Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution:

Count, Count, Count!

2. (1 point) In how many ways can 10 people be seated:

   (a) in a row such that Motu and Patlu sit next to each other (there is only one boy named Motu and only one boy named Patlu in the group)

   Solution: We can take 10 slots, since constraint is given as Motu and Patlu as together, so we can take it as a one pair, so now can consider 9 slots and 9 person.
   We can arrange them in total 9! ways. And we can arrange the Motu and Patlu in a pair in 2! ways, so overall we have:
   $9! * 2!$ ways.

   (b) in a row such that there are 5 engineers and 5 doctors and no two doctors or no two engineers can sit next to each other

   Solution: We can consider 10 slots and arrange all the Engineers on Odd places and Doctors on Even places. And arrange Engineers in odd places in 5! ways and Doctors on even places in 5! ways, so total we have $5! * 5!$. Similarly we can arrange Engineers on Even places and Doctors on Odd places. Overall we have: $5! * 5! * 2$

   (c) in a row such that there are 3 engineers, 3 doctors and 4 lawyers and all people of the same profession should sit in consecutive positions.

   Solution: Since according to the constraint given we can have all same profession people seat together, so we consider together as 3 entities, which we can arrange them in 3! ways. And also inside same profession group we can arrange the Engineers together in 3!, Doctors in 3! and Lawyers in 4! ways. So overall we have $3! * 3! * 4! * 3!$

(d) in a row such that there are 5 married couples and each couple must sit together.

> **Solution:** Constraint given is all couples sit together. So we can arrange all couples in 5! ways, and in a pair each can arrange themselves in 2! ways, so all 5 couples can arrange themselves in $(2!)^5$ ways. So total we have $5! * (2!)^5$

3. (½ point) How many unique 9 letter words can you form using the letters of the word MANMOHANA (the words can be gibberish)?

> **Solution:** We not down the frequencies of each letter as $M- > 2$,$A- > 3$, $N- > 2$, $O- > 1$ and $H- > 1$.
> We can arrange a nine letter word in 9! ways. But since we have all $M's, A's, N's$, as identical. So we have there all arrangements as same, so we divide the total probabilities divide by their respective frequencies. So we have to divide the total probabilities as $\frac{9!}{2!*3!*2!}$.

4. (½ point) Suppose you have a class of 7 students (A,B,C,D,E,F,G) who need to be arranged in a line with the following restrictions:

   1. A has to be in one of the first 3 slots
   2. B and A are very good friends and insist on being next to each other
   3. B doesn't want to stand immediately behind C

   In how many different ways can you arrange them?

> **Solution:** We have total 7 students, now according to 1st constraint we can arrange A in first 3 places.and 2nd constraint we have to place B next to A.
> $\underline{A}$ $\underline{B}$ __ __ __ __ __
> __ $\underline{A}$ $\underline{B}$ __ __ __ __
> $\underline{B}$ $\underline{A}$ __ __ __ __ __
> __ $\underline{B}$ $\underline{A}$ __ __ __ __
> __ __ $\underline{A}$ $\underline{B}$ __ __ __
> So in total these are 5 ways, and rest of the students can be arranged in the 5! ways. So total ways we have $5 * 5!$ ways, now we have to subtract those cases in which C can stand in front of B. So we count the cases:
> $\underline{C}$ $\underline{B}$ $\underline{A}$ __ __ __ __
> This is just one case and all other students can be arranged in the 4! ways. So overall we have the solution as :
> $5 * 5! - 4! = 576 ways$

The boring questions are done. I hope you find the rest of the assignment to be interesting!

The birthday problem

5. (3 points) The days of the year can be numbered 1 to 365 (ignore leap days). Consider a group of n people, of which you are not a member. Any of the 365 days is equally likely to be the birthday of any member of this group. An element of the sample space Ω will be a sequence of n birthdays (one for each person).

   (a) How many elements are there in the sample space?

   > **Solution:** Each person can have his birthday on any of the given 365 days. So each has 365 choices. And there are n persons each having 365 choices. So we have sample space as: $(365)^n$

   (b) Let A be the event that at least one member of the group has the same birthday as you. What is the probability of this event A?

   > **Solution:** the constraint given is "at least 1", so we can subtract the cases when no one has same birthday as mine, so each having 364 choices. Overall we have $\frac{(365)^n-(364)^n}{(365)^n}$,

   (c) What is the minimum value of n such that $P(A) \geq 0.5$?

   > **Solution:** we have even A probability as $\frac{(365)^n-(364)^n}{(365)^n}$, so we choose the value of n such that $P(A) >= 0.5$, so we have $n = 253$

   (d) Let B be the event that at least two members of the group share the same birthday. What is the probability of this event B? (note that the term "you" does not appear in this Q)

   > **Solution:** At least two people can have the same birthday, we take its complement that no people have same birthday, so first people can have 365 days as options to choose from the days given in an year, so second person will have to 364 days, and third person has 363 days, and so on till all people in the group get exhausted.
   > So we will have the $P(B) = 1 - \frac{365!}{(365-n)! * 365^n}$

   (e) What is the minimum value of n such that $P(B) \geq 0.5$?

   > **Solution:** We have probability B as $1 - \frac{365!}{(365-n)! * 365^n}$, now we have to choose n such that $P(B) >= 0.5$, so we have to solve it as:

$1-\frac{365!}{(365-n)!*365^n} >= 0.5$
$\frac{365!}{(365-n)!*365^n} <= 0.5$
we get the minimum value as $n = 23$

(f) [**Ungraded question**] Why is there a big gap between the answers to part (c) and part (e)? (although at "first glance" they look very similar problems)

A biased coin

6. (1 point) Your friend Chaman has a coin which is biased (i.e., $P(H) \neq P(T)$). He proposes that he will toss the coin twice and asks you to bet on one of these events: A: both the tosses will result in the same outcome or B: both the tosses will result in a different outcome. Which event will you bet on to maximize your chance of winning the bet. (I am looking for a precise mathematical answer. No marks for answers which do not have an explanation).

Solution: given $P(H) \neq P(T)$, that is biased coin. let say $P(H) = p$ and $P(T) = 1-p$
we have these outcomes possible on tossing coin twice.
H-H=p*p
H-T=p*(1-p)
T-H=(1-p)*p
T-T=(1-p)*(1-p)
Now we have event A: Both the coin have same outcome that is H-H and T-T
So we have $P(A) = p*p+(1-p)*(1-p) = p^2+1+p^2-2p = 2p^2-2p+1$
Now we have event B: Both the coin have different outcome that is H-T and T-H
So we have $P(B) = p*(1-p)+(1-p)*p = p-p^2+p-p^2 = 2p-2p^2$
Now we will compare the $P(A)$ and $P(B)$
$2p^2-2p+1$.......?........$2p-2p^2$
where ? is the comparison sign to be determined
Add $2p^2$ and $2p$ both the sides, so we have it as:
$4p^2+1$........?............$4p$
Now subtract $4p^2$ from both the sides.
1............?..............$4p-4p^2$
divide by 4 both sides
$\frac{1}{4}$.........?...........$p(1-p)$
0.25...........?..................$p(1-p)$.......eq1
we can find out the maximum value of the equation $p(1-p)$, by finding its maxima:
$f(p) = p-p^2$.....differentaite it
$f'(p) = 1-2p = 0$
$1-2p = 0$

$p = \frac{1}{2}$
Double differentiate to know if a maxima or minima ocuurs:
$f''(p) = -2$
the second derivative is -ve so, a maxima occurs at $p = \frac{1}{2}$
So overall $p(1-p)$ can have maximum value at p that is $p(1-p) = 0.5 * (0.5) = 0.25$
we can conclude that $p(1-p)$ have max value equal to 0.25 and for all values of p it will be less than 0.25
now we look at the eq1
$0.25 >= p(1-p)$
that is the probability of L.H.S. that is $P(A)$ is greater than or equal to $P(B)$
So we place our bet on event A.

Alice in Wonderland

7. (1 point) A bag contains one ball which could either be green on red. You take another red ball and put it in this pouch. You now close your eyes and pull out a ball from the pouch. It turns out to be red. What is the probability that the original ball in the pouch was red?

Solution: Lets say we have A box, it is given that ball in it could be Green or Red, so the probability of each event happening is $\frac{1}{2}$.
case 1. If the ball in the box is Green, and we put one red ball in it, then we can take out the red ball in $\frac{1}{2}$ probability.
case 2. If the ball in the box is Red, and we put one red ball in it, then we can take out the red ball with probability 1, as all balls are red in the box in this case.
$P(case1) * \frac{1}{2} + P(case2) * 1 = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * 1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$
Now we want the probability of the case 2:
$P(case2) * 1 = \frac{1}{2} * 1$
So the probability is: $\frac{\frac{1}{2}*1}{\frac{1}{2}*\frac{1}{2}+\frac{1}{2}*1} = \frac{2}{3}$

Rock, paper and scissors

8. (2 points) Your friend Chaman has 3 strange dice: red, yellow and green. Unlike a standard die whose 6 faces are the numbers 1,2,3,4,5,6 these 3 dice have the following faces: red: 3,3,3,3,3,6, yellow: 5,5,5,2,2,2 and green: 4,4,4,4,4,1. Chaman suggests the following game: (i) You pick any one die (ii) Chaman then "carefully" picks one of the remaining two dice. Each of you will then roll your own die a 100 times. If on a given roll, the score of your die is higher than the score of Chaman's die then you get 1 INR else Chaman gets 1 INR. You play this game for many days and realise that you lose more often than Chaman.

(a) Why are you losing more often? or What is Chaman's "carefully" planned strategy? (the key thing to note is that he lets you choose first)

Solution: lets discover what is this winning strategy, by taking three cases:
Case 1: Pick the Red Dice, which have $P(3) = \frac{5}{6}$ and $P(6) = \frac{1}{6}$
now we pick the Yellow dice, which have $P(2) = \frac{1}{2}$ and $P(5) = \frac{1}{6}$
All possible outcomes:
3-2=$\frac{5}{6} * \frac{1}{2} = \frac{5}{12}$
3-5=$\frac{5}{6} * \frac{1}{2} = \frac{5}{12}$
6-2=$\frac{1}{6} * \frac{1}{2} = \frac{1}{12}$
3-2=$\frac{1}{6} * \frac{1}{2} = \frac{1}{12}$
Here we can see dice Red beats dice Yellow with probability: $\frac{5}{12} + \frac{1}{12} + \frac{1}{12} = \frac{7}{12}$
Case 2: Pick the Yellow Dice, which have $P(2) = \frac{1}{2}$ and $P(5) = \frac{1}{2}$
now we pick the Green dice, which have $P(1) = \frac{1}{6}$ and $P(4) = \frac{5}{6}$
All possible outcomes:
2-1=$\frac{1}{6} * \frac{1}{2} = \frac{1}{12}$
2-4=$\frac{5}{6} * \frac{1}{2} = \frac{5}{12}$
5-1=$\frac{1}{6} * \frac{1}{2} = \frac{1}{12}$
5-4=$\frac{5}{6} * \frac{1}{2} = \frac{5}{12}$
Here we can see dice Yellow beats dice Green with probability: $\frac{5}{12} + \frac{1}{12} + \frac{1}{12} = \frac{7}{12}$
Case 3: Pick the Green Dice, which have $P(1) = \frac{1}{6}$ and $P(4) = \frac{5}{6}$
now we pick the Red dice, which have $P(3) = \frac{5}{6}$ and $P(6) = \frac{1}{6}$
All possible outcomes:
1-3=$\frac{5}{6} * \frac{1}{6} = \frac{5}{36}$
1-6=$\frac{1}{6} * \frac{1}{6} = \frac{1}{36}$
4-3=$\frac{5}{6} * \frac{5}{6} = \frac{25}{36}$
4-6=$\frac{5}{6} * \frac{1}{6} = \frac{5}{36}$
Here we can see dice Green beats dice Red with probability: $\frac{25}{36}$
So we can see the cycle as : dice Red beats dice Yellow, dice Yellow beats dice Green and dice Green beats dice Red. So as soon as we choose first he has an option that can beat our dice in the long run.

(b) You realise what is happening and decide to turn the tables on Chaman. You buy 3 dice which are identical to Chaman's red, yellow and green dice. You now propose that instead of rolling a single die each of you will roll two dice of the same color. The rest of the rules remain the same (i) You pick any one color (ii) Chaman then uses his original strategy to carefully pick a different color (he is overconfident and simply uses the same strategy that he used when you were rolling only one die) (iii) If on a given roll, the sum of your two dice is greater than the sum of Chaman's two dice then you get 1 INR else Chaman gets 1 INR. To his horror Chaman realises that now he is loosing more often. Explain why?

Solution: (i). Lets take two Red dice and see what sum we can have with $P(3) = \frac{5}{6}$ and $P(6) = \frac{1}{6}$
we can all possible combinations as:
3-3=$\frac{5}{6} * \frac{5}{6} = \frac{25}{36}$
3-6=$\frac{5}{6} * \frac{1}{6} = \frac{5}{36}$
6-3=$\frac{1}{6} * \frac{5}{6} = \frac{5}{36}$
6-6=$\frac{1}{6} * \frac{1}{6} = \frac{1}{36}$
So we can conclude that probability of the dice sum: $P(sum(6)) = \frac{25}{36}$, $P(sum(9)) = \frac{10}{36}$ and $P(sum(12)) = \frac{1}{36}$
(ii). Lets take two Yellow dice and see what sum we can have with $P(2) = \frac{1}{2}$ and $P(5) = \frac{1}{2}$
we can all possible combinations as:
2-2=$\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
2-5=$\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
5-2=$\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
5-5=$\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
So we can conclude that probability of the dice sum: $P(sum(4)) = \frac{1}{4}$, $P(sum(7)) = \frac{1}{4}$ and $P(sum(10)) = \frac{1}{4}$
(iii). Lets take two Green dice and see what sum we can have with $P(1) = \frac{1}{6}$ and $P(4) = \frac{5}{6}$
we can all possible combinations as:
1-1=$\frac{1}{6} * \frac{1}{6} = \frac{1}{36}$
1-4=$\frac{1}{6} * \frac{5}{6} = \frac{5}{36}$
4-1=$\frac{5}{6} * \frac{1}{6} = \frac{5}{36}$
4-4=$\frac{5}{6} * \frac{5}{6} = \frac{25}{36}$
So we can conclude that probability of the dice sum: $P(sum(2)) = \frac{1}{36}$, $P(sum(5)) = \frac{10}{36}$ and $P(sum(8)) = \frac{25}{36}$
In previous case we had:
case 1: Red beats dice yellow, now lets implement that strategy now:
All possible sum combinations are:
6-4=$\frac{25}{36} * \frac{1}{4} = \frac{25}{144}$
6-7=$\frac{25}{36} * \frac{2}{4} = \frac{50}{144}$
6-10=$\frac{25}{36} * \frac{1}{4} = \frac{25}{144}$
9-4=$\frac{10}{36} * \frac{1}{4} = \frac{15}{144}$
9-7=$\frac{10}{36} * \frac{2}{4} = \frac{20}{144}$
9-10=$\frac{10}{36} * \frac{1}{4} = \frac{10}{144}$
12-4=$\frac{1}{36} * \frac{1}{4} = \frac{1}{144}$
12-7=$\frac{1}{36} * \frac{2}{4} = \frac{2}{144}$
12-10=$\frac{1}{36} * \frac{1}{4} = \frac{1}{144}$
Yellow beats Red dice with probability as: $\frac{50}{144} + \frac{25}{144} + \frac{10}{144} = \frac{85}{144}$, we can see the case in previous case got reversed.
case 2: Yellow beats dice Green, now lets implement that strategy now:

All possible sum combinations are:
4-2=$\frac{1}{4} * \frac{1}{36} = \frac{1}{144}$
4-5=$\frac{10}{36} * \frac{1}{4} = \frac{10}{144}$
4-8=$\frac{25}{36} * \frac{1}{4} = \frac{25}{144}$
7-2=$\frac{1}{36} * \frac{2}{4} = \frac{2}{144}$
7-5=$\frac{10}{36} * \frac{2}{4} = \frac{20}{144}$
7-8=$\frac{25}{36} * \frac{2}{4} = \frac{50}{144}$
10-2=$\frac{1}{36} * \frac{1}{4} = \frac{1}{144}$
10-5=$\frac{10}{36} * \frac{1}{4} = \frac{10}{144}$
10-8=$\frac{25}{36} * \frac{1}{4} = \frac{1}{144}$
Green beats Yellow dice with probability as: $\frac{50}{144} + \frac{25}{144} + \frac{10}{144} = \frac{85}{144}$, we can see the case in previous case got reversed.
case 3:Green beats dice Red, now lets implement that strategy now:
All possible sum combinations are:
2-6=$\frac{25}{36} * \frac{1}{36} = \frac{25}{1296}$
2-9=$\frac{10}{36} * \frac{1}{36} = \frac{10}{1296}$
2-12=$\frac{10}{36} * \frac{1}{36} = \frac{10}{1296}$
5-6=$\frac{10}{36} * \frac{25}{36} = \frac{250}{1296}$
5-9=$\frac{10}{36} * \frac{10}{36} = \frac{100}{1296}$
5-12=$\frac{10}{36} * \frac{1}{36} = \frac{10}{1296}$
8-6=$\frac{25}{36} * \frac{25}{36} = \frac{625}{1296}$
8-9=$\frac{25}{36} * \frac{10}{36} = \frac{250}{1296}$
8-12=$\frac{25}{36} * \frac{1}{36} = \frac{25}{1296}$
Red beats Green dice with probability as: $1 - \frac{625}{1296} = \frac{671}{1296}$, we can see the case in previous case got reversed.
So the cycle that we had in previous case got reversed, so now "Chaman" looses more often.

Sitting under an apple tree

9. (1 point) Which of the following has a greater chance of success?

   A. Six fair dice are tossed independently and at least one "6" appears.

   B. Twelve fair dice are tossed independently and at least two "6"s appear.

   C. Eighteen fair dice are tossed independently and at least three "6"s appear.

   Explain your answer.

Solution: lets calculate probability:
A: $1 - (\frac{5}{6})^6 = \frac{6^6 - 5^6}{6^6} = \frac{46656 - 15625}{46656} = \frac{31031}{46656} = 0.665102$
B: $1 - (\frac{5}{6})^{12} - \binom{12}{1} * \frac{5}{11}^{11} * \frac{1}{6}$

$1-(\frac{1}{6}^{12})[5^{12}+12*5^{11}]$
$1-(\frac{5}{12})^{12}[1+\frac{12}{5}]$
$1-(\frac{5}{12})^{12}[\frac{17}{5}]$
$1-(0.11215)*\frac{17}{5}$
$1-0.381332=0.61866$
C: $1-(\frac{1}{6})^{18}[5^{18}+18*5^{17}+\binom{18}{2}*5^{16}]$
$1-(\frac{5}{6})^{18}[1+\frac{18}{5}+\frac{17*9}{25}]=0.59734$
Clearly, we can see from the probabilities calculated that A is more likely to happen.

With love from Poland

10. (1 point) A chain smoker carries two matchboxes - one in his left pocket and another in his right pocket. Every time he wants to light a cigarette he randomly selects a matchbox from one of the two pockets and then uses a matchbox from that box to light his cigarette. Suppose he takes out a matchbox and sees for the first time that it is empty, what is the probability that the matchbox in the other pocket has exactly one matchstick left?

Solution: So we have 1 matchstick left in one box and 0 in other.
Lets suppose in starting we have n sticks in both of them. So we have 2n sticks, and the person has taken out 2n-1 sticks already from those, which he can draw from either of the matchbox that is each sticks has 2 options from which it can be drawn, so overall 2n sticks can be drawn in 2^{2n} ways, that is the total sample space.
Now we have to take out the n sticks in 2n-1 trials from one of the boxes, so we can have 0 sticks left in that before the 2n'th trial, this we can do that in $\binom{2n-1}{n}$
Now we can choose the box to be empty from two boxes, so we have total $2*\binom{2n-1}{n}$ ways.
Overall we have $\frac{2*\binom{2n-1}{n}}{2^{2n}}=\frac{\binom{2n-1}{n}}{2^{2n-1}}$

A paradox

11. (1 point) Suppose there are 3 boxes:

    1. a box containing two gold coins,
    2. a box containing two silver coins,
    3. a box containing one gold coin and one silver coin.

    You select one box at random and draw a coin from it. The coin turns out to be a gold coin. You remove this coin and draw another coin from the same box. What is the probability that the second coin is also a gold coin?

Solution: We can apply Bayes theorem in it:
$P(GG|g) = \frac{P(g|GG)*P(GG)}{P(g|GG)*P(GG)+P(g|GS)*P(GS)}$
$P(GG) = \frac{1}{3}$
$P(GS) = \frac{1}{3}$
$P(GG|g) = \frac{\frac{1}{3}*1}{\frac{1}{3}*1+\frac{1}{3}*\frac{1}{2}}$
$P(GG|g) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}}$
$P(GG|g) = \frac{1}{3} * \frac{6}{3} = \frac{2}{3}$

Once upon a time in Goa

12. (1 point) You are in one of the famous casinos in Goa[1]. You are observing the game of roulette. A roulette has 36 slots of which 18 are red and the remaining 18 are black. Each slot is equally likely. The manager places a ball on the roulette and then spins the roulette. When the roulette stops spinning, the ball lands in one of the 36 slots. If it lands in a slot which has the same color as what you bet on then you win. You do not believe in gambling but you are a student of probability[2]. You observe that the ball has landed in a black slot for the 26 consecutive rounds. Based on what you have learned in CS6015 you predict that there is a much higher chance of the ball landing in a red slot in the next round (since the probability of 27 consecutive black slots is very very low). You bet all your life's savings on red. What is the probability that you will win?

Solution: The probability if the winning would be $P(win) = \frac{1}{2}$ and $P(lose) = \frac{1}{2}$
Since i am betting on the 27th trial and not the sequence of 27 consecutive trial, the 27th trial would be independent of the previous trial result, as if i would have been betting on the sequence of 27 consecutive trial then probability would have been dependent on the all 26 previous trials.

Oh Gambler! Thy shall be ruined!

13. (2 points) You play a game in a casino[3] where your chance of winning the game is p. Every time you win, you get 1 rupee and every time you lose the casino gets 1 rupee. You have i rupees at the start of the game and the casino has $N - i$ rupees (obviously, $N >> i$). The game ends when you go bankrupt or the casino goes bankrupt. In either case, the winner will walk away with a total of N rupees.

[1] I know about casinos in Goa purely out of academic interest.
[2] Ah! That's why you are in a casino! That makes perfect sense!
[3] Again, my interest in casinos in purely academic

(a) Find the probability p_i of winning when you start the game with i rupees.

Solution: p_i= Probability to win starting with the amount "i"
Also given is $P(Win) = p$ and $P(Lose) = q$
$p_i = P(i|win) * P(win) + P(i|lose) * P(lose)$
After winning the wealth increase by 1, so P_i goes to P_{i+1} and Afterloosing the wealth decrease by 1, so P_i goes to P_{i-1}
$p_i = p_{i+1} * p + p_{i-1} * q$............(1)
since we know $P(win) + P(lose) = 1$
(1) can be rewitten as:
$p_i(P(win) + P(lose)) = p_{i+1} * p + p_{i-1} * q$
$p_i(p + q) = p_{i+1} * p + p_{i-1} * q$
$q(p_i - p_{i-1}) = p(p_{i+1} - p_i)$
$p_{i+1} - p_i = \frac{p}{q}(p_i - p_{i-1})$
Since we know $p_0 = 0$, as probability of winning is 0 when we have 0 as the starting amount. As we have to stop. Also $p_N = 1$, as probability of winning is 1 when we already have amount N, as we have to stop.
Using the above conclusion in the equation:
$p_0 = 0$, so we put i=1 in the equation as $p_2 - p_1 = \frac{q}{p}(p_1 - p_0) = (\frac{q}{p})p_1$
Now we put i=2 in the equation as $p_3 - p_2 = \frac{q}{p}(p_2 - p_1) = (\frac{q}{p})^2 p_1$
Now we put i=3 in the equation as $p_4 - p_3 = \frac{q}{p}(p_3 - p_2) = (\frac{q}{p})^3 p_1$
Now we can induct from this as: $p_{i+1} - p_i = (\frac{q}{p})^i p_1$
we can write the equation as $p_{i+1} - p_1 = \sum_{k=1}^{i}(p_{k+1} - p_k) = \sum_{k=1}^{i}(\frac{q}{p})^k p_1$
$p_{i+1} = p_1 + p_1 \sum_{k=1}^{i}(\frac{q}{p})^k$
$p_{i+1} = p_1 \sum_{k=0}^{i}(\frac{q}{p})^k = p_1 \frac{1-(\frac{q}{p})^{i+1}}{1-\frac{q}{p}}$
Now we want to know what is the value of p_1, so put $i = n - 1$ in the equation, and we already know $p_N = 1$;
$1 = p_N = p_1 * \frac{1-(\frac{q}{p})^N}{1-\frac{q}{p}}$
from this we can find out $p_1 = \frac{1-(\frac{q}{p})}{1-(\frac{q}{p})^N}$
so we finally we put back the value of p_1 in the equation:
$p_i = \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}$

(b) What happens if $p = \frac{1}{2}$?

Solution: We take the equation $p_{i+1} = p_1 \sum_{k=0}^{i}(\frac{q}{p})^k$
Now we are given $p = q = \frac{1}{2}$
so we have the term $\frac{q}{p} = 1$, so in equation we have:
$p_{i+1} = p_1 \sum_{k=0}^{i}(1)^k = p_1 * (i + 1)$
choose i=N-1 and we know $p_N = 1$

$p_N = 1 = p_1 * N$, this gives $p_1 = \frac{1}{N}$
$p_i = \frac{i}{N}$

(c) **[Ungraded question]** Can you reason why it does not make sense to take on a casino ($N >> i$)? Will you always go bankrupt in the long run?

Solution: Note that in a casino $p < \frac{1}{2}$, i.e, the odds are always in favour of the casino (How does a casino do this without you realising it? We will see this when we discuss the game of roulette!)

The disappointed professor

14. (1 point) A particular class has had a history of low attendance. The dejected professor decides that he will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p if the weather is good, and with probability q if the weather is bad. Given that the probability of bad weather on a given day is r, obtain an expression for the probability that the professor will teach his class on that day. [Bertsekas and Tsitsikilis, Introduction to Probability, 2nd edition.]

Solution: We can apply the formula:
$P(A) = P(B) * P(A|B) + P(B^c) * P(A|B^c)$
here A= event that probability that class will be conducted
B=event that weather is good
C=event that weather is bad
probability that more than or equal to k student will attend the class when weather is good :
$P(A|B) = \sum_{i=k}^{n} \binom{n}{i} (p^i) * (1-p)^{n-i}$
probability that more than or equal to k student will attend the class when weather is bad :
$P(A|B^c) = \sum_{i=k}^{n} \binom{n}{i} (q^i) * (1-q)^{n-i}$
Now 1-r is the probability $P(B^c) = 1 - r$ given that weather is good, and r is probability $P(B) = r$ that weather is bad.
$P(A) = (1-r) * \sum_{i=k}^{n} \binom{n}{i} (p^i) * (1-p)^{n-i} + r * \sum_{i=k}^{n} \binom{n}{i} (q^i) * (1-q)^{n-i}$

The John von architecture

15. (1 point) Suppose you have a biased coin ($P(H) \neq P(T)$). How will you use it to make unbiased decision. (hint: you can toss the coin multiple times)

Solution: $P(H) \neq P(T)$, that is the coin is biased, so we can conduct trials in large numbers to know the bias, and then we can increase the number of attempt accordingly for the part it is less biased.
We can also consider the John-Von architecture, in this we can toss the coin two times instead of one time. Now for example take a coin $P(H) = 0.7$ and $P(T) = 0.3$, now when we toss the coin two times we have 4 possible outcomes as:
H-H=$0.7 * 0.7 = 0.49$
H-T=$0.7 * 0.3 = 0.21$
T-H=$0.3 * 0.7 = 0.21$
T-T=$0.3 * 0.3 = 0.09$
Here we can see that the events H-T and T-H are having equal probabilities, so we will only use these outcomes as the fair. for the H-T case, consider this as Head, and for the T-H case, consider this as Tails. And when H-H or T-T arrives ignore these and again conduct the trial.
For a general case take $P(H) = p$ and $P(T) = 1 - p$, so for all possible outcomes we have:
H-H=p*p................Ignore and re conduct the trial
H-T=p*(1-p)..........This is Head.
T-H=(1-p)*p..........This is Tail.
T-T=(1-p)*(1-p)......Ignore and re conduct the trial

Pascal to the rescue

16. (1 point) A six-side die is rolled three times independently. What is more likely: a sum of 11 or 12?

Solution: Total sample space would be of size $6^3 = 216$
and our favourable outcomes with sum 11, note that the sum 11 can be achieved as follows:
11=6+4+1, 6-4-1 can be arranged in total $3! = 6$ ways.
11=1+5+5, 1-5-5 can be arranged in total $\frac{3!}{2} = 3$ ways.
11=5+4+2, 5-4-2 can be arranged in total $3! = 6$ ways.
11=3+3+5, 3-3-5 can be arranged in total $\frac{3!}{2} = 3$ ways.
11=4+3+4, 4-3-4 can be arranged in total $\frac{3!}{2} = 3$ ways.
11=6+3+2, 6-3-2 can be arranged in total $3! = 6$ ways.
these are total 27 ways.
our favourable outcomes with sum 12, note that the sum 12 can be achieved as follows:
12=6+5+1, 6-5-1 can be arranged in total $3! = 6$ ways.
12=4+3+5, 4-3-5 can be arranged in total $3! = 6$ ways.

12=4+4+4, 4-4-4 can be arranged in total $\frac{3!}{3!} = 1$ ways.
12=5+2+5, 5-2-5 can be arranged in total $\frac{3!}{2} = 3$ ways.
12=6+4+2, 6-4-2 can be arranged in total $3! = 6$ ways.
12=6+3+3, 6-3-3 can be arranged in total $\frac{3!}{2} = 3$ ways.
these are total 25 ways.
So we can calculate that $P(sum(11)) = \frac{27}{216}$ and $P(sum(12)) = \frac{25}{216}$
So clearly $P(sum(11))$ is more likely.

Enemy at the gates

17. (1 point) There are 41 soldiers surrounded by the enemy. They would rather die than get captured. They sit around in a circle and devise the following plan. Each soldier will kill the person to his immediate left. They will continue this till only one soldier remains who would then commit suicide. For example, if there are 7 soldiers numbered 1, 2, 3, 4, 5, 6, 7 sitting in a circle then they proceed as follows: 1 kills 2, 3 kills 4, 5 kills 6, 7 kills 1, 3 kills 5, 7 kills 3, 7 commits suicide.

(a) In how many ways can 41 soldiers be arranged around a circle?

Solution: The soldiers are to be arranged in a circle , so they can be arranged in $(n-1)!$ ways, so total are $(41-1)! = 40!$ ways.

(b) If you were one of the 41 soldiers and the soldiers were randomly arranged in the circle, what is the probability that you would survive?

Solution: There will be only one soldier who will survive in last at a specific position, so if i am that position and rest 40 soldiers are arranged in the circle leaving that position for me. So the probability of i being at that position would be : $\frac{1}{41}$

(c) [**Ungraded question**] Is there a specific position in which you can sit so that you are the last surviving soldier?

Solution:

There could be alternate approaches to solving some of the questions.