Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

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Name and Signature

1. (1 point) Have you read and understood the honor code?

> **Solution:** Yes

Concept: Linear Combinations

2. (2 points) Consider the vectors $[x, y]$, $[a, b]$ and $[c, d]$.

   (a) Express $[x, y]$ as a linear combination of $[a, b]$ and $[c, d]$.

   > **Solution:** $\begin{bmatrix} x \\ y \end{bmatrix} = m \begin{bmatrix} a \\ b \end{bmatrix} + n \begin{bmatrix} c \\ d \end{bmatrix}$ where $m, n \in \mathbb{R}$

   (b) Based on the expression that you have derived above, write down the condition under which $[x, y]$ cannot be expressed as a linear combination of $[a, b]$ and $[c, d]$. (Must: the condition should talk about some relation between the scalars a, b, c, d, x and y)

   > **Solution:** $\begin{bmatrix} x \\ y \end{bmatrix}$ can't be expressed as a linear combination when:
   > $\begin{bmatrix} c \\ d \end{bmatrix} = m \begin{bmatrix} a \\ b \end{bmatrix} + n \begin{bmatrix} x \\ y \end{bmatrix}$ or $\begin{bmatrix} a \\ b \end{bmatrix} = m \begin{bmatrix} c \\ d \end{bmatrix} + n \begin{bmatrix} x \\ y \end{bmatrix}$ where $m, n \in \mathbb{R}$.

Concept: Elementary matrices

3. (1 point) Consider the matrix $E_{r+\alpha q}$ that represents the elementary row operation of adding a multiple of α times row q to row r.

Under what conditions is $E_{r+\alpha q}$

   (a) upper triangular?

   > **Solution:** Only if the elimination performed on row r follows the condition: $r > q$.

(b) lower triangular?

> **Solution:** Only if the elimination performed on row r follows the condition: $q > r$.

4. (1 point) Let $E_1, E_2, E_3, \ldots, E_n$ be n elementary matrices. Let $(i_1, j_1), (i_2, j_2), \ldots (i_n, j_n)$ be the position of the non-zero off-diagonal element in each of these elementary matrices. Further, *if* $k \neq m$ *then* $(i_k, j_k) \neq (i_m, j_m)$ (*i.e.*, no two elementary matrices in the sequence have a non-zero off-diagonal element in the same position). Prove that the product of these n elementary matrices will have all diagonal entries as 1. (Proving this will help you understand why the diagonal elements of L are always equal to 1.)

> **Solution:** Let A be a 3×3 square matrix on which we would like to perform elimination. E_1, E_2, E_3, and E_4 be 4 elementary matrices. We represent our elementary matrices as follows: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ w & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,
> $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix}$. Note here that $(i_1, j_1) = -x, (i_2, j_2) = z, (i_3, j_3) = w, (i_4, j_4) = y$.
>
> Our final elementary matrix $E = E_4 \cdot E_3 \cdot E_2 \cdot E_1 = \begin{bmatrix} 1 & 0 & 0 \\ w - x & 1 & 0 \\ y - zx & z & 1 \end{bmatrix}$.
>
> We can clearly see why we always get the principal diagonal to be all 1 in our final elementary matrix E. If we do matrix multiplication using linear combination of rows (Method 3), we will always get matrix E of the form: $\begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \lambda_2 & \lambda_3 & 1 \end{bmatrix}$ where λ_1, λ_2, and λ_3 equals some combination of our row operations.
>
> Hence, we can say with confidence that product of n elementary matrices $E_1, E_2, E_3, \ldots, E_n$ will always result in a principal diagonal of all ones in the final elementary matrix E.

Concept: Inverse

5. (½ point) If A is a square invertible matrix then prove that the inverse of $A^\top$ is $A^{-1\top}$

Solution: Assuming A is invertible, we have:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I \qquad (\because B^T A^T = (AB)^T)$$

and,

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I \qquad (\because B^T A^T = (AB)^T)$$

This proves that the inverse of A^T is $(A^{-1})^T$.

6. (2 points) Prove that a $n \times n$ matrix A is invertible if and only if Gaussian Elimination of A produces n non-zero pivots.

Solution:
Before we prove, we shall mention a known fact: Elementary matrices are lower triangular matrices having all elements in the principal diagonal equal to 1.

Proof (the if part): If a $n \times n$ matrix A produces n non-zero pivots after Gaussian elimination, then A is invertible.

Let us assume that final elementary matrix E is a product of n elementary matrices, where each elementary matrix performs a row operation. From LU decomposition, we know that $A = LU$ where $L = E^{-1}$. We also know that U has n non-zero pivots, which makes U an upper triangular and invertible matrix. Since both L and U are invertible, their matrix product A must be invertible.

Proof (the only if part): If a $n \times n$ matrix A is invertible, then A produces n non-zero pivots.

We can produce an argument similar to the 'if' part. If A is invertible, then a unique solution exists through back-substitution for $x = U^{-1}c$. For back-substitution to occur, U must have n non-zero pivots, which means Gaussian elimination of A, i.e. EA, produces n non-zero pivots in U.

7. (1 point) If A is a $n \times n$ matrix then what is the cost of:
   (a) Computing A^{-1}

Solution: During elimination of matrix A, we only change the zeros below 1 in each of the columns of the identity matrix. Each column k has only $n-k$ changes, thereby, adding to $\frac{(n-k)^2}{2}$. Summing this fraction to k columns, elimination cost is approx. $\frac{n^3}{6}$. Further, we add the cost of elimination to the A matrix, which is $\frac{n^3}{3}$. Finally, the cost of back substitution: $\frac{n^3}{2}$. In total, the cost of computing A^{-1} is $O(n^3)$.

(b) Computing $A^{-1}b$

Solution: Cost of computing $A^{-1}b$ is $O(n^2)$

Concept: LU factorisation

8. (1 ½ points) In the lecture, we saw that once we do LU factorisation, we can solve $A\mathbf{x} = \mathbf{b}$ by solving two triangular systems $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$.

(a) Prove that $L\mathbf{c} = \mathbf{b}$.

Solution: We know that $U\mathbf{x} = \mathbf{c}$. Substituting $\mathbf{c}$ in $L\mathbf{c} = \mathbf{b}$ gives us: $LU\mathbf{x}$, which is nothing but $A\mathbf{x} = \mathbf{b}$.

(b) What is the cost of solving a triangular system (say $L\mathbf{c} = \mathbf{b}$ or $U\mathbf{x} = \mathbf{c}$)?

Solution: In any triangular system, we perform matrix computations only for non-zero values (above the diagonal in case of upper triangular matrix or below the diagonal in case of lower triangular matrix). Therefore, number of non-zero values in each row of a triangular matrix will be of the form: $\sum_{i=1}^{n} i$, which is equal to $\frac{n(n+1)}{2}$. Hence, the cost effectively becomes $O(n^2)$.

(c) Based on the above results can you comment on the utility of LU factorisation?

Solution:

One time cost of LU factorisation:

Suppose, we call each multiplication-division, one operation. For the first column, there will be n additions and multiplications for $n-1$ rows. Hence, total operations for first column is $n \cdot (n-1)$ operations. Similarly, for second column, total operations $= (n-1) \cdot (n-2)$. Therefore, for each stage i,

$$\sum_{i=1}^{n} (n-i)(n-i+1) = O(\frac{n^3}{3}) \approx O(n^3)$$

Recurring cost of solving $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$:

We know that the cost for $L\mathbf{c} = \mathbf{b}$ or $U\mathbf{x} = \mathbf{c}$ is $O(n^2)$. Therefore, the total cost is $O(2n^2) \approx O(n^2)$.

Utility:

In many engineering applications, when we solve $Ax = b$ system of linear equations, the matrix A remains unchanged, while the RHS vector b varies. The key idea behind LU factorization is to separate row operations on A into a separate matrix L which could be further used to operate on new b vectors. This reduces the computations required to calculate solution $\mathbf{x}$ drastically.

9. (2 points) Consider the following system of linear equations. Find the LU factorisation of the matrix A corresponding to this system of linear equations. Show all the steps involved. (this is where you will see what happens when you have to do more than 1 permutations).

$$\begin{aligned} x + y - 2z &= -3 \\ w + 2x - y &= +2 \\ w - 4x - 7y - z &= -19 \\ 2w + 4x + y - 3z &= -2 \end{aligned}$$

Solution: Let our $A = \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 2 & -1 & 0 \\ 1 & -4 & -7 & -1 \\ 2 & 4 & 1 & -3 \end{bmatrix}$.

First, we switch Row1 and Row2 using permutation matrix.

$$P_1 A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & -4 & -7 & -1 \\ 2 & 4 & 1 & -3 \end{bmatrix}$$

Secondly, we subtract Row1 from Row3.

$$E_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P_1 A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & -6 & -6 & -1 \\ 2 & 4 & 1 & -3 \end{bmatrix}$$

Thirdly, we subtract 2 times Row1 from Row3.

$$E_2E_1P_1A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} E_1P_1A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & -6 & -6 & -1 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

Fourthly, we add 6 times Row2 to Row3.

$$E_3E_2E_1P_1A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} E_2E_1P_1A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & -13 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

Finally, we switch Row3 and Row4.

$$P_2E_3E_2E_1P_1A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} E_3E_2E_1P_1A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -13 \end{bmatrix}$$

From the upper triangular matrix on the RHS, we know that $P_2E_3E_2E_1P_1A = U$.

$$\begin{aligned} P_2E_3E_2E_1P_1A &= U \\ A &= P_1^{-1}E_1^{-1}E_2^{-1}E_3^{-1}P_2^{-1}U \\ A &= LU \\ A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -6 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -13 \end{bmatrix} \end{aligned}$$

We see that matrix L is not lower triangular. We will rectify this by using the Gaussian elimination method: $PA = LU$.

$$P_2P_1A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 2 & 4 & 1 & -3 \\ 1 & -4 & -7 & -1 \end{bmatrix}$$

We perform three row operations: E_1 = Subtract 2 times Row1 from Row3, E_2 = Subtract Row1 from Row 4, E_3 = Add 6 times Row2 from Row 4.

$$E_3E_2E_1P_2P_1A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&6&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\-2&0&1&0\\0&0&0&1\end{bmatrix}P_2P_1A$$
$$E_3E_2E_1P_2P_1A = \begin{bmatrix}1&2&-1&0\\0&1&1&-2\\0&0&3&-3\\0&0&0&-13\end{bmatrix}$$
$$P_2P_1A = E_1^{-1}E_2^{-1}E_3^{-1}U$$
$$PA = LU$$
$$\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\2&0&1&0\\1&-6&0&1\end{bmatrix}\begin{bmatrix}1&2&-1&0\\0&1&1&-2\\0&0&3&-3\\0&0&0&-13\end{bmatrix}$$

10. (1 ½ points) For a square matrix A:

(a) Prove or disprove: LU factorisation is unique.

Solution: Consider A to be a 3×3 matrix. Let A have two different LU factorizations, L_1U_1 and L_2U_2. Then, we can say the following:

$$A = L_1U_1 = L_2U_2$$
$$\implies L_1U_1 = L_2U_2$$
$$\implies U_1U_2^{-1} = L_2L_1^{-1}$$

Here, we see that LHS and RHS will always be a upper triangular matrix and lower triangular matrix respectively. But, for both of them to be equal, LHS and RHS must result in the same diagonal matrix because only the diagonal matrix can be both upper and lower triangular matrix. Since, the principal diagonal of L matrix is all ones, the diagonal matrix must be the identity matrix. Therefore,

$$U_1U_2^{-1} = L_2L_1^{-1} = I$$
$$\implies U_1 = U_2 \text{ and } L_1 = L_2$$

This proves that LU factorization is unique.

(b) Prove or disprove: LDU factorisation is unique.

Solution: Consider A to be a 3×3 matrix. Let A have two different LDU factorizations, $L_1D_1U_1$ and $L_2D_2U_2$. Then, we can say the following:

$$L_1D_1U_1 = L_2D_2U_2$$
$$L_1L_2^{-1}D_1 = D_2U_2U_1^{-1}$$

Since L_1 and L_2^{-1} are both lower triangular, their product is also lower triangular with principal diagonal all ones. This leads to $L_1L_2^{-1}D_1$ to be lower triangular. Similarly, $D_2U_2U_1^{-1}$ will be upper triangular matrix. We know that matrices D_1 and D_2 have non-zero principal diagonals. Hence, LHS = RHS = I.

$$L_1L_2^{-1}D_1 = D_2U_2U_1^{-1} = I$$
$$\implies D_1 = D_2,\ U_1 = U_2 \text{ and } L_1 = L_2$$

This proves that LDU factorization is unique.

11. (1 ½ points) Consider the matrix A which factorises as:

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 5 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\0 & 1 & 5\\0 & 0 & 1\end{bmatrix}$$

Without computing A or A^{-1} argue that

(a) A is invertible (I am looking for an argument which relies on a fact about elementary matrices)

Solution: From above, we know that $L = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 5 & 1\end{bmatrix}$. LU decomposition of matrix A gives us $EA = U$. We know that elementary matrices are invertible. Hence, we can say $A = E^{-1}U = LU$. Since L essentially takes U back to A, A is invertible.

(b) A is symmetric (convince me that $A_{ij} = A_{ji}$ without computing A)

Solution: Let A be a symmetric matrix with its LU decomposition, $A = LU$. We know,

$$A = A^T$$
$$\implies LU = (LU)^T$$
$$\implies LU = U^TL^T$$

Since the factorization is unique,

$$U = L^T$$

In the question, our matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ and matrix $U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$. We can see that $U = L^T$. Hence, our matrix A is symmetric.

(c) A is tridiagonal (again, without computing A convince me that all elements except along the 3 diagonals will be 0.)

Solution: From question 11b, we know that matrix A is symmetric and $U = L^T$. Hence, values below principal diagonal in A will be same as values below principal diagonal in L and values above principal diagonal in A will be same as values above principal diagonal in U. Hence, A is a tridiagonal symmetric matrix: $\begin{bmatrix} a & 2 & 0 \\ 2 & b & 5 \\ 0 & 5 & c \end{bmatrix}$ where a, b, and c are all non-zero.

Concept: Lines and planes

12. (1 ½ points) Consider the following system of linear equations

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 &= 1 \\ a_2x_2 + b_2y_2 + c_2z_2 &= 0 \\ a_3x_3 + b_3y_3 + c_3z_3 &= -1 \end{aligned}$$

Each equation represents a plane, so find out the values for the coefficients such that the following conditions are satisfied:

1. All planes intersect at a line
2. All planes intersect at a point
3. Every pair of planes intersects at a different line.

Solution:

1. $a_2 = a_1 + a_3, b_2 = b_1 + b_3, c_2 = c_1 + c_3$ where $a_1 \neq a_3, b_1 \neq b_3$, and $c_1 \neq c_3$.
2. All real-valued coefficients for which row vectors $\{\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 & c_3 \end{bmatrix}\}$ are linearly independent.

3. This happens when any of the row vector $\{\begin{bmatrix}a_1 & b_1 & c_1\end{bmatrix}, \begin{bmatrix}a_2 & b_2 & c_2\end{bmatrix},$ $\begin{bmatrix}a_3 & b_3 & c_3\end{bmatrix}\}$ is a linear combination of the other two vectors.

13. (1 ½ points) Starting with a first plane $u + 2v - w = 6$, find the equation for
    (a) the parallel plane through the origin.

    Solution: $u + 2v - w = 0$

    (b) a second plane that also contains the points (6,0,0) and (2,2,0).

    Solution: $u + 2v = 6$

    (c) a third plane that meets the first and second in the point (4, 1, 0).

    Solution: $2u + 4v - w = 12$

Concept: Transpose

14. (2 points) Consider the transpose operation.
    (a) Show that it is a linear transformation.

    Solution: Let A, B, and C be matrices of size mxn. Let a_{ij}, b_{ij}, and c_{ij} represent elements of A, B, and C matrices respectively. Also, $C = A + B$ and $c_{ij} = a_{ij} + b_{ij}$. Let $T(Q) = Q^T$.

$$\begin{aligned} T(A+B) &= (A+B)^T \\ &= [(a_{ij}) + (b_{ij})]^T \\ &= [(a_{ij} + b_{ij})]^T \\ &= (c_{ij})^T \\ &= c_{ji} \\ &= (a_{ji} + b_{ji}) \\ &= a_{ji} + b_{ji} \\ &= a_{ij}^T + b_{ij}^T \\ &= A^T + B^T \end{aligned}$$

    Hence, $(A + B)^T = A^T + B^T$. Similarly, for scalar multiplication, we prove:

$$\begin{aligned} T(mA) &= (mA)^T & (m \in \mathbb{R}) \\ &= mA^T \end{aligned}$$

    Therefore, transpose of a matrix is a linear transformation.

(b) Find the matrix corresponding to this linear transformation.

Solution: The matrix which 'transposes' all other matrices is not a linear transformation because linear transformation are only defined on vector spaces. Ideally, if there's a matrix which represents transpose operation, then it will be a permutation matrix P. But, if we permute rows, we won't be expressing transpose. Let's say that we would like to find a transpose of matrix A. For instance, element a_{11} of a $n \times n$ matrix A when transposed results in the same position a_{11}. On the other hand, when we multiply A with the permutation matrix P, element a_{11} will result in a_{k1}. If $k = 1$, then our transpose operation with P is wrong since row1 will stay in place. If $k < n$, then, yet again, our transpose operation is wrong since we're shifting the element a_{11}.

Hence, in order to represent a transpose operation, we write the $n \times n$ matrix A as a column vector of n^2 entries. We then multiply the column vector with a permutation matrix P of $n^2 \times n^2$ shape. The permuted column vector from this matrix operation can be reshaped into $n \times n$ matrix, which essentially is the transpose matrix of A.

A simple example of the above method can be seen below,

Let $A = \begin{bmatrix} 1 & -5 \\ 2 & -3 \end{bmatrix}$. Let $A_{col} = \begin{bmatrix} 1 \\ -5 \\ 2 \\ -3 \end{bmatrix}$ represent the column vector version of A.

Also, let $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then,

$$\begin{aligned} A_{new} &= P \cdot A_{col} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ -5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -5 & -3 \end{bmatrix} && \text{(Restructuring } A_{new} \text{ into } 2 \times 2 \text{ matrix)} \\ &= A^T \end{aligned}$$