多元函数微分法及其应用

多元复合函数的求导法则

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定理 设 $u=u(x,y)$, $v=v(x,y)$ 在点 (x,y) 处有对 x 及对 y 的偏导数，函数 $z=f(u,v)$ 在对应点 (u,v) 处有连续偏导数,则 $z=f[u(x,y),v(x,y)]$ 在点 (x,y) 处的两个偏导数存在，且有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

例1 设 $z=f(x+y,xy)$, 其中 $z=f(u,v)$ 可微，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

（1）设 $z = f(u,v), u = \varphi(x), v = \psi(x)$ 均可微，则

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}.$$

（2）设 $w = f(u), u = \varphi(x, y, z)$ 均可微，

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y}, \quad \frac{\partial w}{\partial z} = \frac{dw}{du}\frac{\partial u}{\partial z}.$$

（3）设 $u = f(x, y, z), z = \varphi(x, y)$ 均可微，

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}.$$

例2 设 $u = f(x, y, z)$ 具有连续的二阶偏导数，$z = x\sin y$，求 $\dfrac{\partial^2 u}{\partial y \partial x}$.

全微分形式的不变性

设函数 $z=f(u,v)$, $u=u(x,y)$ 及 $v=v(x,y)$ 都有连续的一阶偏导数,则复合函数 $z=f[u(x,y),v(x,y)]$ 的全微分

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

例1 设 $z=f(x+y,xy)$, 其中 $z=f(u,v)$ 可微，求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

内容小结

1. 复合函数求导的链式法则

$$z = f(u,v),\ \ u = u(x,y)\ \ v = v(x,y)$$

$$z = f[u(x,y),v(x,y)]$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

2. 全微分形式不变性

$$z = f(u,v),\ \ u = u(x,y)\ \ v = v(x,y)$$

$$z = f[u(x,y),v(x,y)]$$

$$\mathrm{d}\,z = \frac{\partial z}{\partial x}\mathrm{d}\,x + \frac{\partial z}{\partial y}\mathrm{d}\,y = \frac{\partial z}{\partial u}\mathrm{d}\,u + \frac{\partial z}{\partial v}\mathrm{d}\,v.$$

作业

P84　　2 ; 4; 8 ; 9; 12(1)(4);