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Question 3)

we know, $g_N'(x) = g_N(x) + 10.$

Let, $J_N(x) = g_N(x)$ and $J_N'(x) = J_N(x) + 10.$

Now,

$$J_{N-1}(x_{N-1}) = \min_{a_{N-1} \in A(x_{N-1})} E_{x_N}\left[J_N(x_N)\right]$$

and

$$J_{N-1}'(x) = \min_{a_{N-1} \in A(x_{N-1})} E_{x_N}\left[J_N(x_N) + 10\right]$$

$$\Rightarrow J_{N-1}'(x_{N-1}) = \min_{a_{N-1}} E_{x_N}\left[J_N(x_N)\right] + 10.$$

$$\Rightarrow J_{N-1}'(x_{N-1}) = J_{N-1}(x_{N-1}) + 10.$$

Using induction, we can say:

$$J_k'(x) = J_k(x) + 10$$

$$\Rightarrow J_k'(x) \geq J_k(x).$$

Q5)

$J_3(1) = J_3(2) = 0; \quad k = 2, 1, 0.$

For stage 2,

$J_2(1) = \min\{g(1,a), g(1,b)] = \min[1, 0] = 0.$

$J_2(2) = \min\{g(2,a), g(2,b) = \min[2, 5] = 2.$

For stage 1,

$J_1(1) = \min[g(1,a) + p_{11}(a) J_2(1) + p_{12}(a) J_2(2),$

$g(1,b) + p_{11}(b) J_2(1) + p_{12}(b) J_2(2)]$

$\Rightarrow J_1(1) = \min[1 + 0.75 \times 0 + 0.25 \times 2, 0 + 0.25 \times 0 + 0.75 \times 2]$

$\Rightarrow J_1(1) = \min[1.5, 1.5] = \underline{\underline{1.5}}$

$J_1(2) = \min[g(2,a) + p_{21}(a) J_2(1) + p_{22}(a) J_2(2),$

$g(2,b) + p_{21}(b) J_2(1) + p_{22}(b) J_2(2)]$

$\Rightarrow J_1(2) = \min[2 + 0.2 \times 0 + 0.8 \times 2, 5 + 0.8 \times 0 + 0.2 \times 2]$

$\Rightarrow J_1(2) = \min\{3.6, 5.4] = \underline{\underline{3.6}}$

For stage 0,

$$J_0(1) = \min[g(1,a) + p_{11}(a) J_1(1) + p_{12}(a) J_1(2),$$
$$g(1,b) + p_{11}(b) J_1(1) + p_{12}(b) J_1(2)]$$

$$\Rightarrow J_0(1) = \min\{1 + 0.75 \times 1.5 + 0.25 \times 3.6,$$
$$0 + 0.25 \times 1.5 + 0.75 \times 3.6]$$

$$\Rightarrow J_0(1) = \min\{3.025, 3.075\} = 3.025.$$

$$J_0(2) = \min\{g(2,a) + p_{21}(a) J_1(1) + p_{22}(a) J_1(2),$$
$$g(2,b) + p_{21}(b) J_1(1) + p_{22}(b) J_1(2)]$$

$$\Rightarrow J_0(2) = \min[2 + 0.2 \times 1.5 + 0.8 \times 3.6, 5 + 0.8 \times 1.5 + 0.2 \times 3.6]$$

$$\Rightarrow J_0(2) = \min[5.18, 6.92] = 5.18.$$

Therefore, optimal expected costs & policy are:

$\Rightarrow$ For stage 2

$J_2^*(1) = 0$, $J_2^*(2) = 2$, $\mu_2^*(1) = b$, $\mu_2^*(2) = a$.

$\Rightarrow$ For stage 1,

$J_1^*(1) = 1.5$, $J_1^*(2) = 3.6$, $\mu_1^*(1) = a$ or b; $\mu_1^*(2) = a$.

$\Rightarrow$ For stage 0,

$J_0^*(1) = 3.025$, $J_0^*(2) = 5.18$, $\mu_0^*(1) = a$, $\mu_0^*(2) = a$.

Question (6)

(a) From Proposition 3 of (i); we know;

$$J^* = FJ^*$$

or $x^* = f x^*$

Applying f m times on both sides,

$\Rightarrow f^m x^* = f^{m+1} x^*$

$\Rightarrow f^m x^* = x^*$ (Since, $x^* = f x^* = f^2 x^* = \cdots f^m x^*$).

Therefore, x^* is the unique fixed point of f.

(b) From the defn of contraction mapping, we know:

$\|f^m(x^*) - f(x)\| \le \alpha \|x^* - x\|$ ($\because x^* = f x^* = f^m(x^*)$)

$\Rightarrow \|(f(x^*) - x) - (f(x) - x)\| \le \alpha \|x^* - x\|$

$\Rightarrow \big| \|f x^* - x\| - \|f x - x\| \big| \le \alpha \|x^* - x\|$

$\Rightarrow -\alpha \|x^* - x\| \le \|f x^* - x\| - \|f x - x\| \le \alpha \|x^* - x\|$

Since, $f x^* = x^*$

$\Rightarrow -\alpha \|x^* - x\| \le \|x^* - x\| - \|f(x) - x\| \le \alpha \|x^* - x\|$

$\Rightarrow (\alpha + 1) \|x^* - x\| \ge \|f x - x\| \ge (1 - \alpha) \|x^* - x\|$

$\Rightarrow \|x^* - x\| \le \dfrac{1}{1-\alpha} \|f(x) - x\|$

Question 8)

(a) $J_\pi(1) = p_{11}(a)\left[g_0(1,a,1) + \alpha J_\pi(1)\right] +$
$p_{12}(a)\left[g(1,a,2) + \alpha J_\pi(2)\right]$

$J_\pi(2) = p_{21}(c)\left[g(2,c,1) + \alpha J_\pi(1)\right] +$
$p_{22}(c)\left[g(2,c,2) + \alpha J_\pi(2)\right]$

$\Rightarrow J_\pi(1) = 0.5\left[-12 + 0.9(J_\pi(1) + J_\pi(2))\right]$

$J_\pi(2) = 0.4\left[-3 + 0.9 J_\pi(1)\right] + 0.6\left[7 + 0.9 J_\pi(2)\right]$

$\Rightarrow 0.55 J_\pi(1) - 0.45 J_\pi(2) = -6$

$0.36 J_\pi(1) - 0.46 J_\pi(2) = -3$

$\Rightarrow J_\pi(1) = -15.4945, \; J_\pi(2) = -5.6045.$

(b) $J_{\bar\pi}(1) = p_{11}(b)\left[g(1,b,1) + 0.9 \times J_{\bar\pi}(1)\right] +$
$p_{12}(b)\left[g(1,b,2) + 0.9 \times J_{\bar\pi}(2)\right]$

$J_{\bar\pi}(2) = p_{21}(d)\left[g(2,d,1) + 0.9 \times J_{\bar\pi}(1)\right] +$
$p_{22}(d)\left[g(2,d,2) + 0.9 \times J_{\bar\pi}(2)\right]$

$\Rightarrow$

$$J_{\bar{\pi}}(1) = 0.8\left[-4 + 0.9 \times J_{\bar{\pi}}(1)\right] + 0.2\left[-4 + 0.9 \times J_{\bar{\pi}}(2)\right]$$

$$J_{\bar{\pi}}(2) = 0.7\left[-1 + 0.9 \times J_{\bar{\pi}}(1)\right] + 0.3\left[10 + 0.9 \times J_{\bar{\pi}}(2)\right]$$

$$\Rightarrow 0.28\, J_{\bar{\pi}}(1) - 0.18\, J_{\bar{\pi}}(2) = -4$$

$$0.63\, J_{\bar{\pi}}(1) - 0.73\, J_{\bar{\pi}}(2) = -2.3$$

$$\Rightarrow J_{\bar{\pi}}(1) = -27.538$$

$$J_{\bar{\pi}}(2) = -20.615.$$

<u>Question 2)</u>

Let $f(x) = \frac{x}{2}$ where $x \in R^n$

We know that $x^* = 0 = f(x^*) = 0.$

Now, let x be a vector of all ones.

$$\Rightarrow f(x) = \frac{[1 \cdots\cdots 1]^n}{2}$$

$$\Rightarrow f(x) = [0.5 \;\; 0.5 \;\cdots\; 0.5]^n$$

$$\Rightarrow f(x) \le x$$

$\Rightarrow x^* \le x.$ as f is applied infinitely. (i.e $\lim_{k \to \infty} f^k(x) = x^*$)