CS6015 : Linear Algebra and Random Processes
Tutorial #3

Deadline: None

- This tutorial deals with the topics already covered in class till 1st October 2020 (mainly dealing with lectures 8, 9, 10).
- While this is optional, it is strongly recommended that students solve this tutorial.

NAME :
ROLL NUMBER :

1. Reduce matrices A and B to echelon form, to find their ranks. Which variables are free?

   (i) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

   Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

   Solution:

2. Under what conditions on b_1 and b_2 does $Ax = b$ have a solution?

   $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

   Find two vectors in the nullspace of A, and the complete solution to $Ax = b$.

   Solution:

3. Find the complete solutions of the following:

   a. $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$

   b. $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Solution:

4. Suppose $Ax = b$ has infinitely many solutions, is it possible for $Ax = b'$ (where $b' \neq b$) to have:
a. only one solution?
b. no solution?
c. infinite solutions?

Solution:

5. Suppose column 4 of a 3 x 5 matrix is all 0s. Then x_4 is certainly a _____ variable.

Solution:

6. Construct a matrix whose nullspace consists of all combinations of (2,2,1,0) and (3,1,0,1).

Solution:

7. Construct a matrix whose column space contains (1,1,0) and (0,1,1) and whose nullspace contains (1,0,1) and (0,0,1).

Solution:

8. Can a 3 x 3 matrix ever have a nullspace that equals its column space? If yes, give an example when it's possible. If not, argue why.

Solution:

9. Provide counter examples to show that the following statements are false
(a) A and A^T have the same nullspace.
(b) A and A^T have the same free variables.
(c) If R is the reduced form of matrix A, rref(A) then R^T is rref(A^T).

Solution:

10. Find the dimension and construct a basis for the four subspaces associated with each of the matrices:
(i) $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution:

11. Describe the four subspaces in three-dimensional space associated with:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:

12. A is an m x n matrix of rank r. Under what conditions on m, n, and r do the following hold true?
(a) A has a two-sided inverse. That is, $AA^{-1} = A^{-1}A = I$
(b) $Ax = b$ has infinitely many solutions for every b

Solution:

13. Given that $Ax = b$ always has at least one solution, show that the only solution to $A^T y = 0$ is $y = 0$. (Hint: What is the rank?)

Solution:

14. Find a matrix A that has V as its row space, and a matrix B that has V as its nullspace, if V is the subspace spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

Solution:

15. Given a 3 x 3 invertible matrix A, what are the bases for the four subspaces for A? Also give the bases for the four subspaces of the 3 x 6 matrix B = [A A].

Solution:

16. Suppose we exchange the first two rows of a matrix A, does it change any of the four subspaces? Which ones will remain the same?

Solution: