Geometric Algorithms – Beyond Theory

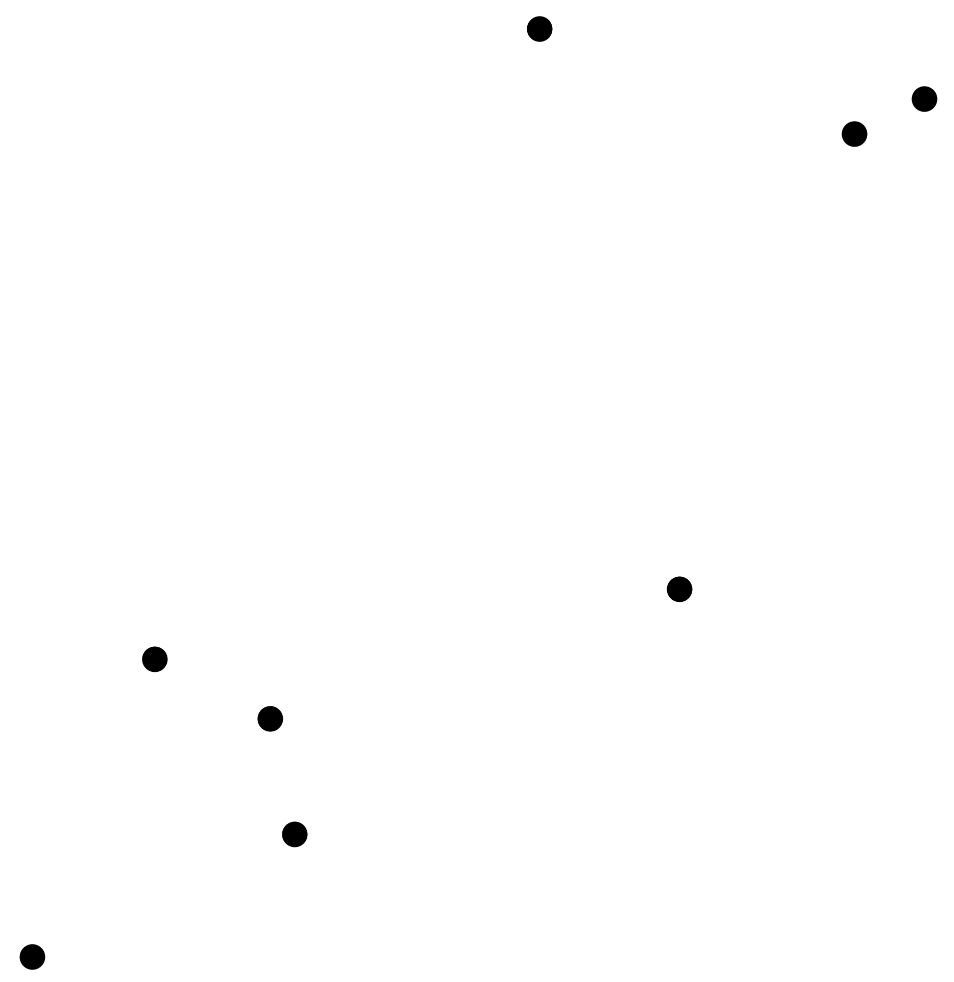
Algorithm Engineering

Robustness of Geometric Algorithms

Partial randomization

Computing convex hulls

We know how to compute convex hulls



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Algorithm INCREMENTALCONVEXHULL(S)

$CH \leftarrow p_1, p_2, p_3$
$S \leftarrow S \backslash \{p_1, p_2, p_3\}$
for all $r \in S$ **do**
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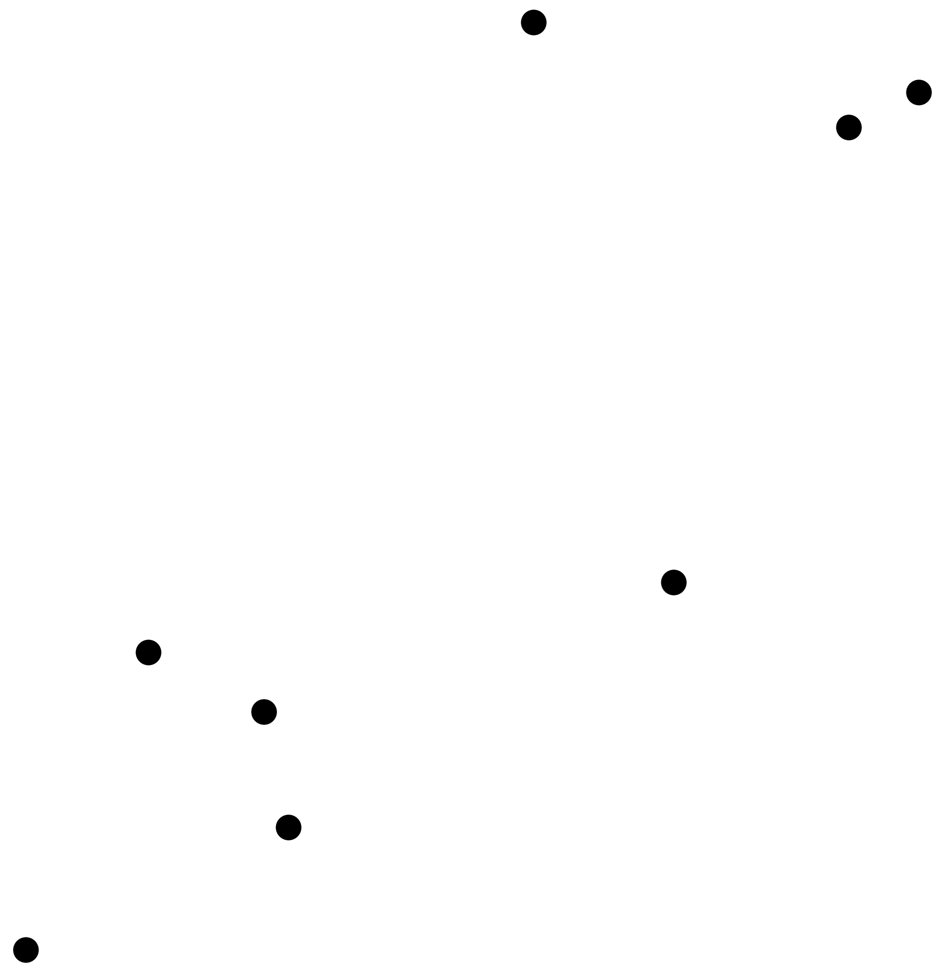

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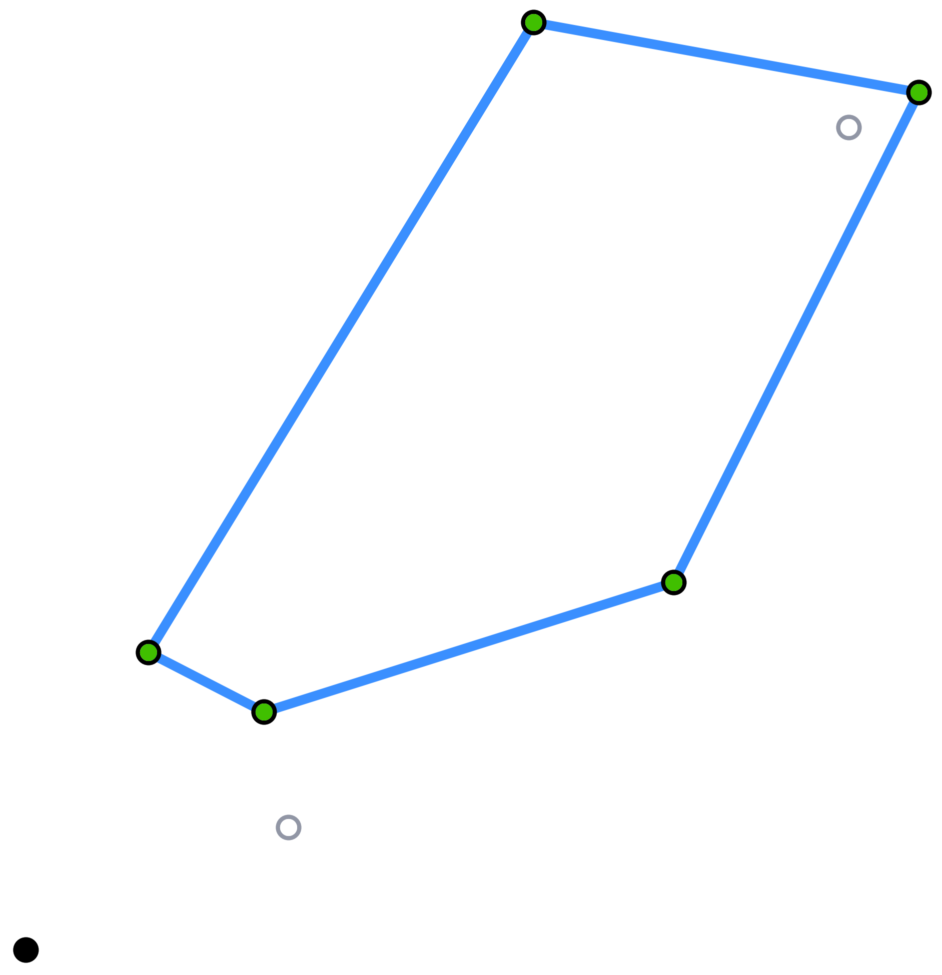

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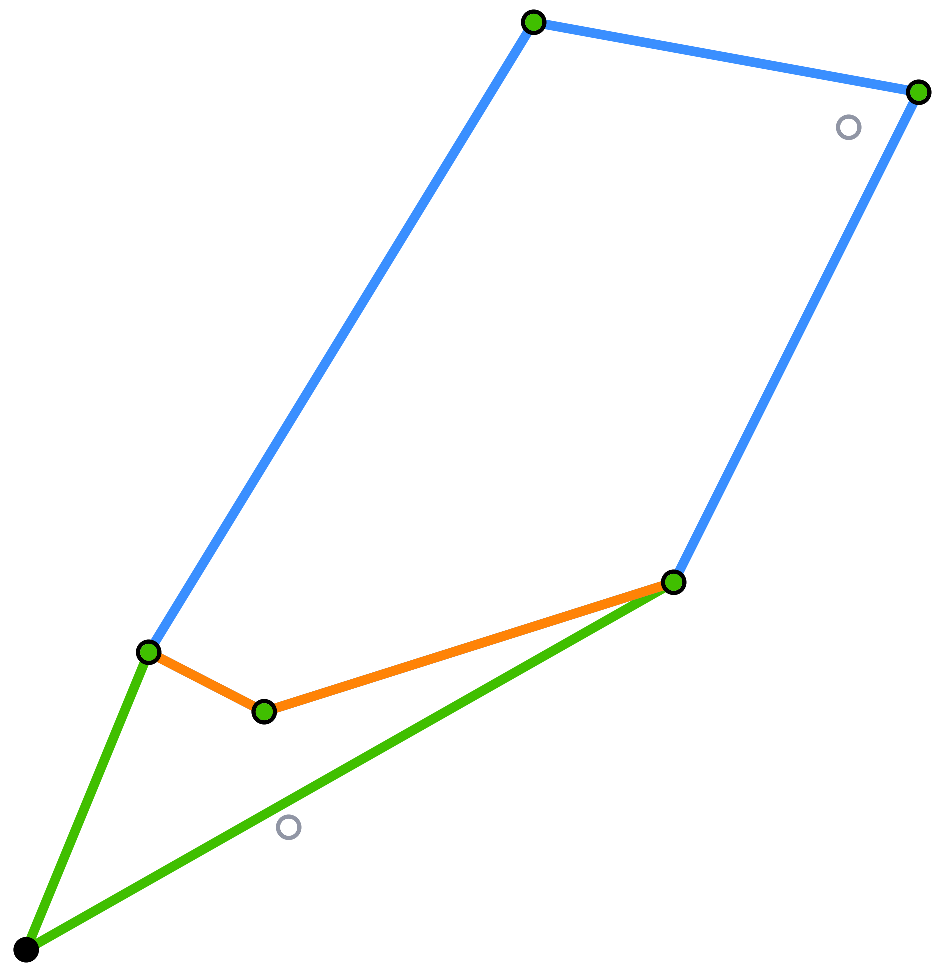

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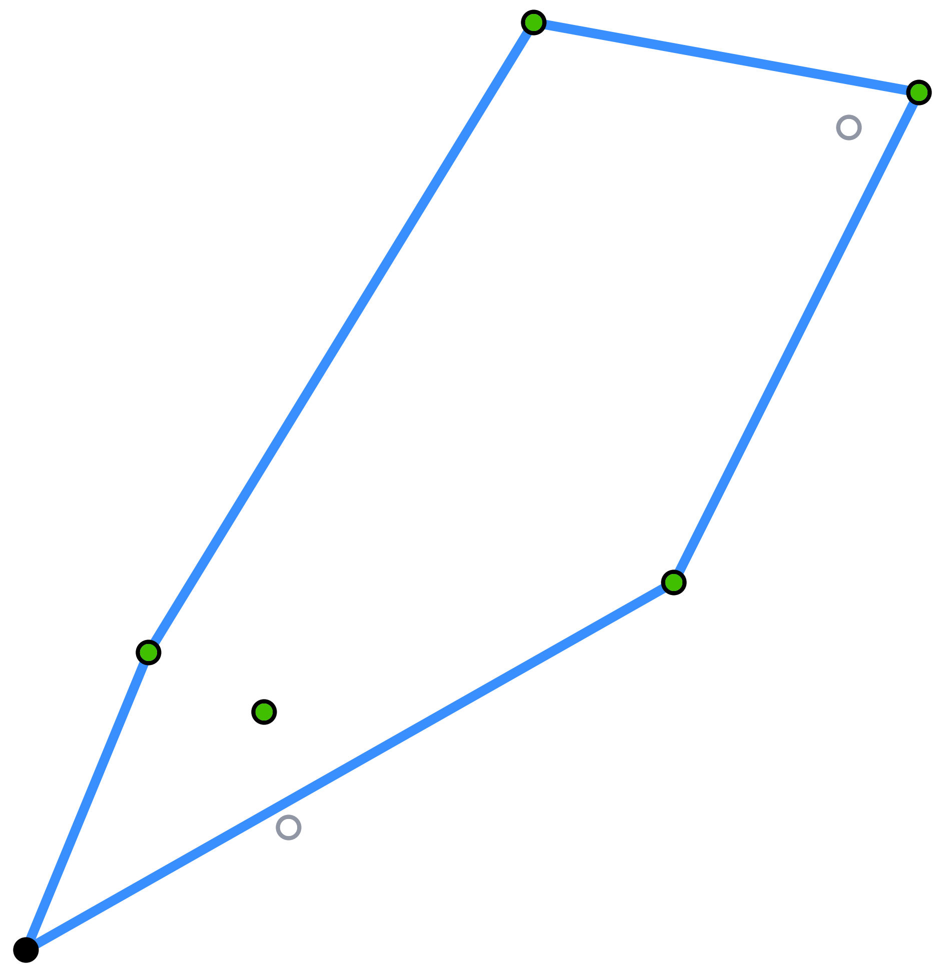

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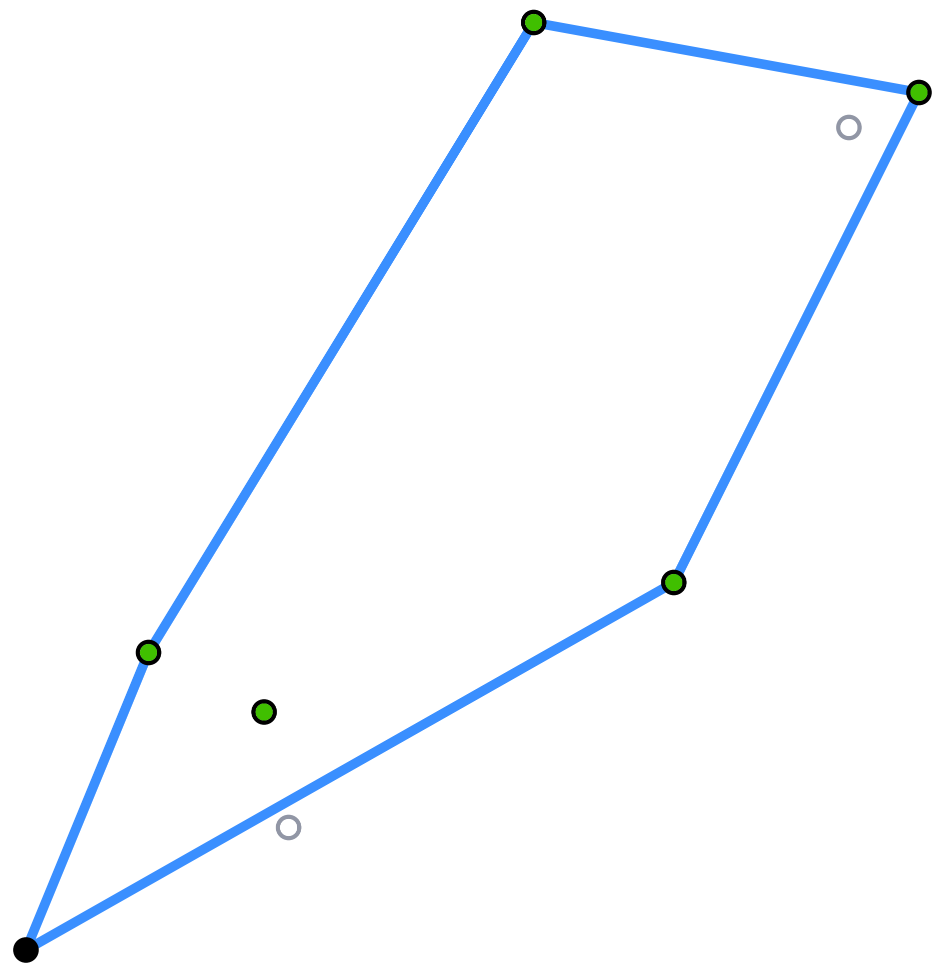

Question: Is this algorithm correct?

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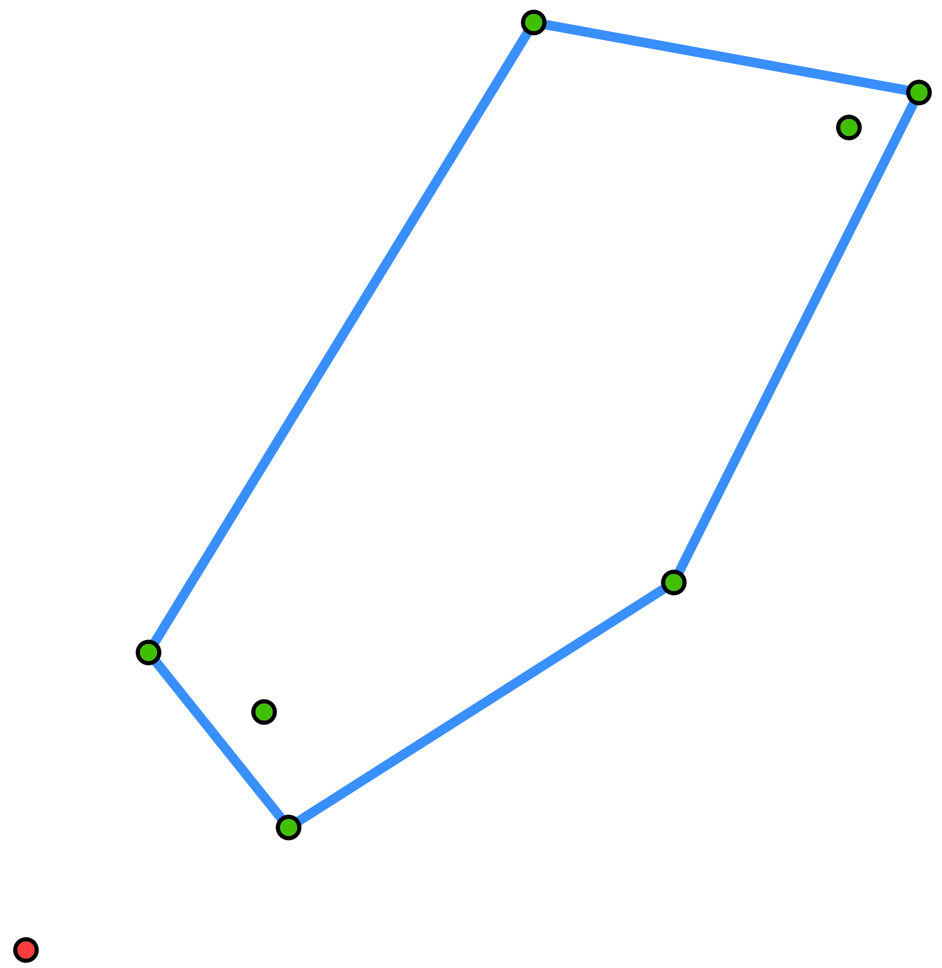
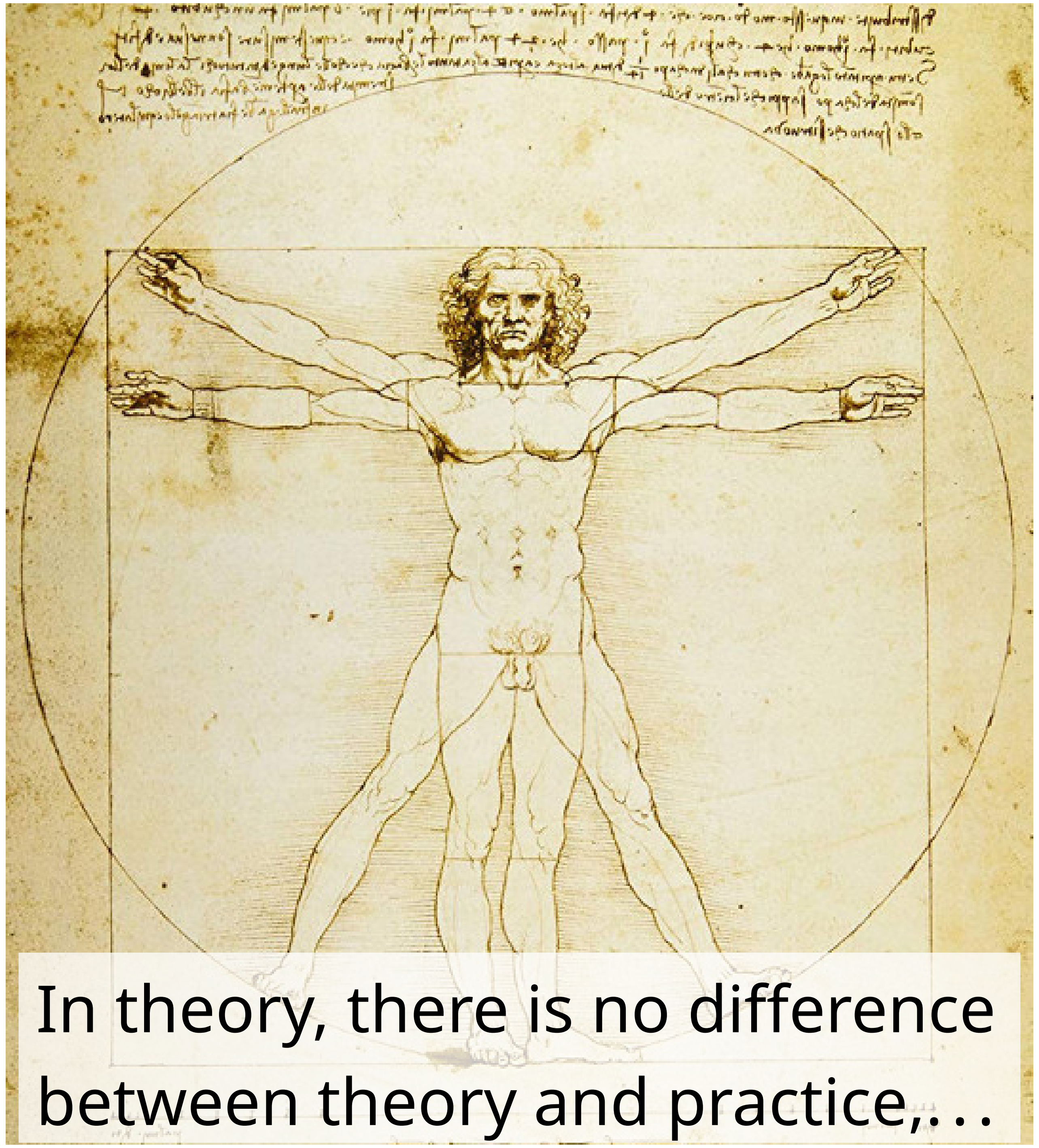


Question: Is this algorithm correct?

This algorithm can sometimes fail!

Why engineering algorithms?

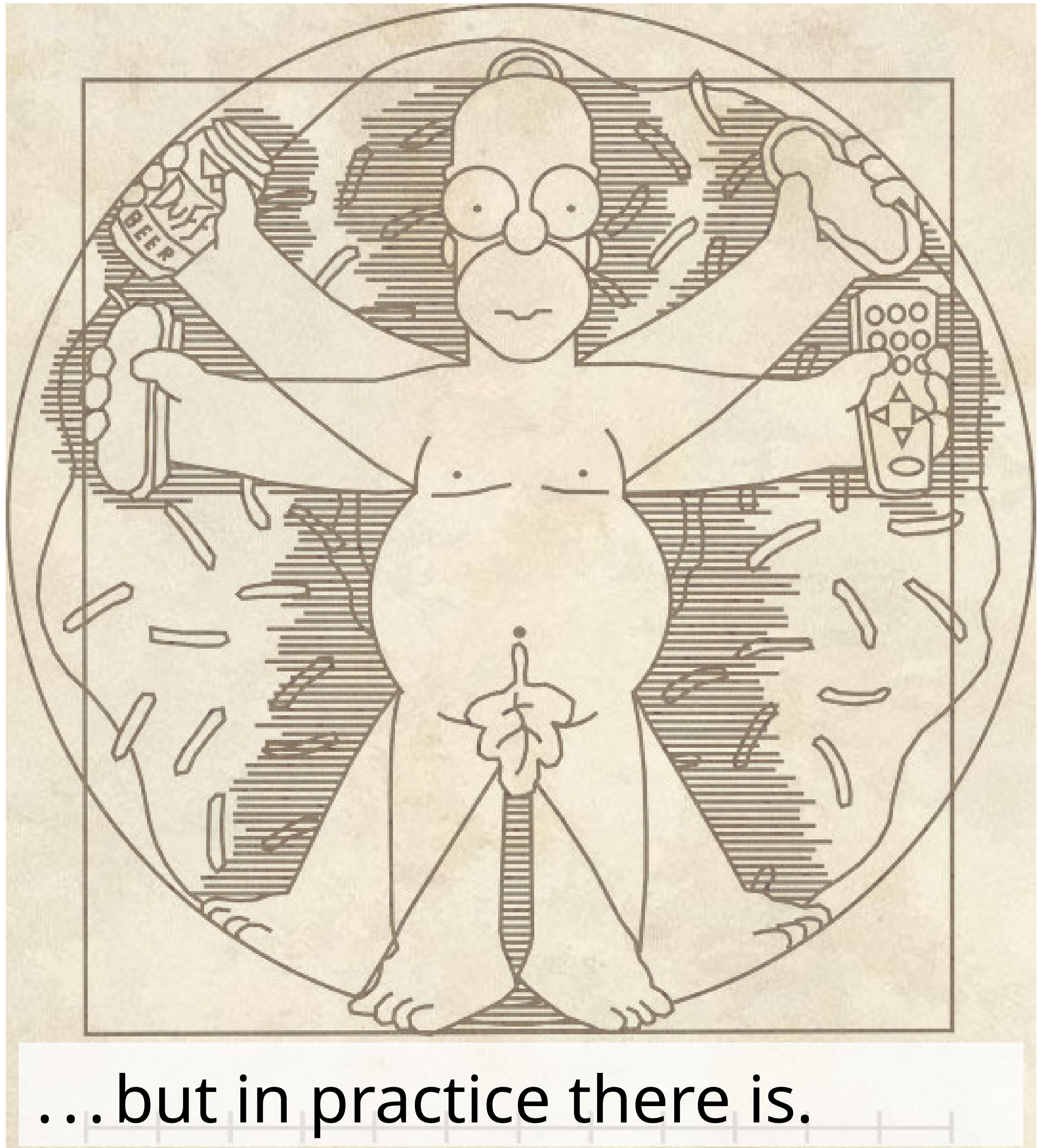
Theory



In theory, there is no difference between theory and practice,. . .

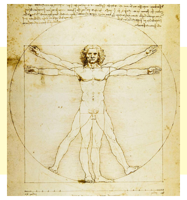
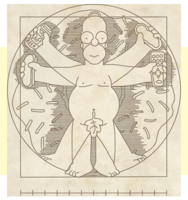
Why engineering algorithms?

The real world out there . . .



. . . but in practice there is.

Why engineering algorithms?

Theory 	Practice 
• number types: $\mathbb{N}, \mathbb{R}$	• number types: int, float, double
• only asymptotics matter	• seconds do matter
• abstract algorithm description, often assuming general position	• non-trivial implementation decisions, error-prone
• unbounded memory, unit access cost	• memory hierarchy / bandwidth
• elementary operations take constant time	• instruction pipelining, . . .

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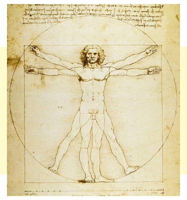
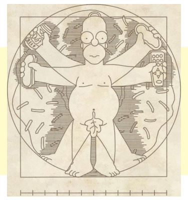
Theory 	Practice 
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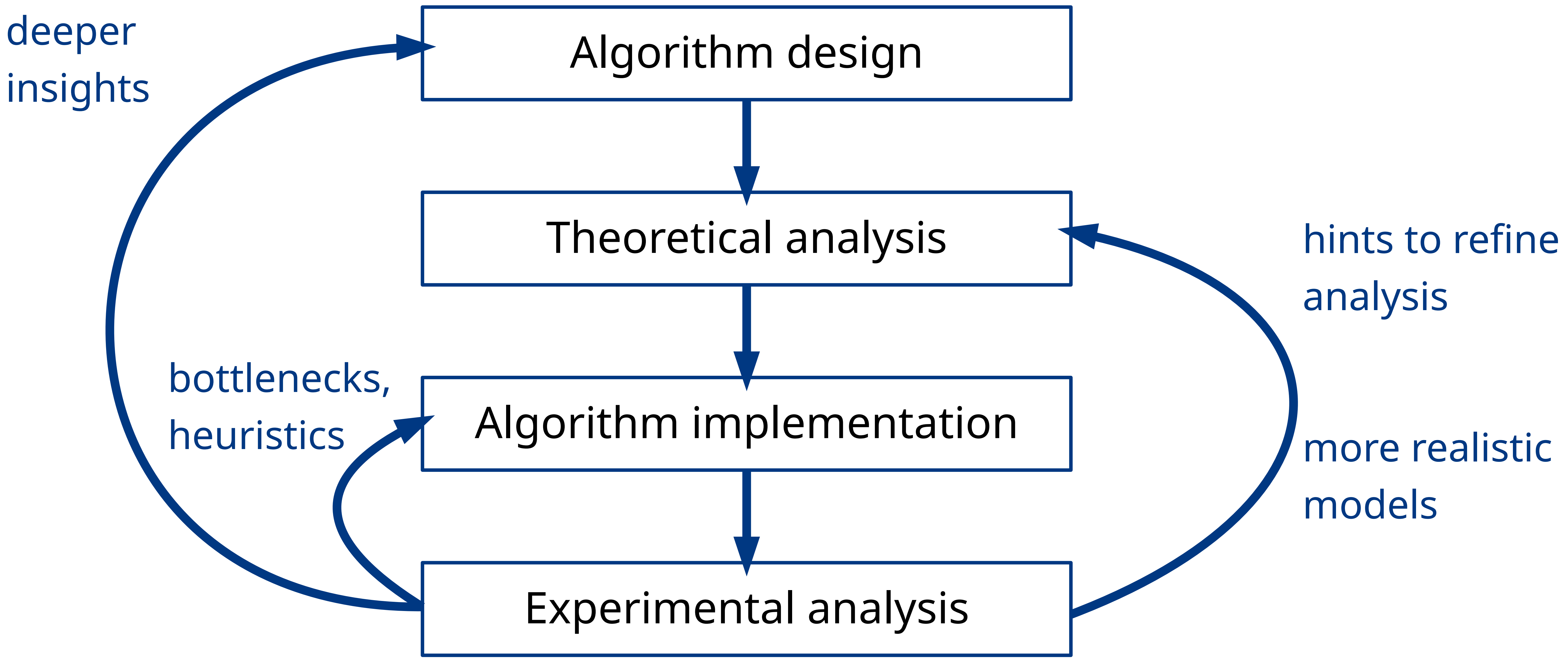
Algorithm engineering cycle



Investigating IncrementalConvexHull

Algorithm IncrementalConvexHull(S)

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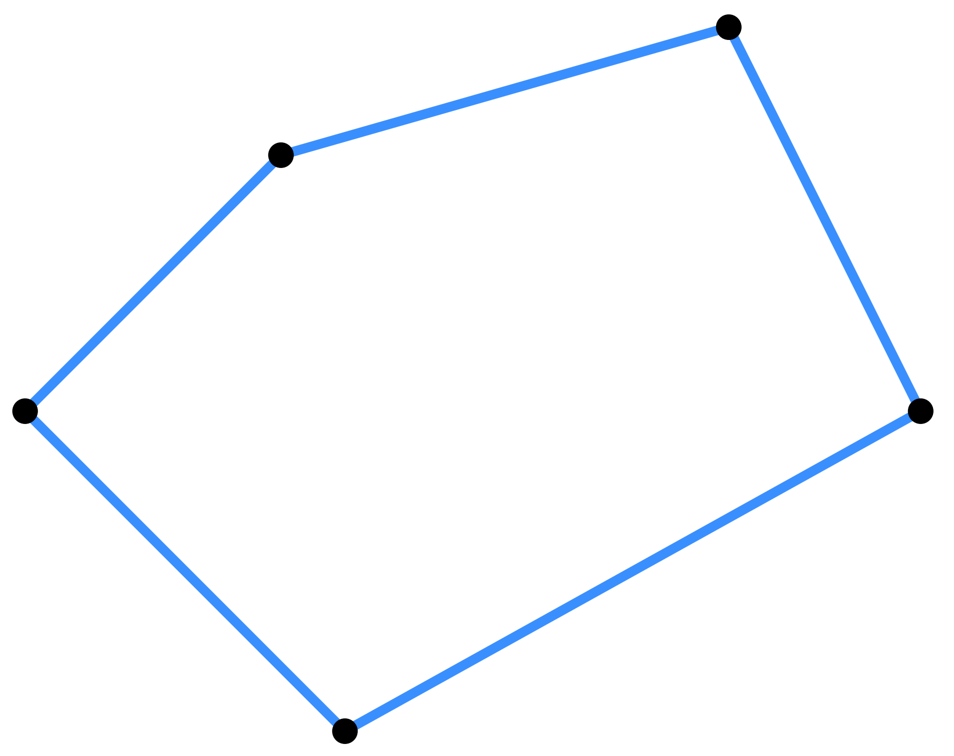

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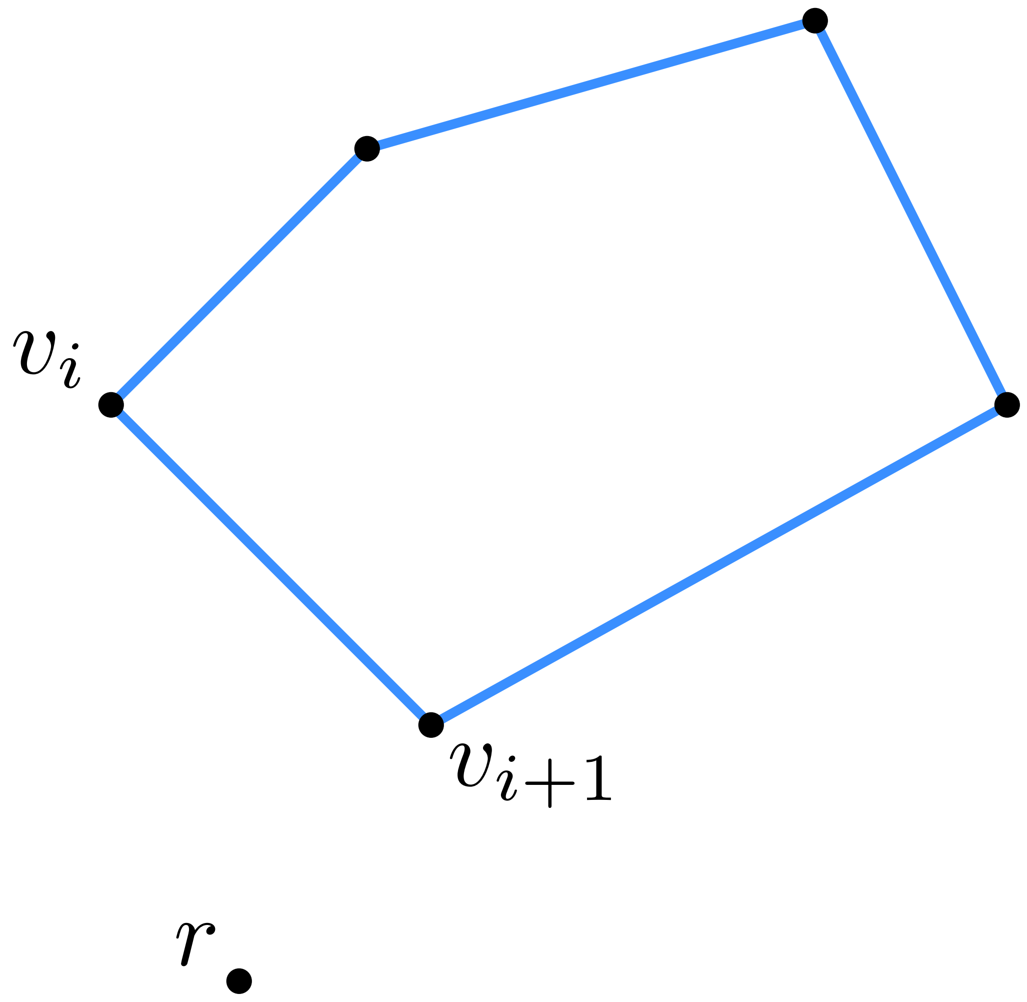

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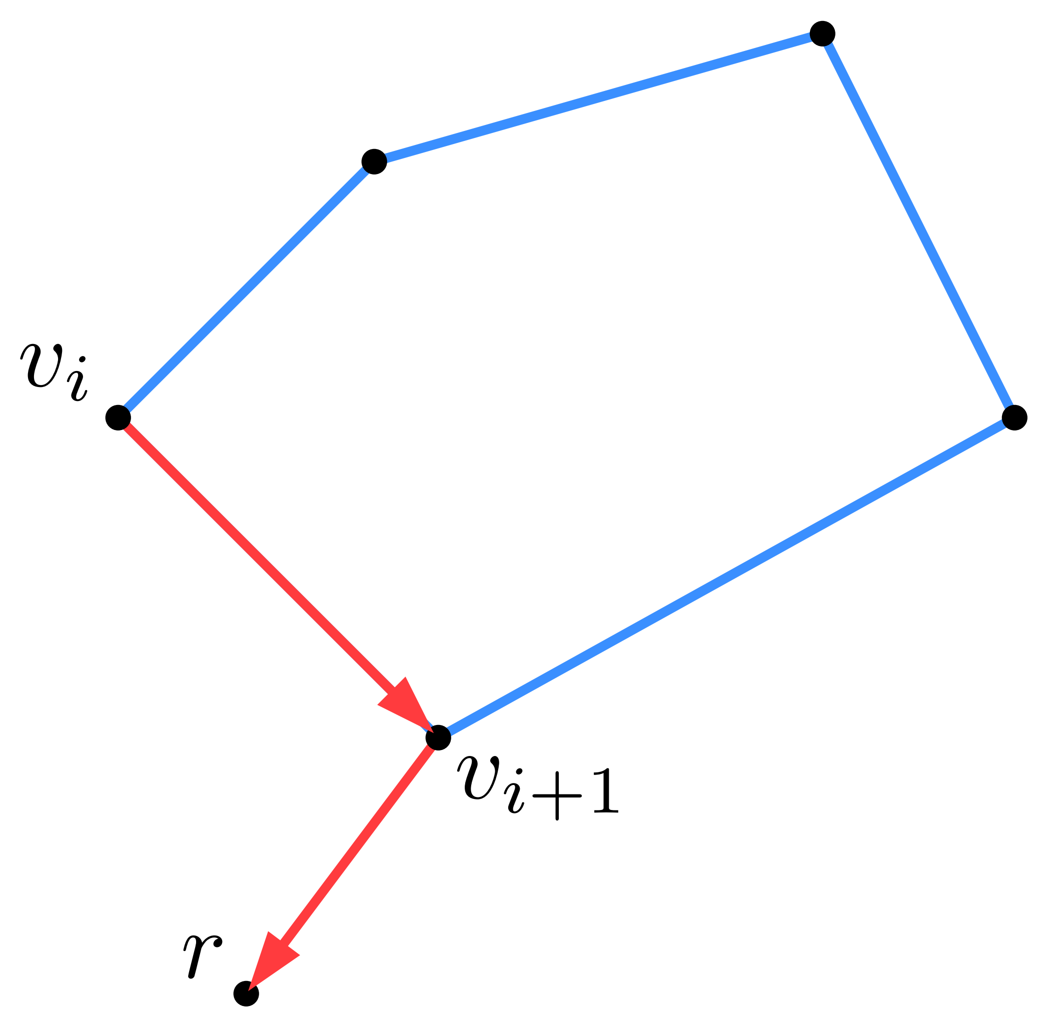
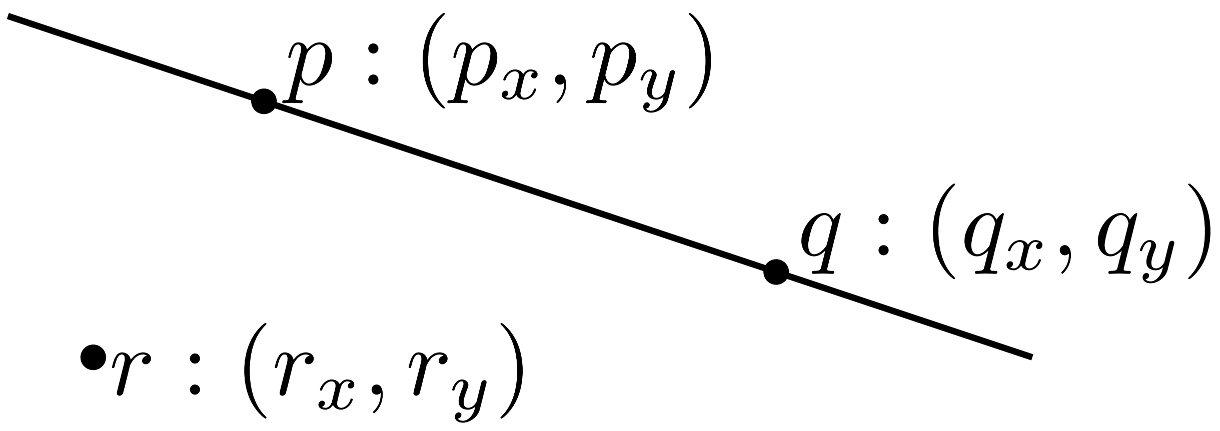


Question: when does r see an edge $\overline{v_i v_{i+1}}$ of CH?

r sees $\overline{v_i v_{i+1}}$ $\Leftrightarrow$ points (v_i, v_{i+1}, r) make a right turn

Orientation

$$orientation(p, q, r) = \begin{cases} +1 & \text{, when } r \text{ lies to the left of } \vec{pq} \\ -1 & \text{, when } r \text{ lies to the right of } \vec{pq} \\ 0 & \text{, when } p, q, \text{ and } r \text{ are collinear} \end{cases}$$



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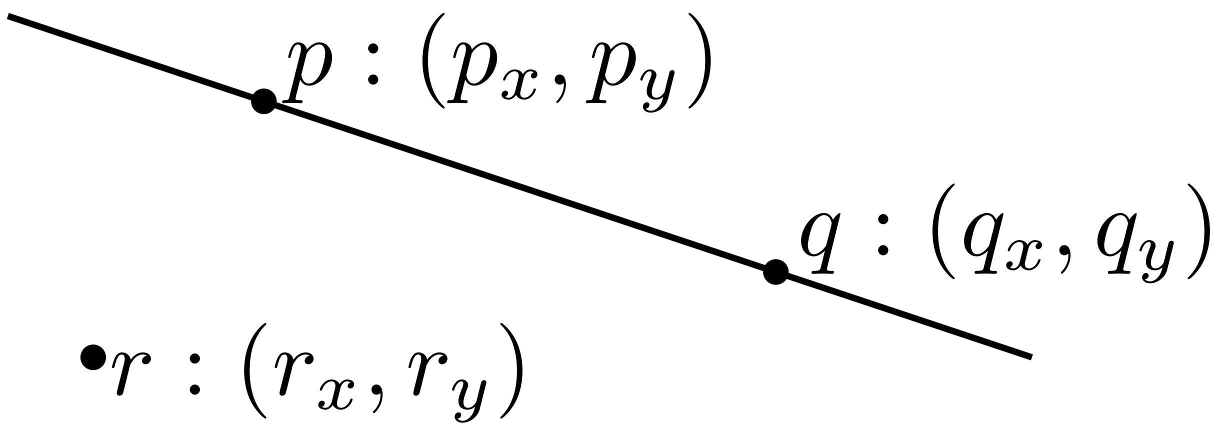
$$= sign\left(\text{det}\begin{bmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{bmatrix}\right)$$



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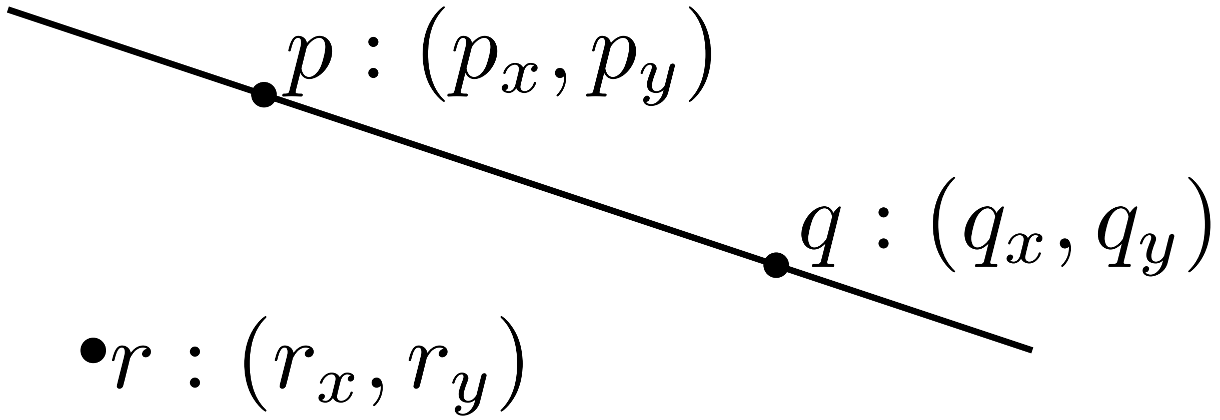
$$= sign\left(\det \begin{bmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{bmatrix}\right)$$

$$= sign\left((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x)\right)$$



Floating point numbers

Machine floating-point numbers:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline s & e_1 & e_2 & \ldots & \ldots & e_j & m_1 & m_2 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & m_k \\ \hline \end{array}$$

$$s\ \underbrace{e_1\ e_2\ \ldots\ \ldots\ e_j}_{e}\ \underbrace{m_1\ m_2\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ m_k}_{m}$$

Normalized:	$(-1)^s \cdot 1.m \cdot 2^{e-(2^{j-1}-1)}$	($e \neq 00\ldots0$, $e \neq 11\ldots1$)
Denormalized:	$(-1)^s \cdot 0.m \cdot 2^{-(2^{j-1}-2)}$	($e = 00\ldots0$)
Special numbers:	infinity and NaN	($e = 11\ldots1$)

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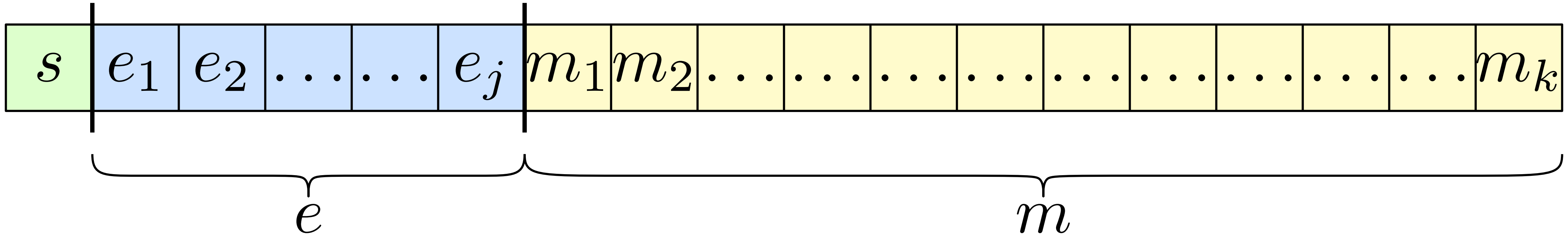

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	Normalized	Denormalized
`float`	$\pm 1.m_1 m_2 \ldots m_{23} \cdot 2^{e-127}$	$\pm 0.m_1 m_2 \ldots m_{23} \cdot 2^{-126}$
`double`	$\pm 1.m_1 m_2 \ldots m_{52} \cdot 2^{e-1023}$	$\pm 0.m_1 m_2 \ldots m_{52} \cdot 2^{-1022}$

Floating point numbers

Consider number type:

$$\texttt{very short float} = \begin{cases} \pm 1.m_1m_2m_3 \cdot 2^{e-3}, & \text{if } 0 < e < 7 \\ \pm 0.m_1m_2m_3 \cdot 2^{-2}, & \text{if } e = 0 \end{cases}$$

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Question: What is the smallest strictly positive normalized number?

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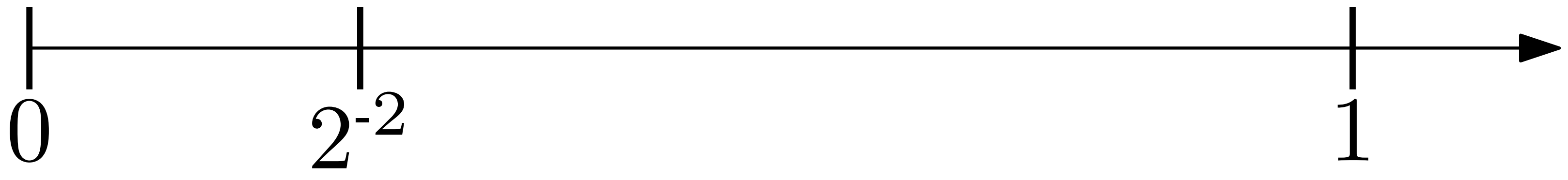
- $1.000_2 \cdot 2^{-2} = 0.25$ smallest strictly positive normalized number



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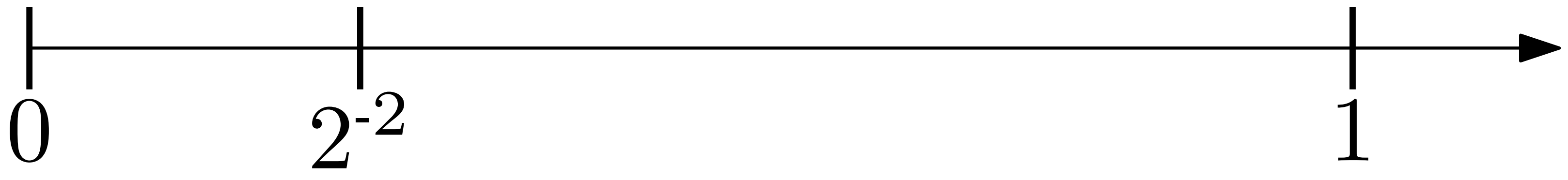
Question: What is the next smallest normalized number?



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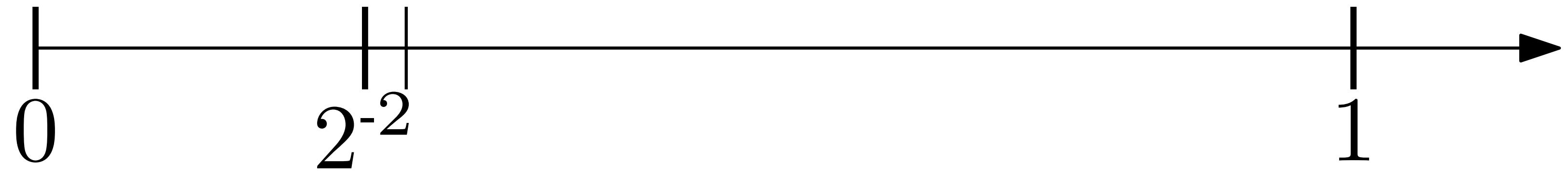
- $1.001_2 \cdot 2^{-2} = 0.28125$ is the next smallest normalized number



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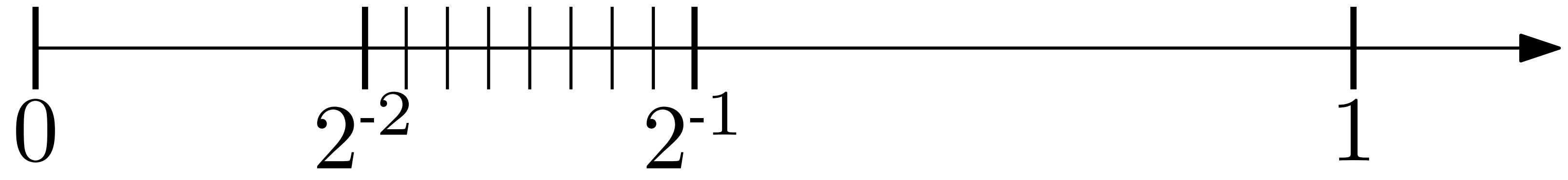

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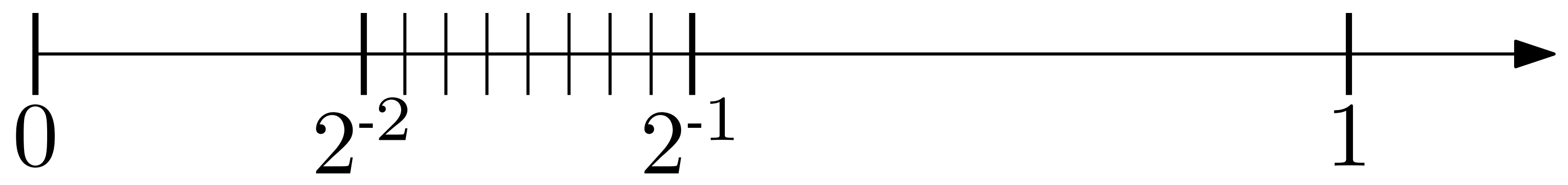

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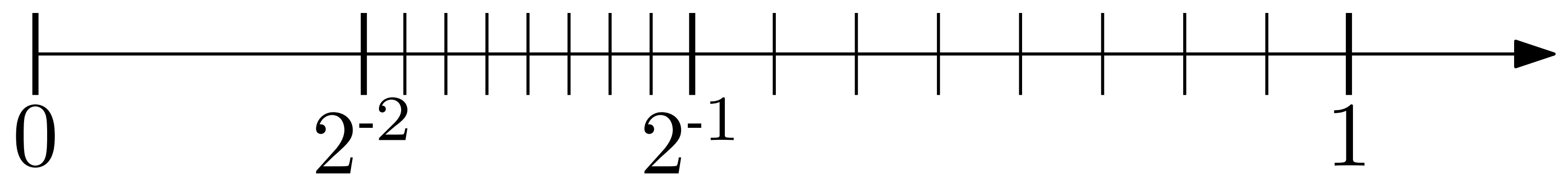

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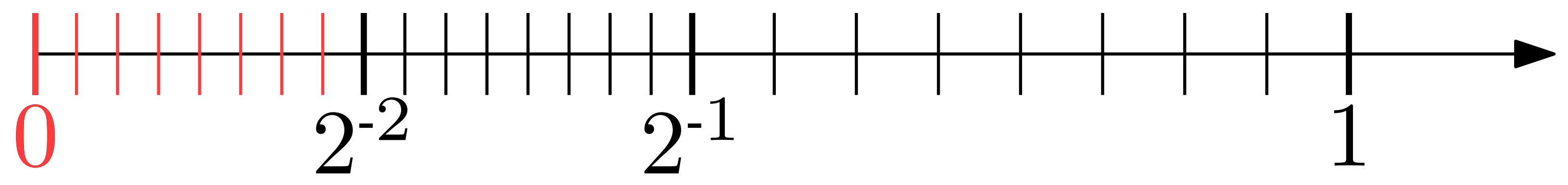
- denormalized numbers lie in $(-2^{-2}, 2^{-2})$ with increment 2^{-5}



Quiz

Consider number type:

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What is $4 + 0.25$?

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A: 4

B: 4.25

C: 4.3

Quiz

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A: 4

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Orientation in doubles

Consider three points p, q, r

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Plot $\mathit{float\_orient}(p', q, r)$

- $p' = p + (u\Delta x, v\Delta y)$,
  whereΔx and Δy are increments between adjacent doubles
- u and v are in $[0, 255]$

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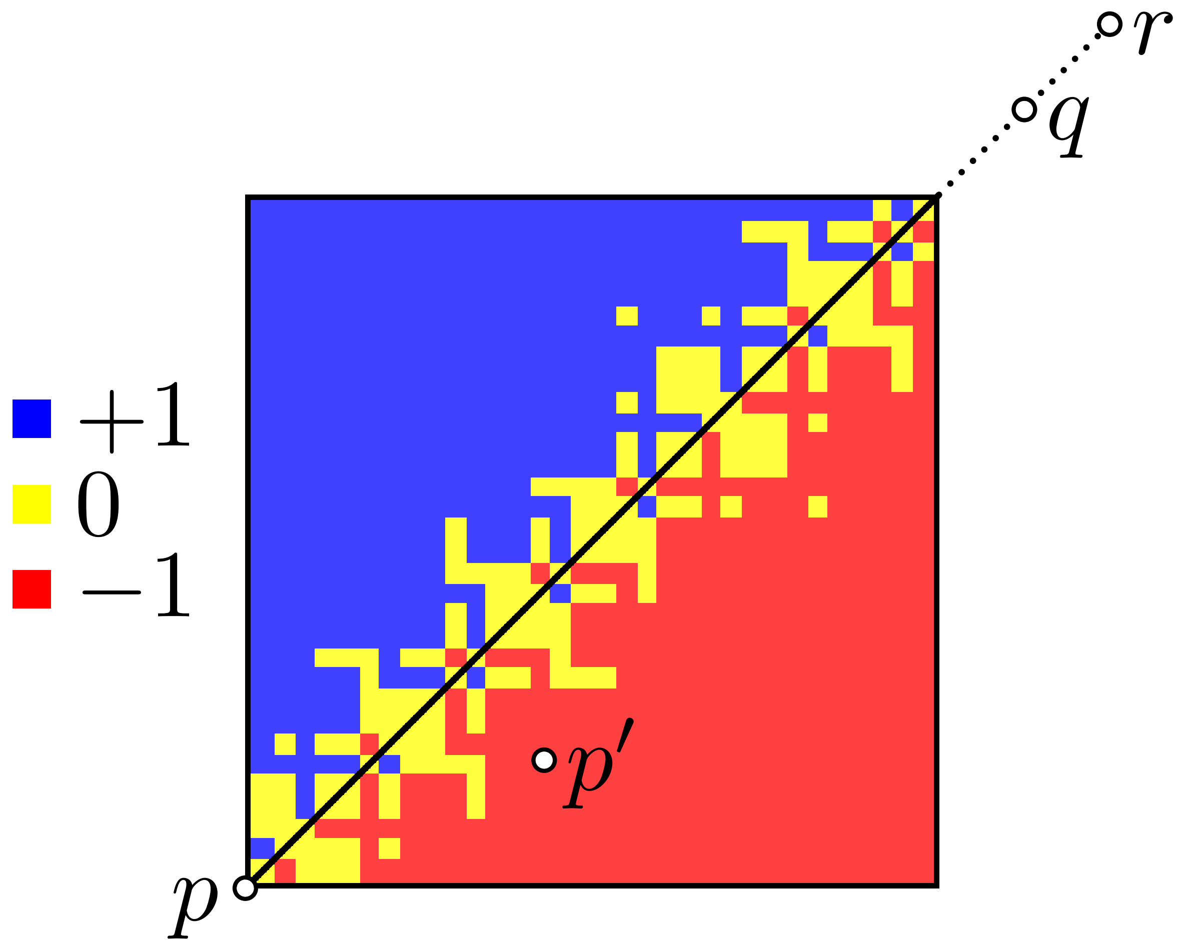
$p = (0.5, 0.5)$

$q = (12, 12)$

$r = (24, 24)$



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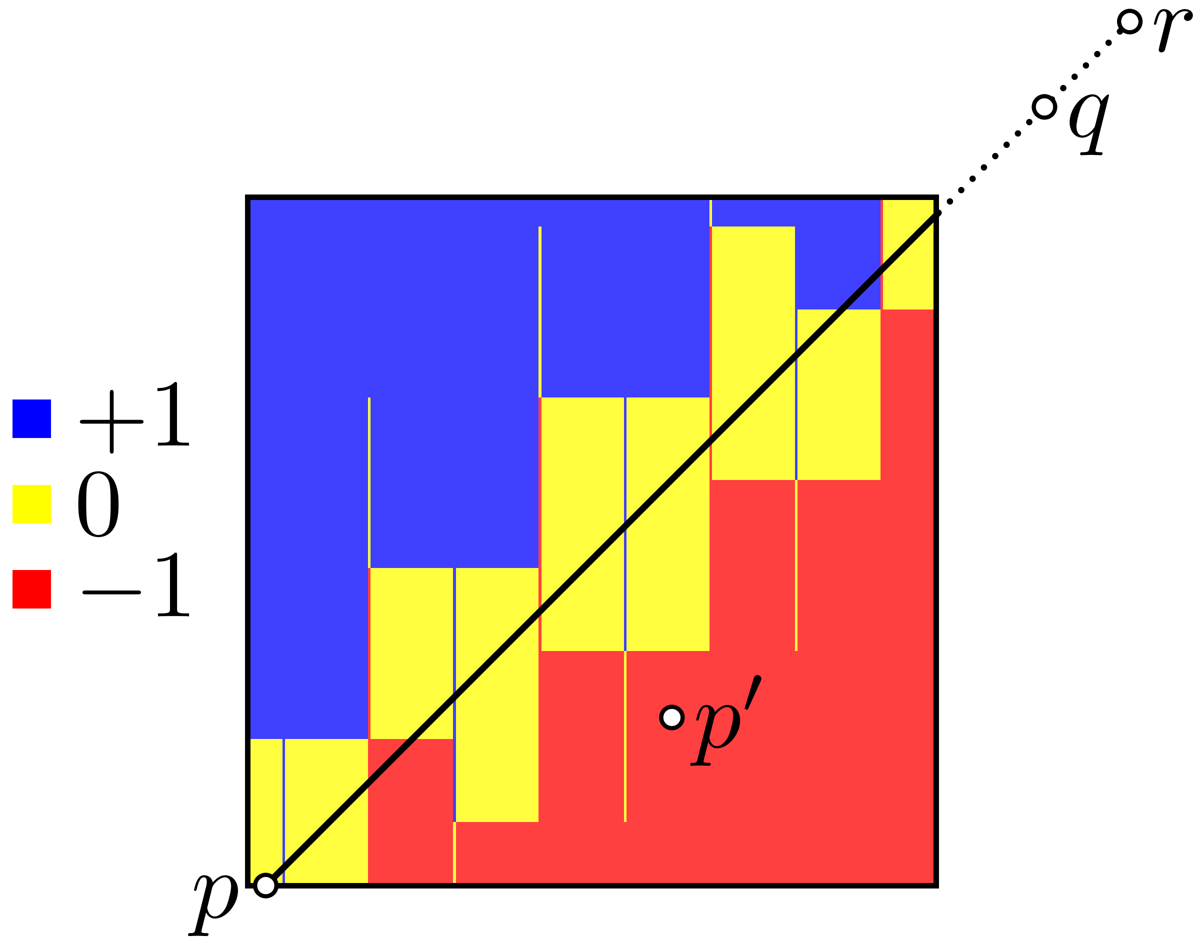
$$p = (0.5000..02531,\ 0.5000..0171)$$
$$q = (17.3000..001,\ 17.3000..001)$$
$$r = (24.000..005,\ 24.000..005)$$



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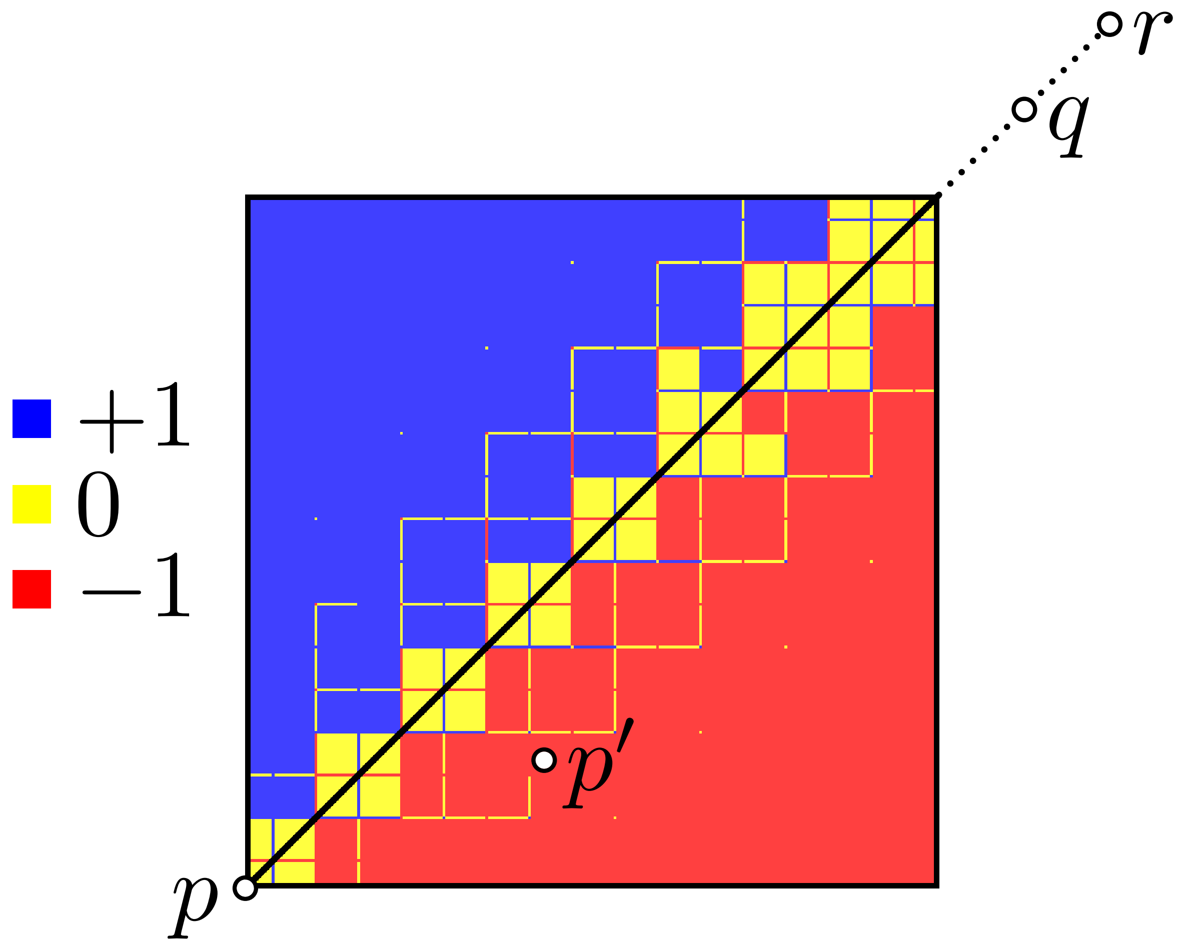
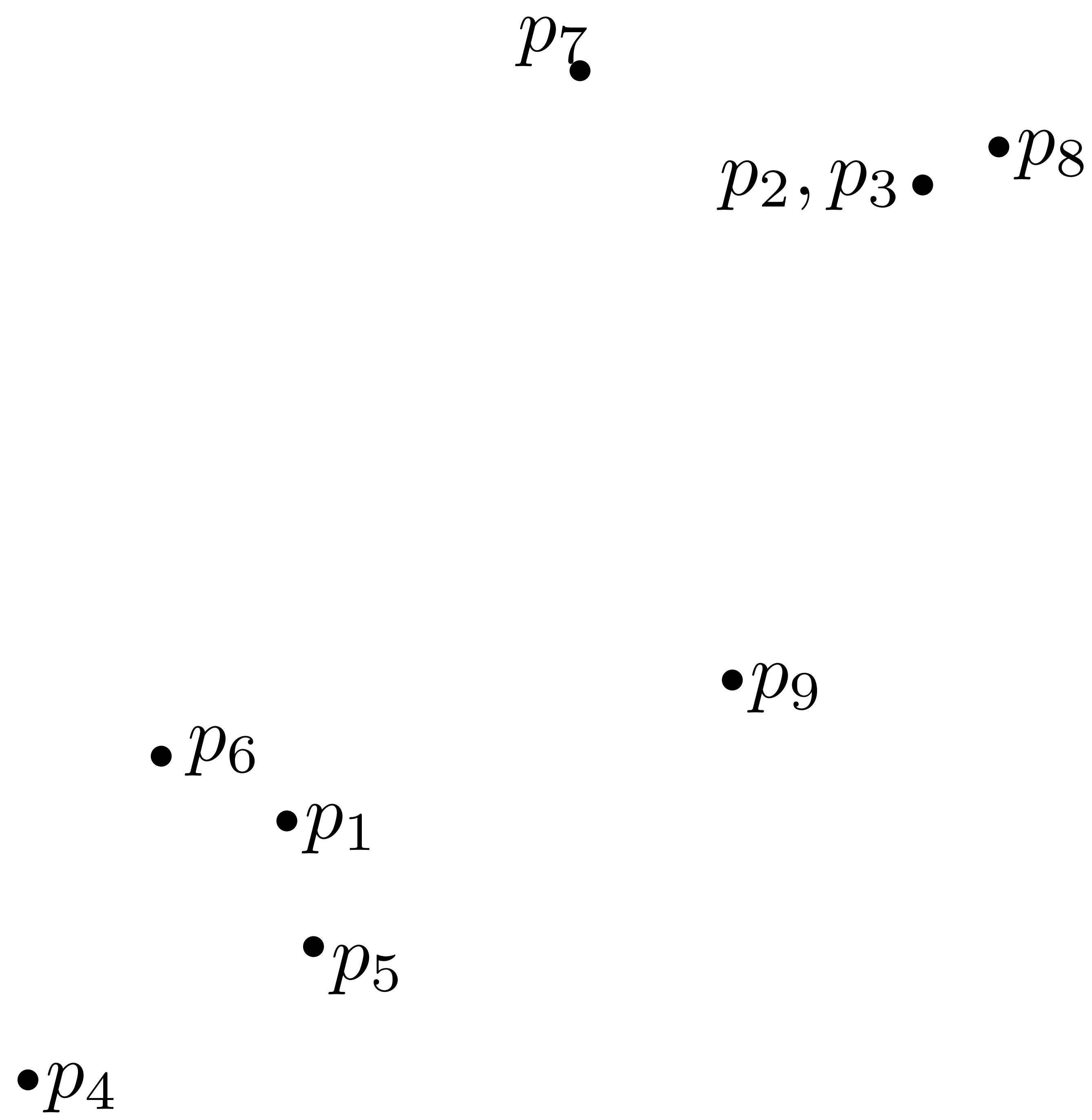
$$p = (0.5, 0.5)$$
$$q = (8.8000..007, 8.8000..007)$$
$$r = (12.1, 12.1)$$



Investigating INCREMENTALCONVEXHULL



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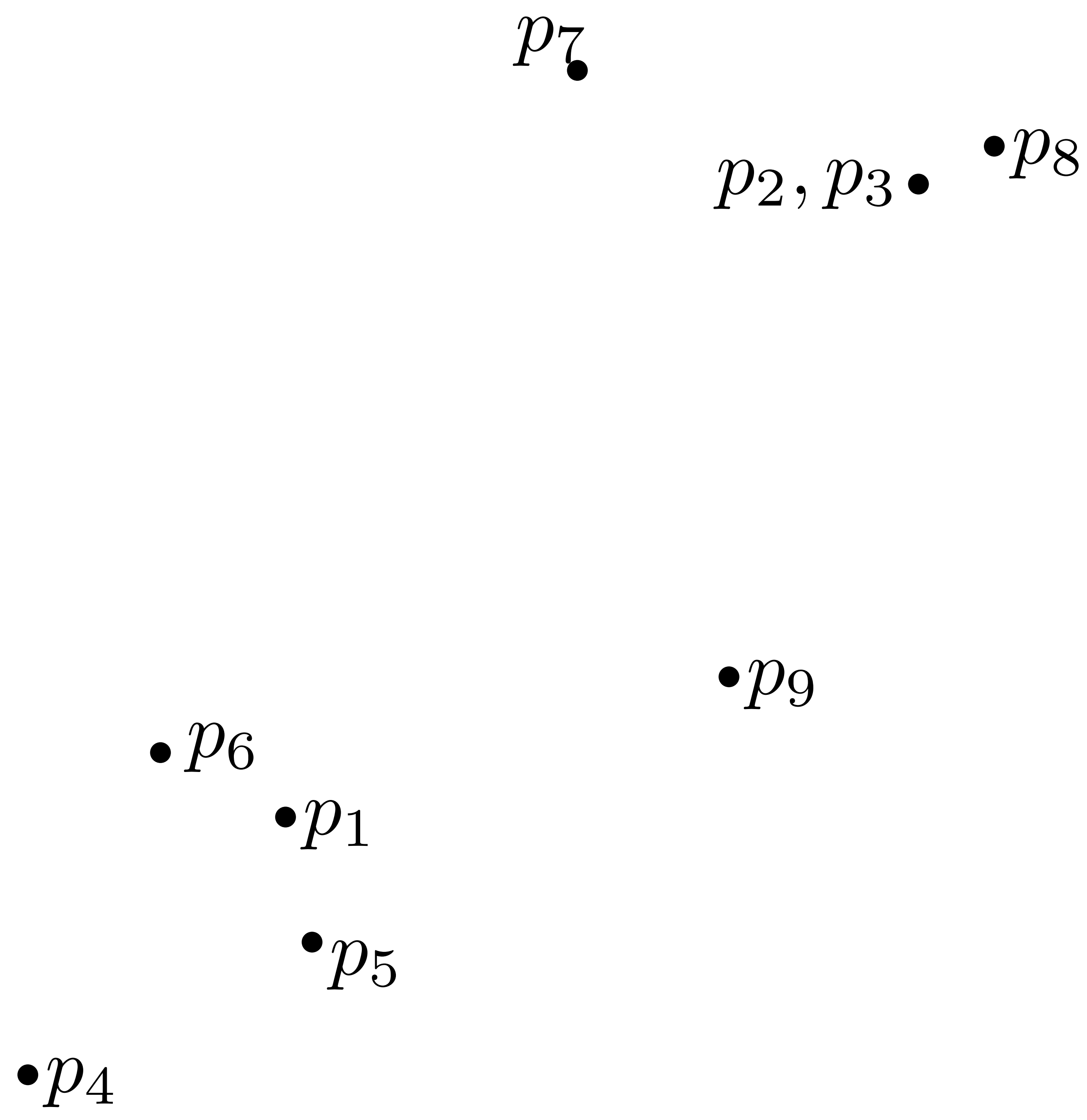
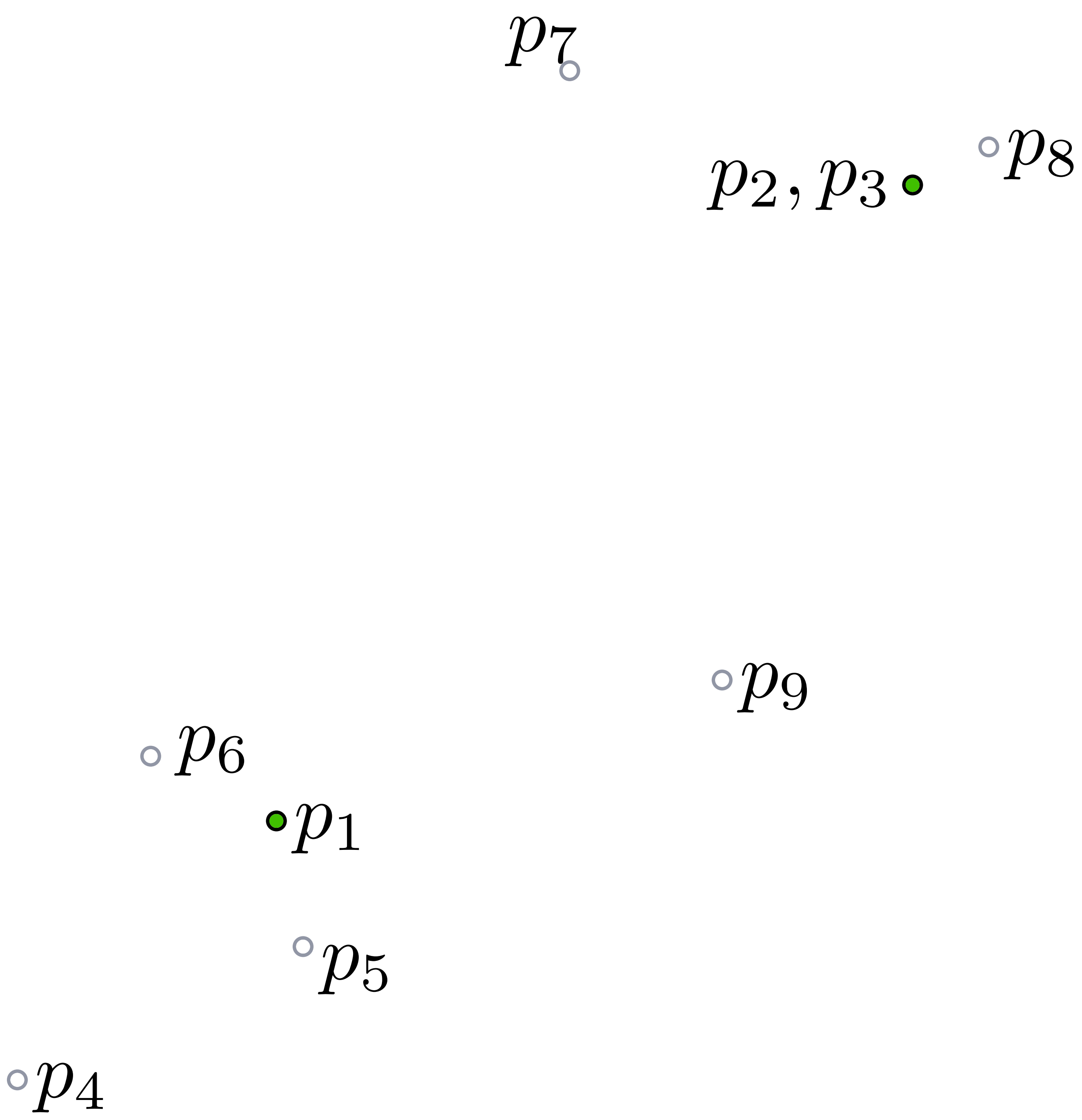


* refer to [Kettner et al.] for exact coordinates

Investigating IncrementalConvexHull



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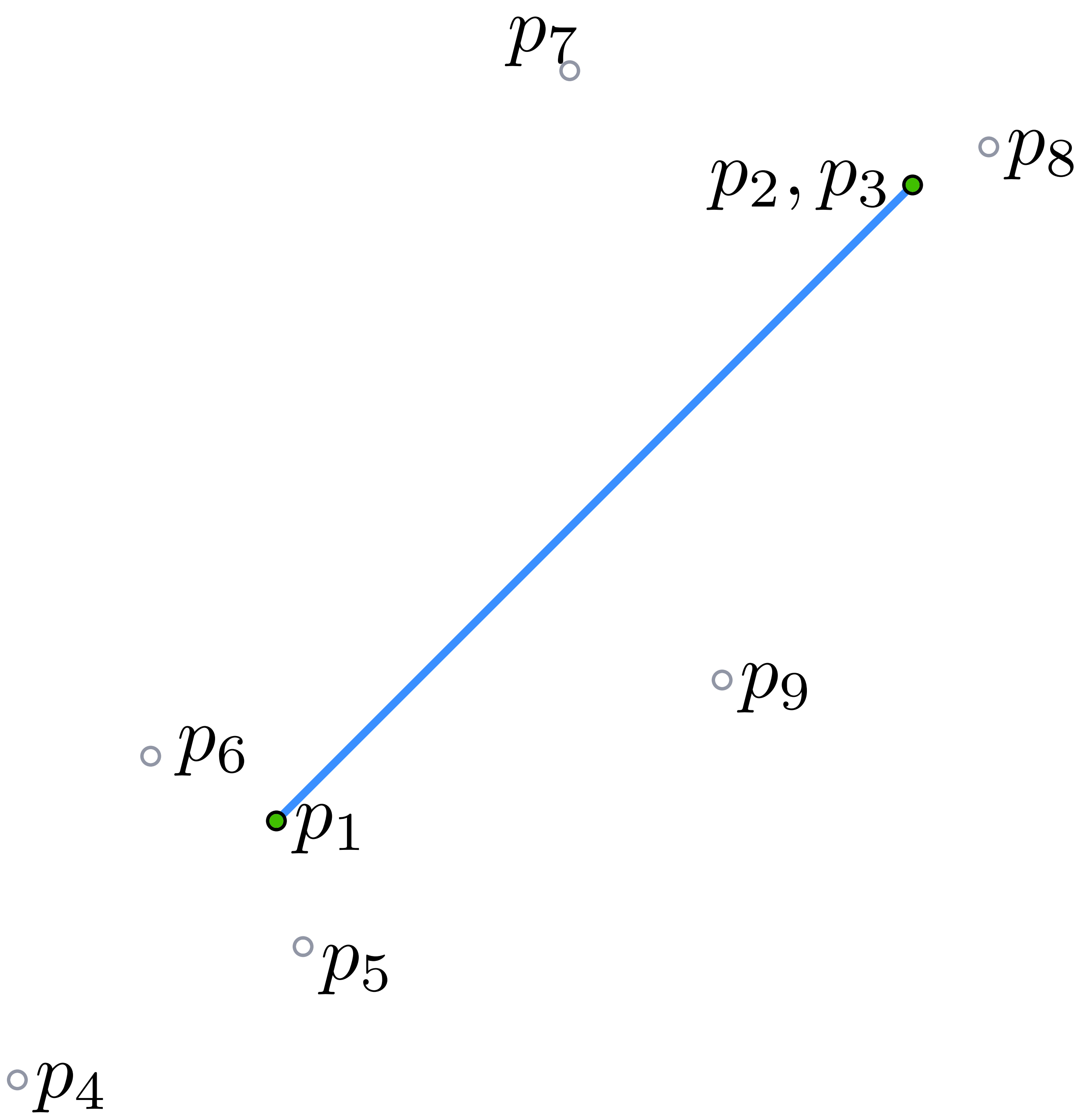

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Investigating IncrementalConvexHull

$$\mathit{float\_orient}(p_1, p_2, p_3) > 0$$
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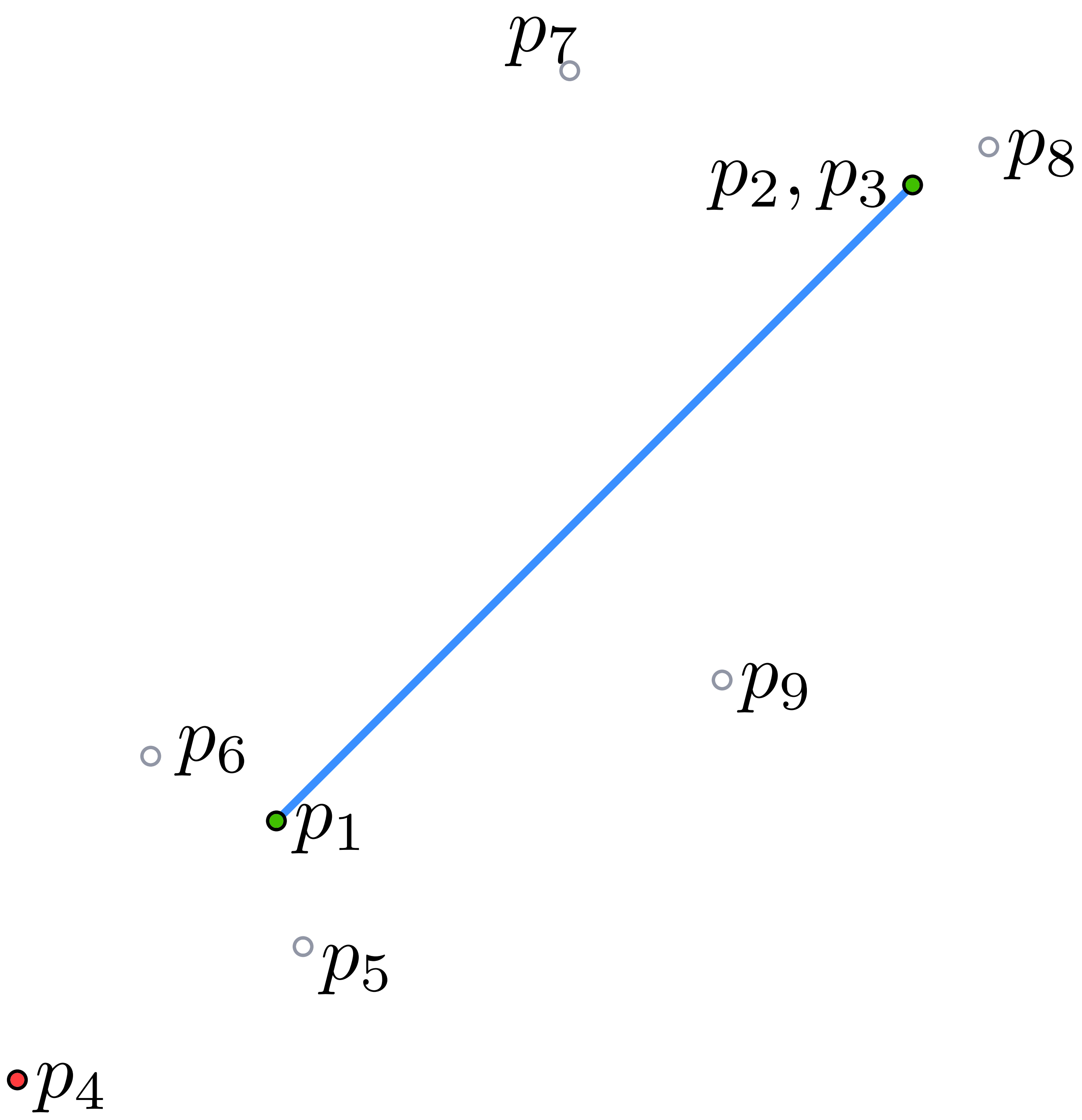

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Investigating IncrementalConvexHull

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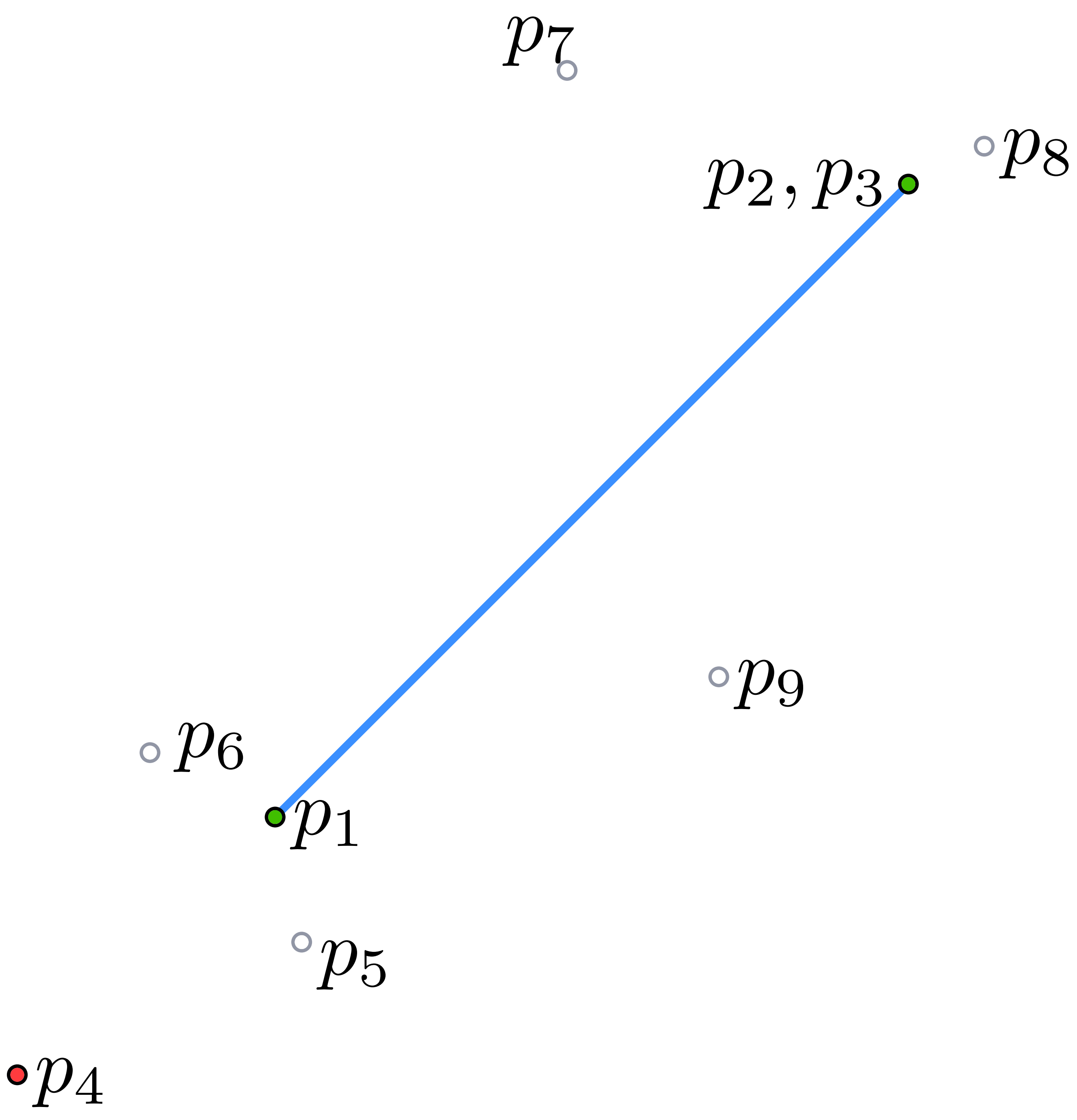
$\Rightarrow$ p_4 does not see any edge!



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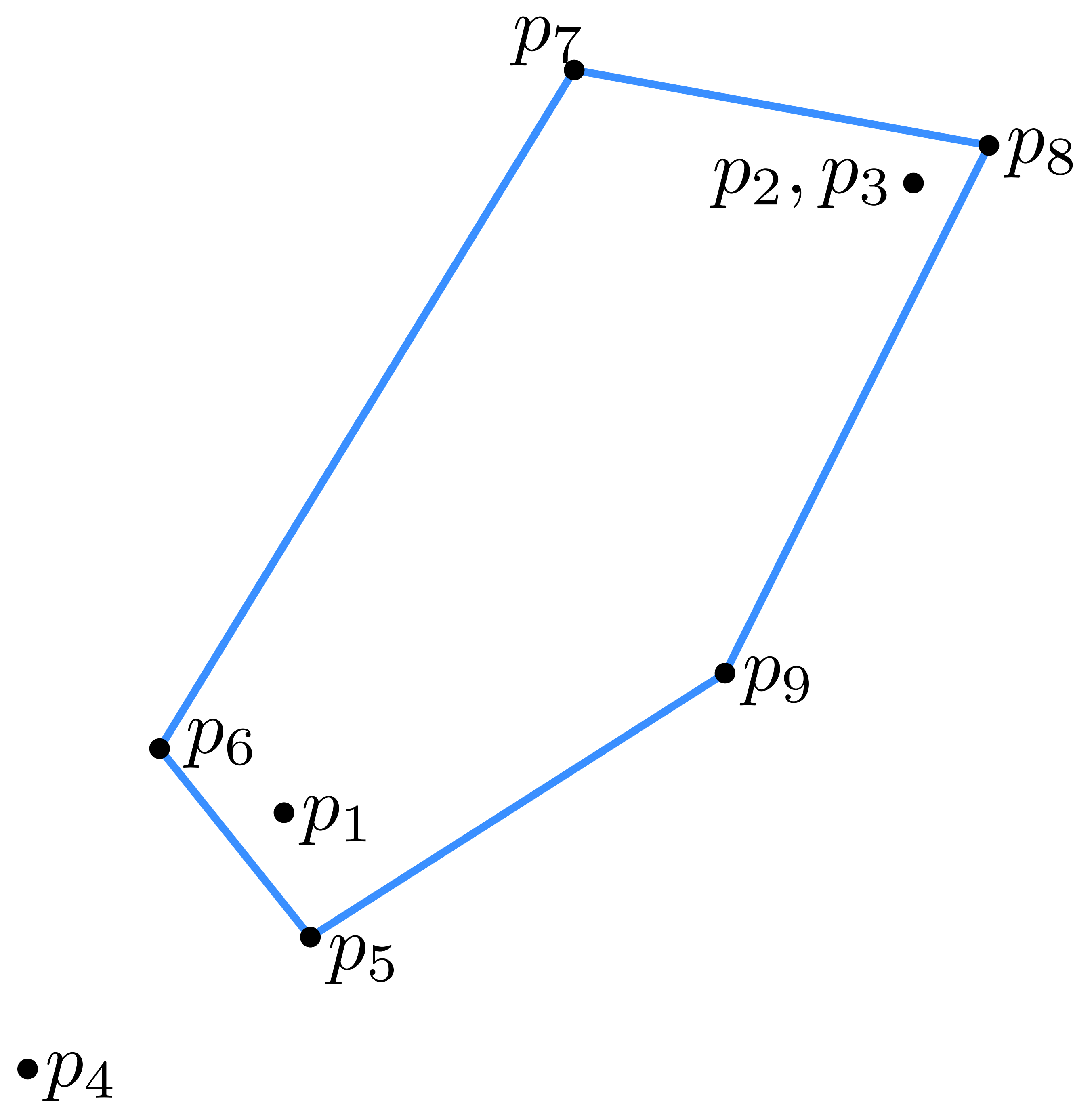

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Investigating INCREMENTALCONVEXHULL

Failure:

- point outside convex hull doesn't see an edge $\Rightarrow$ incorrect solution

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Investigating IncrementalConvexHull

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Other failures include:

- point inside convex hull sees an edge $\Rightarrow$ non-convex solution
- point outside convex hull sees all edges $\Rightarrow$ infinite loop? crash?
- point outside convex hull sees a non-contiguous set of edges $\Rightarrow$ non-convex solution, self-intersecting chain

Geometric predicates

Definition: geometric predicate is a sign of a polynomial evaluated with the coordinates of the input.

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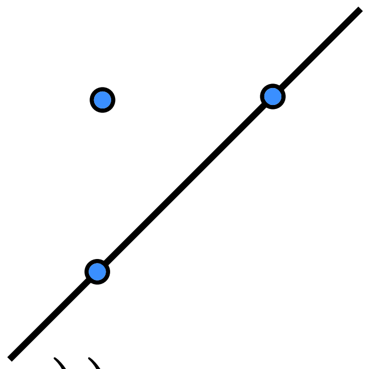
Examples:



Orientation test (in convex hull):

$$orientation(p, q, r) = sign\left((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x)\right)$$

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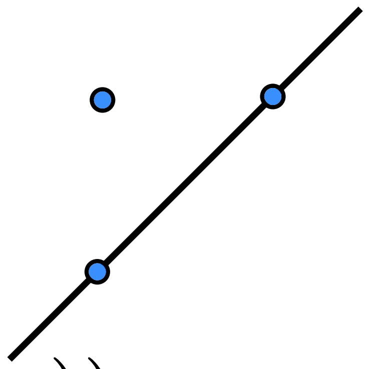

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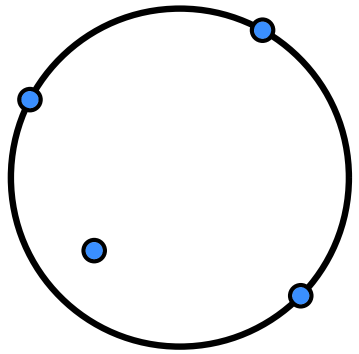

In-circle test (in Delaunay triangulation):

$$in\_circle(a, b, c, d) = \begin{cases} +1 & d \text{ lies inside circle through } a, b, c \\ -1 & d \text{ lies outside circle through } a, b, c \\ 0 & d \text{ lies on circle through } a, b, c \end{cases}$$

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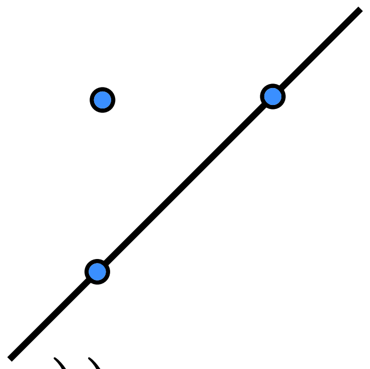

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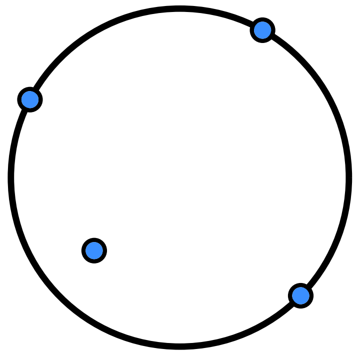

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Question: other examples?

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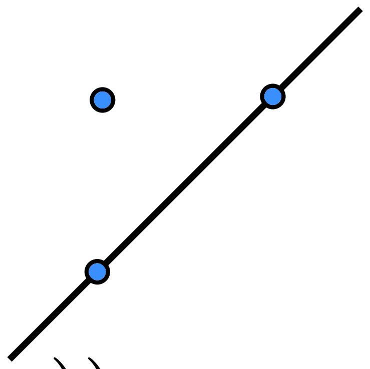

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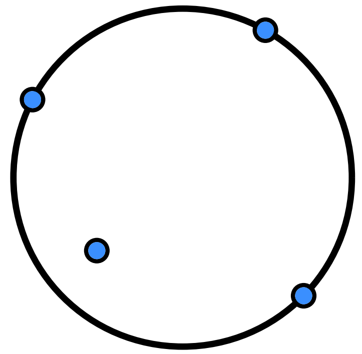

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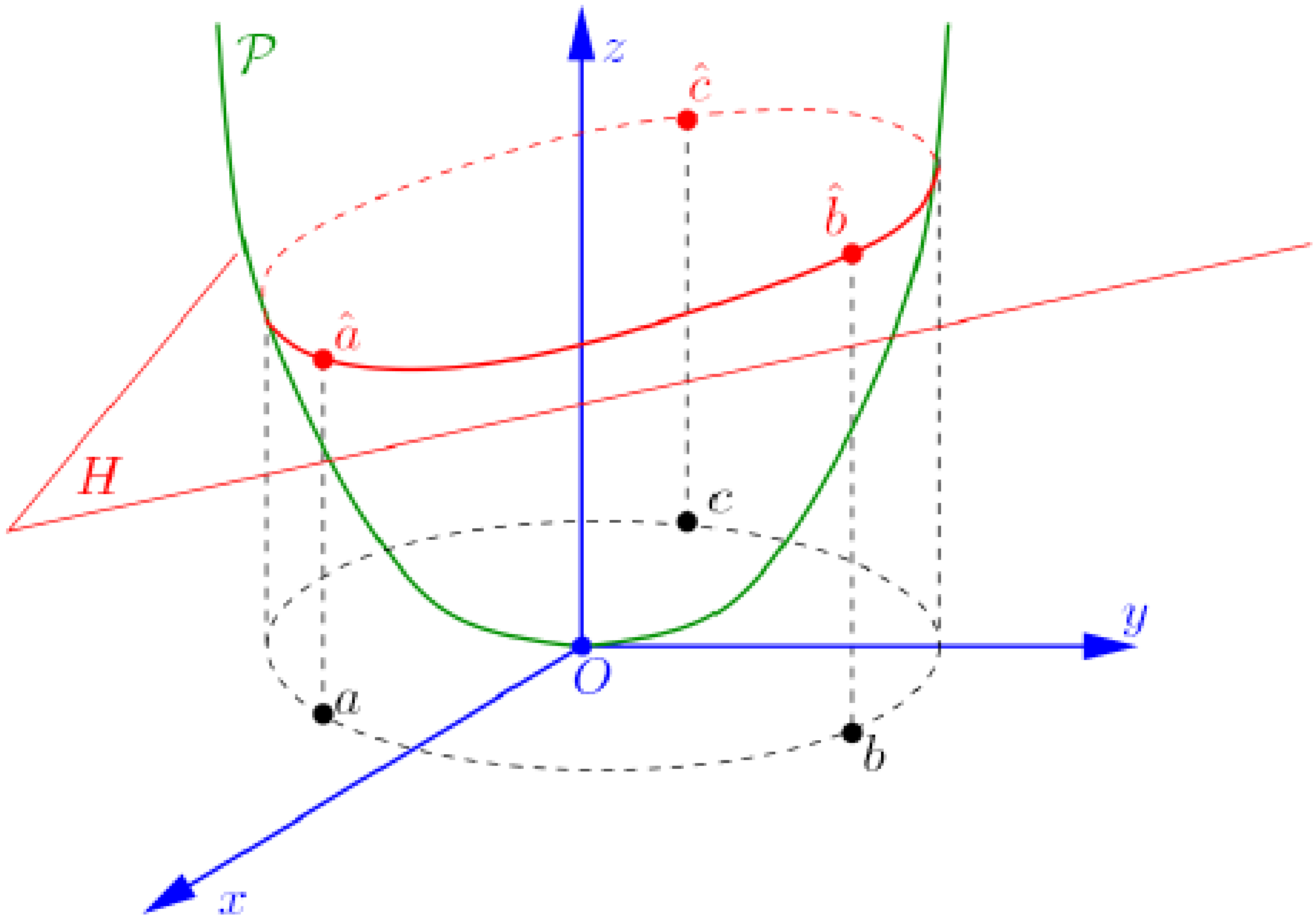
Question: other examples? Many predicates reduce to orientation test!

In-circle test



Observation: in-circle test = orientation test after lifting the point set

$$(a_x, a_y) \mapsto (a_x, a_y, a_x^2 + a_y^2)$$

In-circle test

$$in\_circle(a, b, c, d) =$$

$$= sign\left(\text{det}\begin{bmatrix} a_x & a_y & a_x^2 + a_y^2 & 1 \\ b_x & b_y & b_x^2 + b_y^2 & 1 \\ c_x & c_y & c_x^2 + c_y^2 & 1 \\ d_x & d_y & d_x^2 + d_y^2 & 1 \end{bmatrix}\right)$$

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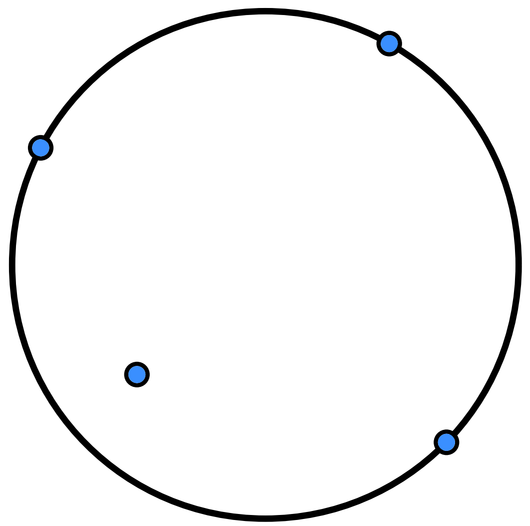

Solutions

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- evaluate predicates exactly
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Floating-point filters

Get correct sign (-1, 0 or 1) of an exact expression E using floating-point!

"filters out" the easy cases

1: Let $F = E(X)$ in **floating point**
2: **if** $F >$ error bound **then**
3: **return** 1
4: **else if** $-F >$ error bound **then**
5: **return** -1
6: **else**
7: increase precision and repeat, or switch to exact arithmetic

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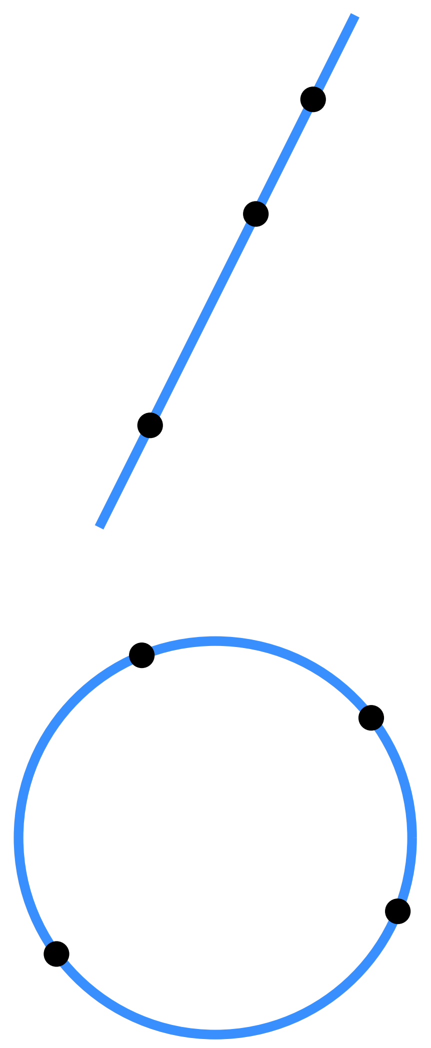
use exact arithmetic $\Rightarrow$ do not limit space to store numbers

Dealing with degeneracies

Option 1: carefully design your algorithm around degenerate cases

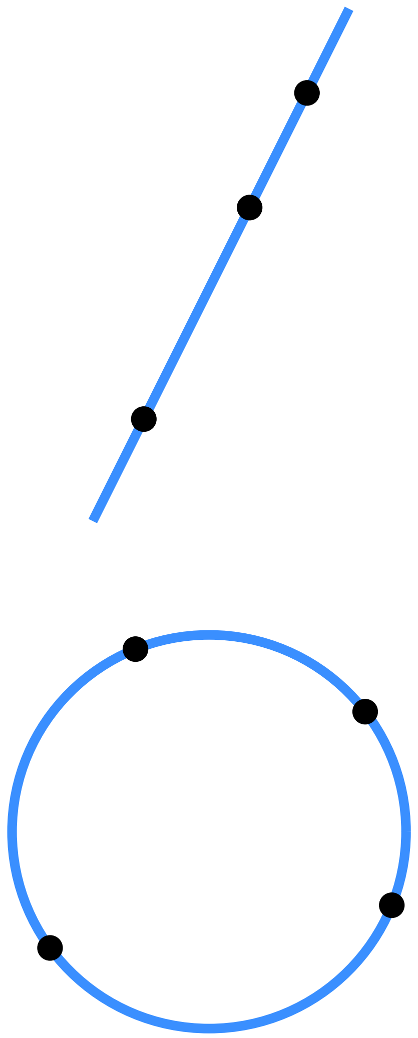


Dealing with degeneracies

Option 1: carefully design your algorithm around degenerate cases

Option 2: randomly perturb your input and solve your problem approximately

- input is precise
- perturbation is “close” to the input
- geometric traits are preserved



Robustness wrap-up

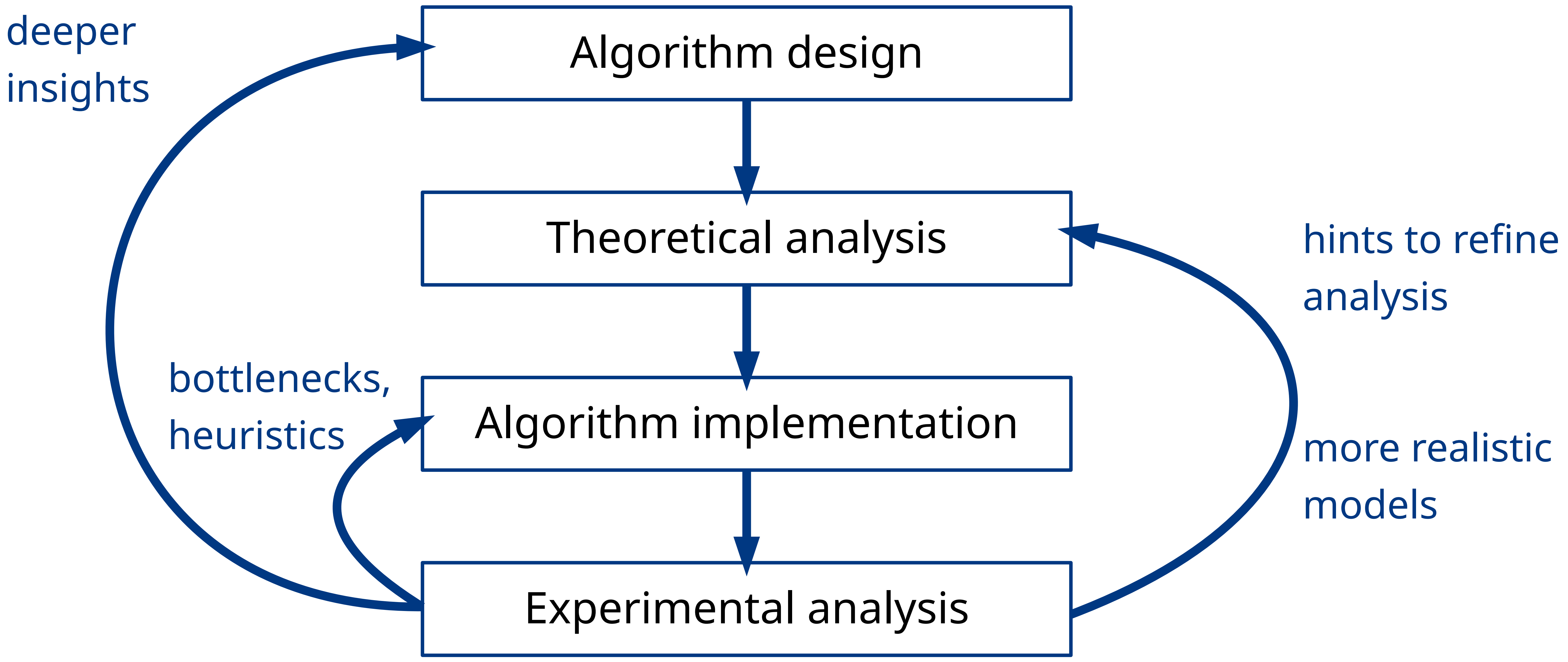
To make a robust implementation of your algorithm:

- trade-off: exact arithmetic vs speed
- often sufficient: answer predicates exactly, but construction may be inexact
- robustness is difficult to achieve
- good news: robust implementations exist

CGAL
Computational Geometry Algorithms Library

Triangle
A Two-Dimensional Quality Mesh Generator and Delaunay Triangulator.

Algorithm engineering cycle



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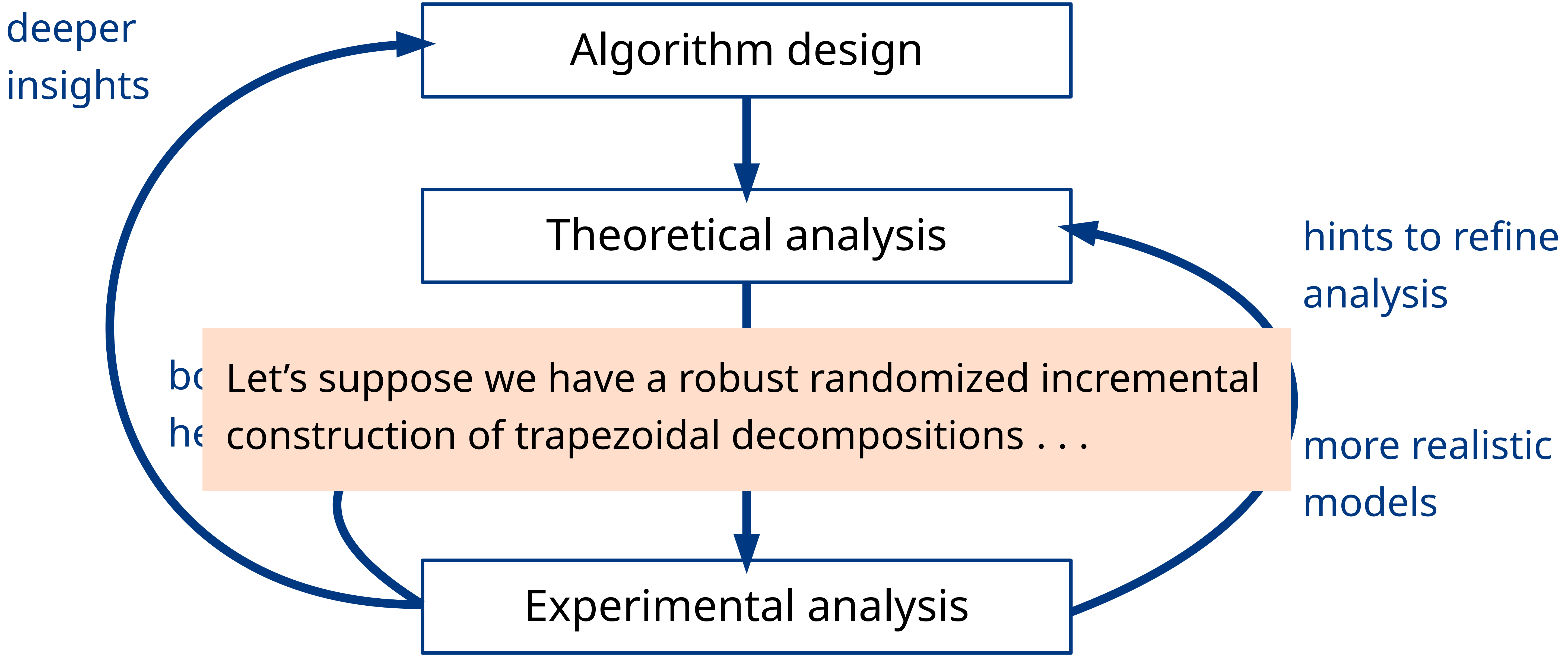
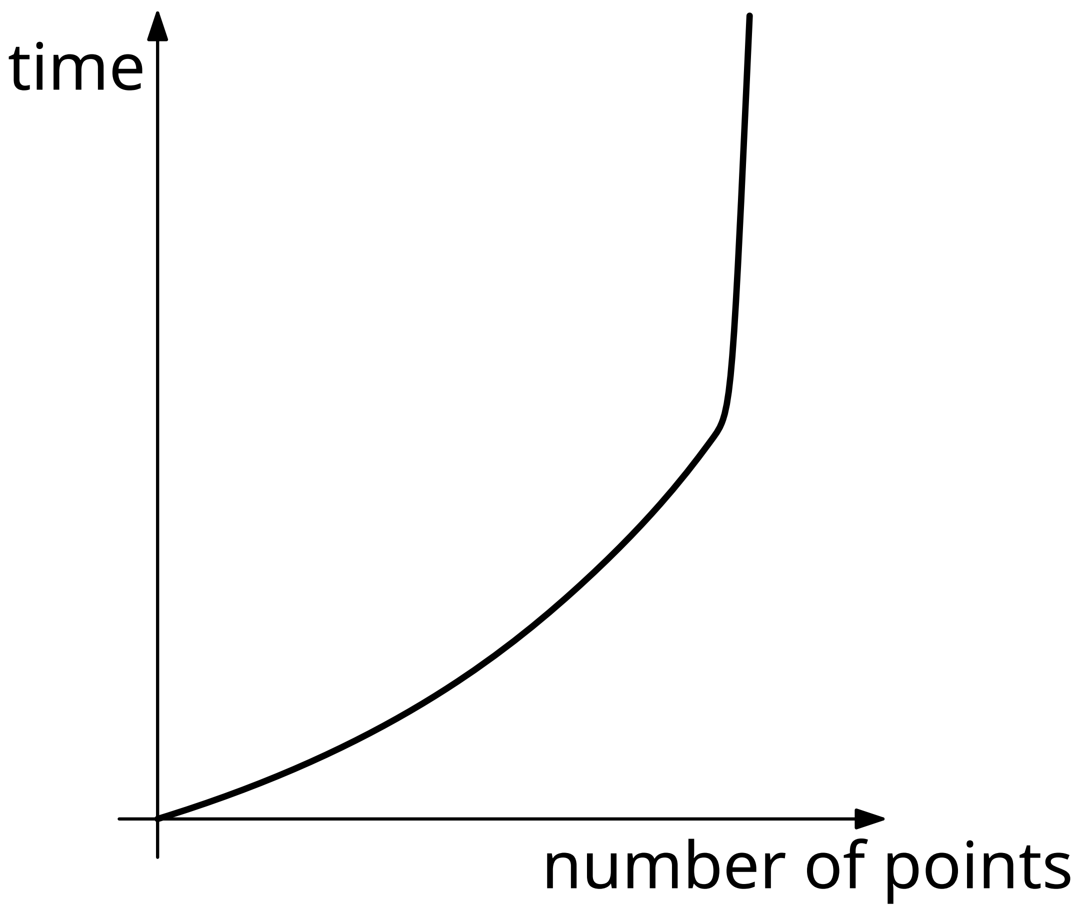


Let’s suppose we have a robust randomized incremental construction of trapezoidal decompositions . . .

Standard RIC in experiments

Experimental results of randomized incremental construction (RIC) of trapezoidal decomposition:

- What went wrong?



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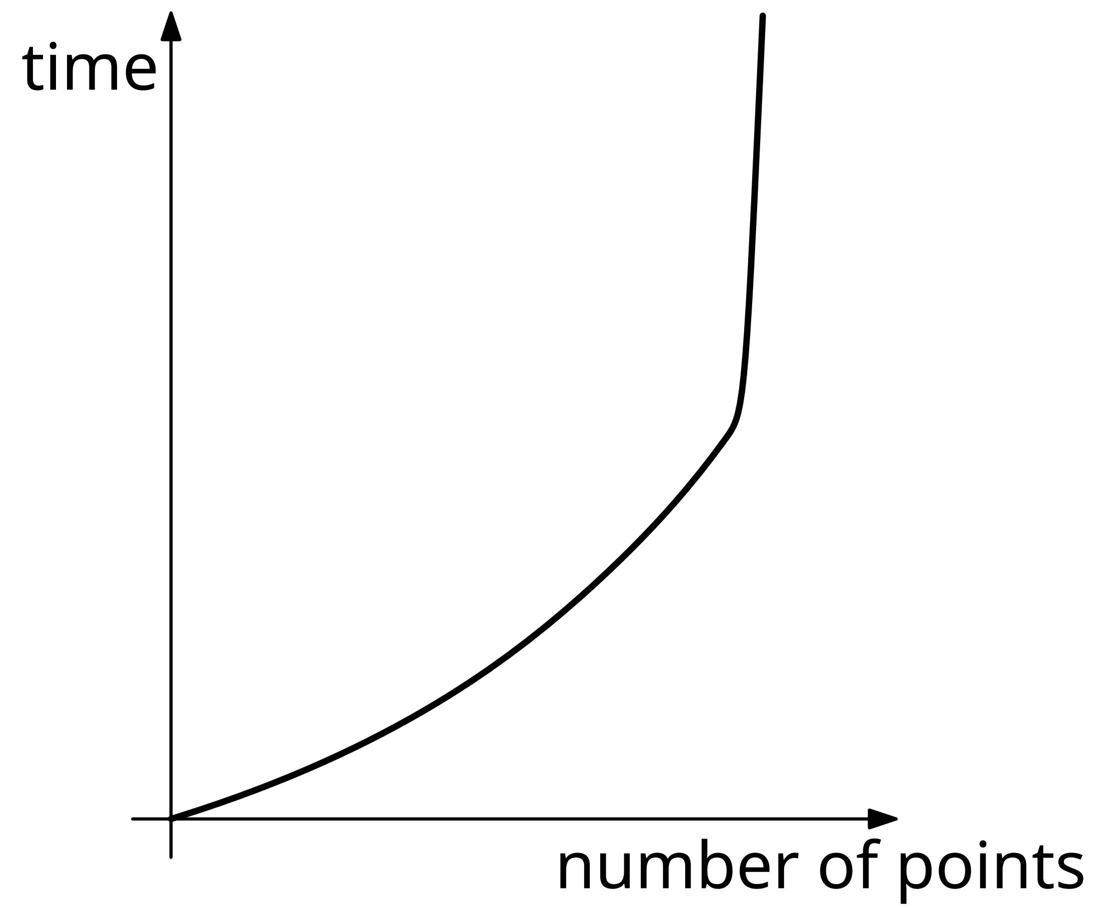
Standard RIC in experiments

Experimental results of randomized incremental construction (RIC) of trapezoidal decomposition:

- What went wrong?
- Thrashing due to random memory access



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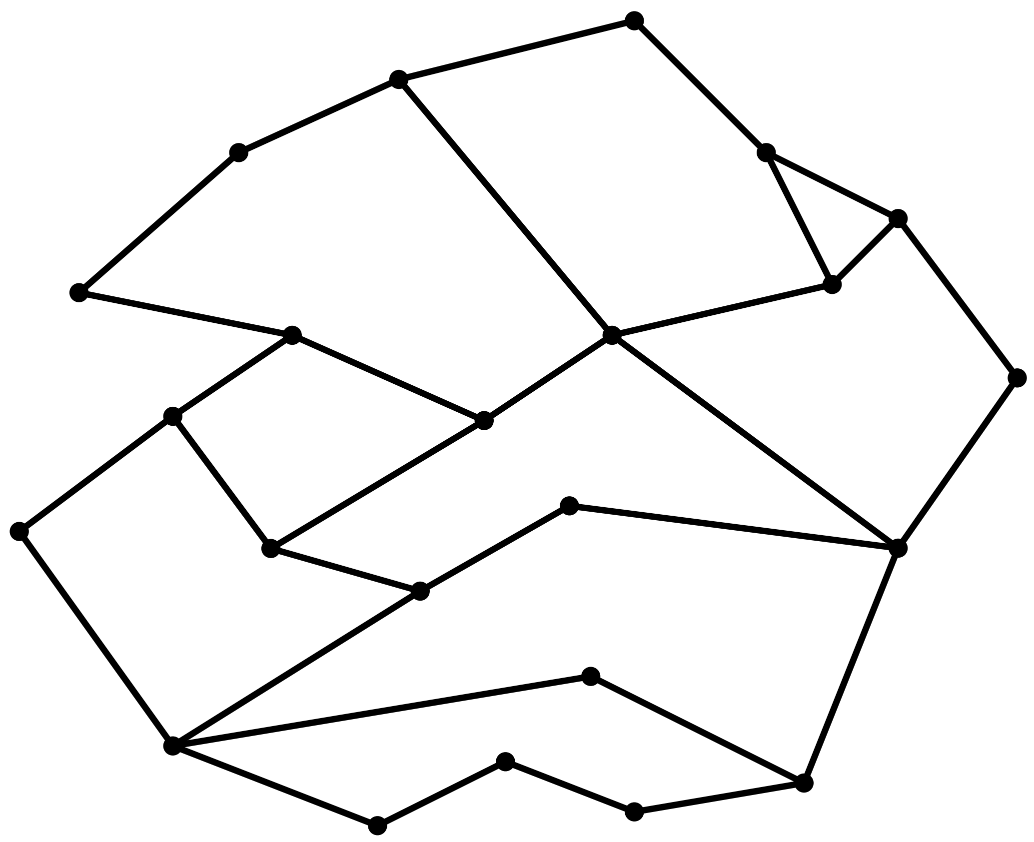
Partial randomization

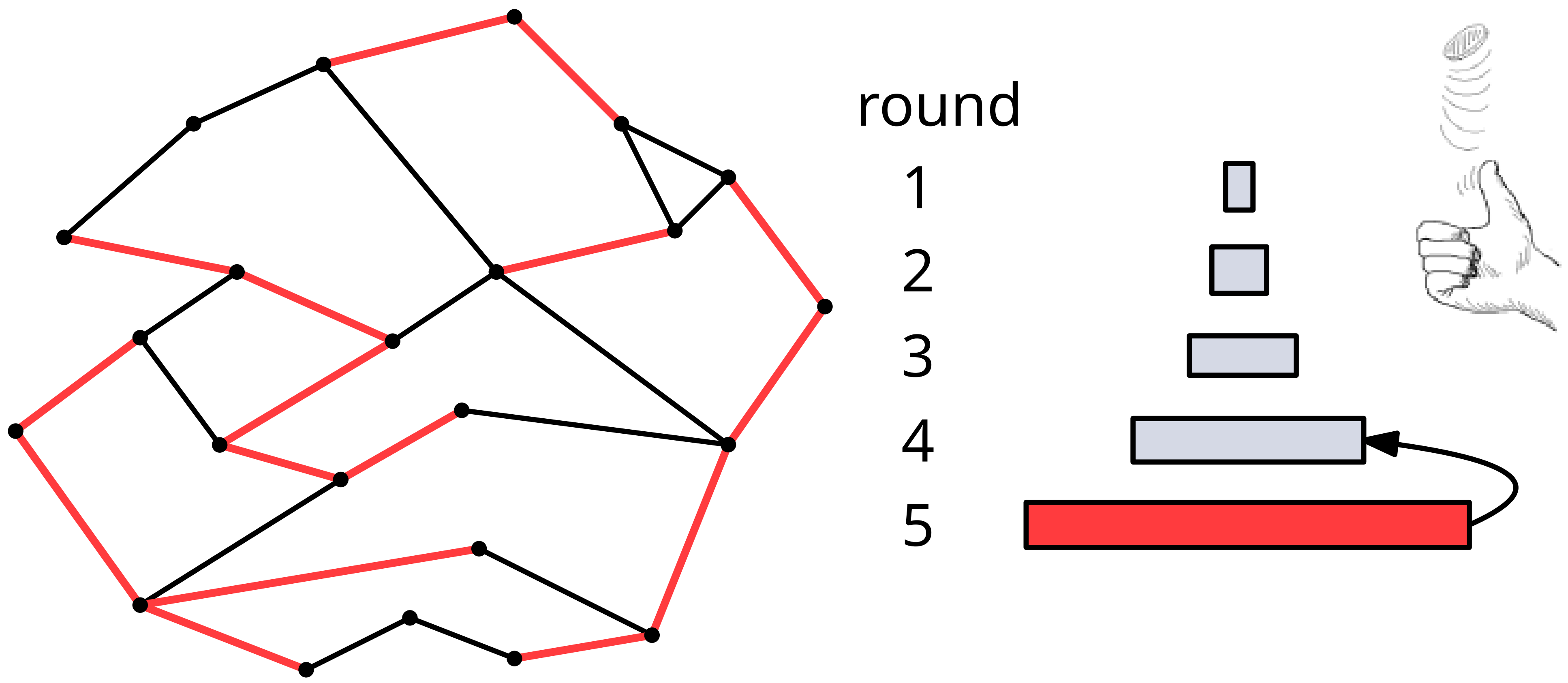
- random access to large data: a bad idea
- don’t randomize? really bad in theory and also causes overhead in experiments
- partially randomized insertion order
  - increase locality of reference, especially as data structure gets large
  - retain enough randomness to guarantee optimality

Partial randomization



- in each round (from last to first) choose each segment with probability $1/2$
- order chosen segments in each round to benefit locality (not random!)

Partial randomization



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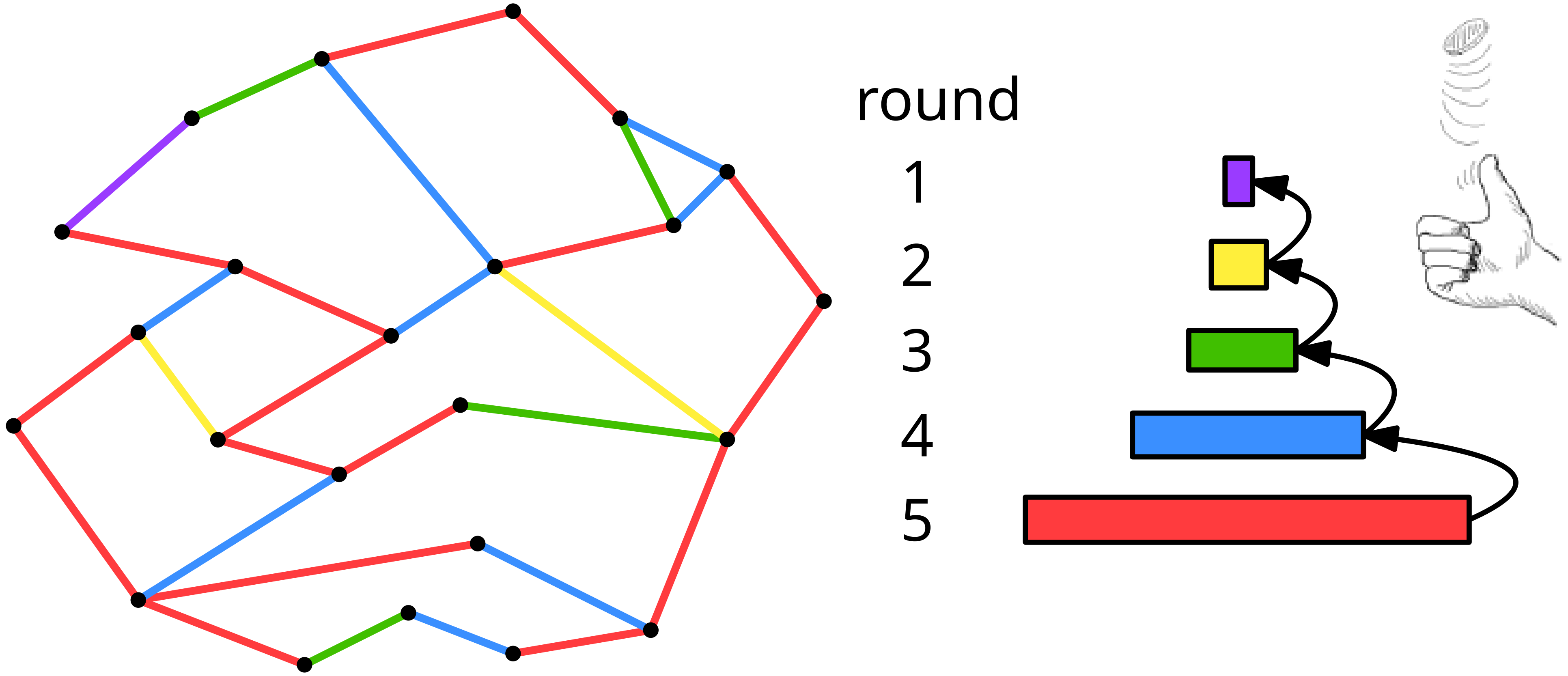

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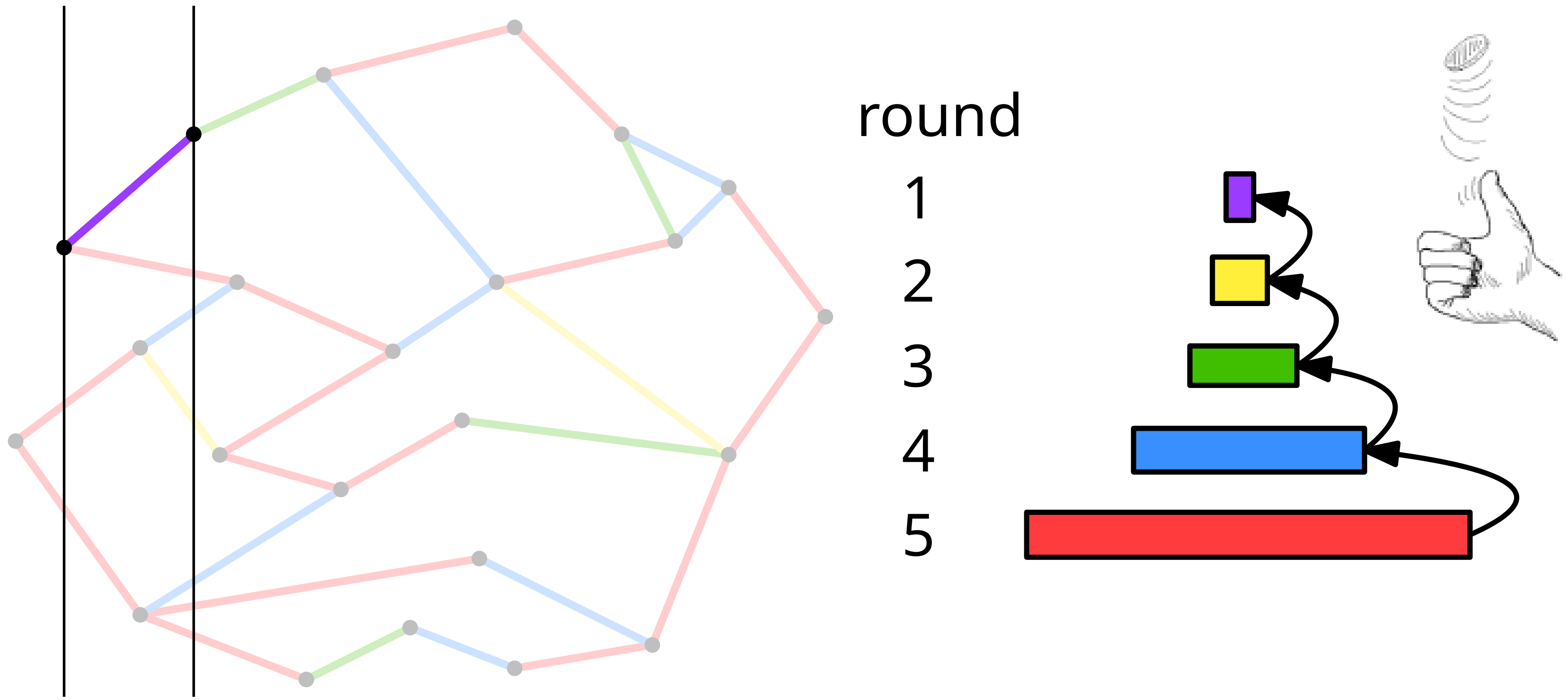

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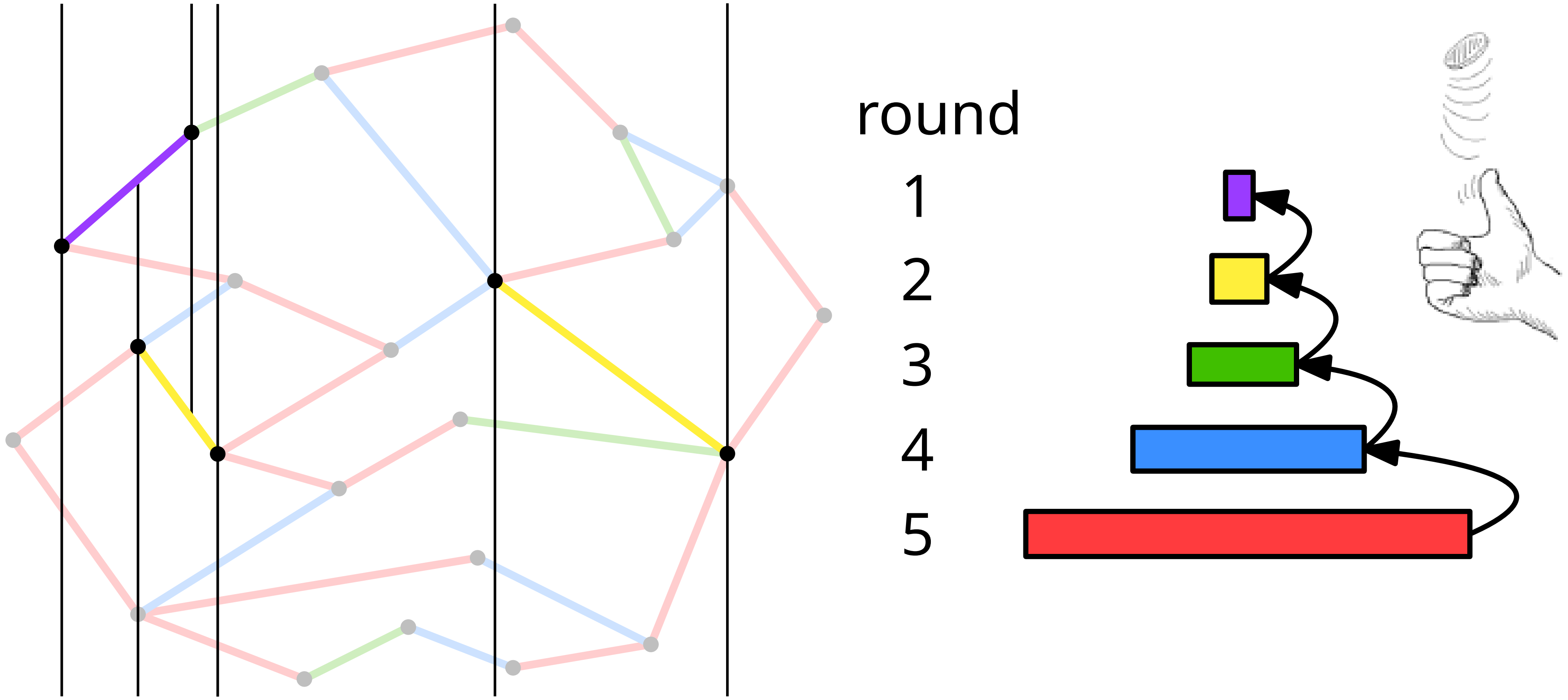

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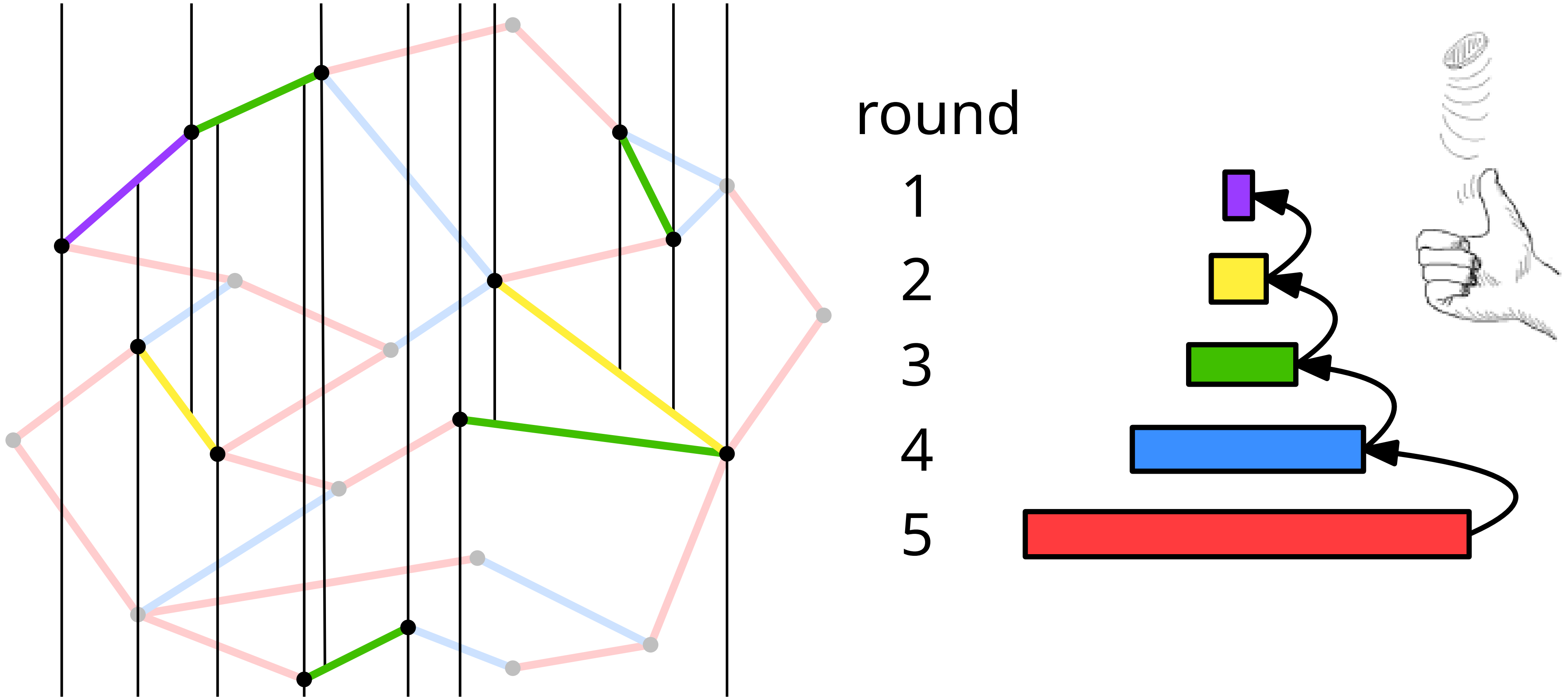

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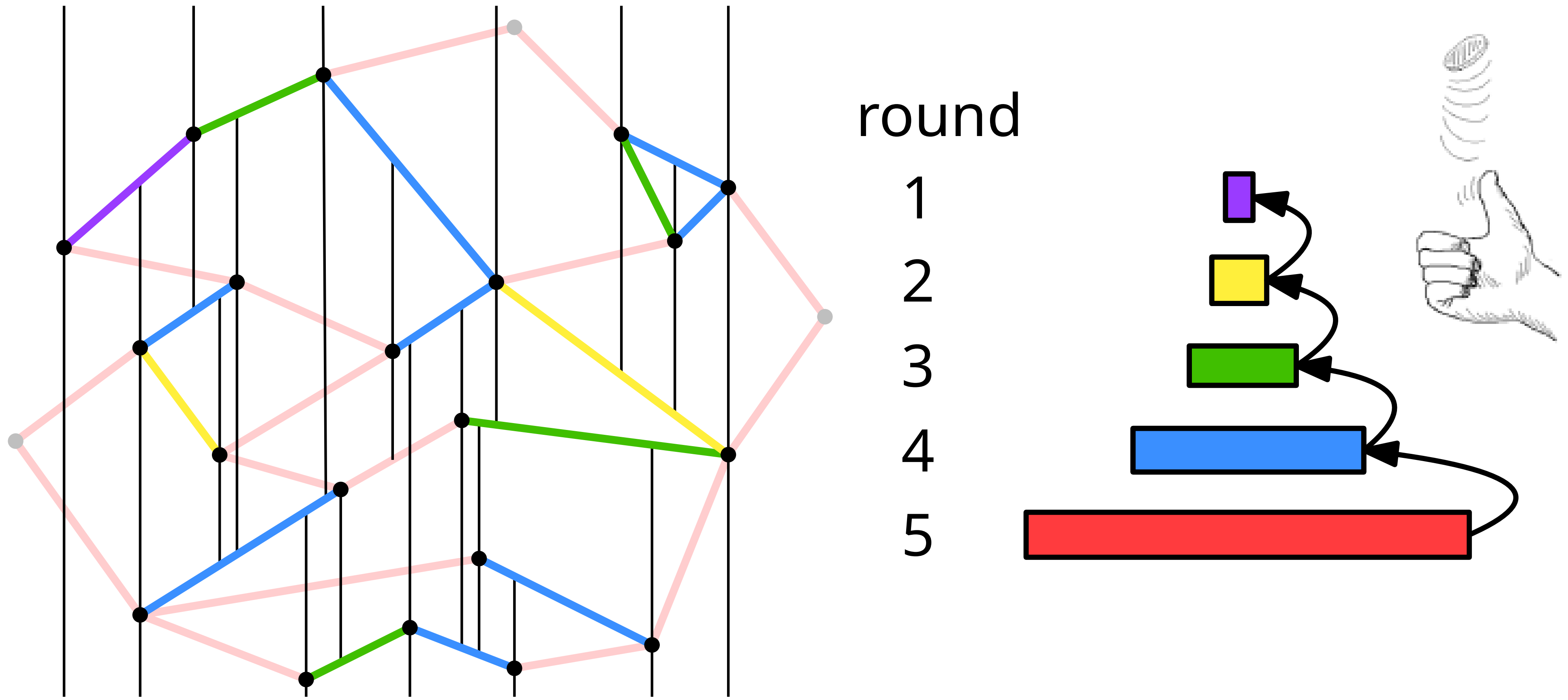

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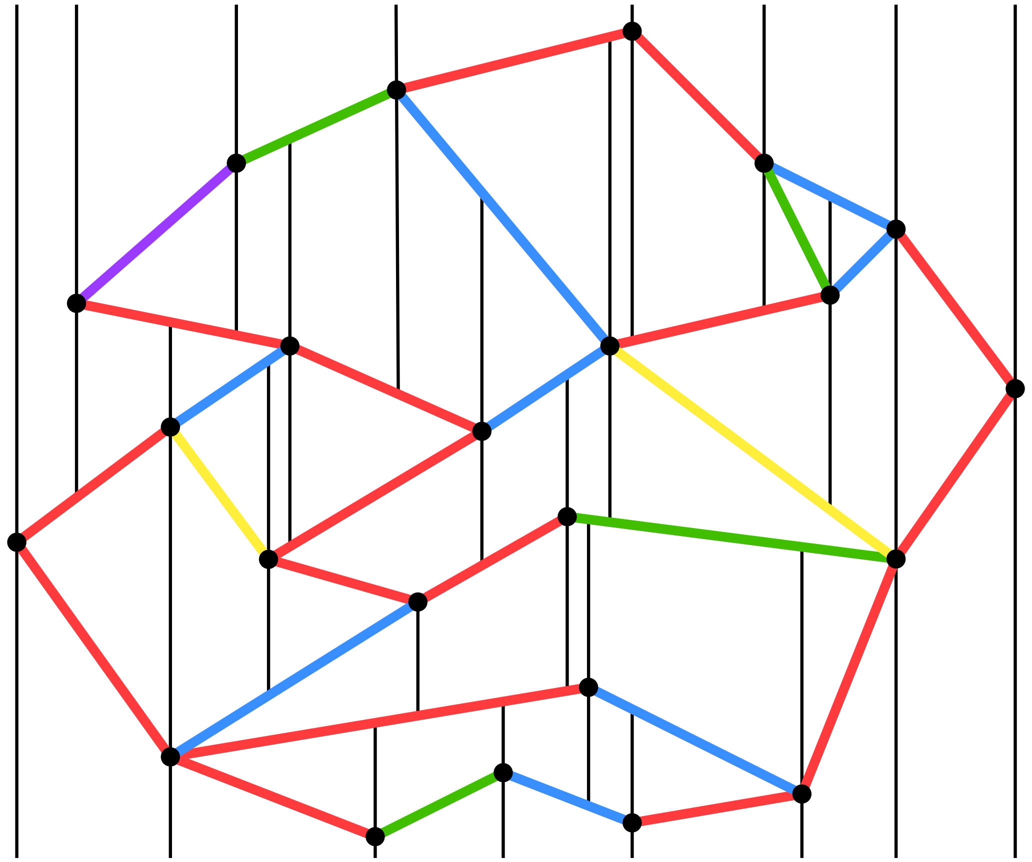

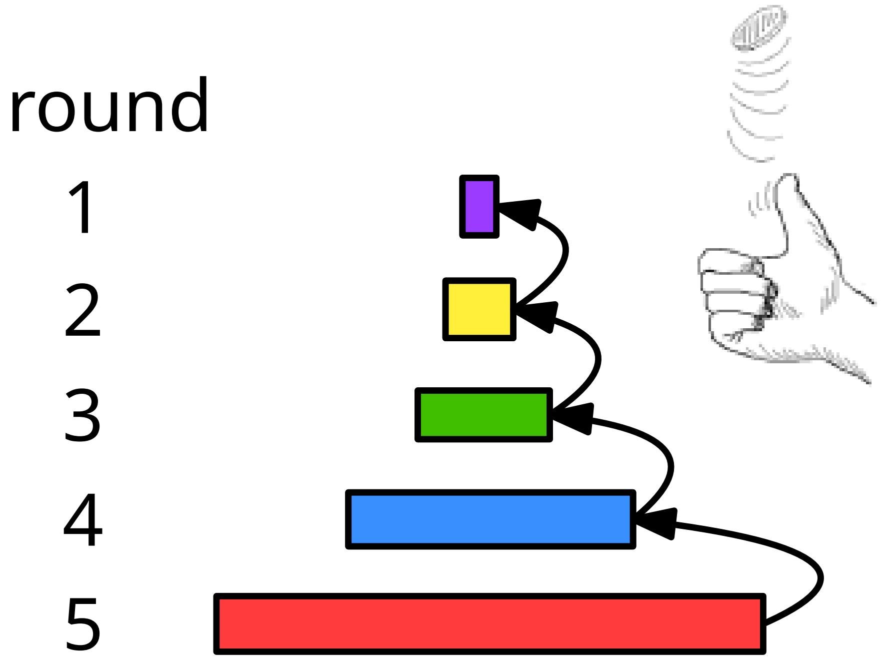

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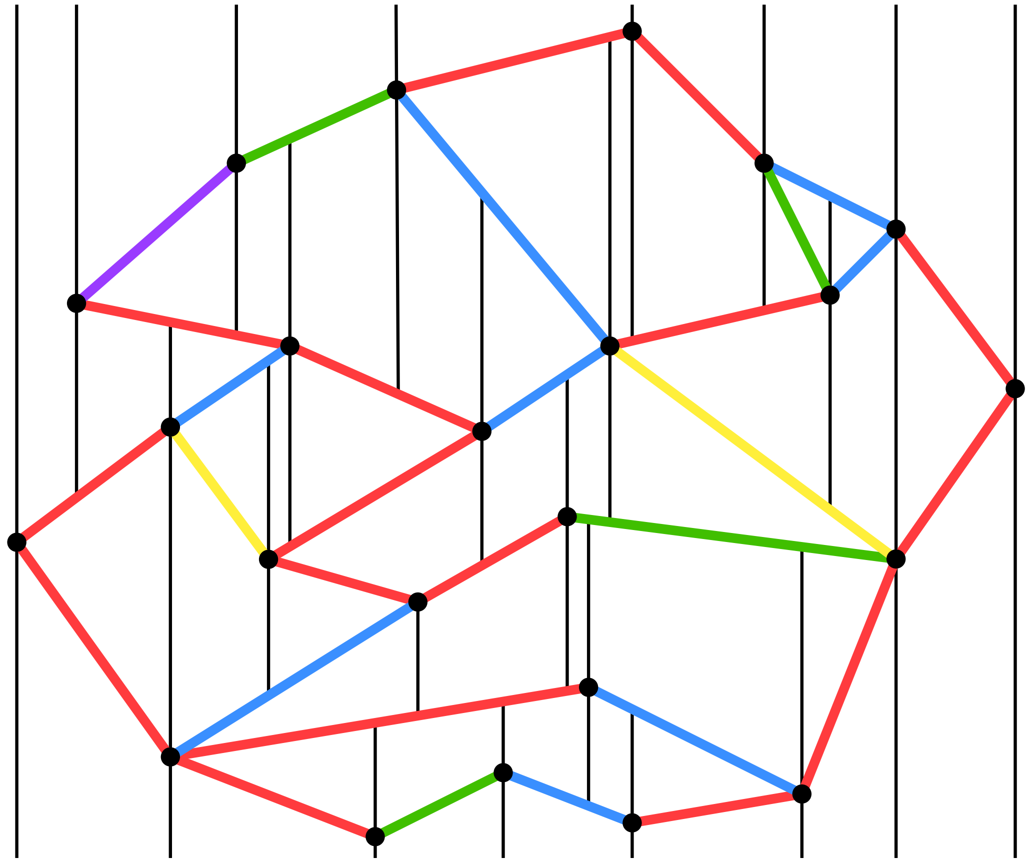

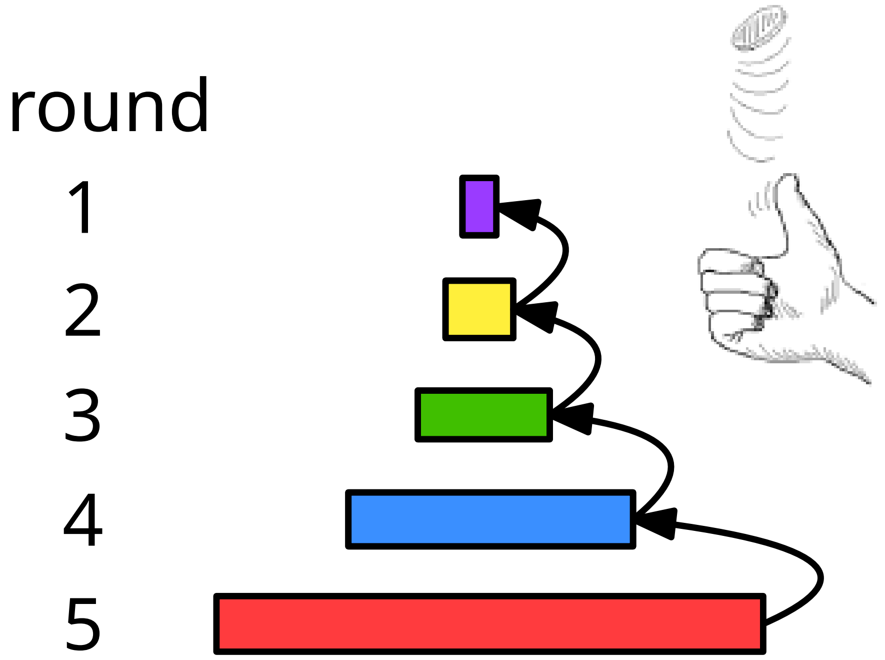

Theorem: The trapezoidal decomposition of n non-intersecting segments in the plane can be constructed in $O(n \log n)$ expected time using partial randomization.

Proof idea: compare probabilities to standard randomized algorithm

Partial randomization

Lemma: For a given trapezoid Δ the probability p_{PRIC} of occurring in a partial RIC is at most 16 times the probability p_{RIC} of occurring in a (standard) RIC.

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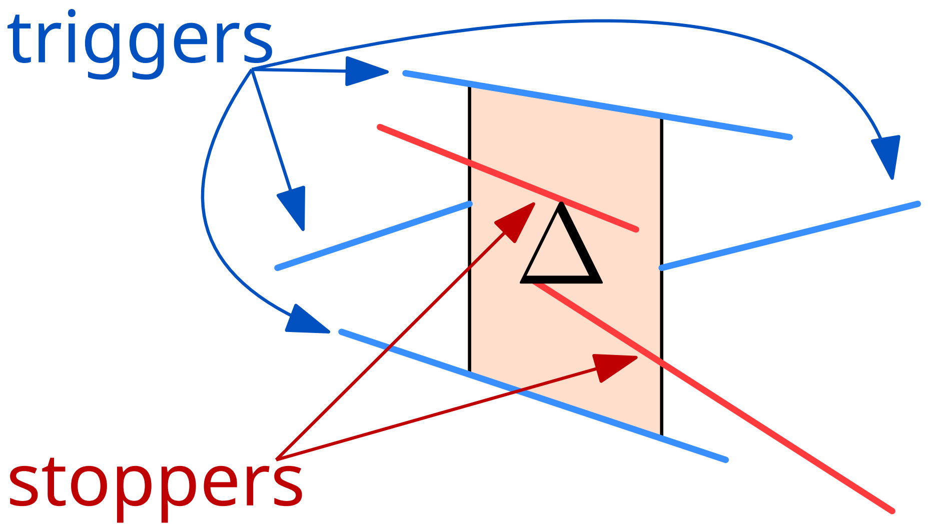

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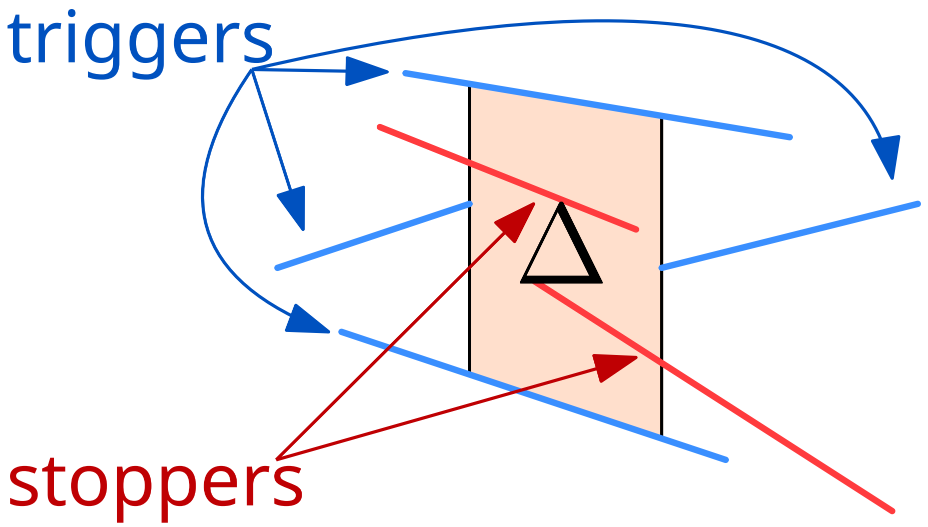
$$p_{PRIC} \le P[\ \boxed{1\,2\,\ldots\,i\text{-}1\ i\ i{+}1\,\ldots\,k}\ ]$$

triggers stoppers



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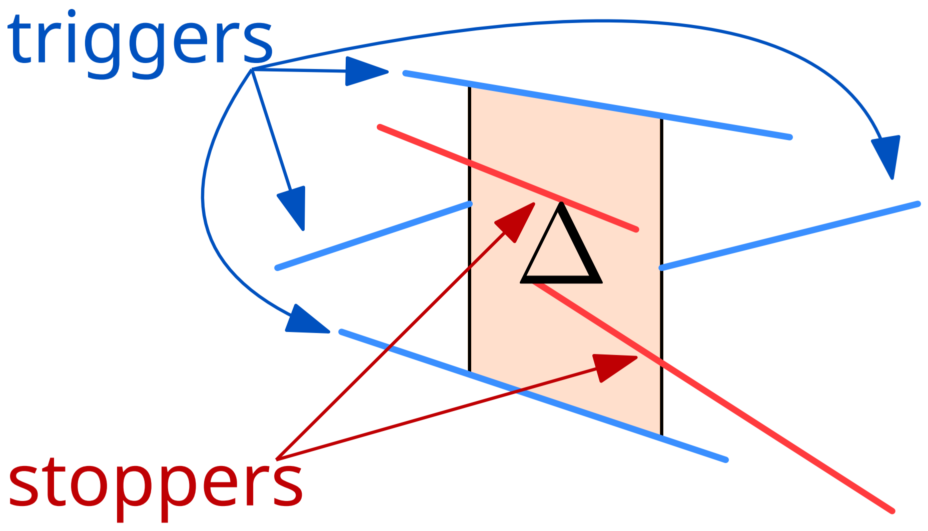

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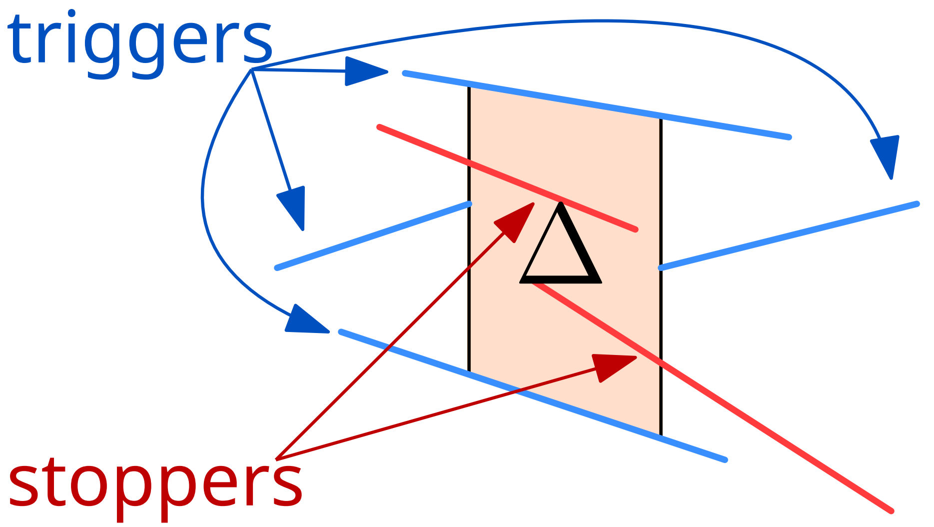

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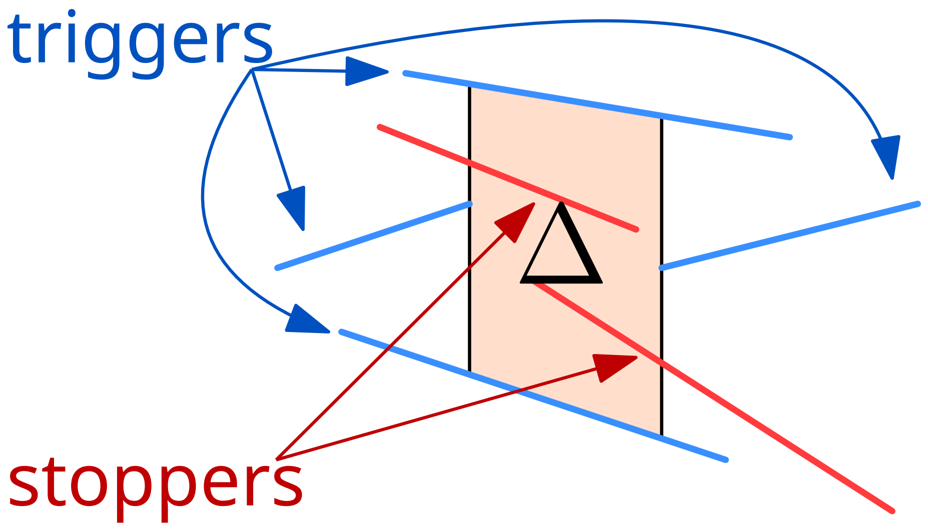

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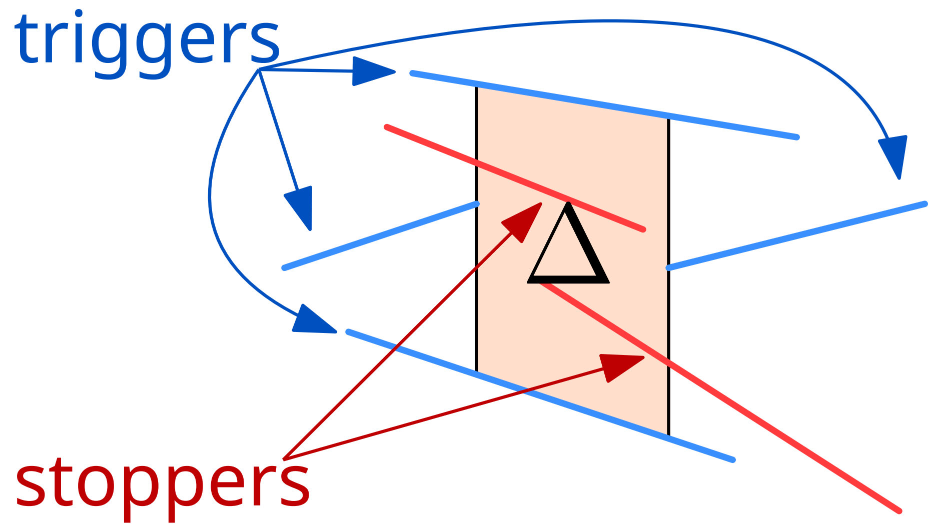

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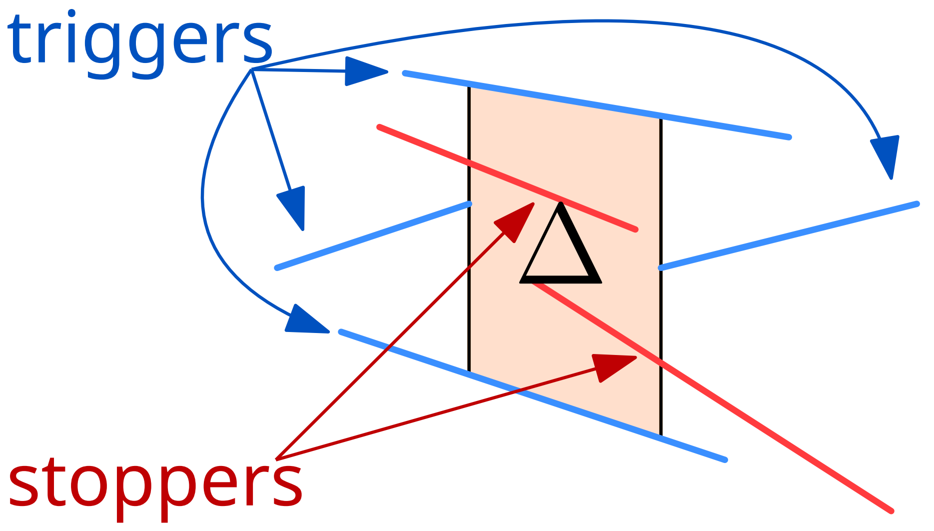

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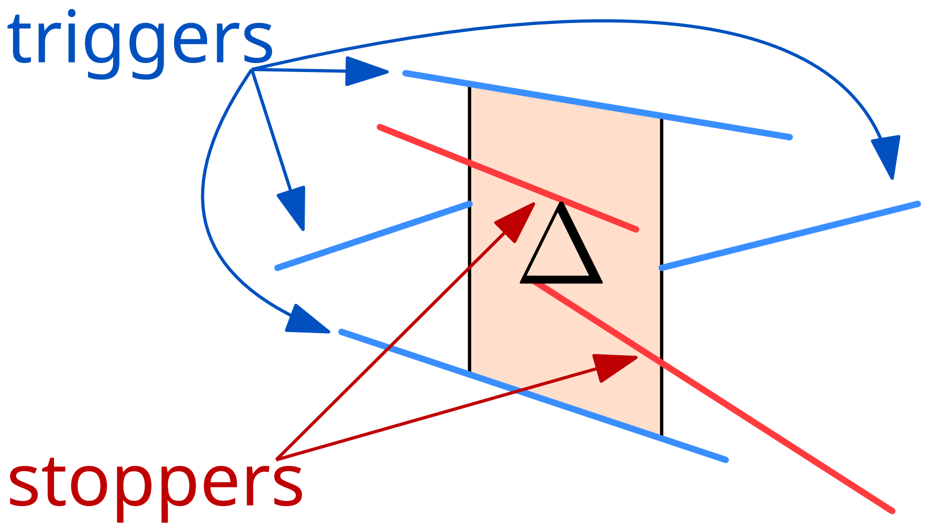

$$p_{PRIC} \leq P[\ \boxed{\underbrace{1\;2\;\ldots\;i\text{-}1\;i}_{\text{triggers}}\;\underbrace{i{+}1\;\ldots\;k}_{\text{stoppers}}}\ ] = P[\bigcup_{i=1}^{k} (S_i \cap T_i)]$$

$T_i =$ All triggers of Δ appear in the i^{th} round or before

$S_i =$ The first stopper of Δ appears in the i^{th} round

$$= \sum_{i=1}^{k} P[S_i \cap T_i] = \sum_{i=1}^{k} P[S_i] \cdot P[T_i]$$

$\{S_i \cap T_i\}$ are disjoint

S_i and T_i are independent

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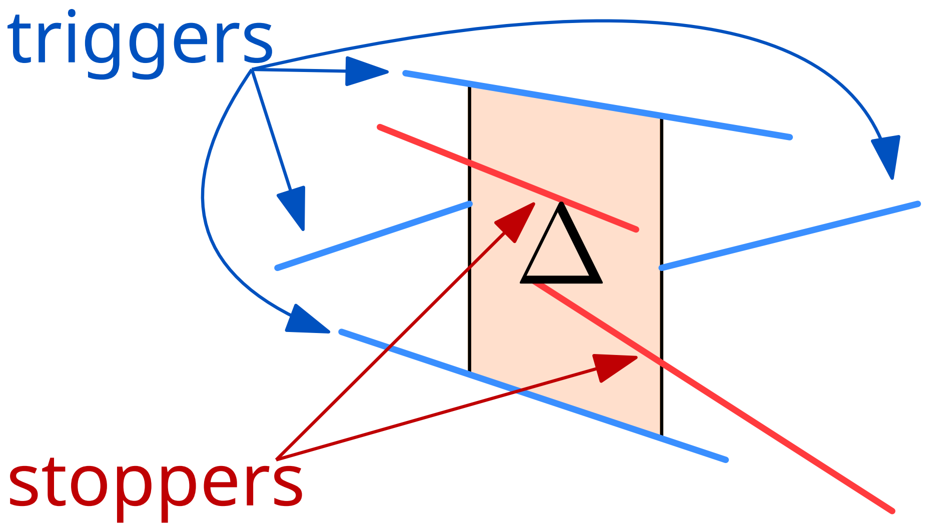
triggers stoppers



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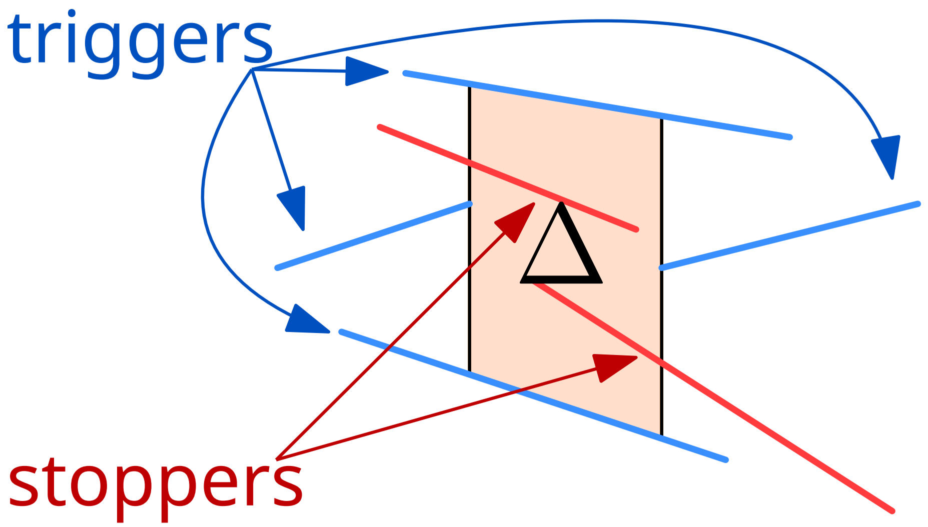

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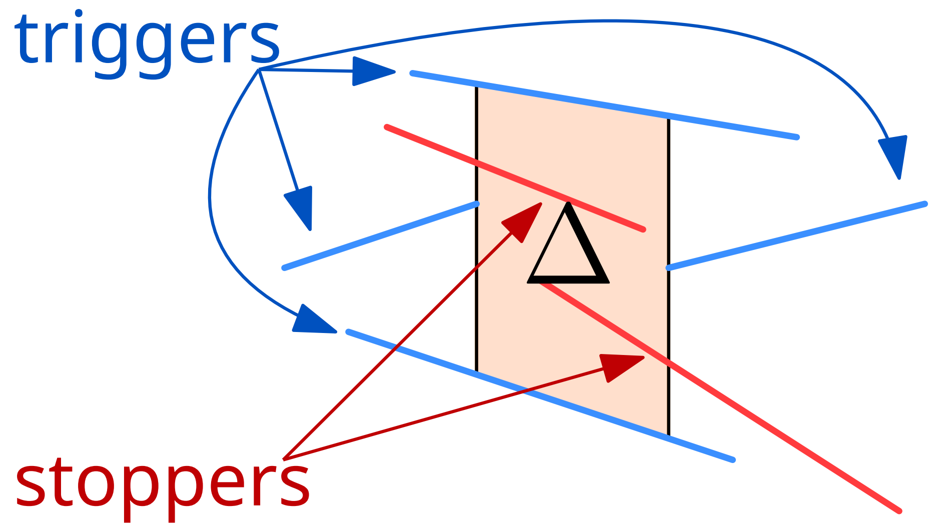

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$$\begin{aligned} P[T_{i-1}] &= P[T_{i-1} \cap T_i] \\ &= P[T_{i-1} | T_i] P[T_i] \\ &\geq 1/2^4 P[T_i] \end{aligned}$$

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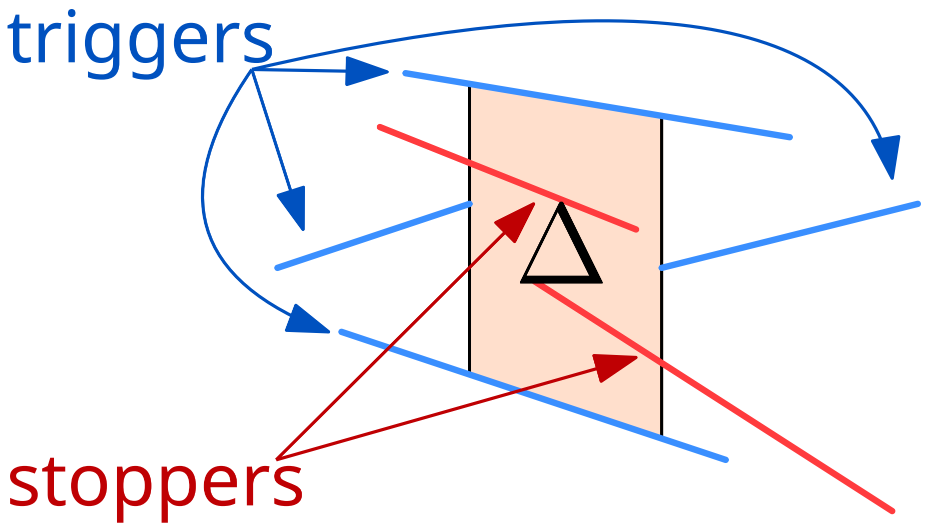

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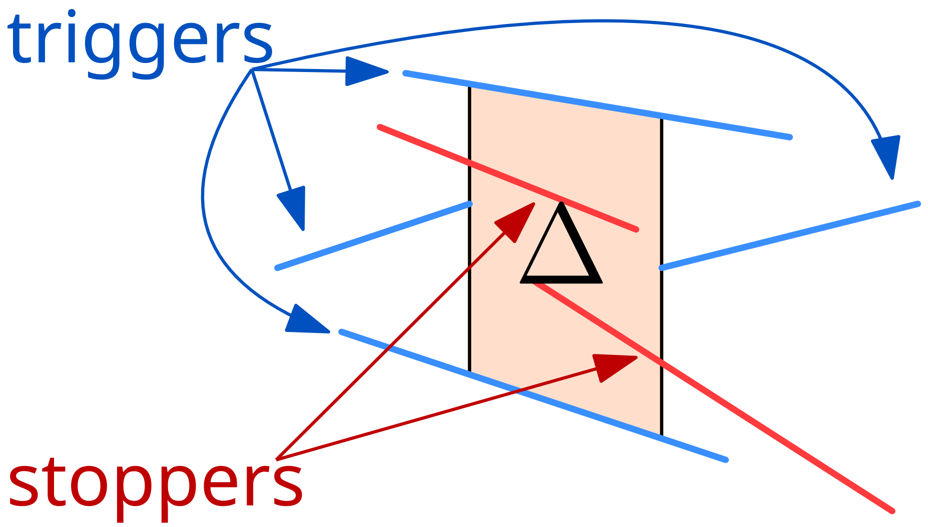

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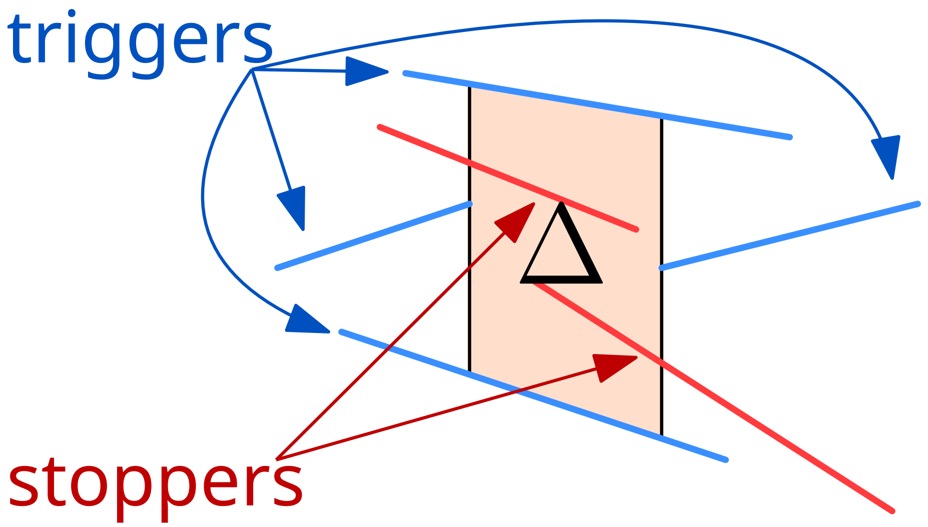

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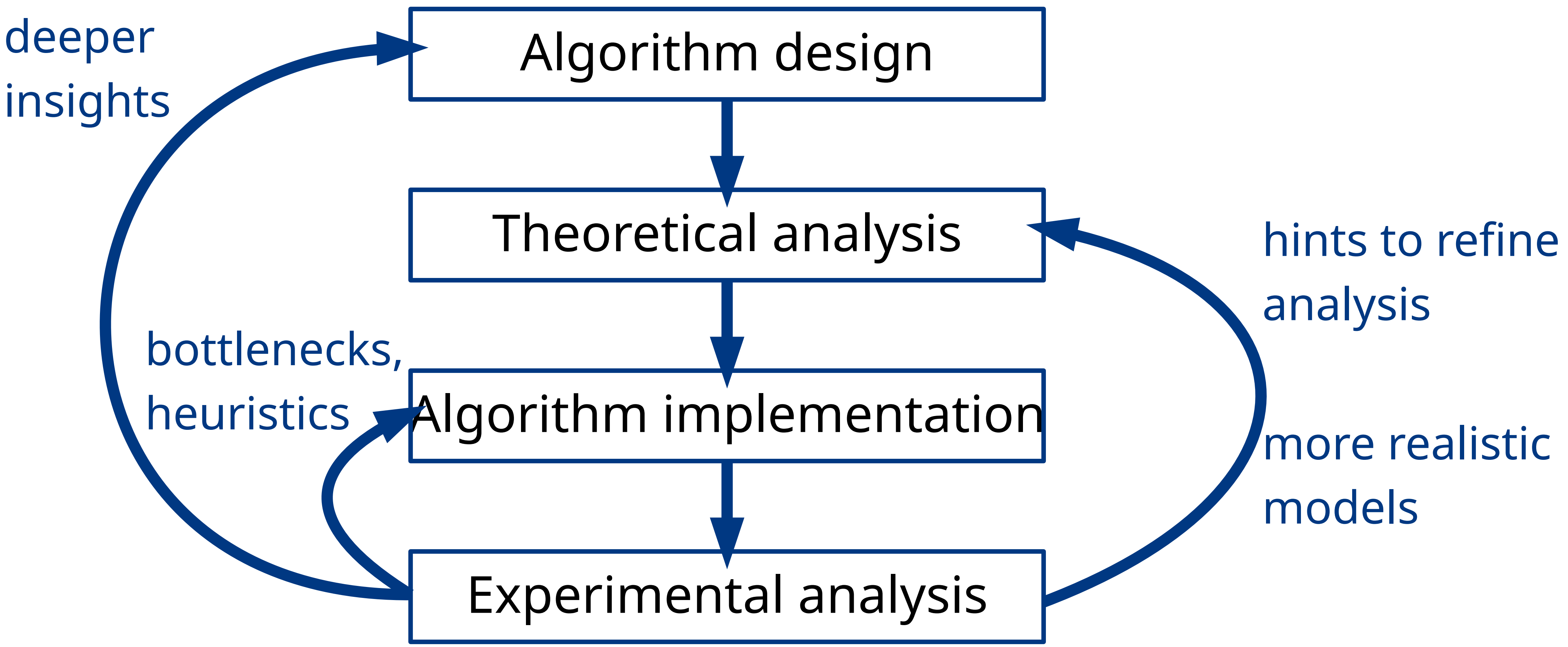
Theorem: The trapezoidal decomposition of n non-intersecting segments in the plane can be constructed in $O(n \log n)$ expected time using partial randomization.

Algorithm engineering cycle

- implementations, experiments, and theory go well together
- robust implementations are challenging
- strong experimental analysis is crucial



Experimental analysis

A Theoretician's Guide to the Experimental Analysis of Algorithms [Johnson]:

- Perform "newsworthy" experiments
- Place work in context
- Use reasonably efficient implementations
- Use testbeds that support general conclusions
- Provide explanations and back them up with experiment
- Ensure reproducibility
- Ensure comparability (and give the full picture)