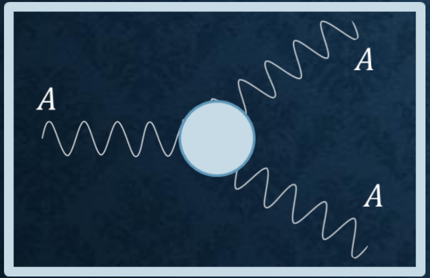
22ND INTERNATIONAL CONFERENCE ON RF SUPERCONDUCTIVITY

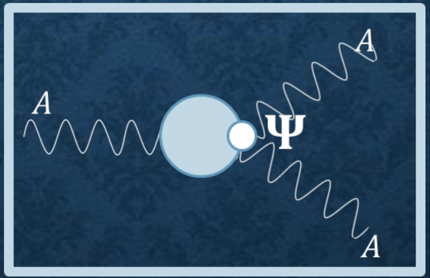
SRF2025 TOKYO

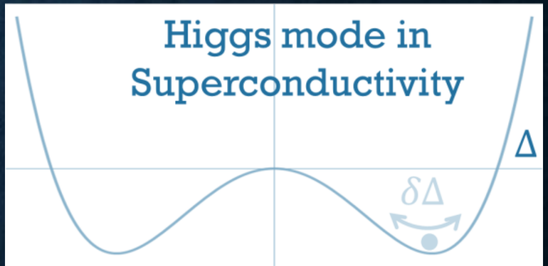
September 21-26, 2025

Nonequilibrium Corrections and Higgs Mode in Superconducting Devices:

Unraveling the Pronounced **Anti-Q Slope** in High-Frequency regime and Current-Dependent Kinetic Inductance







High energy Accelerator Research Organization (KEK)

Takayuki Kubo

This talk is based on the following 3 papers

- T. Kubo, Phys. Rev. Applied **22**, 044042 (2024) DOI: https://doi.org/10.1103/PhysRevApplied.22.044042
- T. Kubo, Phys. Rev. Applied **23**, 054091 (2025) DOI: https://doi.org/10.1103/PhysRevApplied.23.054091
- T. Kubo, arXiv:2509.09766 DOI: https://doi.org/10.48550/arXiv.2509.09766

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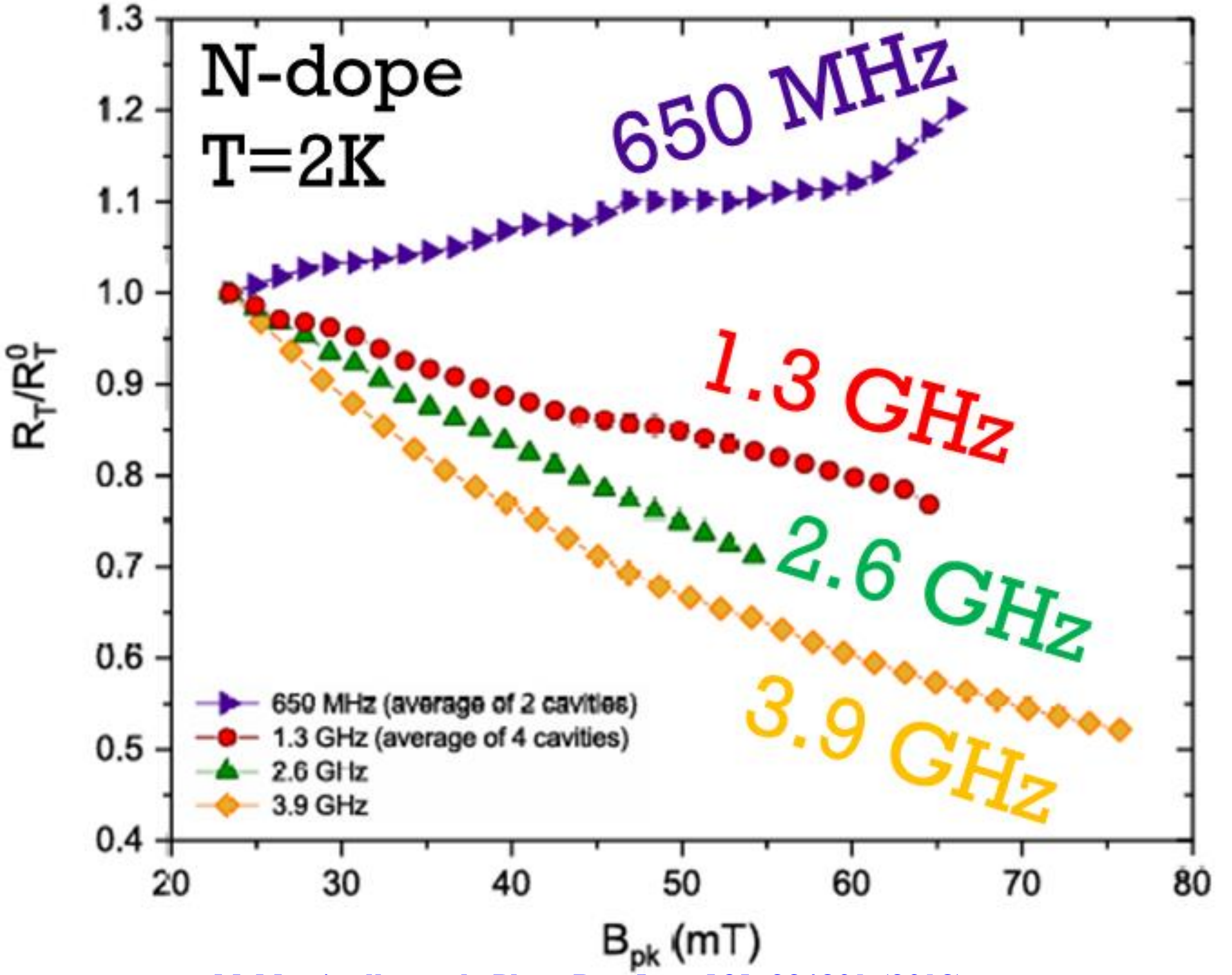
- One of the main theoretical questions in SRF physics is how the $R_s(B)$ curve depends on frequency.



M. Martinello et al., Phys. Rev. Lett. **121**, 224801 (2018)
DOI: https://doi.org/10.1103/PhysRevLett.121.224801

FIG. 4. Normalized data R_T/R_T^0 as a function of the peak magnetic field for N-doped cavities at 2.0 K.

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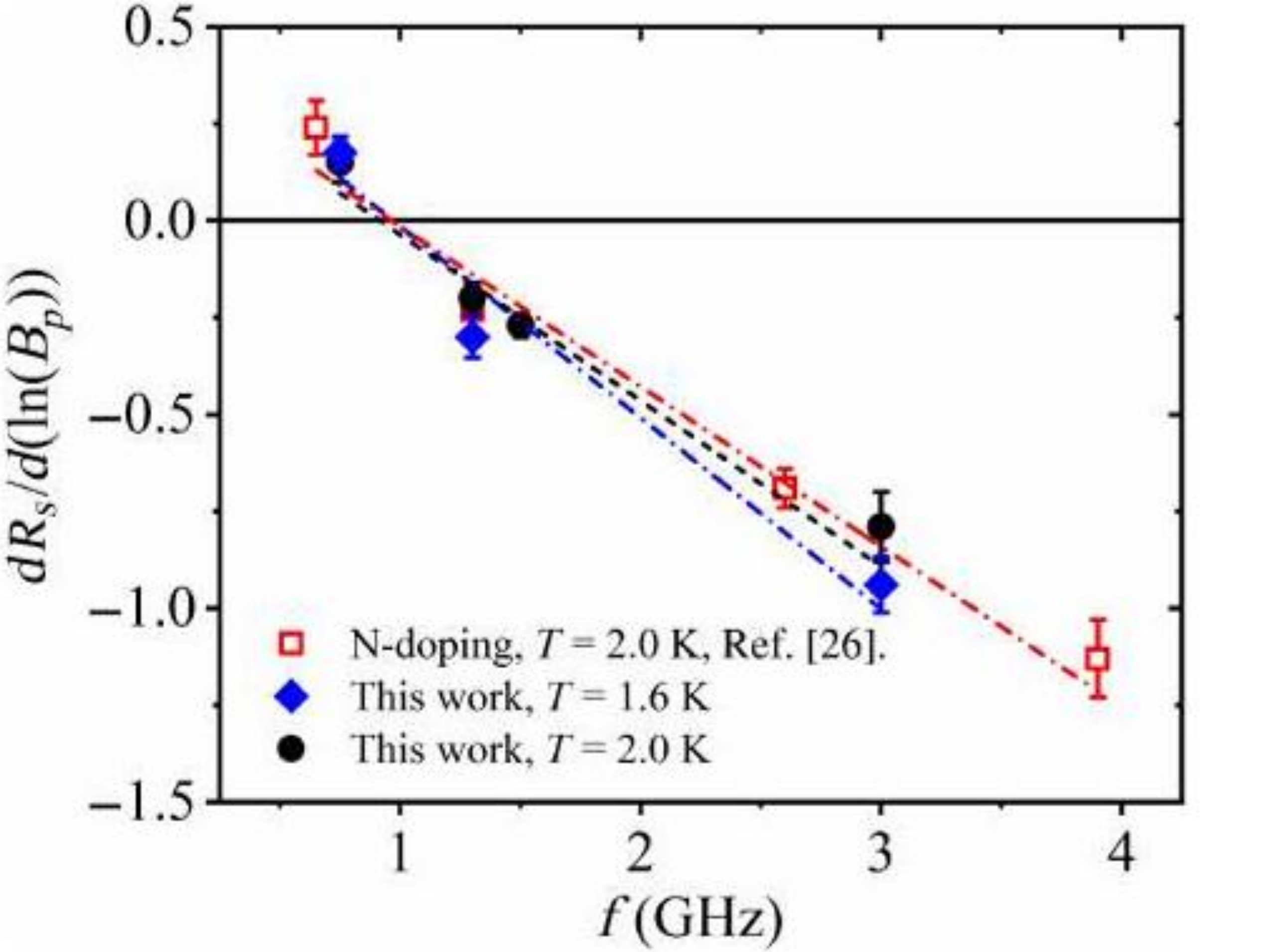


P. Dhakal, B. D. Khanal, A. Gurevich, and G. Ciovati, Phys. Rev. Accel. Beams **27**, 062001 (2024)

DOI: https://doi.org/10.1103/PhysRevAccelBeams.27.062001

FIG. 11. Slope of normalized BCS resistance with respect to $\ln B_p$ from Fig. 10 for this study at 1.6 and 2 K and from Ref. [26] at 2.0 K.

- One of the main theoretical questions in SRF physics is <u>how the $R_s(B)$ curve depends on frequency</u>.
- There is a threshold around 1 GHz: below it, the curve bends upward; above it, the curve bends downward — the so-called anti-Q slope.

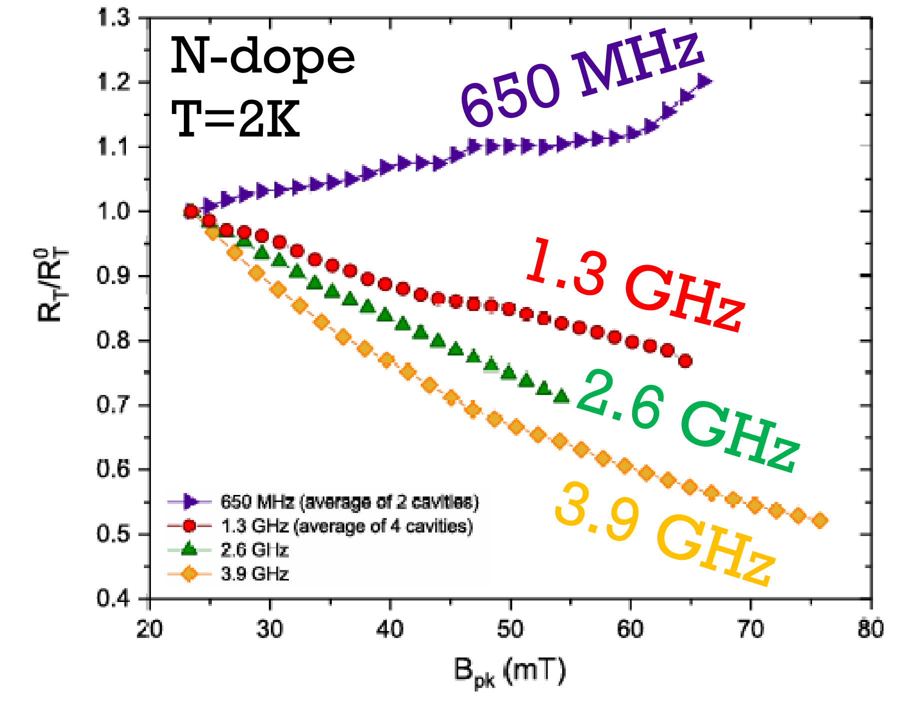
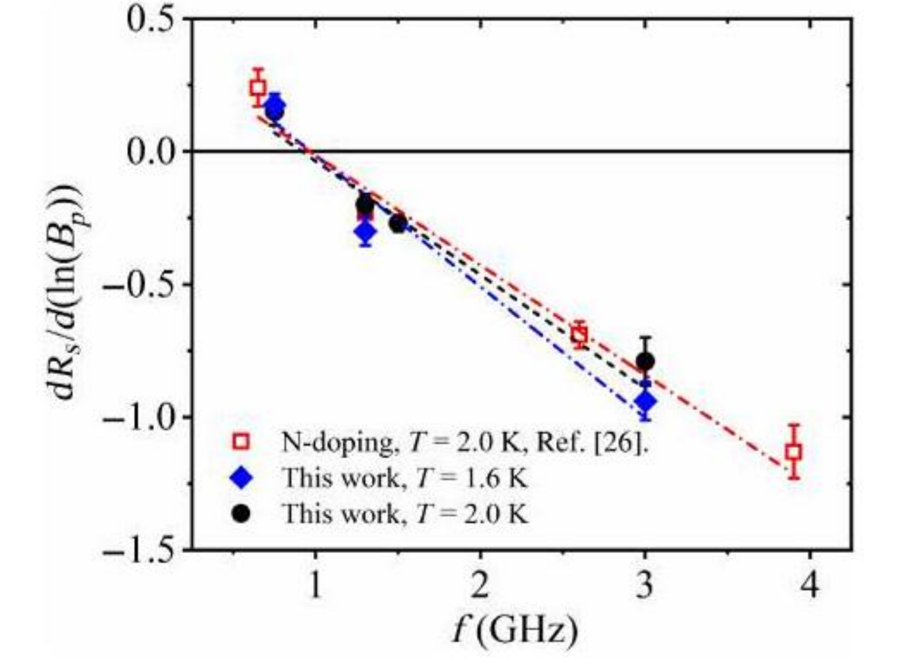


FIG. 4. Normalized data R_T/R_T^0 as a function of the peak magnetic field for N-doped cavities at 2.0 K.



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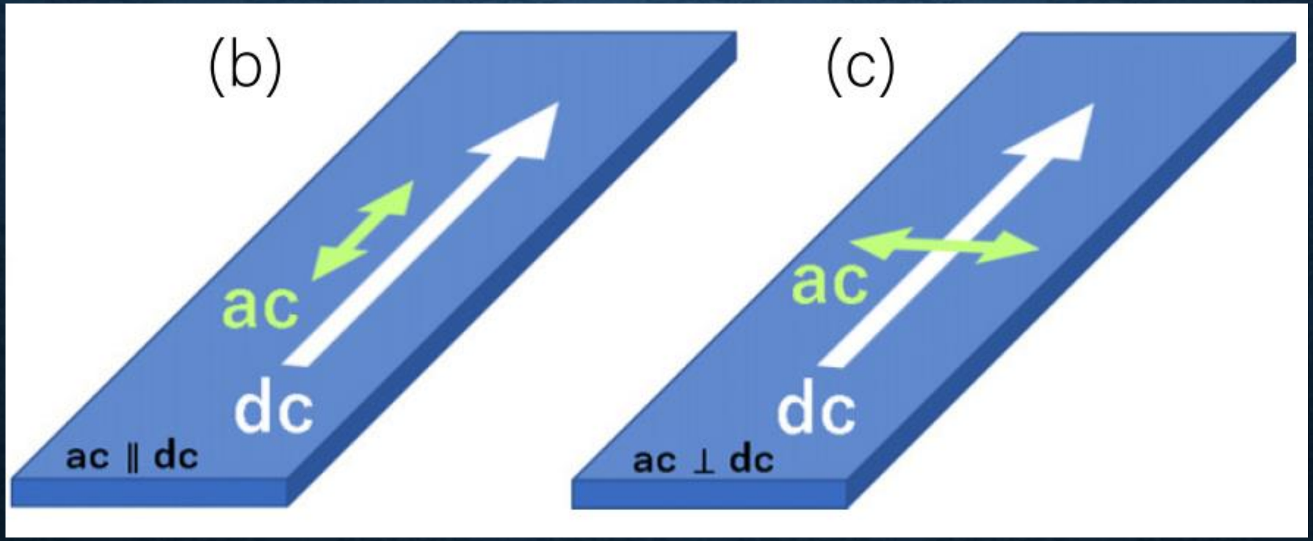
To address this problem, we consider

a superconductor carrying a DC current and perturbed by a weak AC field.

This approach provides a key clue to understanding the issue.

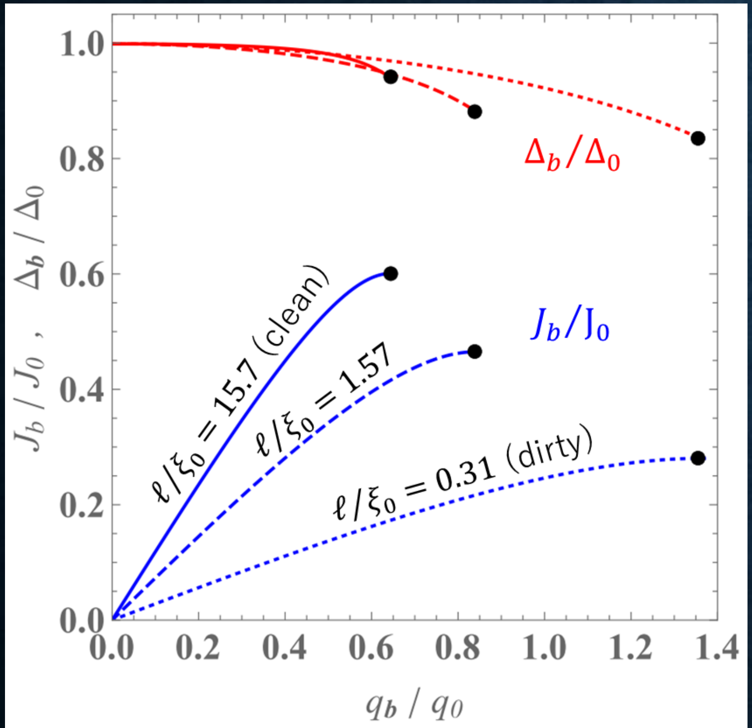


The detailed discussions are found here:

T. Kubo, Phys. Rev. Applied **22**, 044042 (2024) DOI: https://doi.org/10.1103/PhysRevApplied.22.044042

T. Kubo, Phys. Rev. Applied **23**, 054091 (2025) DOI: https://doi.org/10.1103/PhysRevApplied.23.054091

If the AC field is absent, this reduces to a problem of equilibrium superconductivity.
That case was already well understood decades ago.



T. Kubo, Phys. Rev. Applied 22, 044042 (2024) DOI: https://doi.org/10.1103/PhysRevApplied.22.044042

See also, T. Kubo, Phys. Rev. Research **2**, 033203 (2020); F. Pei-Jen Lin and A. Gurevich, Phys. Rev. B **85**, 054513 (2012)

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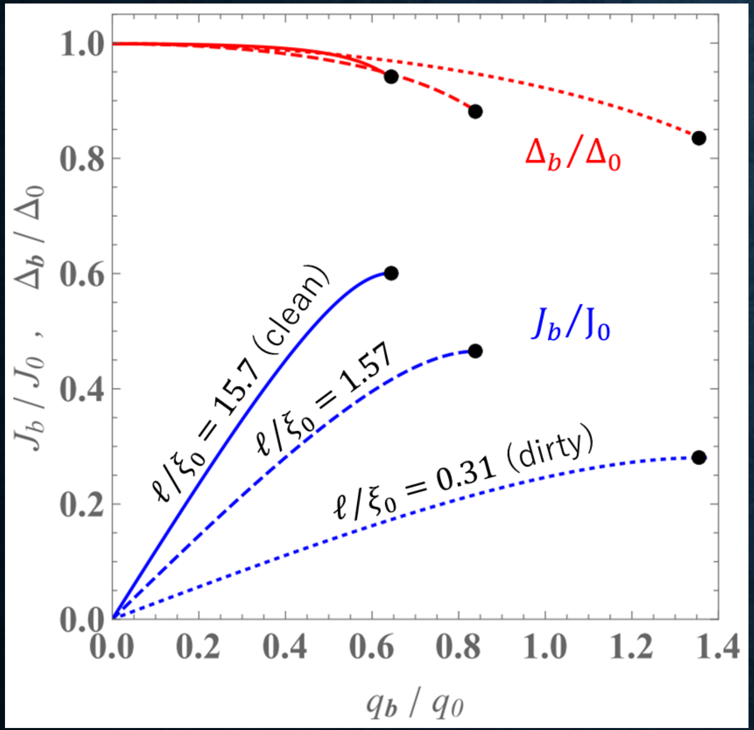

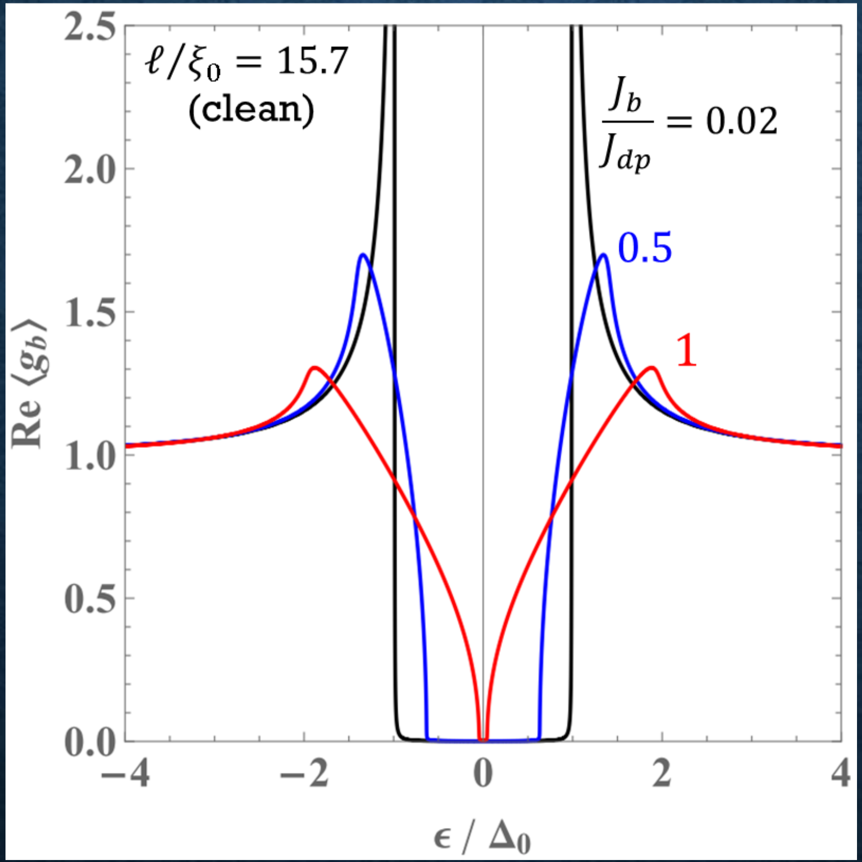

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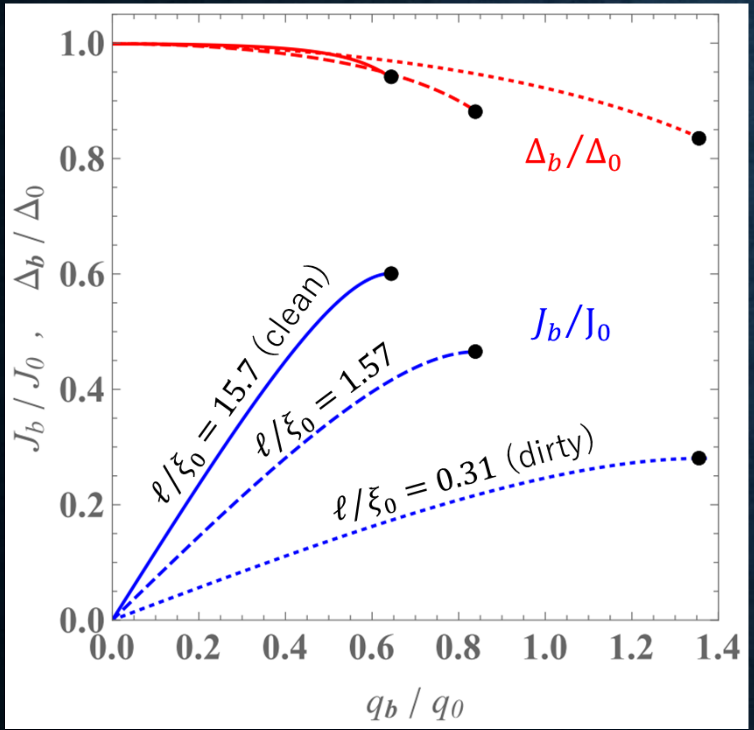

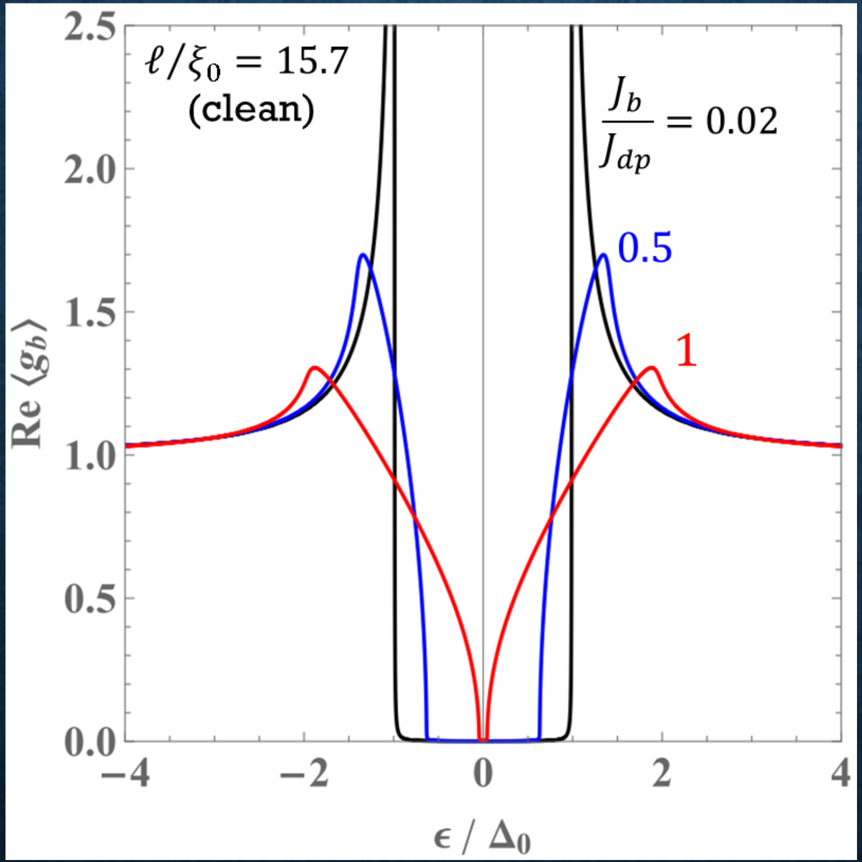

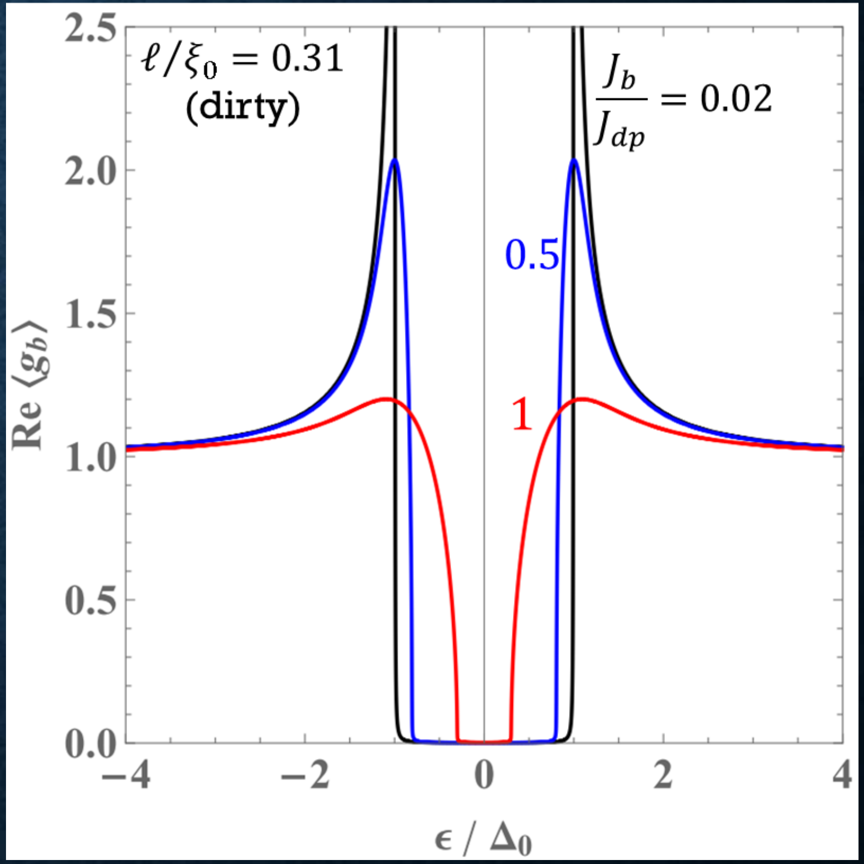

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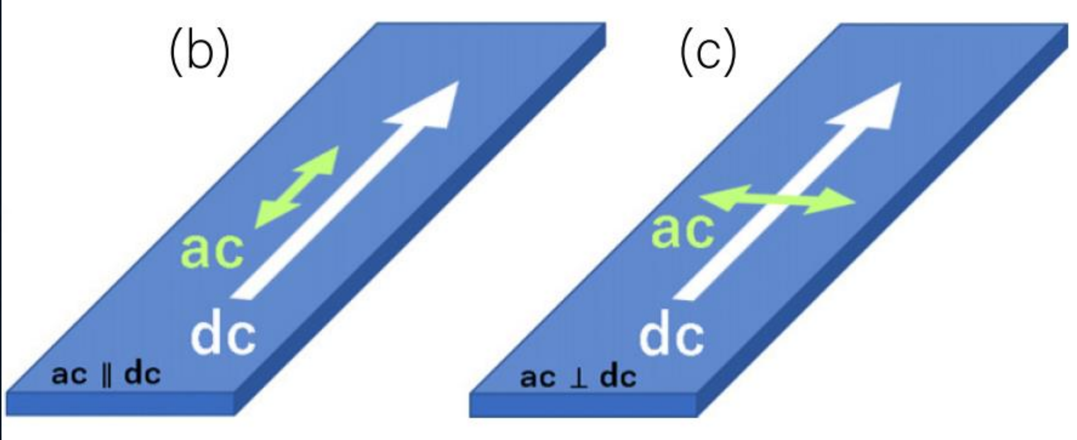

However, once an AC field is present, the problem becomes much more complicated. In that case, we need to use the theory of **nonequilibrium** superconductivity.

A. Moor et al., Phys. Rev. Lett. **118**, 047001 (2017)
T. Jujo, J. Phys. Soc. Jpn. **91**, 074711 (2022)
T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)
T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

- **The Keldysh-Eilenberger theory** is a microscopic theory of **nonequilibrium** superconductivity. It is applicable at any temperature ($0 \le T \le T_c$) and for arbitrary mean free path. In this sense, it serves as the "*theory of everything for conventional superconductivity*".

T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

- **The Keldysh–Usadel theory** represents the dirty-limit reduction of the Keldysh–Eilenberger theory of nonequilibrium superconductivity, applicable at any T ($0 \le T \le T_c$).

T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

Keldysh–Usadel Equation: DC Current with AC Perturbation

T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

$$
\begin{aligned}
&- i(s/2)\left[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega) - \hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\right.\\
&\quad + \hat{\tau}_3\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-) - \delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
&\quad + \hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega) - \hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)\hat{\tau}_3\\
&\quad \left.+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-) - \delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\right]\\
&\quad - i(\delta W/2)\left[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-) - \hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\right.\\
&\quad + \hat{\tau}_3\hat{g}_b^R(\epsilon_-)\hat{\tau}_3\hat{g}_b^K(\epsilon_-) - \hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\\
&\quad + \hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-) - \hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\\
&\quad \left.+\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\hat{g}_b^A(\epsilon_-) - \hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_+)\hat{\tau}_3\right]\\
&= \epsilon_+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega) - \delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
&\quad + [\hat{\Delta}_b, \delta\hat{g}^K(\epsilon,\omega)] + \delta\hat{\Delta}(\omega)\hat{g}_b^K(\epsilon_-) - \hat{g}_b^K(\epsilon_+)\delta\hat{\Delta}(\omega).
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&- i(s/2)\left\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega) - \hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\right.\\
&\quad \left.+ \hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-) - \delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\right\}\\
&\quad - i(\delta W/2)\left\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\right.\\
&\quad - \hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3 + \hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\\
&\quad \left.- \hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\right\} = \epsilon_+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega) - \delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
&\quad + [\hat{\Delta}_b, \delta\hat{g}^r(\epsilon,\omega)] + \delta\hat{\Delta}(\omega)\hat{g}_b^r(\epsilon_-) - \hat{g}_b^r(\epsilon_+)\delta\hat{\Delta}(\omega),
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\delta\Delta(\omega) = -\frac{\mathscr{G}}{8}\int d\epsilon\, \mathrm{Tr}[(-i\tau_2)\delta\hat{g}^K(\epsilon,\omega)]. \tag{17}
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Higgs
↓

We solve these equations to obtain the AC-induced **nonequilibrium corrections** ($\delta\,\hat{g}^{R,A,K}$ and $\delta\Delta$).

Keldysh–Usadel Equation: DC Current with AC Perturbation

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&\quad+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\big]\\
&\quad-i(\delta W/2)\big[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
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&\quad+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\\
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&\quad-i(\delta W/2)\big\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\\
&\quad-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3+\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\\
&\quad-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\big\}=\epsilon_+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)-\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
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\delta\Delta(\omega)=-\frac{\mathscr{G}}{8}\int d\epsilon\,\mathrm{Tr}[(-i\tau_2)\delta\hat{g}^K(\epsilon,\omega)]. \tag{17}
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Higgs
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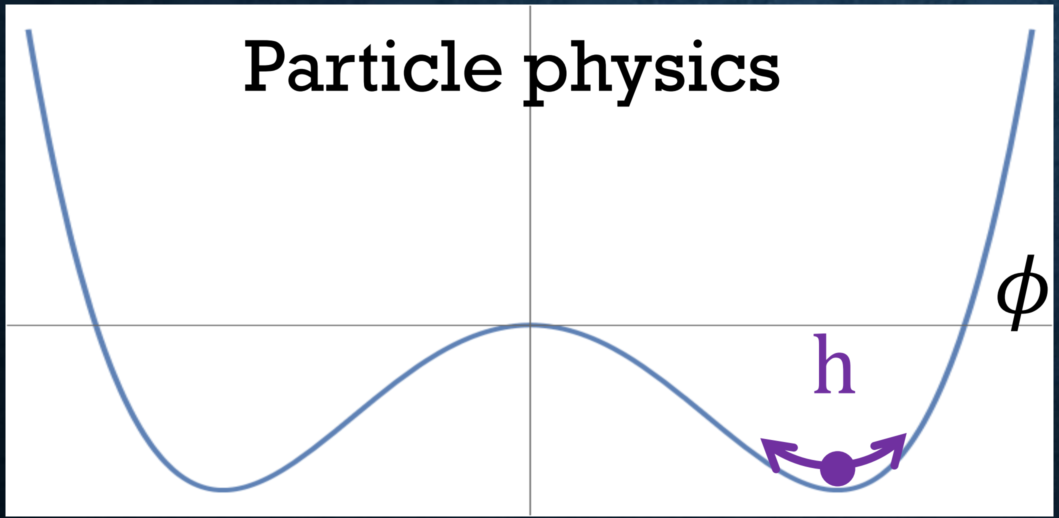
We solve these equations to obtain the AC-induced **nonequilibrium corrections** ($\delta\,\hat{g}^{R,A,K}$ and $\delta\Delta$).

Higgs?



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&\quad+\hat{\tau}_3\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
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&\quad-i(\delta W/2)\big[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
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Higgs

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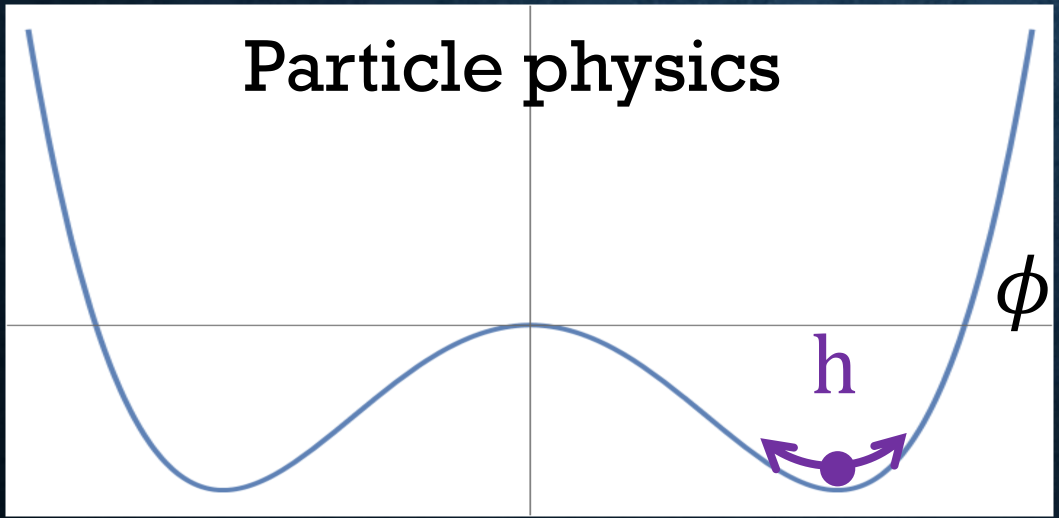
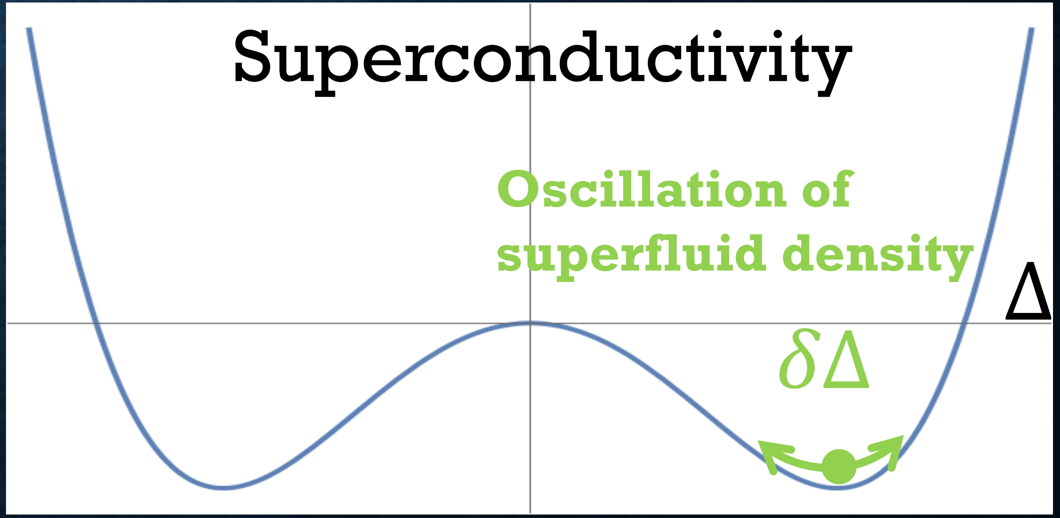
Higgs?





Keldysh–Usadel Equation: DC Current with AC Perturbation

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&\quad + \hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega) - \hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)\hat{\tau}_3 \\
&\quad + \hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-) - \delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\big] \\
&\quad - i(\delta W/2)\big[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-) - \hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3 \\
&\quad + \hat{\tau}_3\hat{g}_b^R(\epsilon_-)\hat{\tau}_3\hat{g}_b^K(\epsilon_-) - \hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3 \\
&\quad + \hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-) - \hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3 \\
&\quad + \hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\hat{g}_b^A(\epsilon_-) - \hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_+)\hat{\tau}_3\big] \\
&= \epsilon_+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega) - \delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\epsilon_- \\
&\quad + [\hat{\Delta}_b, \delta\hat{g}^K(\epsilon,\omega)] + \delta\hat{\Delta}(\omega)\hat{g}_b^K(\epsilon_-) - \hat{g}_b^K(\epsilon_+)\delta\hat{\Delta}(\omega). \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&- i(s/2)\big\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega) - \hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3 \\
&\quad + \hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-) - \delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\big\} \\
&\quad - i(\delta W/2)\big\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-) \\
&\quad - \hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3 + \hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\hat{g}_b^r(\epsilon_-) \\
&\quad - \hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\big\} = \epsilon_+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega) - \delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\epsilon_- \\
&\quad + [\hat{\Delta}_b, \delta\hat{g}^r(\epsilon,\omega)] + \delta\hat{\Delta}(\omega)\hat{g}_b^r(\epsilon_-) - \hat{g}_b^r(\epsilon_+)\delta\hat{\Delta}(\omega), \qquad (13)
\end{aligned}
$$

$$\delta\Delta(\omega) = -\frac{\mathscr{G}}{8}\int d\epsilon\, \mathrm{Tr}[(-i\tau_2)\delta\hat{g}^K(\epsilon,\omega)]. \qquad (17)$$

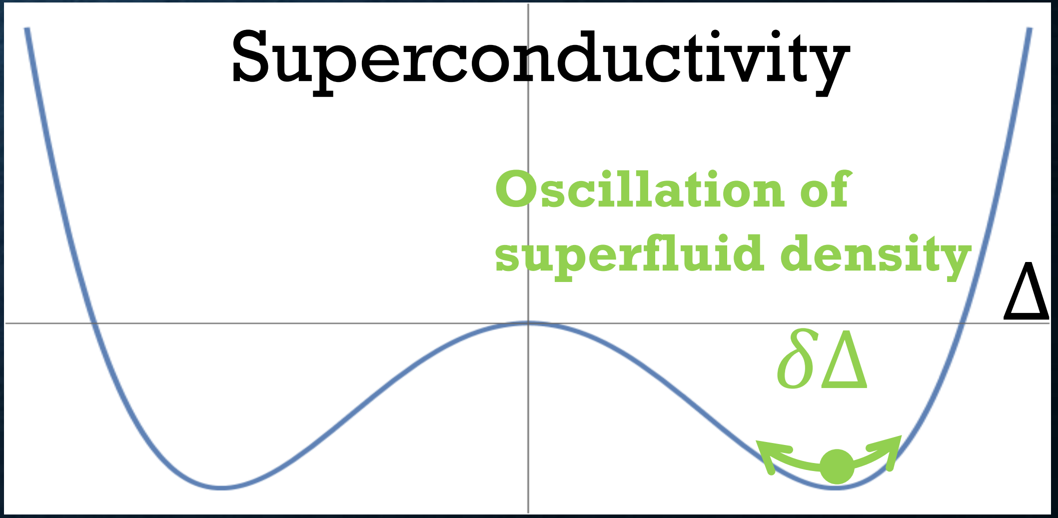
Higgs ↓

We solve these equations to obtain the AC-induced **nonequilibrium corrections** ($\delta\hat{g}^{R,A,K}$ and $\delta\Delta$).

Higgs?

See the famous review paper:
R. Shimano and N. Tsuji,
Annu. Rev. Condens. Matter Phys. **11**, 103 (2020)

The Higgs mode in superconductivity is an $\mathcal{O}(A^2)$ effect. It does not appear in standard linear-response theories such as Mattis–Bardeen, but it does emerge in nonlinear response. In our case, we apply both DC and AC fields. As a result, the Higgs mode shows up as $\delta\Delta \propto A_{dc}\cdot A$, and responds linearly to the AC field.



Keldysh–Usadel Equation: DC Current with AC Perturbation

T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

$$
\begin{aligned}
&-i(s/2)\left[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\right.\\
&\quad+\hat{\tau}_3\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)\hat{\tau}_3\\
&\quad\left.+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\right]\\
&\quad-i(\delta W/2)\left[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\right.\\
&\quad+\hat{\tau}_3\hat{g}_b^R(\epsilon_-)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\\
&\quad\left.+\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_+)\hat{\tau}_3\right]\\
&=\epsilon_+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)-\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
&\quad+[\hat{\Delta}_b,\delta\hat{g}^K(\epsilon,\omega)]+\delta\hat{\Delta}(\omega)\hat{g}_b^K(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\delta\hat{\Delta}(\omega).
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&-i(s/2)\left\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\right.\\
&\quad\left.+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)-\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\right\}\\
&\quad-i(\delta W/2)\left\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\right.\\
&\quad-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3+\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\\
&\quad\left.-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\right\}=\epsilon_+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)-\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
&\quad+[\hat{\Delta}_b,\delta\hat{g}^r(\epsilon,\omega)]+\delta\hat{\Delta}(\omega)\hat{g}_b^r(\epsilon_-)-\hat{g}_b^r(\epsilon_+)\delta\hat{\Delta}(\omega),
\end{aligned}
\tag{13}
$$

$$
\delta\Delta(\omega)=-\frac{\mathscr{G}}{8}\int d\epsilon\,\mathrm{Tr}[(-i\tau_2)\delta\hat{g}^K(\epsilon,\omega)]. \tag{17}
$$

Higgs
↓

We solve these equations to obtain the AC-induced **nonequilibrium corrections** ($\delta\hat{g}^{R,A,K}$ and $\delta\Delta$).

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T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

$$
\begin{aligned}
&-i(s/2)\big[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\big]\\
&\quad-i(\delta W/2)\big[\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\hat{g}_b^R(\epsilon_-)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)-\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_+)\hat{\tau}_3\big]\\
&=\epsilon_+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)-\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
&\quad+[\hat{\Delta}_b,\delta\hat{g}^K(\epsilon,\omega)]+\delta\hat{\Delta}(\omega)\hat{g}_b^K(\epsilon_-)-\hat{g}_b^K(\epsilon_+)\delta\hat{\Delta}(\omega).
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&-i(s/2)\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\\
&\quad+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)-\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\}\\
&\quad-i(\delta W/2)\{\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\\
&\quad-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3+\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\hat{\tau}_3\hat{g}_b^r(\epsilon_-)\\
&\quad-\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\hat{g}_b^r(\epsilon_+)\hat{\tau}_3\}=\epsilon_+\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)-\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3\epsilon_-\\
&\quad+[\hat{\Delta}_b,\delta\hat{g}^r(\epsilon,\omega)]+\delta\hat{\Delta}(\omega)\hat{g}_b^r(\epsilon_-)-\hat{g}_b^r(\epsilon_+)\delta\hat{\Delta}(\omega),
\end{aligned}
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$$

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Higgs

↓

We solve these equations to obtain the AC-induced **nonequilibrium corrections** ($\delta\,\hat{g}^{R,A,K}$ and $\delta\Delta$).

To obtain the ac response,
we substitute the solutions ($\delta\,\hat{g}^{R,A,K}$, $\delta\Delta$) into

$$\delta\mathbf{J}(\omega)=-i\frac{\sigma_n}{e}\int d\epsilon\,\delta\mathbf{S}(\epsilon,\omega), \tag{18}$$

$$
\begin{aligned}
&\delta\mathbf{S}(\epsilon,\omega)=(i/16)\mathrm{Tr}\big[i\mathbf{q}_b\\
&\times\{\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)+\hat{\tau}_3\delta\hat{g}^R(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)\\
&+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\delta\hat{g}^A(\epsilon,\omega)+\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\}\\
&+i\delta\mathbf{q}_\omega\{\hat{\tau}_3\hat{g}_b^R(\epsilon_+)\hat{\tau}_3\hat{g}_b^K(\epsilon_-)+\hat{\tau}_3\hat{g}_b^K(\epsilon_+)\hat{\tau}_3\hat{g}_b^A(\epsilon_-)\}\big].
\end{aligned}
\tag{19}
$$

Instead of showing the rigorous algebraic calculations, we will look at schematic illustrations that represent the final results.

$$J \sim Agg + A_{dc} g \delta g$$

$$\delta \mathbf{J}(\omega) = -i \frac{\sigma_n}{e} \int d\epsilon \, \delta \mathbf{S}(\epsilon, \omega), \tag{18}$$

$$\begin{aligned}
\delta \mathbf{S}(\epsilon, \omega) = {} & (i/16) \mathrm{Tr} \big[i \mathbf{q}_b \\
& \times \big\{ \hat{\tau}_3 \hat{g}_b^R(\epsilon_+) \hat{\tau}_3 \delta \hat{g}^K(\epsilon, \omega) + \hat{\tau}_3 \delta \hat{g}^R(\epsilon, \omega) \hat{\tau}_3 \hat{g}_b^K(\epsilon_-) \\
& + \hat{\tau}_3 \hat{g}_b^K(\epsilon_+) \hat{\tau}_3 \delta \hat{g}^A(\epsilon, \omega) + \hat{\tau}_3 \delta \hat{g}^K(\epsilon, \omega) \hat{\tau}_3 \hat{g}_b^A(\epsilon_-) \big\} \\
& + i \delta \mathbf{q}_\omega \big\{ \hat{\tau}_3 \hat{g}_b^R(\epsilon_+) \hat{\tau}_3 \hat{g}_b^K(\epsilon_-) + \hat{\tau}_3 \hat{g}_b^K(\epsilon_+) \hat{\tau}_3 \hat{g}_b^A(\epsilon_-) \big\} \big].
\end{aligned} \tag{19}$$

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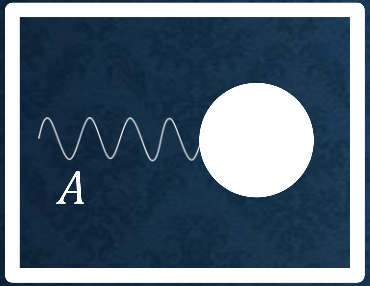
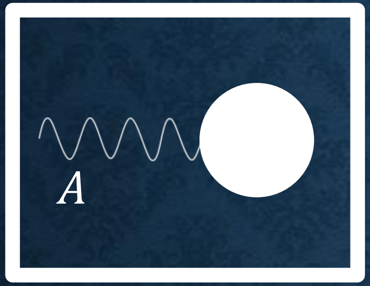
$$J \sim \boxed{Agg} + A_{dc} g \delta g$$



Instead of showing the rigorous algebraic calculations, we will look at schematic illustrations that represent the final results.

$$J \sim \boxed{Agg} + A_{dc} g \delta g \quad \delta g \sim \boldsymbol{A}_{dc} \cdot \boldsymbol{A} + \delta\Delta \text{ and } \delta\Delta \sim \Psi\, \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$$



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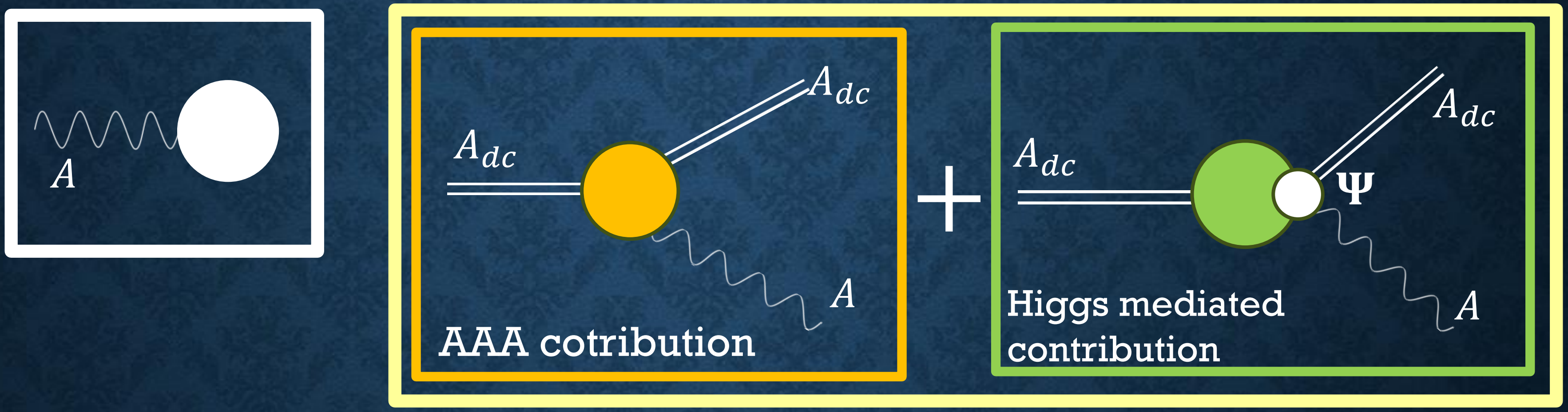

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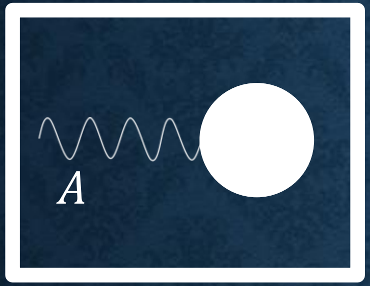
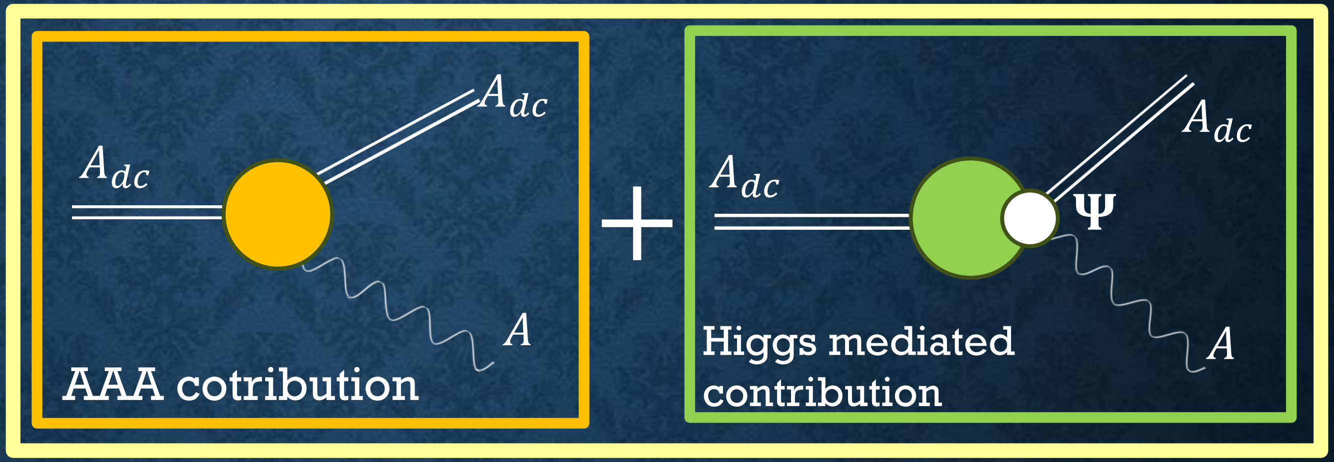


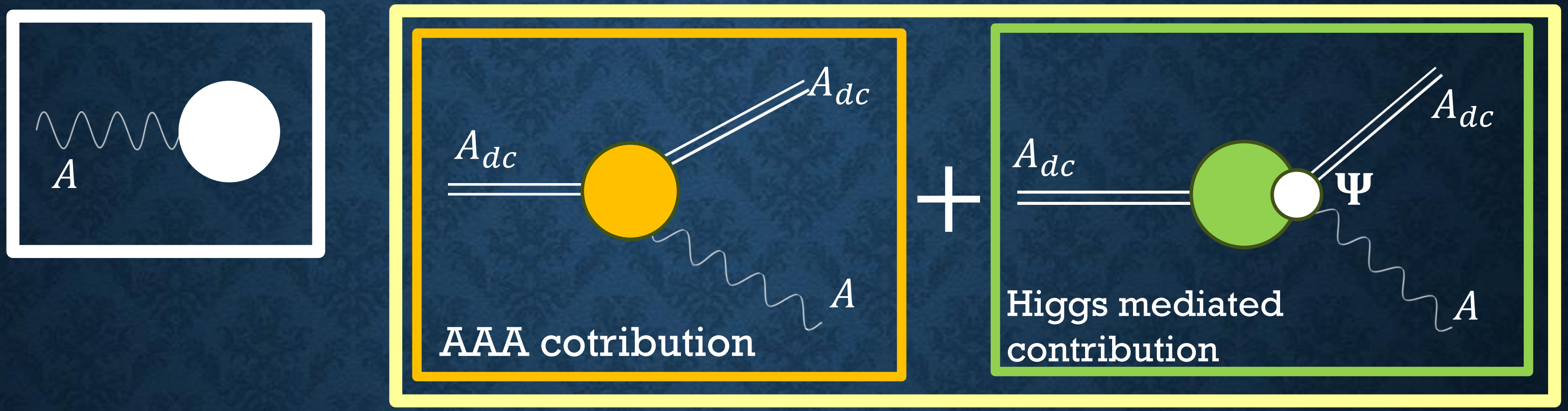


$\propto \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$ $\qquad$ $\propto \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$

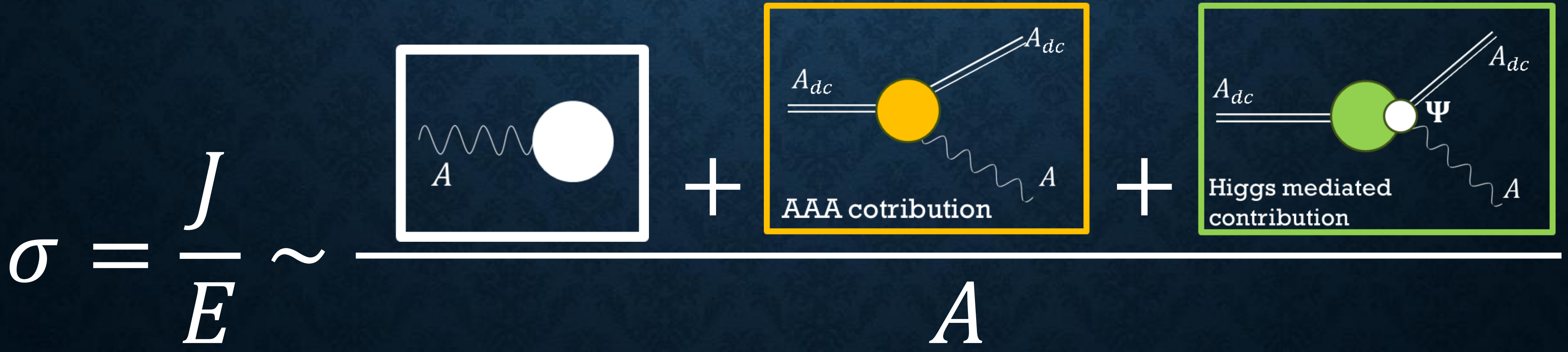
Nonequilibrium corrections due to the Doppler fluctuation of flow $\propto \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$

Instead of showing the rigorous algebraic calculations, we will look at schematic illustrations that represent the final results.

$$J \sim \boxed{Agg} + \boxed{A_{dc} g \delta g} \quad \delta g \sim A_{dc} \cdot A + \delta\Delta \quad \text{and} \quad \delta\Delta \sim \Psi A_{dc} \cdot A$$



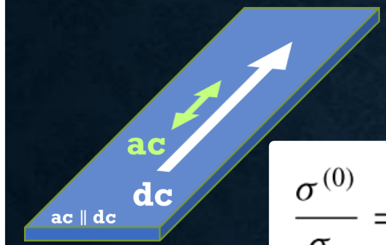
Then, the complex conductivity is given by



Complex conductivity formula

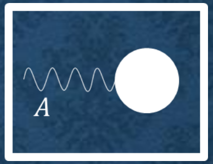
T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

$ac \parallel dc$ case



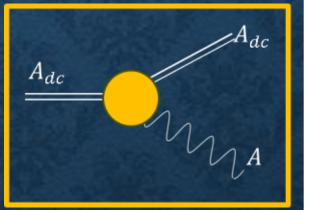
$$\sigma = \sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)}$$

$$\frac{\sigma^{(0)}}{\sigma_n} = \int \frac{d\epsilon}{\hbar\omega} (\mathrm{Re}\, G_b\, \mathrm{Re}\, G_b' + \mathrm{Re}\, F_b\, \mathrm{Re}\, F_b')(f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i \int \frac{d\epsilon}{\hbar\omega} (\mathrm{Re}\, G_b\, \mathrm{Im}\, G_b' + \mathrm{Re}\, F_b\, \mathrm{Im}\, F_b')(2 f_{\mathrm{FD}} - 1), \tag{41}$$



AAA contribution

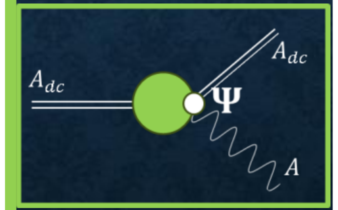
$$\frac{\sigma^{(1)}}{\sigma_n} = \frac{8s}{\hbar\omega} \int \frac{d\epsilon}{\hbar\omega} \mathrm{Re}\, F_b\, \mathrm{Im}\, F_b\, \mathrm{Re}\, G_b' (f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i \frac{2s}{\hbar\omega} \int \frac{d\epsilon}{\hbar\omega} \left[2\, \mathrm{Re}\, F_b\, \mathrm{Im}\, F_b\, \mathrm{Im} \left\{ G_b + G_b' \right\} + \left\{ (\mathrm{Re}\, F_b')^2 - (\mathrm{Re}\, F_b)^2 + (\mathrm{Im}\, F_b)^2 - (\mathrm{Im}\, F_b')^2 \right\} \mathrm{Re}\, G_b \right] (2 f_{\mathrm{FD}} - 1), \tag{42}$$



$\propto A_{dc} \cdot A$

Higgs mediated contribution

$$\frac{\sigma^{(2)}}{\sigma_n} = \frac{2s\Psi}{\hbar\omega} \int \frac{d\epsilon}{\hbar\omega} (\mathrm{Re}\, F_b\, \mathrm{Re}\, G_b' - \mathrm{Re}\, G_b\, \mathrm{Re}\, F_b') \times (f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i \frac{2s\Psi}{\hbar\omega} \int \frac{d\epsilon}{\hbar\omega} \left\{ \mathrm{Re}\, G_b\, \mathrm{Im}(F_b - F_b') + \mathrm{Re}\, F_b\, \mathrm{Im}(G_b + G_b') \right\} (2 f_{\mathrm{FD}} - 1). \tag{43}$$



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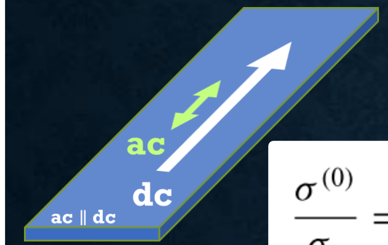
Nonequilibrium corrections due to the Doppler fluctuation of flow $\propto A_{dc} \cdot A$

Complex conductivity formula

T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

$ac \parallel dc$ case



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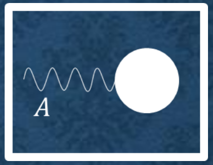

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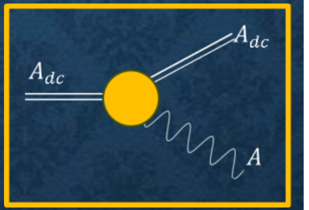

$\propto \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$

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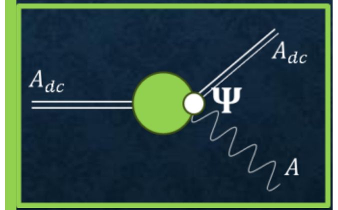
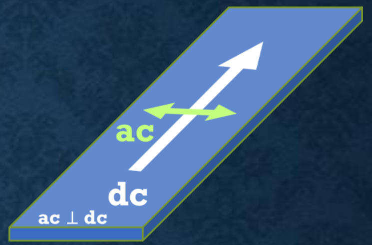
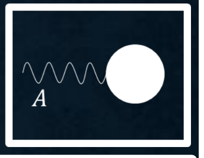


$\propto \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$

$ac \perp dc$ case



$$\sigma = \sigma^{(0)}$$

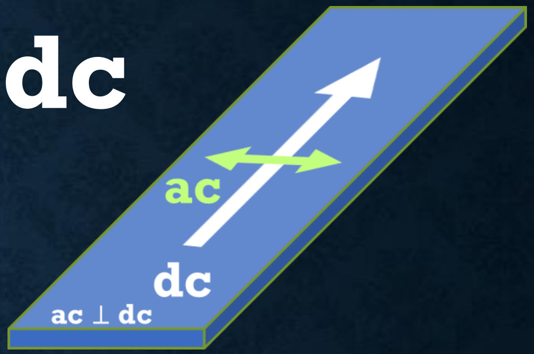


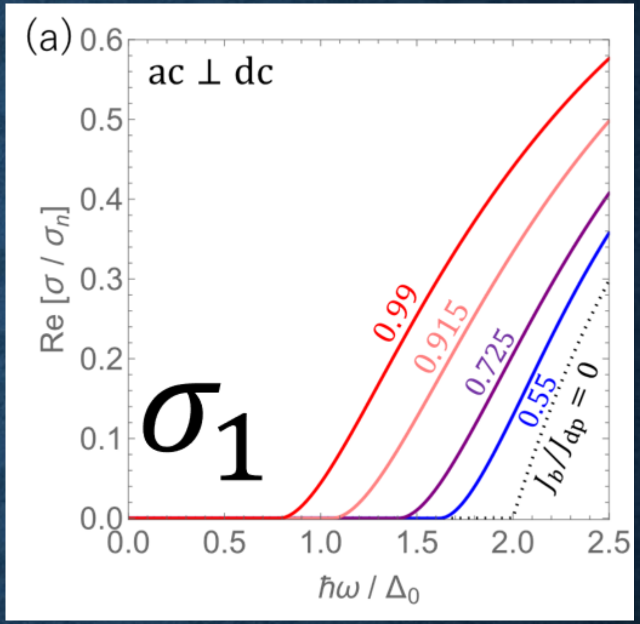
$$\frac{\sigma^{(0)}}{\sigma_n} = \int \frac{d\epsilon}{\hbar\omega}(\mathrm{Re}\,G_b\,\mathrm{Re}\,G_b' + \mathrm{Re}\,F_b\,\mathrm{Re}\,F_b')(f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i\int \frac{d\epsilon}{\hbar\omega}(\mathrm{Re}\,G_b\,\mathrm{Im}\,G_b' + \mathrm{Re}\,F_b\,\mathrm{Im}\,F_b')(2f_{\mathrm{FD}} - 1), \tag{41}$$

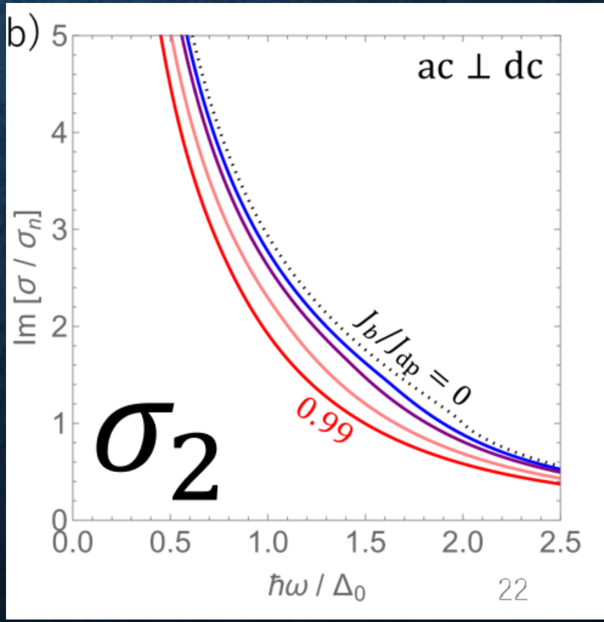
No Doppler fluctuation

Nonequilibrium corrections due to the Doppler fluctuation of flow $\propto \boldsymbol{A}_{dc} \cdot \boldsymbol{A}$

ac ⊥ dc

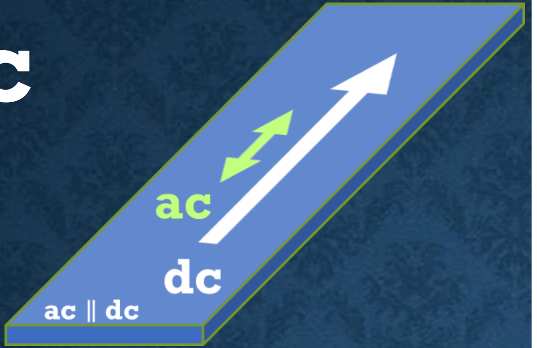


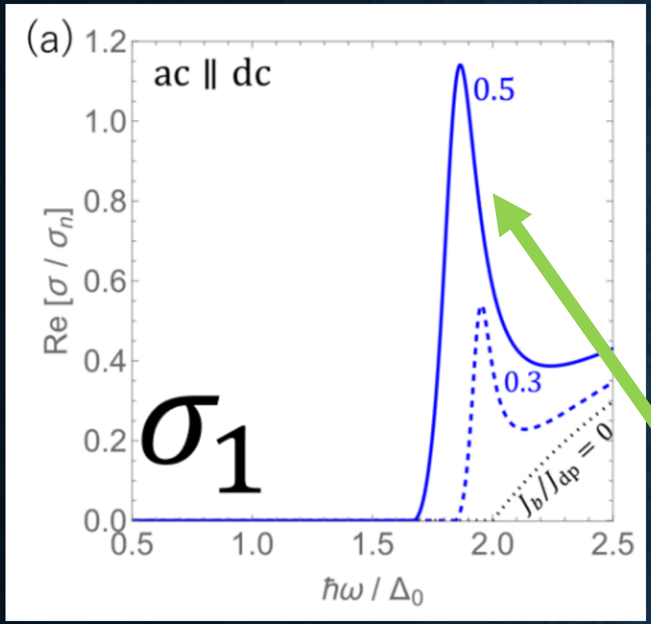


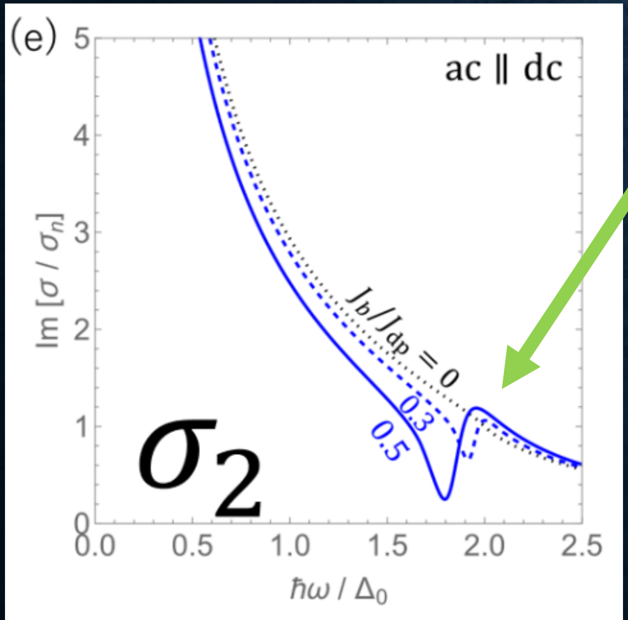


T. Kubo,
Phys. Rev. Applied **23**, 054091 (2025)

ac ∥ dc





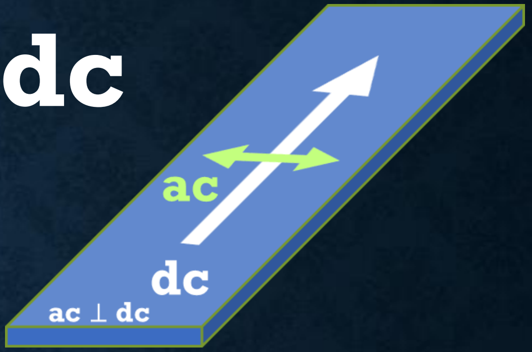


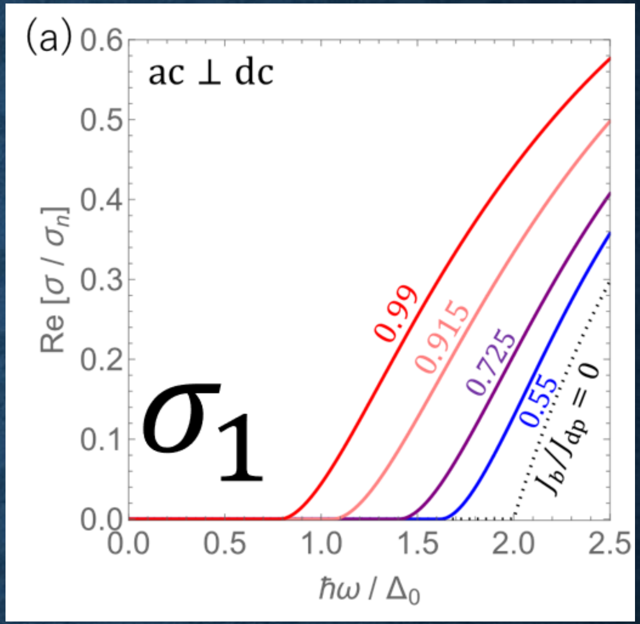
Resonance peak due to the **Higgs** mode.

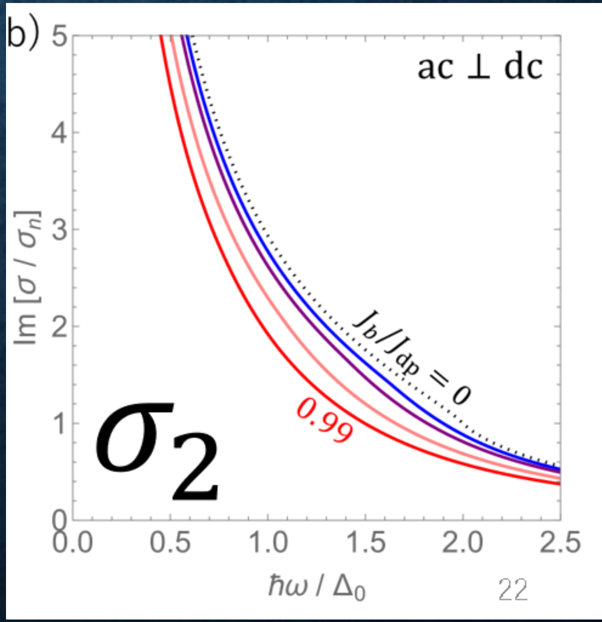
Already observed in experiments!
S. Nakamura et al.,
PRL **122**, 257001 (2019)

T. Kubo,
Phys. Rev. Applied **23**, 054091 (2025)

ac ⊥ dc







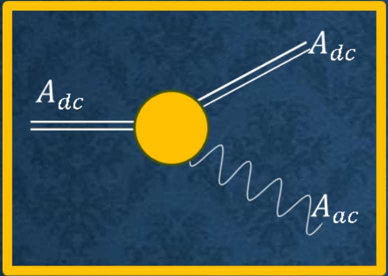
T. Kubo,
Phys. Rev. Applied **23**, 054091 (2025)

The contributions from the nonequilibrium corrections are significant even at low frequencies

$$\sigma = \sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)}$$

Direct AAA photon action

$$\frac{\sigma^{(1)}}{\sigma_n} = \frac{8s}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega} \mathrm{Re}\, F_b \,\mathrm{Im}\, F_b \,\mathrm{Re}\, G_b' (f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i\frac{2s}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}\left[2\,\mathrm{Re}\, F_b \,\mathrm{Im}\, F_b \,\mathrm{Im}\left\{G_b + G_b'\right\} + \left\{(\mathrm{Re}\, F_b')^2 - (\mathrm{Re}\, F_b)^2 + (\mathrm{Im}\, F_b)^2 - (\mathrm{Im}\, F_b')^2\right\}\mathrm{Re}\, G_b\right](2f_{\mathrm{FD}} - 1), \quad (42)$$



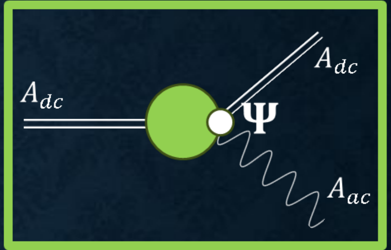
Significantly contributes to

$Re[\sigma]$, $Im[\sigma]$

even at low frequencies ($\hbar\omega \ll \Delta$)

Higgs mediated contribution

$$\frac{\sigma^{(2)}}{\sigma_n} = \frac{2s\Psi}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}(\mathrm{Re}\, F_b \,\mathrm{Re}\, G_b' - \mathrm{Re}\, G_b \,\mathrm{Re}\, F_b') \times (f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i\frac{2s\Psi}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}\left\{\mathrm{Re}\, G_b \,\mathrm{Im}(F_b - F_b') + \mathrm{Re}\, F_b \,\mathrm{Im}(G_b + G_b')\right\}(2f_{\mathrm{FD}} - 1). \quad (43)$$



Significantly contributes to

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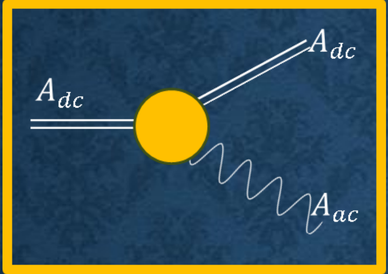


Significantly contributes to

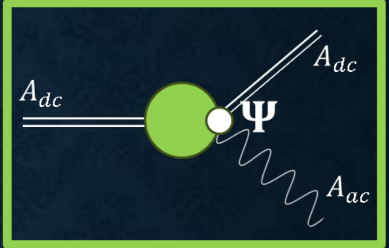
$\boxed{Re[\sigma]}$, $Im[\sigma]$

even at low frequencies $(\hbar\omega \ll \Delta)$

Surface resistance
or
quality factor

Higgs mediated contribution

$$\frac{\sigma^{(2)}}{\sigma_n} = \frac{2s\Psi}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}(\mathrm{Re}\, F_b\, \mathrm{Re}\, G_b' - \mathrm{Re}\, G_b\, \mathrm{Re}\, F_b') \times (f_{\mathrm{FD}} - f_{\mathrm{FD}}') + i\frac{2s\Psi}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}\left\{\mathrm{Re}\, G_b\, \mathrm{Im}(F_b - F_b') + \mathrm{Re}\, F_b\, \mathrm{Im}(G_b + G_b')\right\} (2f_{\mathrm{FD}} - 1). \tag{43}$$



Significantly contributes to

$Im[\sigma]$

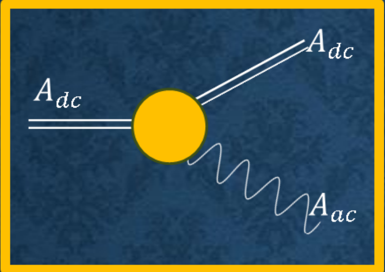
even at low frequencies $(\hbar\omega \ll \Delta)$

The contributions from the nonequilibrium corrections are significant even at low frequencies

$$\sigma = \sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)}$$

Direct AAA photon action

$$\frac{\sigma^{(1)}}{\sigma_n} = \frac{8s}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}\mathrm{Re}\,F_b\,\mathrm{Im}\,F_b\,\mathrm{Re}\,G'_b(f_{\mathrm{FD}} - f'_{\mathrm{FD}})$$
$$+ i\frac{2s}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}\left[2\,\mathrm{Re}\,F_b\,\mathrm{Im}\,F_b\,\mathrm{Im}\left\{G_b + G'_b\right\} + \left\{(\mathrm{Re}\,F'_b)^2\right.\right.$$
$$\left.\left. - (\mathrm{Re}\,F_b)^2 + (\mathrm{Im}\,F_b)^2 - (\mathrm{Im}\,F'_b)^2\right\}\mathrm{Re}\,G_b\right](2f_{\mathrm{FD}} - 1), \tag{42}$$



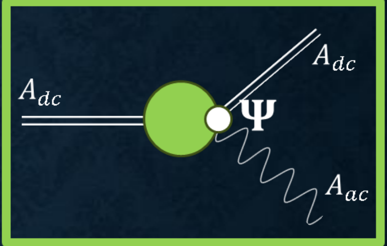
Significantly contributes to

$Re[\sigma]$, $Im[\sigma]$

even at low frequencies ($\hbar\omega \ll \Delta$)

Higgs mediated contribution

$$\frac{\sigma^{(2)}}{\sigma_n} = \frac{2s\Psi}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}(\mathrm{Re}\,F_b\,\mathrm{Re}\,G'_b - \mathrm{Re}\,G_b\,\mathrm{Re}\,F'_b)$$
$$\times (f_{\mathrm{FD}} - f'_{\mathrm{FD}}) + i\frac{2s\Psi}{\hbar\omega}\int \frac{d\epsilon}{\hbar\omega}\left\{\mathrm{Re}\,G_b\,\mathrm{Im}(F_b - F'_b)\right.$$
$$\left. + \mathrm{Re}\,F_b\,\mathrm{Im}(G_b + G'_b)\right\}(2f_{\mathrm{FD}} - 1). \tag{43}$$



Significantly contributes to

$Im[\sigma]$

even at low frequencies ($\hbar\omega \ll \Delta$)

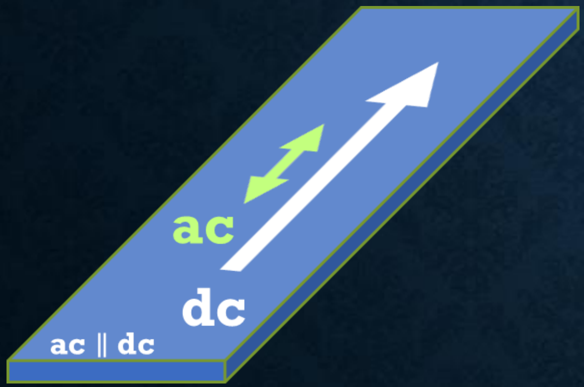
Surface resistance
or
quality factor

Kinetic inductance

Biproduct: the theory of current-dependent kinetic inductance is now established.

$$L_k(J) = L_k(0)\left\{1 + \boxed{C}\left(\frac{J}{J_{dp}}\right)^2 + \cdots\right\}$$

$ac \parallel dc$ case



For $(T, \omega) \to (0, 0)$, we can analytically calculate the coefficient C:

AAA Higgs

$$\boxed{C = C^{(0)} + C^{(1)} + C^{(2)} \simeq 0.409}$$

$\simeq 0.136$

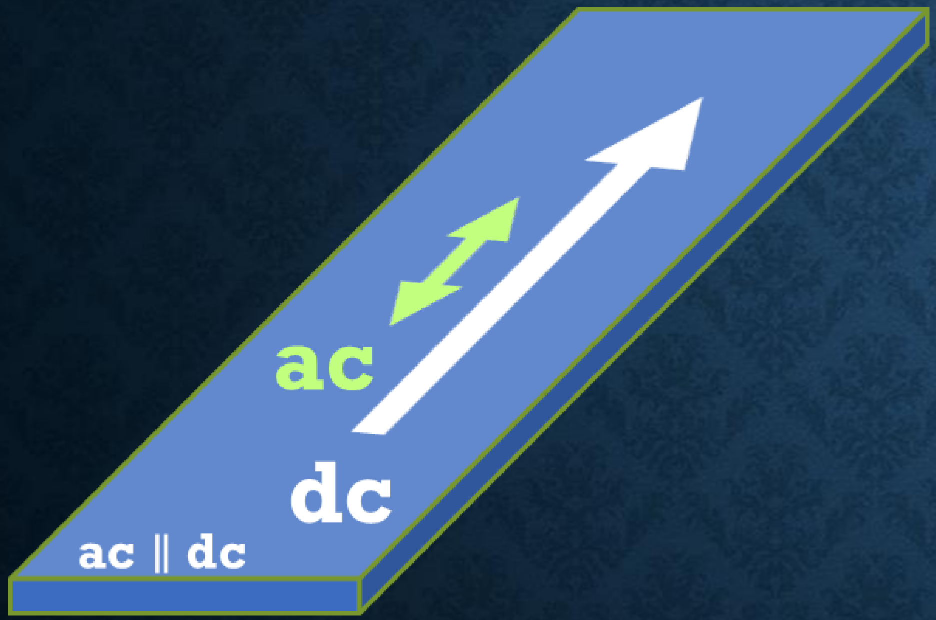
$\simeq 0.0956$ (23%)

$\simeq 0.177$ (43%)

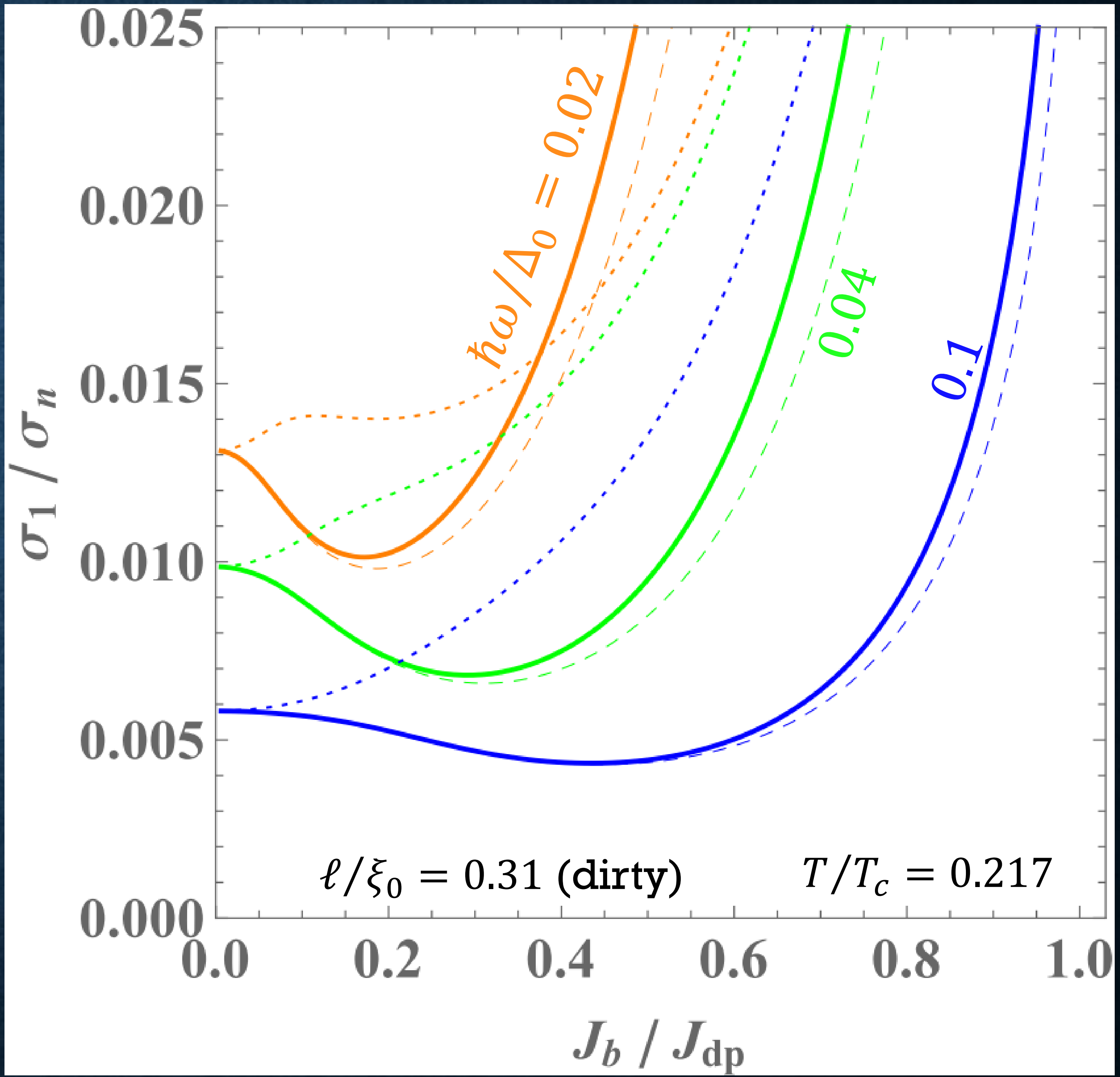
T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)
T. Kubo, Phys. Rev. Applied **23**, 054091 (2025)

43% comes from Higgs!

Analogue $R_s(E)$ curve in ac+dc system

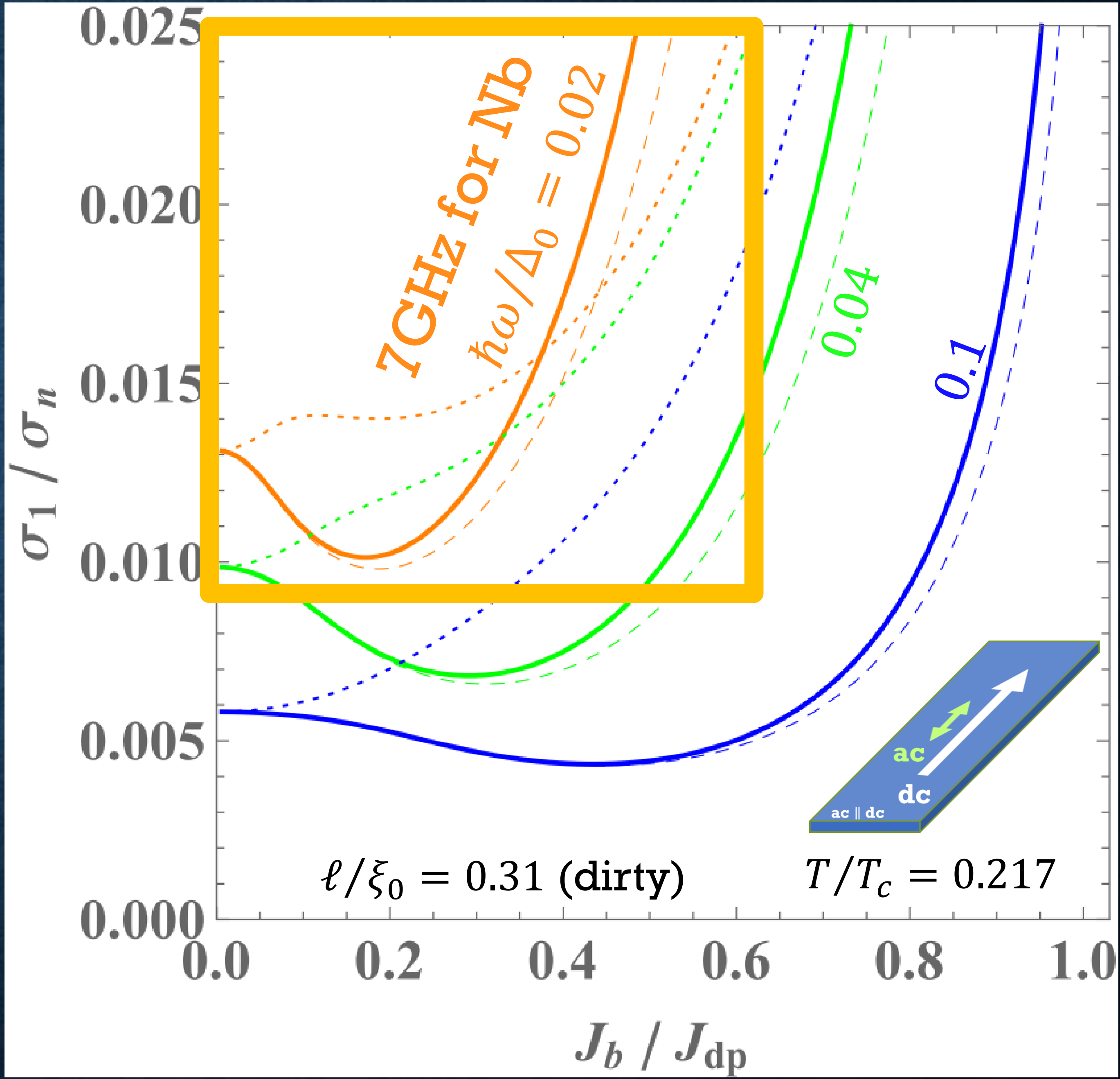


dc: arbitrary strength
ac: perturbation



T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

Let’s see the orange curves ($\hbar\omega/\Delta_0 = 0.02$),which correspond to 7GHz for Nb.



T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

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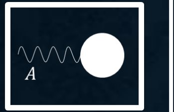
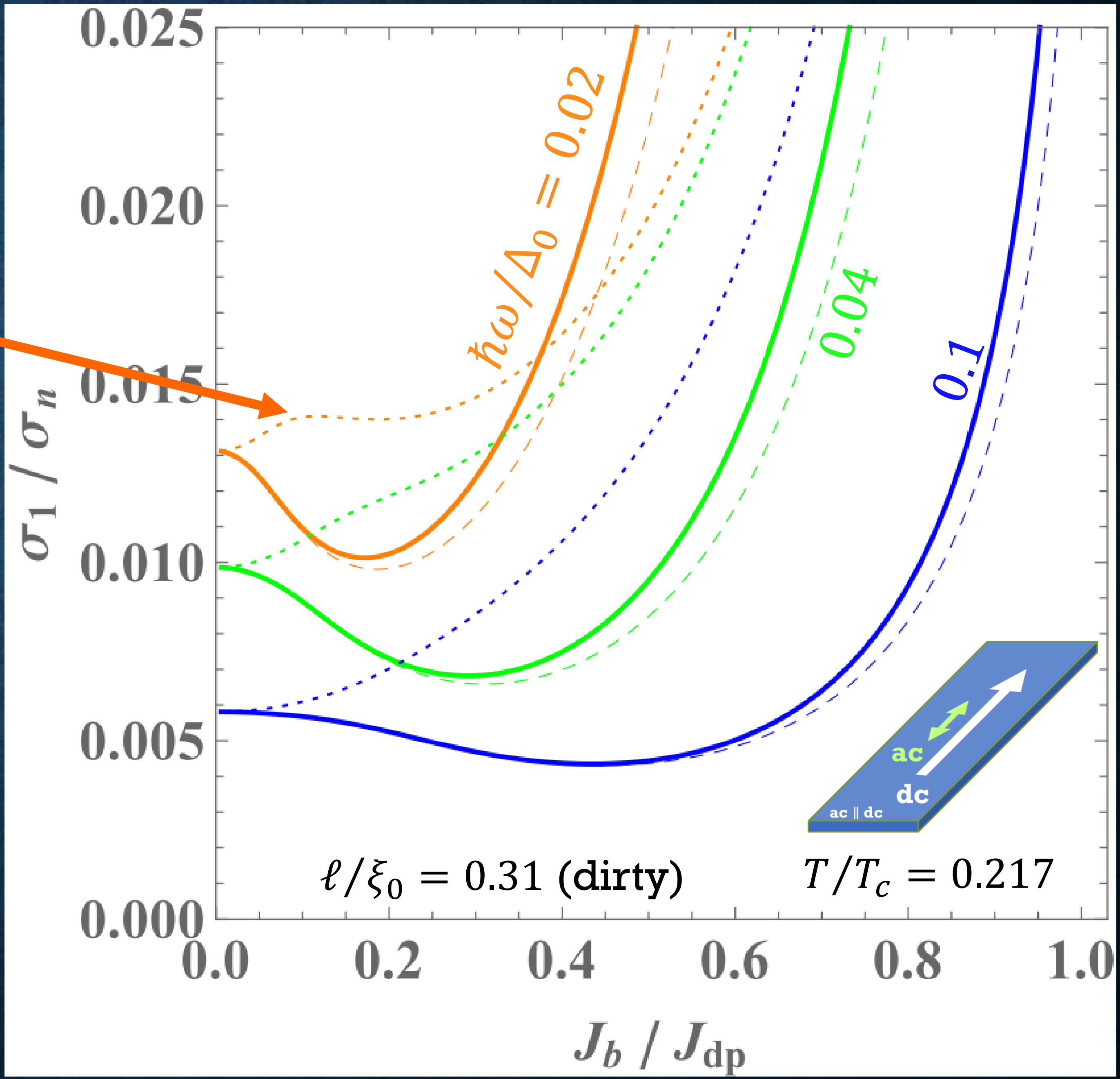


Dotted curve:

$\mathrm{Re}[\sigma^{(0)}]$



T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

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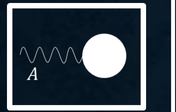
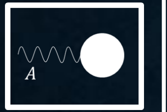
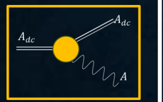
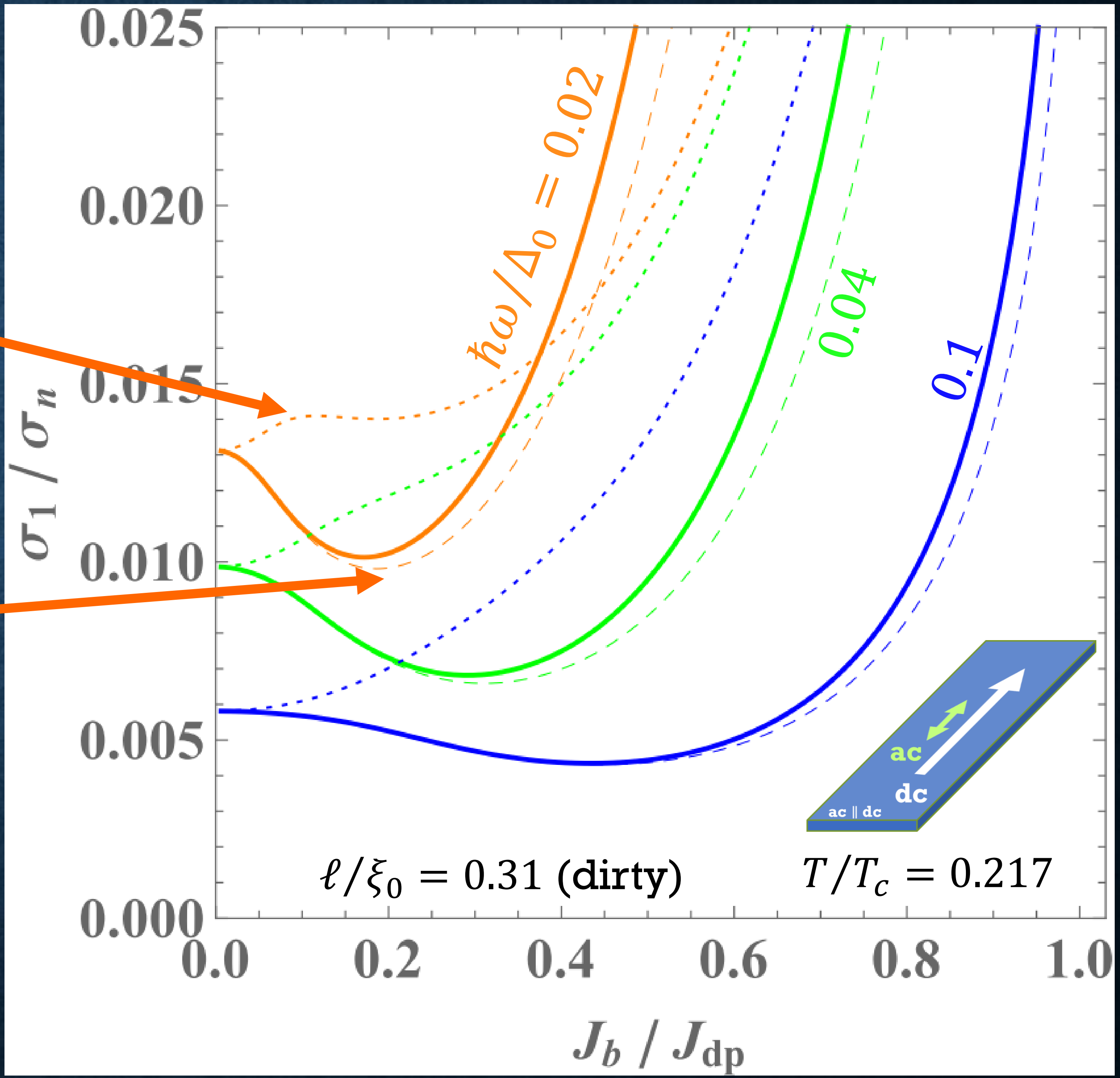


Dotted curve: $\mathrm{Re}[\sigma^{(0)}]$





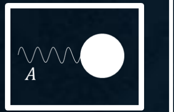
Dashed curve: $\mathrm{Re}[\sigma^{(0)} + \sigma^{(1)}]$



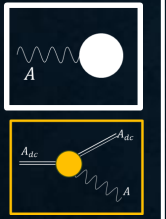
Significant contribution from the direct AAA action term $\sigma^{(1)}$

T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

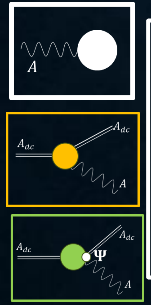
Let’s see the orange curves ($\hbar\omega/\Delta_0 = 0.02$),which correspond to 7GHz for Nb.



Dotted curve: $\mathrm{Re}[\sigma^{(0)}]$

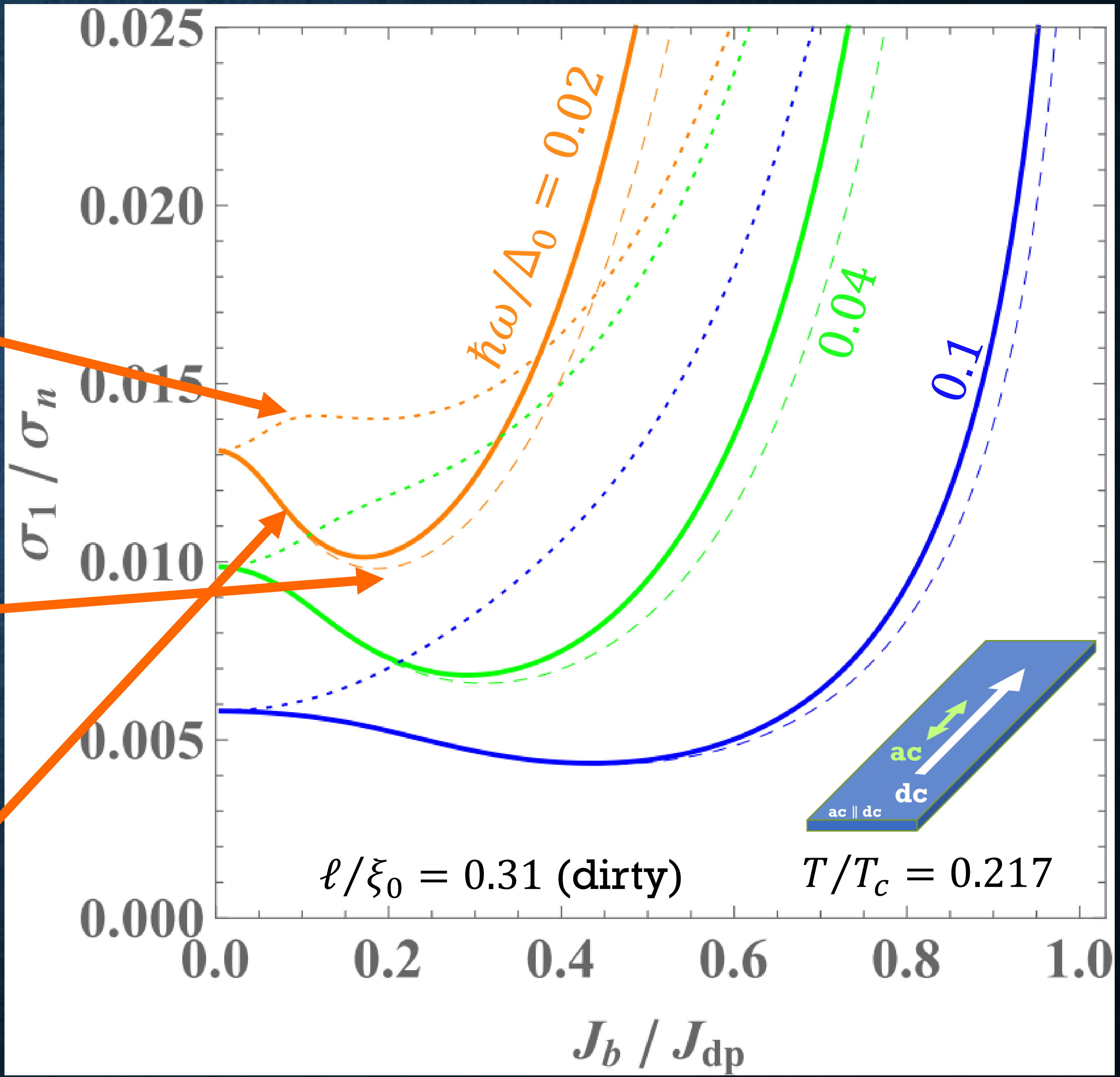


Dashed curve: $\mathrm{Re}[\sigma^{(0)} + \sigma^{(1)}]$



Solid curve: $\mathrm{Re}[\sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)}]$

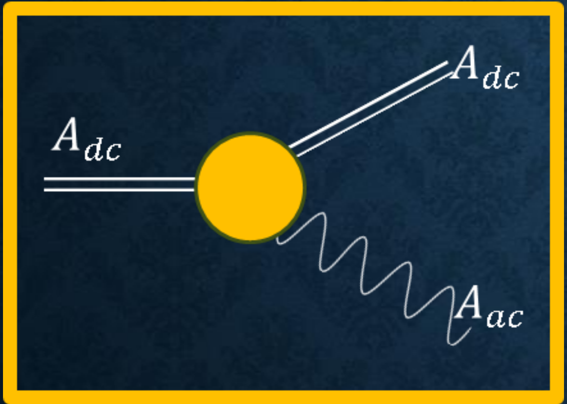
Significant contribution from the direct AAA action term $\sigma^{(1)}$



T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

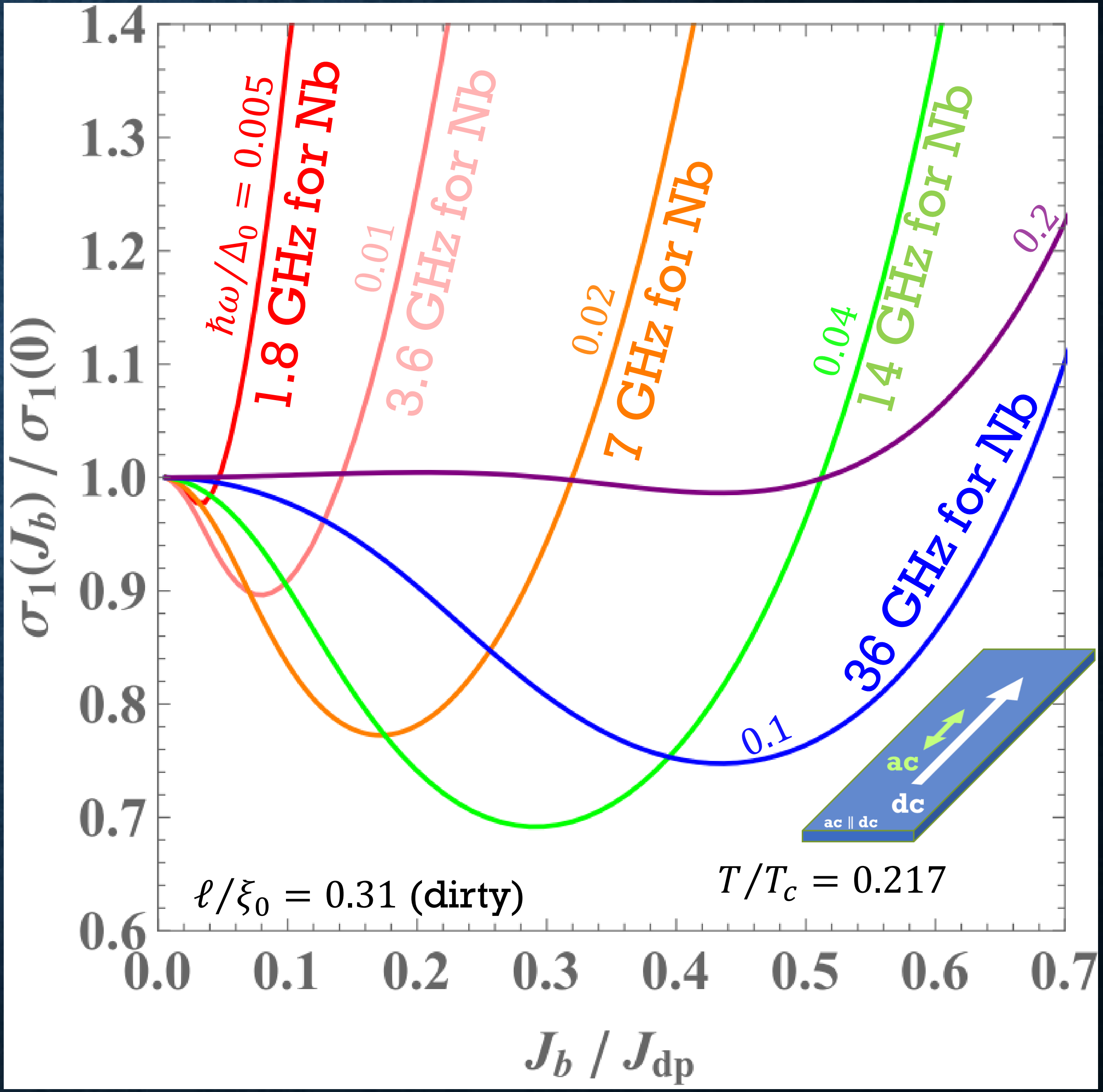
The *analogue anti Q-slope* is pronounced as the frequency increases

The key player is the direct AAA photon action



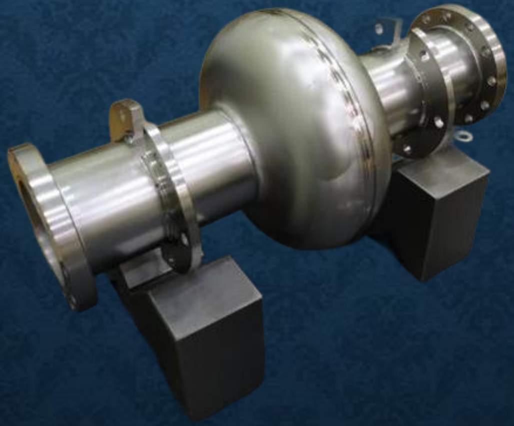
$$\sigma = \sigma^{(0)} + \boxed{\sigma^{(1)}} + \sigma^{(2)}$$

Big clue to understand the pronounced anti Q-slope with increasing frequency



T. Kubo, Phys. Rev. Applied **22**, 044042 (2024)

Nonequilibrium Nonlinear Response theory of amplitude dependent conductivity under strong ac field

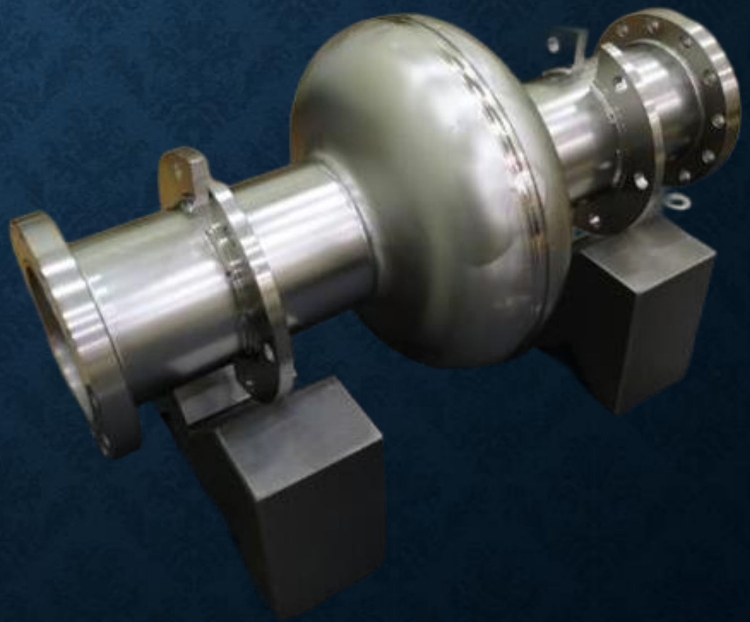


The detailed discussions are found here:

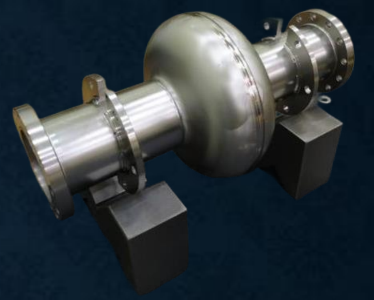
T. Kubo, arXiv:2509.09766 DOI: https://doi.org/10.48550/arXiv.2509.09766

We do not consider any dc bias.

We consider a superconductor under a strong ac field



Keldysh–Usadel Equation for SC under a strong ac field



arXiv:2509.09766

$$
\begin{aligned}
&-i\frac{s}{8}\sum_{\eta=\pm\omega_{\mathrm{ac}}}\sum_{\eta'=\pm\omega_{\mathrm{ac}}} 2\pi\delta(\omega+\eta+\eta')\times \\
&\left[\hat{\tau}_3\hat{g}_e^r\left(\epsilon-\frac{\hbar\eta}{2}+\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\hat{g}_e^r\left(\epsilon+\frac{\hbar\eta}{2}+\frac{\hbar\eta'}{2}\right)\right. \\
&\left.-\hat{g}_e^r\left(\epsilon-\frac{\hbar\eta}{2}-\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\hat{g}_e^r\left(\epsilon+\frac{\hbar\eta}{2}-\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\right] \\
=&\left(\epsilon+\frac{\hbar\omega}{2}\right)\hat{\tau}_3\delta\hat{g}^r(\epsilon,\omega)-\left(\epsilon-\frac{\hbar\omega}{2}\right)\delta\hat{g}^r(\epsilon,\omega)\hat{\tau}_3 \\
&+\delta\hat{\Delta}(\omega)\hat{g}_e^r\left(\epsilon-\frac{\hbar\omega}{2}\right)-\hat{g}_e^r\left(\epsilon+\frac{\hbar\omega}{2}\right)\delta\hat{\Delta}(\omega) \\
&+[\hat{\Delta}_e,\delta\hat{g}^r(\epsilon,\omega)].
\end{aligned} \tag{11}
$$

$$
\delta\Delta(\omega)=-\frac{\mathscr{G}}{8}\int d\epsilon\,\mathrm{Tr}[(-i\tau_2)\delta\hat{g}^K(\epsilon,\omega)]. \tag{15}
$$

$$
\begin{aligned}
&-i\frac{s}{8}\sum_{\eta=\pm\omega_{\mathrm{ac}}}\sum_{\eta'=\pm\omega_{\mathrm{ac}}} 2\pi\delta(\omega+\eta+\eta')\times \\
&\left[\hat{\tau}_3\hat{g}_e^R\left(\epsilon-\frac{\hbar\eta}{2}+\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\hat{g}_e^K\left(\epsilon+\frac{\hbar\eta}{2}+\frac{\hbar\eta'}{2}\right)\right. \\
&-\hat{g}_e^R\left(\epsilon-\frac{\hbar\eta}{2}-\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\hat{g}_e^K\left(\epsilon+\frac{\hbar\eta}{2}-\frac{\hbar\eta'}{2}\right)\hat{\tau}_3 \\
&+\hat{\tau}_3\hat{g}_e^K\left(\epsilon-\frac{\hbar\eta}{2}+\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\hat{g}_e^A\left(\epsilon+\frac{\hbar\eta}{2}+\frac{\hbar\eta'}{2}\right) \\
&\left.-\hat{g}_e^K\left(\epsilon-\frac{\hbar\eta}{2}-\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\hat{g}_e^A\left(\epsilon+\frac{\hbar\eta}{2}-\frac{\hbar\eta'}{2}\right)\hat{\tau}_3\right] \\
=&\left(\epsilon+\frac{\hbar\omega}{2}\right)\hat{\tau}_3\delta\hat{g}^K(\epsilon,\omega)-\left(\epsilon-\frac{\hbar\omega}{2}\right)\delta\hat{g}^K(\epsilon,\omega)\hat{\tau}_3 \\
&+\delta\hat{\Delta}(\omega)\hat{g}_e^K\left(\epsilon-\frac{\hbar\omega}{2}\right)-\hat{g}_e^K\left(\epsilon+\frac{\hbar\omega}{2}\right)\delta\hat{\Delta}(\omega) \\
&+[\hat{\Delta}_e,\delta\hat{g}^K(\epsilon,\omega)].
\end{aligned} \tag{13}
$$

Much different from the ac+dc case

We solve these equations to obtain the **nonequilibrium corrections** ($\delta\hat{g}^{R,A,K}$ and $\delta\Delta$).

To obtain the current response, we substitute the solutions ($\delta\hat{g}^{R,A,K}$, $\delta\Delta$) into

$$
\begin{aligned}
\mathbf{J}(t)&=\mathbf{J}^{(1)}(t)+\mathbf{J}^{(3)}(t) \\
&=\frac{\sigma_n}{8e}\int\left\{\mathbf{S}^{(1)}(\epsilon,t)+\mathbf{S}^{(3)}(\epsilon,t)\right\}d\epsilon,
\end{aligned} \tag{49}
$$

$$
\mathbf{S}^{(1)}(\epsilon,t)=\mathrm{Tr}\left[\hat{\tau}_3\left\{\hat{g}_e^R\circ(\hat{\partial}\circ\hat{g}_e^K)+\hat{g}_e^K\circ(\hat{\partial}\circ\hat{g}_e^A)\right\}\right] \tag{50}
$$

← Linear response

$$
\begin{aligned}
\mathbf{S}^{(3)}(\epsilon,t)=\mathrm{Tr}\Big[\hat{\tau}_3\big\{&\hat{g}_e^R\circ(\hat{\partial}\circ\delta\hat{g}^K)+\hat{g}_e^K\circ(\hat{\partial}\circ\delta\hat{g}^A) \\
&+\delta\hat{g}^R\circ(\hat{\partial}\circ\hat{g}_e^K)+\delta\hat{g}^K\circ(\hat{\partial}\circ\hat{g}_e^A)\big\}\Big].
\end{aligned} \tag{51}
$$

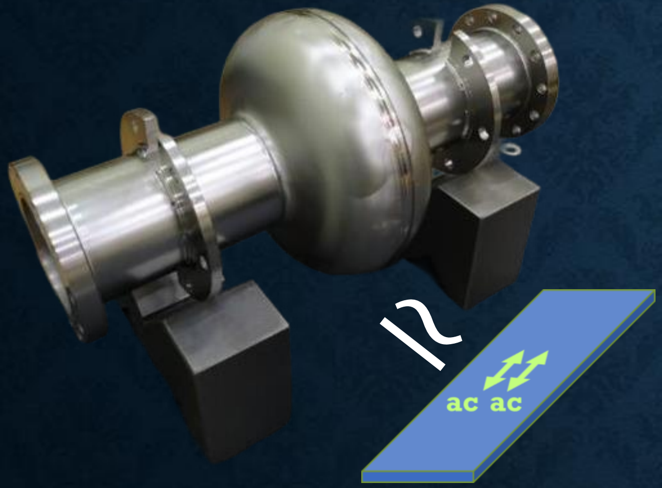
← **Nonlinear** correction of $\mathcal{O}(A^3)$

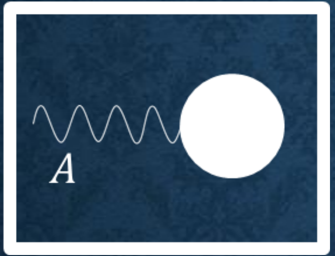
Schematically, the results can be illustrated as

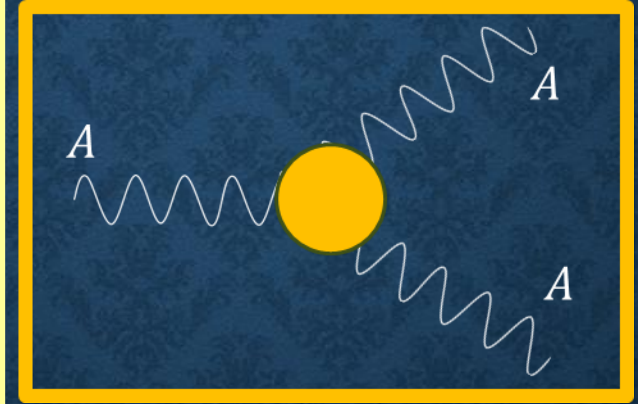
$$J \sim \boxed{Agg} + \boxed{Ag\delta g} \qquad \delta g \sim A^2 + \delta\Delta \quad \text{and} \quad \delta\Delta \sim \Psi\, A^2$$

Linear response

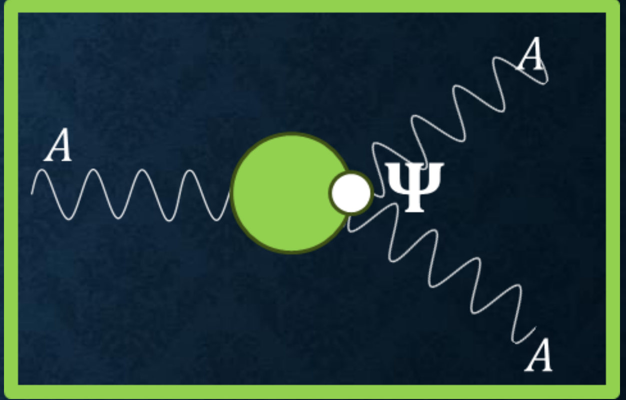
Nonlinear corrections

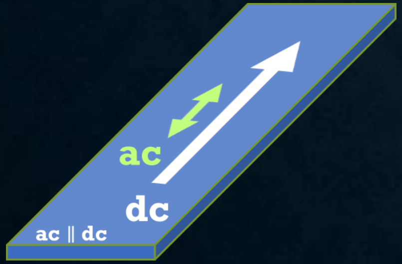






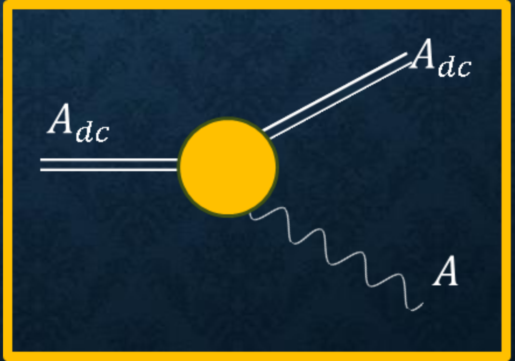
+



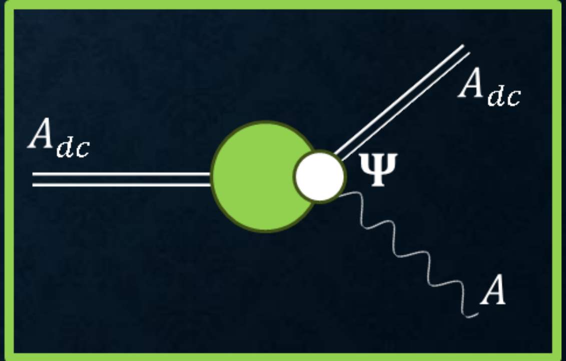


$$J_{ac+dc} \sim$$

+



+

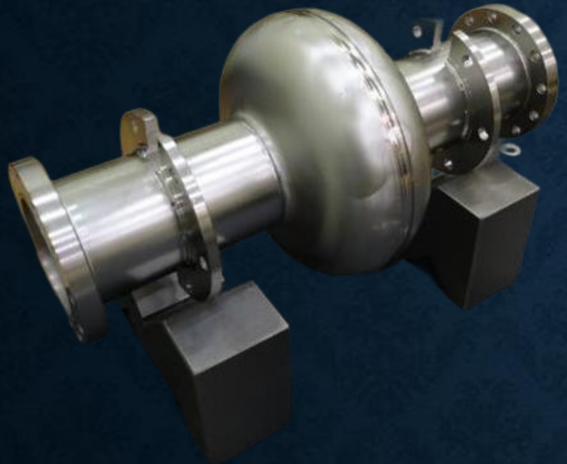


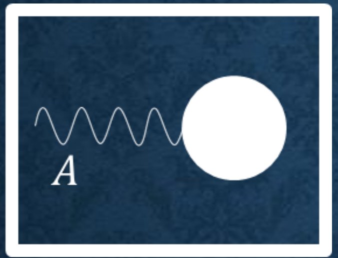
Schematically, the results can be illustrated as

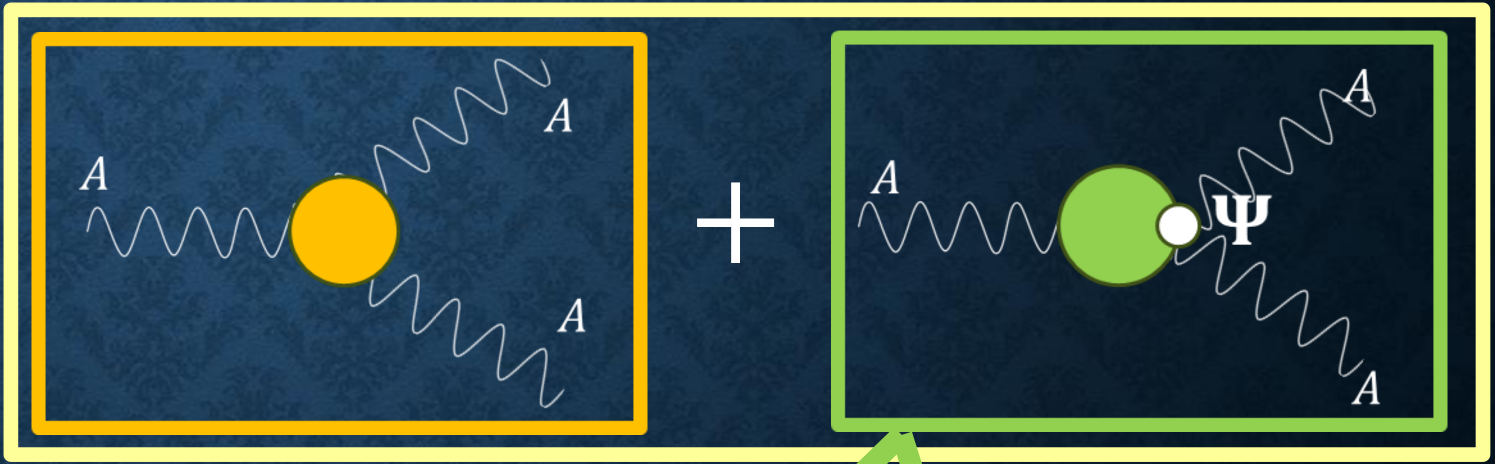
$$J \sim \boxed{Agg} + \boxed{Ag\delta g} \qquad \delta g \sim A^2 + \delta\Delta \quad \text{and} \quad \delta\Delta \sim \Psi\, A^2$$

Linear response

Nonlinear corrections





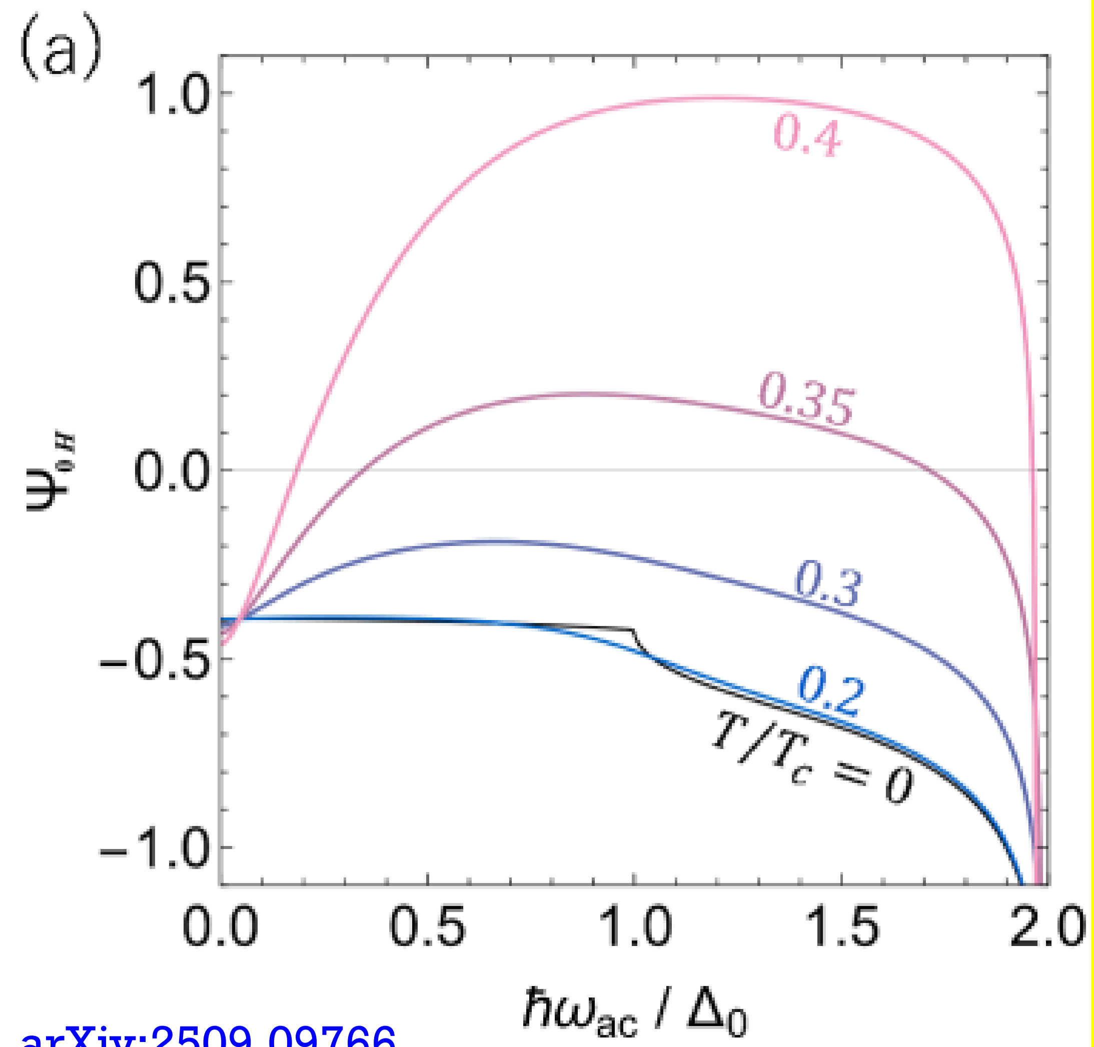


Note: This theory automatically includes the **Eliashberg effect** (microwave-induced gap enhancement) as well as the **Higgs-mode** contribution.

Time averaged $\delta\Delta$:
Eliashberg effect

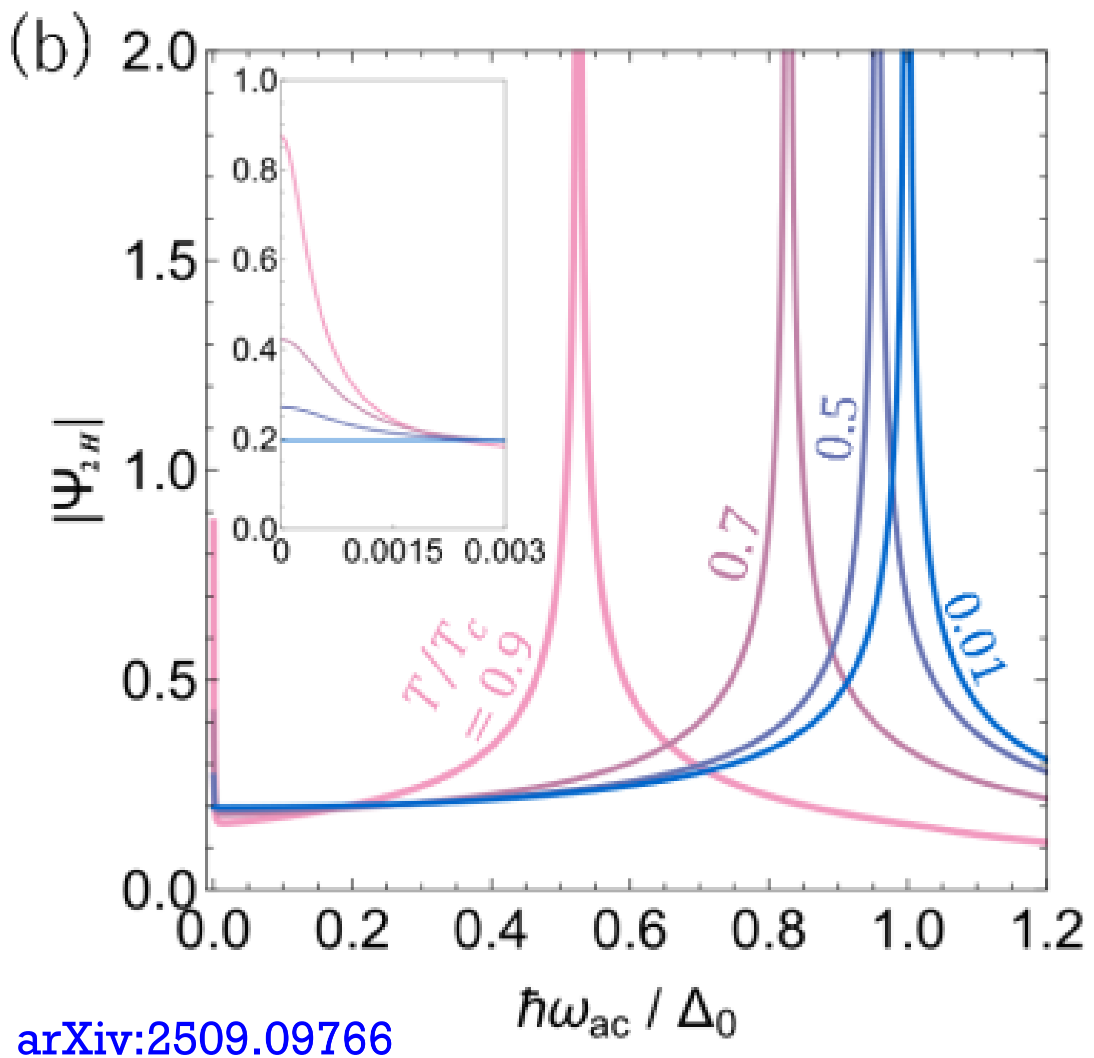
Time dependent $\delta\Delta$:
Higgs

Eliashberg effect



arXiv:2509.09766

Higgs mode



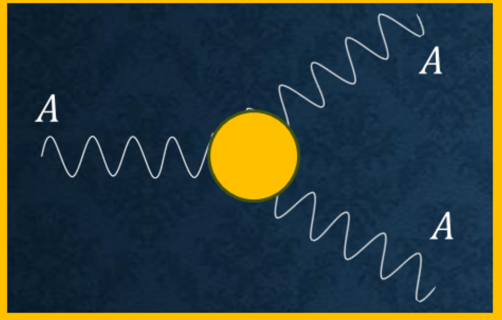
arXiv:2509.09766

See also,
M. Silaev, Phys. Rev. B **99**, 224511 (2019)
P. Derendorf, A. F. Volkov, and I. M. Eremin, Phys. Rev. B **109**, 024510 (2024)

Nonlinear correction

arXiv:2509.09766

$$\delta\sigma_1 = \delta\sigma_1^{qqq} + \delta\sigma_1^{\mathrm{Higgs}} + \delta\sigma_1^{\mathrm{Eliash}}$$



$$\delta\sigma_1^{qqq} = 2\sqrt{\pi}\sigma_n \mathrm{Re}[I_{1\mathrm{H}}^{qqq}]\frac{s}{\hbar\omega_{\mathrm{ac}}}$$

$$I_{1\mathrm{H}}^{\mathrm{qqq}} = \frac{-1}{16\sqrt{\pi}}\int K_{1\mathrm{H}} d\epsilon,$$

$$K_{1\mathrm{H}}(\epsilon,\omega_{\mathrm{ac}}) = \sum_{i=1}^{6} K_{1\mathrm{H},i}, \tag{61}$$

$$\begin{aligned}K_{1\mathrm{H},1} &= i\frac{(F_1 - F_{-3})F_{-1} + (G_1 - G_{-3})G_{-1}}{4\hbar\omega_{\mathrm{ac}}(F_1 + F_{-3})}\\ &\times\Big[\Big\{(F_1 + F_{-3})G^*_{-1} + (G_1 + G_{-3})F^*_{-1}\Big\}(\mathcal{T}_{-1} - \mathcal{T}_{-3})\\ &+\Big\{(F_1 + F_{-3})G_{-1} + (G_1 + G_{-3})F_{-1}\Big\}\mathcal{T}_{-1}\Big],\end{aligned} \tag{62}$$

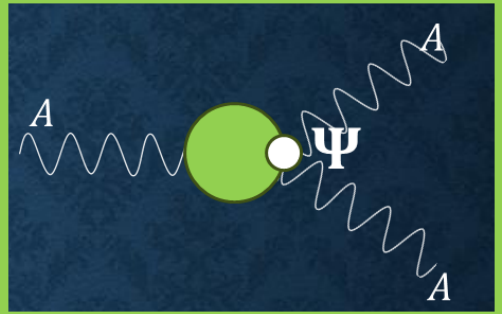
$$\begin{aligned}K_{1\mathrm{H},2} &= i\frac{(G_3 - G_{-1})G_1 + (F_3 - F_{-1})F_1}{4\hbar\omega_{\mathrm{ac}}(F_3 + F_{-1})}\\ &\times\{(F_3 + F_{-1})G_1 + (G_3 + G_{-1})F_1\}\mathcal{T}_{-1},\end{aligned} \tag{63}$$

$$\begin{aligned}K_{1\mathrm{H},3} &= -i\frac{(F_3 - F^*_{-1})G_1 + (G_3 - G^*_{-1})F_1}{4\hbar\omega_{\mathrm{ac}}(F_3 - F^*_{-1})}\\ &\times\Big[\Big\{(G_3 + G^*_{-1})G^*_1 + (F_3 + F^*_{-1})F^*_1\Big\}(\mathcal{T}_3 - \mathcal{T}_1)\\ &+\Big\{(G_3 + G^*_{-1})G_1 + (F_3 + F^*_{-1})F_1\Big\}(\mathcal{T}_{-1} - \mathcal{T}_1)\Big],\end{aligned} \tag{64}$$

$$\begin{aligned}K_{1\mathrm{H},4} &= -i\frac{F_1G_{-1} + G_1F_{-1}}{2(\epsilon_{-1}G_{-1} - F_{-1}\Delta)}\\ &\times\{(F_1 + F_{-3})G_{-1} + (G_1 + G_{-3})F_{-1}\}\mathcal{T}_{-1},\end{aligned} \tag{65}$$

$$\begin{aligned}K_{1\mathrm{H},5} &= -i\frac{(F_3 + F_{-1})G_1 + (G_3 + G_{-1})F_1}{2(\epsilon_1G_1 - F_1\Delta)}\\ &\times\Big\{(G_1F^*_{-1} + F_1G^*_{-1})(\mathcal{T}_{-1} - \mathcal{T}_1)\\ &+(G_1F_{-1} + G_{-1}F_1)\mathcal{T}_{-1}\Big\},\end{aligned} \tag{66}$$

$$\begin{aligned}K_{1\mathrm{H},6} &= -\frac{G^*_{-1}\mathrm{Im}F_1 + F^*_{-1}\mathrm{Im}G_1}{\epsilon_1\mathrm{Im}G_1 - \mathrm{Im}F_1\Delta}\\ &\times\Big\{(\mathrm{Re}G_3\mathrm{Im}F_1 + \mathrm{Im}G_1\mathrm{Re}F_3)(T_3 - T_1)\\ &+(\mathrm{Re}G_{-1}\mathrm{Im}F_1 + \mathrm{Im}G_1\mathrm{Re}F_{-1})(T_{-1} - T_1)\Big\},\end{aligned} \tag{67}$$



Higgs Contribution: Time dependent $\delta\Delta$

$$\delta\sigma_1^{\mathrm{Higgs}} = 2\sqrt{\pi}\sigma_n \mathrm{Re}[I_{1\mathrm{H}}^{\mathrm{Higgs}}]\frac{s}{\hbar\omega_{\mathrm{ac}}}$$

$$I_{1\mathrm{H}}^{\mathrm{Higgs}} = \frac{-1}{16\sqrt{\pi}}\Psi_{2\mathrm{H}}\int Z_{1\mathrm{H}}^{\mathrm{Higgs}} d\epsilon$$

$$Z_{1\mathrm{H}}^{\mathrm{Higgs}}(\epsilon,\omega_{\mathrm{ac}}) = \sum_{i=1,2,3} Z_{1\mathrm{H},i}^{\mathrm{Higgs}}, \tag{71}$$

$$\begin{aligned}Z_{1\mathrm{H},1}^{\mathrm{Higgs}} &= -\frac{2(F_1 - F_{-3})}{\hbar\omega_{\mathrm{ac}}(F_1 + F_{-3})}\\ &\times\Big[\Big\{(F_1 + F_{-3})G^*_{-1} + (G_1 + G_{-3})F^*_{-1}\Big\}(\mathcal{T}_{-1} - \mathcal{T}_{-3})\\ &+\Big\{(F_1 + F_{-3})G_{-1} + (G_1 + G_{-3})F_{-1}\Big\}\mathcal{T}_{-1}\Big],\end{aligned} \tag{72}$$

$$\begin{aligned}Z_{1\mathrm{H},2}^{\mathrm{Higgs}} &= -\frac{2(F_3 - F_{-1})}{\hbar\omega_{\mathrm{ac}}(F_3 + F_{-1})}\\ &\times\Big\{(F_3 + F_{-1})G_1 + (G_3 + G_{-1})F_1\Big\}\mathcal{T}_{-1},\end{aligned} \tag{73}$$

$$\begin{aligned}Z_{1\mathrm{H},3}^{\mathrm{Higgs}} &= -\frac{2(F_3 + F^*_{-1})}{\hbar\omega_{\mathrm{ac}}(F_3 - F^*_{-1})}\\ &\times\Big\{(F_3 - F^*_{-1})G_1 + (G_3 - G^*_{-1})F_1\Big\}(\mathcal{T}_3 - \mathcal{T}_{-1})\end{aligned} \tag{74}$$

Eliashberg Contribution: Time averaged $\delta\Delta$

$$\delta\sigma_1^{\mathrm{Eliash}} = 2\sqrt{\pi}\sigma_n \mathrm{Re}[I_{1\mathrm{H}}^{\mathrm{Eliash}}]\frac{s}{\hbar\omega_{\mathrm{ac}}}$$

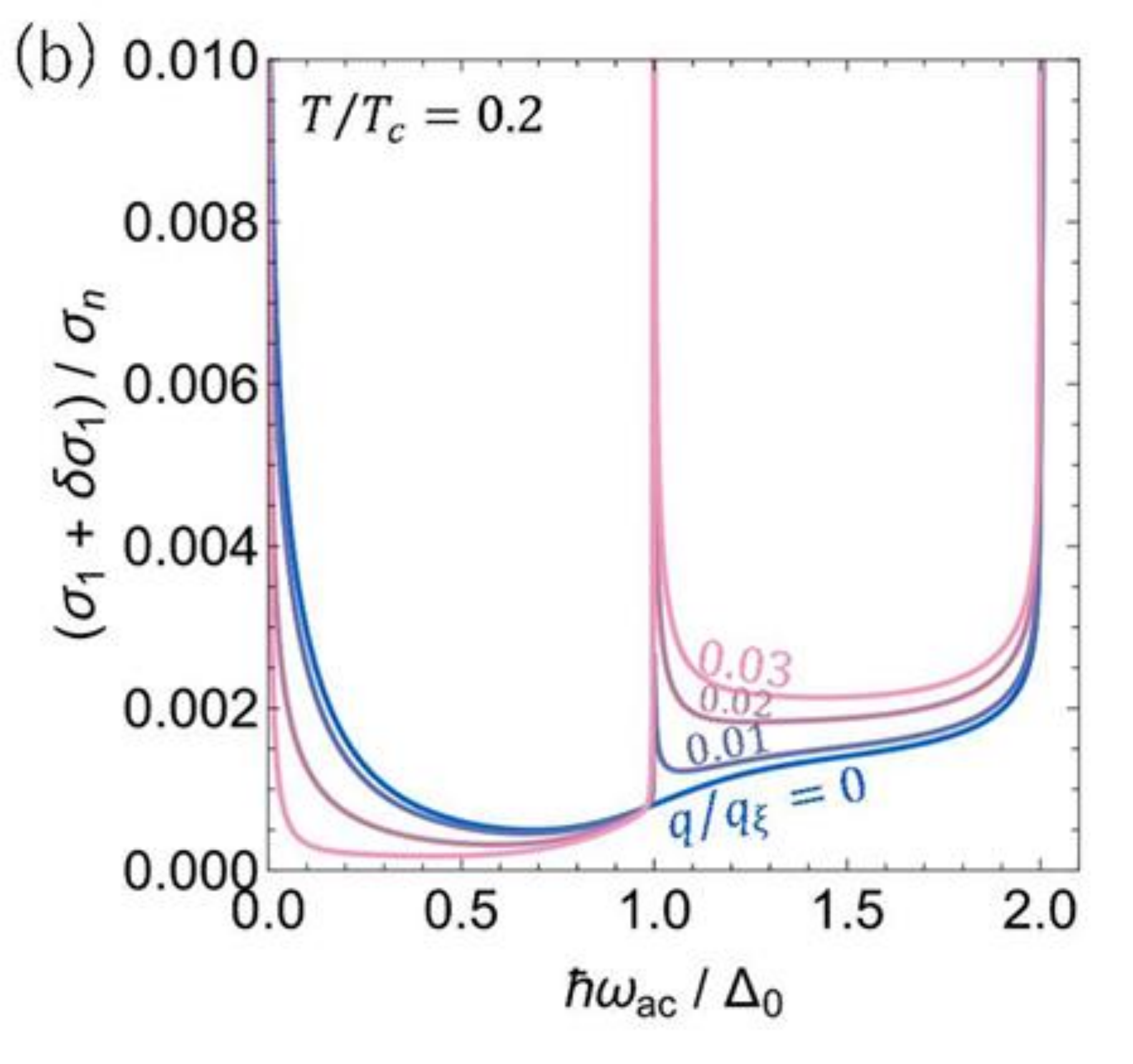
$$I_{1\mathrm{H}}^{\mathrm{Eliash}} = \frac{-1}{16\sqrt{\pi}}\Psi_{0\mathrm{H}}\int Z_{1\mathrm{H}}^{\mathrm{Eliash}} d\epsilon$$

$$Z_{1\mathrm{H}}^{\mathrm{Eliash}}(\epsilon,\omega_{\mathrm{ac}}) = \sum_{i=1,2} Z_{1\mathrm{H},i}^{\mathrm{Eliash}}, \tag{68}$$

$$Z_{1\mathrm{H},1}^{\mathrm{Eliash}} = \frac{4G_{-1}(G_1F_{-1} + F_1G_{-1})}{\epsilon_{-1}G_{-1} - F_{-1}\Delta}\mathcal{T}_{-1}, \tag{69}$$

$$\begin{aligned}Z_{1\mathrm{H},2}^{\mathrm{Eliash}} &= \frac{4G_1}{\epsilon_1G_1 - F_1\Delta}\Big\{(G_1F_{-1} + G_{-1}F_1)\mathcal{T}_{-1}\\ &+(G_1F^*_{-1} + F_1G^*_{-1})(\mathcal{T}_{-1} - \mathcal{T}_1)\Big\},\end{aligned} \tag{70}$$

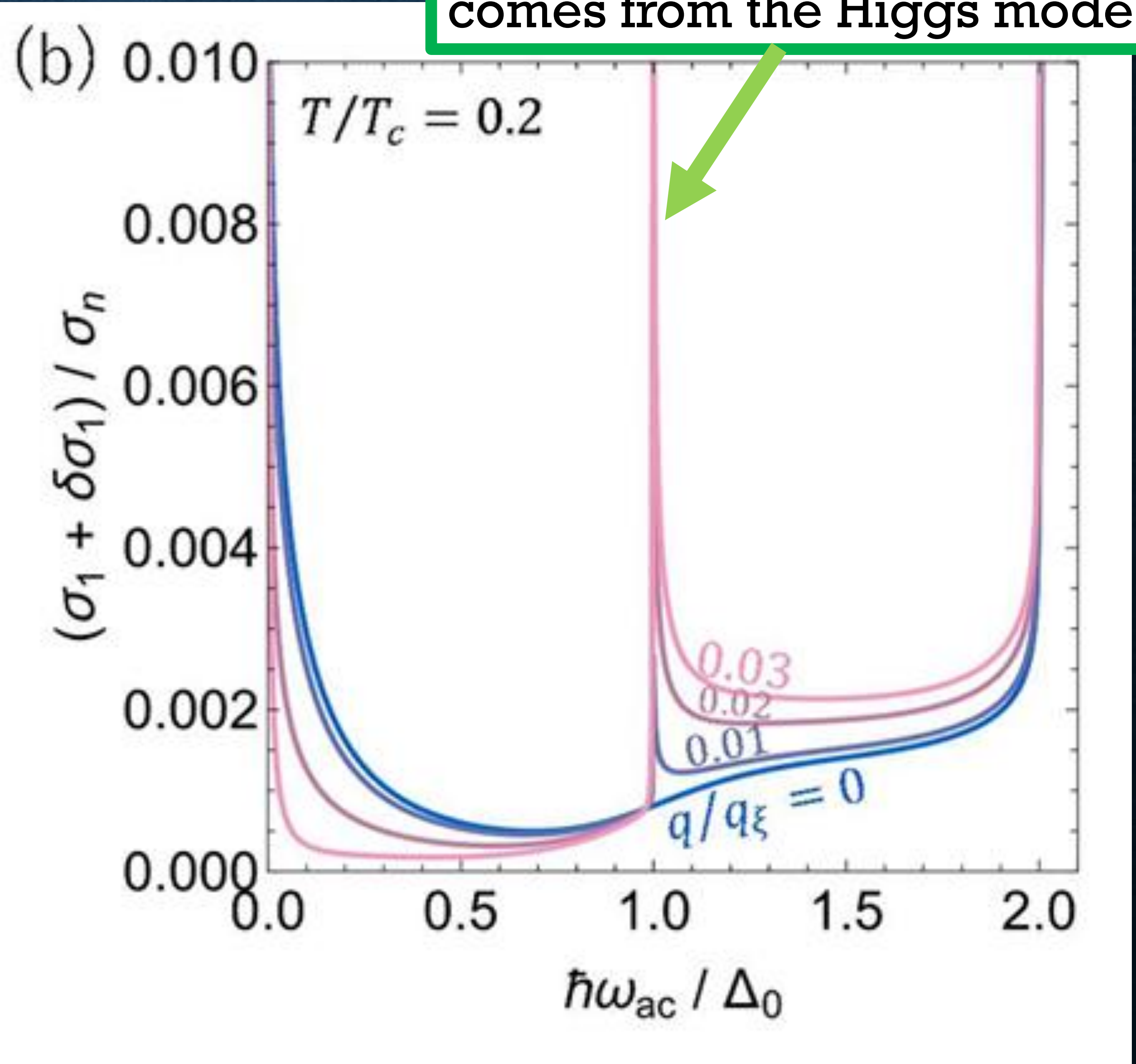
Results



(b)
$T/T_c = 0.2$
0.010
0.008
0.006
0.004
0.002
0.000
0.0
0.5
1.0
1.5
2.0
$(\sigma_1 + \delta\sigma_1) / \sigma_n$
$\hbar\omega_{ac} / \Delta_0$
0.03
0.02
0.01
$q/q_\xi = 0$

Results

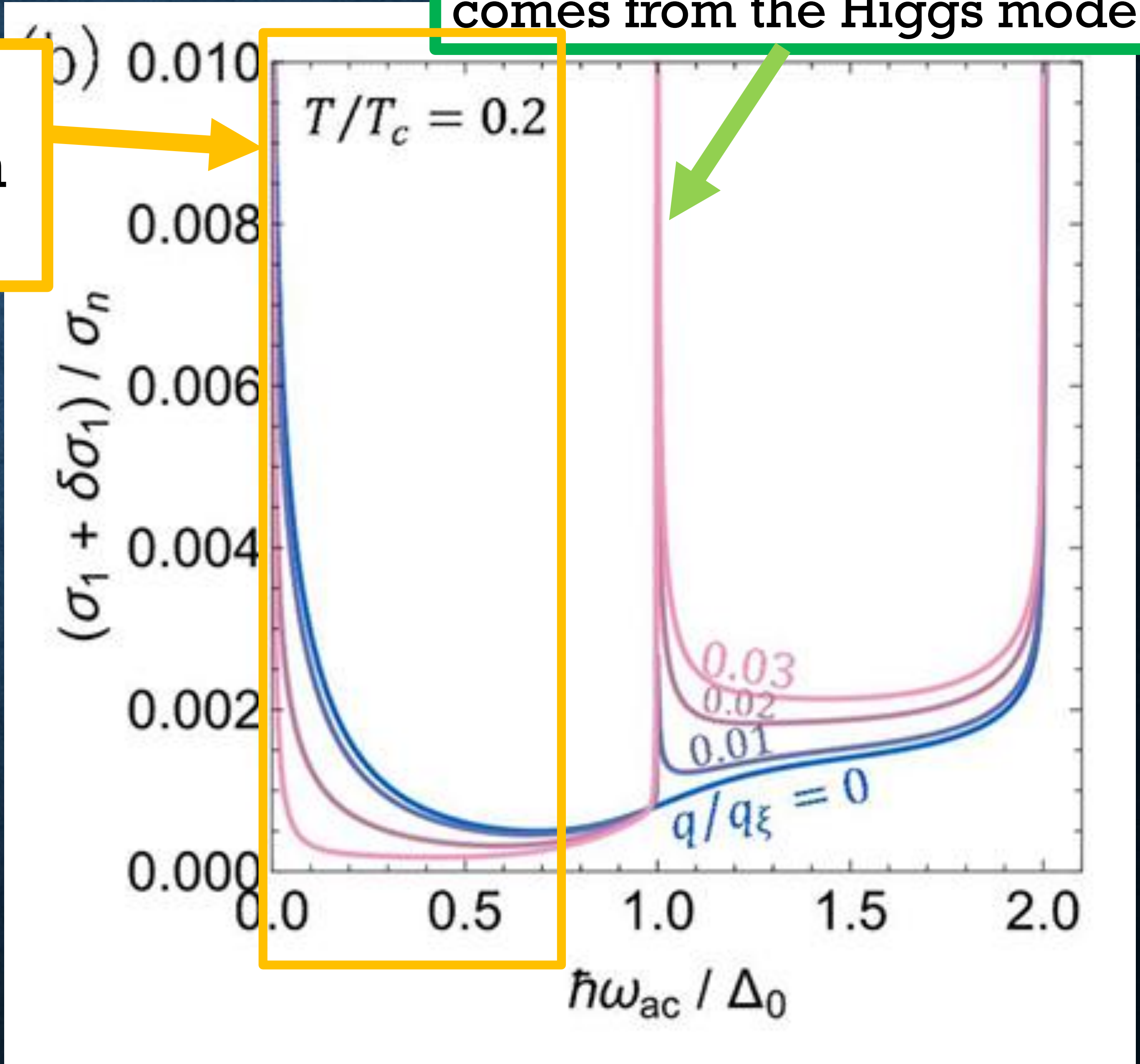
The resonance peak at Δ comes from the Higgs mode



Results

At lower frequencies,
$\sigma_1 + \delta\sigma_1$ is a decreasing function of the ac amplitude (q).

The resonance peak at Δ comes from the Higgs mode



Results

At lower frequencies,
$\sigma_1 + \delta\sigma_1$ is a decreasing function of the ac amplitude (q).

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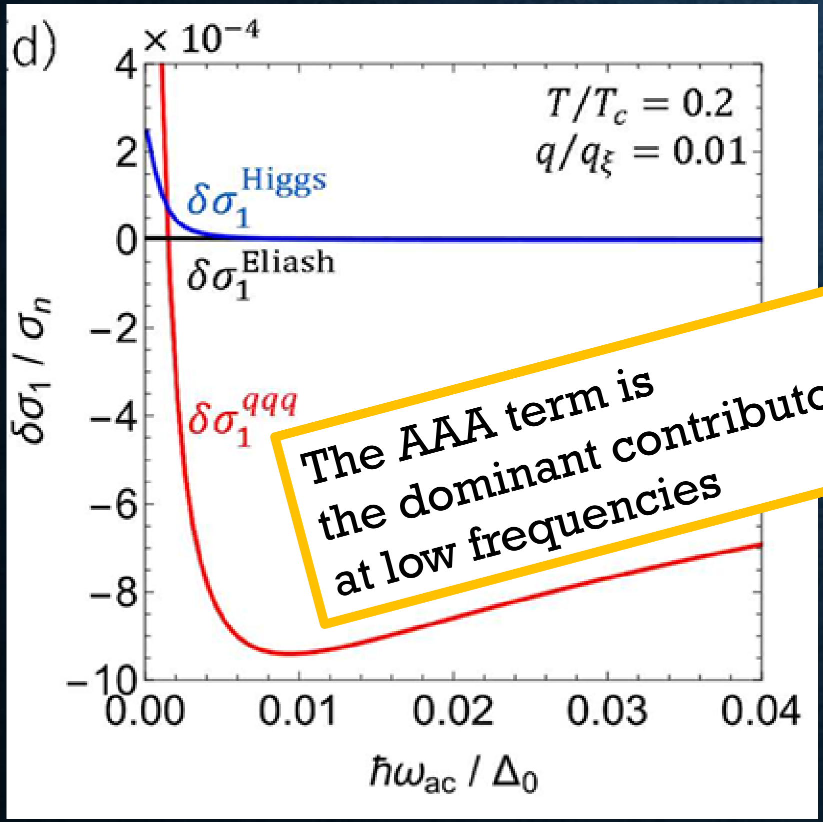
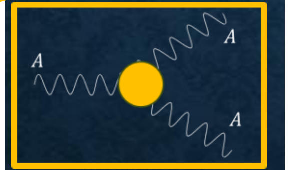
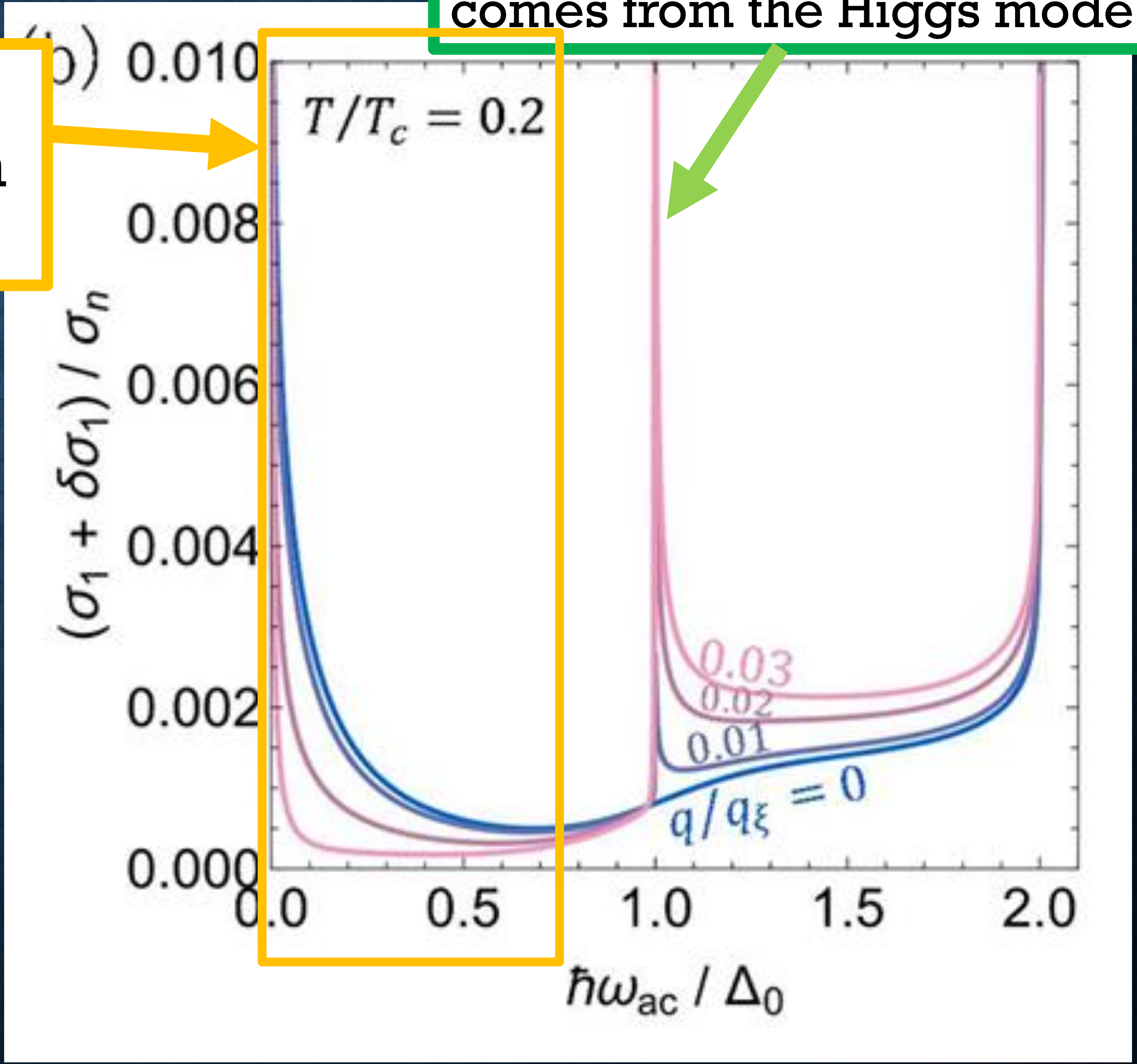
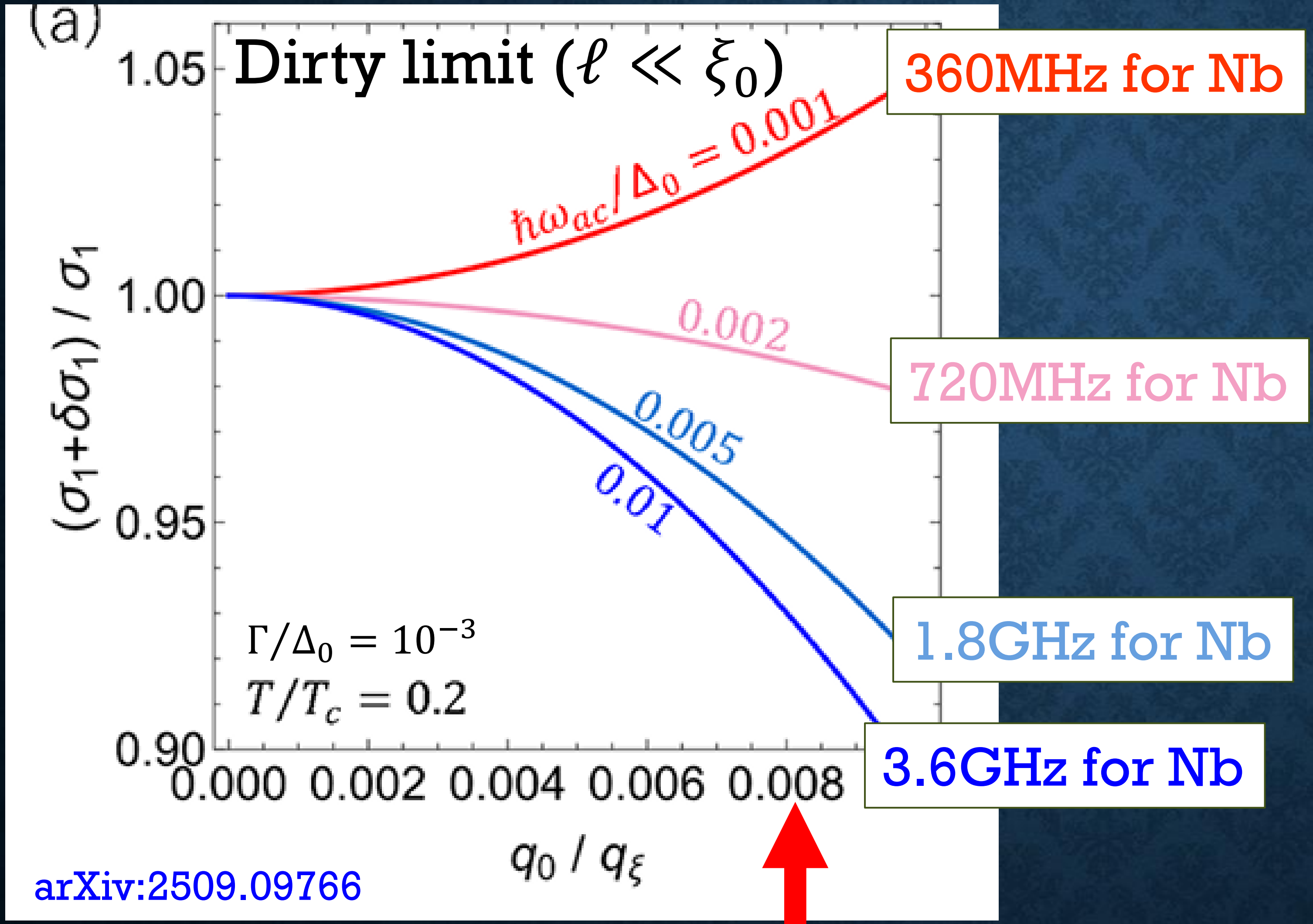


The AAA term is the dominant contributor at low frequencies





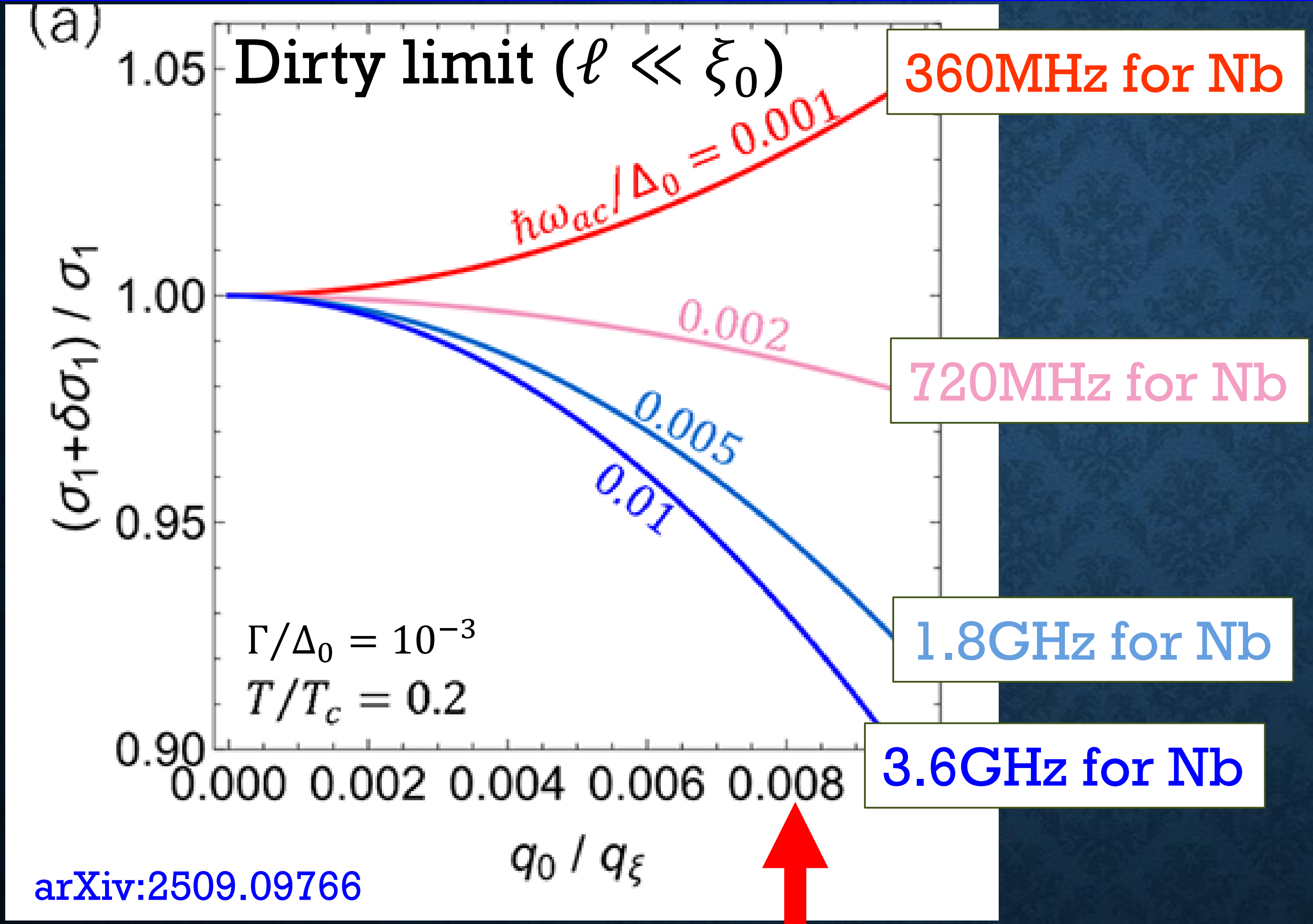
ac amplitude dependent dissipative conductivity



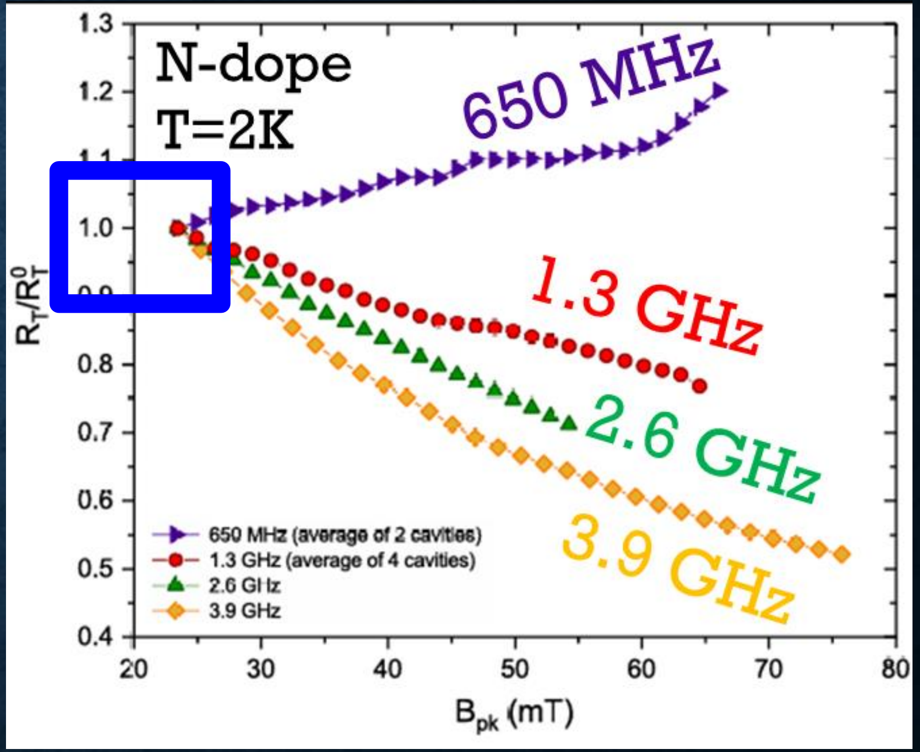
arXiv:2509.09766

Note: this corresponds to **a few (mT)**

ac amplitude dependent dissipative conductivity



Experiments:
Moderately dirty ($\ell \sim \xi_0$)



M. Martinello et al., Phys. Rev. Lett. **121**, 224801 (2018)
DOI: https://doi.org/10.1103/PhysRevLett.121.224801

Note: this corresponds to **a few (mT)**

ac amplitude dependent dissipative conductivity

To deal with **the red-framed region**, the theory must be extended nonperturbatively to capture the quasiparticle spectrum modified by strong AC currents.

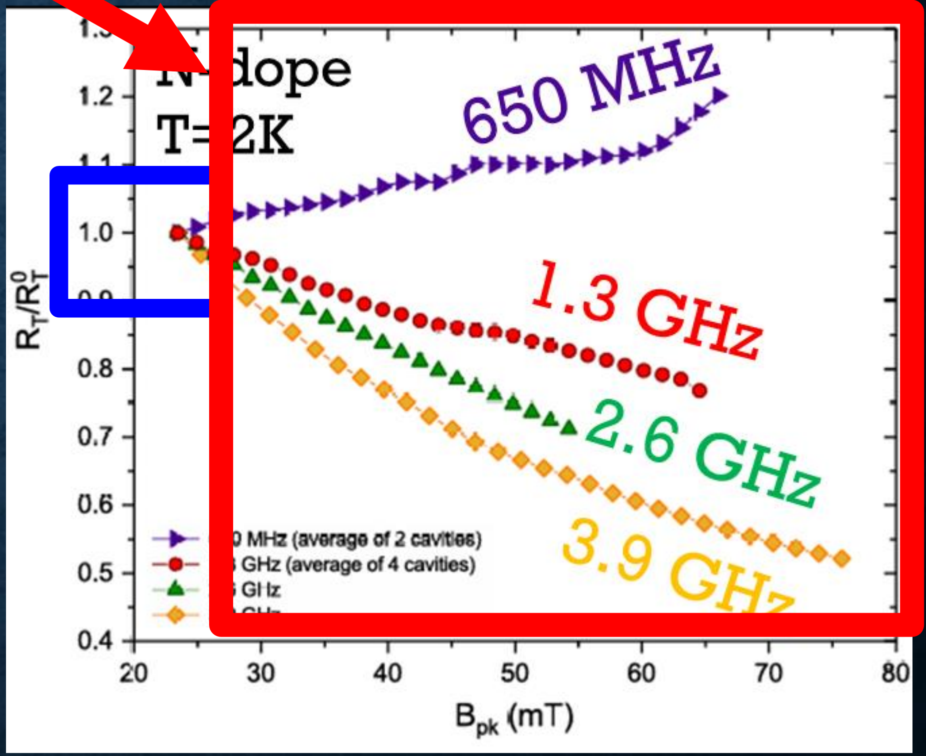
T. Kubo, arXiv:2509.09766 (2025)

Nonperturbative extension like the previous studies:

A. Gurevich, Phys. Rev. Lett. **113**, 087001 (2014)
T. Kubo and A. Gurevich, Phys. Rev. B **100**, 064522 (2019)

The theory of SRF

Experiments:
Moderately dirty ($\ell \sim \xi_0$)

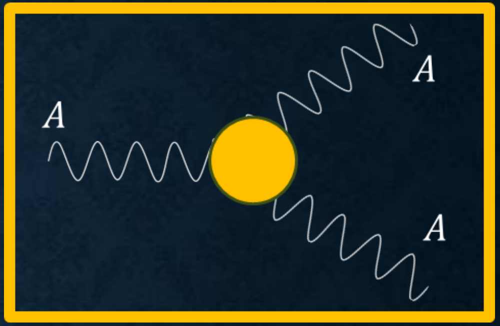


M. Martinello et al., Phys. Rev. Lett. **121**, 224801 (2018)
DOI: https://doi.org/10.1103/PhysRevLett.121.224801

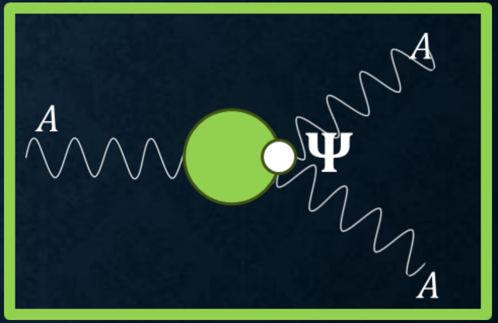
Summary

- The frequency dependent anti-Q slope can be explained by the nonlinear AAA term.
  The Eliashberg effect and the Higgs mode contributions are not significant in SRF regime.

AAA term



Eliashberg and Higgs



- We are only one step away from establishing **the SRF theory**. A nonperturbative extension of the present theory will complete it.

→ **Nonequilibrium Nonlinear Nonperturbative ac response theory**

Acknowledgement

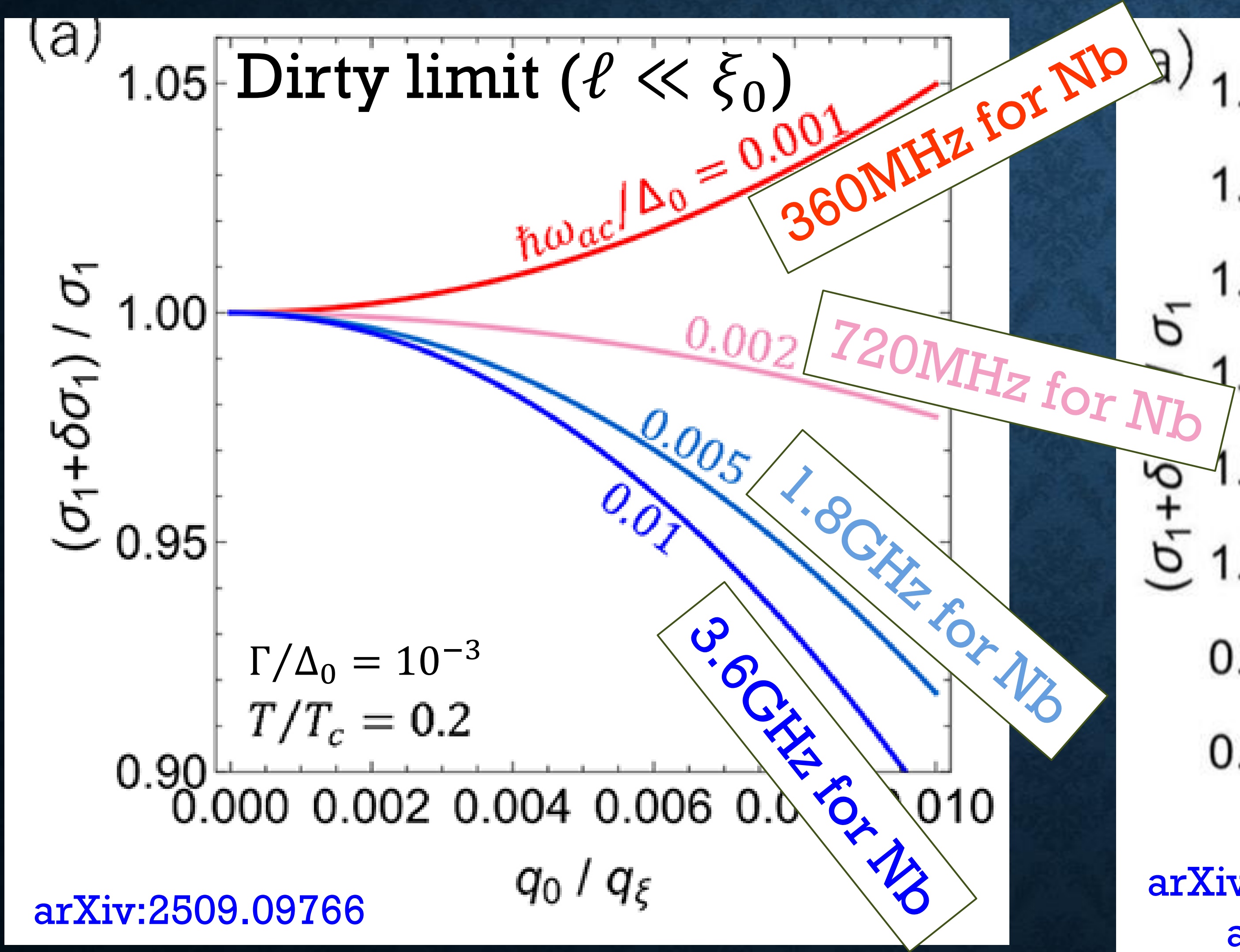
- This work was conducted independently, and I take full responsibility for its content.
- Nevertheless, its foundation stems from my visit to Alex Gurevich at Old Dominion University in 2016 and from my sabbatical there between 2017 and 2019.

I deeply appreciate his warm hospitality and invaluable support during those stays.

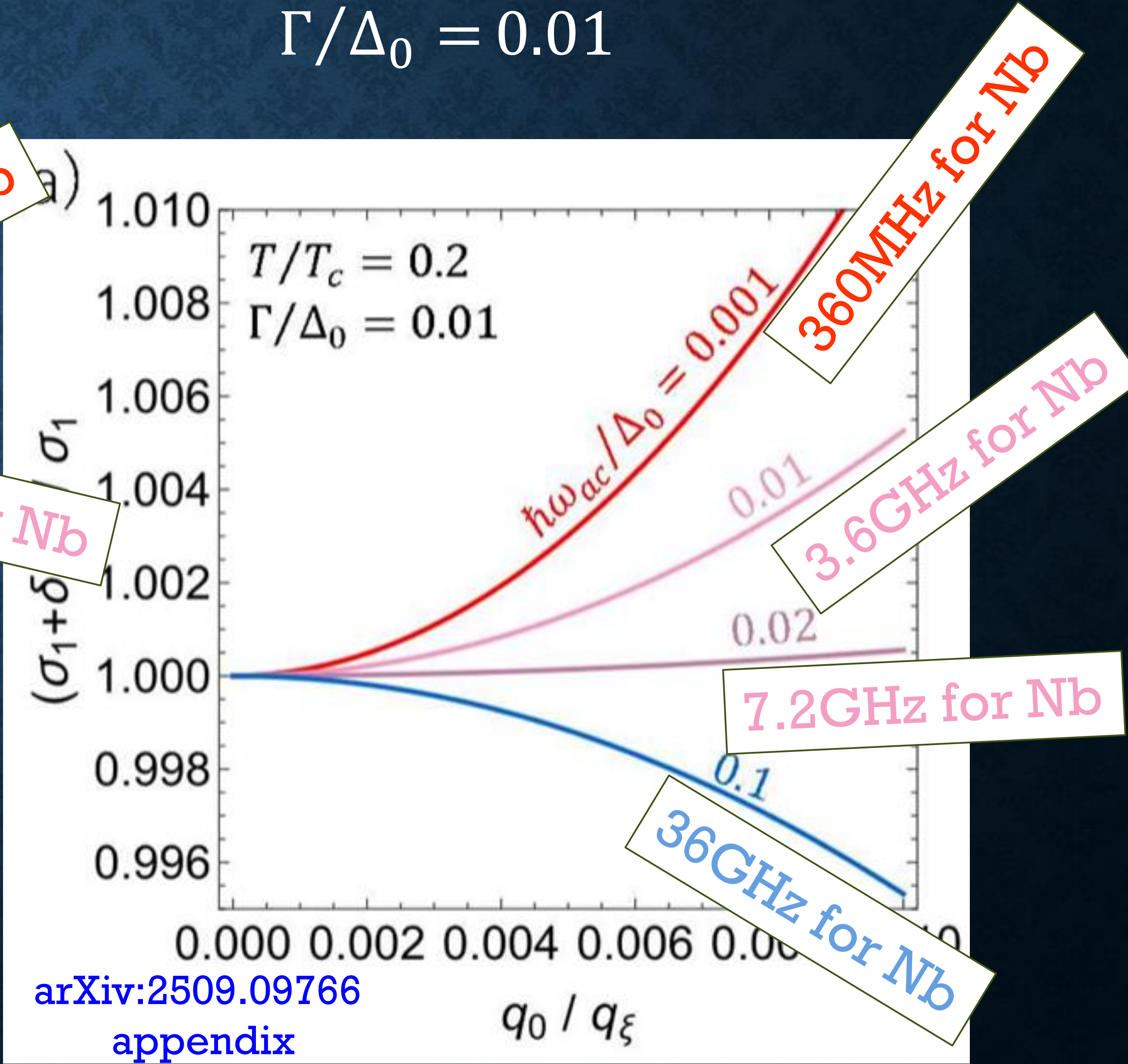
Effects of the inelastic scattering (e.g., electron-phonon) rate

$\Gamma/\Delta_0 = 0.001$

$\Gamma/\Delta_0 = 0.01$



arXiv:2509.09766



arXiv:2509.09766
appendix