passing through any vector $\mathbf{a}$ then P_1 be its projection matrix and P_2 be the projection matrix corresponding to the line perpendicular the line passing through $\mathbf{a}$. From (**eqn3**) in Q4(a), we have $P_1 + P_2 = I$.

Also $P_1 * P_2 = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} * \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Explanation :
Consider line passing through any vector $\mathbf{a}$ then P_1 be its projection matrix and P_2 be the projection matrix corresponding to the line perpendicular the line passing through $\mathbf{a}$. From (**eqn3**) in Q4(a), we have $P_1 + P_2 = I$.

Then $P_1 * P_2 = P_1 * (I - P_1) = P_1 - (P_1)^2$.

We know that for any projection matrix P, $(P)^2 = P$. Hence, $(P_1)^2 = P_1$ then $P_1 * P_2 = \mathbf{0}$.

Concept: Dot product of vectors

5. (1 point) Consider two vectors $\mathbf{u}$ and $\mathbf{v}$. Let θ be the angle between these two vectors. Prove that

$$cos\theta = \frac{\mathbf{u}^\top \mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$$

Solution: Let $\mathbf{p}$ be the projection of $\mathbf{v}$ on $\mathbf{u}$. Then $cos\theta = \frac{||\mathbf{p}||_2}{||\mathbf{v}||_2}$.——(**eqn1**

Consider $||\mathbf{p}||_2$, we know that $||\mathbf{p}||_2 = \sqrt{\mathbf{p}^\top\mathbf{p}}$ and $\mathbf{p} = (\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\mathbf{u}$.

Then, $||\mathbf{p}||_2 = \sqrt{((\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\mathbf{u})^\top(\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\mathbf{u}}$.

Since $(\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})$ is a scalar, $||\mathbf{p}||_2 = \sqrt{(\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\mathbf{u}^\top(\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\mathbf{u}} = \sqrt{(\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})^2(\mathbf{u}^\top\mathbf{u})} = (\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\sqrt{(\mathbf{u}^\top\mathbf{u})}$.

Hence, $||\mathbf{p}||_2 = (\frac{\mathbf{u}^\top\mathbf{v}}{\mathbf{u}^\top\mathbf{u}})\sqrt{(\mathbf{u}^\top\mathbf{u})} = (\frac{\mathbf{u}^\top\mathbf{v}}{\sqrt{(\mathbf{u}^\top\mathbf{u})}}) = \frac{\mathbf{u}^\top\mathbf{v}}{||\mathbf{u}||_2}$.

Substituting $||\mathbf{p}||_2$ in (**eqn1**), we have $cos\theta = \frac{||\mathbf{p}||_2}{||\mathbf{v}||_2} = \frac{\mathbf{u}^\top\mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$.

Therefore,

$$cos\theta = \frac{\mathbf{u}^\top\mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$$

Concept: Vector norms

6. (1 point) The L_p-norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ is defined as:

$$||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

(a) Prove that $||\mathbf{x}||_\infty = max_{1\leq i\leq n}|x_i|$

Solution: Consider $||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$.

Let $|x_m| = max_{1\leq i\leq n}|x_i|$ then $||\mathbf{x}||_p = |x_m|((|\frac{x_1}{x_m}|^p + |\frac{x_2}{x_m}|^p + |\frac{x_3}{x_m}|^p + \cdots + |\frac{x_n}{x_m}|^p)^{\frac{1}{p}}$.

We know that $|\frac{x_i}{x_m}| \leq 1 \ \forall i \in \{1, 2, 3, \ldots, n\}$.

For all j such that $|\frac{x_j}{x_m}| < 1$, as $p \to \infty$ we have $|\frac{x_j}{x_m}|^p \to 0$.

For all l such that $|\frac{x_l}{x_m}| = 1$, as $p \to \infty$ we have $|\frac{x_l}{x_m}|^p = 1$.

So as $p \to \infty$, the sum $(|\frac{x_1}{x_m}|^p + |\frac{x_2}{x_m}|^p + |\frac{x_3}{x_m}|^p + \cdots + |\frac{x_n}{x_m}|^p) \to k$, where $k \in \mathbb{Z}$.

Hence, as $p \to \infty$ we have $\frac{1}{p} \to 0$ and the expression
$(|\frac{x_1}{x_m}|^p + |\frac{x_2}{x_m}|^p + |\frac{x_3}{x_m}|^p + \cdots + |\frac{x_n}{x_m}|^p)^{\frac{1}{p}} \to (k)^{\frac{1}{p}} \to 1$.

Therefore, $||\mathbf{x}||_\infty = |x_m| = max_{1\leq i\leq n}|x_i|$.

(b) True or False (explain with reason): $||\mathbf{x}||_0$ is a norm.

Solution: False.

Let $\mathbf{x} = [x_1] \in \mathbb{R} - \{0\}$ then $||\mathbf{x}||_0 = (|x_1|^0)^{\frac{1}{0}} = 1^\infty = 1$.

Consider $||\alpha\mathbf{x}||_0, \forall\alpha \in \mathbb{R}$.

case 1: $\alpha \neq 0 => ||\alpha\mathbf{x}||_0 = (|\alpha x_1|^0)^{\frac{1}{0}} = 1^\infty = 1$.

case 2: $\alpha = 0 => ||\alpha\mathbf{x}||_0 = (|\alpha x_1|^0)^{\frac{1}{0}} = (|\alpha|^0|x_1|^0)^{\frac{1}{0}} = (0^0)^\infty =$ indeterminate form.

$\alpha||\mathbf{x}||_0 = \alpha 1 = \alpha$.——**(1)**

From **case 1**, **case 2** and **(1)**, we have $||\alpha\mathbf{x}||_0 \neq \alpha||\mathbf{x}||_0$.

Therefore, $||\mathbf{x}||_0$ is not a norm.

Concept: Orthogonal/Orthonormal vectors and matrices

7. (1 point) Consider the following questions:

(a) Construct a 2×2 matrix, such that all its entries are +1 and -1 and its columns are orthogonal.

Solution: Consider the matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

We have $\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 + 1 = 0$. Hence the columns of A are orthogonal.

Therefore, A is a 2×2 orthogonal matrix, such that all its entries are +1 and -1 and its columns are orthogonal.

(b) Now, construct a 4×4 matrix, such that all its entries are +1 and -1, its columns are orthogonal and it contains the above matrix within it.

Solution: Consider the matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and

$$B = \begin{bmatrix} A & & 1 & 1 \\ & & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} & & 1 & 1 \\ & & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

We have,

$$\begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = -2 + 2 = 0.$$

$$\begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

$$\begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

Hence, all the columns of B are orthogonal.

Therefore, B is a 4×4 matrix, such that all its entries are $+1$ and -1, its columns are orthogonal and it contains the above matrix A within it.

8. (1 point) Consider the vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

(a) What multiple of **a** is closest to **b**?

Solution: By definition, the projection of **b** on the **span**(**a**) is the multiple of **a** closest to **b**.

The projection of **b** on the **spanofa** i.e. $\mathbf{p} = (\frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}})\mathbf{a}$

$$=> \mathbf{p} = (\frac{\begin{bmatrix}4 & 5 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\2\\0\\0\end{bmatrix}}{\begin{bmatrix}4 & 5 & 2 & 2\end{bmatrix}\begin{bmatrix}4\\5\\2\\2\end{bmatrix}})\mathbf{a}.$$

$\mathbf{p} = (\frac{4+10}{14+25+4+4})\mathbf{a} = (\frac{14}{49})\mathbf{a} = (\frac{2}{7})\mathbf{a}$.

Therefore, $(\frac{2}{7})\mathbf{a}$ is the multiple of **a** closest to **b**.

(b) Find orthonormal vectors $\mathbf{q_1}$ and $\mathbf{q_2}$ that lie in the plane formed by **a** and **b**?

Solution:

Using Gram-Schmidt Process: Let $\mathbf{a_1} = \mathbf{a}$ and $\mathbf{a_2} = \mathbf{b}$.

(i) $\hat{\mathbf{a_1}} = \mathbf{a_1}$ and $\hat{\mathbf{a_2}} = \mathbf{a_2}$ - $(\frac{\hat{\mathbf{a_1}}^\top \mathbf{a_2}}{\hat{\mathbf{a_1}}^\top \hat{\mathbf{a_1}}})$ $\hat{\mathbf{a_1}}$.

$$\hat{\mathbf{a_1}} = \begin{bmatrix}4\\5\\2\\2\end{bmatrix} \text{ and } \hat{\mathbf{a_2}} = \begin{bmatrix}1\\2\\0\\0\end{bmatrix} - (\frac{\begin{bmatrix}4 & 5 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\2\\0\\0\end{bmatrix}}{\begin{bmatrix}4 & 5 & 2 & 2\end{bmatrix}\begin{bmatrix}4\\5\\2\\2\end{bmatrix}})\begin{bmatrix}4\\5\\2\\2\end{bmatrix} => \hat{\mathbf{a_2}} = \frac{1}{7}\begin{bmatrix}-1\\4\\-4\\-4\end{bmatrix}.$$

(ii) $\hat{\mathbf{q_1}} = \frac{1}{||a_1||_2}\hat{\mathbf{a_1}}$ and $\hat{\mathbf{q_2}} = \frac{1}{||a_2||_2}\hat{\mathbf{a_2}}$.

$$\hat{\mathbf{q_1}} = \frac{1}{||a_1||_2}\hat{\mathbf{a_1}} = (\frac{1}{\sqrt{16+25+4+4}})\begin{bmatrix}4\\5\\2\\2\end{bmatrix} => \hat{\mathbf{q_1}} = (\frac{1}{7})\begin{bmatrix}4\\5\\2\\2\end{bmatrix}.$$

$\hat{\mathbf{q_2}} = \frac{1}{||a_2||_2}\hat{\mathbf{a_2}} = (\frac{1}{7*\sqrt{1}}) \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix} => \hat{\mathbf{q_2}} = (\frac{1}{7}) \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix}$.

$\hat{\mathbf{q_1}}^\top \hat{\mathbf{q_1}} = (\frac{1}{7}) \begin{bmatrix} 4 & 5 & 2 & 2 \end{bmatrix} (\frac{1}{7}) \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \frac{49}{49} = 1.$

$\hat{\mathbf{q_2}}^\top \hat{\mathbf{q_2}} = (\frac{1}{7}) \begin{bmatrix} -1 & 4 & -4 & -4 \end{bmatrix} (\frac{1}{7}) \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix} = \frac{49}{49} = 1.$

$\hat{\mathbf{q_1}}^\top \hat{\mathbf{q_2}} = (\frac{1}{7}) \begin{bmatrix} 4 & 5 & 2 & 2 \end{bmatrix} (\frac{1}{7}) \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix} = 0.$

Hence $\hat{\mathbf{q_1}}$, $\hat{\mathbf{q_2}}$ are orthonormal vectors.

Let the plane formed by $\mathbf{a}$ and $\mathbf{b}$ be P i.e. $\text{P} = \{\alpha\mathbf{a} + \beta\mathbf{b}, \forall\alpha, \beta \in \mathbb{R}\}$.

Notice that P is a subspace.

By construction, $\mathbf{a_1} = \mathbf{a}$ and $\hat{\mathbf{a_2}} = \mathbf{b}$ - $\frac{2}{7}\mathbf{a}$ and so $\hat{\mathbf{a_1}}$ and $\hat{\mathbf{a_2}} \in$ P.

Since $\hat{\mathbf{q_1}} = \frac{1}{||a_1||_2}\hat{\mathbf{a_1}}$ and $\hat{\mathbf{q_2}} = \frac{1}{||a_2||_2}\hat{\mathbf{a_2}}$ then $\hat{\mathbf{q_1}}$, $\hat{\mathbf{q_2}} \in$ P i.e. they lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$.

Therefore, $\hat{\mathbf{q_1}} = (\frac{1}{7}) \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$, $\hat{\mathbf{q_2}} = (\frac{1}{7}) \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix}$ are orthonormal vectors that lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$.

9. (1 point) Suppose $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal vectors. Prove that they are also independent.

Solution:
Given vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal vectors.

Even if at least one of them is a zero vector then the vectors are linearly dependent.

Let $\mathbf{a_t}$ is $\mathbf{0}$ then $\sum_{i=1;i\neq t}^{n} 0\mathbf{a_i} + l\mathbf{a_t} = \mathbf{0}$, $\forall l \in \mathbb{R}$. Hence, they are linearly dependent.

So, given vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal vectors such that $\mathbf{a_i} \neq \mathbf{0}\ \forall\ i \in \{1, 2, \ldots, n\}$.——**(eqn1)**

Proof by contradiction:

Assume $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are linearly dependent.

Then $c_1\mathbf{a_1} + c_2\mathbf{a_2} + \cdots + c_n\mathbf{a_n} = \mathbf{0}$ where $c_i \neq 0$ for at least one $i \in \{1, 2, \ldots, n\}$.

Let $c_k \neq 0$ then consider, $\mathbf{a_k}^\top(c_1\mathbf{a_1} + c_2\mathbf{a_2} + \cdots + c_n\mathbf{a_n}) = \mathbf{0}$.——**(eqn2)**

$c_1\mathbf{a_k}^\top\mathbf{a_1} + \cdots + c_k\mathbf{a_k}^\top\mathbf{a_k} + \cdots + c_n\mathbf{a_k}^\top\mathbf{a_n} = \mathbf{0}$.

Since vectors are orthogonal, $\mathbf{a_i}^\top\mathbf{a_j} = 0$, $\forall i \neq j$ where $i, j \in \{1, 2, \ldots, n\}$.

$c_1\mathbf{a_k}^\top\mathbf{a_1} + \cdots + c_k\mathbf{a_k}^\top\mathbf{a_k} + \cdots + c_n\mathbf{a_k}^\top\mathbf{a_n} = \mathbf{0} => c_k\mathbf{a_k}^\top\mathbf{a_k} = \mathbf{0}$.——**(eqn3)**

From **(eqn1), (eqn2)**, we have $\mathbf{a_k} \neq \mathbf{0}$ and $c_k \neq 0 => c_k\mathbf{a_k}^\top\mathbf{a_k} \neq \mathbf{0}$.——**(eqn4)**

From **(eqn3), (eqn4)**, we have have a contradiction.

Hence, our assumption that $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are linearly dependent is incorrect.

Therefore, given orthogonal vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are linearly independent.

10. (1 point) If Q_1 and Q_2 are orthogonal matrices,show that their product Q_1Q_2 is also an orthogonal matrix.

Solution: Let Q_1 and Q_2 be n $\times$ n orthogonal matrices.

Let $Q_2 = \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \dots & \mathbf{q_n} \end{bmatrix}$ where $\mathbf{q_i}$ is the i^{th} column of Q_2.

Consider $Q_1Q_2 = Q_1 \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \dots & \mathbf{q_n} \end{bmatrix} = \begin{bmatrix} Q_1\mathbf{q_1} & Q_1\mathbf{q_2} & \dots & Q_1\mathbf{q_n} \end{bmatrix}$.

Let $(Q_1Q_2)_i$, $(Q_1Q_2)_j$ be the i^{th} and j^{th} column of Q_1Q_2 respectively.

Consider $((Q_1Q_2)_i)^\top (Q_1Q_2)_j = (Q_1\mathbf{q_i})^\top (Q_1\mathbf{q_j}) = \mathbf{q_i}^\top Q_1^\top (Q_1\mathbf{q_j}) = \mathbf{q_i}^\top (Q_1^\top Q_1)\mathbf{q_j}$.

Since Q_1 is an orthogonal matrix, $Q_1^\top = Q_1^{-1} \Rightarrow \mathbf{q_i}^\top (Q_1^\top Q_1)\mathbf{q_j} = \mathbf{q_i}^\top (I)\mathbf{q_j} = \mathbf{q_i}^\top \mathbf{q_j}$.

Since Q_2 is an orthogonal matrix, $\mathbf{q_i}^\top \mathbf{q_j} = \begin{Bmatrix} 1; \text{ if } i = j \\ 0; \text{ if } i \neq j \end{Bmatrix}$.

Hence, $((Q_1Q_2)_i)^\top (Q_1Q_2)_j = \begin{Bmatrix} 1; \text{ if } i = j \\ 0; \text{ if } i \neq j \end{Bmatrix}$.

Therefore, Q_1Q_2 is also an orthogonal matrix.

Concept: Determinants

11. (2 points) A tri-diagonal matrix is a matrix which has 1's on the main diagonal as well as on the diagonals to the left and right of the main diagonal. For example,

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let A_n be an $n \times n$ tri-diagonal matrix. Prove that $|A_n| = |A_{n-1}| - |A_{n-2}|$

Solution: Consider $A_n = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}$

.

$$det(A_n) = det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right) = (A_n)_{11}C_{11} + (A_n)_{12}C_{12}$$

where $(A_n)_{ij}$ denotes the $(ij)^{th}$ entry of A_n and C_{ij} denotes its corresponding co-factor.

$$det(A_n) = det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right) = (1)C_{11} + (1)C_{12} = C_{11} + C_{12}.$$

On observation we find, $C_{11} = det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right) = det(A_{n-1}).$

Also $C_{12} = (-1)\ det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right).$

Let $B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}.$

$$det(B) = det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right) = (B)_{11}D_{11} + (B)_{12}D_{12} = D_{11} + D_{12}$$

where $(B)_{ij}$ denotes the $(ij)^{th}$ entry of B and D_{ij} denotes its corresponding co-factor.

$$det(B) = det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right) = (B)_{11}D_{11} + (B)_{12}D_{12} = D_{11} + D_{12}.$$

On observation we find, $D_{11} = det\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right) = det(A_{n-2})$.

Also $D_{12} = (-1)\ det\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}\right)$.

From property(10) / Q13(b) we have, $D_{12}= (-1)\ det\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}^{\top}\right)$.

So $D_{12} = (-1)\ det\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}\right) = 0.$ { From property (6) }

Hence, $C_{12} = (-1)\ det(B) = (-1)\ (D_{11} + D_{12}) = (-1)\ D_{11} = -(det(A_{n-2}))$.

So, $det(A_n) = C_{11} + C_{12} = det(A_{n-1}) + (-(det(A_{n-2}))) = det(A_{n-1}) - (det(A_{n-2}))$.

Therefore, $|A_n| = |A_{n-1}| - |A_{n-2}|$.

12. (1 point) State True or False and explain your answer: $det(A + B) = det(A) + det(B)$

Solution: False.

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

So, $det(A) = 1$, $det(B) = 1 => det(A) + det(B) = 2$ and $det(A + B) = 0$.

Therefore, $det(A + B) \neq det(A) + det(B)$.

13. (1 point) This question is about properties 9 and 10 of determinants.

(a) Prove that $det(AB) = det(A)det(B)$

Solution:
Let A, B be n $\times$ n matrices.

case(i): $det(A) = 0$.

If $det(A) = 0 =>$ A is not invertible $=>$ rank of $A <$ n.

We know that rank of $AB \leq$ rank of $A =>$ rank of $AB <$ n

$=> \exists$ a zero pivot in U_{AB}, so $det(U_{AB}) = 0$ where U_{AB} is the U in LU factorization of AB obtained by Gaussian Elimination.

Since row operations don't change the determinant, $det(U_{AB}) = det(AB) = 0$.

Hence, $det(AB) = det(A)det(B)$.

case(ii): $det(A) \neq 0$ and $det(B) \neq 0$.

If $det(A) = 0 => A$ is invertible $=> A^{-1}$ exists.

Since A^{-1} exists => RREF$(A) = I$.

Also we know that RREF$(A) = EA$ where $E = (M * E_t * E_{t-1} * \cdots * E_1)$ where each E_i is an elementary matrix and M is a diagonal matrix with non-zero entries. Also E is invertible since each E_i and M are invertible.

Then, $EA = I => A = E^{-1} => A = (M * E_t * E_{t-1} * \cdots * E_1)^{-1}$

$=> A = E_1^{-1} * \cdots * E_t^{-1} * M^{-1} => A = F_1 * F_2 * \cdots * F_t * N$ where $E_i^{-1} = F_i$ which is also an elementary matrix and $M^{-1} = N$ which is also a diagonal matrix.

claim (1) : If A is a diagonal matrix then $det(AB) = det(A)det(B)$.

Proof : Let a_{ij} denote the $(ij)^{th}$ entry of A and $B_i^\top$ denote the $(i)^{th}$ row of B.

Then $det(AB) = det\left(\begin{bmatrix} a_{11}B_1^\top \\ a_{22}B_2^\top \\ \cdots \\ a_{nn}B_n^\top \end{bmatrix}\right)$. By property(2) of determinants we have,

$$det\left(\begin{bmatrix} a_{11}B_1^\top \\ a_{22}B_2^\top \\ \cdots \\ a_{nn}B_n^\top \end{bmatrix}\right) = (a_{11} * a_{22} * \cdots * a_{nn})\ det\left(\begin{bmatrix} B_1^\top \\ B_2^\top \\ \cdots \\ B_n^\top \end{bmatrix}\right).$$

By property(7) of determinants we have,

$$(a_{11} * a_{22} * \cdots * a_{nn})\ det\left(\begin{bmatrix} B_1^\top \\ B_2^\top \\ \cdots \\ B_n^\top \end{bmatrix}\right) = det(A)det(B).$$

Hence, $det(AB) = det(A)det(B)$.

claim (2) : If A is an elementary matrix then $det(AB) = det(A)det(B)$.

Proof : We know that any elementary matrix A corresponding to some row operation is obtained by performing the same row operation on I.

Also since A is an elementary matrix corresponding to some row operation then AB is obtained by performing the same row operation on B.

Since row operations do not change the determinant of a matrix, $det(A) = det(I) = 1$ and $det(AB) = det(B)$.

Then, $det(A)det(B) = det(B) = det(AB)$.

Hence, $det(AB) = det(A)det(B)$.

Since we have $A = F_1 * F_2 * \cdots * F_t * N$ then,

Now, $det(AB) = det((F_1 * F_2 * \cdots * F_t * N) * B) = det(F_1 * F_2 * \cdots * F_t * N * B)$.

By **claim(2)** we have,

$det(F_1 * F_2 * \cdots * F_t * N * B) = det(F_1) * det(F_2 * \cdots * F_t * N * B)$.

By repeatedly using **claim(2)** we have,

$det(F_1) * det(F_2 * \cdots * F_t * N * B) = det(F_1) * det(F_2) * \cdots * det(F_t) * det(N * B)$.

By **claim(1)** we have,

$det(F_1) * det(F_2) * \cdots * det(F_t) * det(N * B) = det(F_1) * det(F_2) * \cdots * det(F_t) * det(N) * det(B)$.

Again by repeatedly using **claim(2)** we have,

$det(F_1) * det(F_2) * \cdots * det(F_t) * det(N) * det(B) = det(F_1) * det(F_2) * \cdots * (det(F_t) * det(N)) * det(B) = det(F_1 * F_2 * \cdots * F_t * det(N)) * det(B)$

$det(F_1 * F_2 * \cdots * F_t * det(N)) * det(B) = det(A) * det(B)$.

Hence, $det(AB) = det(A)det(B)$.

Therefore from **case(i), case(ii)** we have $det(AB) = det(A)det(B)$.

(b) (2 points) Prove that $det(A^\top) = det(A)$

Solution: Let A be an n $\times$ n matrix.

case(i): $det(A) = 0$.

If $det(A) = 0 => A$ is not invertible => column rank of $A <$ n.

We know that column rank of A = row rank of A => row rank of $A <$ n.

Also, row rank of A = column rank of $A^\top$ => column rank of $A^\top <$ n.

=> $\exists$ a zero pivot in $U_{A^\top}$, so $det(U_{A^\top}) = 0$ where $U_{A^\top}$ is the U in LU factorization of $A^\top$ obtained by Gaussian Elimination.

Since row operations don't change the determinant, $det(U_{A^\top}) = det(A^\top) = 0$.

Hence, $det(A^\top) = det(A)$.

case(ii): $det(A) \neq 0$.

From **Q13(a)'s case(ii)** we have $A = F_1 * F_2 * \cdots * F_t * N$ then,

Now, $det(A^\top) = det((F_1 * F_2 * \cdots * F_t * N)^\top) = det(N^\top * F_t^\top * \cdots * F_2^\top * F_1^\top)$.

Since N is diagonal matrix we have,

$det(N^\top * F_t^\top * \cdots * F_2^\top * F_1^\top) = det(N * F_t^\top * \cdots * F_2^\top * F_1^\top)$.

By **Q13(a)'s case(ii)'s claim(1)** we have,

$det(N * F_t^\top * \cdots * F_2^\top * F_1^\top) = det(N) * det(F_t^\top * \cdots * F_2^\top * F_1^\top)$.

By repeatedly using **Q13(a)'s case(ii)'s claim(2)** we have,

$det(N) * det(F_t^\top * \cdots * F_2^\top * F_1^\top) = det(N) * det(F_t^\top) * (F_{t-1}^\top * \cdots * F_2^\top * F_1^\top)$

$det(N) * det(F_t^\top) * (F_{t-1}^\top * \cdots * F_2^\top * F_1^\top) = det(N) * det(F_t^\top) * det(F_{t-1}^\top) * \cdots * det(F_2^\top) * det(F_1^\top)$

Since transpose of an elementary matrix is also an elementary matrix,

$det(N) * det(F_t^\top) * det(F_{t-1}^\top) * \cdots * det(F_2^\top) * det(F_1^\top) = det(N) * 1 * \cdots * 1.$

$det(N) * 1 * \cdots * 1 = det(N) * 1 * \cdots * 1 = 1 * \cdots * 1 * det(N) = det(F_1) * det(F_2) * \cdots * det(F_{t-1}) * det(F_t) * det(N)$

By repeatedly using **Q13(a)'s case(ii)'s claim(2)** we have,

$det(F_1) * det(F_2) * \cdots * det(F_{t-1}) * det(F_t) * det(N) = det(F_1) * det(F_2) * \cdots * det(F_{t-1}) * det(F_t * N) = det(F_1 * F_2 * \cdots * F_{t-1} * F_t * N) = det(A).$

Hence, $det(A^\top) = det(A)$.

Therefore, from **case(i), case(ii)** we have $det(A^\top) = det(A)$.

14. (1 point) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

(a) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$

Solution: Let $\mathbf{o}$ be the origin and $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

The area of the parallelogram with vertices $\{\mathbf{o}, \mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}\}$ = The area of the parallelogram with vectors $\mathbf{u}, \mathbf{v}$ as adjacent sides.

We know that area of the parallelogram with vectors $\mathbf{u}, \mathbf{v}$ as adjacent sides = absolute value of determinant of the matrix with $\mathbf{u}, \mathbf{v}$ as columns = $|det(A)|$.

Hence, area of the parallelogram with vectors $\mathbf{u}, \mathbf{v}$ as adjacent sides = $|det(A)|$ = $|(4 * 3) - (2 * 1)| = 10$ sq. units.

We know that, area of triangle with vertices $\{\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}\}$ +
area of triangle with vertices $\{\mathbf{v}, \mathbf{u}, \mathbf{o}\}$
= area of the parallelogram with vertices $\{\mathbf{o}, \mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}\}$
= area of the parallelogram with vectors $\mathbf{u}, \mathbf{v}$ as adjacent sides = 10 sq. units.

The triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$ is congruent to the triangle with vertices $\mathbf{v}, \mathbf{u}, \mathbf{o}$ by side-side-side congruence. Hence, their areas are equal.

So, (area of triangle with vertices $\{\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}\}) * 2 = 10$ sq. units.

Therefore, area of triangle with vertices $\{\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}\} = 5$ sq. units.

(b) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u}-\mathbf{v}$

Solution: Given $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ then $\mathbf{u}-\mathbf{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

Let $\mathbf{o}$ be the origin and $B = \begin{bmatrix} \mathbf{v} & \mathbf{u}-\mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}$

The area of the parallelogram with vertices $\{\mathbf{o}, \mathbf{v}, \mathbf{u}, \mathbf{u}-\mathbf{v}\}$ = The area of the parallelogram with vectors $\mathbf{v}, \mathbf{u}-\mathbf{v}$ as adjacent sides.

We know that area of the parallelogram with vectors $\mathbf{v}, \mathbf{u}-\mathbf{v}$ as adjacent sides = absolute value of determinant of the matrix with $\mathbf{v}, \mathbf{u}-\mathbf{v}$ as columns $= |det(B)|$.

Hence, area of the parallelogram with vectors $\mathbf{v}, \mathbf{u}-\mathbf{v}$ as adjacent sides = $|det(B)| = |(1*(-2)) - (4*2)| = 10$ sq. units.

We know that, area of triangle with vertices $\{\mathbf{u}, \mathbf{v}, \mathbf{u}-\mathbf{v}\}$ +
area of triangle with vertices $\{\mathbf{o}, \mathbf{u}-\mathbf{v}, \mathbf{v}\}$
= area of the parallelogram with vertices $\{\mathbf{o}, \mathbf{v}, \mathbf{u}, \mathbf{u}-\mathbf{v}\}$
= area of the parallelogram with vectors $\mathbf{v}, \mathbf{u}-\mathbf{v}$ as adjacent sides
= 10 sq. units.

The triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u}-\mathbf{v}$ is congruent to the triangle with vertices $\{\mathbf{o}, \mathbf{u}-\mathbf{v}, \mathbf{v}\}$ by side-side-side congruence. Hence, their areas are equal.

So, (area of triangle with vertices $\mathbf{u}, \mathbf{v}, \mathbf{u}-\mathbf{v}) * 2 = 10$ sq. units.

Therefore, area of triangle with vertices $\{\mathbf{u}, \mathbf{v}, \mathbf{u}-\mathbf{v}\} = 5$ sq. units.

15. (2 points) The determinant of the following matrix can be computed as a sum of 120 (5!) terms.

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

State true or false with an appropriate explanation: All the 120 terms in the determinant will be 0.

Solution: True.

Explanation :

Claim : The determinant any $n \times n$ matrix is the sum of $n!$ terms,$\forall n \geq 1$ where $n \in \mathbb{Z}$.

Proof: We use principle of mathematical induction to prove the claim.

Base case: The claim is true for $n = 1$ because any matrix with only one element will have only one term in its determinant.

Induction Hypothesis: Assume the claim is true for $n = k$ where $k \geq 1$ and $k \in \mathbb{Z}$.

Induction Step: To prove the claim is true for $n = k + 1$.

Let B be a $(k+1) \times (k+1)$ matrix. Then $|B| = \sum_{l=1}^{k+1} b_{1l}(C_{1l})$ where b_{ij} is the $(ij)^{th}$ entry of B and C_{ij} is its corresponding co-factor.

Each co-factor C_{ij} is the determinant of a $k \times k$ matrix and so from induction hypothesis we can say it is a sum of $k!$ terms.

We can see that $|B|$ is the linear combination of $(k + 1)$ co-factors each of which is in itself a sum of $k!$ terms, so $|B|$ is the sum of $(k + 1)(k!)$ i.e. $(k + 1)!$ terms.

Hence, proved that the claim is true for $n = k + 1$.

By principle of mathematical induction the claim is true $\forall \mathbf{n} \geq \mathbf{1}$.

Now consider $|A| = det \left(\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix} \right)$.

Apply row operation ($R_1 : R_1 - R_2$) and from property(5) of determinants we know that row operations do not change the determinant of a matrix.

$$\text{So, } |A| = det\left(\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}\right) = det\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}\right).$$

Hence, $|A| = 0(C_{11}) + 0(C_{12}) + 0(C_{13}) + 0(C_{14}) + 0(C_{15})$, where C_{ij} is the co-factor corresponding to the $(ij)^{th}$ entry of A.

We can observe that each C_{ij} is the determinant of a 4×4 matrix, so from our **claim** we can say that each C_{ij} is a sum of 4! i.e. 24 terms.

We observe that $|A|$ is sum of 5 co-factors each of which is multiplied by 0 , so $|A|$ is sum of 5 x 24 i.e. 120 terms each of which is 0.

Therefore, all the 120 terms in the determinant of A are 0.