Aerothermodynamic Design Sensitivities for a Reacting Gas Flow Solver on an Unstructured Mesh Using a Discrete Adjoint Formulation

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Outline

Introduction and Motivation

- Exploring design space using high-fidelity CFD is challenging
- zero-order methods (sampling) are prohibitively expensive
- Need to be intelligent about techniques for evaluating sensitivity to design parameters
- Gradient-based optimization much more efficient than sampling, but requires calculating sensitivity derivatives

Introduction and Motivation

- How to compute sensitivity of design variables?
- Direct differentiation approach
  - Navier-Stokes equations can be directly differentiated to yield sensitivity derivatives necessary for gradient-based optimization
  - Requires a minimum of one flow solution for each design variable sensitivity
  - Prohibitively expensive for large number of design parameters
- Adjoint approach
  - Solve adjoint equations in addition to Navier Stokes flow equations to obtain sensitivity derivatives
  - One flow and adjoint solution needed for each cost function, regardless of number of design variables
  - Considerably more efficient than direct differentiation approach for large number of design parameters

Introduction and Motivation

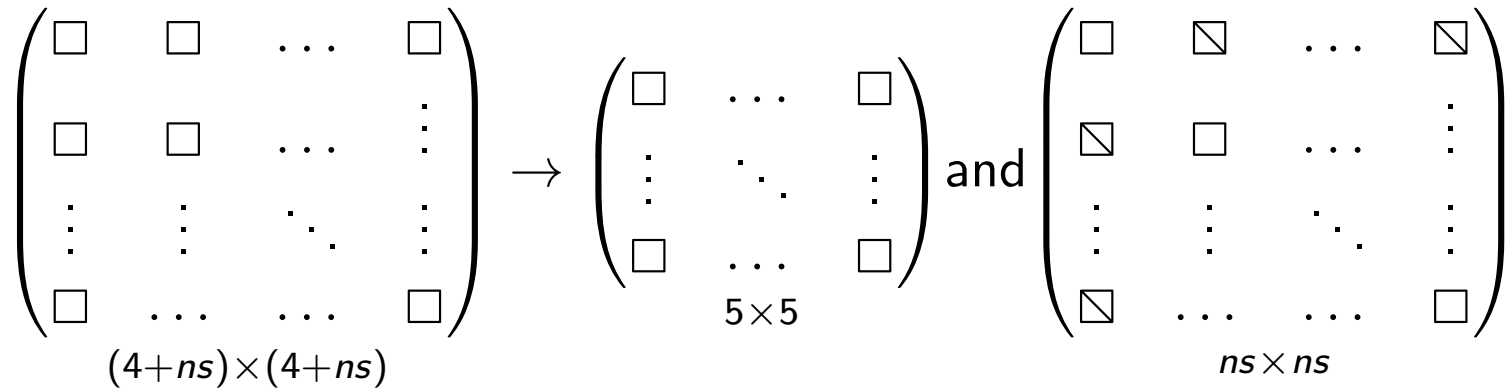
- Adjoint-based design optimization has recieved considerable attention in compressible, perfect gas CFD solvers, but very little in reacting flow solvers
- Difficulty of adjoint approach lies in implementating exact linearizations for 2nd-order flux construction scheme
- Particularly difficult for reacting flows, due to
  - complexity of linearizing the additional equations for multi-species chemical kinetics
  - Serious memory and computational cost concerns

Introduction and Motivation

- Reacting gas simulations require solving a large number of conservation equations
- Memory concerns
  - Size of Jacobians scales quadratically with number species in gas mixture
  - Solving system of equations in a tightly-coupled fashion can be limited by memory constraints
- Cost concerns
  - Cost of solving the linear system scales quadratically with number of species in gas mixture
- Efficiently solving adjoint problem is a primary motivator
  - Solving adjoint system particularly costly if linear solver is slow
  - Can be necessary to store jacobian twice $\rightarrow$ large memory overhead

Introduction and Motivation

- Loosely-coupled solvers have become popular in the combustion community.
  - Decouple species conservation equations from meanflow equations, and solve two smaller systems



- Beneficial for adjoint formulation
  - Two smaller systems are considerably easier to linearize
  - Storing jacobian for adjoint solve becomes practical
- Candler, et al. originally derived this for Steger-Warming scheme, this work extends to Roe FDS scheme
  - Candler, G. V., Subbareddy, P. K., and Nompelis, I. "Decoupled Implicit Method for Aerothermodynamics and Reacting Flows." *AIAA Journal*, Vol. 51, no. 5, pp.

Fully-Coupled Point Implicit Method

- All work presented is for inviscid flows in chemical non-equilibrium, using a one-temperature model, but is extendable to viscous flows.
- Beginning with the semi-discrete form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{1}{V}\sum_{f}(\mathbf{F}\cdot\mathbf{S})^{f} = \mathbf{W}$$

$$\mathbf{U} = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_{ns} \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \quad \mathbf{F}\cdot\mathbf{S} = \begin{pmatrix} \rho_1 \overline{U} \\ \vdots \\ \rho_{ns} \overline{U} \\ \rho u \overline{U} + p s_x \\ \rho u \overline{U} + p s_y \\ \rho u \overline{U} + p s_z \\ (\rho E + p)\overline{U} \end{pmatrix} S, \quad \mathbf{W} = \begin{pmatrix} \dot{\rho}_1 \\ \vdots \\ \dot{\rho}_{ns} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Fully-Coupled Point Implicit Method

- Using the Roe FDS scheme to compute the inviscid flux at the face, $\mathbf{F}^f$, and linearizing the system results in

$$\frac{\delta \mathbf{U}^n}{\Delta t} + \frac{1}{V}\sum_f \left(\frac{\partial \mathbf{F}^f}{\partial \mathbf{U^L}}\delta \mathbf{U}^L + \frac{\partial \mathbf{F}^f}{\partial \mathbf{U^R}}\delta \mathbf{U}^R\right)^n \mathbf{S}^f - \frac{\partial \mathbf{W}}{\partial \mathbf{U}}\delta \mathbf{U}^n = -\frac{1}{V}\sum_f (\mathbf{F}^f \cdot \mathbf{S}^f)^n + \mathbf{W}^n$$

- Which can be thought of more simply as

$$\mathbf{Au} = \mathbf{b}$$

$$\mathbf{A} \rightarrow \begin{array}{c}(4+ns)\times(4+ns)\\ \text{Jacobian Block}\end{array}$$

$$\mathbf{b} \rightarrow \begin{array}{c}(4+ns)\times 1\\ \text{Residual}\end{array}$$

Fully-Coupled Point Implicit Method

- Constructing the Jacobian in a fully-coupled fashion results in large, dense block matricies
- Using a stationary iterative method (i.e., Gauss-Seidel, SSOR, etc.), work is dominated by matrix-vector products

$$\text{Cost} \rightarrow O((4 + ns)^2)$$

- Leads to onerous quadratic scaling with respect to number of species

Decoupled Point Implicit Method

- The main idea is to separate the meanflow and species composition equations, adding a new equation for the total mixture density
- Leads to two sets of conserved variables

$$\mathbf{U}' = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \qquad \hat{\mathbf{U}} = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_{ns} \end{pmatrix}$$

Meanflow Species Composition

Decoupled Point Implicit Method

- The fluxes are solved in two sequential steps
  - The mixture fluxes are first solved as

$$\frac{\partial \mathbf{U}'}{\partial t} + \frac{1}{V}\sum_f (\mathbf{F}' \cdot \mathbf{S})^f = 0$$

  - Followed by the species fluxes

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + \frac{1}{V}\sum_f (\hat{\mathbf{F}} \cdot \mathbf{S})^f = \hat{\mathbf{W}}$$

- Since the mixture density was determined in the first step, step two actually solves for the species mass fractions

$$\delta\hat{\mathbf{U}}^n = \rho^{n+1}\hat{\mathbf{V}}^{n+1} - \rho^n\hat{\mathbf{V}}^n = \rho^{n+1}\delta\hat{\mathbf{V}}^n + \hat{\mathbf{V}}^n\delta\rho^n$$

$$\hat{\mathbf{V}} = (c_1, \ldots, c_{ns})^T, c_s = \rho_s/\rho$$

Decoupled Point Implicit Method

- The Roe FDS scheme species mass fluxes can be rewritten as

$$\hat{\mathbf{F}}_{\rho_s} = c_s \mathbf{F}'_{\rho} + (c_s^L - \tilde{c}_s)\rho^L \lambda^+ + (c_s^R - \tilde{c}_s)\rho^R \lambda^-$$

$$\frac{\partial \hat{\mathbf{F}}_{\rho_s}}{\partial c_s^L} = w\mathbf{F}_{\rho} + (1-w)\rho^L \lambda^+ - w\rho^R \lambda^-$$

$$\frac{\partial \hat{\mathbf{F}}_{\rho_s}}{\partial c_s^R} = (1-w)\mathbf{F}_{\rho} + (w-1)\rho^L \lambda^+ + w\rho^R \lambda^-$$

- Jacobian Approximations

$$\text{Step 1:} \quad \left.\frac{\partial \mathbf{F}}{\partial \mathbf{U}'}\right|_{\hat{\mathbf{V}}} = 5 \times 5 \underset{c_s=\text{Constant}}{\text{Roe FDS Jacobian}}$$

$$\text{Step 2:} \quad \left.\frac{\partial \mathbf{F}}{\partial \hat{\mathbf{V}}}\right|_{\hat{\mathbf{U}}'} = \begin{pmatrix} \frac{\partial F_{\rho_1}}{\partial c_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial F_{\rho_{ns}}}{\partial c_{ns}} \end{pmatrix}$$

Decoupled Point Implicit Method

- Chemical source term linearized via

$$\hat{\mathbf{W}}^{n+1} = \hat{\mathbf{W}}^{n} + \left.\frac{\partial \hat{\mathbf{W}}}{\partial \mathbf{U}}\right|_{\mathbf{U}'} \frac{\partial \mathbf{U}}{\partial \hat{\mathbf{V}}}$$

$$\mathbf{C} = \left.\frac{\partial \hat{\mathbf{W}}}{\partial \mathbf{U}}\right|_{\mathbf{U}'} \frac{\partial \mathbf{U}}{\partial \hat{\mathbf{V}}}$$

- Full system to be solved in step two

$$\rho^{n+1}\frac{\delta \hat{\mathbf{V}}^{n}}{\Delta t} + \frac{1}{V}\sum_{f}\left(\frac{\partial \hat{\mathbf{F}}^{f}}{\partial \mathbf{V}^{L}}\delta \mathbf{V}^{L} + \frac{\partial \hat{\mathbf{F}}^{f}}{\partial \hat{\mathbf{V}}^{R}}\delta \hat{\mathbf{V}}^{R}\right)^{n,n+1}\mathbf{S}^{f} - \mathbf{C}^{n,n+1}\delta \mathbf{V}^{n}$$

$$= -\frac{1}{V}\sum_{f}(\hat{\mathbf{F}}^{n,n+1}\cdot \mathbf{S})^{f} + \mathbf{W}^{n,n+1} - \hat{\mathbf{V}}^{n}\frac{\delta \rho^{n}}{\Delta t} - R_{\rho}$$

$$R_{\rho} = -\frac{1}{V}\sum_{f}\sum_{s}(\hat{F}_{\rho_s}^{n,n+1}\cdot \mathbf{S})$$

- R_{ρ} is included to preserve $\sum_{s} c_s = 1$, $\sum_{s} \delta c_s = 0$.

- The derivation of the adjoint approach to compute design sensitivities begins with forming the Lagrangian as

$$L(\mathbf{D}, \mathbf{Q}, \mathbf{X}, \mathbf{\Lambda}) = f(\mathbf{D}, \mathbf{Q}, \mathbf{X}) + \mathbf{\Lambda}^T \mathbf{R}(\mathbf{D}, \mathbf{Q}, \mathbf{X})$$

$$f = \text{cost function}$$

Cost and Memory Savings of the Decoupled Implicit Problem

- Most significant savings comes from the source term linearization being purely node-based
  - Convective contributions to block Jacobians are diagonal
  - Source term jacobian is dense block Jacobian
  - In the global system (w/chemistry), all off-diagonal block jacobians are diagonal

$$\begin{pmatrix} \square & & & \\ & \ddots & & \\ & & \square & \\ & & & \ddots \\ & & & & \square \end{pmatrix} \begin{pmatrix} \delta\hat{\mathbf{V}}_1 \\ \vdots \\ \delta\hat{\mathbf{V}}_i \\ \vdots \\ \delta\hat{\mathbf{V}}_{nodes} \end{pmatrix} = \begin{pmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_i \\ \vdots \\ \hat{b}_{nodes} \end{pmatrix} - \begin{pmatrix} (\sum_{j=1}^{N_{nb}} [\diagdown] \delta\hat{\mathbf{V}}_j)_1 \\ \vdots \\ (\sum_{j=1}^{N_{nb}} [\diagdown] \delta\hat{\mathbf{V}}_j)_i \\ \vdots \\ (\sum_{j=1}^{N_{nb}} [\diagdown] \delta\hat{\mathbf{V}}_j)_{nodes} \end{pmatrix}$$

- Matrix-vector products $\rightarrow$ inner products: $O(ns^2) \rightarrow O(ns)$

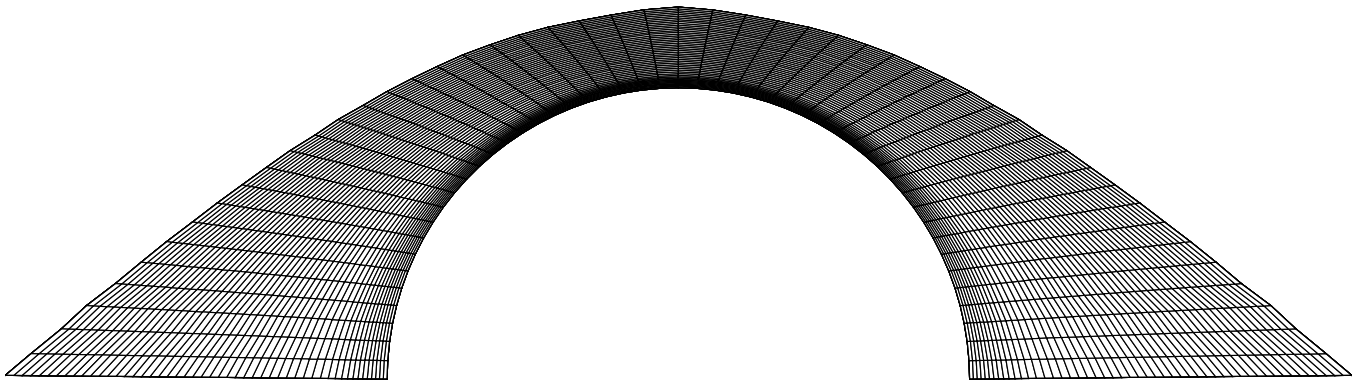
Cost and Memory Savings of the Decoupled Implicit Problem

- Comparing size of Jacobian systems, using Compressed Row Storage

$$\mathbf{A}_d = \text{Decoupled system Jacobians}$$
$$\mathbf{A} = \text{Fully-coupled system Jacobians}$$

$$\begin{aligned} \textit{Relative Memory Cost} &= \frac{size(\mathbf{A}_d)}{size(\mathbf{A})} \\ &= \lim_{ns \to \infty} \frac{(ns^2 + 5^2)(N_{nodes}) + (ns + 5^2)(N_{nbrs})}{(ns + 4)^2(N_{nodes} + N_{nbrs})} \\ &= \frac{N_{nodes}}{N_{nodes} + N_{nbrs}} \end{aligned}$$

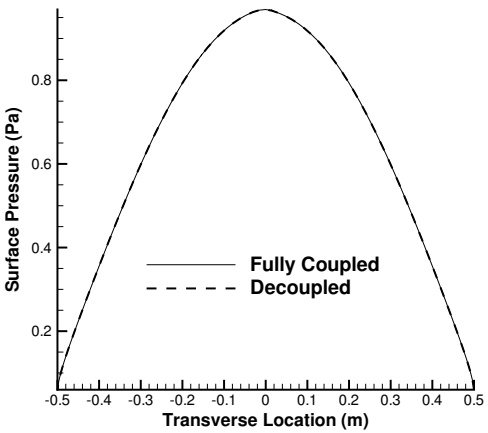
Numerical Results: 2D Cylinder

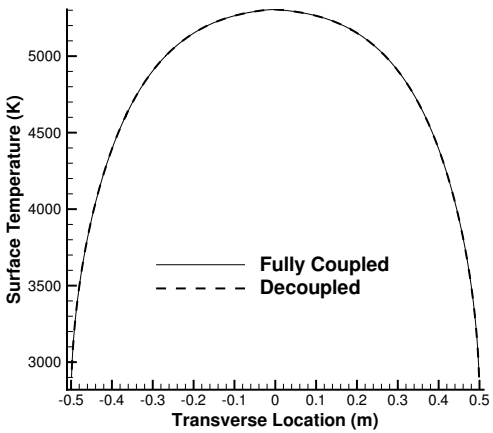


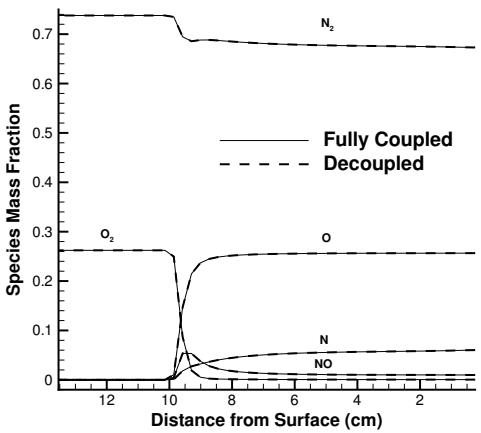
- Fully-coupled and decoupled methods both implemented in the Generic Gas Path of FUN3D
- Tested on 2D cylinder case
  - $V_\infty = 5000\ m/s$, $\rho_\infty = 0.001\ kg/m^3$, and $T_\infty = 200\ K$

Numerical Results: 2D Cylinder

- Verification of implementation
  - 5-species air model: N, N_2, O, O_2, and NO with five reactions

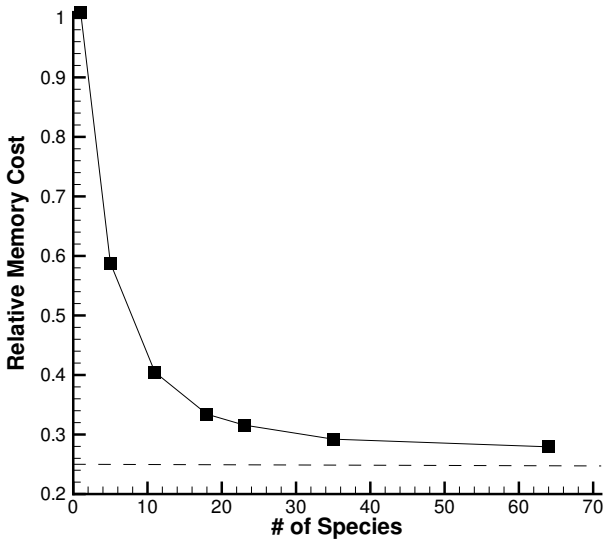


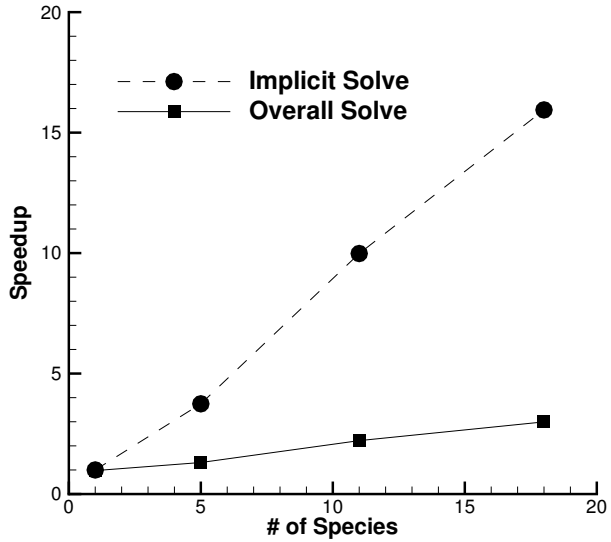




  - Surface pressure and surface temperature agree discretely to 8 significant figures
  - Mass fractions on stagnation line agree to 4 significant figures

Numerical Results: 2D Cylinder



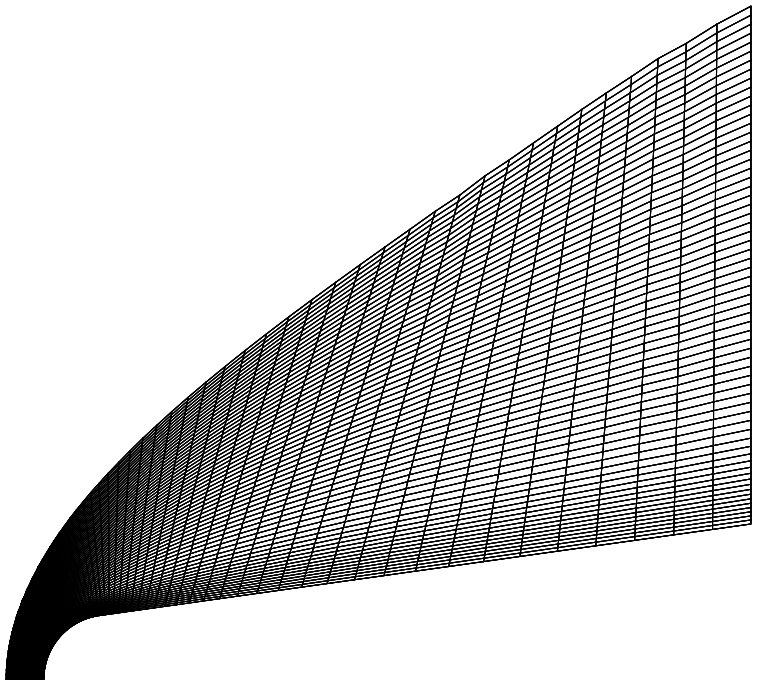


- On structured grids $N_{nbrs} \approx 6N_{nodes}$
  - Half precision off-diagonal $N_{nbrs} = \frac{6N_{nodes}}{2}$

$$Memory\ \ Cost \approx \frac{N_{nodes}}{N_{nodes} + N_{nbrs}} = \frac{N_{nodes}}{N_{nodes} + 6N_{nodes}/2} = \frac{1}{4}$$

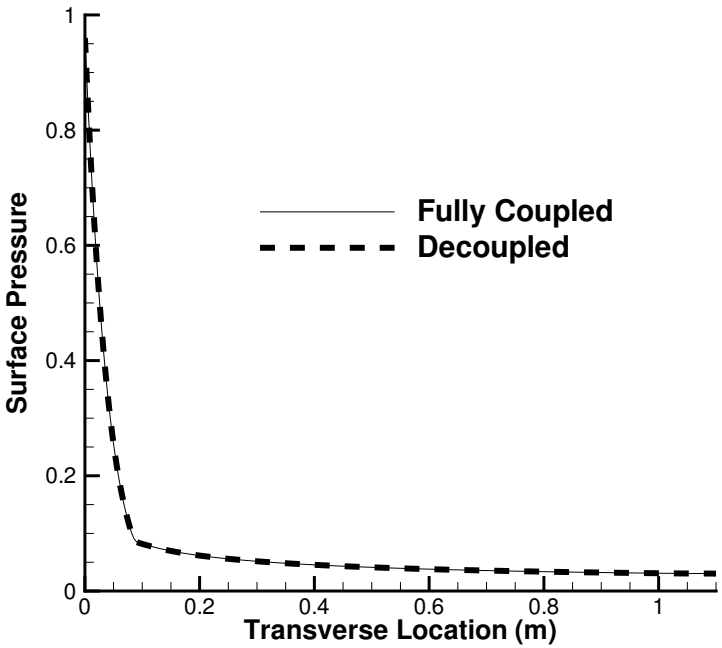
  - Linear speedup in solver: $\frac{O(N^2)}{O(N)} = O(N)$

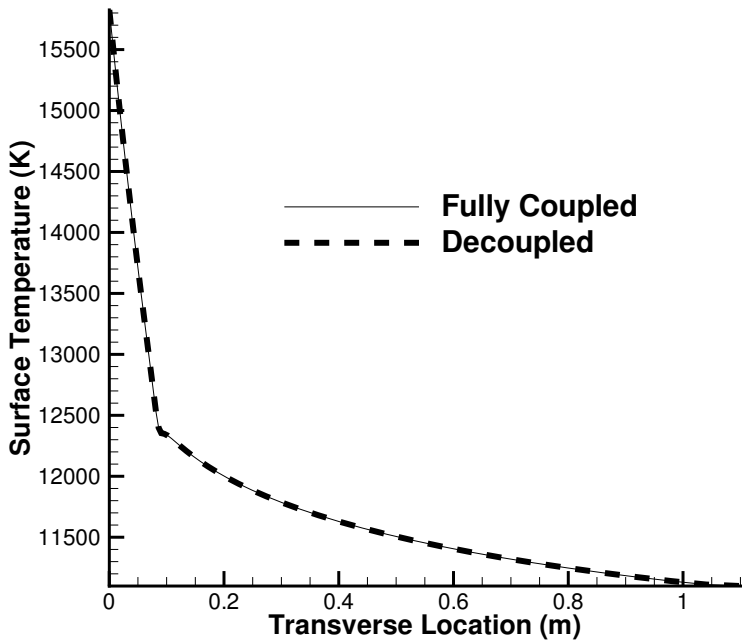
Numerical Results: Axisymmetric Spherically Capped Cone



- Verify that the decoupled scheme is robust at high velocities
  - $V_\infty = 15000\ m/s$, $\rho_\infty = 0.001\ kg/m^3$, $T_\infty = 200\ K$.
  - 11-species air model N, N_2, O, O_2, NO, N^+, N_2^+, O^+, O_2^+, NO^+, and electrons, with 22 possible reactions.

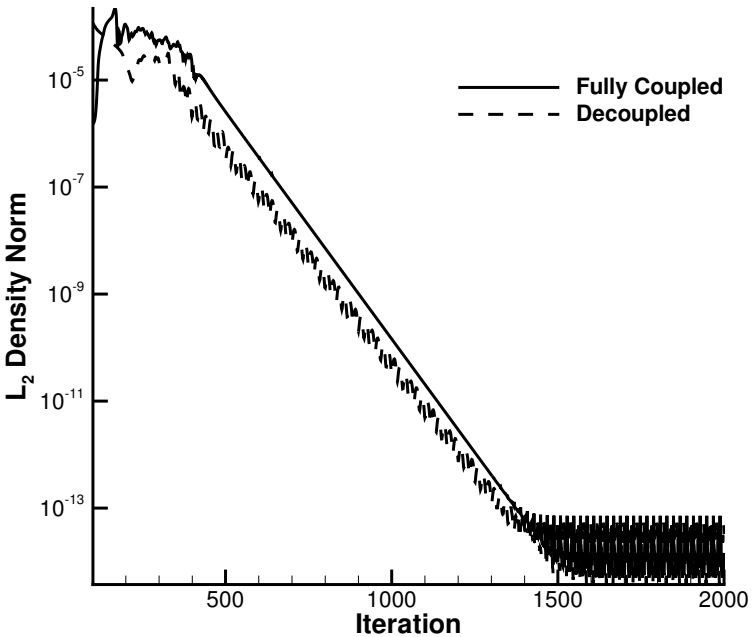
Numerical Results: Axisymmetric Spherically Capped Cone

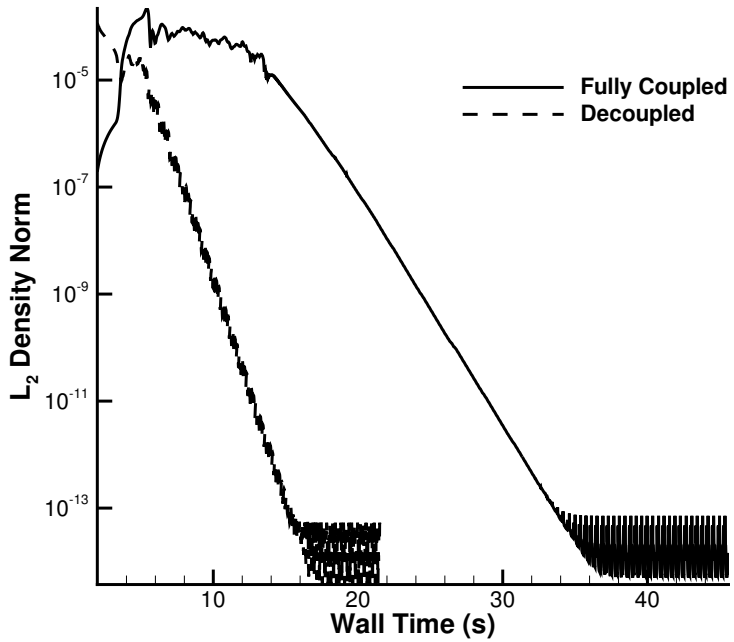




- Surface pressure and surface temperature agree discretely to 8 significant figures

Numerical Results: Axisymmetric Spherically Capped Cone





- Necessary to scale source term magnitude by $0.001 \leq w \leq 1$ for the first 500 iterations, due to extreme reaction rates
- Both schemes converge in a similar number of iterations
- Decoupled scheme $\approx$ 2x faster

Conclusions

- Decoupling the species equations yield impressive benefits at minimal cost in robustness
  - 2 times faster and 1/3 required memory for both 2D Cylinder and Sphere-Cone 11-species cases
  - Convergence issues at very high velocities can be offset by scaling source term as solution progresses
- Improvements valuable for adjoint work
  - Preliminary testing has shown that memory overhead in adjoint is significantly reduced with decoupled scheme
  - Can expect similar speedup in adjoint solve
  - Differences between fully-coupled and decoupled method results may impact adjoint and require further study

Acknowledgements

- The authors would like to recognize the FUN3D team at NASA Langley Research Center, for their support in integrating aspects of the compressible gas path into the reacting gas path of FUN3D.
- Thanks to the Entry Systems Modeling Project within the NASA Game Changing Development Program for their funding and support of this research.

Backup: Derivation

- For the Roe flux difference splitting scheme, the species mass fluxes are given by

$$F_{\rho_s} = \frac{\rho_s^L \mathbf{U}^L + \rho_s^R \mathbf{U}^R}{2} - \frac{\tilde{c}_s(\lambda_1 dv_1 + \lambda_2 dv_2) + \lambda_3 dv_{3_s}}{2}$$

$$dv_1 = \frac{p^R - p^L + \tilde{\rho}\tilde{a}(\mathbf{U}^R - \mathbf{U}^L)}{\tilde{a}^2}$$

$$dv_2 = \frac{p^R - p^L - \tilde{\rho}\tilde{a}(\mathbf{U}^R - \mathbf{U}^L)}{\tilde{a}^2}$$

$$dv_{3_s} = \frac{\tilde{a}^2(\rho_s^R - \rho_s^L) - \tilde{c}_s(p^R - p^L)}{\tilde{a}^2}$$

$$\lambda_1 = \mid \tilde{\mathbf{U}} + \tilde{a} \mid, \quad \lambda_2 = \mid \tilde{\mathbf{U}} - \tilde{a} \mid, \quad \lambda_3 = \mid \tilde{\mathbf{U}} \mid$$

Backup: Derivation

- The $\tilde{}$ notation signifies a Roe-averaged quantity

$$\tilde{\mathbf{U}} = wU^L + (1-w)\mathbf{U}^R$$

$$w = \frac{\tilde{\rho}}{\tilde{\rho} + \rho^R}$$

$$\tilde{\rho} = \sqrt{\rho^R \rho^L}$$

- The species mass fluxes must sum to the total mass flux

$$F_\rho = \sum_s F_{\rho_s} = \frac{\rho^L \mathbf{U}^L + \rho^R \mathbf{U}^R}{2} - \frac{\tilde{c}_s(\lambda_1 dv_1 + \lambda_2 dv_2) + \lambda_3 dv_3}{2}$$

$$dv_3 = \frac{\tilde{a}^2(\rho^R - \rho^L) - (p^R - p^L)}{\tilde{a}^2}$$

Backup: Derivation

- Substituting back into species mass flux equation

$$F_{\rho_s} = \tilde{c}_s F_\rho + \frac{(c_s^L - \tilde{c}_s)\rho^L(\mathbf{U}^L + \mid \tilde{\mathbf{U}} \mid)}{2} + \frac{(c_s^R - \tilde{c}_s)\rho^R(\mathbf{U}^R - \mid \tilde{\mathbf{U}} \mid)}{2}$$

- This can be simplified to yield a form similar to that derived by Candler, et. al for the Steger-Warming scheme

$$F_{\rho_s} = \tilde{c}_s F_\rho + (c_s^L - \tilde{c}_s)\rho^L \lambda^+ + (c_s^R - \tilde{c}_s)\rho^R \lambda^-$$

$$\lambda^+ = \frac{\mathbf{U}^L + \mid \tilde{\mathbf{U}} \mid}{2}, \quad \lambda^- = \frac{\mathbf{U}^R - \mid \tilde{\mathbf{U}} \mid}{2}$$

Backup: Derivation

- Differentiating with respect to the mass fraction, c_s, the left and right state contributions are

$$\frac{\partial F_{\rho_s}}{\partial c_s^L} = wF_\rho + (1-w)\rho^L\lambda^+ - w\rho^R\lambda^-$$
$$\frac{\partial F_{\rho_s}}{\partial c_s^R} = (1-w)F_\rho + (w-1)\rho^L\lambda^+ + w\rho^R\lambda^-$$

- Again, where w is the Roe-averaged density weighting

$$w = \frac{\tilde{\rho}}{\tilde{\rho}+\rho^R}, \quad \tilde{\rho} = \sqrt{\rho^R\rho^L}$$