18.330 Pset 7

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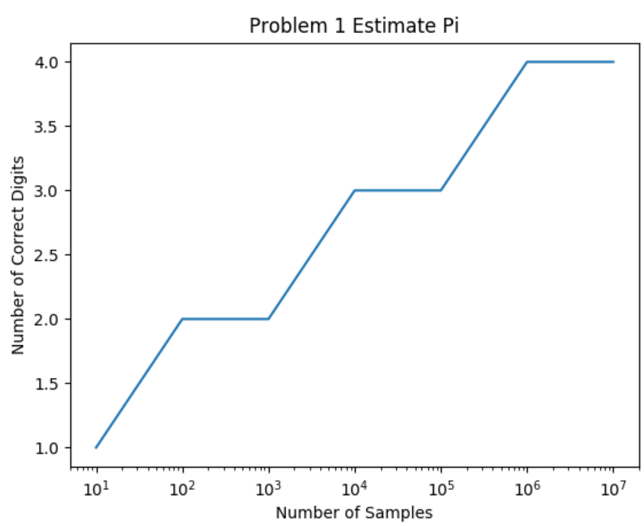
1 Problem 1

1.1 Part A

The region $B_2 = \{(x, y) \in R^2 | x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$ over which we are integrating with Monte Carlo is simply just one quarter of the unit circle. The area of this region, which is also the value of the integral when $f(x) = 1$, will be $\frac{\pi}{4}$. Therefore, by symmetry, we can see that $I = 4\int_{B_2} dx = 4(\frac{\pi}{4}) = \pi$.

1.2 Part B

Here is a plot showing the number of correct digits found by Monte Carlo integration when comparing the result of the integral to the exact solution π:



2 Problem 2

2.1 Part A

We know that $\sigma(< f >') = \frac{1}{2N}(\sigma_A^2(f) + \sigma_B^2(f))$, we can use the relationship $\sigma(< f >) = \sigma(f)/N$ and rules of variance to get the following:

$$\sigma^2(< f >') = \sigma^2(\tfrac{1}{2}(< f >_A + < f >_B))$$

$$= \tfrac{1}{4}\sigma^2(< f >_A) + \tfrac{1}{4}\sigma^2(< f >_B)$$

$$= \tfrac{1}{4}(\tfrac{\sigma_A^2(f)}{N/2} + \tfrac{\sigma_B^2(f)}{N/2})$$

$$= \tfrac{1}{2N}(\sigma_A^2(f) + \sigma_B^2(f))$$

2.2 Part B

We know that $\sigma^2(f) = (< f^2 > - < f >^2)$, and also we can write $< f >= \frac{1}{N}\sum_{i\in A} f_i + \frac{1}{N}\sum_{j\in B} f_j$:

$$\sigma^2(f) =$$

$$.5(\tfrac{1}{N/2}\textstyle\sum_{i\in A}(f_i)^2 + \tfrac{1}{N/2}\sum_{j\in B}(f_j)^2) - .25(\tfrac{1}{N/2}\sum_{i\in A}(f_i) + \tfrac{1}{N/2}\sum_{j\in B}(f_j))^2$$

$$= \tfrac{1}{N}(\textstyle\sum_{i\in A}(f_i)^2 + \sum_{j\in B}(f_j)^2) - (\tfrac{1}{N^2}\sum_{i\in A}(f_i)\sum_{i\in A}(f_i) + \tfrac{2}{N^2}\sum_{i\in A}(f_i)\sum_{j\in B}(f_j) + \tfrac{1}{N^2}\sum_{j\in B}(f_j)\sum_{j\in B}(f_j))$$

$$=< f^2 > -(\tfrac{1}{N}\textstyle\sum_{i\in A} f_i + \tfrac{1}{N}\sum_{j\in B} f_j)^2$$

$$=< f^2 > -(< f >)^2$$

And this is the standard expression for variance, so we have shown that this expression for $\sigma^2(f)$ is valid.

2.3 Part C

The variance of the stratified estimator, $\sigma^2(< f >')$, is never larger than that of the one obtained from simple Monte Carlo. This can be seen because simple Monte Carlo has a variance of $\sigma^2(< f >) = \frac{\sigma^2(f)}{N} = \frac{1}{N}(\frac{1}{2}(\sigma_A^2(f) + \sigma_B^2(f)) + \frac{1}{4}(< f >_A - < f >_B)^2)$, as shown in part b. The variance for the stratified estimator is $\sigma^2(< f >') = \frac{1}{2N}(\sigma_A^2(f) + \sigma_B^2(f))$. Therefore, $\sigma^2(< f >') \leq \sigma^2(< f >)$, and they will be equal when the means of the stratified samples are the same, so when $\frac{1}{4}(< f >_A - < f >_B)^2 = 0$.

2.4 Part D

Let N_A = samples in A, $N_B = N - N_A$ = samples in B. We know that $\sigma^2(< f >') = \frac{1}{2N}(\sigma_A^2(f) + \sigma_B^2(f)) = \frac{1}{4}(\frac{\sigma_A^2(f)}{N/2} + \frac{\sigma_B^2(f)}{N/2})$ when the samples are evenly distributed between A and B. Therefore, we can conclude that it would be $\sigma^2(< f >') = \frac{1}{2(N_A+(N-N_A))}(\sigma_A^2(f) + \sigma_B^2(f)) = \frac{1}{4}(\frac{\sigma_A^2(f)}{N_A} + \frac{\sigma_B^2(f)}{N-N_A})$.
Further, we can see that this will be minimized with respect to N_A:

$$\frac{d}{dN_A}\sigma^2(< f >') = \frac{-.25\sigma_A^2}{N_A^2} + \frac{.25\sigma_B^2}{(N-N_A)^2} = 0$$

$$(\sigma_A^2)((N - N_A)^2) = (N_A^2)(\sigma_B^2)$$

$$(\sigma_A)(N - N_A) = N_A(\sigma_B)$$

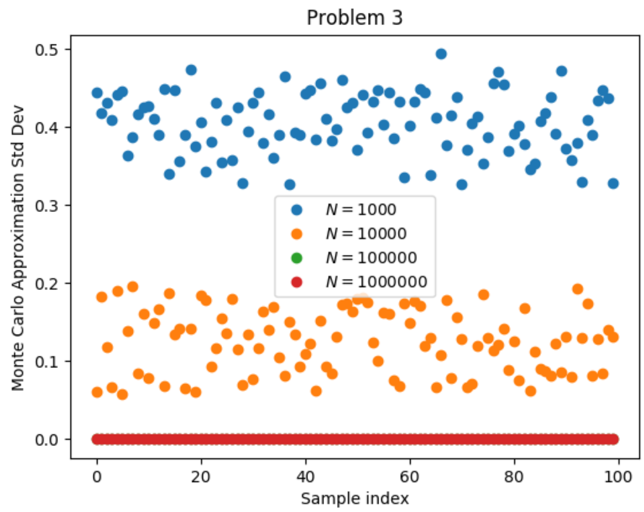
This gives us the expression $\frac{N_A}{N} = \frac{\sigma_A(f)}{\sigma_A(f)+\sigma_B(f)}$ as the minimizer for the estimator variance. It obtains a minimum value of:

$$\sigma^2(< f >') = \frac{1}{4}(\frac{\sigma_A^2(f)(\sigma_A(f)+\sigma_B(f))}{N\sigma_A(f)} + \frac{\sigma_B^2(f)(\sigma_A(f)+\sigma_B(f))}{N\sigma_B(f)}) = \frac{(\sigma_A(f)+\sigma_B(f))}{4N}(\frac{\sigma_A^2(f)}{\sigma_A(f)} + \frac{\sigma_B^2(f)}{\sigma_B(f)}) = \frac{(\sigma_A(f)+\sigma_B(f))}{4N}(\sigma_A(f) + \sigma_B(f))$$

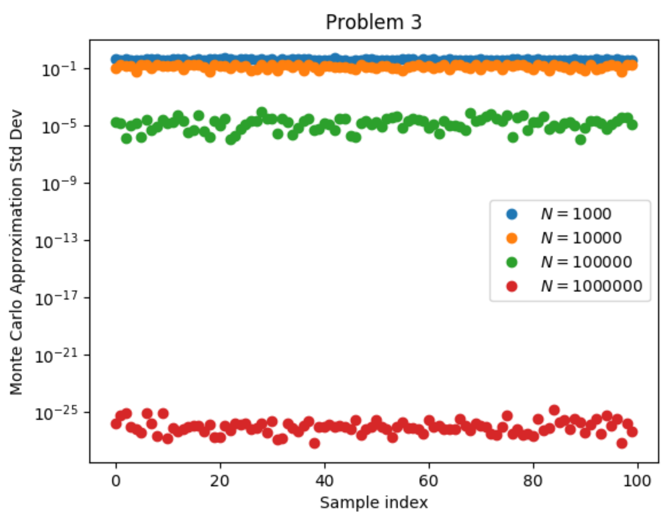
This gives us the desired result of $\sigma^2(< f >') = \frac{(\sigma_A(f)+\sigma_B(f))^2}{4N}$

3 Problem 3

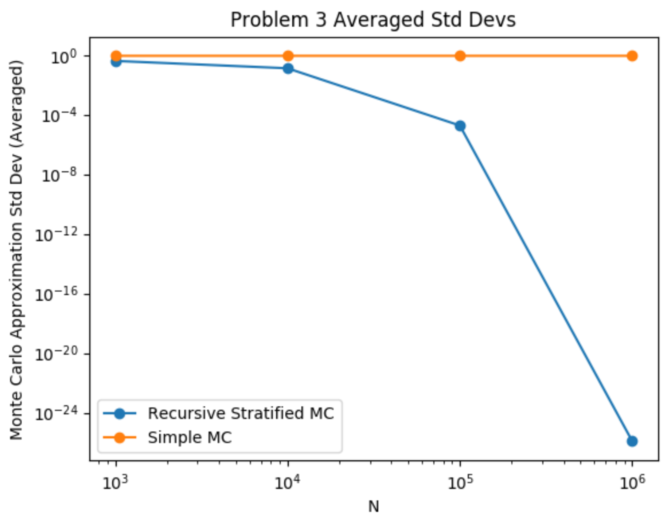
My implementation for the recursive stratified sampling Monte Carlo algorithm is attached on stellar in problem3.jl file. From running my recursive stratified sampling MC algorithm with values of $N = [1000, 10000, 100000, 1000000]$ for 100 iterations for each value of N, I generated the plots of the total estimated standard deviation for each iteration. The plot showing a linear axis scale for y is:



And then, since the standard deviations for values of $N = 100000$ and $N = 1000000$ were significantly smaller, I also plotted it on log-scale for the y axis:



Lastly, I plotted the average standard deviation for each different N value computed from the stratified sampling MC and also the average standard deviation for each N value computed from the simple MC algorithm:



This shows that standard deviation falls off like $\sigma \sim N^{-\alpha}$ for values of $\alpha > 1/2$ when using the recursive stratified sampling MC algorithm.

4 Problem 4

4.1 Part A

The Fourier transform for $\alpha f(t)$ will be:

$$\hat{f}'(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\alpha f(t)dt = \alpha\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt = \alpha\hat{f}(\omega)$$

4.2 Part B

The Fourier transform for $f(\alpha t)$ will be:

$$\hat{f}'(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(\alpha t)dt$$

Then, do a change of variables where $u = \alpha t$, $du = \alpha dt$.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\frac{u}{\alpha}}\frac{1}{\alpha}f(u)du = \frac{1}{\alpha}\hat{f}(\frac{\omega}{\alpha})$$

4.3 Part C

The Fourier transform for $f(t+t_0)$, $t_0 \in R$ will be:

$$\hat{f}'(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(t+t_0)dt$$

Then, do a change of variables where $u = t + t_)$, $du = 1dt$.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(u-t_0)}f(u)du = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(u)}f(u)e^{-i\omega(-t_0)}du = \frac{1}{2\pi}e^{i\omega(t_0)}\int_{-\infty}^{\infty} e^{-i\omega(u)}f(u)du = e^{i\omega(t_0)}\hat{f}(\omega)$$

4.4 Part D

The Fourier transform for $k(t) = f(t)g(t)$ will be:

$$\hat{k}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}k(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}f(t)g(t)dt$$

Then, I will rewrite $f(t)$ as the inverse fourier transform, so $f(t) = \int_{-\infty}^{\infty} e^{i\omega' t}\hat{f}(\omega')d\omega'$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}[\int_{-\infty}^{\infty} e^{i\omega' t}\hat{f}(\omega')d\omega']g(t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega')\int_{-\infty}^{\infty} e^{-i\omega t+i\omega' t}g(t)d\omega' dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega')\hat{g}(\omega' - \omega)d\omega'$$

$$= \frac{1}{2\pi}\hat{f}(\omega)\hat{g}(\omega)$$