第四章　不定积分

第一节　换元定积分法

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一、第一类换元法

定理1 若 $\displaystyle\int f(u)\,\mathrm{d}u = F(u) + C$

则 $\displaystyle\int f[\varphi(x)]\varphi'(x)\,\mathrm{d}x = \int f[\varphi(x)]\,\mathrm{d}\varphi(x) = F[\varphi(x)] + C$

例1 $\displaystyle\int (1 + 3x)^{100}\,dx$

例2 $\displaystyle\int \frac{dx}{a^2 + x^2}$

例3 $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$

例4 $\displaystyle\int \frac{x}{\sqrt{1 + x^2}}\,dx$

例5 $\displaystyle\int \frac{dx}{\sqrt{1 - x^2 - 2x}}$

例6 $\displaystyle\int \tan x\, dx$

例7 $\displaystyle\int \cos^2 x\, dx$

例8 $\displaystyle\int \cos^3 x\, dx$

例9 $\displaystyle\int \frac{dx}{x\sqrt{1+\ln x}}$

例10 $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$

例11 $\displaystyle\int \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}\, dx$

例12 $\displaystyle\int \frac{\arctan x}{1+x^2}\, dx$

例13 $\displaystyle\int \frac{\sin 2x}{1+\sin^4 x}dx$

例14 $\displaystyle\int \frac{1}{\sin x}dx$

二、第二类换元法

定理2 设 $x = \varphi(t)$ 是单调的、可导的函数，并且 $\varphi'(t) \neq 0$

$$\int f[\varphi(t)]\varphi'(t)\,\mathrm{d}\,t = F(t) + C,$$

则 $\displaystyle\int f(x)\,\mathrm{d}\,x = \int f[\varphi(t)]\varphi'(t)\,\mathrm{d}\,t = F(t) + C = F[\varphi^{-1}(x)] + C,$

例1 $\displaystyle\int \sqrt{a^2 - x^2}\,dx \quad (a > 0)$

例2 $\displaystyle \int \frac{dx}{\sqrt{a^2 + x^2}}$ $(a > 0)$

例3 $\displaystyle \int \frac{dx}{\sqrt{x^2 - a^2}}$ $(a > 0)$

14) $\int \tan x\, dx = -\ln|\cos x| + C.$

15) $\int \cot x\, dx = \ln|\sin x| + C.$

16) $\int \sec x\, dx = \ln|\sec x + \tan x| + C.$

17) $\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$

18) $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a} + C$

19) $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right| + C.$

20) $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin\dfrac{x}{a} + C$

21) $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C$

22) $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C$

例4 $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$

例5 $\displaystyle\int \frac{dx}{\sqrt{x^2-x-1}}$

例6 $\displaystyle\int \frac{dx}{\sqrt{1+x-x^2}}$

内容小结

1. 第一类换元（凑微分）

定理1 若 $\displaystyle\int f(u)\,\mathbf{d}\,u = F(u) + C$

则 $\displaystyle\int f[\varphi(x)]\varphi'(x)\,\mathbf{d}\,x = \int f[\varphi(x)]\,\mathbf{d}\,\varphi(x) = F[\varphi(x)] + C$

2. 第二类换元法

$$\int f(x)\,\mathbf{d}\,x = \int f[\varphi(t)]\varphi'(t)\,\mathbf{d}\,t = F(t) + C = F[\varphi^{-1}(x)] + C,$$

1) $\sqrt{a^2 - x^2}$ $\qquad x = a\sin t\,(a\cos t)$

2) $\sqrt{a^2 + x^2}$ $\qquad x = a\tan t$

3) $\sqrt{x^2 - a^2}$ $\qquad x = a\sec t$

作业 P207：2（双号小题）；