CS6015 : Linear Algebra and Random Processes
Tutorial #2

Deadline: None

- This tutorial covers topics already covered in class. [Lecture slide 4-7]
- While this is optional, it is strongly recommended that students solve this tutorial.

Name :
Roll Number :

1. Find if A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is invertible and compute A^{-1} by **Gauss-Jordan** method if it exists.

Solution:

2. For which right hand sides (find a condition on b_1, b_2, b_3) are these systems solvable? Solve using **Gaussian Elimination**.

(a) $$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution:

3. Describe the column spaces (lines or planes) of the following two matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution:

4. Comment on each of the following collection of vectors (in R^3) and state if they are linearly independent.

   (a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} \right\}$

   (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

   (d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$

   (e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution:

5. Whether the following set of vectors span the entire R^3? If not, then what does the span of these vectors represent?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \right\}$$

Solution:

6. The span of vectors is the set of all their linear combinations. Consider two 2-D vectors v and w. What will be the spans of these vectors in the following cases:

   (a) v and w lie on the same line

   (b) v and w do not lie on the same line

   (c) v and w are zero

Solution:

7. Comment on the following set of transformations and state if they are linear transformations or not. The input is of the form, v = (v_1, v_2).

(a) $T(v) = (v_2, v_1)$

(b) $T(v) = (v_1, v_1)$

(c) $T(v) = (0, v_1)$

(d) $T(v) = (0, 1)$

Solution:

8. Which of the following subsets of R^3 are actually subspaces?

   (a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

   (b) The plane of vectors with $b_1 = 1$

   (c) The vectors with $b_1 b_2 b_3 = 0$

   (d) All linear combinations of v = [1,4,0] and w = [2,2,2]

   (e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$

   (f) All vectors with $b_1 \leq b_2 \leq b_3$

Solution:

9. How many (0, 1, ∞) solutions does this system of linear equation have? Write the specific answer if there exists a unique solution, or write the closed form expression if infinitely many solutions exist for this system of equations.

$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ 2x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

Solution:

10. For a given set of vectors (v = $[v_1, v_2, v_3 ... v_n]$) in a vector space S, prove that the span(v) is a subspace of S.

Solution:

11. Prove that a square matrix can have atmost one inverse.

Solution:

12. Mark the following statements as true/false:

    (a) The columns of a matrix are a basis for the column space.

    (b) In Ax = v, let x_1, x_2 and v_1, v_2 be the basis vectors of the input and output spaces respectively. In the input space, if a vector is = c(x_1) + d(x_2), then the transformed vector in output space will be = c(v_1) + d(v_2).

    (c) Every basis for a particular space have equal number of vectors and this number is called the dimension of the space.

Solution:

13. For the following set of statements, state whether they are True or False. Provide a valid reasoning or example/counter-example to justify your judgement.

    (a) A and A^T have the same left nullspace.

    (b) A and A^T have the same number of pivots.

    (c) If the row space equals the column space then A^T = A.

    (d) If A^T = -A then the row space of A equals the column space.

Solution:

14. Consider a plane x - 2y + 3z = 0 in R^3. Find the basis vectors which span this plane. Further, find a basis for the following :-

    (a) intersection of this plane with the standard XY plane.

    (b) all vectors perpendicular to the plane.

Solution:

15. Which of the following vectors are linearly independent:

    (a) (1,0) and (1,0.001)
    (b) (2,1) and (0,0)
    (c) (-1,-1) and (1,1)
    (d) (5,0) and (0,5)

Solution:

16. Compute the rank and nullity of the given matrix M = $\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Solution: