Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Kartik Bharadwaj
Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Concept: Linear Transformation

2. (1 point) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose $T(\begin{bmatrix}4 & 8 & 12\end{bmatrix}^\top) = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ and $T(\begin{bmatrix}3 & 12 & 27\end{bmatrix}^\top) = \begin{bmatrix}2 & 2 & 2\end{bmatrix}^\top$. Find $T(\begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top)$

Solution:

$$\begin{aligned} T(\begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top) &= (\frac{-1}{4})\, T(\begin{bmatrix}4 & 8 & 12\end{bmatrix}^\top) + (\frac{-1}{3})\, T(\begin{bmatrix}3 & 12 & 27\end{bmatrix}^\top) \\ &= (\frac{-1}{4})\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + (\frac{-1}{3})\begin{bmatrix}2 & 2 & 2\end{bmatrix}^\top \\ &= \begin{bmatrix}\frac{-11}{12} & \frac{-11}{12} & \frac{-11}{12}\end{bmatrix}^\top \end{aligned}$$

3. (1 point) Prove that if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ then $T(b\mathbf{x}+c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$.

Solution: Let us assume $T(\mathbf{q}) = A\mathbf{q}$ where $\mathbf{q}$ is a vector and A is a matrix.

For $\mathbf{q} = b\mathbf{x} + c\mathbf{y}$, where b, c are constants and $\mathbf{x}, \mathbf{y}$ are vectors.

$$\begin{aligned} T(b\mathbf{x}+c\mathbf{y}) &= A(b\mathbf{x}+c\mathbf{y}) \\ &= Ab\mathbf{x} + Ac\mathbf{y} \\ &= T(b\mathbf{x}) + T(c\mathbf{y}) \\ &= bT(\mathbf{x}) + cT(\mathbf{y}) \end{aligned}$$

4. (2 points) In the lecture, we mentioned that a system of linear equations can have 0, 1 or ∞ solutions. Can you formally argue why a system of linear equations cannot have exactly 2 solutions? (Hint: If $\mathbf{x}$ and $\mathbf{y}$ are two solutions then ...)

> **Solution:** Let's say that for a linear system $\mathbf{A}x = \mathbf{b}$, we have two solutions $\mathbf{x}$ and $\mathbf{y}$. Also, let $\mathbf{z} = c\mathbf{x} + (1-c)\mathbf{y}$, for c $\in$ R. For $\mathbf{x}$ and $\mathbf{y}$ to be the only two solutions, we want to prove $\mathbf{Az} \neq \mathbf{b}$, thereby, concluding that $\mathbf{x}$ and $\mathbf{y}$ are the only two solutions.
>
> $$\begin{aligned} \mathbf{A}z &= \mathbf{A}(c\mathbf{x} + (1-c)\mathbf{y}) \\ &= c\mathbf{Ax} + (1-c)\mathbf{Ay} \\ &= cb + (1-c)b \\ &= b \end{aligned}$$
>
> We see that $\mathbf{Az} = \mathbf{b}$ which violates what we want. This means that when a system of linear equations $\mathbf{A}x = \mathbf{b}$ has two solutions, any linear combination of those solutions is also a solution for the linear system. In this scenario, the linear system will have infinitely many solutions.

5. (2 points) Suppose $A \in \mathbf{R}^{3\times 3}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3 (\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0})$. Further, suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. If $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ is one solution for $A\mathbf{x} = b$, write down at least one more solution (you are welcome to write down all the infinite solutions if you want :-)).

> **Solution:** We know that $\mathbf{A} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top = \mathbf{b}$ and $\mathbf{Ay} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$.
>
> Adding the above two system of linear equations we get: $\mathbf{A}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbf{y}) = \mathbf{b}$. This means that any linear combination of $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ and $\mathbf{y}$ will give us the vector $\mathbf{b}$ for all values of $\mathbf{y}$ which satisfy $\mathbf{Ay} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$.
>
> Hence, our set of solutions for $\mathbf{Ax} = \mathbf{b}$ will be: $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, m\mathbf{y} \}$ where $m \in \mathbb{R}$.

Concept: Matrix multiplication

6. (1 point) True or False: If A, B, C are matrices and if AC = BC then A = B. **Explain your answer**.

Solution: The above sentence is True only when matrix C is invertible. One way to prove the following is: If $AC - BC = 0$ and $C \neq 0$, then $(A - B)C = 0$. This implies $A - B = 0$, which leads to $A = B$.

Another way would be to multiply $(A - B)C = 0$ by C^{-1}, if C is invertible. This leads to $(A - B)CC^{-1} = 0$, which equals to $A - B = 0$ or $A = B$.

7.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & -2 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

For each of the equations below, find $\mathbf{x}$

(a) (½ point) $A\mathbf{x} = \begin{bmatrix} 1 & 4 & -1 & 2 \end{bmatrix}^\top$

Solution: $\mathbf{x} = \begin{bmatrix} \frac{9}{23} & \frac{25}{23} & \frac{13}{23} & \frac{-1}{23} \end{bmatrix}^\top$

(b) (½ point) $A\mathbf{x} = \begin{bmatrix} 1 & 2 & 0.5 & 0 \end{bmatrix}^\top$

Solution: $\mathbf{x} = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \end{bmatrix}^\top$

8. (1 point) Prove that $(AB)^\top = B^\top A^\top$

Solution: Let $A = (a_{ij})_{m\times p}$ and $B = (b_{jk})_{p\times n}$. Also, let $C = AB$ and $Q = B^T A^T$.

$$\begin{aligned} C &= (AB)^\top \\ &= \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \dots & c_{mn} \end{bmatrix}^\top \end{aligned}$$

where $c_{ik} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \dots & a_{ip} \end{bmatrix} \begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \\ \vdots \\ b_{pk} \end{bmatrix}$ for $1 \leq i \leq m$ and $1 \leq k \leq n$.

$$\begin{aligned} &= ((c_{ik})_{m\times n})^\top \\ &= (c_{ki})_{n\times m} \end{aligned}$$

$$\begin{aligned} Q &= B^T A^T \\ &= ((b_{jk})_{p\times n})^\top ((a_{ij})_{m\times p})^\top \\ &= (b_{kj})_{n\times p} (a_{ji})_{p\times m} \\ &= \begin{bmatrix} q_{11} & q_{12} & q_{13} & \dots & q_{1m} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ q_{n1} & q_{n2} & q_{n3} & \dots & q_{nm} \end{bmatrix} \end{aligned}$$

where $q_{ki} = \begin{bmatrix} b_{k1} & b_{k2} & b_{k3} & \dots & b_{kp} \end{bmatrix} \begin{bmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \\ \vdots \\ a_{pi} \end{bmatrix}$ for $1 \leq i \leq m$ and $1 \leq k \leq n$.

$$\begin{aligned} &= (c_{ki})_{n\times m} \\ &= C \end{aligned}$$

Hence, $(AB)^\top = B^\top A^\top$.

9. If A, B, C are matrices (assume appropriate dimensions) prove that
   (a) (½ point) $A(B + C) = AB + AC$

Solution:

$$
\begin{aligned}
(A(B+C)) &= \sum_{k=1}^{n} A_{ik}(B+C)_{kj} \\
&= \sum_{k=1}^{n} A_{ik}(B_{kj}+C_{kj}) \\
&= \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}C_{kj} \\
&= (AB)_{ij} + (AC)_{ij} = (AB+AC)
\end{aligned}
$$

By proof of induction and equality, LHS = RHS.

(b) (½ point) $(AB)C = A(BC)$

Solution: Assuming all dimensions are appropriate, let C be multiplied by a unit vector x. Then,

$$
\begin{aligned}
(AB)Cx &= (AB)y && Cx = y \\
&= A(By) \\
&= A(BCx) \\
&= A(BC)x
\end{aligned}
$$

By proof of induction and equality, LHS = RHS.

10. (1 point) Let A be any matrix. In the lecture we saw that $A^\top A$ is a square symmetric matrix. Is $AA^\top$ also a square symmetric matrix? (Hint: The answer is either "Yes, except when ..." or "No, except when ...".)

Solution: Let $Q = AA^T$. Then,

$$
\begin{aligned}
Q^T &= (AA^T)^T \\
&= (A^T)^T A^T && \text{(Using the property, } (AB)^T = B^T A^T\text{)} \\
&= AA^T \\
&= Q
\end{aligned}
$$

Hence, AA^T will always be symmetric matrix.

Concept: Inverse

11. (1 point) Let A and B be square invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

We know that a matrix $\mathbf{A}$ is invertible if there exists a matrix $\mathbf{A^{-1}}$ such that:

$$A^{-1}A = AA^{-1} = I$$

From the above property, we can say the following:

$$B^{-1}A^{-1}AB = B^{-1}IB = I$$
$$ABB^{-1}A^{-1} = AIA^{-1} = I$$

Hence,

$$(AB)^{-1} = B^{-1}A^{-1}$$

12. What is the inverse of the following two matrices? (Hint: I don't want you to compute the inverse using some method. Instead think of the linear transformation that these matrices do and think how you would reverse that transformation. **You will have to explain your answer in words clearly stating the linear transformations being performed.**)

(a) (½ point)

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Solution: Initially, if we consider A to be an 4x4 identity matrix, then the above given matrix essentially scales down the column vectors by 2. Hence, to get back the identity matrix, A^{-1} will be:

$$A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(b) (½ point)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The given matrix could be seen as a form of elementary matrix where the operation $R_2 \rightarrow R_2 + 2R_1$ applies. To reverse this operation, we simply subtract $2R_1$ from R_2, thereby, making the operation as: $R_2 \rightarrow R_2 - 2R_1$. Hence,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) (1 point)

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Solution: The above matrix A is a rotation matrix which rotates the co-ordinate space anti-clockwise. The inverse will simply be the clockwise operation.

$$A^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

Concept: System of linear equations

13. (1 point) Argue why the following system of linear equations will not have any solutions.

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

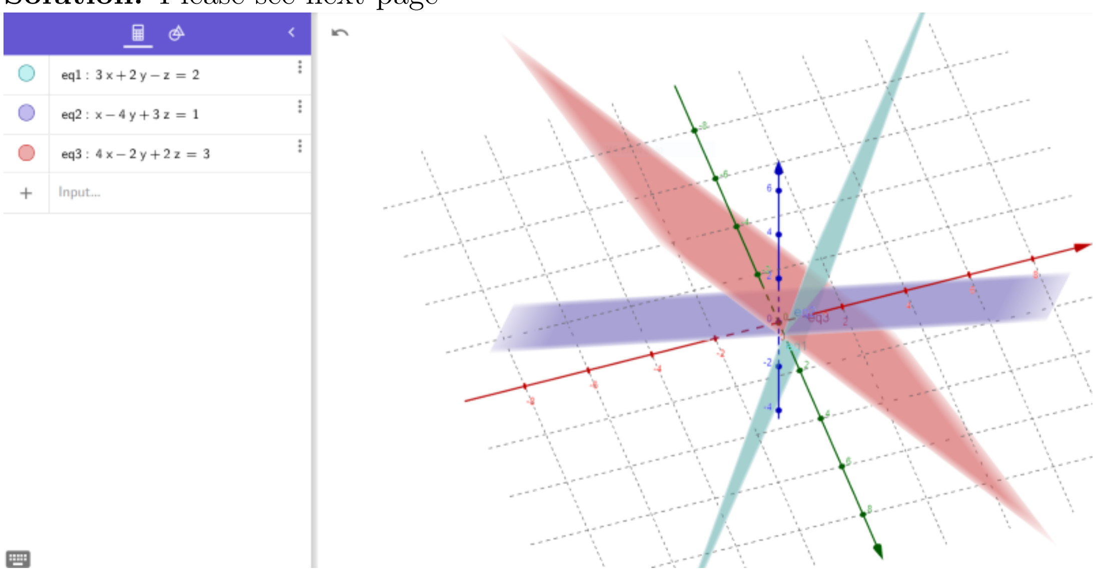
Solution: The above matrix-vector equation is of the form: $\mathbf{Ax} = \mathbf{B}$. The matrix $\mathbf{A}$ is a 4x4 matrix representing column vectors in 4-D. The first three equations will represent three 3-D hyperplanes in 4-D. The intersection or non-intersection of these hyperplanes is dependent on the last equation: $0\mathbf{x} = 4$. If the last equation had been $0\mathbf{x} = 0$, the only information it will add is that the intersection of the three 3-D hyperplanes intersect on a line. Since that is not the case and the last equation essentially gets us: $0 = 4$, the three hyperplanes will not intersect on a line.

14. Consider the following 3 planes

$$\begin{aligned} 3x + 2y - z &= 2 \\ x - 4y + 3z &= 1 \\ 4x - 2y + 2z &= 3 \end{aligned}$$

(a) (½ point) Plot these planes in geogebra and paste the resulting figure here (you can download the figure as .png and paste it here)

Solution: Please see next page



(b) (½ point) How many solutions does the above system of linear equations have? (based on visual inspection in geogebra)

Solution: Infinitely many solutions

(c) (1 point) Notice that the third equation can be obtained by adding the first two equations. Based on this observation, can you explain your answer for the number of solutions in the previous part of the question. (Note that I am looking for an answer in plain English which does not include terms like "linear independence" or "dependence of columns/rows". In other words, your answer should be based only on concepts/ideas which have already been discussed in the class)

Solution: If we plot only the first two planes, they will intersect at a line, resulting in infinitely many solutions. These solutions satisfies the first two equations. These solutions will also satisfy an equation of a plane which is a linear combination of the first two equations.

15. Consider the following system of linear equations:

$$x + y - z = 1$$
$$x - y + z = 2$$

Add one more equation to the above system such that the resulting system of 3 linear equations has

(a) (½ point) 0 solutions

Solution: $4x - 2y + 2z = 3$

(b) (1 point) exactly 1 solution

Solution: $-x + y + z = 3$

(c) (½ point) infinite solutions

Solution: $2x = 3$