Classes of convex programs

March 11, 2016

Agenda

LP

- A Linear Program (LP) is an optimization problem of the form

$$\begin{aligned} \min\ & c'x \\ \text{s.t. }\ & Ax = b \\ & x \geq 0 \end{aligned}$$

- Can be solved very fast with the simplex method or interior point methods

QP: Formulation

- A convex Quadratic Program (QP) is an optimization problem of the form

$$\begin{aligned} \min\ & c'x + x'Qx \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

where $Q \succeq 0$

QP: Mean-variance portfolio problem (Markowitz)

- Classical optimization problem in finance
- We are a given a set of assets with normally distributed returns
  - μ_i: expected return of asset i
  - Σ_{ij}: covariance of the returns of asset i and j
  - Goal: find the fraction x_i of the available capital to invest in each asset
- Find a portfolio solving

$$
\begin{aligned}
\min \; & x'\Sigma x && \text{(minimize risk)} \\
\text{s.t. } & \mu' x = \alpha && \text{(desired return)} \\
& 1' x = 1 && \text{(budget constraint)} \\
& x \geq 0 &&
\end{aligned}
$$

QP: Solution approaches

- Newton method if constraints are simple
- Interior point algorithms
- Simplex method:

$$
\begin{aligned}
\min\ & c'x + x'Qx \\
\text{s.t.}\ & Ax = b && (\lambda) \\
& x \geq 0 && (\mu)
\end{aligned}
$$

KKT conditions:

$$
\begin{aligned}
-c' - 2x'Q &= \lambda' A - \mu && \text{(linear in } x, \lambda, \mu\text{)} \\
\mu &\geq 0 \\
\mu' x &= 0
\end{aligned}
$$

QCQP: Formulation

- A convex Quadratically Constrained Quadratic Program (QCQP) is an optimization problem of the form

$$\begin{aligned} &\min c'x \\ \text{s.t. } & a_i'x + x'Q_ix \leq b_i \quad i = 1, \ldots, m \end{aligned}$$

  where $Q_i \succeq 0$
- Solved with interior point methods
- KKT conditions are not linear
- More general, and harder, than QPs

QCQP: Mean-variance portfolio problem (Markowitz)

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$$
\begin{aligned}
\max \mu' x & \qquad \text{(maximize return)} \\
\text{s.t. } x' Q x \leq \beta & \qquad \text{(desired risk)} \\
1' x = 1 & \qquad \text{(budget constraint)} \\
x \geq 0 &
\end{aligned}
$$

SOCP: Formulation

- A second order cone program (SOCP) (also called a conic quadratic program) is an optimization problem of the form

$$\begin{aligned} &\min c'x \\ \text{s.t. } &a_i'x + \sqrt{x'Q_ix} \le b_i \quad i = 1, \dots, m \end{aligned}$$

where $Q_i \succeq 0$

- Note that

$$\begin{aligned} \sqrt{x'Qx} = \sqrt{\left(Q^{\frac{1}{2}}x\right)' Q^{\frac{1}{2}}x} &= \left\| Q^{\frac{1}{2}}x \right\|_2 \\ &= \|x\|_{2,Q} \\ &= \|y\|_2 \, , y = Q^{\frac{1}{2}}x \end{aligned}$$

SOCP: Applications

- Robust programming
- Value-at-risk minimization
- Chance constraints (i.e. $P(X \leq T) \geq \alpha$)
- Stochastic joint location-inventory problems

SOCP: Robust counterpart of LP

- Consider an LP in which the coefficients are unknown and we want to optimize the worst case

$$\min_{x \in \mathbb{R}^n} c'x$$

$$\text{s.t.} \quad \max_{a \in K} \sum_{i=1}^{n} a_i x_i \leq b$$

- How to choose K?
  - Box constraints (e.g. $0 \leq a_i \leq 2$ for $i = 1, \ldots, n$) is easy to solve but too pessimistic.
  - Ellipsoid is a reasonable alternative (e.g. $\sum_{i=1}^{n}(a_i - 1)^2 \leq 1$)
  - General form of ellipsoid: $(a - \mu)Q^{-1}(a - \mu) \leq 1$

SOCP: Robust counterpart of LP

We can prove that

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c'x \\ \text{s.t.} \quad &\max_{a \in K} \sum_{i=1}^{n} a_i x_i \le b \end{aligned}$$

$$\text{with } K = \left\{ a \in \mathbb{R}^n : (a-\mu)' Q^{-1} (a-\mu) \le 1 \right\}$$

and

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c'x \\ \text{s.t.} \quad &\mu' x + \sqrt{x' Q x} \le b \end{aligned}$$

are equivalent

SOCP-representable

- QCQP

$$c'x + x'Qx + r \leq 0 \Leftrightarrow \|Q^{\frac{1}{2}}x + Q^{-\frac{1}{2}}c\| \leq \sqrt{c'Q^{-1}c - r}$$

- Rotated cone constraint (hyperbolic constraint)

$$w'w \leq xy, \; x \geq 0, \; y \geq 0$$

$$\Leftrightarrow \sqrt{4\sum_{i=1}^{n} w_i^2 + (x-y)^2} \leq x + y$$

- Quadratic/linear fractional

$$\min \sum_{i=1}^{m} \frac{\|F_i x + g_i\|^2}{a_i' x + b_i}$$

$$\Leftrightarrow \min \left\{ \sum_{i=1}^{m} t_i : (F_i x + g_i)' (F_i x + g_i) \leq t_i \left(a_i' x + b\right) \right\}$$

Solving SOCP

- Specialized SOCP interior point methods.
- Using CPLEX

$$\begin{aligned} & \min c'x \\ \text{s.t. } & a_i'x + t_i \leq b_i, \quad x'Qx \leq t_i^2, t_i \geq 0, \quad i = 1, \ldots, m \end{aligned}$$

SDP: Formulation

- Given two matrices C and X of the same dimension, let $C \cdot X = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} \cdot X_{ij}$.
- A semidefinite program (SDP) is an optimization problem of the form

$$\begin{aligned} \min\ & C \cdot X \\ \text{s.t.}\ & A_i \cdot X = b_i \quad i = 1, \ldots, k \\ & X \succeq 0 \end{aligned}$$

- Variable is the matrix X.
- Includes SOCP as special cases

$$\|y\|_2 \leq t \Leftrightarrow \begin{bmatrix} tI & y \\ y' & t \end{bmatrix} \succeq 0$$