Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

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Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Concept: Projection

2. (2 points) Consider a matrix A and a vector $\mathbf{b}$ which does not lie in the column space of A. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on to the column space of A. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \\ 11 \\ 8 \end{bmatrix}$, find $\mathbf{b}$.

Solution: We know that $A^T\mathbf{e} = 0$ where $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Using this fact, we will find all $\mathbf{e} \in \mathcal{N}(A^T)$. By adding $\mathbf{e}$ to $\mathbf{p}$, we will get $\mathbf{b}$.

$$A^T\mathbf{e} = 0$$

Getting the RREF of A^T,

$$RREF = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Getting $\mathbf{e}$ by using RREF form $R = \begin{bmatrix} I & F \end{bmatrix}$,

$$\mathbf{e} = m \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + n \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (m, n \in \mathbb{R})$$

Hence,

$$\mathbf{b} = \mathbf{p} + \mathbf{e}$$
$$\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 11 \\ 8 \end{bmatrix} + m \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + n \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3. (2 points) Consider the following statement: Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

   (a) Give one example where the above statement is True.

Solution: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$. Let $b_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Then, our projection matrix is,

$$\begin{aligned}
P &= A(A^T A)^{-1} A^T \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} (\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{5}{6} & \dfrac{1}{6} & \dfrac{1}{3} \\ \dfrac{1}{6} & \dfrac{5}{6} & \dfrac{-1}{3} \\ \dfrac{1}{3} & \dfrac{-1}{3} & \dfrac{1}{3} \end{bmatrix}
\end{aligned}$$

Projection of vector b_1 onto the $\mathcal{C}(A)$ is,

$$p_1 = Pb_1$$

$$p_1 = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

Projection of vector b_2 onto the $\mathcal{C}(A)$ is,

$$p_2 = Pb_2$$

$$p_2 = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} \frac{7}{6} \\ \frac{-1}{6} \\ \frac{2}{3} \end{bmatrix}$$

Here, we see that $p_1 \neq p_2$.

(b) Give one example where the above statement is False.

Solution: Let $b_1 = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Projection of vector b_1 onto the $\mathcal{C}(A)$ is,

$$p_1 = Pb_1$$

$$p_1 = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ \frac{-1}{3} \end{bmatrix}$$

Projection of vector b_2 onto the $\mathcal{C}(A)$ is,

$$p_2 = Pb_2$$

$$p_2 = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ \frac{-1}{3} \end{bmatrix}$$

Here, we see that $p_1 = p_2$.

(c) Based on the above examples, state the generic condition under which the above statement will be True or False.

Solution: Given two vectors b_1 and b_2, let the difference between the two vectors be represented by $d = b_1 - b_2$. If $d \in \mathcal{N}(A^T)$, then projections of b_1 and b_2 onto the $\mathcal{C}(A)$ will be the same and the condition mentioned in the question will be False.
Alternatively, if $d \notin \mathcal{N}(A^T)$, then projections of b_1 and b_2 onto the $\mathcal{C}(A)$ will be different and the condition mentioned in the question will be True.

4. (2 points) (a) Find the projection matrix P_1 that projects onto the line through $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to $\mathbf{a}$.

Solution: For projection matrix P_1,

$$\begin{aligned} P_1 &= \frac{\mathbf{a}\mathbf{a}^\mathbf{T}}{\mathbf{a}^\mathbf{T}\mathbf{a}} \\ &= \frac{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}}{\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}} \\ &= \frac{\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}}{10} \\ &= \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix} \end{aligned}$$

The line $\mathbf{b} \perp \mathbf{a}$ will satisfy the condition: $\mathbf{b}^T\mathbf{a} = \mathbf{a}^T\mathbf{b} = 0$. Hence, $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

For projection matrix P_2,

$$\begin{aligned}
P_2 &= \frac{\mathbf{b}\mathbf{b}^{\mathbf{T}}}{\mathbf{b}^{\mathbf{T}}\mathbf{b}} \\
&= \frac{\begin{bmatrix}-3\\1\end{bmatrix}\begin{bmatrix}-3 & 1\end{bmatrix}}{\begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}-3\\1\end{bmatrix}} \\
&= \frac{\begin{bmatrix}9 & -3\\-3 & 1\end{bmatrix}}{10} \\
&= \begin{bmatrix}\frac{9}{10} & \frac{-3}{10}\\ \frac{-3}{10} & \frac{1}{10}\end{bmatrix}
\end{aligned}$$

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain the result.

Solution:

$$\begin{aligned}
P_1 + P_2 &= \frac{\mathbf{a}\mathbf{a}^{\mathbf{T}}}{\mathbf{a}^{\mathbf{T}}\mathbf{a}} + \frac{\mathbf{b}\mathbf{b}^{\mathbf{T}}}{\mathbf{b}^{\mathbf{T}}\mathbf{b}} \\
&= \begin{bmatrix}\frac{1}{10} & \frac{3}{10}\\ \frac{3}{10} & \frac{9}{10}\end{bmatrix} + \begin{bmatrix}\frac{9}{10} & \frac{-3}{10}\\ \frac{-3}{10} & \frac{1}{10}\end{bmatrix} \\
&= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}
\end{aligned}$$

For $P_1 P_2$,

$$\begin{aligned}
P_1 P_2 &= \frac{\mathbf{a}\mathbf{a}^{\mathbf{T}}}{\mathbf{a}^{\mathbf{T}}\mathbf{a}} \frac{\mathbf{b}\mathbf{b}^{\mathbf{T}}}{\mathbf{b}^{\mathbf{T}}\mathbf{b}} \\
&= \frac{\mathbf{a}\mathbf{a}^{\mathbf{T}}\mathbf{b}\mathbf{b}^{\mathbf{T}}}{(\mathbf{a}^{\mathbf{T}}\mathbf{a})(\mathbf{b}^{\mathbf{T}}\mathbf{b})} \\
&= 0 && \text{(Since } \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = 0 \text{ i.e., } \mathbf{a} \perp \mathbf{b}.\text{)}
\end{aligned}$$

For two vectors $\mathbf{a}$ and $\mathbf{b}$, the above result tells us that if $\mathbf{a} \perp \mathbf{b}$, then their respective projection matrices P_1 and P_2 are also perpendicular. Also, projection matrices that sum to identity matrix are orthogonal to each other. In other words, $\mathcal{C}(P_1) \perp \mathcal{C}(P_2)$.

Concept: Dot product of vectors

5. (1 point) Consider two vectors $\mathbf{u}$ and $\mathbf{v}$. Let θ be the angle between these two vectors. Prove that

$$cos\theta = \frac{\mathbf{u}^\top \mathbf{v}}{||\mathbf{u}||_2 ||\mathbf{v}||_2}$$

Solution: Let vector $u = \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \end{bmatrix}$ and $v = \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix}$ make an angle α and β respectively with the x-axis where $\beta > \alpha$ or $\beta - \alpha = \theta$.
Hence, $\sin\alpha = \frac{\mathbf{u_2}}{||\mathbf{u}||_\mathbf{2}}$ and $\cos\alpha = \frac{\mathbf{u_1}}{||\mathbf{u}||_\mathbf{2}}$. Also, $\sin\beta = \frac{\mathbf{v_2}}{||\mathbf{v}||_\mathbf{2}}$ and $\cos\beta = \frac{\mathbf{v_1}}{||\mathbf{v}||_\mathbf{2}}$.
Now,

$$\theta = \beta - \alpha$$

Multiplying throughout the equation by cos,

$$\begin{aligned}
\cos\theta &= \cos(\beta - \alpha) \\
\cos\theta &= \cos\beta\cos\alpha + \sin\beta\sin\alpha \\
\cos\theta &= \frac{\mathbf{v_1}}{||\mathbf{v}||_\mathbf{2}}\frac{\mathbf{u_1}}{||\mathbf{u}||_\mathbf{2}} + \frac{\mathbf{v_2}}{||\mathbf{v}||_\mathbf{2}}\frac{\mathbf{u_2}}{||\mathbf{u}||_\mathbf{2}} \\
\cos\theta &= \frac{\mathbf{v_1 u_1 + v_2 u_2}}{||\mathbf{v}||_\mathbf{2}||\mathbf{u}||_\mathbf{2}} \\
cos\theta &= \frac{\mathbf{u}^\top \mathbf{v}}{||\mathbf{u}||_2 ||\mathbf{v}||_2}
\end{aligned}$$

Concept: Vector norms

6. (1 point) The L_p-norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ is defined as:

$$||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

(a) Prove that $||\mathbf{x}||_\infty = max_{1\leq i\leq n}|x_i|$

Solution: Let's assume that $x_1 = \max\{|x_1|, |x_2|, |x_3|, \cdots, |x_n|\}$.

$$\begin{aligned}
||\mathbf{x}||_\infty &= \lim_{p\to\infty} (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \\
&= \lim_{p\to\infty} \sqrt[p]{|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p} \\
&= \lim_{p\to\infty} \sqrt[p]{|x_1|^p\Big(1 + \frac{|x_2|^p}{|x_1|^p} + \frac{|x_3|^p}{|x_1|^p} + \cdots + \frac{|x_n|^p}{|x_1|^p}\Big)} \\
&= \lim_{p\to\infty} \sqrt[p]{|x_1|^p\Big(1 + \Big(\frac{|x_2|}{|x_1|}\Big)^p + \cdots + \Big(\frac{|x_n|}{|x_1|}\Big)^p\Big)}
\end{aligned}$$

Since $|x_1|$ is the maximum value, $\Big(\frac{|x_i|}{|x_1|}\Big) \leq 1$. Also, as $p \to \infty$, $\Big(\frac{|x_i|}{|x_1|}\Big)^p \to 0$.

$$\begin{aligned}
&= \lim_{p\to\infty} \sqrt[p]{|x_1|^p \cdot 1} \\
&= |x_1| \\
&= max_{1\leq i\leq n}|x_i|
\end{aligned}$$

(b) True or False (explain with reason): $||\mathbf{x}||_0$ is a norm.

Solution: False.

$$||\mathbf{x}||_0 = (|x_1|^0 + |x_2|^0 + |x_3|^0 + \cdots + |x_n|^0)^{\frac{1}{0}}$$

Assuming $x_1, x_2, \cdots, x_n$ are non-zero, $|x_i|^0 = 1$.

$$= (n)^{\frac{1}{0}}$$

The above equation is not defined for the zeroth-root. Hence, $||\mathbf{x}||_0$ is not a norm.

Concept: Orthogonal/Orthonormal vectors and matrices

7. (1 point) Consider the following questions:

   (a) Construct a 2×2 matrix, such that all its entries are +1 and -1 and its columns are orthogonal.

Solution: $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

(b) Now, construct a 4×4 matrix, such that all its entries are +1 and -1, its columns are orthogonal and it contains the above matrix within it.

Solution: $\frac{1}{2}\begin{bmatrix}1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & -1 & -1 & -1\\ 1 & 1 & 1 & -1\end{bmatrix}$

8. (1 point) Consider the vectors $\mathbf{a} = \begin{bmatrix}4\\5\\2\\2\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}$

(a) What multiple of $\mathbf{a}$ is closest to $\mathbf{b}$?

Solution:

$$\mathbf{p} = \frac{\mathbf{a}\mathbf{a}^\mathbf{T}}{\mathbf{a}^\mathbf{T}\mathbf{a}}\mathbf{b}$$

$$\mathbf{p} = \frac{\begin{bmatrix}4\\5\\2\\2\end{bmatrix}\begin{bmatrix}4 & 5 & 2 & 2\end{bmatrix}}{\begin{bmatrix}4 & 5 & 2 & 2\end{bmatrix}\begin{bmatrix}4\\5\\2\\2\end{bmatrix}}\begin{bmatrix}1\\2\\0\\0\end{bmatrix}$$

$$\mathbf{p} = \frac{\begin{bmatrix}16 & 20 & 8 & 8\\ 20 & 25 & 10 & 10\\ 8 & 10 & 4 & 4\\ 8 & 10 & 4 & 4\end{bmatrix}\begin{bmatrix}1\\2\\0\\0\end{bmatrix}}{49}$$

$$\mathbf{p} = \frac{\begin{bmatrix}56\\70\\28\\28\end{bmatrix}}{49}$$

Since $\mathbf{p} = \hat{\mathbf{x}}\mathbf{a}$,

$$\mathbf{p} = \hat{\mathbf{x}} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ \frac{10}{7} \\ \frac{4}{7} \\ \frac{4}{7} \end{bmatrix}$$

Hence, the multiple $\hat{\mathbf{x}} = \frac{2}{7}$ for which the above equation holds.

(b) Find orthonormal vectors $\mathbf{q_1}$ and $\mathbf{q_2}$ that lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$?

Solution: Let $\mathbf{a_1} = a$ and $\mathbf{a_2} = b$. Let us retain $\mathbf{a_1}$ as the first orthogonal vector. Then, $\hat{\mathbf{a_1}} = a_1$. Let $\mathbf{p}$ be the component of $\mathbf{a_2}$ along a_1. Let $\mathbf{e}$ be the component of $\mathbf{a_2}$ orthogonal to a_1. We want to delete $\mathbf{p}$ and retain $\mathbf{e}$. We have the projection $\mathbf{p}$ from the previous question. Therefore,

$$\hat{\mathbf{a_2}} = \mathbf{e} = \mathbf{a_2} - \mathbf{p}$$

$$\hat{\mathbf{a_2}} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{8}{7} \\ \frac{10}{7} \\ \frac{4}{7} \\ \frac{4}{7} \end{bmatrix}$$

$$\hat{\mathbf{a_2}} = \begin{bmatrix} \frac{-1}{7} \\ \frac{4}{7} \\ \frac{-4}{7} \\ \frac{-4}{7} \end{bmatrix}$$

After normalizing $\hat{\mathbf{a_1}}$ and $\hat{\mathbf{a_2}}$, we get $\mathbf{q_1} = \begin{bmatrix} \frac{4}{7} \\ \frac{5}{7} \\ \frac{2}{7} \\ \frac{2}{7} \end{bmatrix}$ and $\mathbf{q_2} = \begin{bmatrix} \frac{-1}{7} \\ \frac{4}{7} \\ \frac{-4}{7} \\ \frac{-4}{7} \end{bmatrix}$.

9. (1 point) Suppose $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal vectors. Prove that they are also independent.

Solution: We know that $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal. To prove they are all independent, we have to show that only for $c_1 = c_2 = \cdots = c_k = 0$ in the following equation:

$$0 = c_1\mathbf{a_1} + c_2\mathbf{a_2} + \cdots + c_n\mathbf{a_n}$$

Compute the dot product of $\mathbf{a_i^T}$ throughout the equation for $i = 1, 2, \ldots, n$.

$$0 = c_1\mathbf{a_i^T}\mathbf{a_1} + c_2\mathbf{a_i^T}\mathbf{a_2} + \cdots + c_n\mathbf{a_i^T}\mathbf{a_n}$$

Since $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal vectors, $\mathbf{a_i^T}\mathbf{a_j} = 0$ for $i \neq j$. Hence,

$$\begin{aligned} 0 &= c_i\mathbf{a_i^T}\mathbf{a_i} \\ 0 &= c_i\|\mathbf{a}_i\|^2 \\ c_i &= 0 && \text{(Since } \mathbf{a_i} \text{ is a non-zero vector, } \|\mathbf{a}_i\| > 0\text{)} \end{aligned}$$

Since $c_i = 0$ for $i = 1, 2, \ldots, n$, vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal and independent to each other.

10. (1 point) If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also an orthogonal matrix.

Solution: Let Q_1 and Q_2 be two $n \times n$ matrices where $Q_1 = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \end{bmatrix}$ and $Q_2 = \begin{bmatrix} Q_{21} & Q_{22} & \cdots & Q_{2n} \end{bmatrix}$. Here, Q_{11} to Q_{1n} are columns of Q_1 and Q_{21} to Q_{2n} are columns of Q_2. Then,

$$Q_1Q_2 = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \end{bmatrix} \begin{bmatrix} Q_{21} & Q_{22} & \cdots & Q_{2n} \end{bmatrix}$$

Since each column of Q_2 will linearly combine columns of Q_1 to give the respective column of Q_1Q_2,

$$Q_1Q_2 = \begin{bmatrix} (\sum_{i=1}^{n} Q_{1i})Q_{21} & (\sum_{i=1}^{n} Q_{1i})Q_{22} & \cdots & (\sum_{i=1}^{n} Q_{1i})Q_{2n} \end{bmatrix}$$

$$Q_1Q_2 = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$$

where $C_j = (\sum_{i=1}^{n} Q_{1i})Q_{2j}$ for $j = 1, 2, ..., n$. For Q_1Q_2 to be an orthogonal matrix, columns of Q_1Q_2 must be orthogonal to each other. We will show that $C_r \perp C_s$ for $r,\ s = 1, 2, ..., n$ and $r \neq s$.

$$\begin{aligned}
C_1^T C_2 &= ((\sum_{i=1}^{n} Q_{1i})Q_{21})^T (\sum_{i=1}^{n} Q_{1i})Q_{22} \\
&= (Q_{21}^T (\sum_{i=1}^{n} Q_{1i})^T (\sum_{i=1}^{n} Q_{1i})Q_{22}) \\
&= (Q_{21}^T (Q_{11}^T + Q_{12}^T + \cdots + Q_{1n}^T)(Q_{11} + Q_{12} + \cdots + Q_{1n})Q_{22})
\end{aligned}$$

Since columns of Q_1 are orthogonal, $Q_{1r} \perp Q_{1s} = Q_{1r}^T Q_{1s} = 0$ for $r,\ s = 1, 2, ..., n$ and $r \neq s$.

$$= (Q_{21}^T (Q_{11}^T Q_{11} + Q_{12}^T Q_{12} + \cdots + Q_{1n}^T Q_{1n})Q_{22})$$

Since for any vector x, $x^T x = ||x||_2^2$,

$$\begin{aligned}
&= (Q_{21}^T (||Q_{11}||_2^2 + ||Q_{12}||_2^2 + \cdots + ||Q_{1n}||_2^2)Q_{22}) \\
&= \sum_{i=1}^{n} ||Q_{1i}||_2^2 (Q_{21}^T Q_{22})
\end{aligned}$$

Since columns of Q_2 are orthogonal, $Q_{2r} \perp Q_{2s} = Q_{2r}^T Q_{2s} = 0$ for $r,\ s = 1, 2, ..., n$ and $r \neq s$.

$$\begin{aligned}
&= (\sum_{i=1}^{n} ||Q_{1i}||_2^2) \cdot 0 \\
&= 0
\end{aligned}$$

Hence, columns of Q_1Q_2 are orthogonal, thereby, making Q_1Q_2 an orthogonal matrix.

Concept: Determinants

11. (2 points) A tri-diagonal matrix is a matrix which has 1's on the main diagonal as well as on the diagonals to the left and right of the main diagonal. For example,

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let A_n be an $n \times n$ tri-diagonal matrix. Prove that $|A_n| = |A_{n-1}| - |A_{n-2}|$

Solution: To prove the above equation, we will use a basic fact: Every determinant can be written as a linear combination of any row i times its co-factors.

Let $D_i = det(A_i)$ for an $i \times i$ matrix.

Without loss of generality,

$$D_0 = 1$$
$$D_1 = a_1$$
$$D_2 = det(\begin{bmatrix} a_2 & b_1 \\ c_1 & a_1 \end{bmatrix}) = a_2a_1 - b_1c_1 = a_2D_1 - b_1c_1D_0$$
$$D_3 = det(\begin{bmatrix} a_3 & b_2 & 0 \\ c_2 & a_2 & b_1 \\ 0 & c_1 & a_1 \end{bmatrix}) = a_3D_2 - b_2c_2a_1 = a_3D_2 - b_2c_2D_1$$
$$D_4 = det(\begin{bmatrix} a_4 & b_3 & 0 & 0 \\ c_3 & a_3 & b_2 & 0 \\ 0 & c_2 & a_2 & b_1 \\ 0 & 0 & c_1 & a_1 \end{bmatrix}) = a_4D_3 - b_3c_3(a_2a_1 - b_1c_1) = a_4D_3 - b_3c_3D_2$$
$$D_5 = det(\begin{bmatrix} a_5 & b_4 & 0 & 0 & 0 \\ c_4 & a_4 & b_3 & 0 & 0 \\ 0 & c_3 & a_3 & b_2 & 0 \\ 0 & 0 & c_2 & a_2 & b_1 \\ 0 & 0 & 0 & c_1 & a_1 \end{bmatrix}) = a_5D_4 - b_4c_4D_3$$

Using the structure of a tri-diagonal matrix, we see a pattern here. For a $n \times n$ matrix, $D_n = a_nD_{n-1} - b_{n-1}c_{n-1}D_{n-2}$. In our tri-diagonal matrix A, $a_1 = b_1 = c_1 = 1$.

Therefore, $D_n = D_{n-1} - D_{n-2}$.

Another way to look at this is using block determinants.

Let $A_n = \begin{bmatrix} A_{n-1} & u \\ v & a_1 \end{bmatrix}$ where $u = [0, 0, 0, \ldots, b_1]^T$ is a $(n-1) \times 1$ vector and $v = [0, 0, 0, \ldots, c_1]^T$ is a $(n-1) \times 1$ vector. Then, by using formula for determinant of block matrices,

$$det(A_n) = det(A_{n-1}) \times det(a_1 - v(A_{n-1})^{-1}u)$$
$$det(A_n) = det(A_{n-1}) \times (a_1 - b_1c_1(A_{n-1})^{-1}) \tag{1}$$

Using $A^{-1} = \frac{C^T}{det(A)}$ and $det(C^T) = det(C)$,

$$\because (A_{n-1})^{-1}_{i,i} = \frac{(-1)^{i+i}\det(A_{n-2})}{\det(A_{n-1})} = \frac{\det(A_{n-2})}{\det(A_{n-1})} \tag{2}$$

Using (2) in (1), we get

$$\begin{aligned} det(A_n) &= det(A_{n-1}) \times (a_1 - b_1 c_1 \frac{\det(A_{n-2})}{\det(A_{n-1})}) \\ det(A_n) &= a_1 det(A_{n-1}) - b_1 c_1 \det(A_{n-2}) \end{aligned}$$

In our tri-diagonal matrix A, $a_1 = b_1 = c_1 = 1$. Therefore, $det(A_n) = det(A_{n-1}) - \det(A_{n-2})$.

12. (1 point) State True or False and explain your answer: $det(A+B) = det(A) + det(B)$

Solution: False.

Property 2 says that we can add two determinants only one row at a time with all except the added row being the same in both same matrices. This is different from finding the determinant of matrix $A+B$ where addition of two matrices adds all the rows at once. Therefore, $det(A+B) \neq det(A) + det(B)$.

For instance, let's take an example of a 2×2 matrix. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Hence, $A+B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$. Since $det(A) = ad - bc$ and $det(B) = eh - gf$, we claim that $det(A) + det(B) = (ad + eh) - (bc + gf)$.

$$\begin{aligned} det(A+B) &= \begin{vmatrix} a+e & b+f \\ c+g & d+h \end{vmatrix} \\ &= \begin{vmatrix} a & b \\ c+g & d+h \end{vmatrix} + \begin{vmatrix} e & f \\ c+g & d+h \end{vmatrix} && \text{(Property 3b)} \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ g & h \end{vmatrix} + \begin{vmatrix} e & f \\ c & d \end{vmatrix} + \begin{vmatrix} e & f \\ g & h \end{vmatrix} && \text{(Property 3b)} \\ &= (ad - bc) + (ah - bg) + (ed - cf) + (eh - gf) \\ &= (ad + ah + ed + eh) - (bc + bg + cf + gf) \\ &\neq (ad + eh) - (bc + gf) \\ &\neq det(A) + det(B) \end{aligned}$$

13. (1 point) This question is about properties 9 and 10 of determinants.

(a) Prove that $det(AB) = det(A)det(B)$

Solution: We know that $det(EA) = det(E)det(A)$. This is because:

1. Let E be an elementary matrix performing row exchange on I. Then, matrix EA is the result of row exchange of the corresponding two rows of A. Then, det(EA) = (−1) det(A), where $det(E) = -1$.

2. Let E be an elementary matrix obtained by multiplying nonzero scalar k to the entries of a row of I. Then, EA results in a matrix where entries of a specific row of A is scaled by k. Then, $det(EA) = kdet(A)$, since $det(E) = det(kI) = kdet(I) = k$.

3. Let E be an elementary matrix which scales one row of I and adds it to another row of I. Then, matrix EA results in adding a multiple of a row of A to another row of A. Then, $det(EA) = det(A)$, since scaled addition of one row to another preserves value of determinant.

There are two scenarios which we would like to prove: A is invertible, and A is not invertible.

Part(a): First, let's assume that A is invertible. Then, Gauss-Jordan of A tells us that:

$$\begin{aligned} A &= E_n E_{n-1} \cdots E_1 \\ det(A) &= det(E_n E_{n-1} \cdots E_1) && \text{(Taking determinant on both sides)} \\ det(A) &= det(E_n)det(E_{n-1}) \cdots det(E_1) && (\because det(EA) = det(E)det(A)) \end{aligned}$$

Now, let's claim that,

$$\begin{aligned} det(AB) &= det(E_n E_{n-1} \cdots E_1 B) \\ det(AB) &= det(E_n)det(E_{n-1}) \cdots det(E_1)det(B) && (\because det(EA) = det(E)det(A)) \\ det(AB) &= det(E_n E_{n-1} \cdots E_1)det(B) \\ det(AB) &= det(A)det(B) \end{aligned}$$

Part(b): Now, assume that A is not invertible. Then, $det(A) = 0$. Also, $det(AB) = 0$ since matrix AB is linear combination of columns of A, thereby, making matrix AB singular. Hence, $det(AB) = det(A)det(B) = 0$.

(b) (2 points) Prove that $det(A^\top) = det(A)$

Solution: We know that $det(EA) = det(E)det(A)$. Also, using Property 7, $det(E^T) = det(E)$.
Gauss-Jordan of A gives the following:

$$\begin{aligned} A &= E_n E_{n-1} \cdots E_1 \\ det(A) &= det(E_n E_{n-1} \cdots E_1) && \text{(Taking determinant on both sides)} \end{aligned}$$

Since $det(EA) = det(E)det(A)$,

$$det(A) = det(E_n)det(E_{n-1}) \cdots det(E_1) \tag{1}$$

Now, for A^T,

$$\begin{aligned} A^T &= (E_n E_{n-1} \cdots E_1)^T \\ A^T &= E_1^T E_2^T \cdots E_n^T \\ det(A^T) &= det(E_1^T E_2^T \cdots E_n^T) && \text{(Taking determinant on both sides)} \\ det(A^T) &= det(E_1^T)det(E_2^T) \cdots det(E_n^T) && (\because det(EA) = det(E)det(A)) \\ det(A^T) &= det(E_1)det(E_2) \cdots det(E_n) && (\because det(E^T) = det(E)) \\ det(A^T) &= det(A) && \text{(From (1))} \end{aligned}$$

14. (1 point) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

(a) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$

Solution: For a 2×2 matrix A in the $2D$ plane, $det(A)$ is equal to the area of a parallelogram made by two vectors emerging from the same vertex. These two vectors make up the columns of the matrix A. Hence,

$$\begin{aligned} A &= \begin{bmatrix} u & v \end{bmatrix} \\ A &= \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \\ det(A) &= \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} \\ det(A) &= 12 - 2 = 10 && (det(A) = ad - bc) \end{aligned}$$

Therefore, Area of triangle $= \dfrac{det(A)}{2} = 5$.

(b) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$

Solution: Same as above, Area of triangle $= \dfrac{det(A)}{2} = 5$, since the parallelogram is made by the two vectors emerging from the same vertex.

15. (2 points) The determinant of the following matrix can be computed as a sum of 120 (5!) terms.

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

State true or false with an appropriate explanation: All the 120 terms in the determinant will be 0.

Solution: True.

We know that,

$$det(\text{matrix}_{n\times n}) = \sum_{P \,\in\, n!\text{ Permutations}} a_{1\alpha} a_{1\beta} \cdots a_{1\omega}\ det(P)$$

where $\{\alpha, \beta, \cdots, \omega\}$ = some permutation of {1,2,3,$\cdots$, n}

$$det(\text{matrix}_{n\times n}) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \tag{1}$$

Using (1), $det(\text{matrix}_{3\times 3}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$. Since each co-factor is made up of two terms, the determinant for a 3×3 matrix can be calculated using only 3 co-factors. Similarly, number of co-factors for $n \times n$ is $\frac{n!}{2}$. Therefore, for a 5×5 matrix, there are 120 terms or 60 co-factors in its determinant. If we prove all 60 co-factors are 0, then all 120 terms are zero too.

$$\begin{aligned} det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} + a_{15}C_{15} \\ det(A) &= xC_{11} + xC_{12} + xC_{13} + xC_{14} + xC_{15} \end{aligned} \tag{2}$$

We can say more about these co-factors. The determinant of each co-factor is made up of combination of its own co-factors. For instance,

$$\begin{aligned} det(C_{11}) &= a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24} + a_{25}C_{25} \\ det(C_{11}) &= xC_{22} + xC_{23} + xC_{24} + xC_{25} \end{aligned}$$

Similarly, for $det(C_{22})$,

$$det(C_{22}) = a_{33}C_{33} + a_{34}C_{34} + a_{35}C_{35}$$
$$det(C_{22}) = 0C_{33} + xC_{34} + xC_{35}$$

Similarly, for $det(C_{33})$,

$$det(C_{33}) = a_{44}C_{44} + a_{45}C_{45}$$
$$det(C_{33}) = xC_{44} + xC_{45}$$

Since $C_{44} = x$ and $C_{45} = -x$,

$$det(C_{33}) = x(x) + x(-x)$$
$$det(C_{33}) = 0$$

What does this continuous train of determinant of co-factors tell us? It tells us that each of the co-factors in (2) is made of 12 smaller co-factors. The determinant of smaller co-factors (here, it is the 2×2 matrix at the bottom right corner) affects the determinant of its parent co-factors. Therefore, all co-factors in (2) is made up of combination of permutations of 2×2 co-factors in the last two rows. If we can prove that the determinant of all permutations of 2×2 co-factors in the last two rows is 0, then all combinations of permutations of these 2×2 co-factors which make up the parent co-factors will also be zero.

The last two rows of A consist of only two distinct columns $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \end{bmatrix}$. Therefore, there are only 3 distinct type of 2×2 co-factors which could be made from the two distinct columns. They are:

$$\det(\begin{bmatrix} 0 & x \\ 0 & x \end{bmatrix}) = 0 \qquad \text{(Property 6)}$$
$$\det(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 0 \qquad \text{(Property 4, 6)}$$
$$\det(\begin{bmatrix} x & x \\ x & x \end{bmatrix}) = 0 \qquad \text{(Property 4)}$$

Combination of any of the above co-factors make the determinant of parent co-factor equal to 0. This means that the determinant of all 12 co-factors for each of the parent co-factor in (2) is 0, thereby, making determinant of all parent co-factors in (2) equal to zero. Since all 60 (5×12) co-factors are zero, all 120 terms are equal to 0.