# Problem Set 6

18.330 Intro to Numerical Analysis (MIT, Spring 2019)

Due: April 4. To be submitted *online* on Stellar.

## Problem 1. Solving linear systems (25 points)

Following the algorithms shown in the lecture, implement a function that returns the LU decomposition of a general invertible square matrix $A$, and one that returns the Cholesky decomposition of a symmetric positive definite matrix $A = A^\top$. Then, implement functions for solving a linear system using forward and backsubstitution, respectively.

Use your functions to solve the following linear systems with the fastest appropriate algorithm:

(a)

$$\begin{pmatrix} 1.0 & 3.0 & 2.0 \\ 2.4 & -3.3 & 1.1 \\ -1.0 & 0.0 & 2.0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \end{pmatrix},$$

(b)

$$\begin{pmatrix} 5.0 & -3.0 & 2.0 \\ -3.0 & 6.0 & -1.0 \\ 2.0 & -1.0 & 5.0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \end{pmatrix},$$

(c)

$$\begin{pmatrix} 2.0 & -3.0 & 2.0 \\ -3.0 & 3.0 & -1.0 \\ 2.0 & -1.0 & 2.0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \end{pmatrix}.$$

## Problem 2. Uniqueness of the QR decomposition. (25 points)

In class we saw an algorithm to compute the QR decomposition of a general matrix $A = QR$, where $Q^{-1} = Q^\top$ is orthogonal and $R$ is upper triangular. Here you will show that this decomposition is unique, provided we fix the signs of the diagonal elements of $R$. We will assume that the diagonal elements of $R$ are all positive, $R_{ii} > 0$.

(a) First, show that the inverse of a square upper triangular matrix $R$ is again upper triangular. *Hint:* The $i$'th column of the inverse $R^{-1}$ is the solution to $R\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ is a vector with all zeros, except one $1$ in the $i$'th position.

(b) Then, show that the product $R_1 R_2$ of two square upper triangular matrices is again upper triangular.

(c) Now show that the $QR$ decomposition is unique. To do this, assume that the matrix $A$ has two different $QR$ decompositions:

$$A = Q_1 R_1 = Q_2 R_2.$$

Multiply this by $Q_2^\top$ from the left and $R_1^{-1}$ from the right,

$$Q_2^\top Q_1 = R_2 R_1^{-1}.$$

First, argue that the right hand side is again an upper triangular matrix. Then, argue that the left hand side is again orthogonal. Finally, argue that the only upper triangular matrix with positive diagonal entries that is also orthogonal is the identity matrix. (*Hint:* Orthogonal matrices have orthonormal columns. Compute the inner products between all the columns.) From this it follows that the $Q$ factor of the QR decomposition is unique.

(d) Finally, conclude that the $R$ factor is unique as well.

## Problem 3. Finding eigenvectors with the QR algorithm (25 points)

We saw in class that for a matrix $A$, the QR algorithm

$$\begin{aligned} A_k &= Q_k R_k \\ A_{k+1} &= R_k Q_k \end{aligned}$$

(often) converges to an upper triangular matrix with the eigenvalues of $A$ on the diagonal.

(a) Use what you know about Orthogonal Iteration to improve the `eig_qr` function from class such that it returns a list of eigenvalues and a matrix whose columns are the corresponding orthonormal eigenvectors.

(b) Test your implementation on the matrix

$$A = \begin{pmatrix} 2 & -3 & 2 \\ -3 & 1 & 4 \\ 2 & 4 & -1 \end{pmatrix}.$$

Check that (i) the eigenvectors your function returns are indeed eigenvectors of $A$ corresponding to the eigenvalues your function returns, (ii) the eigenvectors are orthonormal.

(c) Test your implementation in the same way on the matrix

$$B = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 1 & 4 \\ 2 & 4 & -1 \end{pmatrix}.$$

Explain your observations.

## Problem 4. Eigenvectors using Inverse Iteration (25 points)

In Problem 3 you saw that we can only use the QR algorithm to obtain eigenvectors if they are orthogonal. But what if our matrix does not have orthogonal eigenvectors? Here you will see one way of getting at any eigenvector, provided you know its eigenvalue at least approximately.

Similar to power iteration, consider *inverse iteration*

$$\mathbf{b}_{k+1} = \frac{(A - \mu \mathbb{1})^{-1} \mathbf{b}_k}{\|(A - \mu \mathbb{1})^{-1} \mathbf{b}_k\|},$$

for some number $\mu$.

(a) Show that if

$$A\mathbf{v} = \lambda\, \mathbf{v}$$

for some square matrix $A$ and eigenvalue $\lambda$, then

$$(A - \mu \mathbb{1})^{-1} \mathbf{v} = \frac{1}{\lambda - \mu} \mathbf{v}.$$

(b) Conclude that inverse iteration converges to the eigenvector for the eigenvalue *closest to* $\mu$, provided that the initial guess $\mathbf{b}_0$ has a component in that eigenspace, that $\mu$ is *not* an eigenvalue of $A$, and that the eigenvalue of $A$ closest to $\mu$ is not repeated.

(c) Write a function that computes all eigenvectors of a generic matrix $A$ with no repeated eigenvalues given $A$ and a list of its eigenvalues. Test your function on the matrix

$$B = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 1 & 4 \\ 2 & 4 & -1 \end{pmatrix}.$$

Check that it returns the correct eigenvectors.