Problem Set 2

18.330 Intro to Numerical Analysis (MIT, Spring 2019)

Due: February 26. To be submitted *online* on Stellar.

Problem 1. Adaptive Quadrature (20 points)

In this problem you will implement a generic adaptive quadrature scheme to numerically integrate

$$\int_a^b f(x)\,dx.$$

It will work with any basic quadrature algorithm. Your adaptive scheme should be a recursive function taking the following parameters: The integrand $f(x)$, the integration interval $[a,b]$, another function implementing a basic (i.e., Newton-Cotes) quadrature rule, the number of coarse subdivisions N for the coarse quadrature, and a tolerance parameter ε.

Your algorithm should basically work like this:

1. Approximate $Q_N \approx \int_a^b f(x)\,dx$ using the basic quadrature rule and N subdivisions.
2. Approximate again $Q_{2N} \approx \int_a^b f(x)\,dx$ using a finer subdivision, for example $2N$.
3. Estimate the error

$$e = |Q_N - Q_{2N}|.$$

   If $e < \varepsilon$, we are done and return Q_{2N}.

   Otherwise, run the adaptive integrator again on the subintervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$ individually, and return the sum.

Test your adaptive algorithm on

$$f(x) = x^{10} e^{4x^3 - 3x^4}$$

using the rectangular rule, the trapezoid rule, and Simpson's rule as basic algorithms. Pick $N = 1$ and plot the relative error of your adaptive integration scheme against ε between 10^{-1} and 10^{-8}, integrating on the interval $[0,3]$.

For the "true value" you can for instance use the standard Simpson's rule with a very large number of subdivisions.

Problem 2. ODE solvers and numerical quadrature. (20 points)

Show that the evaluation of the integral

$$I = \int_a^b f(t)\,dt$$

can we written equivalently as the solution of an ODE,

$$\frac{du}{dt} = f(t).$$

from $t = a$ to $t = b$ subject to the initial condition $u(a) = 0$.

Use both the (i) Euler method and (ii) the Improved Euler method to integrate $\int_a^b f(x)\,dx$ using stepsize $h = (b-a)/N$. In each case, write down the approximation formula and compare to the Newton-Cotes rules.

Problem 3. Error analysis for the midpoint rule (20 points)

Consider the midpoint rule for integrating the ODE

$$\frac{du}{dt} = f(t,u),$$

which is defined as

$$\begin{aligned} t_{n+1} &= t_n + h \\ u_{n+1} &= u_n + hf\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} f(t_n, u_n)\right). \end{aligned}$$

The midpoint rule samples the slope using an Euler step with half stepsize $h/2$, and then takes a full step of size h in the direction of that slope.

By proceeding the same way we did in class for the Euler and Improved Euler method, perform an error analysis for the midpoint rule. Find both the local and the overall error incurred when integrating over a fixed interval. How does the error behave as the stepsize h decreases?

Problem 4. Leapfrog integration. (20 points)

In physics problems, we often need to integrate Newton's laws,

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),$$

which are second-order in time. Solutions are very often oscillatory in nature, for example in the very simple problem

$$\ddot{x} = -x.$$

For this simple equation, fix the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$.

(a) Numerically compute the solution to this IVP using the Euler method and plot the trajectory in the x–$\dot{x}$ plane. Compare your results to the exact solution.

(b) The Euler method does not respect conservation of energy. To see this, for the trajectory computed above, compute the total energy as a function of time

$$E_n = \frac{1}{2}x_n^2 + \frac{1}{2}v_n^2,$$

and plot it against the value for the exact solution.

Apparently, the Euler method fails quite miserably! Luckily, this problem can be fixed by the *Leapfrog method*, which computes positions and velocities independently, and at half-stepsizes apart from each other (hence the name). It is defined as

$$\begin{aligned} t_{n+1} &= t_n + h \\ v_{n+1/2} &= v_n + \frac{h}{2}F(x_n) \\ x_{n+1} &= x_n + h\, v_{n+1/2} \\ v_{n+1} &= v_{n+1/2} + \frac{h}{2}F(x_{n+1}). \end{aligned}$$

(c) Implement the Leapfrog algorithm and check that the test ODE from above is now integrated qualitatively correctly. Also plot the energy of the Leapfrog solution and check that it is now approximately conserved.

(d) The Euler method fails in a second way as well: It does not respect time-reversal symmetry. See this by integrating our test ODE from (x_0, v_0) at $t_0 = 0$ to some t', yielding (x', v'), and then backwards from (x', v') at t' to t_0. Do the trajectories agree as they should when you plot them? Does the Leapfrog method fix this problem as well?

(e) Show analytically that the Leapfrog method respects time-reversal symmetry, i.e., taking one step forward from (t_n, u_n) to $(t_n + h, u_{n+1})$ and then one backward step using $-h$ from $(t_n + h, u_{n+1})$ to (t_n, u_{n+2}) brings us back to the initial condition, $u_{n+2} = u_n$. Show similarly that the Euler method does not respect time-reversal symmetry.

Problem 5. Choreographed Orbits (20 points)

The Kepler problem—finding the trajectories of two gravitationally bound bodies—is well known to be analytically solvable. However, as soon as there are three massive bodies (say, Earth, Moon, and the Sun), all hope is lost and the the motions must be investigated numerically. Many interesting effects in this 3-body-problem remain to be uncovered. Here, you will study a relatively new phenomenon: *choreographed orbits*.

(a) Write down the equations of motion (Newton's laws) for three bodies of equal mass m at positions $\mathbf{r}_{1,2,3}$, interacting through the force of gravity.

(b) We will now assume that the motion is entirely confined to the x–y plane, and that $Gm = 1$, with G the gravitational constant. Using the Leapfrog method from the previous problem, integrate the system of ODEs from $t = 0$ to $t = 20$ subject to the following initial conditions:

$$\begin{aligned} (x_1, y_1) &= (-0.7, 0.35) & (\dot{x}_1, \dot{y}_1) &= (0.99, 0.078) \\ (x_2, y_2) &= (1.1, -0.07) & (\dot{x}_2, \dot{y}_2) &= (0.1, 0.47) \\ (x_3, y_3) &= (-0.4, -0.3) & (\dot{x}_3, \dot{y}_3) &= (-1.1, -0.53) \end{aligned}$$

Plot the solution orbits in the x–y plane and comment on the shape.

(c) Now perturb the initial conditions slightly, by changing a few by, say, 25%. Solve the equations of motion for at least two additional, perturbed initial conditions and plot the trajectories. Comment on the fragility of the original orbit pattern.

You can read a pedagogical introduction to the subject of choreographed orbits on Scholarpedia.