Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

> **Solution:** Yes

Concept: System of linear equations

2. (2 points) This question has two parts as mentioned below:

   (a) Find a 2 x 3 system Ax = b whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

> **Solution:** From the given complete solution, can conclude that the dimension of null space is 1 and that $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ lies in the null space of A.
> Any matrix A can be constructed by ensuring that the following hold:
> (i) The rank of the matrix needs to be 2 (i.e., construct two independent columns of 2x1)
> (ii) (col 1 of A) + 3x(col 2 of A) + (col 3 of A) = $\mathbf{0}$
> A corresponding **b** vector can be computed using the formulated matrix, A with the particular solution of $x_p = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
> That is, (col 1 of A) + 2x(col 2 of A) gives **b**

   (b) Now find a 3 x 3 system which has these solutions exactly when $b_1 + b_2 = b_3$. (Note: $b = [b_1\ b_2\ b_3]^T$.)

Solution: A simple way to construct the 3 x 3 matrix, is to now create a third entry in the two independent columns from previous part by adding the first two entries to form the third. (So we now have 3 x 1 columns)
The remaining column is again computed to satisfy (col 1 of A) + 3x(col 2 of A) + (col 3 of A) = **0**
The vector **b** can be formed either by adding the first two entries or solving with the particular solution.

3. (2 points) Consider the matrices A and B below

$$\text{(i) A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \text{ (ii) B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) Write down the row reduced echelon form of matrices A and B (also mention the steps involved).

Solution: (i) By operation, R3: R3-R1 => $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

By operation, R1: R1 - 2R2 => $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (RREF form)

(ii) By operation, R2: R2 - 4R1 => $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$

R3: R3 - 7R1 => $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ and R2: $\frac{-1}{3}$R2 => $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}$

R3: R3 + 6R2 => $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

R1: R1 - 2R2 => $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (RREF form)

(b) Find all solutions to $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.

Solution: (i) $A\mathbf{x} = 0$

Using the RREF derived in (a), $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get $x_1 - 2x_3 + x_4 = 0$ and $x_2 + x_3 = 0$. This is the case for infinitely many solutions, Also Rank(A) = Rank($A|B$) = $2 < 4$

We arrive at the conditions: $x_2 = -x_3$ and $x_1 + x_4 = 2x_3$. There are 2 free variables (x_3, x_4). General form for the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - t \\ -s \\ s \\ t \end{bmatrix}$$

One specific solution (of the infinitely many) can be when s = t = 1,

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

(ii) $B\mathbf{x} = 0$

Using the RREF derived in (a), $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get $x_1 = x_3$ and $x_2 = -2x_3$. This is the case for infinetely many solutions with 1 free variable (x_3). Also, Rank(A) = Rank($A|B$) = $2 < 3$.

General form for the solution is: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ -2s \\ s \end{bmatrix}$

One specific solution (of the infinitely many) can be when s = 1 is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(c) Write down the basis for the four fundamental subspaces of A.

Solution: Four fundamental subspaces of a matrix are C(A) (Column space), C(A^T) (Row space), N(A) (Null space), N(A^T) (Left nullspace).

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{C(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad C(\mathrm{A}^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad N(\mathrm{A}^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) Write down the basis for the four fundamental subspaces of B.

Solution: $\mathrm{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\mathrm{C(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad C(\mathrm{A}^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$N(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \qquad N(\mathrm{A}^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Concept: Rank

4. (1 ½ points) Consider the matrices A and B as given below:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & x \end{bmatrix} \text{ and B} = \begin{bmatrix} 3 & 1 & 3 \\ y & 2 & y \end{bmatrix}$$

Give the values for entries x and y such that the ranks of the matrices A and B are

(a) 1

Solution: For rank 1:
For matrix A:
$\mathbf{x = 3}$ so that row 3 = (3/2)*row 1
For matrix B:
$\mathbf{y = 6}$ so that row 2 = 2*row 1

(b) 2

> **Solution:** For rank 2:
> $x \in R - \{3\}$ and $y \in R - \{6\}$, so that both A and B will have 2 pivots each.

(c) 3

> **Solution:** In matrix A: row 2 = row 1/-2. So, rank 3 is not possible for A.
> Matrix B is a 2*3 matrix, so the maximum possible rank is 2.
> Hence, rank 3 is not possible for both A and B.

Concept: Nullspace and column space

5. (½ point) State True or False and explain you answer: The nullspace of R is the same as the nullspace of U (where R is the row reduced echelon form of A and U is the matrix in LU decomposition of A).

> **Solution:** True.
> We go from Ax=0 to Ux=0 to Rx=0 by performing elementary row transformations, and during this process the solutions of the original system Ax=0 remain unaltered. Hence, the null spaces of A, U and R remain the same.

6. (1 point) Suppose column 1 + column 2 + column5 = $\mathbf{0}$ in a 4×5 matrix A.

(a) What is a special solution for $A\mathbf{x} = \mathbf{0}$

> **Solution:**
> We have to find solution for $Ax = 0$,
> Given column 1 + column 2 + column5 = 0, thus linear combination of col1, col2, col5 is a solution.
>
> Therefore special solution is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) Describe the null space of A.

Solution:
The rank of A is 4, thus $dim(\mathcal{N}(A)) = 1$, a line

i.e k $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, where k $\in R$

7. (2 points) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. The column space of this matrix is a 2 dimensional plane. What is the equation of this plane? (You need to write down the steps you took to arrive at the equation)

Solution: Let the two vectors be $u = [1, 1, 1]^T)$ and $v = [1, 2, 3]^T$. The plane that represents the column space of these two vectors, contains these vectors as well. Therefore the normal of the plane is also orthogonal to the vectors given. We know the cross product of two vectors gives us a new vector, which is orthogonal to both the vectors. Since cross product is orthogonal to both the vectors, it can serve as the normal vector for the plane as well. In fact any scalar multiple of the cross product of the two vectors is a valid normal for the plane that represents the column space.
Let the vector n be the cross product of vectors u and v.
$n = u$ X v
$n = [1, 1, 1]^T$ X $[1, 2, 3]^T$
$n = [1, -2, 1]^T$
Let vector $w = [x, y, z]^T$ represent any generic vector in the above mentioned plane. Since n is normal to the plane, it is also normal to every vector lying on the plane. Therefore n is perpendicular to w. Therefore their dot product is 0.
This implies $w \cdot n = 0$
$[x, y, z] \cdot [1, -2, 1] = 0$
$x - 2y + z = 0$, which is the equation of the plane which is the represents the column space of the two above mentioned vectors.

8. (1 point) True or false? (If true give logical, valid reasoning or give a counterexample if false)
a. If the row space equals the column space then $A^T = A$

Solution:
False.

Counter Example:
$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ The row space and column space spans R^2, but $A \neq$ A^T

b. If $A^T = -A$ then the row space of A equals the column space.

Solution:
True.
We know $rowspace(A) = colspace(A^T)$.
Given $A^T = -A$
thus $rowspace(A) = colspace(-A)$, the colspace spanned by -A is same as rowspace of A as it is just a scalar (-1) multiplcation, by linear transformations they are equal.

9. (1 point) Which of the four fundamental subspaces are the same for the following pairs of matrices of different sizes? (Assume A is a $m \times n$ matrix)

(a) $\begin{bmatrix} A \end{bmatrix}$ and $\begin{bmatrix} A \\ A \end{bmatrix}$

Solution:
(i) C($\begin{bmatrix} A \end{bmatrix}$) $\neq C(\begin{bmatrix} A \\ A \end{bmatrix})$
(ii) N($\begin{bmatrix} A \end{bmatrix}$) $= N(\begin{bmatrix} A \\ A \end{bmatrix})$
(iii) C(A^T) = C($[A^T \ A^T]$) .
(iv) N(A^T) $\neq$ N($[A^T \ A^T]$) .
Row space(i.e. Column space of the transpose) and Null space of the two matrices are same.

(b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$

Solution:
(i) C($\begin{bmatrix} A \\ A \end{bmatrix}$) $= C(\begin{bmatrix} A & A \\ A & A \end{bmatrix})$
(ii) N($\begin{bmatrix} A \\ A \end{bmatrix}$) $\neq N(\begin{bmatrix} A & A \\ A & A \end{bmatrix})$
(iii) C($\begin{bmatrix} A \\ A \end{bmatrix}^T$) $\neq C(\begin{bmatrix} A & A \\ A & A \end{bmatrix}^T)$
(iv) N($\begin{bmatrix} A \\ A \end{bmatrix}^T$) $= N(\begin{bmatrix} A & A \\ A & A \end{bmatrix}^T)$

Column space and Left Null space(Null space of the transpose) of the two matrices are same.

10. (2 points) For each of the questions below, construct a matrix A which satisfies the given condition or argue why the given condition cannot be satisfied?

   (a) A matrix whose row space is equal to its column space

   Solution: If a Matrix A satisfies the condition $A^T = \pm A$(Symmetric or Skew-Symmetric), then the *rowspace* will be same as *columnspace*

   Example : $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$

   (b) A matrix whose null space is equal to its column space

   Solution: The *rowspace* and the *nullspace* together span the **domain** of the linear transformation R^n and the *columnspace* and the left *nullspace* together span the **co-domain** of the linear transformation R^m for a $m \times n$ matrix.

   Consider a $m \times n$ matrix A, if its *nullspace* is equal to the *columnspace* then it should satisfy $m = n$, $n = 2k$ and *nullspace*, *columnspace* are k dimensional. Therefore, if the *nullspace* and *columnspace* of A coincide, A must be similar to a matrix of the form $A = \begin{bmatrix} 0 & \mathrm{B}_{k\times k} \\ 0 & 0 \end{bmatrix}$, where B is invertible.

   For instance, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, when $n = 2$.

   (c) A matrix for which all the four fundamental subspaces are equal

   Solution: Not Possible

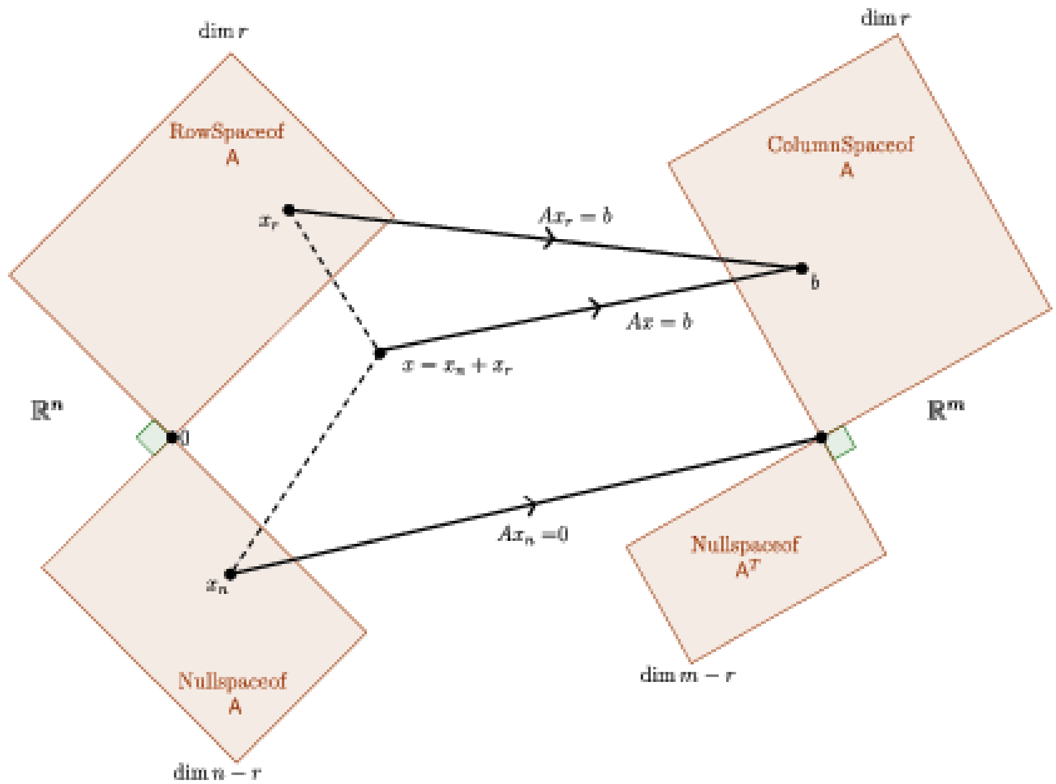

Figure : Fundamental subspaces of a $m \times n$ ***matrix*** A

Given a real $m \times n$ matrix A, there are four associated vector subspaces which are known colloquially as its fundamental subspaces, namely the column spaces and the null spaces of the matrices A and its transpose A^T.
The row space and the nullspace together span the domain of the linear transformation: R^n. Their intersection only contains 1 element: the n component 0 vector.
Similarly, the column space and the left nullspace together span the co-domain of the linear transformation: R^m. Their intersection only contains 1 element: the m component 0 vector.
For any $m \times n$ matrix A, the *rowspace* of A and *nullspace* of A are always orthogonal, similarly the same applies for *columnspace* of A and *nullspace* of A^T as shown in the Figure.Thus *rowspace* of A and *nullspace* of A cannot be equal and the *columnspace* of A and *nullspace* of A^T cannot be equal.Therefore constructing a matrix for which all four fundamental subspaces are equal is not possible.

11. (1 point) True or false? If A is a $m \times m$ square matrix then $\mathcal{N}(A) = \mathcal{N}(A^2)$ (If true give logical, valid reasoning or give a counterexample if false)

Solution: False. Counterexample:

A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} k \\ k \end{bmatrix}$ lies in the Null space of A^2, but not in the Null space of A.

12. (2 points) Consider matrices A and B and their product AB. For each of the questions below fill in the blanks with one of the following options: $<, >, =, \leq, \geq$, *can't say*. Explain your answer.

(a) $dim(\mathcal{C}(AB))$_______________$dim(\mathcal{C}(A))$

Solution: $\leq$

Multiplication of AB means the getting the linear combinations of A in each columns of AB matrix.

Thus $C(AB) \subseteq C(A)$

Also each rows of matrix AB are combination of rows of B .

Thus $R(AB) \subseteq R(B)$

From Gaussian elimination, we know that dimension of column space is same as row space so which is equal to the rank of the matrix.

Hence, $dim(C(AB)) \leq \min\{dim(C(A)),\ dim(C(B))\}$
dim(C(AB)) $\leq$ dim(C(A).

(b) $dim(\mathcal{C}(AB))$_______________$dim(\mathcal{C}(B))$

Solution: $\leq$
From the above proof we can write that
$dim(C(AB)) \leq \min\{dim(C(A)),\ dim(C(B))\}$
Hence, dim(C(AB)) $\leq$ dim(C(B)).

(c) $dim(\mathcal{C}((AB)^\top))$_______________$dim(\mathcal{C}(A^\top))$

Solution: $\leq$

From the above proof we can write that

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

$$dim(C(B^{\mathrm{T}}A^{\mathrm{T}}) \leq dim(C(\mathrm{A}^{\mathrm{T}})$$

$$\mathrm{dim}(\mathrm{C}((AB)^{\mathrm{T}})) \leq \mathrm{dim}(\mathrm{C}(A^{\mathrm{T}}))\ .$$

(d) $dim(\mathcal{C}((AB)^\top))$_______________$dim(\mathcal{C}(B^\top))$

Solution: $\leq$

From the above proof we can write that

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

$$dim(C(B^{\mathrm{T}}A^{\mathrm{T}}) \leq dim(C(\mathrm{B}^{\mathrm{T}})$$

$$\mathrm{dim}(\mathrm{C}((AB)^{\mathrm{T}})) \leq \mathrm{dim}(\mathrm{C}(B^{\mathrm{T}}))\ .$$

Concept: Free variables

13. (2 ½ points) True or False (with reason if true or example to show it is false).

(a) A square matrix has no free variables

Solution: False. A counter example is matrix A = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Matrix A has one free variable.

(b) An invertible matrix has no free variables

Solution: True.

A is invertible matrix iff Ax = 0 has the trivial solution only.

Now, if we assume that invertible matrix to have free variables, then there would have been a non-zero vector x for which Ax = 0, but that can't happen because A is an invertible matrix.

(c) An $m \times n$ matrix has no more than n pivot variables.

Solution: True, because there are only n variables in total.

(d) An $m \times n$ matrix has no more than m pivot variables.

Solution: True, because there are only m rows and each row has atmost one pivot variable.

(e) Matrices A and A^T have the same number of pivots.

Solution: True, because it is the rank of matrix. And we know that rank(A) = rank(A^T)

Concept: Reduced Echelon Form

14. (½ point) Suppose R is $m \times n$ matrix of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a right-inverse B with $RB = I$ if $r = m$.

Solution: If the matrix R has rank r and r = m (i.e. number of rows), then the zeros in the last row vanish and we can write R as:
R = $\begin{bmatrix} I_{r\times r} \; F_{r\times(n-r)} \end{bmatrix}$

Now, $R_{r\times n}B_{n\times r} = I_{r\times r}$

$B = \begin{bmatrix} I_{r\times r} \\ 0_{(n-r)\times r} \end{bmatrix}$

An example of R with r=m=3 and n=5:
R = $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 5 & 4 \\ 0 & 0 & 1 & -2 & 4 \end{bmatrix}$
We need a $B_{5\times 3}$ such that RB = I. Basically, we need a linear combination of the columns of R in such a way that the first three columns come as it is, whereas the last 2 columns do not contribute at all.

So, B = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$