Problem Set 1

18.330 Intro to Numerical Analysis (MIT, Spring 2019)

Due: February 19

Problem 1. A simple sum (20 points)

Consider the following infinite sum and its partial sum

$$S = \sum_{n=1}^{\infty}, \qquad S_N = \sum_{n=1}^{N} f(n)$$

with $f(n) = 1/n^4$. Define the relative error of the Nth partial sum as $E_N = |S_N - S|/S$.

(a) How large must we choose N to ensure that S_N agrees with S to 9-digit precision (i.e., $E_N < 10^{-9}$)?

(b) Write a JULIA program to evaluate S_N. Plot E_N against N and assess the accuracy of the prediction from part (a).

Problem 2. Another sum (20 points)

Sometimes, we need to evaluate a sum that involves many summands of almost equal mangnitude. As an extreme example, consider

$$P_N = \sum_{i=1}^{N} \frac{\pi}{N}.$$

Note: it is not a typo that the summands are independent of i.

Consider the following Julia program that computes the partial sums:

```
function P_N(N)
    summand = pi/N

    S = 0.0
    for i=1:N
        S += summand
    end

    return S
end
```

(a) Consider the relative error

$$E_N = \frac{|P_N - \pi|}{\pi}.$$

In words, state how you expect E_N to depend on N for values of N in the range $10^2 < N < 10^9$.

(b) Use the code above to write a Julia program which plots E_N against N for values of N in the range $10^2 < N < 10^9$. How does the result compare to your expectation?

Problem 3. Simpson's rule (20 points)

In this problem you will derive Simpson's rule yourself.

(a) As a first step, consider a general quadrature rule on the interval $[-1, 1]$,

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{n=1}^{N} w_n f(x_n).$$

From this rule $\{w_n, x_n\}$, construct *new* $\{w_n', x_n'\}$ for integrating over a general interval $[a, b]$, i.e.,

$$\int_{a}^{b} f(x)\,dx \approx \sum_{n=1}^{N} w_n' f(x_n').$$

(b) Now derive the non-composite Simpson's rule for integrating a function on $[-1, 1]$. To do this, construct the unique 2nd-order polynomial $P(x) = a + bx + cx^2$ that agrees with $f(x)$ at $-1, 0, 1$. You will find a, b, c in terms of $f(-1), f(0), f(1)$.

Then, integrate $P(x)$ from -1 to 1 and express the result as a quadrature rule for $f(x)$.

(c) Combine your results to write down the Simpson's rule for integrating on the interval $[a, b]$.

(d) Subdivide an interval $[u, v]$ into N equal parts and apply the Simpson's rule to each subinterval to obtain the composite Simpson's rule. How many function evaluations does this quadrature rule need?

Problem 4. Comparing quadrature rules (20 points)

Write JULIA programs that implement the composite 0th, 1st and 2nd order Newton-Cotes quadrature rules for arbitrary functions. For the integrals below, plot the relative error $E_N = |I_{\text{approx}} - I_{\text{exact}}| / I_{\text{exact}}$ as a function of the number of function evaluations N for each quadrature rule. Choose N between 10 and 10^7.

(a) $I_a = \int_0^{\pi} e^{\cos\left[(x+1)^2 + 2\sin(4x+1)\right]} dx$

(b) $I_b = \int_0^{\pi} e^{\cos\left[\cos^2(x+1) + 2\sin(4x+1)\right]} dx$

(c) $I_c = \int_0^{2\pi} \frac{\tanh x}{\sqrt{|x-\pi|}} dx$

(d) $I_d = \int_0^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x} dx$

The "exact" values for these integrals are

$$\begin{aligned}
I_a &= 2.5193079820307612557 \\
I_b &= 4.4889560612699568830 \\
I_c &= 6.6388149923287733132 \\
I_d &= 1.7981374998645790990
\end{aligned}$$

How do the results compare to your expectations?

Problem 5. Simpson's rule revisited (20 points)

Recall the (non-composite) Simpson's rule:

$$I = \int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

In class we saw that the Simpson's rule was able to integrate the function $f(x) = x^2$ exactly. This was because it uses a quadratic interpolation of the integrand.

Show that Simpson's rule actually integrates all *cubic* polynomials $Q(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ exactly.