18.330 Pset 4

Kathleen Brandes

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1 Problem 1

1.1 Part A

To develop a backwards difference stencil with order 2 convergence for h, I first used the standard backwards difference stencil:

$$\frac{df}{dx} = \frac{f(x)-f(x-h)}{h}$$

This is a first order method, with a step size of h. To convert it to a second order method, I applied richardson's theorem, with $p = 1$, $t = 2$, and $\Delta = h$, which resulted in the following stencil:

$$F_{richardson}(\Delta) = \frac{t^p F(\Delta/2) - F(\Delta)}{t^p - 1} = \frac{2^1(\frac{f(x)-f(x-h/2)}{h/2}) - \frac{f(x)-f(x-h)}{h}}{2^1 - 1}$$

$$F_{richardson}(h) = \frac{3f(x)-4f(x-h/2)+f(x-h)}{h}$$

This stencil exactly followed the procedure outlined by richardson, so therefore it will have convergence of the order $p + 1$, which in this case will be order 2 convergence, as desired.

1.2 Part B

To develop a stencil for the mixed variable second derivative $\frac{d^2 f(x,y)}{dxdy}$, first I let $h_x = h_y = h$. Then we can apply the central difference method for the partial derivative with respect to y:

$$\frac{df}{dy} = \frac{f(x,y+h)-f(x,y-h)}{2h}$$

Similarly, we can get the first derivative with respect to x:

$$\frac{df}{dx} = \frac{f(x+h,y)-f(x-h,y)}{2h}$$

These results can be combined as follows to get the second derivative:

$$\frac{d}{dx}\left(\frac{df}{dy}\right) = \frac{1}{2h}\left(\frac{f(x+h,y+h)-f(x+h,y-h)}{2h} - \frac{f(x-h,y+h)-f(x-h,y-h)}{2h}\right)$$
$$= \frac{1}{4h}(f(x+h,y+h) - f(x+h,y-h) - f(x-h,y+h) + f(x-h,y-h)$$

To determine the convergence of h, I have taylor expanded each term:

$$f(x+h,y+h) = f(x,y) + hf_x + hf_y + \frac{h^2}{2}f_{xx} + h^2 f_{xy} + \frac{h^2}{2}f_{yy} + \mathcal{O}(h^3)f'''(x,y)$$
$$f(x+h,y-h) = f(x,y) + hf_x - hf_y + \frac{h^2}{2}f_{xx} + h^2 f_{xy} + \frac{h^2}{2}f_{yy} \pm \mathcal{O}(h^3)f'''(x,y)$$
$$f(x-h,y+h) = f(x,y) - hf_x + hf_y + \frac{h^2}{2}f_{xx} + h^2 f_{xy} + \frac{h^2}{2}f_{yy} \pm \mathcal{O}(h^3)f'''(x,y)$$
$$f(x-h,y-h) = f(x,y) - hf_x - hf_y + \frac{h^2}{2}f_{xx} + h^2 f_{xy} + \frac{h^2}{2}f_{yy} + \mathcal{O}(h^3)f'''(x,y)$$

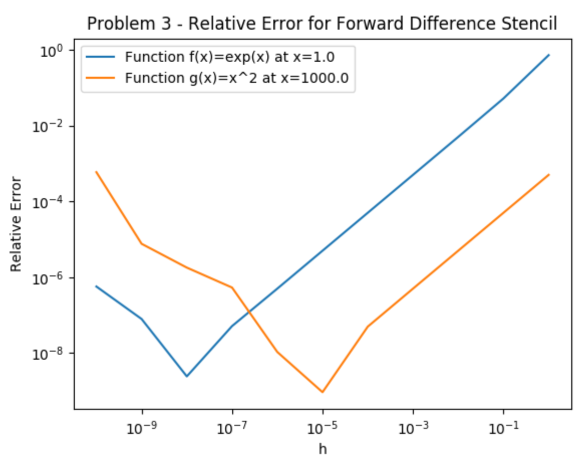
These can be substituted into the initial expression for the second derivative:

$$\frac{d^2 f(x,y)}{dxdy} = \frac{1}{4h}((2hf_y \pm \mathcal{O}(h^3)f'''(x)) - (2hf_y \pm \mathcal{O}(h^3)f'''(x)) = \frac{1}{4h}\mathcal{O}(h^3)f'''(x)$$

Therefore, we can see that this stencil for the multivariable second derivative will converge with $\mathcal{O}(h^2)$ speed.

2 Problem 2

The code for this problem is attached on stellar in the file problem2.jl. Let $f(x) = e^x$ and $g(x) = x^2$. The forward difference stencil will be $f'(x,h) = \frac{f(x+h)-f(x)}{h}$. Here is a graph showing the relative error of the forward difference stencil compared to the exact solution for function $f(x,h)$ evaluated at $x = 1.0$ and function $g(x,h)$ evaluated at $x = 1000.0$, and with varying values of h:



The relative error decreases at approximately an order one rate, as expected for the forward difference stencil, then it begins to increase again as h continues to get smaller. This is likely because the forward difference stencil divides by h, so when h gets very small it is essentially dividing by zero which will cause for rounding and precision errors in computing the derivative estimate. This loss in precision impacts $g(x)$ for larger values of h than $f(x)$ because the value of the numerator in the stencil for $g(x)$ will be larger causing greater differences in precision.

3 Problem 3

3.1 Part A

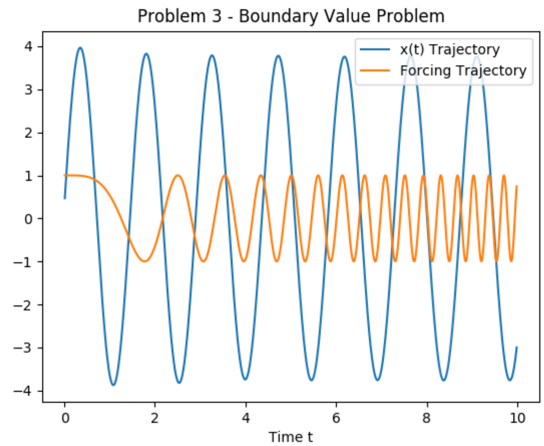
This problem is a second order differential equation with boundary values for $x(t_0)$ and $x(t_f)$. We could use the shooting method to guess $x'(t_0)$ and convert this into an initial value problem and numerical methods from previous lectures, like Euler's method or Runge-Kutta, then check if there is a root for a function representing the difference between the numerical solved for value of $x(t_f)$ and the boundary value at x_f. If there is a root, then the solution to the initial value problem can be a solution to the boundary value problem.

3.2 Part B

The given equation of motion is $\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = cos(t^2)$ for $\omega_0 = 4.3$ and with boundary values of $x(0) = .3$ and $x(10) = -2.9$. To evaluate this expression, I used the central difference method for second derivatives. Let $h = \frac{10-0}{N+1}$ for a large N, and then we can get an expression for $\frac{d^2x(t)}{dt^2} = \frac{x(t-h)-2x(t)+f(t+h)}{h^2}$. Then, I can discretize the space into $N+1$ samples of $x(t)$ at times $t_i = 0 + ih$. Therefore, we can write a vector, $\boldsymbol{x}$, representing the desired solutions at each time step, and the finite difference matrix with $\{1, -2, 1\}$ along the diagonal (with -2 being the primary diagonal). Including the term from ω^2 and the boundary conditions in a sparse vector $\boldsymbol{\Delta}$, where the first entry is $x(0)$, the last entry is $x(10)$ and all other entries are zero. Then, we can get an expression for the value of $x(t)$ using only the boundary conditions and the finite central difference stencil:

$$\boldsymbol{x} = (\tfrac{1}{h^2}\boldsymbol{A} + \omega_0^2 \boldsymbol{I})^{-1}(cos(t^2) - \tfrac{1}{h^2}\boldsymbol{\Delta})$$

This stencil has been coded in julia, and is attached to stellar in the file problem3.jl. I evaluated this stencil for the given boundaries, and the forcing function of $cos(t^2)$ to produce a plot showing both the trajectory $x(t)$ found by the stencil and the forcing function at each time step:

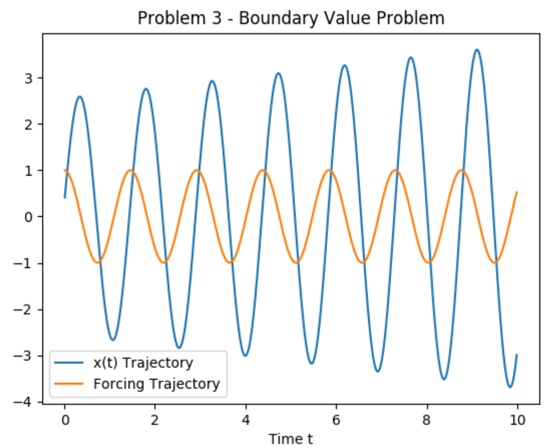


3.3 Part C

Using same stencil and boundary values and equation, but a different forcing function $cos(4.3t)$, the trajectory will be:

$$\boldsymbol{x} = (\frac{1}{h^2}\boldsymbol{A} + \omega_0^2\boldsymbol{I})^{-1}(cos(4.3t) - \frac{1}{h^2}\boldsymbol{\Delta})$$

I then plotted the trajectory $x(t)$ and the forcing function at each time step:



This phenomenon is an example of resonance since the frequency ω_0 of the forcing function matches that of the natural frequency.

4 Problem 4

4.1 Part A

Let the matrices be subdivided into blocks of size NxN.The solution vector is of the format: $\phi(x,y) = [\phi_{11}...\phi_{1N}\phi_{21}...\phi_{2N}...\phi_{NN}]$ where $\phi_{ij} = \phi(x_i, y_j)$. Let the finite difference matrices then be subdivided into blocks of size NxN. Given the ordering of the solution vector, each row of block matrices within the finite difference matrix represents a set of solutions ϕ_{ij} with a single i value (constant x) and all possible $j \in [1, N]$ (all variants of y); I will call these block-row-i.

The central difference for the first derivative with respect to x is $\frac{d}{dx}\phi(x,y) = \frac{\phi(x+h,y)-\phi(x-h,y)}{2h} = \frac{\phi_{i+1,j}-\phi_{i-1,j}}{2h}$. Therefore, the matrix D_x will have I_n(where I is the identity matrix) in the $(i+1)$-th column block for a given block-row-i. Similarly, it will have $-I_n$ in the $(i-1)$-th column block for a given block-row-i. All other blocks in the matrix will be the zeros matrices for nxn. This will all be multiplied by $\frac{1}{2h}$. For example, when $N = 3$, then the block matrix will be:

$$D_x = \frac{1}{2h}\begin{bmatrix} 0 & I & 0 \\ -I & 0 & I \\ 0 & -I & 0 \end{bmatrix}$$

. The central difference for the first derivative with respect to y is $\frac{d}{dy}\phi(x,y) = \frac{\phi(x,y+h)-\phi(x,y-h)}{2h} = \frac{\phi_{i,j+1}-\phi_{i,j-1}}{2h}$. The resulting finite difference matrix D_y will have nxn matrix M with -1 in all entries below the main diagonal, and a 1 in all entries above the main diagonal, and zeros everywhere else. This matrix M will be along the block-matrix diagonal for D_y, and then all multiplied by $\frac{1}{2h}$. For example, when $N = 3$, then the block matrix (and matrix M) will be:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$D_y = \frac{1}{2h}\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}$$

. The central difference for the second derivative with respect to x will be $\frac{d^2}{dx^2}\phi(x,y) = \frac{1}{h^2}(\phi(x-h,y)-2\phi(x,y)+\phi(x+h,y)) = \frac{1}{h^2}(\phi_{i-1,j}-2\phi_{i,j}+\phi_{i+1,j})$. This results in a finite difference matrix L_x where for a given block-row-i, there will be I_n in the $(i-1)$-th and $(i+1)$-th column blocks, and then $-2I_n$ in the i-th column block. This will all be multiplied by $\frac{1}{h^2}$. For example, when $N = 3$, then the block matrix will be:

$$L_x = \frac{1}{h^2}\begin{bmatrix} -2I & I & 0 \\ I & -2I & I \\ 0 & I & -2I \end{bmatrix}$$

. The central difference for the second derivative with respect to y will be $\frac{d^2}{dy^2}\phi(x,y) = \frac{1}{h^2}(\phi(x,y-h)-2\phi(x,y)+\phi(x,y+h)) = \frac{1}{h^2}(\phi_{i,j-1}-2\phi_{i,j}+\phi_{i,j+1})$. This results in a finite difference matrix L_y where the block diagonal is the nxn finite difference matrix for a standard double derivative (let this be A). For example, when $N = 3$, then the block matrix will be:

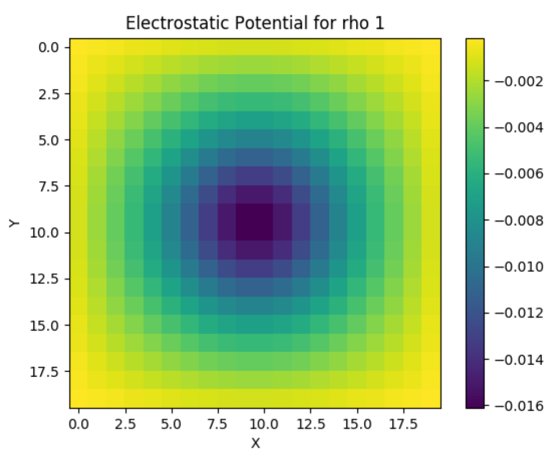
$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$L_y = \frac{1}{h^2}\begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$$

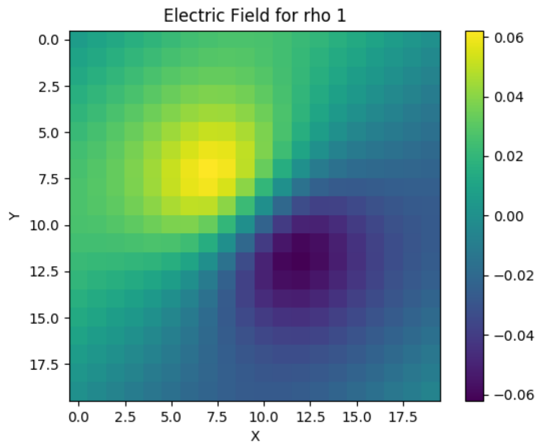
. All of these finite difference matrices factor in the boundary conditions because the results of $\phi_{0,j} = \phi_{i,0} = \phi_{N+1,j} = \phi_{i,N+1}$ are factored into the finite difference matrices by having the boundaries incorporated into the results for $\phi_{1,j}/\phi i, 1/\phi_{N,j}/\phi_{i,N}$.

4.2 Part B

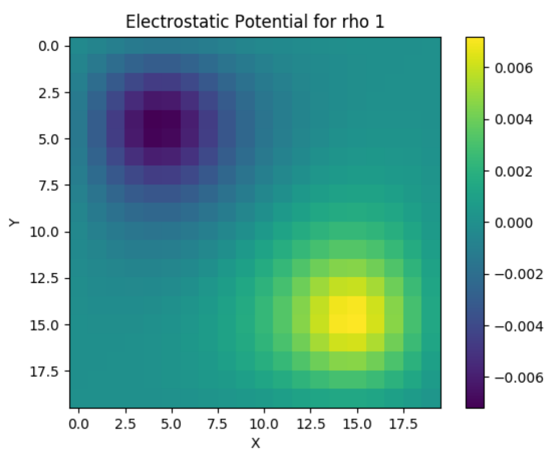
The code for this problem is attached on stellar in the file problem4.jl. For the first equation $\rho_1(x,y) = e^{-50(x-.5)^2+-50(y-.5)^2}$, we can use the above matrices to make $L = L_x + L_y$ and then solve for the electrostatic potential by doing: $\boldsymbol{\phi} = \boldsymbol{L}^{-1}\boldsymbol{\rho}$. This results in the following electrostatic potential when $N = 20$:



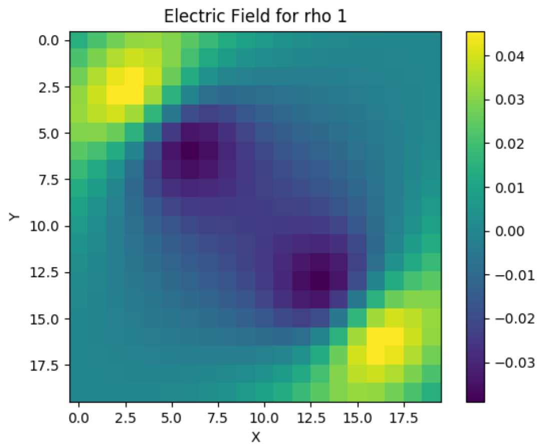
And similarly, using the result from this evaluation, we can get the electric field with $D = D_x + D_y$ by evaluating $\boldsymbol{E} = -\boldsymbol{D\phi}$. This yields the following graph of the field:



The same two evaluations were done with $\rho_2(x, y) = e^{-100(x-.25)^2-100(y-.25)^2} - e^{-100(x-.75)^2-100(y-.75)^2}$. This had the resulting electrostatic potential graph:



And the electric field graph:


Electric Field for rho 1
Y
X
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
0.04
0.03
0.02
0.01
0.00
−0.01
−0.02
−0.03