CS6015 : Linear Algebra and Random Processes
Tutorial #10

Deadline: None

- While this is optional, it is strongly recommended that students solve this tutorial.
- Questions marked with an asterisk are hard questions which thoroughly test your concepts.

Name :
Roll Number :

1. In India, 30 percent of the population has a blood type of O+, 33 percent has A+, 12 percent has B+, 6 percent has AB+, 7 percent has O-, 8 percent has A-, 3 percent has B-, and 1 percent has AB-. If 15 Indians are chosen at random, what is the probability that 3 have a blood type of O+, 2 have A+, 3 have B+, 2 have AB+, 1 has O-, 2 have A-, 1 has B-, and 1 has AB-?

   Solution:

2. You have gone to a casino with the hopes of winning a fortune. You decide to play a special game of throwing dice. In this game, you throw 5 dice consecutively. If at least two of the dice have the same number, you win the normal prize. Also if you decide on a number and say it before, and that is the number that is present on at least three of the dice, you win the jackpot. For example, if you say 6 before throwing the 5 dice, and then 6 comes up on atleast 3 of the dice, you win the jackpot. Find:

   1. the probability of winning a normal prize
   2. the probability of winning a jackpot
   3. the probability of winning a normal prize with the number on 2 of the dice being 1

3. A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses at which the experiment terminates.

   Solution:

4. A lost tourist arrives at a point with 3 roads. The first road brings him back to the same point after 1 hours of walk. The second road brings him back to the same point after 6 hours of travel. The last road leads to the city after 2 hours of walk. There are no signs on the roads. Assuming that the tourist chooses a road equally likely at all times, (because the tourist doesn't remember which route was taken in any previous scenario), what is the mean time until the tourist arrives to the city.

Solution:

5. A family has 5 natural children and has adopted 2 girls. Each natural child has equal probability of being a girl or a boy, independent of the other children. Let X denote the random variable which represents the number of girl children in the family. What is the expectation of X?

Solution:

6. Let X be a random variable that takes integer values from -9 to 9 with uniform probability. Find the expectation of Random variable Y when

   1. Y = X mod (3)
   2. Y = $X^2 - |X|$

Solution:

7. Let X be a random variable with probability distribution function given by $f_X(x) = \frac{1}{2}e^{-|x|}$ and Y be a random variable be a random variable with probability distribution function given by $f_Y(y) = 3y^2$ for $0 \leq y \leq 1, 0$ otherwise. Find the expectation of a random variable Z given as:

   1. Z = X + Y
   2. Z = X * Y

8. Let X and Y be jointly continuous random variable with joint PDF :
   $f_{X,Y}(x,y) = \begin{bmatrix} 6e^{(-2x+3y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{bmatrix}$

   1. Are X and Y independent?
   2. Find E[Y—X>2]
   3. Find P(X>Y)

9. X and Y are jointly continuous with joint pdf
$$f(x,y) = \begin{bmatrix} cxy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{bmatrix}$$

   1. Find c
   2. Find the marginal PDFs of X and Y.

10. * Let N be the RV denoting the requests per day a web server at IIT-M receives with a distribution of N $\sim$ Poisson(λ). The request is either from a human with probability p or from a bot with probability (1-p). Let X be the RV denoting number of requests from humans per day, and Y be the RV denoting number of requests from bots per day. Find the Joint probability of X and Y.

    Solution:

11. * Let there be an experiment 1 where two fair dice are thrown simultaneously. The experiment is considered a success, if the sum of the results of the two dice is a prime number. Let X denote the random variable representing the number of independent trials of the experiment required to achieve the first success. Find the Probability Distribution of X.
    Let there be experiment 2 as follows. A biased coin with probability p of coming heads is tossed once. Let Y be the RV that denotes the result of the experiment. Given these two experiments, find the joint probability distribution of X and Y in a tabular form (take X $\leq$ 5).
    Also find the expectation of X.

    Solution:

12. It is estimated that 50 percent of emails are spam emails. Some software has been applied to filter these spam emails before they reach your inbox. A certain brand of software claims that it can detect 99 percent of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5 percent.
    Now if an email is detected as spam, then what is the probability that it is in fact a non-spam email?

    Solution:

13. * We know, MGF of RV X is
    $M_X(s) = E[e^{sX}]$
    Remember the Taylor series expansion for e^x , $\forall x \in \mathbb{R}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \Sigma_{k=0}^{\infty} \frac{x^k}{k!}$

Therefore we can write,
$e^{sX} = \Sigma_{k=0}^{\infty} \frac{(sX)^k}{k!} = \Sigma_{k=0}^{\infty} \frac{X^k s^k}{k!}$

Finally we have,
$M_X(s) = E[e^{sX}] = \Sigma_{k=0}^{\infty} E[X^k] \frac{s^k}{k!}$

Therefore if we have the Taylor Series expansion of $M_X(s)$, the coefficient of the term $\frac{s^k}{k!}$ is the k^{th} moment of RV X.

Let Y be a Uniform(0,1) Random Variable. What is the moment generating function of Y? Using the result from above, find the k^{th} moment of Y.

Solution:

14. The profit for a company has a moment-generating function M(t) = $\frac{0.15}{0.15-t}, \forall t < 0.15$. The company pays a bonus equal to 20 percent of the profit. What is the moment generating function of the bonus?

Solution:

15. * Let $X \sim Exponential(\lambda)$. Find the MGF and the generic form for kth moment of X. Hint: You might need this result:

$$\sum_{k=0}^{k} (\frac{a}{b})^k = \frac{b}{b-a} \ , \forall \ |\frac{a}{b}| < 1$$

Solution:

16. A random person in Chennai has a 1 percent chance of having cancerous tumour. The popular test in Chennai for cancer correctly classifies 80 percent of cancerous tumour and 90 percent of benign tumour. Based on this information, find out

    1. Probability that the person has cancer given a positive cancer result in the test.
    2. Probability that the person has cancer given a negative result.

Solution:

17. A bulb factory has three machines A,B and C. On any given day A produces 40 percent of the bulbs, B produces 35 percent of the bulbs, and C produces 25 percent. Machine A produces N_a defective bulbs which follows the distribution Poisson($\lambda_a = 10$), Machine B produces N_a defective bulbs which follows the distribution Poisson($\lambda_b = 4$), Machine A produces N_c defective bulbs which follows the distribution Poisson($\lambda_c = 1.5$). Find the following:

    1. Probability of selecting a bulb at random and it being defective.
    2. Probability of a randomly chosen defective bulb being made from Machine A.
    3. Probability of a randomly chosen non-defective bulb being made from Machine C.

Solution:

18. Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side is colored black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Solution:

19. There is a fish packaging industry that classifies a fish into categories Fish A and Fish B based on the length of the fish. Let X be the random variable that denotes the length of Fish A following the Gaussian distribution with mean = 14 inches and standard deviation of 2.1 inch. Similarly random variable Y denotes the length of Fish B following a Gaussian distribution with mean 34 inches and standard deviation of 3.3 inch. Assuming any fish is equally likely to be either Fish A or Fish B, find:

    1. The expression of the probability of fish being Fish A given its length is n inches, in terms of n.
    2. The expression of the probability of fish being Fish B given its length is n inches, in terms of n.

20. * The number of miles that a particular car can run before its battery wears out is exponentially distributed with an average of 10,000 miles. The owner of the car needs to take a 5000-mile trip. What is the probability that he will be able to complete the trip without having to replace the car battery?

Solution:

Please solve the following questions from the book "Introduction To Probability, 2nd edition, by Dimitri P. Bertsekas and John N. Tsitsiklis ", Chapter 3:

21. Question 19

22. Question 29

23. Question 32

24. Question 35

25. Let X and Y be independent r.v.s taking values 1, 2, 3, 4, each with probability 1/4. Let Z = max(X, Y). Find

    1. joint distribution of X and Z
    2. E(X) and E(Y)
    3. Cov(X, Z)

26. Let X $\sim$ Unif [0, 3] and Y = $(X-1)^2$. Find the distribution, density and mean of Y.

27. We operate communication channel where we are trying to transmit a binary signal K which can have 2 levels (-1 or +1). It is known that the signal gets corrupted in-transit by a Gaussian noise. The received signal Y at the receiver is hence a continuous signal, Y = K + W, where W $\sim$ N(0,1). Answer the following questions based on this.

    1. Write down the closed form expression for $\mathrm{P}_{K|Y}$ (k | y) using mixed Bayes theorem (when one r.v is continuous, while other is discrete).
    2. Find the conditional PDF of Y | K, and plot P(Y | K = +1) and P(Y | K = -1).
    3. Finally, find P (K = 1 | Y = y). Also, draw/plot the joint PDF of X and Y assuming that X follows a uniform distribution.

28. You have two fair coins. You toss the first coin till you get 5 heads. Let X be the random variable indicating the number of tosses required to get 5 heads. Let x be the value of X in one such experiment. You now toss the second coin x times. Let Y be the random variable indicating the number of heads in this experiment. Plot $p_{X,Y}(x,y)$. (Disclaimer: I have not really though about the answer to this question).

29. A factory produces biased coins. Let P be the random variable indicating the probability of heads of a coin selected from this factory (support of P is of course the interval $[0, 1]$).
    (a) What would be an appropriate distribution for P? (Hint: A 1992 film starring Anil Kapoor and Madhuri Dixit).
    (b) Once you have selected a coin from the factory you toss it n times. Let Y be the random variable indicating the number of heads in these n tosses. Plot $f_{X,Y}(x, y)$.
    (c) Plot $p_Y(y)$, i.e., the marginal distribution of Y
30. Consider the problem of breaking a stick twice and the random variables X and Y as we defined in class. Plot $f_{XY}(x, y)$.