Convergence analysis for infinite sums

$$S = \sum_{n=1}^{\infty} f(n)$$

ow accurate is the partial sum

$$S_N = \sum_{n=1}^{N} f(n)$$

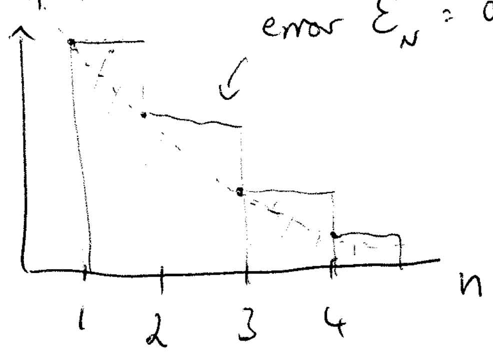
rror estimate

$$\mathcal{E}_N = |S - S_N| = \left|\sum_{n=N+1}^{\infty} f(n)\right|$$

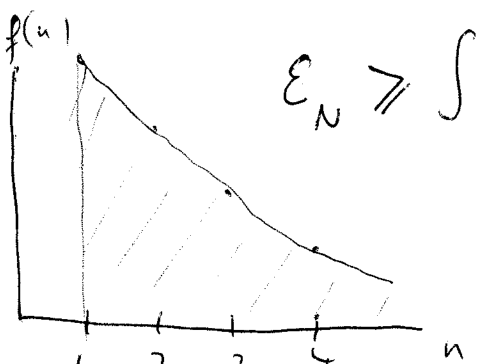
ften $f(n)$ monotonically decreasing, continuous fn of n.

x: $f(n)$

error $\mathcal{E}_N$ = area under rectangles

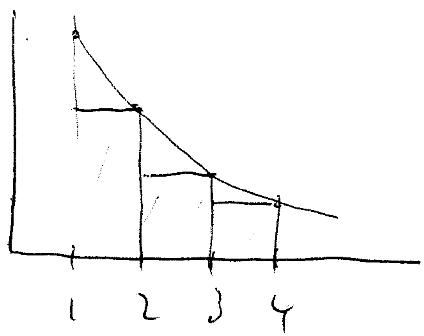


$$\mathcal{E}_N \geqslant \int f(n)\,dn$$



$$\Rightarrow \sum_{n=N}^{M-1} f(n) > \int_N^M f(x)\,dx$$

lso:



$$\rightarrow \sum_{n=N}^{M+1} f(n+1) < \int_N^M f(x)\,dx$$

$$\Leftrightarrow \sum_{n=N+1}^{M} f(n) < \int_N^M f(x)\,dx$$

ake $M \to \infty$ $\rightsquigarrow$ $\boxed{\mathcal{E}_N = S - S_N < \int_N^\infty f(x)\,dx}$

xample

$$S = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$, not positive and decreasing.

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$$

$$\Rightarrow \mathcal{E}_N < \int_N^\infty \frac{dx}{2x(2x-1)} = -\frac{1}{2} \log\left(1 - \frac{1}{2N}\right)$$

partial sum correct to 6 digits:

$$\underset{N}{\mathcal{E}} < 10^{-6} \underbrace{\log(2)}_{\text{exact value}} \Rightarrow N \gtrsim 360{,}000$$

between 10^5 and 10^6, as expected from numerical experiments.

In the fly analysis

If the true S is unknown but $f(n)$ positive and decreasing:

$$\underbrace{\frac{\mathcal{E}_N}{S_N}}_{\text{upper bound on true error}} > \frac{\mathcal{E}_N}{S}$$

uler - MacLaurin summation formula

r general $f(n)$

$$\sum_{n=M+1}^{N} f(n) - \int_{M}^{N} f(x)\,dx = \sum_{p=0}^{\infty} C_p \left[f^{(p)}(N) - f^{(p)}(M) \right]$$

rapidly decreasing

periodic functions: RHS $\equiv 0$, sum and integral coincide!