1 Problem 641

Consider a row of n dice all showing 1.

First turn every second die, $(2, 4, 6, \dots)$, so that the number showing is increased by 1. Then turn every third die. The sixth die will now show a 3. Then turn every fourth die and so on until every n'th die (only the last die) is turned. If the die to be turned is showing a 6 then it is changed to show a 1.

Let $f(n)$ be the number of dice that are showing a 1 when the process finishes. You are given $f(100) = 2$ and $f(10^8) = 69$.

Find $f(10^{36})$.

Upon inspection, one sees that the problem is asking us to compute

$$\#\{k \in \{1, \dots, n\} \mid \sigma(k) \bmod 6 = 1\}, \tag{1.1}$$

where $\sigma : \mathbb{N} \to \mathbb{N}$ is the function that counts how many divisors a given integer has, e.g. $\sigma(12) = 6$ and $\sigma(p) = 2$, where p is a prime number.

In words, we are tasked to find numbers satisfying

$$\sigma(k) = 1, 7, 13, 19, \dots \tag{1.2}$$

Since the above numbers are all odd, we may search through the squares:

$$\#\{k = \widetilde{k}^2 \in \{1, \dots, n\} \mid \sigma(k) \bmod 6 = 1\}, \tag{1.3}$$

A quick and brute-force approach easily yields the results given in the table below.

Upon inspecting the table, we see that the n of interest has representation

$$n = a^6 b^4, \qquad b = \prod_{i=1}^{2m} p_i, \qquad a \in \mathbb{N}, \qquad m \in \mathbb{N}, \qquad p_i\text{'s are distinct is primes.} \tag{1.4}$$

Most of the values in the table fit the representation above. A few needs to adjusted slightly, e.g. $n = 85525504 = 2^{10} \cdot 17^4 = 2^6 \cdot (2 \cdot 17)^4$. Note that the numbers whose prime factorization contains an even number of distinct primes are those numbers whose Möbius value is equal to one.

For each value of b, there are

$$\frac{n^{1/4}}{b^{3/2}} \tag{1.5}$$

choices for a, because $a^6 b^4 \leq n$ must hold. If we set $b = n^{\frac{1}{6}}$ in the above, then we get 1.

Let $\mu : \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function and let $M_+ : \mathbb{R}_+ \to \mathbb{N}$, given by

$$M_+(x) := \sum_{i=1}^{\lfloor x \rfloor} \mu^2(i), \tag{1.6}$$

be the function that counts the natural numbers $i \leq \lfloor x \rfloor$ such that $\mu(i) = 1$.

Then

$$f(n) = \sum_{a=1}^{n^{1/6}} M_+ \left(\frac{n^{1/4}}{a^{3/2}} \right). \tag{1.7}$$

The values of Möbius function can easily be computed using a sieve (the Sieve of Eratosthenes works fine), and the rest is trivial.

$f(n)$	n	
1	1	1^1
2	64	2^6
3	729	3^6
4	1296	$2^4 \cdot 3^4$
5	4096	2^{12}
6	10000	$2^4 \cdot 5^4$
7	15625	5^6
8	38416	$2^4 \cdot 7^4$
9	46656	$2^6 \cdot 3^6$
10	50625	$3^4 \cdot 5^4$
11	82944	$2^{10} \cdot 3^4$
12	117649	7^6
13	194481	$3^4 \cdot 7^4$
14	234256	$2^4 \cdot 11^4$
15	262144	2^{18}
16	456976	$2^4 \cdot 13^4$
17	531441	3^{12}
18	640000	$2^{10} \cdot 5^4$
19	944784	$2^4 \cdot 3^{10}$
20	1000000	$2^6 \cdot 5^6$
21	1185921	$3^4 \cdot 11^4$
22	1336336	$2^4 \cdot 17^4$
23	1500625	$5^4 \cdot 7^4$
24	1771561	11^6
25	2085136	$2^4 \cdot 19^4$
26	2313441	$3^4 \cdot 13^4$
27	2458624	$2^{10} \cdot 7^4$
28	2985984	$2^{12} \cdot 3^6$
29	3240000	$2^6 \cdot 3^4 \cdot 5^4$
30	4477456	$2^4 \cdot 23^4$
31	4826809	13^6
32	5308416	$2^{16} \cdot 3^4$
33	6765201	$3^4 \cdot 17^4$
34	7290000	$2^4 \cdot 3^6 \cdot 5^4$
35	7529536	$2^6 \cdot 7^6$
36	9150625	$5^4 \cdot 11^4$
37	10556001	$3^4 \cdot 19^4$
38	11316496	$2^4 \cdot 29^4$
39	11390625	$3^6 \cdot 5^6$
40	12446784	$2^6 \cdot 3^4 \cdot 7^4$
41	14776336	$2^4 \cdot 31^4$
42	14992384	$2^{10} \cdot 11^4$
43	16777216	2^{24}
44	17850625	$5^4 \cdot 13^4$
45	20250000	$2^4 \cdot 3^4 \cdot 5^6$
46	22667121	$3^4 \cdot 23^4$
47	24137569	17^6
48	28005264	$2^4 \cdot 3^6 \cdot 7^4$
49	29246464	$2^{10} \cdot 13^4$
50	29986576	$2^4 \cdot 37^4$
51	34012224	$2^6 \cdot 3^{12}$
52	35153041	$7^4 \cdot 11^4$
53	36905625	$3^{10} \cdot 5^4$
54	40960000	$2^{16} \cdot 5^4$
55	45212176	$2^4 \cdot 41^4$
56	47045881	19^6
57	52200625	$5^4 \cdot 17^4$
58	54700816	$2^4 \cdot 43^4$
59	57289761	$3^4 \cdot 29^4$
60	60466176	$2^{10} \cdot 3^{10}$
61	64000000	$2^{12} \cdot 5^6$
62	68574961	$7^4 \cdot 13^4$
63	74805201	$3^4 \cdot 31^4$
64	75898944	$2^6 \cdot 3^4 \cdot 11^4$
65	78074896	$2^4 \cdot 47^4$
66	81450625	$5^4 \cdot 19^4$
67	85525504	$2^{10} \cdot 17^4$
68	85766121	$3^6 \cdot 7^6$
69	96040000	$2^6 \cdot 5^4 \cdot 7^4$