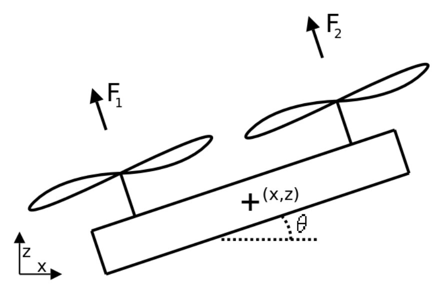
6.832: Problem Set #1

Due on Wednesday, February 13, 2019 at 17:00. See course website for submission details.

1. Definition of Underactuated

The following problem explores the definition of underactuated as described in the lecture notes. You should not need to derived detailed equations of motion for any of the problems in this section.

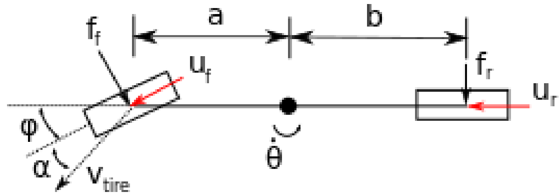
1.1 Simple Helicopter



1. **[2 pts]** A helicopter with two rotors is constrained to move in a vertical plane. Assume gravity acting on the helicopter. The task is to control the position (x, z) and pitch (θ) by varying the thrust produced by the two rotors. **Explain whether this system is fully-actuated (in all states), or if there are any states in which this problem is underactuated.** Use the definition of underactuated provided in lecture.

2. **[3 pts] If you said "Underactuated" above, then please provide an expression for an acceleration that cannot be instantaneously achieved by the system.** Assume that F_1 and F_2 are unbounded (and can be negative) and that the current state of the system is $x = 5, z = 1, \theta = 0.5$ radians. (Your answer should consist of three numerical values: $\ddot{x}, \ddot{y}$, and $\ddot{\theta}$.)

1.2 Bicycle



Consider the simple model of a vehicle known as the bicycle model, illustrated above. Let x, y be the position of the vehicle in inertial coordinates and θ be the heading angle. Lateral tire forces are typically modeled as being proportional to the lateral slip angle α, which defines the angle between the angle of the tire and the velocity of the tire v_{tire}, which depends on the speed and angular velocity of the vehicle, giving $f = C\alpha$ for some constant C.

Generating the equations of motion can be tedious, and we will often use a software package to do it automatically. For this problem, we have done it for you – but, as is often the case, the equations are pretty messy! One way to write the system of equations is, for some constants C_r, C_f:

$$
\begin{aligned}
f_r &= C_r \arctan\left(\frac{\dot{y}\cos\theta - \dot{x}\sin\theta - \dot{\theta}b}{\dot{x}\cos\theta + \dot{y}\sin\theta}\right) \\
f_f &= C_f \left(\arctan\left(\frac{\dot{y}\cos\theta - \dot{x}\sin\theta + \dot{\theta}a}{\dot{x}\cos\theta + \dot{y}\sin\theta}\right) - \phi\right) \\
I\ddot{\theta} &= -bf_r + a(f_f\cos\phi + u_f\sin\phi) \\
m\ddot{x} &= -f_r\sin\theta - f_f\sin(\theta+\phi) + u_r\cos\theta + u_f\cos(\theta+\phi) \\
m\ddot{y} &= f_r\cos\theta + f_f\cos(\theta+\phi) + u_r\sin\theta + u_f\sin(\theta+\phi)
\end{aligned}
$$

1. **[2 pts]** For the purposes of this problem, assume that the driver has control over the steering angle ϕ and has rear wheel drive. Treat the drive torque as a simple ground reaction force u_r acting at the tire and let $u_f = 0$. **Is this system is fully-actuated or underactuated? Explain.**

2. **[2 pts]** Now, suppose the the driver has control of both the front and rear longitudinal tire forces u_r and u_f and so has 3 total control inputs. **Is this system is fully-actuated or underactuated? Give an intuitive explanation.**

3. **[3 pts]** Since these dynamics are not control affine, consider a simplified system of equations linearized about $\phi = \phi_0$ and $u_f, u_r = 0$. For simplicity, without loss of generality, let $\theta = 0$. Recalling the general dynamics form,

$$\ddot{q} = f_1(q, \dot{q}, t) + f_2(q, \dot{q}, t)u$$

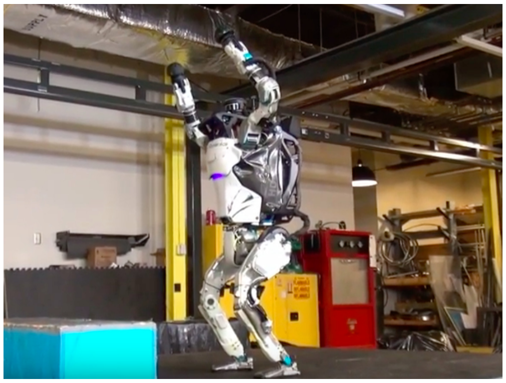
We can write:

$$f_2(q, \dot{q}, t) = \begin{bmatrix} -aC_f\left(\left(\arctan\left(\frac{\dot{y}+\dot{\theta}a}{\dot{x}}\right) - \phi_0\right)\sin(\phi_0) + \cos(\phi_0)\right) & 0 & a\sin(\phi_0) \\ -C_f\left(\left(\arctan\left(\frac{\dot{y}+\dot{\theta}a}{\dot{x}}\right) - \phi_0\right)\cos(\phi_0) - \sin(\phi_0)\right) & 1 & \cos(\phi_0) \\ -C_f\left(\left(\arctan\left(\frac{\dot{y}+\dot{\theta}a}{\dot{x}}\right) - \phi_0\right)\sin(\phi_0) + \cos(\phi_0)\right) & 0 & \sin(\phi_0) \end{bmatrix}$$

Find the rank of f_2 when $\phi_0 = 0$.

4. **[2 pts] Are there values for ϕ_0 for which the system is fully-actuated? If so, what are they? If not, explain why.**

1.3 Humanoid



(a) Standing

(b) Backflipping

Figure 1: (a) The atlas robot stands on the floor. (b) The atlas robot is performing a backflipping.

You may already know the Atlas humanoid backflipping demo (if not, take a look here). Suppose the dynamics of the Atlas humanoid is governed by the manipulator equation taught in the class. **Describe whether or not each of the following statements is true, and justify your answer.**

1. **[1 pt] It is sufficient to represent the state of the humanoid as the angles and angular velocities of all joints.**

1. **[1 pt] Assume there is no torque limit, the robot at the state of backflipping (b) is fully-actuated.**

1. **[2 pt] Again, assume there is no torque limit, the robot at the state of standing (a) is fully-actuated.**

2. Nonlinear Dynamics

Consider a system with dynamics given by

$$\dot{x} = -x^3 + 6x^2 - 5x - 12$$

2.1 [5 pts] Sketch a phase diagram x vs $\dot{x}$ for this system, and label its equilibrium points. (Feel free to plot this with a tool of your choice to check your work.)

2.2 [6 pts] For each equilibrium point you indicated above, write and justify whether it is stable, unstable, or marginally stable – and if it is stable, write and justify its region of attraction.

3. Discrete Stability

For a univariate dynamic system $\dot{x} = f(x)$ we have seen via graphical analysis that x^* is a locally stable equilibrium if the following conditions hold:

- $f(x^*) = 0$
- $\frac{\partial f}{\partial x} x^* < 0$

In other words, x^* is a locally stable equilibrium if $f(x)$ has a zero-crossing at x^* with negative slope.

Now, consider a simple discretization of this continuous system, where for some fixed time step h we have:

$$x[k+1] = x[k] + hf(x[k])$$

3.1 [5 pts] For arbitrary h, the two conditions above are not sufficient for stability of the discrete system. **Provide a counterexample demonstrating this by giving values for x^*, $f(x)$, and h below.**

3.2 [5 pts] Find the upper bound h^* such that all $h < h^*$ results in a stable discrete system. Write your answer in terms of G, where $G = \left|\frac{\partial f}{\partial x}(x^*)\right| > 0$

4. Feedback Linearization

4.1 [2 pts] Explain whether this statement is true or false: for any underactuated system of the form $\ddot{q} = f_1(q, \dot{q}) + f_2(q, \dot{q})u$, one can choose $u(x, u')$ so that $\dot{x} = Ax + Bu'$, where u' is a new control input.

4.2 Take a robot whose dynamics are given by the manipulator equations, $M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau_g(q) + B(q)u$ for $q \in \mathbb{R}^n$. The robot starts in a given initial configuration $q(0) = q_0$ and with a given initial velocity $\dot{q}(0) = \dot{q}_0$. Suppose $B(q)$ is rank n for all q.
Note: this problem was revised on 2018/02/12 replace $H(q)$ with $M(q)$, following our convention in class for the inertia matrix being designated M.

Write (with a brief explanation) whether each of the following statements are true for *any* twice-differentiable desired trajectory $q_d : \mathbb{R} \to \mathbb{R}^n$.

1. **[1 pt]** Can feedback linearization be used to make it so that $q(t) = q_d(t)$ for all $t \geq 0$?

2. **[1 pt]** Can feedback linearization be used to make it so that $\dot{q}(t) = \dot{q}_d(t)$ for all $t \geq 0$?

3. **[1 pt]** Can feedback linearization can be used to make it so that $\ddot{q}(t) = \ddot{q}_d(t)$ for all $t \geq 0$?

4.3 [5 pts] Consider an actuated pendulum, where the base is forced to oscillate horizontally in simple harmonic motion, $C\sin(\omega t)$. Then, the dynamics of the pendulum angle θ are:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{C}{l}\omega^2 \sin(\omega t)\cos\theta + \frac{u}{ml^2} \tag{1}$$

Even with the base shaking, we would like the pendulum to spin at a constant speed, $\dot{\theta} = 1$. To achieve this, we should choose $\ddot{\theta}_{des}$ to stabilize any velocity error.

Use feedback linearization to find the control law such that $\ddot{\theta} = -\dot{\theta} + 1$, in terms of the relevant terms in Equation 1, including state variables θ and $\dot{\theta}$.
Note: this problem was revised on 2018/02/09 to place a negative sign in front of $\frac{g}{l}\sin\theta$ *in order to match the conventions of class.*

5. Lab Assignment: Install Drake

Throughout the course, we will use the software toolbox Drake (drake.mit.edu) and its Python bindings. Drake contains our best implementations of many of the algorithms taught in the course, as well as code to generate the equations of motion from very simple descriptions of robots.

Follow the instructions at underactuated.csail.mit.edu/Spring2019/install_drake_docker.html on the computer you will use to complete this and future problem sets to get up and running with Drake (via its Python bindings, PyDrake). Once you have successfully followed those instructions to the end, continue this problem.

These following two exercises will expose you to simulating dynamical systems in Drake. They both rely on a pendulum model – one on a simple damped pendulum, and one on the moving-base pendulum from the Feedback Linearization problem. A simulator that can handle both is demonstrated in the included code. Follow the instructions at the Docker install page to launch a notebook at a directory containing the included code, using tag **drake-20190129**. Your command should look something like this, depending on your OS:

Listing 1: Linux, Mac, and Docker Toolbox users

```
./docker_run_notebook.sh drake-20190129 ./set_1_code
```

Listing 2: Docker for Windows users

```
./docker_run_notebook_win10.bat drake-20190129 ./set_1_code
```

Open a browser and log in to the notebook, again following the instructions on that web pge. Open up the *PendulumSimulation* file. Try reading through and running the code it contains (you can run code in a cell by clicking it and pressing Control+Enter).

5.1 Recall that the second-order dynamics of a damped pendulum are

$$ml^2\ddot{\theta} + mgl\sin\theta = -b\dot{\theta} + u$$

Consider the case where the control input u takes on a constant value. For this problem, use constants $m = 3$, $l = 1$, $g = 10$, and $b = 2$. You can simulate this system using tools by following the examples in the supplied Python code (remember to set $C = w = 0$ to keep the base from moving), and can change its controller by editing the *feedback_rule* function. Use the simulation (in combination with your understanding of the system's equation of motion) to answer these questions.

1. **[3 pts]** Try simulating the system with different fixed control input u. Try values both above and below 30 in magnitude. **What changes when $|u| > 30$?**

2. **[5 pt] Sketch the bifurcation diagram $\dot{\theta}$ vs. u, showing the equilibrium point(s) for a fixed u. From the equations of motion, derive an approximate analytic expression for the steady-state $\dot{\theta}$ as a function of a fixed u when $u > 30$.**

5.2 [3 pts] Recall the undamped pendulum with moving base used in the Feedback Linearization problem earlier in this assignment. You can simulate it using the supplied code by changing the base amplitude C and frequency w to nonzero values, and by disabling damping by setting $b = 0$. Change *feedback_rule* to implement your feedback linearization rule from before. If it works correctly, the pendulum should slowly spin with constant velocity $\dot{\theta} \approx 1$.

While the pendulum visualization is useful for developing an intuitive understanding of the system, be sure to pay attention to the trace plots of θ and $\dot{\theta}$ across time, too. **Sketch the $\dot{\theta}$ vs t trace here. Briefly comment on the tracking performance of your feedback rule. (How well is $\dot{\theta} = 1$ enforced? Can you come up with a way to improve the tracking?)** Use $m = 3, l = 1, g = 10, b = 0, C = 0.5, w = 2$ for answering this part, but feel free to experiment with other values.

6. Survey

Please up help improve the course by answering a few survey questions!
You are always welcome to submit feedback to *underactuated-tas@mit.edu*, directly to the instructor, or on Piazza (including anonymously).

6.1 About how long did you spend on this problem set?

6.2 How have you found the pace of the lectures so far?

Way too slow *A little too slow* *Just right* *A little too fast* *Much too fast*

6.3 Is there a particular type of system (robot, vehicle, something else entirely, etc.) that you're hoping we'll cover in the class?

6.4 Any other comments? (How has your day been going?)