Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Kartik Bharadwaj
Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Count, Count, Count!

2. (1 point) In how many ways can 10 people be seated:

(a) in a row such that Motu and Patlu sit next to each other (there is only one boy named Motu and only one boy named Patlu in the group)

Solution: Clubbing Motu and Patlu as a single entity, we have 9 slots which can be arranged in 9! ways. Since Motu and Patlu can themselves be set in 2! ways, total number of ways is $9! \times 2$.

(b) in a row such that there are 5 engineers and 5 doctors and no two doctors or no two engineers can sit next to each other

Solution: We can have either DEDEDEDEDE or EDEDEDEDED. For each such arrangement, there are 5! ways to arrange the doctors, and 5! ways to arrange the engineers, so the answer is $(2)(5!)(5!) = 28800$.

(c) in a row such that there are 3 engineers, 3 doctors and 4 lawyers and all people of the same profession should sit in consecutive positions.

Solution: Clubbing engineers, doctors, and lawyers, each as a separate entity. These entities can be arranged in 3! ways. Further, each entity in itself can be arranged in 3!, 3!, and 4! ways respectively. Hence, total number of ways is $3! \times 3! \times 3! \times 4! = 5184$.

(d) in a row such that there are 5 married couples and each couple must sit together.

Solution: 5 married couples can be arranged in 5! ways. And each couple can be arranged in 2! ways. Hence, total number of ways is $5! \times (2!)^5 = 3840$.

3. (½ point) How many unique 9 letter words can you form using the letters of the word MANMOHANA (the words can be gibberish)?

> **Solution:** All 9 letters can be arranged in 9! ways. Then, we remove repeating words by dividing 3! (A's), 2! (M's), and 2! (N's). Therefore, total unique words are $\dfrac{9!}{3! \times 2! \times 2!}$.

4. (½ point) Suppose you have a class of 7 students (A,B,C,D,E,F,G) who need to be arranged in a line with the following restrictions:

   1. A has to be in one of the first 3 slots
   2. B and A are very good friends and insist on being next to each other
   3. B doesn't want to stand immediately behind C

   In how many different ways can you arrange them?

> **Solution:** When A is in 1^{st} position, only one possibility exist: $A, B, -, -, -, -, -$ leading to 5! ways.
>
> When A is in 2^{nd} position, two possibilities exists: $B, A, -, -, -, -, - \implies 5!$ and $-, A, B, -, -, -, - \implies 4 \times 4!$.
>
> When A is in 3^{rd} position, $-, B, A, -, -, -, -$ and $-, -, A, B, -, -, -$ are the only possibilities, each giving us $4 \times 4!$ and $3 \times 4!$ ways.
>
> Hence, total number ways is: $5! \times 2 + 2 \times 4 \times 4! + 3 \times 4!$.

The boring questions are done. I hope you find the rest of the assignment to be interesting!

The birthday problem

5. (3 points) The days of the year can be numbered 1 to 365 (ignore leap days). Consider a group of n people, of which you are not a member. Any of the 365 days is equally likely to be the birthday of any member of this group. An element of the sample space Ω will be a sequence of n birthdays (one for each person).

   (a) How many elements are there in the sample space?

Solution: 365^n since there are n position to fill and each position can take value from 1 to 365 with repetition.

(b) Let A be the event that at least one member of the group has the same birthday as you. What is the probability of this event A?

Solution: Let the number of choices to fix my birthday be ${}^{365}C_1$. Then, the total number of possibilities such that none of the n members have the same birthday as me is $(364)^n$. Therefore,

$$\begin{aligned} P(A) &= 1 - P(\overline{A}) \\ &= 1 - \frac{364^n}{365^n} \end{aligned}$$

(c) Write a formula for computing the probability of the event that any two members of the group will have the same birthday?

Solution: Invalid question as per the Prof's instruction.

(d) What is the minimum value of n such that $P(A) \geq 0.5$?

Solution: From (b) we have,

$$\begin{aligned} P(A) &\geq 0.5 \\ 1 - \frac{364^n}{365^n} &\geq 0.5 \\ 0.5 &\geq \frac{364^n}{365^n} \\ log(0.5) &\geq n \times log(\frac{364}{365}) \\ n &\geq 252.652 \end{aligned}$$

Hence, minimum value of $n = 253$.

(e) Let B be the event that at least two members of the group share the same birthday. What is the probability of this event B?

Solution: Let event $\overline{B}$ be that none of the members have the same birthday.

Then, $P(B) = 1 - P(\overline{B})$. Hence, probability of such an event is:

$$\begin{aligned} P(B) &= 1 - \frac{365 \times 364 \times 363 \times \cdots \times (366-n)}{365^n} \\ &= 1 - \frac{365!}{(365-n)! \times 365^n} \end{aligned}$$

(f) What is the minimum value of n such that $P(B) \geq 0.5$?

Solution:

$$\begin{aligned} P(B) &\geq 0.5 \\ 1 - \frac{365!}{(365-n)! \times 365^n} &\geq 0.5 \\ log(365!) - n \times log(365) - log(365-n)! &\leq log(0.5) \\ n &\geq 23 \end{aligned}$$

Hence, the minimum value of n for which $P(B) \geq 0.5$ is 23.

(g) **[Ungraded question]** Why is there a big gap between the answers to part (d) and part (f)? (although at “first glance” they look very similar problems)

A biased coin

6. (1 point) Your friend Chaman has a coin which is biased (i.e., $P(H) \neq P(T)$). He proposes that he will toss the coin twice and asks you to bet on one of these events: A: both the tosses will result in the same outcome or B: both the tosses will result in a different outcome. Which event will you bet on to maximize your chance of winning the bet. (I am looking for a precise mathematical answer. No marks for answers which do not have an explanation).

Solution: Let $P(H) = p$ and $P(T) = 1 - p$. Then, $P(A) = P(HH \text{ or } TT) = p^2 + (1-p)^2$ and $P(B) = P(HT \text{ or } TH) = 2p(1-p)$. Let's assume that,

$$\begin{aligned} P(A) &> P(B) \\ p^2 + (1-p)^2 &> 2p(1-p) \\ p^2 + 1 + p^2 - 2p &> 2p - 2p^2 \\ 4p^2 - 4p + 1 &> 0 \\ (p - 0.5)^2 &> 0 \end{aligned} \tag{1}$$

Now, let's assume that,

$$\begin{aligned} P(A) &< P(B) \\ p^2 + (1-p)^2 &< 2p(1-p) \\ p^2 + 1 + p^2 - 2p &< 2p - 2p^2 \\ 0 &< -4p^2 + 4p - 1 \\ -(p-0.5)^2 &> 0 \end{aligned} \tag{2}$$

For (2), no solution exists. From (1), when $p > 0.5$ and $p < 0.5$, for all p values can take, $P(A) > P(B)$. Hence, I will bet on A.

Alice in Wonderland

7. (1 point) A bag contains one ball which could either be green on red. You take another red ball and put it in this pouch. You now close your eyes and pull out a ball from the pouch. It turns out to be red. What is the probability that the original ball in the pouch was red?

Solution: Let's assume that $P(G) = P(R) = \frac{1}{2}$. Hence, we would like to find:

$$\begin{aligned} P(R|Draw1_{red}) &= \frac{P(Draw1_{red}|R) \cdot P(R)}{P(Draw1_{red}|R) \cdot P(R) + P(Draw1_{red}|G) \cdot P(G)} \\ &= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + 0.5 \cdot \frac{1}{2}} \\ &= \frac{1}{3} \end{aligned}$$

Rock, paper and scissors

8. (2 points) Your friend Chaman has 3 strange dice: red, yellow and green. Unlike a standard die whose 6 faces are the numbers 1,2,3,4,5,6 these 3 dice have the following faces: red: 3,3,3,3,3,6, yellow: 5,5,5,2,2,2 and green: 4,4,4,4,4,1. Chaman suggests the following game: (i) You pick any one die (ii) Chaman then "carefully" picks one of the remaining two dice. Each of you will then roll your own die a 100 times. If on a given roll, the score of your die is higher than the score of Chaman's die then you get 1 INR else Chaman gets 1 INR. You play this game for many days and realise that you lose more often than Chaman.

(a) Why are you losing more often? or What is Chaman's "carefully" planned strategy? (the key thing to note is that he lets you choose first)

Solution: Let's assume that the probability of me picking any of the three dices is equally likely. For instance, let p_{RY} represent my winning probability or Chaman's losing probability if I choose red first and Chaman chooses yellow after me. Similarly, q_{RY} will represent Chaman's winning probability or my losing probability if I choose red first and Chaman chooses yellow after me.

$$p_{RY} = \frac{7}{12}, q_{RY} = \frac{5}{12}$$
$$p_{RG} = \frac{11}{36}, q_{RG} = \frac{25}{36}$$
$$p_{YR} = \frac{5}{12}, q_{YR} = \frac{7}{12}$$
$$p_{YG} = \frac{7}{12}, q_{YG} = \frac{5}{12}$$
$$p_{GR} = \frac{25}{36}, q_{GR} = \frac{11}{36}$$
$$p_{GY} = \frac{5}{12}, q_{GY} = \frac{7}{12}$$

Clearly, we can see Chaman's strategy. If I pick the red die first, Chaman will pick the green die. This is because $q_{RG} > p_{RY}$, i.e., I have higher chances of losing or Chaman has higher chances of winning with him picking a green die than a yellow die. Similarly, if I pick a yellow die, Chaman will pick a red die. Finally, if I pick a green die, Chaman will pick a yellow die.

(b) You realise what is happening and decide to turn the tables on Chaman. You buy 3 dice which are identical to Chaman's red, yellow and green dice. You now propose that instead of rolling a single die each of you will roll two dice of the same color. The rest of the rules remain the same (i) You pick any one color (ii) Chaman then uses his original strategy to carefully pick a different color (he is overconfident and simply uses the same strategy that he used when you were rolling only one die) (iii) If on a given roll, the sum of your two dice is greater than the sum of Chaman's two dice then you get 1 INR else Chaman gets 1 INR. To his horror Chaman realises that now he is loosing more often. Explain why?

Solution: Now, when we add 3 identical dices, we have new winning probability

P_{ij} and losing probability Q_{ij} change.

$$P_{RY} = \frac{531}{1296}, Q_{RY} = \frac{765}{1296}$$
$$P_{RG} = \frac{671}{1296}, Q_{RG} = \frac{625}{1296}$$
$$P_{YR} = \frac{765}{1296}, Q_{YR} = \frac{531}{1296}$$
$$P_{YG} = \frac{531}{1296}, Q_{YG} = \frac{765}{1296}$$
$$P_{GR} = \frac{625}{1296}, Q_{GR} = \frac{671}{1296}$$
$$P_{GY} = \frac{765}{1296}, Q_{GY} = \frac{531}{1296}$$

Initially, when we didn't add the new dices, and I picked the red die, Chaman would pick the green die. His probability of winning would be q_{RG}. However, with new dies added, if I picked red, Chaman will pick yellow dice because $q_{RG} > Q_{RY} > Q_{RG}$. This means adding new diesf dices decreases his winning probability and hence, he loses more often.

Sitting under an apple tree

9. (1 point) Which of the following has a greater chance of success?

   A. Six fair dice are tossed independently and at least one "6" appears.

   B. Twelve fair dice are tossed independently and at least two "6"s appear.

   C. Eighteen fair dice are tossed independently and at least three "6"s appear.

   Explain your answer.

Solution: For A, there are 6 slots where each slot can be filled 6 numbers leading to sample space of 6^6. Also, $P(A) = 1 - P(\text{No 6's appears})$.

Similarly, for B, there are now 12 slots to fill, thereby, making sample space equal to 6^{12}. Also, $P(B) = 1 - P(\text{a single 6 appears}) - P(\text{No 6's appears})$.

Finally, for C, there are 18 slots to fill, thereby, making sample space equal to 6^{18}. Also, $P(C) = 1 - P(\text{No 6's appears}) - P(\text{a single 6 appears}) - P(\text{Two 6's appears})$.

Hence,

$$P(A) = 1 - \left(\frac{5}{6}\right)^6$$
$$P(B) = 1 - \sum_{r=0}^{1} {}^{12}C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{12-r}$$
$$P(C) = 1 - \sum_{r=0}^{2} {}^{18}C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{18-r}$$

Hence, in all three probabilities, the one which is highest will have the smallest value subtracted from 1. After calculation, we see that $P(A) = 0.665$, $P(B) = 0.618$, and, $P(C) = 0.597$. We can intuitively see why A has the highest probability. For A, a 6 has to appear only in one of the slots. However, for B, two 6's has to appear in two of the 12 slots. That is, one 6 has to appear in a slot of 6, and second 6 has to appear in another slot of 6. This means, as our sample space grows, we are making it harder for higher number of 6's to appear. Similar argument can be raised for C. Hence, $P(A)$ has the highest probability.

With love from Poland

10. (1 point) A chain smoker carries two matchboxes - one in his left pocket and another in his right pocket. Every time he wants to light a cigarette he randomly selects a matchbox from one of the two pockets and then uses a matchbox from that box to light his cigarette. Suppose he takes out a matchbox and sees for the first time that it is empty, what is the probability that the matchbox in the other pocket has exactly one matchstick left?

Solution: Let there be n_1 matches in the left matchbox and n_2 matches in the right matchbox. We can represent every selection of a matchstick as a Bernoulli event without replacement. Since we want to get only 1 matchstick in the right matchbox at the end of the experiment, we have $(n_1 + n_2 - 1)$ positions to fill. There are $\frac{(n_1 + n_2 - 1)!}{(n_2 - 1)! \cdot n_1!}$ to fill these positions.

Assuming probability of choosing either the left or right pocket is equally likely, we have:

$$\begin{aligned} P(1 \text{ matchstick is left}) &= \frac{(n_1 + n_2 - 1)!}{(n_2 - 1)! \cdot n_1!} \cdot \left(\frac{1}{2}\right)^{n_1} \cdot \left(\frac{1}{2}\right)^{n_2 - 1} \\ &= \frac{(n_1 + n_2 - 1)!}{(n_2 - 1)! \cdot n_1!} \cdot \left(\frac{1}{2}\right)^{n_1 + n_2 - 1} \end{aligned}$$

A paradox

11. (1 point) Suppose there are 3 boxes:

    1. a box containing two gold coins,
    2. a box containing two silver coins,
    3. a box containing one gold coin and one silver coin.

    You select one box at random and draw a coin from it. The coin turns out to be a gold coin. You remove this coin and draw another coin from the same box. What is the probability that the second coin is also a gold coin?

    Solution: Let's assume that $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$. Hence, we would like to find:

    $$\begin{aligned} P(Draw2_{Gold}) &= P(B_1|Draw1_{Gold}) \\ &= \frac{P(Draw1_{Gold}|B_1) \cdot P(B_1)}{\sum_{i=1}^{3} P(Draw1_{Gold}|B_i) \cdot P(B_i)} \\ &= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 0.5 \cdot \frac{1}{3}} \\ &= \frac{2}{3} \end{aligned}$$

Once upon a time in Goa

12. (1 point) You are in one of the famous casinos in Goa[1]. You are observing the game of roulette. A roulette has 36 slots of which 18 are red and the remaining 18 are black. Each slot is equally likely. The manager places a ball on the roulette and then spins the roulette. When the roulette stops spinning, the ball lands in one of the 36 slots. If it lands in a slot which has the same color as what you bet on then you win. You do not believe in gambling but you are a student of probability[2]. You observe that the ball has landed in a black slot for the 26 consecutive rounds. Based on what you have learned in CS6015 you predict that there is a much higher chance of the ball landing in a red slot in the next round (since the probability of 27 consecutive black slots is very very low). You bet all your life's savings on red. What is the probability that you will win?

[1] I know about casinos in Goa purely out of academic interest.
[2] Ah! That's why you are in a casino! That makes perfect sense!

Solution: Given that there are 36 slots, $P(R) = P(B) = \frac{18}{36} = \frac{1}{2}$. We are told that the ball has landed in the black slot for 26 consecutive times.

Our initial assumption about the probability of 27 consecutive black slots is very low and that the probability of the ball landing at a red slot is high, is incorrect. This is because the outcome of each trial is independent. Therefore, $P(B_{26}) = \left(\frac{1}{2}\right)^{26}$ and $P(B_{27}) = \left(\frac{1}{2}\right)^{27}$. Now, let's assume that the 27th trial results in a red slot. Then, $P(B_{26}R_{27}) = \left(\frac{1}{2}\right)^{26} \cdot \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{27}$. This means $P(B_{26}R_{27})$ and $P(B_{27})$ are equally likely. Hence, $P(winning) = \frac{1}{2}$.

Oh Gambler! Thy shall be ruined!

13. (2 points) You play a game in a casino[3] where your chance of winning the game is p. Every time you win, you get 1 rupee and every time you lose the casino gets 1 rupee. You have i rupees at the start of the game and the casino has $N - i$ rupees (obviously, $N >> i$). The game ends when you go bankrupt or the casino goes bankrupt. In either case, the winner will walk away with a total of N rupees.

(a) Find the probability p_i of winning when you start the game with i rupees.

Solution: Let E_i be the event of winning the total of N rupees when I start with i rupees and the casino starts with $N - i$ rupees. Then, on the first game,

$$\begin{aligned} P(E_i) &= P(E_i|win) \cdot P(win) + P(E_i|lose) \cdot P(lose) \\ P(E_i) &= P(E_i|win) \cdot p + P(E_i|lose) \cdot q \\ P(E_i) &= P(E_{i+1}) \cdot p + P(E_{i-1}) \cdot q \\ p_i &= p_{i+1} \cdot p + p_{i-1} \cdot q \qquad (1) \end{aligned}$$

(b) What happens if $p = \frac{1}{2}$?

Solution: Since $p = \frac{1}{2}$, then $q = \frac{1}{2}$. Then, our equation (1) becomes,

$$p_i = \frac{p_{i+1} + p_{i-1}}{2} \qquad (2)$$

[3]Again, my interest in casinos in purely academic

We know that $p_0 = 0$ and $p_N = 1$. Substituting initial values in (2), we get, $p_1 = \dfrac{1}{N}$. Therefore, $p_i = \dfrac{i}{N}$. This means for $p = \dfrac{1}{2}$, the probability of winning the game essentially depends on how close i is to N. If $N >> i$, then chances of winning is very low.

(c) **[Ungraded question]** Can you reason why it does not make sense to take on a casino ($N >> i$)? Will you always go bankrupt in the long run?

Solution: Note that in a casino $p < \frac{1}{2}$, i.e, the odds are always in favour of the casino (How does a casino do this without you realising it? We will see this when we discuss the game of roulette!)

The disappointed professor

14. (1 point) A particular class has had a history of low attendance. The dejected professor decides that he will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p if the weather is good, and with probability q if the weather is bad. Given that the probability of bad weather on a given day is r, obtain an expression for the probability that the professor will teach his class on that day. [Bertsekas and Tsitsikilis, Introduction to Probability, 2nd edition.]

Solution: Let the event A be the event that the professor teaches, and let B be the event that the weather is bad. Then,

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

where,

1. $P(B) = r$
2. $P(B^c) = 1 - r$
3. $P(A|B) = \sum_{i=k}^{n} {}^nC_i \cdot q^i \cdot (1-q)^i$
4. $P(A|B^c) = \sum_{i=k}^{n} {}^nC_i \cdot p^i \cdot (1-p)^i$

The John von architecture

15. (1 point) Suppose you have a biased coin ($P(H) \neq P(T)$). How will you use it to make unbiased decision. (hint: you can toss the coin multiple times)

> **Solution:** Suppose, our experiment is to toss the unbiased coin two times. If we get HT, we can label it 'HEADS'. If we get TH, we label it 'TAILS'. If we get either HH or TT, we repeat the experiment again. In this way, we can make an unbiased decision.

Pascal to the rescue

16. (1 point) A six-side die is rolled three times independently. What is more likely: a sum of 11 or 12?

> **Solution:** The mean for a single six-sided die is 3.5. Hence, the mean for 3 six-sided die is $3 \times 3.5 = 10.5$. Since, 10.5 is closer to 11 than 12, a sum of 11 is more likely.

Enemy at the gates

17. (1 point) There are 41 soldiers surrounded by the enemy. They would rather die than get captured. They sit around in a circle and devise the following plan. Each soldier will kill the person to his immediate left. They will continue this till only one soldier remains who would then commit suicide. For example, if there are 7 soldiers numbered 1, 2, 3, 4, 5, 6, 7 sitting in a circle then they proceed as follows: 1 kills 2, 3 kills 4, 5 kills 6, 7 kills 1, 3 kills 5, 7 kills 3, 7 commits suicide.

   (a) In how many ways can 41 soldiers be arranged around a circle?

   > **Solution:** By fixing the initial killing position, other soldiers can be arranged in 40! ways.

   (b) If you were one of the 41 soldiers and the soldiers were randomly arranged in the circle, what is the probability that you would survive?

   > **Solution:** In every unique arrangement, there are 41 soldiers. Since we have 40! unique arrangements, total number of dead plus alive positions $= 40! \cdot 41$. This happens to be our sample space. Now, in every unique arrangement, there is a survivable position. We have 40! unique arrangements. Therefore,
   >
   > $$\begin{aligned} P(surviving) &= \frac{\text{Total number of survivable positions in all unique arrangements}}{\text{Total number of dead + alive positions in all unique arrangements}} \\ &= \frac{40!}{40! \cdot 41} \\ &= \frac{1}{41} \end{aligned}$$

(c) **[Ungraded question]** Is there a specific position in which you can sit so that you are the last surviving soldier?

> **Solution:** If we choose position 1 as the initial killing position, then 19 is the survivable position.