Euler-Maclaurin formula

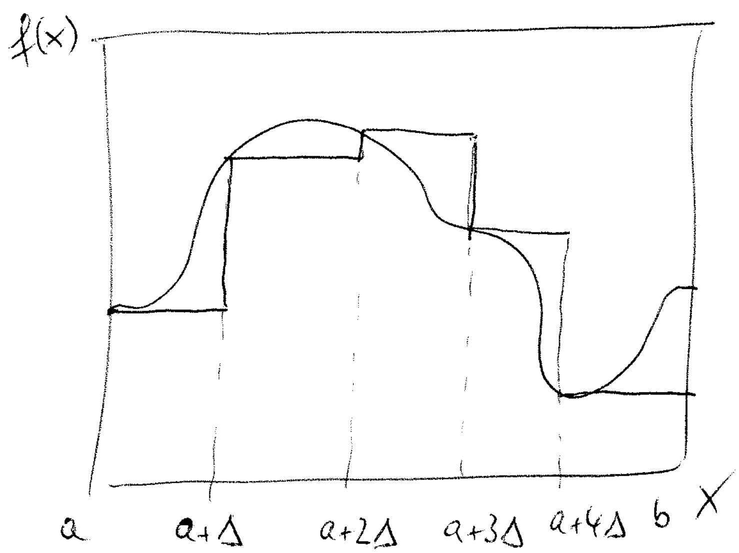
$$\sum_{n=M+1}^{N} f(n) - \int_M^N f(x)\,dx = \sum_{p=0}^{\infty} C_p \left[f^{(p)}(N) - f^{(p)}(M) \right]$$

true if $\boxed{N - M \geq 2}$

Jewton - Cotes quadrature

$$\int_a^b f(x)\,dx \approx \sum_{n=1}^{N} w_n\, f(x_n)$$

ectangular rule ($p=0$)



niform sub-intervals of length $\Delta = \frac{b-a}{N}$

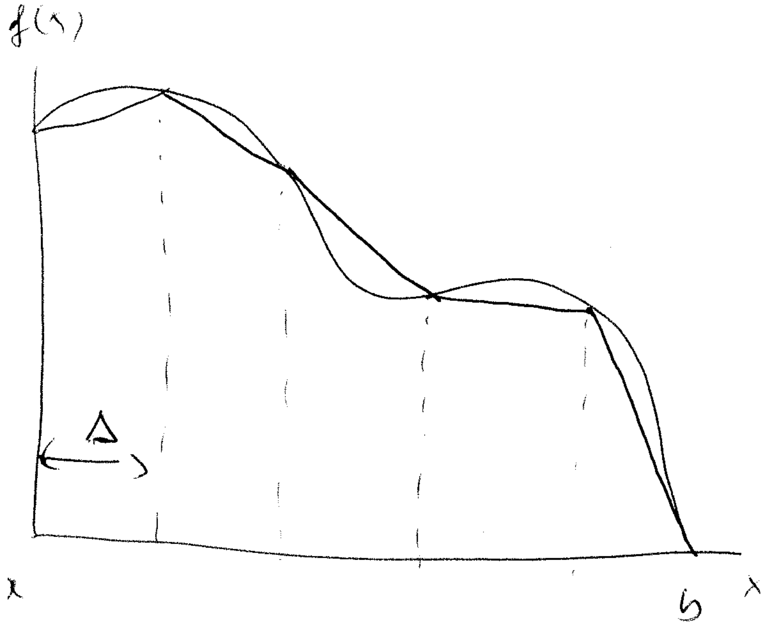
pproximate $f(x)$ by $f(x_n) = f(a+n\Delta)$

ach rectangle has area $A_n = \Delta \cdot f(a+n\Delta)$

$$\Rightarrow \quad \boxed{I_N^{rect} = \sum_{n=0}^{N-1} \Delta\, f(a+n\Delta)}$$

here: $w_n = \Delta$, $x_n = a+n\Delta$)

rapezoidal rule ($p = 1$)



rea of trapezoid $A_0 = \Delta f(a) + \frac{1}{2}\Delta\left(f(a+\Delta) - f(a)\right)$

$$A_1 = \Delta f(a+\Delta) + \frac{1}{2}\Delta\left(f(a+2\Delta) - f(a+\Delta)\right)$$

$\Rightarrow$ $I^{trap} = \frac{1}{2}\Delta f(a) + \frac{1}{2}\Delta f(a+\Delta)$

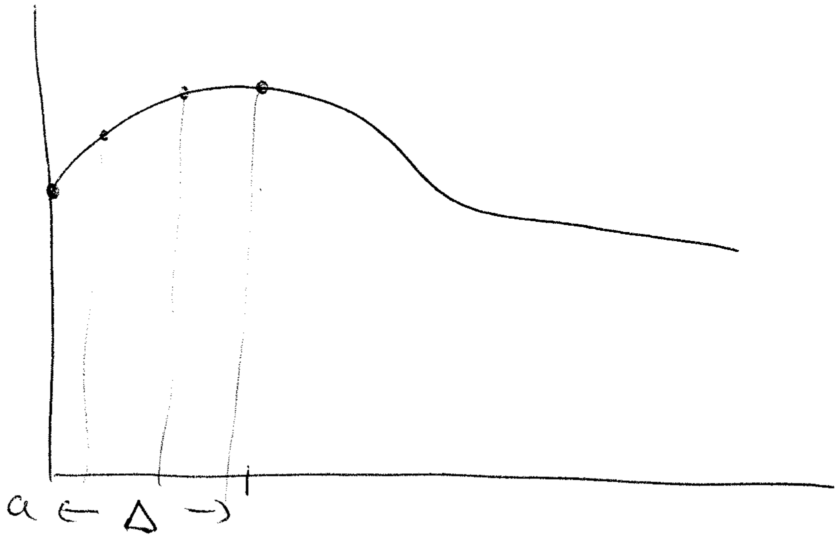
$$+ \frac{1}{2}\Delta f(a+\Delta) + \frac{1}{2}\Delta f(a+2\Delta)$$

$$+ \frac{1}{2}\Delta f(a+2\Delta) + \frac{1}{2}\Delta f(a+ \text{[illegible]})$$

$$+ \dots$$

$$\boxed{= \frac{1}{2}\Delta f(a) + \Delta \sum_{n=1}^{N-1} f(a+n\Delta) + \frac{1}{2}\Delta f(b)}$$

[H]igher order rules



Take $p+1$ evenly spaced points on the sub interval $[x_n, x_{n+1}]$

compute the p-th order polynomial $P_p(x)$ that agrees with $f(x)$ on those points.

$$P_p(x_n) = f(x_n)$$

$$P_p\left(x_n + \frac{x_{n+1} - x_n}{p}\right) = f\left(x_n + \frac{x_{n+1} - x_n}{p}\right)$$

$$\vdots$$

integrate $\int_{x_n}^{x_{n+1}} P_n(x)\, dx$ exactly

sum over all integrals.

x: $p = 2$: Simpson's rule (Kepler's barrel rule)

nique parabola through 3 points

$P(x) = a + bx + cx^2$

$$\left.\begin{aligned} P(x_n) &= f_0 \\ P(x_n + \tfrac{\Delta}{2}) &= f_1 \\ P(x_n + \Delta) &= f_2 \end{aligned}\right\} \Rightarrow \int_{x_n}^{x_{n+1}} P(x)\,dx = \frac{\Delta}{3}\left[f_0 + 4f_1 + f_2\right]$$

Remember Runge's phenomenon : Higher order

Newton-Cotes becomes less useful due to

massive oscillations at the boundaries of the intervals.

rror analysis

not rigorous)

ingle subinterval of width Δ:

$$f(x) \approx P(x) = C_0 + C_1 x + \cdots + C_p x^p$$

can show: exists $x_0 \in$ subinterval such that the first $p+1$ terms of the Taylor expansion of $f(x)$ agree with $P(x)$.

$$\Rightarrow \quad f(x) - P(x) = C_{p+1} x^{p+1} + C_{p+2} x^{p+2} + \cdots$$

$$\Rightarrow \text{error} = \int (f(x) - P(x))\, dx$$

$$= \int (C_{p+1} x^{p+1} + \cdots)\, dx$$

$$\sim \Delta^{p+2} + \mathcal{O}(\Delta^{p+3})$$

$$\sim \frac{1}{N^{p+2}}$$

$$\text{tal error} \sim N \cdot \frac{1}{N^{p+2}} \sim \frac{1}{N^{p+1}}$$

$$\Rightarrow \quad \text{error}_{\text{rect}} \sim \frac{1}{N}$$

$$\text{error}_{\text{trap}} \sim \frac{1}{N^2}$$

$$\text{error}_{\text{simps}} \sim \frac{1}{N^3}$$

$$\vdots$$

Integration tricks

Improper integrals

$$\int_0^\infty f(x)\,dx$$

change variables. Ex: $x = \dfrac{u}{1-u}$

$$dx = \frac{du}{(1-u)^2}$$

$$\Rightarrow \int_0^\infty f(x)\,dx = \int_0^1 f\left(\frac{u}{1-u}\right)\frac{du}{(1-u)^2}$$

can only work if $f(x) \longrightarrow 0$ as $x \longrightarrow \infty$ so the singularity cancels.

Integrable singularities

$$I = \int_0^1 \frac{e^{+x}}{\sqrt{x}}\,dx$$

is well-defined and finite despite integrand $\to \infty$ at $x=0$!

Singularity subtraction: (analytical technique)

$$I = \underbrace{\int_0^1 \frac{1}{\sqrt{x}}\,dx}_{=\left[2\sqrt{x}\right]_0^1 = 2} + \underbrace{\int_0^1 \frac{e^{+x}-1}{\sqrt{x}}\,dx}_{\text{nonsingular}}$$

nonsingular, $\frac{e^{+x}-1}{\sqrt{x}} \approx \frac{x - \frac{1}{2}x^2 + \cdots}{\sqrt{x}}$

$$\sim \sqrt{x} - \frac{1}{2}x^{3/2} + \cdots$$

Singularity cancellation:

$$u = \sqrt{x}, \quad du = \frac{dx}{2\sqrt{x}}$$

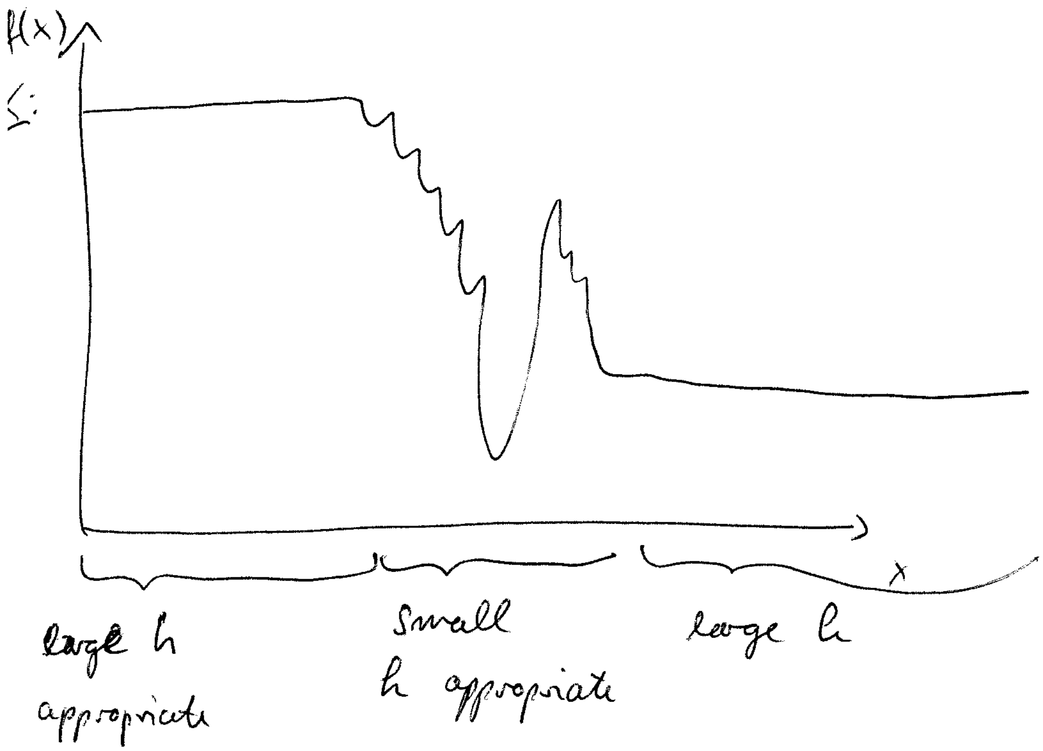
$$\to \int_0^1 \frac{e^{+x}}{\sqrt{x}}\,dx = 2\underbrace{\int_0^1 e^{u^2}\,du}_{\text{no singularities}}$$

regularization (ε expansion)

Introduce $$I_\varepsilon = \int_0^1 \frac{e^x}{\sqrt{x+\varepsilon}}\, dx$$

and try to find the limit $\varepsilon \to 0$ numerically.

Adaptive Quadrature



many functions have regions where they barely change and region where they change <u>a lot</u>

Integration becomes expensive when minimum necessary h is used everywhere

Idea: Change h locally depending on behavior of f(x)

Implementation (Recursive)

- Subdivide [a,b] into coarsest grid (e.g. N = 100)
- on each subinterval, estimate the error of the integration by locally performing <u>a finer</u> quadrature (say, N = 200).
- if the two quadratures do not differ much, we are done (large h region)
- otherwise, refine the discretization (small h region)