**Honor code**: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Kartik Bharadwaj
**Name and Signature**

1. (1 point) Have you read and understood the honor code?

**Solution:** Yes

**Concept**: System of linear equations

2. (2 points) This question has two parts as mentioned below:

   (a) Find a 2 x 3 system Ax = b whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

**Solution:** For matrix $A$, we can use the RREF form $R\mathbf{x} = 0$ where $R = \begin{bmatrix} I & F \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -F \\ I \end{bmatrix}$. Given only one basis of nullspace $\mathbf{x}_{nullspace} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $F = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix}$.

Since row space of $R$ is equal to row space of $A$ due to RREF, we can use,

$$R\mathbf{x}_{particular} = d$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = d$$

$$d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, our $2 \times 3$ system is: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Now find a 3 x 3 system which has these solutions exactly when $b_1 + b_2 = b_3$. (Note: $b = [b_1\ b_2\ b_3]^T$.)

**Solution:** First, we use the RREF form $R\mathbf{x} = 0$ where $R = \begin{bmatrix} I & F \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -F \\ I \end{bmatrix}$. Given only one basis of nullspace $\mathbf{x}_{nullspace} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $F = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

As row space of $R$ is equal to row space of $A$ due to RREF, we can use,

$$R\mathbf{x}_{particular} = d$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = d$$

$$d = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

We now add Row1 and Row2 to Row3, to get $b_1 + b_2 = b_3$ (or $d_1 + d_2 = d_3$). Therefore, our $3 \times 3$ system becomes: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

3. (2 points) Consider the matrices $A$ and $B$ below

(i) A = $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ (ii) B = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(a) Write down the row reduced echelon form of matrices $A$ and $B$ (also mention the steps involved).

**Solution:** For matrix $A$, we will first get reduced echelon form through Gaussian elimination. For $A$, we only need one operation: subtract Row1 from Row3.

Hence, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

$$REF = E_{31}A$$
$$REF = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we will subtract 2 times Row2 from Row1.

$$RREF = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$RREF = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, for matrix $B$, we will first get reduced echelon form through Gaussian elimination. We will perform 3 row operations:

1. $E_{21} \implies$ Subtract 4 times Row1 from Row2.
2. $E_{31} \implies$ Subtract 3 times Row1 from Row3.
3. $E_{32} \implies$ Subtract 2 times Row2 from Row3.

$$REF = E_{32}E_{31}E_{21}B$$
$$REF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B$$
$$REF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$REF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$
$$REF = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we divide Row2 by $-3$ and then subtract 2 times Row2 from Row1.

$$RREF = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$
$$RREF = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$RREF = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Find all solutions to $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.

**Solution:** For matrix $A$, $R\mathbf{x} = 0$ where $R = \begin{bmatrix} I & F \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -F \\ I \end{bmatrix}$. Hence,

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first 2 columns of $R$ are pivot columns and the last 2 columns are free columns.

$$x_{nullspace} = m \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + n \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(where } m, n \in \mathbb{R}\text{)}$$

For matrix $B$,

$$R\mathbf{x} = 0$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first 2 columns of $R$ are pivot columns and the last column is a free column.

$$x_{nullspace} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \text{(where } c \in \mathbb{R}\text{)}$$

(c) Write down the basis for the four fundamental subspaces of $A$.

**Solution:**

$$Basis(\mathcal{C}(\mathcal{A})) = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \}$$

$$Basis(\mathcal{N}(\mathcal{A})) = \{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$$

$$Basis(\mathcal{C}(\mathcal{A}^{\mathcal{T}})) = \{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \}$$

$$Basis(\mathcal{N}(\mathcal{A}^{\mathcal{T}})) = \{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$$

(d) Write down the basis for the four fundamental subspaces of $B$.

**Solution:**

$$Basis(\mathcal{C}(\mathcal{A})) = \{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \}$$

$$Basis(\mathcal{N}(\mathcal{A})) = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \}$$

$$Basis(\mathcal{C}(\mathcal{A}^{\mathcal{T}})) = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \}$$

$$Basis(\mathcal{N}(\mathcal{A}^{\mathcal{T}})) = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \}$$

**Concept**: Rank

4. (1 ½ points) Consider the matrices $A$ and $B$ as given below:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & x \end{bmatrix} \text{ and B} = \begin{bmatrix} 3 & 1 & 3 \\ y & 2 & y \end{bmatrix}$$

Give the values for entries $x$ and $y$ such that the ranks of the matrices $A$ and $B$ are

(a) 1

> **Solution:** $x = 3, y = 6$

(b) 2

> **Solution:** $x \in \mathbb{R}$ and $x \neq 3$; $y \in \mathbb{R}$ and $y \neq 6$

(c) 3

> **Solution:** There can be no values of $x$ and $y$ for which ranks of matrices $A$ and $B$ become 3.

**Concept**: Nullspace and column space

5. (½ point) State True or False and explain you answer: The nullspace of $R$ is the same as the nullspace of $U$ (where $R$ is the row reduced echelon form of $A$ and $U$ is the matrix in $LU$ decomposition of $A$).

> **Solution:** True because any kind of row operation performed on $U$ and $R$ doesn't affect the domain $\mathbf{x}$ for which $U\mathbf{x} = R\mathbf{x} = 0$.

6. (1 point) Suppose column 1 + column 2 + column5 = $\mathbf{0}$ in a $4 \times 5$ matrix $A$.

(a) What is a special solution for $A\mathbf{x} = \mathbf{0}$

> **Solution:** For a $4 \times 5$ matrix $A$, the RREF form will have 4 pivot columns and 1 free column. In particular, column 1 and column 2 will be two of the 4 pivot columns. Column 5 will be the free column. Note that, column 1 and column 2 can't be free column as it will violate the rule of RREF. Hence, $R\mathbf{x} = 0$ is as follows,
>
> $$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since column 1 + column 2 + column 5 = $\mathbf{0}$ and column 5 is free column, $x_5$ will be free variable and $x_1, x_2, x_3, x_4$ will be pivot variables. Using $R = \begin{bmatrix} I & F \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -F \\ I \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, our special solution is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Describe the null space of $A$.

**Solution:** $x_{nullspace} = \text{Span}\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$

7. (2 points) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. The column space of this matrix is a 2 dimensional plane. What is the equation of this plane? (You need to write down the steps you took to arrive at the equation)

**Solution:** We know that $\mathcal{C}(\mathcal{A})$ is a subspace which is the linear combinations of the basis of $A$ (or columns of $A$). To find the pivot columns, we can compute the reduced echelon form of $A$. We perform three row operations:

1. $E_{21} \implies$ Subtract Row1 from Row2.
2. $E_{31} \implies$ Subtract Row1 from Row3.
3. $E_{32} \implies$ Subtract 2 times Row2 from Row3.

We will assume that our $b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Also, we will perform these operations on $b$ as it will be useful further. Hence, our REF is:

$$REF = E_{32}E_{31}E_{21}A|b$$

$$REF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 2 & y \\ 1 & 3 & z \end{array}\right]$$

$$REF = \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y - x \\ 0 & 0 & x - 2y + z \end{array}\right]$$

Since we have only independent columns, $Basis(\mathcal{C}(\mathcal{A})) = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$. This means $\mathcal{C}(\mathcal{A})$ is a plane passing through zero vector $\in \mathbb{R}^3$. Hence, equation of the plane will be of the form $ax + by + cz = 0$. For the solution to exist in REF, the last row should be equal to 0.

Therefore, $x - 2y + z = 0$, where $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are points in the plane which is spanned by the $Basis(\mathcal{C}(\mathcal{A}))$.

8. (1 point) True or false? (If true give logical, valid reasoning or give a counterexample if false)

a. If the row space equals the column space then $A^T = A$

**Solution:** False. For instance, consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Although $\mathcal{C}(\mathcal{A}^T) = \mathcal{C}(\mathcal{A})$, $A \neq A^T$.

b. If $A^T = -A$ then the row space of A equals the column space.

**Solution:** True. We would like to prove that $\mathcal{C}(\mathcal{A}^T) = \mathcal{C}(\mathcal{A})$. But, since $A^T = -A$, we can prove $\mathcal{C}(-\mathcal{A}) = \mathcal{C}(\mathcal{A})$.

*Proof*: Since $A^T = -A$, we can say that $\mathcal{N(A)} = \mathcal{N(A^T)} = \mathcal{N(-A)}$. This is because, for any $x \in \mathcal{N(A)}$, $Ax = 0 \iff (-A)x = 0$. Therefore, if the nullspaces are the same, it is evident from rank-nullity theorem that $\mathcal{C(-A)} = \mathcal{C(A)}$.

9. (1 point) Which of the four fundamental subspaces are the same for the following pairs of matrices of different sizes? (Assume $A$ is a $m \times n$ matrix)

(a) $\begin{bmatrix} A \end{bmatrix}$ and $\begin{bmatrix} A \\ A \end{bmatrix}$

**Solution:** For a $m \times n$ matrix $P = \begin{bmatrix} A \end{bmatrix}$, we know that $\mathcal{C}(\mathrm{P})$ and $\mathcal{N}(\mathrm{P}^{\mathcal{T}}) \in \mathbb{R}^m$ whereas $\mathcal{C}(\mathrm{P}^{\mathcal{T}})$ and $\mathcal{N}(\mathrm{P}) \in \mathbb{R}^n$.

On the other hand, matrix $Q = \begin{bmatrix} A \\ A \end{bmatrix}$ is of the shape $2m \times n$. Hence, $\mathcal{C}(\mathrm{Q})$ and $\mathcal{N}(\mathrm{Q}^{\mathcal{T}}) \in \mathbb{R}^{2m}$ whereas $\mathcal{C}(\mathrm{Q}^{\mathcal{T}})$ and $\mathcal{N}(\mathrm{Q}) \in \mathbb{R}^n$.

Since the row space and null space belong to same vector space $\mathbb{R}^n$, we can clearly see that,

1. $\mathcal{C}(\mathrm{P}) \neq \mathcal{C}(\mathrm{Q})$
2. $\mathcal{N}(\mathrm{P}^{\mathcal{T}}) \neq \mathcal{N}(\mathrm{Q}^{\mathcal{T}})$
3. $\mathcal{C}(\mathrm{P}^{\mathcal{T}}) = \mathcal{C}(\mathrm{Q}^{\mathcal{T}})$
4. $\mathcal{N}(\mathrm{P}) = \mathcal{N}(\mathrm{Q})$

(b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$

**Solution:** For a $2m \times n$ matrix $P = \begin{bmatrix} A \\ A \end{bmatrix}$, we know that $\mathcal{C}(\mathrm{P})$ and $\mathcal{N}(\mathrm{P}^{\mathcal{T}}) \in \mathbb{R}^{2m}$ whereas $\mathcal{C}(\mathrm{P}^{\mathcal{T}})$ and $\mathcal{N}(\mathrm{P}) \in \mathbb{R}^n$.

On the other hand, matrix $Q = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ is of the shape $2m \times 2n$. Hence, $\mathcal{C}(\mathrm{Q})$ and $\mathcal{N}(\mathrm{Q}^{\mathcal{T}}) \in \mathbb{R}^{2m}$ whereas $\mathcal{C}(\mathrm{Q}^{\mathcal{T}})$ and $\mathcal{N}(\mathrm{Q}) \in \mathbb{R}^{2n}$.

Since the column space and left null space belong to same vector space $\mathbb{R}^{2m}$, we can clearly see that,

1. $\mathcal{C}(\mathrm{P}) = \mathcal{C}(\mathrm{Q})$

2. $\mathcal{N}(\mathrm{P}^{\mathcal{T}}) = \mathcal{N}(\mathrm{Q}^{\mathcal{T}})$

3. $\mathcal{C}(\mathrm{P}^{\mathcal{T}}) \neq \mathcal{C}(\mathrm{Q}^{\mathcal{T}})$

4. $\mathcal{N}(\mathrm{P}) \neq \mathcal{N}(\mathrm{Q})$

10. (2 points) For each of the questions below, construct a matrix $A$ which satisfies the given condition or argue why the given condition cannot be satisfied?

(a) A matrix whose row space is equal to its column space

**Solution:** If $A^T = -A$, then row space is equal to its column space. Consider, a matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We can see that $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}^{\mathcal{T}}) = Span(\{\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

(b) A matrix whose null space is equal to its column space

**Solution:** Let matrix $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Here, $\mathcal{C}(\mathcal{A}) = \mathcal{N}(\mathcal{A}) = Span(\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\})$, where the first 2 columns are free columns and the last 2 columns are pivot columns. Hence, any matrix of the form $A = \begin{bmatrix} O_{n\times n} & I_{n\times n} \\ O_{n\times n} & O_{n\times n} \end{bmatrix}$ where $O_{n\times n}$ represents a null matrix of size $n \times n$, $I_{n\times n}$ represents an identity matrix of size $n \times n$, and $n \in \mathbb{N}$.

(c) A matrix for which all the four fundamental subspaces are equal

**Solution:**
Let's assume that we have a matrix $A$ for which $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}^{\mathcal{T}}) = \mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}^{\mathcal{T}})$. This means that number of basis vectors required to span each of the subspaces are equal to each other. Hence,

$$dim(\mathcal{C}(\mathcal{A})) = dim(\mathcal{C}(\mathcal{A}^{\mathcal{T}})) = dim(\mathcal{N}(\mathcal{A})) = dim(\mathcal{N}(\mathcal{A}^{\mathcal{T}})) \qquad (1)$$

This happens only if $A$ is a $n \times n$ square matrix.
However, we know that $\mathcal{C}(\mathcal{A}) \perp \mathcal{N}(\mathcal{A}^{\mathcal{T}})$. This means they basis can't span the same subspace. The only common subspace is the zero vector. However, we know that:

1. If $Basis(\mathcal{C}(\mathcal{A}))$ is equal to the zero vector, then we have all dependent columns.
2. If $Basis(\mathcal{N}(\mathcal{A}^{\mathcal{T}}))$ is equal to the zero vector, then we have all independent columns(rows).

Both these conditions violates (1) and (2). Hence, $\mathcal{C}(\mathcal{A}) \neq \mathcal{N}(\mathcal{A}^{\mathcal{T}})$. Similarly, $\mathcal{C}(\mathcal{A}^{\mathcal{T}}) \neq \mathcal{N}(\mathcal{A})$ since $\mathcal{C}(\mathcal{A}^{\mathcal{T}}) \perp \mathcal{N}(\mathcal{A})$. Therefore, no matrix $A$ exists for which all the four fundamental subspaces are equal.

11. (1 point) True or false? If $A$ is a $m \times m$ square matrix then $\mathcal{N}(A) = \mathcal{N}(A^2)$ (If true give logical, valid reasoning or give a counterexample if false)

**Solution:** False. Let matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

We can see that $\mathcal{N}(A^2) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ whereas $\mathcal{N}(A) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

Hence, $\mathcal{N}(A) \neq \mathcal{N}(A^2)$.

12. (2 points) Consider matrices $A$ and $B$ and their product $AB$. For each of the questions below fill in the blanks with one of the following options: $<, >, =, \leq, \geq$, *can't say*. Explain your answer.

(a) $dim(\mathcal{C}(AB))$________________$dim(\mathcal{C}(A))$

**Solution:** Each column in the matrix product $AB$ is the linear combination of the columns of $A$. This implies that $\mathcal{C}(\mathcal{AB}) \subseteq \mathcal{C}(\mathcal{A})$. Hence,

$$dim(\mathcal{C}(\mathcal{AB})) \leq dim(\mathcal{C}(\mathcal{A}))$$

(b) $dim(\mathcal{C}(AB))$________________$dim(\mathcal{C}(B))$

**Solution:** Each row in the matrix product $AB$ is the linear combination of the rows of $B$. This implies that $Rowspace(\mathcal{AB}) \subseteq Rowspace(\mathcal{B})$. Since $rank = dim(Rowspace) = dim(columnspace)$,

$$dim(\mathcal{C}(\mathcal{AB})) \leq dim(\mathcal{C}(\mathcal{B}))$$

(c) $dim(\mathcal{C}((AB)^\top))$________________$dim(\mathcal{C}(A^\top))$

**Solution:** Each row in the matrix product $(AB)^T$ is the linear combination of the rows of $A^T$. This implies that $Rowspace(\mathcal{AB})^\mathcal{T} \subseteq Rowspace(\mathcal{A}^\mathcal{T})$. Since $rank = dim(Rowspace) = dim(columnspace)$,

$$dim(\mathcal{C}(\mathcal{AB})^\mathcal{T}) \leq dim(\mathcal{C}(\mathcal{A})^\mathcal{T})$$

(d) $dim(\mathcal{C}((AB)^\top))$_______________$dim(\mathcal{C}(B^\top))$

**Solution:** Each column in the matrix product $(AB)^T$ is the linear combination of the column of $B^T$. This implies that $\mathcal{C}(\mathcal{AB})^\mathcal{T} \subseteq \mathcal{C}(\mathcal{B}^\mathcal{T})$. Hence,

$$dim(\mathcal{C}(\mathcal{AB})^\mathcal{T}) \leq dim(\mathcal{C}(\mathcal{B})^\mathcal{T})$$

**Concept**: Free variables

13. (2 ½ points) True or False (with reason if true or example to show it is false).

(a) A square matrix has no free variables

**Solution:** False. In the square matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $C_3 = 2C_2 - C_1$. This means column 3 is dependent and will lead to a free column in RREF as seen below:

$$RREF = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In the above RREF, $x_3$ is a free variable for the last column which happens to be a free column.

(b) An invertible matrix has no free variables

**Solution:** True. If a full rank matrix $A$ has its RREF equal to an identity matrix, then $A$ is invertible. This is because identity matrix are invertible and have only independent/pivot columns without any free columns and free variables. Hence, only the zero vector will exist in its nullspace.

(c) An $m \times n$ matrix has no more than $n$ pivot variables.

**Solution:** True. In the RREF of an $m \times n$ matrix $A$ where $n$ is the number of columns of $A$,

$$n = \text{number of pivot variables} + \text{number of free variables}$$

This means number of pivot variables $\not> n$.

(d) An $m \times n$ matrix has no more than $m$ pivot variables.

**Solution:** True. In the RREF of an $m \times n$ matrix $A$ every row contains at most one pivot.

(e) Matrices $A$ and $A^T$ have the same number of pivots.

**Solution:** True. We know that a transpose operation essentially converts column vectors into row vectors and vice-versa. Hence, if a $m \times n$ matrix $A$ had $k$ independent columns (or $k$ pivots), then $A^T$ will have $k$ independent rows with the same number of pivots. This means that $A$ and $A^T$ both have the same rank.

**Concept**: Reduced Echelon Form

14. (½ point) Suppose R is $m \times n$ matrix of rank $r$, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a right-inverse $B$ with $RB = I$ if $r = m$.

**Solution:** $R$ is full row rank matrix since $r = m$. Then, our $R = \begin{bmatrix} I & F \end{bmatrix}$. We can find the right-inverse $B$ in the following way:

$$
\begin{aligned}
B &= R^T(RR^T)^{-1} \\
B &= \begin{bmatrix} I & F \end{bmatrix}^T (\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} I & F \end{bmatrix}^T)^{-1} \\
B &= \begin{bmatrix} I \\ F \end{bmatrix} (\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix})^{-1} \\
B &= \begin{bmatrix} I \\ F \end{bmatrix} \begin{bmatrix} I^2 + F^2 \end{bmatrix}^{-1} \\
B &= \begin{bmatrix} I \\ F \end{bmatrix} \begin{bmatrix} I + F^2 \end{bmatrix}^{-1} \\
B &= \begin{bmatrix} \begin{bmatrix} I + F^2 \end{bmatrix}^{-1} \\ F \begin{bmatrix} I + F^2 \end{bmatrix}^{-1} \end{bmatrix}
\end{aligned}
$$

Hence, right-inverse $B = \begin{bmatrix} \begin{bmatrix} I + F^2 \end{bmatrix}^{-1} \\ F \begin{bmatrix} I + F^2 \end{bmatrix}^{-1} \end{bmatrix}$, where $\begin{bmatrix} I + F^2 \end{bmatrix}^{-1}$ is a $m \times m$ matrix and $B$ is a $n \times m$ matrix.