Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

1. (1 point) Have you read and understood the honor code?

> **Solution:**
> Yes

Eigenstory: Special Properties

2. (1 point) Prove that for any square matrix A the eigenvectors corresponding to distinct eigenvalues are always independent.

> **Solution:**
> Lets say x_1, x_2 be the eigenvectors corresponding to distinct eigenvalues λ_1, λ_2.
> We say linear combination of these eigen vectors produces zero, that is they are dependent.
> Therefore, $c_1x_1 + c_2x_2 = 0$ ..(1)
> $c_1Ax_1 + c_2Ax_2 = 0$ Multiplying by A.
> $c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0$ ..(2) Since $Ax = \lambda x$
> subtract λ_2 times (1) from (2)
> $c_1(\lambda_1 - \lambda_2)x_1 = 0$
> But, $\lambda_1 \neq \lambda_2$ and $X_1 \neq 0$, Hence c_1 is forced to be 0.
> Similarly $c_2 = 0$, therefore, only trivial combination is zero.
> We can extend this argument to k eigen vectors corresponding to distinct k eigen values.
> Therefore, For any square matrix A the eigenvectors corresponding to distinct eigenvalues are always independent.

3. (2 points) Prove the following.

   (a) The sum of the eigenvalues of a matrix is equal to its trace.

Solution:

Let A be the matrix $\begin{bmatrix} a_{11} \; ......a_{1n} \\ .. \\ .. \\ \vdots \\ a_{n1} .....a_{nn} \end{bmatrix}$

Characteristic polynomial to find eigenvalues of A is $|A - \lambda I|$

We know that if we have a polynomial $x^n + b_{n-1}x^{n-1} + ... + b_1 x + b_0$, then $(-1)^{n-1}b_{n-1}$ is the sum of the roots of this polynomial. In our case, the polynomial is $det(A - \lambda I)$ and roots are the eigenvalues, we have $(-1)^{n-1}b_{n-1} = \lambda_1 + \lambda_2 + +\lambda_n$.

By defination of charcteristic polynomial of $n \times n$ Matrix A ,

$det(A - \lambda I) = (-\lambda)^n + trace(A)(-\lambda)^{n-1} + ... + det(A)$

The roots of n-order polynomial are $\lambda_1, \lambda_2..., \lambda_n$

Therefore,By comparing the coefficient trace(A) is equal to the sum of eigenvalues of A.

(b) The product of the eigenvalues of a matrix is equal to its determinant.

Solution:

A be the matrix $\begin{bmatrix} a_{11} \; ......a_{1n} \\ .. \\ .. \\ \vdots \\ a_{n1} .....a_{nn} \end{bmatrix}$

Characteristic polynomial to find eigenvalues of A is $|A - tI|$

By defination of charcteristic polynomial of $n \times n$ Matrix A ,

$det(A - tI) = det(A) + ... + trace(A)(-t)^{n-1} + (-t)^n$

The roots of n-order polynomial are $\lambda_1, \lambda_2..., \lambda_n$

$P(t) = (\lambda_1 - t)(\lambda_2 - t)...(\lambda_n - t)$

$P(t) = \prod_{i=1}^{n} \lambda_i + ... + \sum_{i=1}^{n} \lambda_i(-\lambda)^{n-1} + (-\lambda)^n$

Therefore, $det(A) = \prod_{i=1}^{n} \lambda_i$ which is product of eigen values of a matrix.

4. (2 points) What is the relationship between the rank of a matrix and the number of non-zero eigenvalues? Explain your answer.

Solution: Let us first state the proposition that if A and C are non-singular matrices, then:

$Rank(ABC) = Rank(B)$

Proof: Let $D = ABC$. From matrix multiplications, $Rank(D) \leq Rank(AB) \leq Rank(B)$. Also since $B = A^{-1}DC^{-1}$, we have $Rank(B) \leq Rank(A^{-1}D \leq Rank(D)$

Consider an $n \times n$ matrix A with real eigenvalues. Diagonalising A, we get $A = PDP^{-1}$
Applying the preceding proposition, we get $Rank(A) = Rank(D)$. Suppose $Rank(A) = k \leq n$
Thus $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $v_1, v_2, \ldots, v_k$ are the eigenvalues and orthonormal eigenvectors of A respectively.
Let $v_1, v_2, \ldots, v_k$ correspond to the orthonormal eigenvectors and $v_{k+1}, \ldots, v_n$ correspond to eigenvectors of 0 eigenvalues of A
We define ϵ to be the span of $v_1, \ldots, v_k$. That is:
$\epsilon{:=}c_1v_1 + \cdots + c_kv_k : c_1, \ldots, c_k \in \mathbb{R}$

We want to prove that $\epsilon = C(A)$, which would be to prove that span of eigenvectors of A = Span of column vectors of A. That is, $Rank(A)$ = Number of non-zero eigenvalues.

(a) We write Ax as:
$c_1v_1 + \cdots + c_kv_k = Ax$, where $x = \sum_{i=1}^{k}(\frac{c_i}{\lambda_i}v_i)$. Thus $\epsilon \subseteq C(A)$
If $k = n$, this proof suffices since $v_1, \ldots, v_k$ becomes an orthonormal basis for $\mathbb{R}^n$

(b) If $k < n$:
We write every $x \in \mathbb{R}^n$ as $a_1v_1 + \cdots + a_nv_n$, such that:
$Ax = \sum_{j=1}^{k} a_j\lambda_jv_j \in \epsilon$. Thus, $C(A) \subseteq \epsilon$.

Nullspace of A: $N(A) := x \in \mathbb{R}^n : Ax = 0$.
Non-zero elements of $N(A)$ are orthogonal to non-zero elements of $C(A)$ as well as ϵ. And finally, we get:
$Dim(N(A) + Rank(A) = n =$ Number of columns of A.

$\therefore$ the rank of a matrix is equal to the number of non-zero eigenvalues, including repetitions.

5. (1 point) If A is a square symmetric matrix then prove that the number of positive pivots it has is the same as the number of positive eigenvalues it has.

Solution: Consider a real symmetric matrix A = $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$
Eigen values of A are $\lambda_1 = 4, \lambda_2 = -2$.
LDU decomposition of A gives

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

where $U = L^\top$
$A = LDL^\top$
sign of pivot variables of D = sign of eigenvalues.
consider a matrix B such that:

$$B = IDI^\top = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -8\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

B has the same diagonal matrix as A with eigenvalues equal to the pivot variables of D. Eigen values of B are 1 and -8.
Intuitively, the off diagonal elements of L = D = 3 edges closer to 0. the eigen value of A converges towards eigenvalue of B.
However, if an eigen value was to change sign(move from positive to negative or viceversa) it would entail crossing 0,at which point the matrix would become singular. This is not possible as pivot variables in our example are non changing and non-zero.
Therefore,If A is a square symmetric matrix then prove that the number of positive pivots it has is the same as the number of positive eigenvalues it has.

Eigenstory: Special Matrices

6. (2 points) Consider the matrix $R = I - 2\mathbf{u}\mathbf{u}^\top$ where $\mathbf{u}$ is a unit vector $\in \mathbb{R}^n$.
   (a) Show that R is symmetric and orthogonal. (How many independent vectors will R have?)

Solution:

$$R = I - 2uu^\top$$
$$R^\top = (I - 2uu^\top)^\top$$
$$= I^\top - 2(uu^\top)^\top$$
$$= I - 2(u^\top)^\top u^\top$$
$$= I - 2uu^\top$$
$$= R$$

Therefore, R = $R^\top$, R is symmetric matrix.
To prove R is orthogonal matrix, we need to prove $RR^{-1} = I = RR^\top$ By symmetricity of R , $RR^\top = RR$

$$RR = (I - 2uu^\top)(I - 2uu^\top)$$

$$= I - 2uu^\top - 2uu^\top + 4(uu^\top)(uu^\top)$$

$$= I - 4uu^\top + 4u(u^\top u)u^\top$$
$$= I - 4uu^\top + 4uu^\top$$
$$= I$$

Therefore, R is symmetric and orthogonal. Since, an orthogonal matrix is invertible, R will have n independent vectors.

(b) Let $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Draw the line passing through this vector in geogebra (or any tool of your choice). Now take any vector in $\mathbf{R}^3$ and multiply it with the matrix R (i.e., the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$). What do you observe or what do you think the matrix R does or what would you call matrix R? (Hint: the name starts with R)

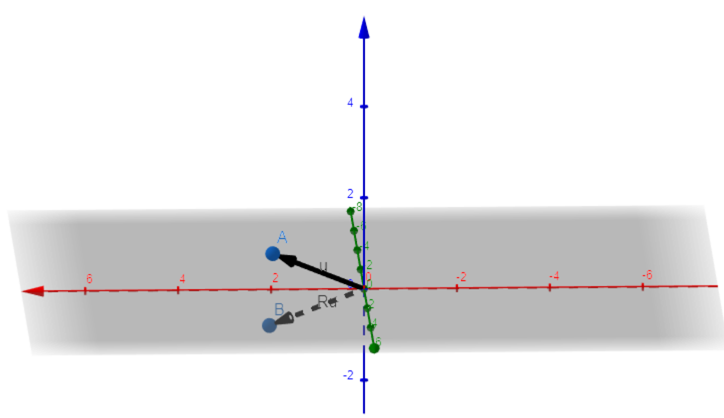
Solution: For $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $R = I - 2\mathbf{u}\mathbf{u}^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

Let a be a vector in R^3

$a = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$Ra = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

R is a reflection matrix it reflects a about x axis.



(c) Compute the eigenvalues and eigenvectors of the matrix R as defined above with

$\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Solution: Eigenvalues and Eigenvectors of R is computed as follows:

$R = I - 2\mathbf{u}\mathbf{u}^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$|R - \lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{bmatrix} = 0$

$(1-\lambda)(\lambda^2 - 1) = 0$

$(1-\lambda)(1+\lambda)(-1+\lambda) = 0$

$\lambda = 1, -1, 1$

To calculate eigenvectors:

For $\lambda = 1$

$|R - \lambda I| = 0$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$-x_2 - x_3 = 0, x_2 = -x_3$

$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ will be the eigen vectors for $\lambda = 1$

For $\lambda = -1$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$x_2 - x_3 = 0, x_2 = x_3$

$x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

So eigenvalues are 1,1,-1 and eigenvector are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(d) I believe that irrespective of what $\mathbf{u}$ is any such matrix R will have the same eigenvalues as you obtained above (with one of the eigenvalues repeating). Can you reason why this is the case? (Hint: think about how we reasoned about the eigenvectors of the projection matrix P even without computing them.)

Solution:

If eigen vector x is in the column space of hyperplane then its reflection will be in the same plane direction and in such case:

$$Rx = \lambda x = x$$

Since $\lambda = 1$

If eigen vector x is orthogonal to column space of hyperplane then its reflection will be in the opposite direction (as that of the original direction) to the plane and in such case:

$$Rx = \lambda x = -x$$

Since $\lambda = -1$

Therefore, Reflector matrix only two eigenvalues are possible 1 and -1.

7. (2 points) Let Q be a $n \times n$ real orthogonal matrix (i.e., all its elements are real and its columns are orthonormal). State with reason whether the following statements are True or False (provide a proof if the statement is True and a counter-example if it is False).

   (a) If λ is an eigenvalue of Q then $|\lambda| = 1$

Solution: True,

Let λ and v be eigenvalue and eigenvector of Q, respectively. Then $Qv = \lambda v$

Taking L_2 norm of the above:

$$||Qv||^2 = ||\lambda v||^2$$

$$= |\lambda|^2 ||v||^2$$

LHS:

$$||Qv||^2 = (Qv)^\top (Qv)$$

eqaution of length.

$$= v^\top Q^\top Q v$$

$$= v^\top I v$$

Since Q is orthogonal.

$$= v^\top v$$

$$= ||v||^2$$

Equation of length. The equation therefore,

$$||v||^2 = |\lambda|^2 ||v||^2$$

v is a non-zero eigenvector, and $||v|| \neq 0$.

$$|\lambda|^2 = 1$$

$$|\lambda| = 1$$

Since length is non-negative.

(b) The eigenvectors of Q are orthogonal

Solution: This statement is false,
Consider a Rotation matrix with $\theta = 90$

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigenvectors of this matrix are $x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ But, $x_1.x_2 = 2 \neq 0$
Eigenvector x_1, x_2 are not orthogonal.

(c) Q is always diagonalizable.

Solution: This statement is True
From Singular value Decomposition (SVD) we have

$$QV = U\Sigma$$

$$Q = U\Sigma V^\top$$

where U and V are orthonormal eigen vector matrices, Σ^2 is a diagonal matrix with entries as the eigenvalues of $Q^\top Q$ and $QQ^\top$.
But Q is a orthogonal matrix we know,

$$Q^\top Q = QQ^\top = I$$

Therefore, U and V are the same. So, $QV = V\Sigma \longrightarrow Q = V\Sigma V^\top$ and Hence, Q is always Diagonalizable.

8. (1½ points) Any rank one matrix can be written as $\mathbf{u}\mathbf{v}^\top$.

(a) Prove that the eigenvalues of any rank one matrix are $\mathbf{v}^\top\mathbf{u}$ and 0.

Solution: Let $A = \mathbf{u}\mathbf{v}^\top$. Let us assume that $v^\top u = \lambda_1 = \mathbf{u}^\top\mathbf{v}$ is an eigenvalue of A.
We want to prove that $Au = \lambda_1 u$ where u is a column vector in R^n.
LHS :

$$\begin{aligned} Au &= uv^\top u \\ &= u(v^\top u) \\ &= u(\lambda_1) \end{aligned}$$

$$= \lambda_1 u$$

Therefore, $v^\top u$ is an eigenvalue of A.
$A = \mathbf{u}\mathbf{v}^\top$ is rank 1 since columns of A are linear combinations of the $n \times 1$ column vector u.
It therefore follows that since A is a singular matrix, 0 is an eigenvalue. Therefore, $\lambda = 0$ is an eigenvalue of $A = uv^\top$.

(b) How many times does the value 0 repeat?

Solution: By rank-nullity theorem:

$$n = r + Nullity$$

$$Nullity = n - r$$

$$Nullity = n - 1$$

Therefore, the eigenspace of A has dimension n - 1.
The value 0 repeats n - 1 times.

(c) What are the eigenvectors corresponding to these eigenvalues?

Solution:
For $\lambda = \mathbf{v}^\top\mathbf{u}$
Let $A = \mathbf{u}\mathbf{v}^\top$

$$Au = uv^\top u$$

$$= u(v^\top u)$$

$$= u(\lambda_1)$$

$$= \lambda_1 u$$

Therefore, u is the eigenvector for eigenvalue $v^\top u$.
For $\lambda = 0$
Let $A = \mathbf{u}\mathbf{v}^\top$
Let a column vector w exist in the same dimension as u and v. such that $v^\top u = 0$ ...(1)

$$Aw = uv^\top w$$

$$Aw = u(v^\top w)$$

$$= u.0 \; from(1)$$

Therefore, eigen vector for (n-1) eigenvalues = 0 is the n-1 dimensional hyper plane comprising of (n-1) orthogonal vectors.

9. (2 points) Consider a $n \times n$ Markov matrix.

(a) Prove that the dominant eigenvalue of a Markov matrix is 1

Solution:
Proof (part 1): 1 is an eigenvalue of a Markov matrix
Let x be an $n \times 1$ column vector such that:
$x = \begin{bmatrix} 1 & 1 & 1.. & 1 \end{bmatrix}^\top$

$$Ax = \begin{bmatrix} (a_{11} * 1) + (a_{12} * 1) + ... + (a_{1n} * 1) \\ (a_{21} * 1) + (a_{22} * 1) + ... + (a_{2n} * 1) \\ .... \\ (a_{n1} * 1) + (a_{n2} * 1) + ... + (a_{nn} * 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ .. \\ .. \\ 1 \end{bmatrix}$$

$= x$
$= 1.x$
Therefore, 1 is the eigen value of a Markov matrix

Proof (part 2): all other eigenvalues are less than 1
We will prove this by contradiction. Let assume there exist a λ and an x such that $ax = \lambda x$.
Let there be an x_i such that it is the largest element in the column vector x.
Since any scalar multiple of x may satisfy the above equation, we assume without loss of generality that $x_i > 0$
Since A is a Markov Matrix, elements of the rows of A are non-negative and sum up to 1. Each entry in λx is a convex combination of elements of rows of A and elements of x.
LHS : As a result of the above statement, the largest possible element in λx (as a result of some $a_{ij} * x_i$) is x_i. Largest element $\leq x_i$
RHS : $\lambda x_i > x_i$, since $\lambda > 1$
This is a contradiction.
Dominant eigen value of A is 1.

(b) Consider any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$. Show that one of the eigenvalues of such a matrix is 1. (I hope you notice that a Markov matrix is a special case of such a matrix where $a + b = c + d = 1$.)

Solution:
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To prove that x is an eigen vector of A

$Ax = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} a+b \\ c+d \end{bmatrix}$

Multiplying and Dividing by (c+d)

$Ax = (c+d) \begin{bmatrix} a+b/c+d \\ c+d/c+d \end{bmatrix}$

Since, $a+b = c+d$, $\frac{a+b}{c+d} = 1$

$Ax = (c+d) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a eigen vector with eigen value $(c+d) = (a+b)$

(c) Does the result extend to $n \times n$ matrices where the sum of the elements of a row is the same for all the n rows? (Explain with reason)

Solution:

It can be shown for an $n \times n$ matrix A where the ssum of the elements of a row is thee same for all n rows, $x = \begin{bmatrix} 1 & 1 & 1....1 \end{bmatrix}^\top$ is an $n \times 1$ column vector which is an eigenvector to A.

(d) What is the corresponding eigenvector?

Solution:

Let each row of A sum to c. Thus:

$a_{11} + a_{12} + ... + a_{1n} = a_{21} + a_{22} + ... + a_{2n}$

$= a_{n1} + a_{n2} + ... + a_{nn}$

$= c$

$$Ax = \begin{bmatrix} (a_{11} * 1) + (a_{12} * 1) + ... + (a_{1n} * 1) \\ (a_{21} * 1) + (a_{22} * 1) + ... + (a_{2n} * 1) \\ .... \\ (a_{n1} * 1) + (a_{n2} * 1) + ... + (a_{nn} * 1) \end{bmatrix}$$

$$= \begin{bmatrix} c \\ c \\ c \\ .. \\ .. \\ c \end{bmatrix}$$

$$= c \begin{bmatrix} 1 \\ 1 \\ 1 \\ .. \\ .. \\ 1 \end{bmatrix}$$

$= cx$

$x = \begin{bmatrix} 1 & 1 & 1....... & 1 \end{bmatrix}^\top$ will always be an eigen vector of A with eigen value $= c$ where c is sum of each row.

Eigenstory: Special Relations

10. (4 points) For each of the statements below state True or False with reason.

(a) The eigenvalues of A^T are **always** the same as that of A.

Solution: The above statement is True.
For a Matrix A. We know from properties of determinants that $|A| = |A^\top|$
Characteristic Equation for eigenvalues of A is

$$|A - \lambda I| = 0$$

Transposing both the side

$$|(A - \lambda I)^\top| = 0$$

$$|A^\top - \lambda I^\top| = 0$$

$$|A^\top - \lambda I| = 0$$

Since, both A and A^T have the same characteristic equation, they will have the same eigenvalues.

(b) The eigenvectors of A^T are **always** the same as that of A

Solution: The above statement is False.
Let A be an $n \times n$ non-symmetric matrix, then $A^\top \neq A$. Assume both A and $A^\top$ has at least 1 real eigenvalues. Let λ be the eigenvalue of A.
$Au = \lambda u$ and $A^{=}\lambda v$, where u, v are eigen vectors of A and $A^\top$ From the proposition:

$$u = v$$

$$\lambda u = \lambda v$$

From equation of eigen vectors and eigen values

$$Au = A^\top v$$

if we say, u = v and we know that $A^\top \neq A$ this must imply that u = v = 0, which by defination is not eigen vector
Hence, proposition doesn't hold.

(c) The eigenvalues of A^{-1} are **always** the reciprocal of the eigenvalues of A.

Solution: The above statement is true iff A^{-1} exist.
Let A be a non-singular, and let x be an eigenvector for A with eigenvalue λ

$$A^{-1}x = A^{-1}(\frac{1}{\lambda}\lambda x)$$

$$A^{-1}x = \frac{1}{\lambda}A^{-1}(\lambda x)$$

$$A^{-1}x = \frac{1}{\lambda}A^{-1}(Ax)$$

$$A^{-1}x = \frac{1}{\lambda}x$$

If λ is a eigenvalue of A then $\frac{1}{\lambda}$ is eigen value of A^{-1}

(d) The eigenvectors of A^{-1} are **always** the same as the eigenvectors of A.

Solution: The above statement is true iff A^{-1} exist.
Let A be a non-singular, and let x be an eigenvector for A with eigenvalue λ

$$Ax = \lambda x$$

Multiplying by A^{-1}

$$A^{-1}Ax = A^{-1}\lambda x$$

$$Ix = \lambda A^{-1}x$$

$$\frac{1}{\lambda}x = A^{-1}x$$

$$A^{-1}x = \frac{1}{\lambda}x$$

Therefore, x is an eigenvector of A^{-1} and A are always same.

(e) If $\mathbf{x}$ is an eigenvector of A and B then it is also an eigenvector of both AB and BA, even if the eigenvalues of A and B corresponding to $\mathbf{x}$ are different.

Solution: The above statement is true.
Let x be the eigenvector of A and B such that $Ax = \alpha x$ and $Bx = \beta x$ where $\alpha \neq \beta$

$$ABx = A(Bx)$$
$$ABx = A(\beta x)$$
$$ABx = \beta(Ax)$$
$$ABx = \beta\alpha x$$
$$ABx = \alpha\beta x$$

$$BAx = B(Ax)$$
$$BAx = B(\alpha x)$$
$$BAx = \alpha(Bx)$$
$$BAx = \alpha\beta x$$

Thus x is also a eigen vectoor of AB and BA.

(f) If $\mathbf{x}$ is and eigenvector of A and B then it is also an eigenvector of $A + B$

Solution: The above solution is true
Let x be the eigenvector of A such that $Ax = \alpha x$ and $Bx = \beta x$

$$(A + B)x = Ax + Bx$$
$$= \alpha x + \beta x$$
$$= (\alpha + \beta)x$$

Thus x is also an eigenvector of $A + B$.

(g) If λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$.

Solution: The above solution is true
Let x be the eigenvector of A such that $Ax = \lambda x$.

$$(A + KI)x = Ax + K(Ix)$$
$$= \lambda x + kx$$
$$= (\lambda + k)x$$

Thus for the same eigenvector x ,$\lambda + k$ is an eigenvalue of $A + kI$.

(h) The non-zero eigenvalues of $AA^\top$ and $A^\top A$ are equal.

Solution: The above statement is true.
Let A be $m \times n$ matrix such that $A^\top A$ and $AA^\top$ are $n \times n$ and $m \times m$ matrix respectively.
Let λ be a non-zero eigenvalue of $A^\top A$ and x is non-zero eigenvector x.

$$A^\top Ax = \lambda x$$

Multiplying A from both the sides.

$$AA^\top Ax = A\lambda x$$

$$AA^\top(Ax) = \lambda(Ax)$$

Therefore, we get λ as non-zero eigenvalue for the matrix $AA^\top$ with eigenvector Ax

Eigenstory: Change of basis

11. (2 points) Consider the following two basis. Basis 1: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, and Basis 2: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}$,. Consider a vector $\mathbf{x} = \begin{bmatrix}a\\b\end{bmatrix}$ in Basis 1 (i.e., $\mathbf{x} = a\mathbf{u_1} + b\mathbf{u_2}$). How would you represent it in Basis 2?

Solution: $x = \begin{bmatrix}a\\b\end{bmatrix}$ in Basis 1. Let $\mathbf{x_2}$ and $\mathbf{x_1}$ be the representation of x in Basis 2 and Basis 1 respectively.
Let the matrix that represents Basis 1 be A_1:

$$A_1 = \begin{bmatrix}\mathbf{u_1} & \mathbf{u_2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\mathbf{x_1} = A_1\mathbf{x}$$

$$= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$x_1 = \begin{bmatrix}\frac{a+b}{\sqrt{2}}\\ \frac{a-b}{\sqrt{2}}\end{bmatrix}$$

Let the matrix that represents Basis 2 be A_2:

$$A_2 = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Representing x_1 in Basis 2:

$$x_2 = A^{-1}x_1$$
$$x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{bmatrix}$$
$$x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{bmatrix}$$
$$x_2 = \begin{bmatrix} b \\ -a \end{bmatrix}$$

This is the representation of $\mathbf{x}$ in Basis 2.

12. (1 point) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the standard basis. Let $C(\mathbf{u})$ and $C(\mathbf{v})$ be the representation of these vectors in a different basis. Prove that $\mathbf{u} \cdot \mathbf{v} = C(\mathbf{u}) \cdot C(\mathbf{v})$ if and only if the basis represented by C is an orthonormal basis (i.e., dot products are preserved only when the new basis is orthonormal).

Solution:
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the standard basis.And Let $C(\mathbf{u})$ and $C(\mathbf{v})$ be the representation of these vectors in a different basis (Say C).
If C is a orthonormal Basis,

$$C^\top C = I$$
$$c_i^\top c_j = 0,\ if\ i \neq j$$
$$c_i^\top c_j = 1,\ if\ i = j$$

$\mathbf{u}$ in standard basis transformed to vector in orthonormal basis.So vectors $\mathbf{u}$ and $\mathbf{v}$ in C's basis are $C\mathbf{u}$ and $C\mathbf{v}$.
We claim that the length and angles between $\mathbf{u}$ and $\mathbf{v}$ is preserved in C's basis. which is a orthonormal basis.
Claim 1: length is preserved.

$$||\mathbf{u}|| = ||C\mathbf{u}||$$

Proof:

$$||C\mathbf{u}||_2^2 = C\mathbf{u}.C\mathbf{u}$$

,

$$||C\mathbf{u}||_2^2 = (C\mathbf{u})^\top.C\mathbf{u}$$

Since, Dot product of $y.y = y^\top y$

$$||C\mathbf{u}||_2^2 = \mathbf{u}^\top(C^\top.C)\mathbf{u}$$

$$||C\mathbf{u}||_2^2 = \mathbf{u}^\top\mathbf{u}$$

$$||C\mathbf{u}||_2^2 = ||\mathbf{u}||_2^2$$

$$||C\mathbf{u}|| = ||\mathbf{u}||$$

Therefore, length is preserved.

Claim 2: Angles are preserved.

$$cos\theta = \frac{\mathbf{u}^\top\mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$$

$$cos\theta = cos\theta_c$$

Proof:
In C's basis,

$$cos\theta_c = \frac{(C\mathbf{u})^\top C\mathbf{v}}{||C\mathbf{u}||_2||C\mathbf{v}||_2}$$

$$cos\theta_c = \frac{\mathbf{u}^\top(C^\top.C)\mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$$

As length are preserved.

$$cos\theta_c = \frac{\mathbf{u}^\top\mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$$

$$cos\theta_c = cos\theta$$

Therefore, $\mathbf{u} \cdot \mathbf{v} = C(\mathbf{u}) \cdot C(\mathbf{v})$ if and only if the basis represented by C is an orthonormal basis.

Eigenstory: PCA and SVD

13. (1 point) How are PCA and SVD related? (no vague answers please, think and answer very precisely with mathematical reasoning)

Solution:
Let X be the $n \times d$ data matrix where n is the number of data points and d is the dimension of each data point.
Let us assume that it is centered, i.e. column means have been subtracted and are now equal to zero.
In PCA, we do daigonalization of the Covariance Matrix C as it is a symmetric matrix. Which is given by
$C = \frac{1}{n} X^\top X$
$C = V \sum V^\top$
where V is a matrix of eigenvectors (each column is an eigenvector). The eigenvectors are called principal directions of the data. $\sum$ is a diagonal matrix with eigenvalues λ_i in the decreasing order on the diagonal.
Singular value decomposition of X
$X = USV^\top$
$C = \frac{1}{n} X^\top X$
$C = \frac{1}{n}(USV^\top)^\top USV^\top$
where U is a unitary matrix and S is the diagonal matrix of singular values s_i
$C = \frac{1}{n} VS^\top U^\top USV^\top$
$C = \frac{1}{n} VS^2 V^\top$
By observation, we can say that right singular vectors V are principal directions and that singular values are related to the eigenvalues of covariance matrix via $\lambda_i = (s_i)^2/n$.
Principal components are given by $XV = USV^\top V = US$

14. (1½ points) Consider the matrix $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

(a) Find Σ and V, *i.e.*, the eigenvalues and eigenvectors of $A^\top A$

Solution:
A $= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$
$A^\top A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$
Eigen values of $A^\top A$
$|A^\top A - \lambda I| = 0$
$\begin{bmatrix} 25 - \lambda & 7 \\ 7 & 25 - \lambda \end{bmatrix} = 0$
$(25 - \lambda)^2 - 49 = 0$
$\lambda^2 - 50\lambda + 576 = 0$
$\lambda_1 = 18, \lambda_2 = 32$
$Eigenvector for \lambda = 18, \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} x = \lambda x$

$\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} x = 18x$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Similarly, $Eigenvector for \lambda = 32$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \sum^\top \sum = \begin{bmatrix} 18 & 0 \\ 0 & 32 \end{bmatrix}$

Since, $A = U\Sigma V^\top$, $A^\top A = V \sum^\top \sum V^\top$

Normalize the V, since V is orthonormal

$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \sum = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{32} \end{bmatrix}$

(b) Find Σ and U, *i.e.*, the eigenvalues and eigenvectors of $AA^\top$

Solution:

A $= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$AA^\top = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

Eigen values of $AA^\top$

$|AA^\top - \lambda I| = 0$

$\begin{bmatrix} 32 - \lambda & 0 \\ 0 & 18 - \lambda \end{bmatrix} = 0$

$(32 - \lambda)(18 - \lambda) = 0$

$\lambda_1 = 18, \lambda_2 = 32$

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sum^\top \sum = \begin{bmatrix} 18 & 0 \\ 0 & 32 \end{bmatrix}$

Since, $A = U\Sigma V^\top$, $AA^\top = U \sum \sum^\top U^\top$

Normalize the U, since U is orthonormal

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sum = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{32} \end{bmatrix}$

(c) Now compute $U\Sigma V^\top$. Did you get back A? If yes, good! If not, what went wrong?

Solution:

$U\Sigma V^\top = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{32} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 0 & \sqrt{32} \\ \sqrt{18} & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{32}/\sqrt{2} & \sqrt{32}/\sqrt{2} \\ -\sqrt{18}/\sqrt{2} & \sqrt{18}/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$
$$= \text{A}$$

15. (2 points) Prove that the matrices U and V that you get from the SVD of a matrix A contain the basis vectors for the four fundamental subspaces of A. (this is where the whole course comes together: fundamental subspaces, basis vectors, orthonormal vectors, eigenvectors, and our special symmetric matrices $AA^\top$, $A^\top A$!)

Solution:
The Singular Value Decomposition of a matrix A with rank r is $A = U\Sigma V^\top$ where U is a $m \times m$ orthogonal matrix, Σ is a $m \times n$ matrix containing the singular values of the matrix A on the main diagonal, and V is a $n \times n$ orthogonal matrix.

$$A = U\Sigma V^\top = \begin{bmatrix} U_{1_{m\times r}} & U_{2_{m\times m-r}} \end{bmatrix} \begin{bmatrix} \Sigma_{rr\times r} & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r} \end{bmatrix} \begin{bmatrix} V_{1_{n\times r}} & V_{2_{n\times n-r}} \end{bmatrix}^\top \text{ ....(1)}$$
$$= \begin{bmatrix} U_{1_{m\times r}}\Sigma_r & 0_{m\times n-r} \end{bmatrix} \begin{bmatrix} V_{1_{r\times n}}^\top \\ V_{2_{n-r\times n}}^\top \end{bmatrix}$$
$$= U_{1_{m\times r}} \Sigma r_{r\times r} V_{1_{r\times n}}^\top \text{ ....(2)}$$

The matrix in (2) is called the Full Rank Singular Value Decomposition. Further, notice that because U_1 and V_1 contain orthonormal vectors, where $r \leq n$, it follows that then $U_1^\top U_1 = I_r$ and $V_1^\top V_1 = I_r$

Column Space of A is the columns of U_1
Proof : Suppose $b \in C(A)$ then $b = Az = U_1\Sigma_r V_1^\top z = U_1 z*$
Therefore, $b \in C(U_1)$.
Since, Σ_r is invertible and $V_1^\top V_1 = I_r$ then,
$U_1\Sigma_r V_1^\top = A \longrightarrow U_1 = AV_1\Sigma_r^{-1}$
Suppose $b \in C(U_1)$, $b = U_1 z = AV_1\Sigma_r^{-1} z = Az*$
Therefore, $b \in C(A)$
It follows that $C(A) = C(U_1)$. Therefore, since U_1 is an orthonormal set of vectors, then an orthonormal basis for C(A) are the columns of U1.

Similar proof for Row Space of A i.e Column Space of $A^\top$
where $A^\top = V_1\Sigma_r U_1^\top z$
Therefore, $C(A^\top) = C(V_1)$. Therefore, since V_1 is an orthonormal set of vectors, then an orthonormal basis for C(A) are the columns of V_1.

Null Space of A is Columns of V_2
Proof: Solving equation (1),

$$\Sigma = \begin{bmatrix} \Sigma_{r_{r\times r}} & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r} \end{bmatrix}$$
$$= \begin{bmatrix} U_1^\top{}_{r\times m} \\ U_2^\top{}_{m-r\times m} \end{bmatrix} A \begin{bmatrix} V_{1_{n\times r}} & V_{2_{n\times n-r}} \end{bmatrix}$$
$$= \begin{bmatrix} U_1^\top AV_1 & U_1^\top AV_2 \\ U_2^\top AV_1 & U_2^\top AV_2 \end{bmatrix}$$
It follows : $\begin{bmatrix} U_1^\top AV_2 \\ U_2^\top AV_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow U^\top AV_2 = 0 \longrightarrow AV_2 = 0$

It follows that $V_2 \in N(A)$. Since the columns of V_2 are orthonormal, then they are linearly independent of each other. Since dim N(A) = n r and there are n r columns in V_2, then it follows that the columns of V_2 form a basis for N(A).

Similarly, Left Null space of A is columns of U_2

$A = U \sum V^\top = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \sum \begin{bmatrix} V_1 & V_2 \end{bmatrix}^\top$

Therefore,

U_1 forms a basis for $C(A)$

U_2 forms a basis for $N(A^\top)$

V_1 forms a basis for $C(A^\top)$

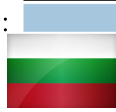
V_2 forms a basis for $N(A)$

16. (2 points) Fun with flags.

(a) Browse through the flags of all countries and paste 5 rank one flags below.

Solution:
Rank 1 flags:

Botswana : 

Bulgaria : 

Colombia : 

Estonia : 

Armenia : 

(b) What is the rank of the flag of Greece?

Solution:
Flag of Greece (400 x 267):

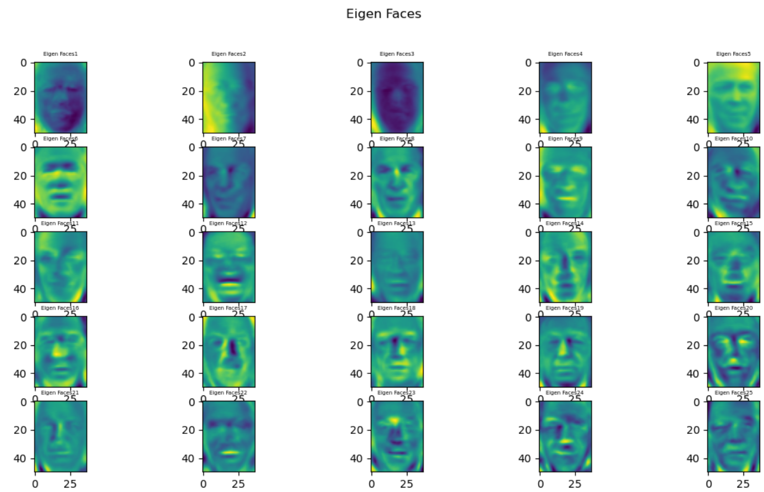
Rank of the Flag of Greece is 3.
The no of independent vectors in 400 x 267 matrix that represent the flag are 3.
First column, last column, and the column that include the plus i.e only last two blue lines

17. (2 points) Consider the LFW dataset (Labeled Faces in the Wild).

    (a) Perform PCA using this dataset and plot the first 25 eigenfaces (in a 5×5 grid)

Solution: Here is something to get you started.

```
import matplotlib.pyplot as plt
from sklearn.datasets import fetch_lfw_people
from sklearn.decomposition import PCA
from PIL import Image
import numpy as np


# Load data
lfw_dataset = fetch_lfw_people(min_faces_per_person=100)
_, h, w = lfw_dataset.images.shape
X = lfw_dataset.data

# Compute a PCA
n_components = 100
pca = PCA(n_components=n_components, whiten=True).fit(X)
principal_components = pca.components_
fig, ax = plt.subplots(5,5)
i = 0
fig.suptitle("Eigen Faces", fontsize = 12)
for row in ax:
    for col in row:
        col.imshow(principal_components[i].reshape((h,w)))
        i += 1
        col.set_title("Eigen Faces" + str(i), fontsize = 5)

plt.show()
```



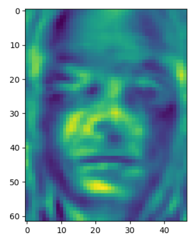
(b) Take your close-up photograph (face only) and reconstruct it using the first 25 eigenfaces :-). If due to privacy concerns, you do not want to to use your own photo then feel free to use a publicly available close-up photo (face only) of your favorite celebrity.

Solution:

```
import matplotlib.pyplot as plt
from sklearn.datasets import fetch_lfw_people
from sklearn.decomposition import PCA
from PIL import Image
import numpy as np


# Load data
lfw_dataset = fetch_lfw_people(min_faces_per_person=100)
_, h, w = lfw_dataset.images.shape
X = lfw_dataset.data

# Compute a PCA
n_components = 25
pca = PCA(n_components=n_components, whiten=True).fit(X)
principal_components = pca.components_

image = Image.open('I1.jpg')
im = np.array(image.resize((w,h)).convert('L'))
im1 = im.reshape((1,h*w))
z = pca.transform(im1)
```

```
reconstructed = np.dot(z,principal_components).reshape(h,w)
plt.imshow(reconstructed)..
plt.show()
```

Projected on 25 eigen basis



...And that concludes the story of *How I Met Your Eigenvectors :-)* (I hope you enjoyed it!)