FEBio Theory Manual(extended 2.0)[1]

0.1 Frictional Sliding Contact

0.1.1 Contact Integral

First we need to define the tangential gap function $g_t(X, \bar{Y}(X))$ that measures the (tangential)distance between a given point $X \in \Gamma_c^{(1)}$ on the slave reference contact surface and $\bar{Y}(X) \in \Gamma_c^{(2)}$, the closest point to X on the master contact surface, expressed as

$$\begin{cases} g_t(X, \bar{Y}(X)) = m_{\alpha\beta}(y_\gamma^{\vdash} - y_{\gamma\rho}^{\vdash}) \qquad \gamma \in \{\alpha, \beta\} \\ g_t^\alpha = \xi_{n+1}^\alpha - \xi_n^\alpha \qquad (shorthand\, notation), \end{cases} \tag{1}$$

where $\alpha, \beta = 1, ..., d-1$ with d for the dimension in the problem, $m_{\alpha\beta}, y_\gamma^{\vdash} - y_{\gamma\rho}^{\vdash}$ are the metric tensor and the component(current and previous) of the closest point. ξ_n^α is the isoparametric coordinate of the projected slave node onto the master element.

The mortar frictional contact integral defined between surfaces(master and slave) is given by

$$G^c(\varphi, w) = \int_{\Gamma_c^{(1)}} t_{T_\alpha} \delta\xi^\alpha d\Gamma, \tag{2}$$

where $\Gamma_c^{(1)}$ is the part of the boundary that will be in contact with the other body and $t_{T_\alpha}$the frictional traction calculated based upon the following determination

$$t_{T_\alpha}^{n+1} = \begin{cases} T_{T_\alpha}^{n+1} = t_{T_\alpha}^n + \epsilon_T m_{\alpha\beta}(\xi_{n+1}^\beta - \xi_n^\beta) \quad if \ \parallel T_T^{n+1} \parallel - \mu t_N^{n+1} \le 0\,(stick) \\ \mu t_N^{n+1} \frac{T_{T_\alpha}^{n+1}}{\| T_T^{n+1} \|} \quad otherwise\,(slip), \end{cases} \tag{3}$$

where ϵ_T the friction penalty value, $\parallel T_T^{n+1} \parallel = [t_{T_\alpha} m^{\alpha\beta} t_{T_\beta}]^{\frac{1}{2}}$, $m_{\alpha\beta} = \tau_\alpha.\tau_\beta$, $\tau^\alpha = m^{\alpha\beta}\tau_\beta$, $\tau_\alpha = \varphi_t(\bar{Y}(X))_{,\alpha} = \frac{\partial \varphi_t(\bar{Y}(X))}{\partial \alpha}$, τ^α, τ_α the contra and co variant base vectors, μ the friction coefficient, $\delta\xi^\alpha$ the variation of the tangential gap function is given by

$$A_{\alpha\beta}\delta\xi^\alpha = [w^{(1)}(X) - w^{(2)}(\bar{Y}(X))].\tau_\alpha - gv.[\frac{\partial w^{(2)}(\bar{Y}(X))}{\partial \alpha}], \tag{4}$$

where $A_{\alpha\beta} = m_{\alpha\beta} + g\kappa_{\alpha\beta}$, $\kappa_{\alpha\beta} = v.\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\beta}$ the curvature of the surface at $\varphi_t^{(2)}(\bar{Y}(X))$.
Note that, $[w^{(1)}(X) - w^{(2)}(\bar{Y}(X))] = [\delta\varphi^{(1)}(X) - \delta\varphi^{(2)}(\bar{Y}(X))]$.
Assuming a perfect sliding ($gap_n = \frac{\partial gap_n}{\partial t} = 0$), the equation (4) simplifies to

$$\delta\xi^\alpha = [w^{(1)}(X) - w^{(2)}(\bar{Y}(X))].\tau^\alpha \tag{5}$$

Thus, the contact virtual work can be expressed in the following compact form(in term of t_N the contact pressure and $t_{T_\alpha}$the frictional traction)

$$G^c(\varphi, w) = \int_{\Gamma_c^{(1)}} [t_N \delta g + t_{T_\alpha} \delta\xi^\alpha] d\Gamma \tag{6}$$

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0.1.2 Linearization of the Contact Integral

The linearization of the equation (2) is given by the following equation

$$\triangle G^c(\varphi, w) = \triangle \int_{\Gamma_c^{(1)}} t_{T_\alpha} \delta\xi^\alpha d\Gamma = \int_{\Gamma_c^{(1)}} [\triangle t_{T_\alpha} \delta\xi^\alpha + t_{T_\alpha} \triangle(\delta\xi^\alpha)] d\Gamma, \tag{7}$$

where

$$\triangle t_{T_\alpha}^{n+1} = \begin{cases} \triangle T_{T_\alpha}^{n+1} = \epsilon_T \{ m_{\alpha\beta} \triangle\xi^\beta + [(\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma}.\tau_\beta + \varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma}.\tau_\alpha) \triangle\xi^\gamma + \\ \triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha}.\tau_\beta + \triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\beta}.\tau_\alpha](\xi_{n+1}^\beta - \xi_n^\beta)\} \quad (stick) \\ \mu p_{T_\alpha} \epsilon_N H(g) \triangle g + \mu t_N^{n+1} \frac{\triangle T_{T_\beta}^{n+1}}{\| T_T^{n+1} \|} [\delta_\alpha^\beta - p_T^\beta p_{T_\alpha}] + \\ \mu t_N^{n+1} p_T.[\triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\beta} + \triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma} \triangle\xi^\gamma] p_T^\beta p_{T_\alpha} \quad (slip), \end{cases} \tag{8}$$

where $p_{T_\alpha} = \frac{T_{T_\alpha}^{n+1}}{\| T_T^{n+1} \|}$ and $\delta_\alpha^\beta = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$ stands for the Kronecker delta.

And the quantity $\triangle(\delta\xi^\alpha)$ is given by

$$[m_{\alpha\beta} + g\kappa_{\alpha\beta}] \triangle(\delta\xi^\alpha) = -\tau_\alpha.\delta\varphi_t^{(2)}(\bar{Y}(X))_{,\beta} \triangle\xi^\beta - \tau_\alpha.\triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\beta} \delta\xi^\beta -$$

$$[\tau_\alpha.\varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma} + gv.\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\beta\gamma}] \delta\xi^\beta \triangle\xi^\beta - \delta\xi^\beta \tau_\beta.[\triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha} +$$

$$\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma} \triangle\xi^\gamma] - \triangle\xi^\beta \tau_\beta.[\delta\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha} + \varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma} \delta\xi^\gamma] - gv.[\delta\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\beta} \triangle\xi^\beta +$$

$$\triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\beta} \delta\xi^\beta] + [\delta\varphi^{(1)}(X) - \delta\varphi^{(2)}(\bar{Y}(X))].[\triangle\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha} + \varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma} \triangle\xi^\gamma] +$$

$$[\triangle\varphi^{(1)}(X) - \triangle\varphi^{(2)}(\bar{Y}(X))].[\delta\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha} + \varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma} \delta\xi^\gamma] \tag{9}$$

0.1.3 Discretization of the Contact Integral

The discretization of the equation (2) can be written as the following expression

$$G^c(\varphi^h, w^h) \approx \sum_{e=1}^{n_{sel}} \{ \sum_{k=1}^{n_{int}} W_k j(\eta_k) t_{T_\alpha}^h(\eta_k) \delta\xi^{\alpha^h}(\eta_k) \}, \tag{10}$$

where φ^h, w^h are the finite dimensional counterparts of φ, w, n_{sel} is the number of surface elements (i.e., the slave surface elements), n_{int} is the number of integration points, W_k is the quadrature weight corresponding to local quadrature points k and $j(\eta_k) = \| \frac{\partial X^h(\eta_k)}{\partial\eta_\alpha} \times \frac{\partial X^h(\eta_k)}{\partial\eta_\beta} \|$ is the Jacobian of the transformation at each point η_k.

The equation (9) can be also expressed as follows

$$G^c(\varphi^h, w^h) \approx \sum_{e=1}^{n_{sel}} \{ \sum_{k=1}^{n_{int}} W_k j(\eta_k) \delta\Phi_k^c.f_k^c \}, \tag{11}$$

where $\delta\Phi^c = \begin{bmatrix} C_s^{(1)} \\ C_1^{(2)} \\ . \\ . \\ C_{mnode}^{(2)} \end{bmatrix}$, $f^c = t_{T_\alpha} D_\alpha + t_{T_\beta} D_\beta$ $(t_{T_\alpha} = t_{T_1}, t_{T_\beta} = t_{T_2}, D_\alpha = D_1, D_\beta = D_2)$ are the vector of nodal variations(slave and master nodes) and the local contact force vector corresponding to local quadrature points .The *mnode* refers to the number of nodes per element on the master surface (e.g. for a quadrilateral surface element we have $mnode = 4$)

Where D_1, D_2 have been defined by equation (6.57).

0.1.4 Discretization of the Contact Stiffness

The discretization of the equation (7) can be written as the following expression

$$\triangle G^c(\varphi^h, w^h) \approx \sum_{e=1}^{n_{sel}} \{ \sum_{k=1}^{n_{int}} W_k j(\eta_k) [\triangle t_{T_\alpha}^h(\eta_k) \delta\xi^{\alpha^h}(\eta_k) + t_{T_\alpha}^h(\eta_k) \triangle\delta\xi^{\alpha^h}(\eta_k)] \} =$$

$$\sum_{e=1}^{n_{sel}} \{ \sum_{k=1}^{n_{int}} W_k j(\eta_k) \delta\Phi_k^c . k_k^c \triangle\Phi_k^c \}, \tag{12}$$

where $\triangle\Phi^c = \begin{bmatrix} \triangle d_s^{(1)} \\ \triangle d_1^{(2)} \\ . \\ . \\ \triangle d_{mnode}^{(2)} \end{bmatrix}$, $k^c = k_{direct}^c + t_{T_\alpha} A^{\alpha\beta} k_{T_\beta}^c$ are the vector of local nodal values and the global contact stiffness defined as the following expressions

$$k_{direct}^c = \begin{cases} \epsilon_T [m_{\alpha\beta} D_\alpha D_\beta^T + D_\alpha G_\alpha^T] \ (stick) \\ -\mu p_{T_\alpha} \epsilon_N H(g) D_\alpha \mathbf{N}^T + \frac{\mu \epsilon_T t_N}{\| T_T^{n+1} \|} [\delta_\alpha^\beta - p_T^\beta p_{T_\alpha}][m_{\beta\gamma} D_\alpha D_\gamma^T + D_\alpha G_\beta^T] - \\ \mu t_N p_T^\alpha p_{T_\beta} D_\beta \bar{P}_\alpha^{\ T} \quad (slip), \end{cases} \tag{13}$$

where

$$G_\alpha = \{ -\mathbf{T}_{\alpha\beta} - \mathbf{T}_{\beta\alpha} + [\varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma} . \tau_\alpha + \varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma} . \tau_\beta] D_\gamma \} g_t^\beta \tag{14}$$

The quantity $k_{T_\alpha}^c$ is given by

$$k_{T_\alpha}^c = \mathbf{T}_{\alpha\beta} D_\beta^T + D_\beta \mathbf{T}_{\alpha\beta}^T - (\varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma} . \tau_\alpha) D_\beta D_\gamma^T + \bar{T}_{\beta\alpha} D_\beta^T + D_\beta \bar{T}_{\beta\alpha}^T +$$

$$g(\mathbf{N}_{\alpha\beta} D_\beta^T + D_\beta \mathbf{N}_{\alpha\beta}^T) - \mathbf{N}\bar{\mathbf{N}}_\alpha^T - \mathbf{T}_\beta m^{\beta\gamma} \bar{T}_{\gamma\alpha}^T - \bar{\mathbf{N}}_\alpha \mathbf{N}^T - \bar{T}_{\gamma\alpha} m^{\beta\gamma} \mathbf{T}_\beta^T, \tag{15}$$

where $\mathbf{N}_{\alpha\beta}, \mathbf{T}_{\alpha\beta}, \mathbf{P}_\alpha$ are defined as follows

$$\mathbf{T}_{\alpha\beta} = \begin{bmatrix} 0 \\ -N_{1,\beta}(\bar{Y}(X))\tau_\alpha \\ . \\ . \\ -N_{4,\beta}(\bar{Y}(X))\tau_\alpha \end{bmatrix}, \mathbf{N}_{\alpha\beta} = \begin{bmatrix} 0 \\ -N_{1,\alpha\beta}(\bar{Y}(X))v \\ . \\ . \\ -N_{4,\alpha\beta}(\bar{Y}(X))v \end{bmatrix}, \mathbf{P}_\alpha = \begin{bmatrix} 0 \\ -N_{1,\alpha}(\bar{Y}(X))p_T \\ . \\ . \\ -N_{4,\alpha}(\bar{Y}(X))p_T \end{bmatrix} \tag{16}$$

and

$$
\begin{cases}
\bar{T}_{\alpha 1} = \mathbf{T}_{\alpha 1} - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.\tau_\alpha)D_2 \\
\bar{T}_{\alpha 2} = \mathbf{T}_{\alpha 2} - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.\tau_\alpha)D_1 \\
\bar{P}_1 = \mathbf{P}_1 - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.p_T)D_2 \\
\bar{P}_2 = \mathbf{P}_2 - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.p_T)D_1
\end{cases}
\tag{17}
$$

Where also $\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{N}, D_1, D_2, \bar{\mathbf{N}}_\mathbf{1}, \bar{\mathbf{N}}_\mathbf{2}$ vectors have been defined by equations (6.56) and (6.57).

0.2 Thermodynamically coupled Frictional Sliding Contact

0.2.1 Contact Integral

The formulation of the thermodynamical part of the friction stresses requires new definitions, namely heat fluxes(assumed to be positive if the heat flow out of the contacting body into the interface region and zero in the case of the out of contact), the entropy evolution on the contact interfaces and the mechanical dissipation.

The constitutive relation of heat fluxes across the contact interfaces can be shown as:

$$
\begin{cases}
q_c^s = H_c^s \nabla \Theta^s & slave\, interface \\
q_c^m = H_c^m \nabla \Theta^m & master\, interface,
\end{cases}
\tag{18}
$$

where $\nabla\Theta^s, \nabla\Theta^m, H_c^s, H_c^m$ temperature gradients and heat transfer coefficients(slave and master interfaces).

The entropy evolution is given by

$$
\aleph_c = \frac{C_c(T_c - T_0)}{T_0},
\tag{19}
$$

where C_c, T_c, T_0 are the heat capacity per unit surface of the trapped debris, interface characteristic temperature and reference temperature.

> ***Remark:***
> According to the equation (19), aside the heat capacity C_c, we still need to define the two values T_c, T_0. Since both of them can be eliminated algebraically, therefore, in the case when $\exists \alpha, \beta > 0 \in R \mid T_c = \alpha T_0$, the equation (19) simplified to
>
> $$\aleph_c = \frac{C_c(\alpha T_0 - T_0)}{T_0} = \frac{C_c(\alpha - 1)T_0}{T_0} = C_c(\alpha - 1) = \underbrace{C_c\beta}_{scaled value of heat capacity} \tag{20}$$
>
> Thus, we just need to set a scaled value of the real heat capacity per unit surface (master and slave) of the trapped debris. Note that, due to the thermodynamically consistent reason, the entropy evolution, suppose to be positive all the time.

At this point, it is worth mentioning that, the formulation of the heat fluxes across the contact interfaces (q_c^s, q_c^m), depend on the normal pressure t_N , can be done by replacement of the quantities H_c^s, H_c^m with $t_N h_c^s, t_N h_c^m$ and the quantity$\nabla\Theta$ with $(T^{s|m} - T_c)$ in the equation (18), where $T^{s|m}[T^s, T^m]$ stands for the temperature of two bodies(which the master and slave interfaces belong to) at the contact activation stage(contact approach time).

Using the definition of the frictional traction and time derivative of tangential relative displacement (tangential relative velocity) one obtains the following expression for the mechanical dissipation

$$D_{mech} = | \overbrace{t_{T_\alpha}}^{frictionaltraction} . \underbrace{\frac{\partial(\xi^\alpha)}{\partial t}}_{tangentialrelativevelocity} |\approx| t_{T_\alpha}.\frac{(\xi^\alpha_{n+1}-\xi^\alpha_n)}{\Delta t} | \tag{21}$$

Note that, in the case of non perfect sliding($gap_n \neq 0$), the tangential relative velocity term $\frac{\partial(\xi^\alpha)}{\partial t} = \dot{\xi}^\alpha$ should be calculated as follows

$$A_{\alpha\beta}\frac{\partial(\xi^\alpha)}{\partial t} = \underbrace{[V^{(1)}(X) - V^{(2)}(\bar{Y}(X))]}_{relativematerialvelocity}.\tau_\alpha - gv.[\frac{\partial V^{(2)}(\bar{Y}(X))}{\partial \alpha}], \tag{22}$$

where $A_{\alpha\beta} = m_{\alpha\beta} + g\kappa_{\alpha\beta}$, $\kappa_{\alpha\beta} = v.\varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\beta}$ the curvature of the surface at $\varphi_t^{(2)}(\bar{Y}(X))$.

Combining the definition of the frictional stresses, the entropy evolution, the mechanical dissipation and the local energy balance on the contact interfaces, the heat fluxes equation for the contact interfaces can be expressed as:

$$\begin{cases} q_c^s = t_N R(T^s - T^m) + (R1)D_{mech} + t_N[(R3)T^s - (R4)] \\ q_c^m = t_N R(T^s - T^m) + (R2)D_{mech} + t_N[(R5)T^m - (R6)], \end{cases} \tag{23}$$

where

$$\begin{cases} R = \frac{h_c^s h_c^m}{F_1 + h_c^s + h_c^m} \,;\, R1 = \frac{R}{h_c^m} \,;\, R2 = \frac{R}{h_c^s} \,; \\ R3 = F_1 R1 \,;\, R4 = F_2 R1 \,;\, R5 = F_1 R2 \,;\, R6 = F_2 R2 \,; \\ F_1 = \frac{C_c T_{c(s|m)}}{\Delta t T_0} \,;\, F_2 = F_1 T_{c(s|m)} \\ t_N \;\; contact\, pressure \end{cases} \tag{24}$$

Using above definitions, the thermodynamic integral defined between master and slave surfaces is given by:

$$G^c(\varphi, T, \delta T) = \int_{\Gamma_c^{(1)}} \{q_c^{slave}\delta T^S + q_c^{master}\delta T^M\} d\Gamma = \int_{\Gamma_c^{(1)}} \{\underbrace{t_N R(T^s - T^m)(\delta T^S - \delta T^M)}_{conduction} -$$

$$\underbrace{D_{mech}(R1\delta T^S + R2\delta T^M)}_{dissipation} + \underbrace{t_N[(R3T^S + R4)\delta T^S - (R5T^M + R6)\delta T^M]}_{heatsinks}\} d\Gamma_s, \tag{25}$$

where $\delta T^S, \delta T^M$ are the (virtual) temperature variation of slave and master interfaces.

Note that, due to the contact inequality constraints: the Kuhn-Tucker conditions $\begin{cases} T_n(\lambda, g) \geqq 0 \\ g_n(x, y^{\Vdash}) \leqq 0 \\ T_n(\lambda, g) g_n(x, y^{\Vdash}) = 0 \end{cases}$,
in the case of the out of contact $t_N = 0$, the conduction ,the heat sinks and the dissipation integral will tend to be zero which will have implied zero heat fluxes across the interfaces.

0.2.2 Linearization of the Contact Integral

The linearization of the equation (24) is given by the following equation

$$\triangle G^c(\varphi, T, \delta T) = \int_{\Gamma_c^{(1)}} \triangle\{q_c^{slave}\delta T^S + q_c^{master}\delta T^M\}d\Gamma = \int_{\Gamma_c^{(1)}} \{\triangle t_N R[(T^s - T^m)(\delta T^S - \delta T^M)]+$$

$$t_N R[\triangle(T^s - T^m)(\delta T^S - \delta T^M)] + t_N R[(T^s - T^m)\triangle(\delta T^S - \delta T^M)] - \triangle D_{mech}(R1\delta T^S + R2\delta T^M)-$$

$$D_{mech}\triangle(R1\delta T^S + R2\delta T^M) + \triangle t_N[(R3T^S + R4)\delta T^S - (R5T^M + R6)\delta T^M]+$$

$$t_N[\triangle(R3T^S + R4)\delta T^S - \triangle(R5T^M + R6)\delta T^M]\}d\Gamma_s, \tag{26}$$

where

$$\begin{cases} \triangle t_N = \triangle\{\varepsilon_N g\} = H(g)\varepsilon_N \triangle g \\ \triangle(T^s(X) - T^m(\bar{Y}(X))) = \triangle T^s(X) - \triangle T^m(\bar{Y}(X)) - T^m(\bar{Y}(X))_{,\alpha}\triangle\xi^\alpha \\ \triangle(\delta T^S(X) - \delta T^M(\bar{Y}(X))) = -\delta T^M(\bar{Y}(X))_{,\alpha}\triangle\xi^\alpha \\ \triangle D_{mech} = (\triangle t_{T_\alpha}\frac{\partial(\xi^\alpha)}{\partial t} + t_{T_\alpha}\triangle\frac{\partial(\xi^\alpha)}{\partial t})sign(t_{T_\alpha}.\frac{\partial(\xi^\alpha)}{\partial t}) \\ \triangle(R1\delta T^S + R2\delta T^M) = -R2\delta T^M(\bar{Y}(X))_{,\alpha}\triangle\xi^\alpha \\ \triangle(R3T^S + R4)\delta T^S - \triangle(R5T^M + R6)\delta T^M = \triangle R3T^S\delta T^S - R5[\triangle T^M\delta T^M + \\ T^M(\bar{Y}(X))_{,\alpha}\triangle\xi^\alpha\delta T^M] - (R5T^M - R6)\delta T^M(\bar{Y}(X))_{,\alpha}\triangle\xi^\alpha \end{cases} \tag{27}$$

0.2.3 Discretization of the Contact Integral

The discretization of the equation (24) can be written as the following expression

$$G^c(\varphi^h, T^h, \delta T^h) \approx \sum_{e=1}^{n_{sel}}\{\sum_{k=1}^{n_{int}} W_k j(\eta_k)[t_N(\eta_k)R(T^s(\eta_k) - T^m(\eta_k))(\delta T^S(\eta_k) - \delta T^M(\eta_k))-$$

$$D_{mech}(\eta_k)(R1\delta T^S(\eta_k) + R2\delta T^M(\eta_k)) + t_N(\eta_k)((R3T^S(\eta_k) + R4)\delta T^S(\eta_k)-$$

$$(R5T^M(\eta_k) + R6)\delta T^M(\eta_k))]\} = \sum_{e=1}^{n_{sel}}\{\sum_{k=1}^{n_{int}} W_k j(\eta_k)\delta\Phi_k^c.fther_k^c\}, \tag{28}$$

where $\delta\Phi^c = \begin{bmatrix} C_\varphi^{(1)} \\ C_T^{(1)} \\ --- \\ C_{\varphi 1}^{(2)} \\ C_{T1}^{(2)} \\ . \\ . \\ C_{\varphi mnode}^{(2)} \\ C_{Tmnode}^{(2)} \end{bmatrix}$, $fther_k^c$ are the vector of nodal variations(slave and master nodes) and the thermal part of the local contact force vector corresponding to local quadrature point k.The *mnode* refers to the number of nodes per element on the master surface (e.g. for a quadrilateral surface element we have $mnode = 4$).

Where also

$$T^h := T^{s|m^h}(\eta) = \sum_{i=1}^{n_{int}} N_i(\eta) T_i^{s|m}(t) \tag{29}$$

,

$$\delta T^h := \delta T^{s|m^h}(\eta) = \sum_{i=1}^{n_{int}} N_i(\eta) c_i^{s|m} \tag{30}$$

$$\varphi^h := \varphi^{s|m^h}(\eta) = \sum_{i=1}^{n_{int}} N_i(\eta) d_i^{s|m}(t),\ \delta\varphi^h := \delta\varphi^{s|m^h}(\eta) = \sum_{i=1}^{n_{int}} N_i(\eta) C_i^{s|m}, \tag{31}$$

where $T_i^{s|m}(t)$, $d_i^{s|m}(t)$ are nodal values, $N_i(\eta)$ denotes a standard shape function and $c_i^{s|m}, C_i^{s|m}$ stand for the time independent nodal values of the associated variational fields.

The thermal part of the contact contribution to the residual can be also expressed as

$$fther^c = -\underbrace{Rt_N(\mathbf{a}.\mathbf{c})\mathbf{a}}_{conduction} + \underbrace{D_{mech}\mathbf{b}}_{dissipation} + \underbrace{t_N[(R3(\mathbf{e}.\mathbf{c}) - R4)\mathbf{e} - (R5(\mathbf{f}.\mathbf{c}) - R6)\mathbf{f}]}_{heatsinks}, \tag{32}$$

where (the temperature vectors)

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ --- \\ 0 \\ 0 \\ -N_1(\bar{\xi}) \\ . \\ . \\ 0 \\ 0 \\ -N_{mnode}(\bar{\xi}) \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ R1 \\ --- \\ 0 \\ 0 \\ R2N_1(\bar{\xi}) \\ . \\ . \\ 0 \\ 0 \\ R2N_{mnode}(\bar{\xi}) \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ T^s \\ --- \\ 0 \\ 0 \\ T_1^m \\ . \\ . \\ 0 \\ 0 \\ T_{mnode}^m \end{bmatrix} \tag{33}$$

and the vectors

$$\mathbf{d_{1,2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ --- \\ 0 \\ 0 \\ -N_{1,[1,2]}(\bar{\xi}) \\ . \\ . \\ 0 \\ 0 \\ -N_{mnode,[1,2]}(\bar{\xi}) \end{bmatrix},\ \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ --- \\ 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ --- \\ 0 \\ 0 \\ N_1(\bar{\xi}) \\ . \\ . \\ 0 \\ 0 \\ N_{mnode}(\bar{\xi}) \end{bmatrix} \tag{34}$$

Additionally, we need to (re)define the following vectors [defined in the equation (6.56)]

$$\mathbf{N} = \begin{bmatrix} v \\ 0 \\ --- \\ -N_1(\bar{\xi})v \\ 0 \\ . \\ . \\ -N_{mnode}(\bar{\xi})v \\ 0 \end{bmatrix}, \mathbf{T}_\alpha = \begin{bmatrix} \tau_\alpha \\ 0 \\ --- \\ -N_1(\bar{\xi})\tau_\alpha \\ 0 \\ . \\ . \\ -N_{mnode}(\bar{\xi})\tau_\alpha \\ 0 \end{bmatrix}, \mathbf{N}_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ --- \\ -N_{1,\alpha}(\bar{\xi})v \\ 0 \\ . \\ . \\ -N_{mnode,\alpha}(\bar{\xi})v \\ 0 \end{bmatrix}$$

and the following vectors

$$\mathbf{T}_{\alpha\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ --- \\ -N_{1,\beta}(\bar{Y}(X))\tau_\alpha \\ 0 \\ . \\ . \\ -N_{mnode,\beta}(\bar{Y}(X))\tau_\alpha \\ 0 \end{bmatrix}, \mathbf{N}_{\alpha\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ --- \\ -N_{1,\alpha\beta}(\bar{Y}(X))v \\ 0 \\ . \\ . \\ -N_{4,\alpha\beta}(\bar{Y}(X))v \\ 0 \end{bmatrix} \tag{35}$$

0.2.4 Discretization of the Contact Stiffness

The discretization of the equation (25) is given by

$$\triangle G^c(\varphi^h, T^h, \delta T^h) \approx \sum_{e=1}^{n_{sel}} \{ \sum_{k=1}^{n_{int}} W_k j(\eta_k)[\triangle t_N(\eta_k) R((T^s(\eta_k) - T^m(\eta_k))(\delta T^S(\eta_k) - \delta T^M(\eta_k)))+$$

$$t_N(\eta_k)R(\triangle(T^s(\eta_k) - T^m(\eta_k))(\delta T^S(\eta_k) - \delta T^M(\eta_k))) + t_N(\eta_k)R((T^s(\eta_k) - T^m(\eta_k))\triangle(\delta T^S(\eta_k) - \delta T^M(\eta_k))) -$$

$$\triangle D_{mech}(\eta_k)(R1\delta T^S(\eta_k) + R2\delta T^M(\eta_k)) - D_{mech}(\eta_k)\triangle(R1\delta T^S(\eta_k) + R2\delta T^M(\eta_k))+$$

$$\triangle t_N(\eta_k)((R3T^S(\eta_k) + R4)\delta T^S(\eta_k) - (R5T^M(\eta_k) + R6)\delta T^M(\eta_k))+$$

$$t_N(\eta_k)(\triangle(R3T^S(\eta_k) + R4)\delta T^S(\eta_k) - \triangle(R5T^M(\eta_k) + R6)\delta T^M(\eta_k))]\} =$$

$$\sum_{e=1}^{n_{sel}} \{ \sum_{k=1}^{n_{int}} W_k j(\eta_k) \delta\Phi_k^c . kther_k^c \triangle\Phi_k^c \}, \tag{36}$$

where $\triangle\Phi^{c}=\begin{bmatrix}\triangle d_s^{(1)}\\ \triangle T_s^{(1)}\\ ---\\ \triangle d_1^{(2)}\\ \triangle T_1^{(2)}\\ .\\ .\\ \triangle d_{mnode}^{(2)}\\ \triangle T_{mnode}^{(2)}\end{bmatrix}$, $kther^{c}=k^{c}_{conduction}+k^{c}_{dissipation}+k^{c}_{heatsinks}+k^{c}_{direct}$ are the vector of local nodal values and the global contact stiffness defined as the following expressions

$$k^{c}_{conduction}=Rt_N(\mathbf{a.c})\mathbf{d}D_1^T-RH(g)\varepsilon_N(\mathbf{a.c})\mathbf{aN^T}+Rt_N\mathbf{aa^T}-Rt_N(\mathbf{d.c})\mathbf{a}D_1^T \tag{37}$$

$$k^{c}_{dissipation}=R2\mid t_{T_1}\mid\mid\frac{(\xi^1_{n+1}-\xi^1_n)}{\Delta t}\mid\mathbf{d}D_1^T+\frac{\mid t_{T_1}\mid}{\Delta t}\mathbf{b}D_1^T \tag{38}$$

$$k^{c}_{heatsinks}=H(g)\varepsilon_N(-R3(\mathbf{c.e})+R4)\mathbf{eN^T}+H(g)\varepsilon_N(-R5(\mathbf{c.f})+R6)\mathbf{fN^T}$$

$$+R3t_N\mathbf{ee^T}+R5t_N\mathbf{ff^T}+(R5(\mathbf{c.f})-R6)t_N\mathbf{d}D_1^T \tag{39}$$

$$k^{c}_{direct}=R^1_{extra}D_1\mathbf{e^T}\quad(slip), \tag{40}$$

where D_1, D_2 vectors have been defined by equations (6.57) and k^c_{direct} denotes the linearziation of the term $\triangle t_{T_\alpha}\frac{\partial(\xi^\alpha)}{\partial t}$ with respect to temperature degree of freedom(in the case of slip) and

$$R^{\alpha}_{extra}=W_\alpha\frac{1}{1+\frac{\epsilon_T\Delta t}{t_N\eta}}t_N[K(\alpha+T_m)+\mu(T_m)], \tag{41}$$

where η is a fluidity parameter (associated with viscoplastic shearing effects), T_m is the maximum absolute temperature on the two contact surfaces, $K(\alpha+T_m)$ is a user defined control function of the evaluation of the frictional stress under steady state conditions and

$$W_\alpha=\frac{\epsilon_T\Delta t}{t_N\eta}\frac{T^{n+1}_{T_\alpha}}{\parallel T^{n+1}_T\parallel} \tag{42}$$

$$\mu(T_m)=\mu_0\frac{(T_d-T_0)^2}{(T_m-T_d)^2}, \tag{43}$$

where T_d, T_0 are the damage temperature on the two contact surfaces and the reference temperature and $\mu_0$the static coefficient of friction at reference temperature.

Note that, in the case of stick, due to the fact that the frictional stress is not dependent on interface temperature, therfore, the linearziation of the term $\triangle t_{T_\alpha}\frac{\partial(\xi^\alpha)}{\partial t}$ with respect to temperature degree of freedom will tend to be zero.

0.3 Sliding Contact with Cohesive Force(Cohesive Fracture Model formulation through Contact Mechanics)

0.3.1 Contact Integral

The mathematical formulation of sliding contact boundary value problem combined with the mixed mode cohesive law, in the form of contact integral can be shown as:

$$G^c(\varphi, w) = \underbrace{\int_{\Gamma_c^{(1)}} t_N \delta g d\Gamma}_{g_N(X)>0} + \overbrace{\underbrace{\int_{\Gamma_c^{(1)}} t_N^{coh} \delta g d\Gamma + \int_{\Gamma_c^{(1)}} t_{T_\alpha}^{coh} \delta \xi^\alpha d\Gamma}_{g_N(X)\leq 0}}^{totalworkofseparation}, \tag{44}$$

where

$$\begin{cases} t_N = \lambda_N + \varepsilon_N g_N \\ \delta g = -v.[w^{(1)}(X) - w^{(2)}(\bar{Y}(X))] \\ A_{\alpha\beta}\delta\xi^\alpha = [w^{(1)}(X) - w^{(2)}(\bar{Y}(X))].\tau_\alpha - gv.[\frac{\partial w^{(2)}(\bar{Y}(X))}{\partial \alpha}] \end{cases} \tag{45}$$

and $t_N^{coh}, t_{T_\alpha}^{coh}$ are the normal and tangential cohesive traction have been defined based on exponential mixed mode cohesive zone law as follows

$$\begin{cases} \Pi(\delta_n, \delta_t) = \Pi_0[1 - (1 + \frac{\delta_n}{\delta_n^c}).exp[-(\frac{\delta_n}{\delta_n^c} + \frac{\delta_t^2}{(\delta_t^c)^2})]] \\ T_n(\delta_n, \delta_t) = \frac{\partial \Pi}{\partial \delta_n} \mid_{\Pi_0 = \Pi_n} = \frac{\Pi_n}{\delta_n^c}.(\frac{\delta_n}{\delta_n^c}).exp[-(\frac{\delta_n}{\delta_n^c} + \frac{\delta_t^2}{(\delta_t^c)^2})] \\ T_t(\delta_n, \delta_t) = \frac{\partial \Pi}{\partial \delta_t} \mid_{\Pi_0 = \Pi_t} = 2\frac{\Pi_t}{\delta_t^c}.(\frac{\delta_t}{\delta_t^c}).(1 + \frac{\delta_n}{\delta_n^c}).exp[-(\frac{\delta_n}{\delta_n^c} + \frac{\delta_t^2}{(\delta_t^c)^2})] \\ \delta_n^c = \frac{\Pi_n}{T_n^{max}.exp(1)}, \ \delta_t^c = \frac{\Pi_t}{T_t^{max}.\sqrt{\frac{1}{2}exp(1)}} \end{cases} \tag{46}$$

where $T_n, T_t, T_n^{max}, T_t^{max}, \delta_n^c, \delta_t^c, \Pi(\delta_n, \delta_t), \delta_n, \delta_t$ are the normal opening traction (coupled mode I cracks), the tangential opening traction (coupled mode II,III cracks), the maximum normal opening traction, the maximum tangential opening traction, the characteristic opening length of the normal direction, the characteristic opening length of the tangential direction, the potential function, the relative displacement of normal opening and the relative displacement of tangential opening.

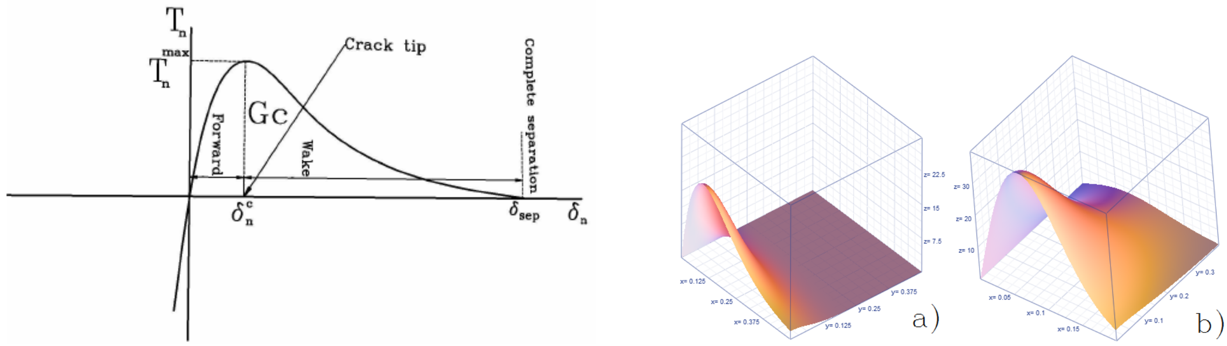

Figure 1:Two dimensional illustration of the normal opening traction versus normal opening(pure normal opening).Three dimensional illustration of the normal and tangential traction-opening $[a) : T_n(\delta_n, \delta_t) = z, \delta_n = x, \delta_t = y, \ b) : T_t(\delta_n, \delta_t) = z, \delta_n = y, \delta_t = x \ ]$

Using the above definition, $t_N^{coh}, t_{T_\alpha}^{coh}$ have been calculated based upon the following determination

$$
\begin{cases}
t_N^{coh} &= t_N^{max}.(\frac{g_N(X)}{\varepsilon_n^{coh}}).e^{(1-(\frac{g_N(X)}{\varepsilon_n^{coh}}))} e^{-(\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})} \\
t_T^{coh} &= 2\sqrt{\frac{1}{2}e}.t_T^{max}.(\frac{g_T(X)}{\varepsilon_t^{coh}}).(1+\frac{g_N(X)}{\varepsilon_n^{coh}}).e^{-(\frac{g_N(X)}{\varepsilon_n^{coh}}+\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})} \\
t_{T_\alpha}^{coh} &= 2\sqrt{\frac{1}{2}e}.t_T^{max}.(\frac{g_T^\alpha(X)}{\varepsilon_t^{coh}}).(1+\frac{g_N(X)}{\varepsilon_n^{coh}}).e^{-(\frac{g_N(X)}{\varepsilon_n^{coh}}+\frac{(g_T^\alpha(X))^2}{(\varepsilon_t^{coh})^2})},
\end{cases} \tag{47}
$$

where $g_N(X)[-v.(\varphi^{(1)}(X)-\varphi^{(2)}(\bar{Y}(X)))]$, $g_T(X)[g_T^\alpha = m_{\alpha\beta}(\xi_{n+1}^\beta - \xi_n^\beta)]$ are the normal and tangential gap functions, t_N^{max}, t_T^{max} are the maximum normal tangential opening tractions, $\varepsilon_n^{coh}, \varepsilon_t^{coh}$ are the characteristic opening length of the normal and the tangential directions.

0.3.2 Linearization of the Contact Integral

The linearization of the equation (43) is given by the following equation

$$
\triangle G^c(\varphi,w) = \triangle\int_{\Gamma_c^{(1)}} t_N\delta g d\Gamma + \triangle\int_{\Gamma_c^{(1)}} t_N^{coh}\delta g d\Gamma + \triangle\int_{\Gamma_c^{(1)}} t_{T_\alpha}^{coh}\delta\xi^\alpha d\Gamma =
$$

$$
\int_{\Gamma_c^{(1)}}(\triangle t_N\delta g + t_N\triangle\delta g)d\Gamma + \int_{\Gamma_c^{(1)}}(\triangle t_N^{coh}\delta g + t_N^{coh}\triangle\delta g)d\Gamma + \int_{\Gamma_c^{(1)}}(\triangle t_{T_\alpha}^{coh}\delta\xi^\alpha + t_{T_\alpha}^{coh}\triangle\delta\xi^\alpha)d\Gamma =
$$

$$
\underbrace{\int_{\Gamma_c^{(1)}}(\triangle t_N\delta g + t_N\triangle\delta g)d\Gamma}_{section(6.3.3)} + \int_{\Gamma_c^{(1)}}[(\frac{\partial t_N^{coh}}{\partial g_N}.\triangle g_N + \frac{\partial t_N^{coh}}{\partial g_T}.\triangle g_T)\delta g + t_N^{coh}\triangle\delta g)]d\Gamma +
$$

$$
\int_{\Gamma_c^{(1)}}[(\frac{\partial t_T^{coh}}{\partial g_N}.\triangle g_N + \frac{\partial t_T^{coh}}{\partial g_T}.\triangle g_T)\delta\xi^\alpha + t_{T_\alpha}^{coh}\triangle\delta\xi^\alpha]d\Gamma, \tag{48}
$$

where $\triangle\delta g, \triangle\delta\xi^\alpha$ have been defined by equations (6.45) and (9), and $\triangle t_N^{coh}, \triangle t_{T_\alpha}^{coh}$ are defined as follows

$$
\triangle t_{N|T}^{coh} := \begin{cases}
\frac{\partial t_N^{coh}}{\partial g_N}.\triangle g_N = \{t_N^{max}.e^{-(\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})}[(\frac{e^{(1-(\frac{g_N(X)}{\varepsilon_n^{coh}}))}}{\varepsilon_n^{coh}}) + (\frac{g_N(X)}{\varepsilon_n^{coh}})(\frac{-1}{\varepsilon_n^{coh}})e^{(1-(\frac{g_N(X)}{\varepsilon_n^{coh}}))}]\}\triangle g_N \\
\frac{\partial t_N^{coh}}{\partial g_T}.\triangle g_T = \{t_N^{max}.(\frac{g_N(X)}{\varepsilon_n^{coh}}).e^{(1-(\frac{g_N(X)}{\varepsilon_n^{coh}}))}[(\frac{-2g_T(X)}{(\varepsilon_t^{coh})^2})e^{-(\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})}]\}\triangle g_T \\
\frac{\partial t_T^{coh}}{\partial g_N}.\triangle g_N = \{2\sqrt{\frac{1}{2}e}.t_T^{max}.(\frac{g_T(X)}{\varepsilon_t^{coh}}).e^{-(\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})}[(\frac{e^{-(\frac{g_N(X)}{\varepsilon_n^{coh}})}}{\varepsilon_n^{coh}})+ \\
(1+\frac{g_N(X)}{\varepsilon_n^{coh}}).(\frac{-1}{\varepsilon_n^{coh}}).e^{-(\frac{g_N(X)}{\varepsilon_n^{coh}})}]\}\triangle g_N \\
\frac{\partial t_T^{coh}}{\partial g_T}.\triangle g_T = \{2\sqrt{\frac{1}{2}e}.t_T^{max}.(1+\frac{g_N(X)}{\varepsilon_n^{coh}}).e^{-(\frac{g_N(X)}{\varepsilon_n^{coh}})}[(\frac{e^{-(\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})}}{\varepsilon_t^{coh}})+ \\
(\frac{g_T(X)}{\varepsilon_t^{coh}}).(\frac{-2g_T(X)}{(\varepsilon_t^{coh})^2}).e^{-(\frac{(g_T(X))^2}{(\varepsilon_t^{coh})^2})}]\}\triangle g_T
\end{cases} \tag{49}
$$

in matrix form

$$
\triangle t_{N|T}^{coh} := \begin{bmatrix} \frac{\partial t_N^{coh}}{\partial g_N} & \frac{\partial t_N^{coh}}{\partial g_T} \\ \frac{\partial t_T^{coh}}{\partial g_N} & \frac{\partial t_T^{coh}}{\partial g_T} \end{bmatrix} . \begin{bmatrix} \triangle g_N \\ \triangle g_T \end{bmatrix}, \tag{50}
$$

where
$[setting\ \Delta\ for\ \delta]$

$$\begin{cases} \triangle g_N = -\nu.\{\Delta\varphi(X) - \Delta\varphi(\bar{Y}(X))\} \\ \triangle g_T^{\alpha} = \Delta m_{\alpha\beta}(\xi_{n+1}^{\beta} - \xi_n^{\beta}) = \Delta m_{\alpha\beta}(\xi_{n+1}^{\beta} - \xi_n^{\beta}) + m_{\alpha\beta}\Delta(\xi_{n+1}^{\beta} - \xi_n^{\beta}) = \\ [((\tau_\beta \frac{\partial^2(\varphi(\bar{Y}(X)))}{\partial\alpha\partial\gamma}) + (\tau_\alpha \frac{\partial^2(\varphi(\bar{Y}(X)))}{\partial\beta\partial\gamma}))\tau^{\gamma}(\Delta\varphi(X) - \Delta\varphi(\bar{Y}(X))) + (\tau_\beta \frac{\Delta\partial(\varphi(\bar{Y}(X)))}{\partial\alpha}) + \\ (\tau_\alpha \frac{\Delta\partial(\varphi(\bar{Y}(X)))}{\partial\beta})](\xi_{n+1}^{\beta} - \xi_n^{\beta}) + m_{\alpha\beta}\tau^{\beta}(\Delta\varphi(X) - \Delta\varphi(\bar{Y}(X)) \end{cases} \tag{51}$$

0.3.3 Discretization of the Contact Integral

The discretization of the equation (43) can be written as the following expression

$$G^c(\varphi^h, w^h) \approx \sum_{e=1}^{n_{sel}}\{\sum_{k=1}^{n_{int}} W_k j(\eta_k)[t_N^h(\eta_k)\delta g^h(\eta_k) + t_N^{coh^h}(\eta_k)\delta g^h(\eta_k) + t_{T_\alpha}^{coh^h}(\eta_k)\delta\xi^{\alpha^h}(\eta_k)]\} =$$

$$\sum_{e=1}^{n_{sel}}\{\sum_{k=1}^{n_{int}} W_k j(\eta_k)\delta\Phi_k^c.f_k^c\}, \tag{52}$$

where $\delta\Phi^c = \begin{bmatrix} C_s^{(1)} \\ C_1^{(2)} \\ . \\ . \\ C_{mnode}^{(2)} \end{bmatrix}$, $f^c = t_N\mathbf{N} + \underbrace{t_N^{coh}\mathbf{N}}_{t_{N_\alpha}^{coh}\mathbf{N}+t_{N_\beta}^{coh}\mathbf{N}} + t_{T_\alpha}^{coh}D_\alpha + t_{T_\beta}^{coh}D_\beta$ are the vector of nodal variations(slave and master nodes) and the local contact force vector corresponding to local quadrature points .

Where $D_1, D_2(D_\alpha, D_\beta)$ and $\mathbf{N}$ have been defined by equations (6.56) and (6.57).

0.3.4 Discretization of the Contact Stiffness

The discretization of the equation (47) can be written as the following expression

$$\triangle G^c(\varphi^h, w^h) \approx \sum_{e=1}^{n_{sel}}\{\sum_{k=1}^{n_{int}} W_k j(\eta_k)[(\triangle t_N^h(\eta_k)\delta g^h(\eta_k) + t_N^h(\eta_k)\triangle\delta g^h(\eta_k)) +$$

$$((\frac{\partial t_N^{coh^h}}{\partial g_N}(\eta_k).\triangle g_N^h(\eta_k) + \frac{\partial t_N^{coh^h}}{\partial g_T}(\eta_k).\triangle g_T^h(\eta_k))\delta g^h(\eta_k) + t_N^{coh^h}(\eta_k)\triangle\delta g^h(\eta_k))) +$$

$$((\frac{\partial t_T^{coh^h}}{\partial g_N}(\eta_k).\triangle g_N^h(\eta_k) + \frac{\partial t_T^{coh^h}}{\partial g_T}(\eta_k).\triangle g_T^h(\eta_k))\delta\xi^{\alpha^h}(\eta_k) + t_{T_\alpha}^{coh^h}(\eta_k)\triangle\delta\xi^{\alpha^h}(\eta_k))]\} =$$

$$\sum_{e=1}^{n_{sel}}\{\sum_{k=1}^{n_{int}} W_k j(\eta_k)\delta\Phi_k^c.k_k^c\triangle\Phi_k^c\}, \tag{53}$$

where $\triangle\Phi^c = \begin{bmatrix} \triangle d_s^{(1)} \\ \triangle d_1^{(2)} \\ . \\ . \\ \triangle d_{mnode}^{(2)} \end{bmatrix}$, $k^c = \underbrace{k_N^c}_{definedbyequation(6.55)} + k_N^{coh^c} + t_{T_\alpha}^{coh}A^{\alpha\beta}k_{T_\beta}^{coh^c} + k_T^{coh^c}$ are the vector of local nodal values and the global contact stiffness defined as the following expressions$(\alpha, \beta = 1, 2)$

$$k_N^{coh^c} = H(g_N)(t_{N,N}^{coh'})\mathbf{N}\mathbf{N}^{\mathbf{T}} + (t_{N,\tau_\alpha}^{coh'})(m_{\alpha\beta}\mathbf{N}D_\beta^T + \mathbf{N}G_\alpha^T) + (t_{N,\tau_\beta}^{coh'})(m_{\beta\alpha}\mathbf{N}D_\alpha^T + \mathbf{N}G_\beta^T) +$$

$$t_N^{coh}\{g_N[m^{11}\bar{\mathbf{N}}_{\mathbf{1}}\bar{\mathbf{N}}_{\mathbf{1}}^{\mathbf{T}} + m^{12}(\bar{\mathbf{N}}_{\mathbf{1}}\bar{\mathbf{N}}_{\mathbf{2}}^{\mathbf{T}} + \bar{\mathbf{N}}_{\mathbf{2}}\bar{\mathbf{N}}_{\mathbf{1}}^{\mathbf{T}}) + m^{22}\bar{\mathbf{N}}_{\mathbf{2}}\bar{\mathbf{N}}_{\mathbf{2}}^{\mathbf{T}}] -$$

$$D_1\mathbf{N}_{\mathbf{1}}^{\mathbf{T}} - D_2\mathbf{N}_{\mathbf{2}}^{\mathbf{T}} - \mathbf{N}_{\mathbf{1}}D_1^T - \mathbf{N}_{\mathbf{2}}D_2^T + \kappa_{12}(D_1D_2^T + D_2D_1^T)\}, \tag{54}$$

where $t_{N,N}^{coh'} = \frac{\partial t_N^{coh}}{\partial g_N}$, $t_{N,\tau_\alpha}^{coh'} = \frac{\partial t_N^{coh}}{\partial g_T^\alpha}$, $t_{N,\tau_\beta}^{coh'} = \frac{\partial t_N^{coh}}{\partial g_T^\beta}$ and

$$G_\alpha = \{-\mathbf{T}_{\alpha\beta} - \mathbf{T}_{\beta\alpha} + [\varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma}.\tau_\alpha + \varphi_t^{(2)}(\bar{Y}(X))_{,\alpha\gamma}.\tau_\beta]D_\gamma\}g_T^\beta \tag{55}$$

The quantity $k_{T_\alpha}^{coh^c}$ is given by

$$k_{T_\alpha}^{coh^c} = \mathbf{T}_{\alpha\beta}D_\beta^T + D_\beta\mathbf{T}_{\alpha\beta}^T - (\varphi_t^{(2)}(\bar{Y}(X))_{,\beta\gamma}.\tau_\alpha)D_\beta D_\gamma^T + \bar{T}_{\beta\alpha}D_\beta^T + D_\beta\bar{T}_{\beta\alpha}^T +$$

$$g_N(\mathbf{N}_{\alpha\beta}D_\beta^T + D_\beta\mathbf{N}_{\alpha\beta}^T) - \mathbf{N}\bar{\mathbf{N}}_\alpha^T - \mathbf{T}_\beta m^{\beta\gamma}\bar{T}_{\gamma\alpha}^T - \bar{\mathbf{N}}_\alpha\mathbf{N}^T - \bar{T}_{\gamma\alpha}m^{\beta\gamma}\mathbf{T}_\beta^T \tag{56}$$

and the quantity $k_T^{coh^c}$ is given by

$$k_T^{coh^c} = (t_{T,N}^{coh'})D_\alpha\mathbf{N}^{\mathbf{T}} + H(g_N)(t_{T,N}^{coh'})D_\beta\mathbf{N}^{\mathbf{T}} + (t_{T,\tau_\alpha}^{coh'})(m_{\alpha\beta}D_\alpha D_\beta^T + D_\alpha G_\alpha^T) + (t_{T,\tau_\beta}^{coh'})(m_{\beta\alpha}D_\beta D_\alpha^T + D_\beta G_\beta^T), \tag{57}$$

where $t_{T,N}^{coh'} = \frac{\partial t_T^{coh}}{\partial g_N}$, $t_{T,\tau_\alpha}^{coh'} = \frac{\partial t_T^{coh}}{\partial g_T^\alpha}$, $t_{T,\tau_\beta}^{coh'} = \frac{\partial t_T^{coh}}{\partial g_T^\beta}$ and $\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{N}, D_1, D_2, \bar{\mathbf{N}}_{\mathbf{1}}, \bar{\mathbf{N}}_{\mathbf{2}}$ vectors have been defined by equations (6.56) and (6.57) and $\mathbf{N}_{\alpha\beta}, \mathbf{T}_{\alpha\beta}, \mathbf{P}_\alpha$ are defined as follows

$$\mathbf{T}_{\alpha\beta} = \begin{bmatrix} 0 \\ -N_{1,\beta}(\bar{Y}(X))\tau_\alpha \\ . \\ . \\ -N_{4,\beta}(\bar{Y}(X))\tau_\alpha \end{bmatrix}, \mathbf{N}_{\alpha\beta} = \begin{bmatrix} 0 \\ -N_{1,\alpha\beta}(\bar{Y}(X))v \\ . \\ . \\ -N_{4,\alpha\beta}(\bar{Y}(X))v \end{bmatrix}, \mathbf{P}_\alpha = \begin{bmatrix} 0 \\ -N_{1,\alpha}(\bar{Y}(X))p_T \\ . \\ . \\ -N_{4,\alpha}(\bar{Y}(X))p_T \end{bmatrix} \tag{58}$$

and

$$\begin{cases} \bar{T}_{\alpha 1} = \mathbf{T}_{\alpha 1} - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.\tau_\alpha)D_2 \\ \bar{T}_{\alpha 2} = \mathbf{T}_{\alpha 2} - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.\tau_\alpha)D_1 \\ \bar{P}_1 = \mathbf{P}_1 - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.p_T)D_2 \\ \bar{P}_2 = \mathbf{P}_2 - (\varphi_t^{(2)}(\bar{Y}(X))_{,12}.p_T)D_1 \end{cases} \tag{59}$$

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