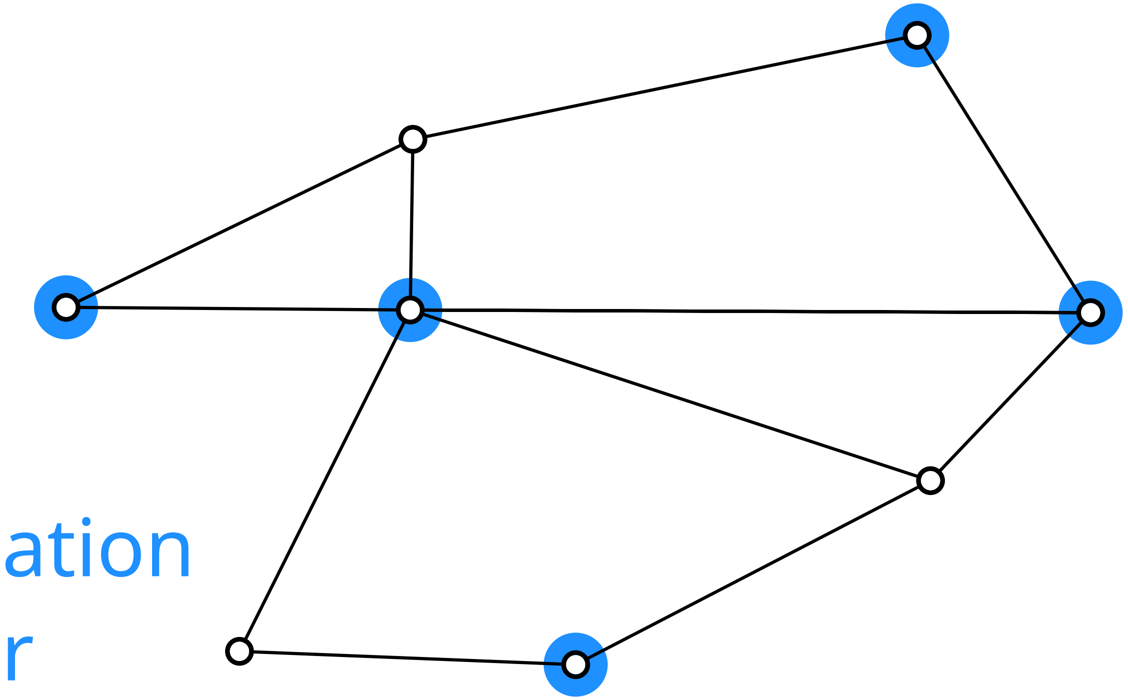
Approximation Algorithms

Introduction
NP-Optimization Problems and Approximation
Approximation Algorithm for Vertex Cover



"All exact science is dominated by the idea of approximation."

– Bertrand Russell (1872 – 1970)

Approximation Algorithms

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  examples: traveling salesperson problem, maximum satisfiability

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today:

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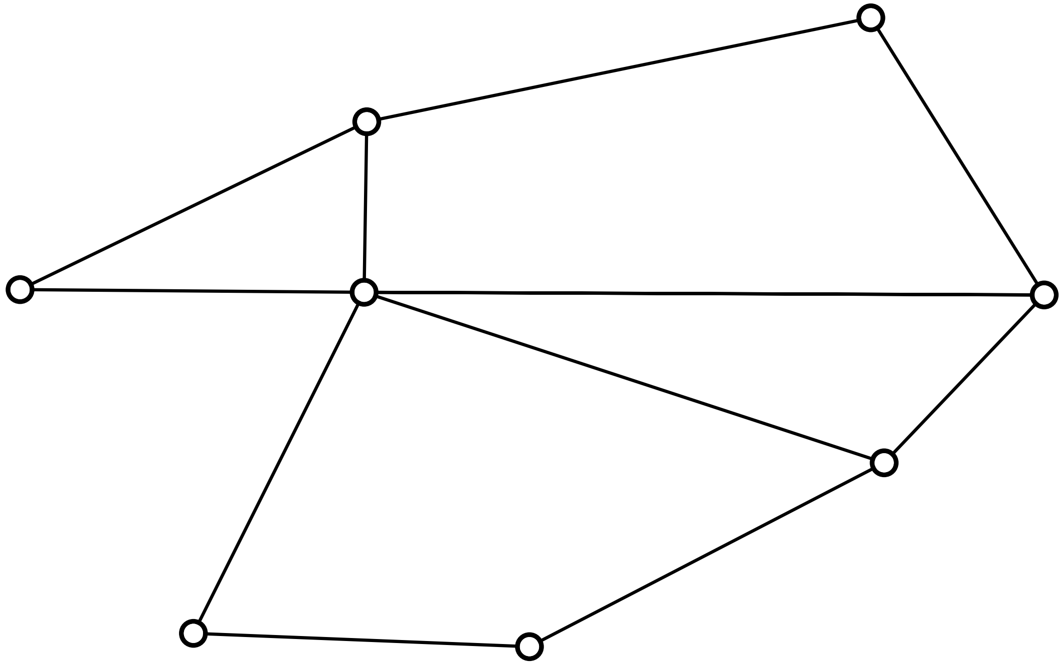
today:
technique: lower bounding optimal solution (key ingredient for approximation!)
optimization problem: vertex cover

VertexCover

Input: Graph $G = (V, E)$

Output:



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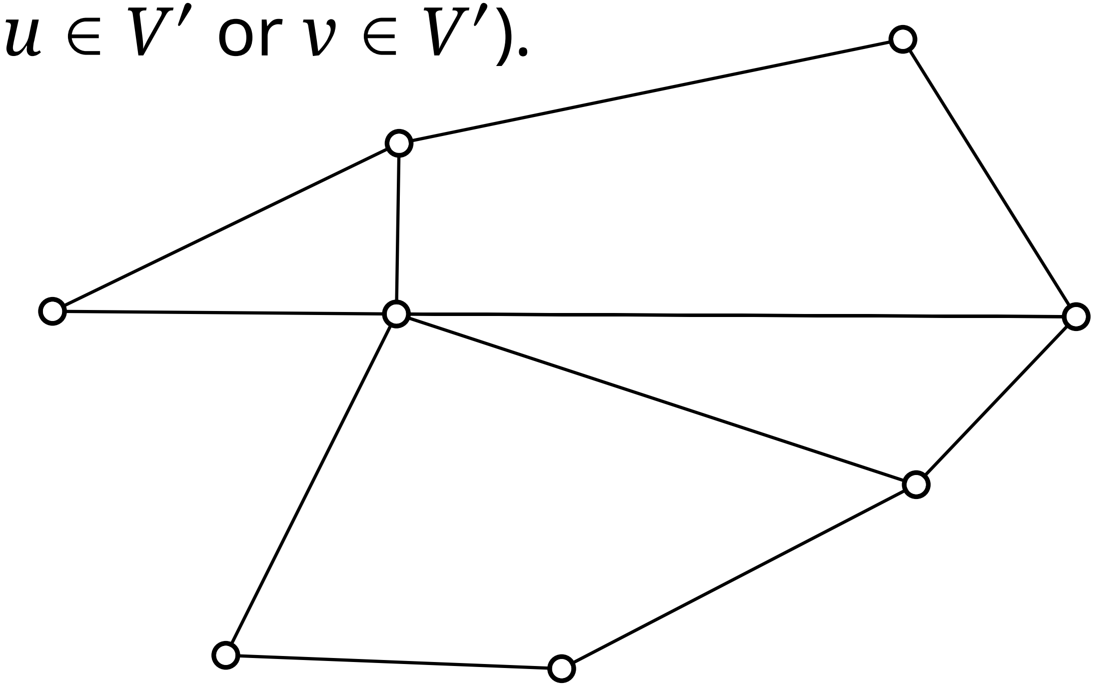

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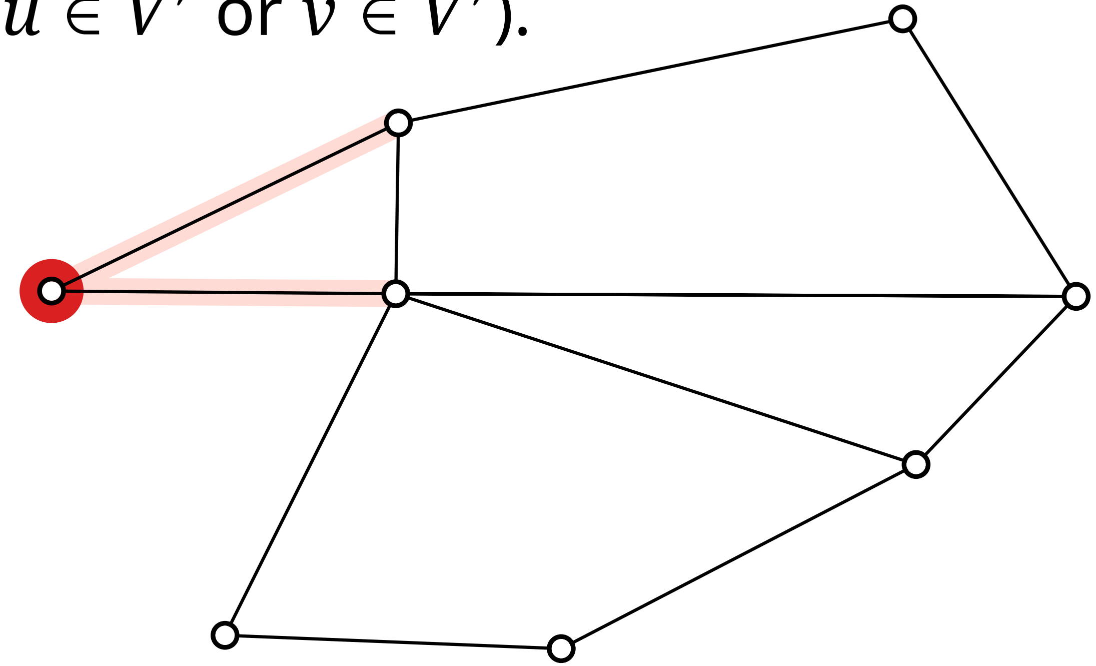

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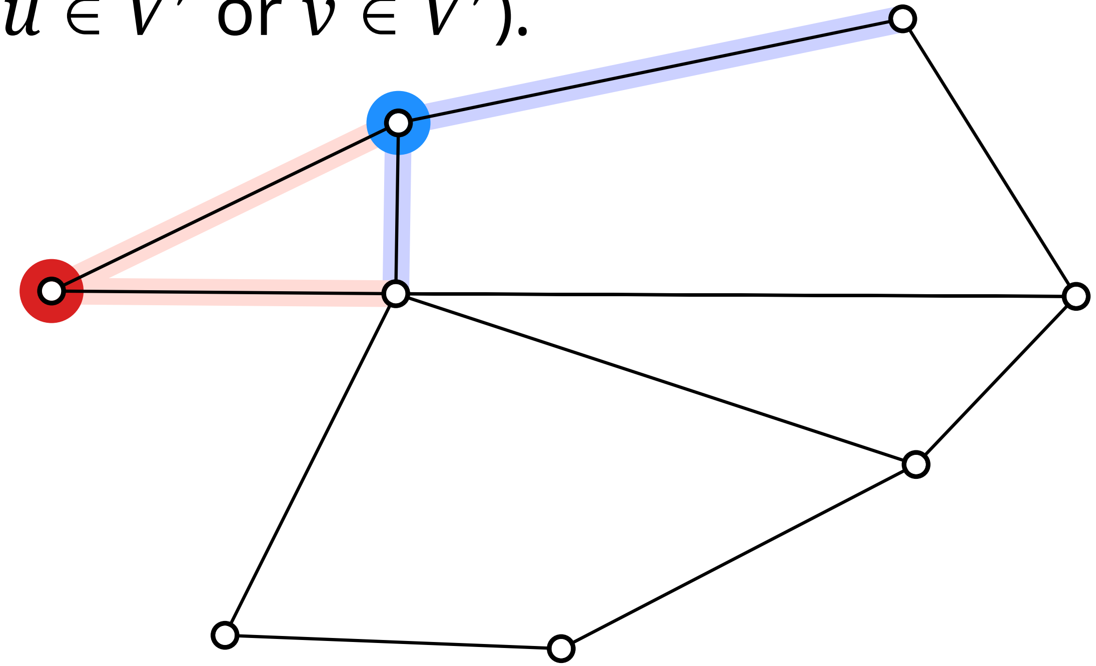

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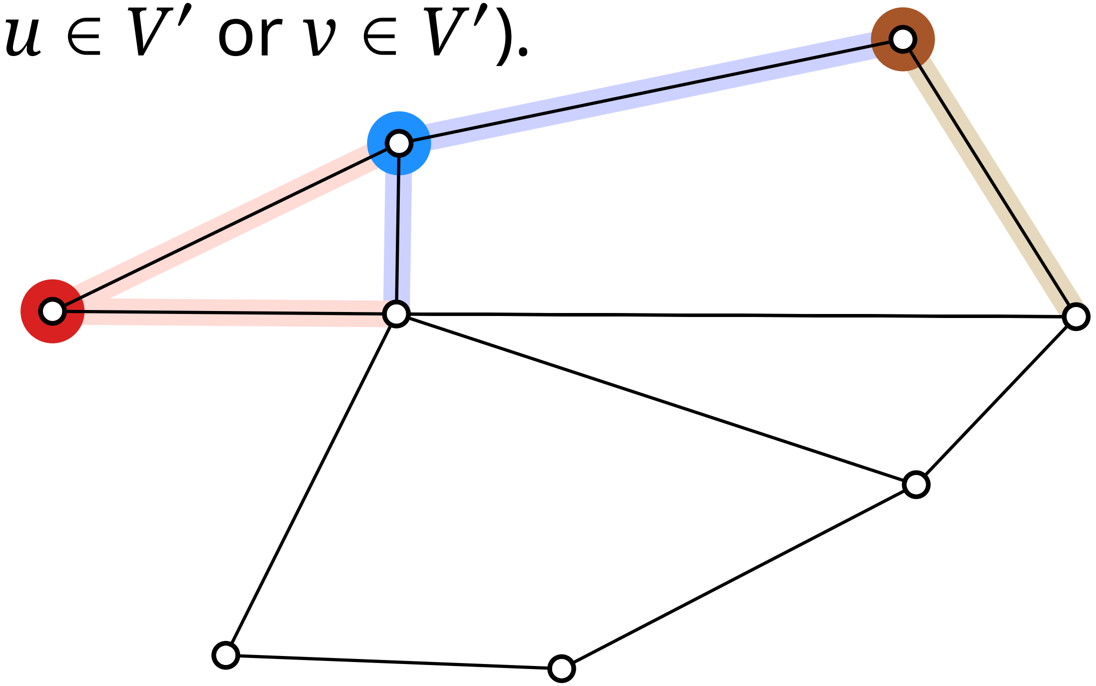

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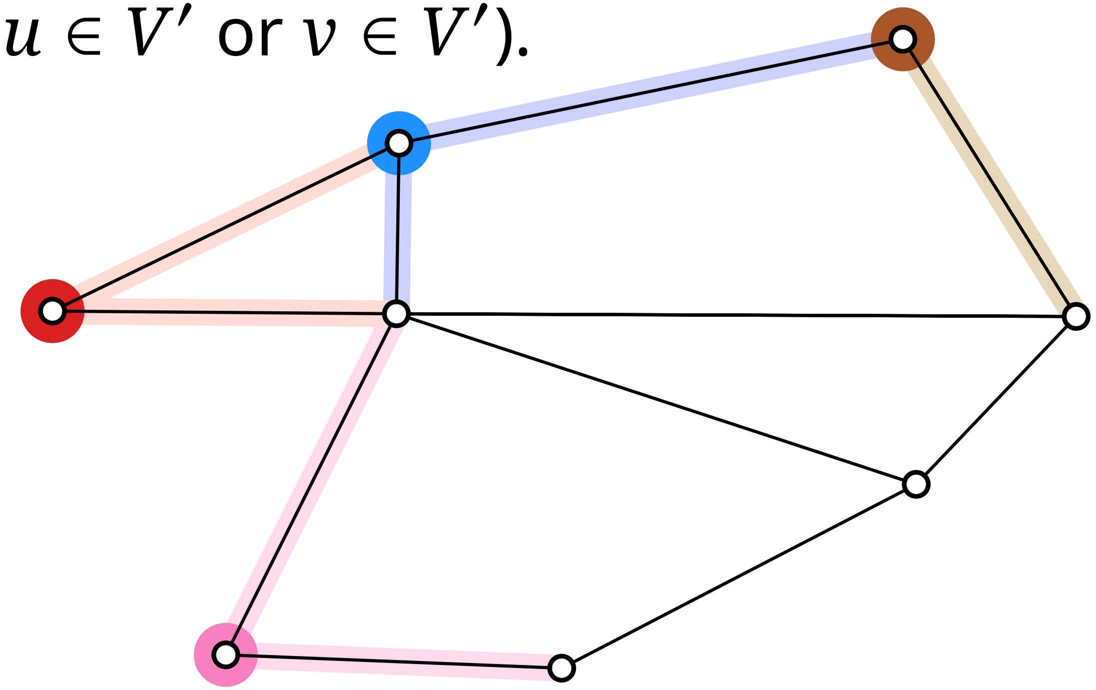

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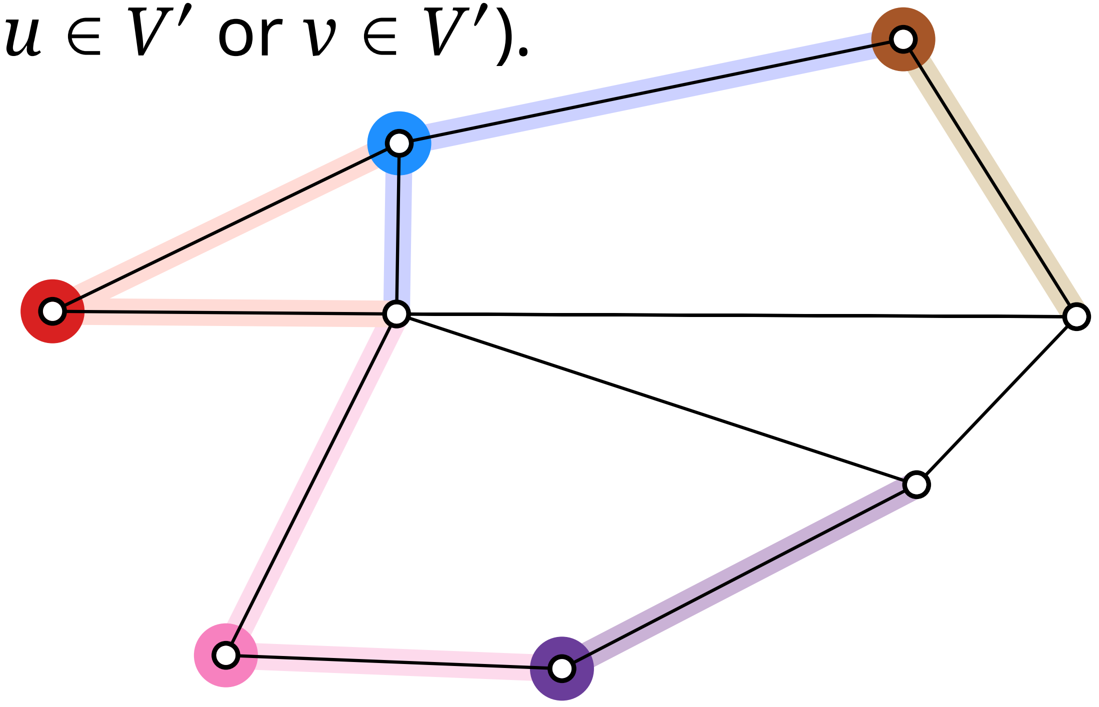

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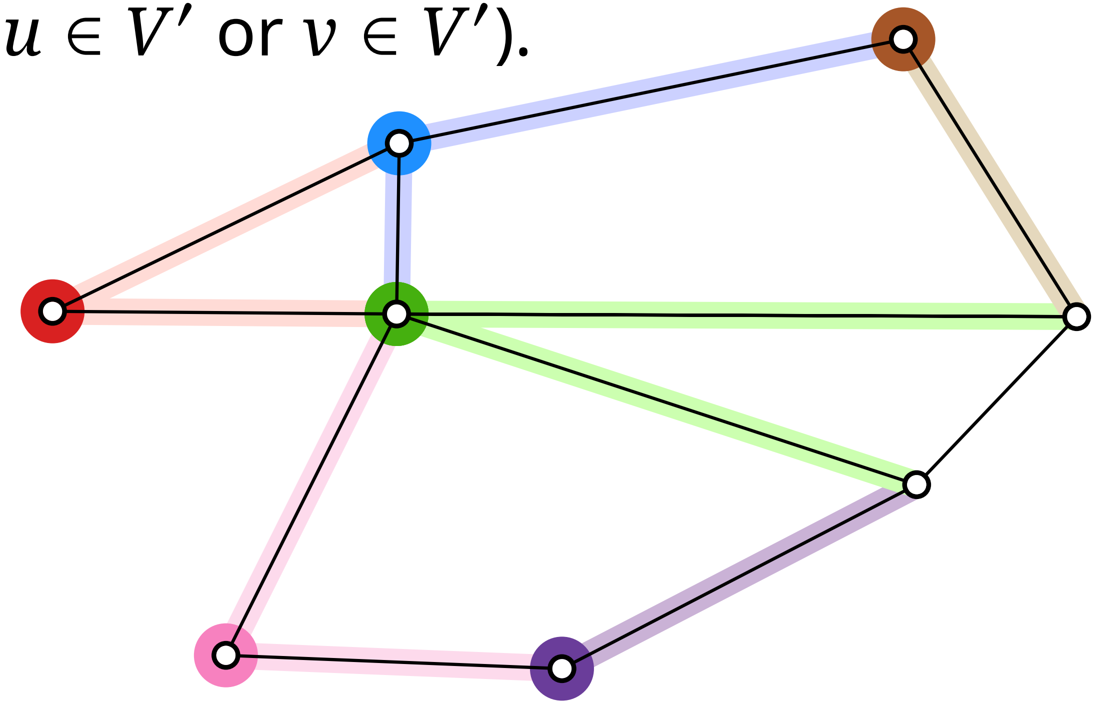

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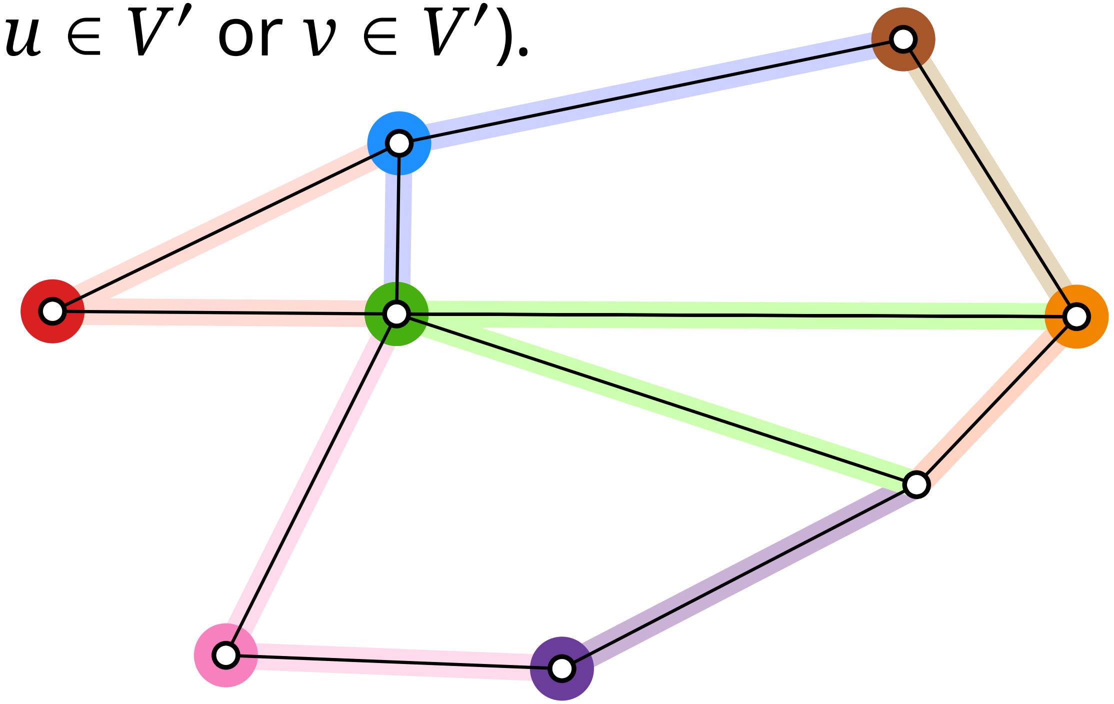

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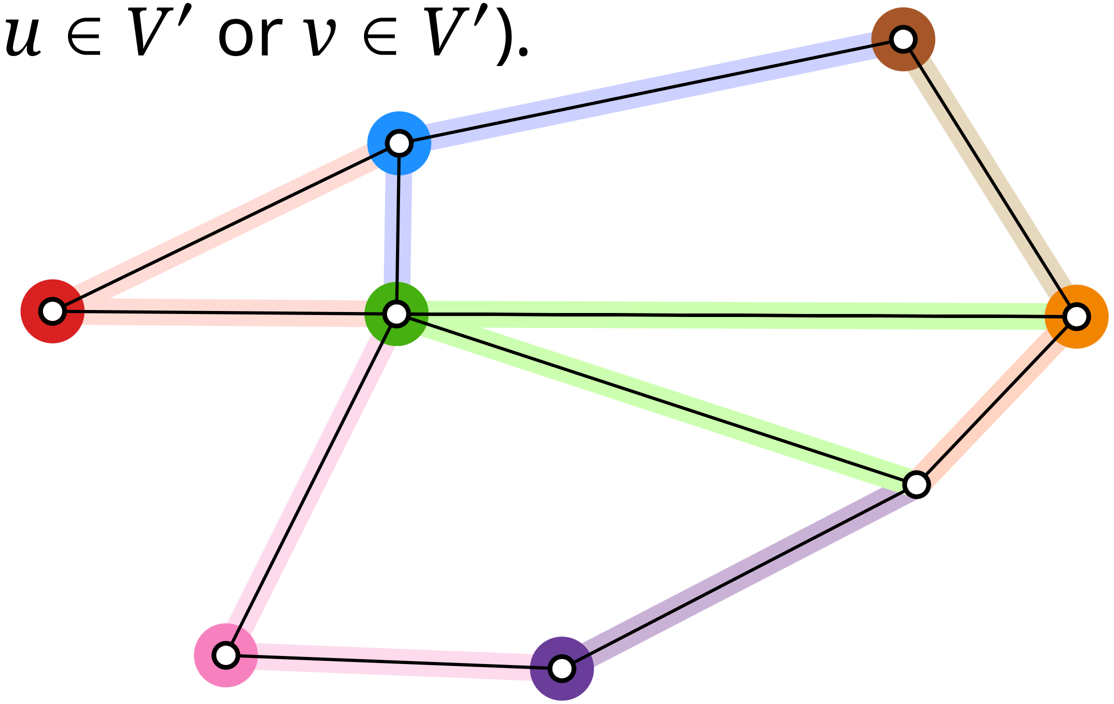

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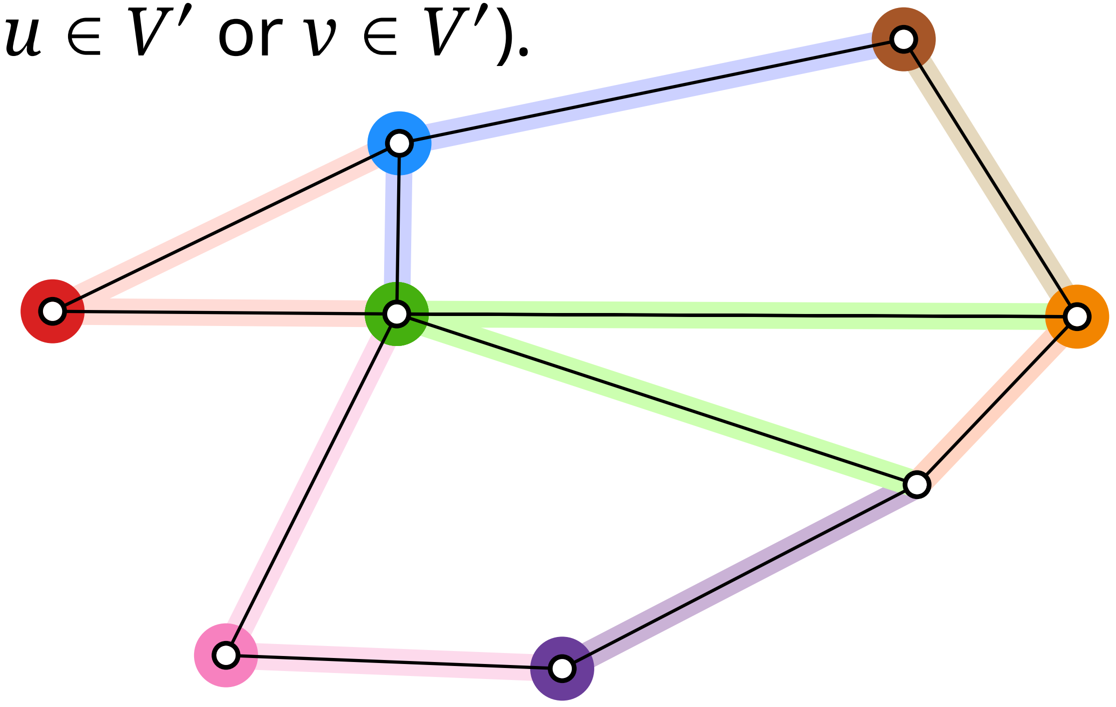

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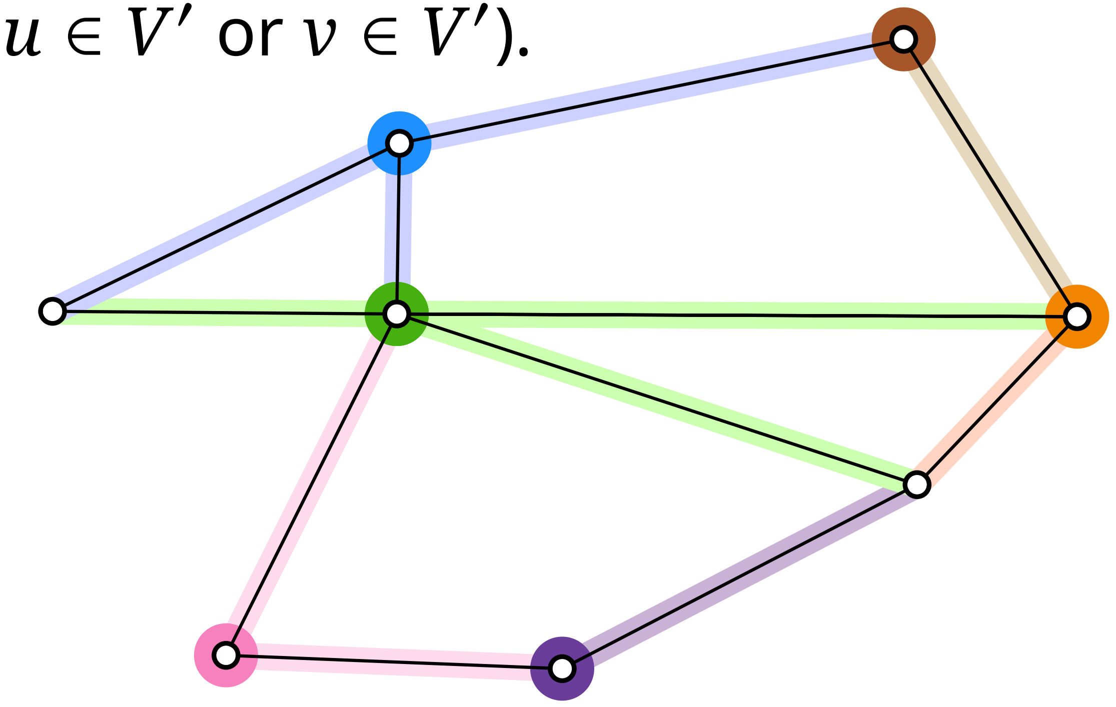

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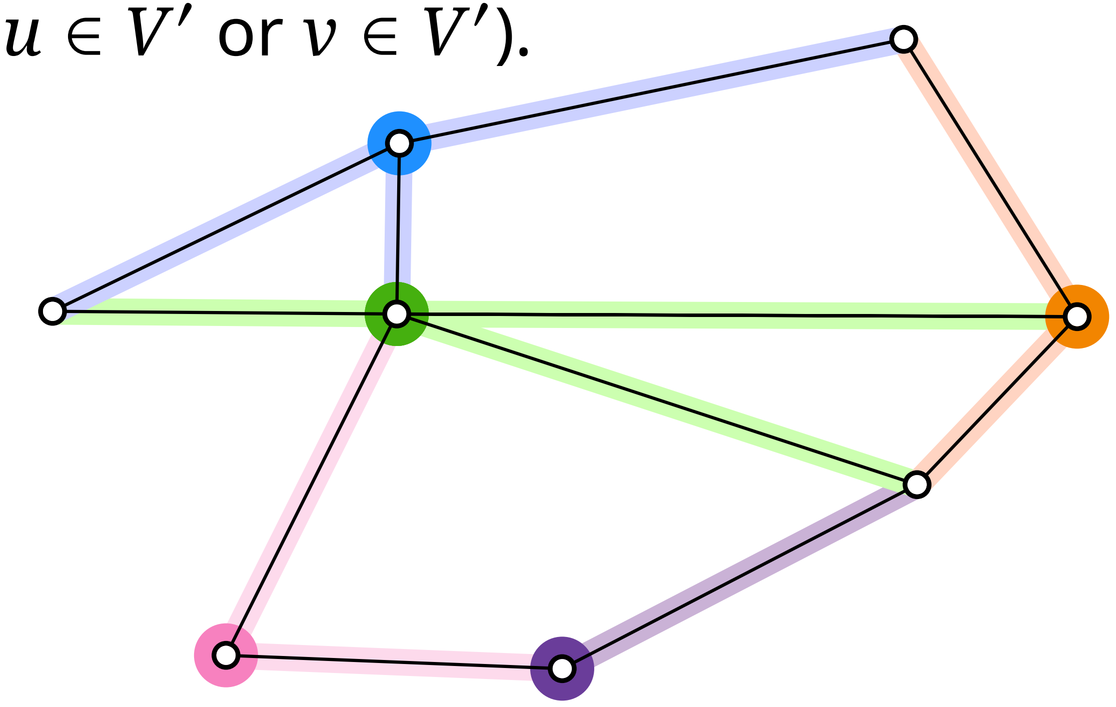

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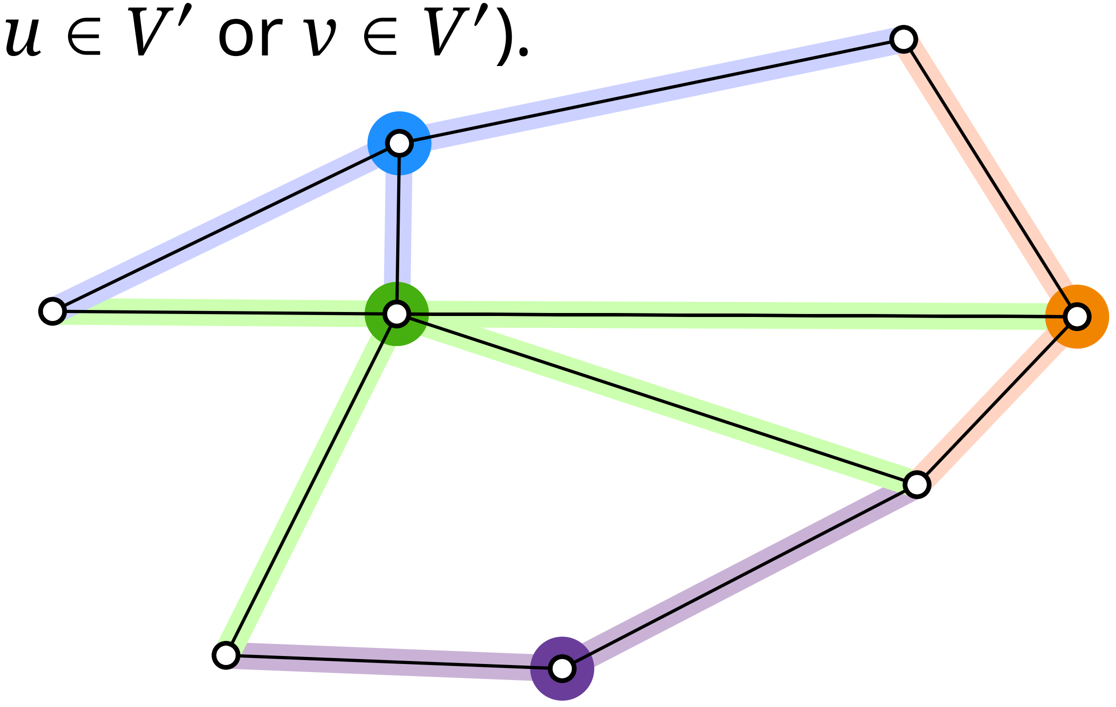

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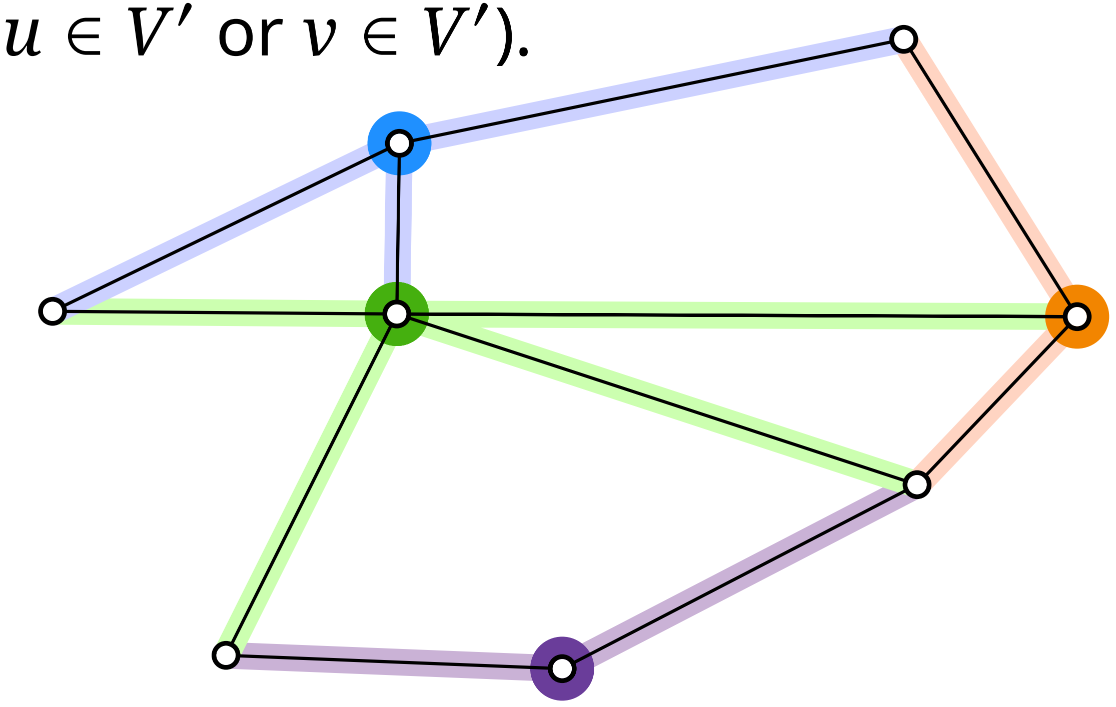

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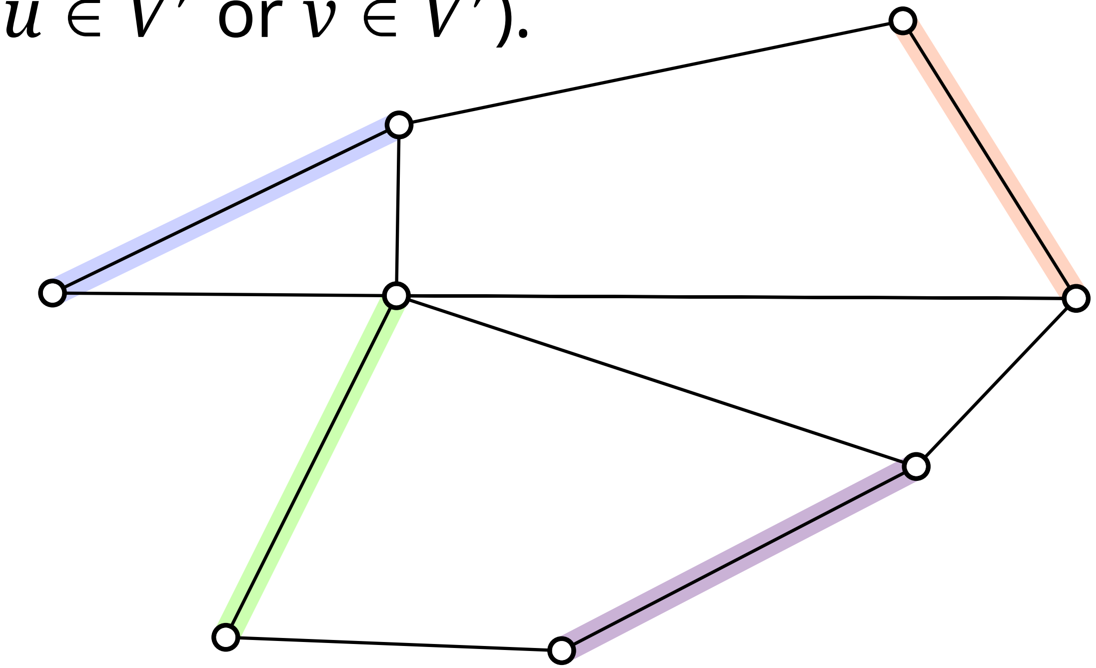

this is a vertex cover 1 vertex per colored edge needed

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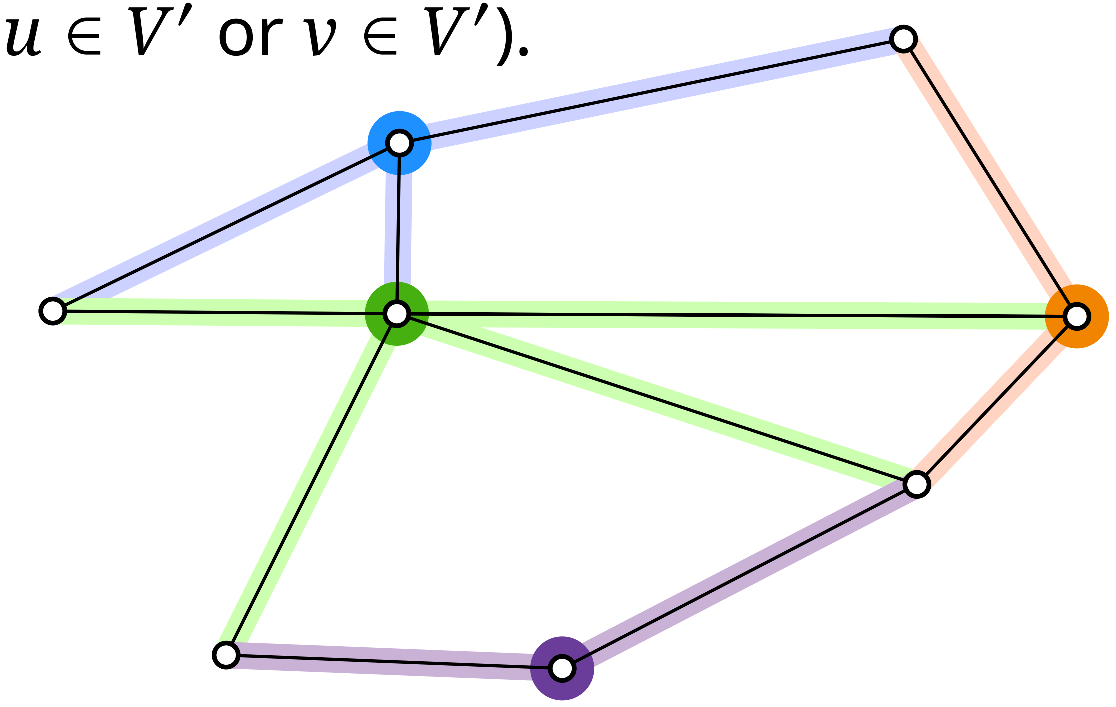

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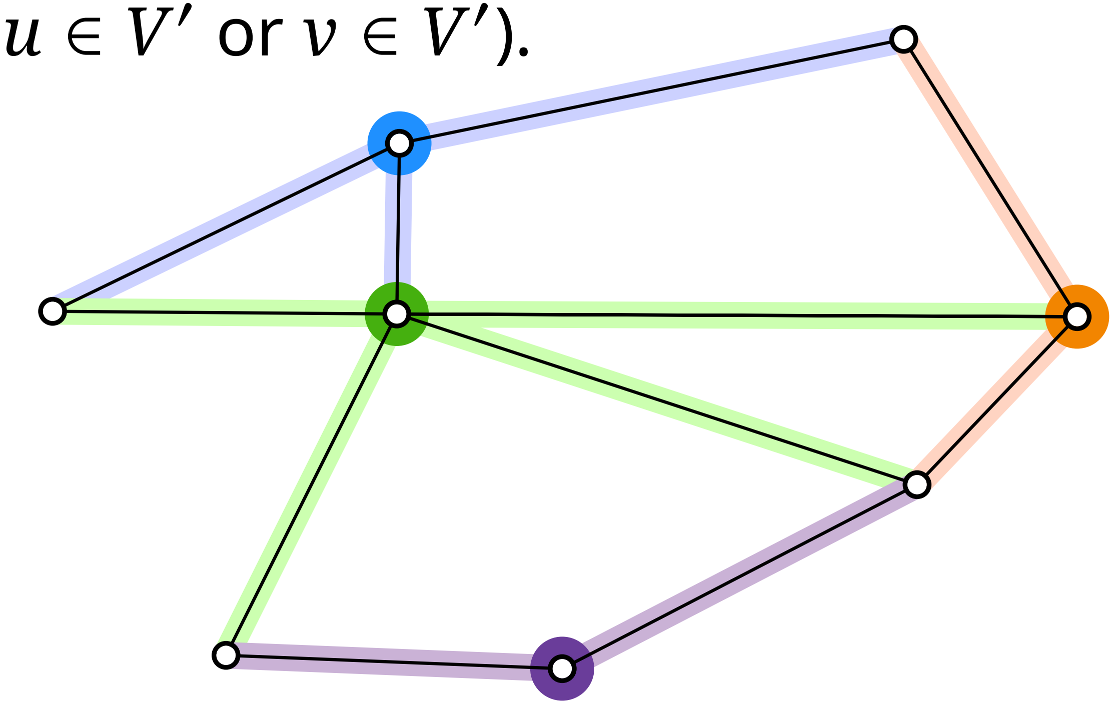

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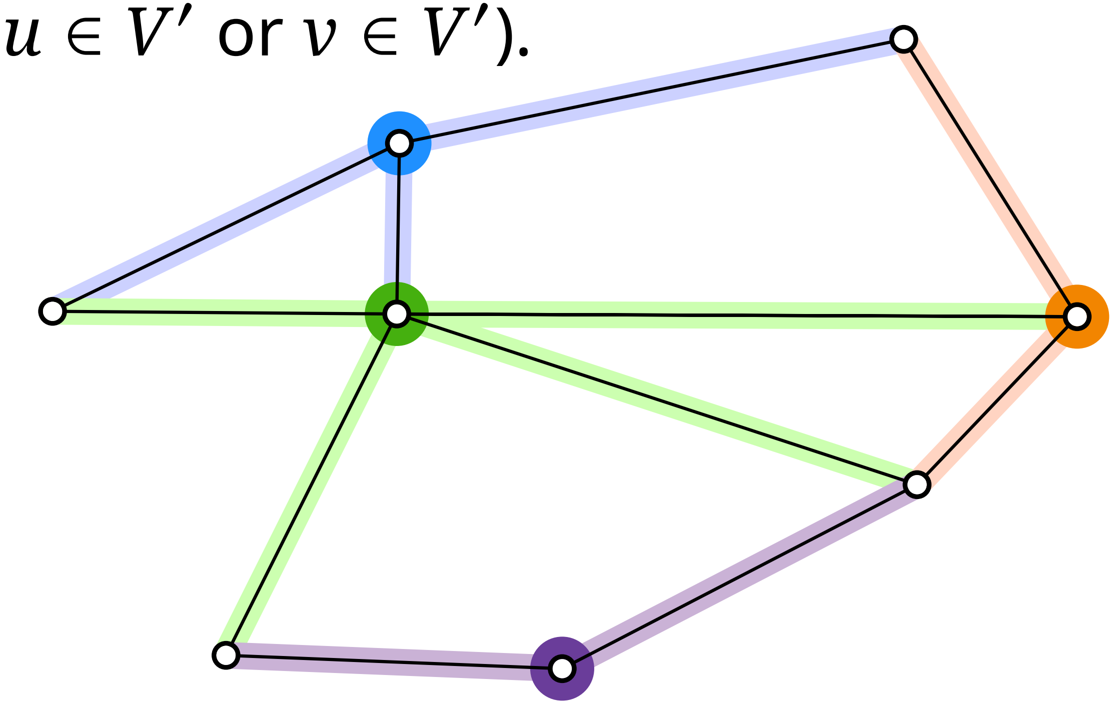

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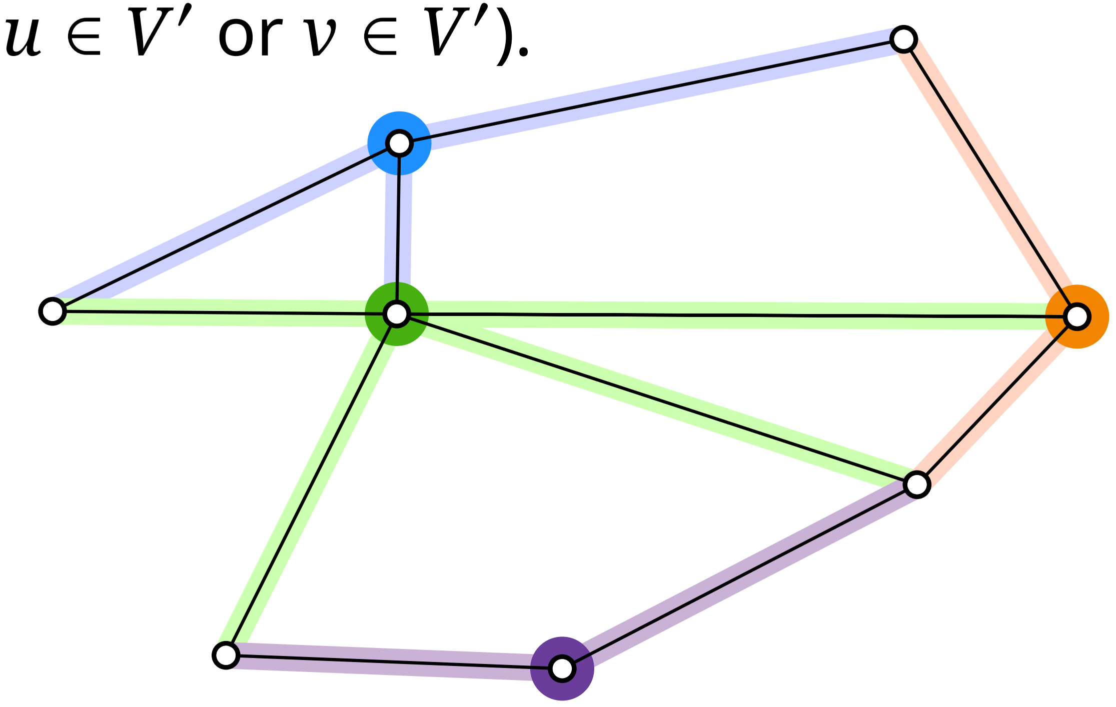

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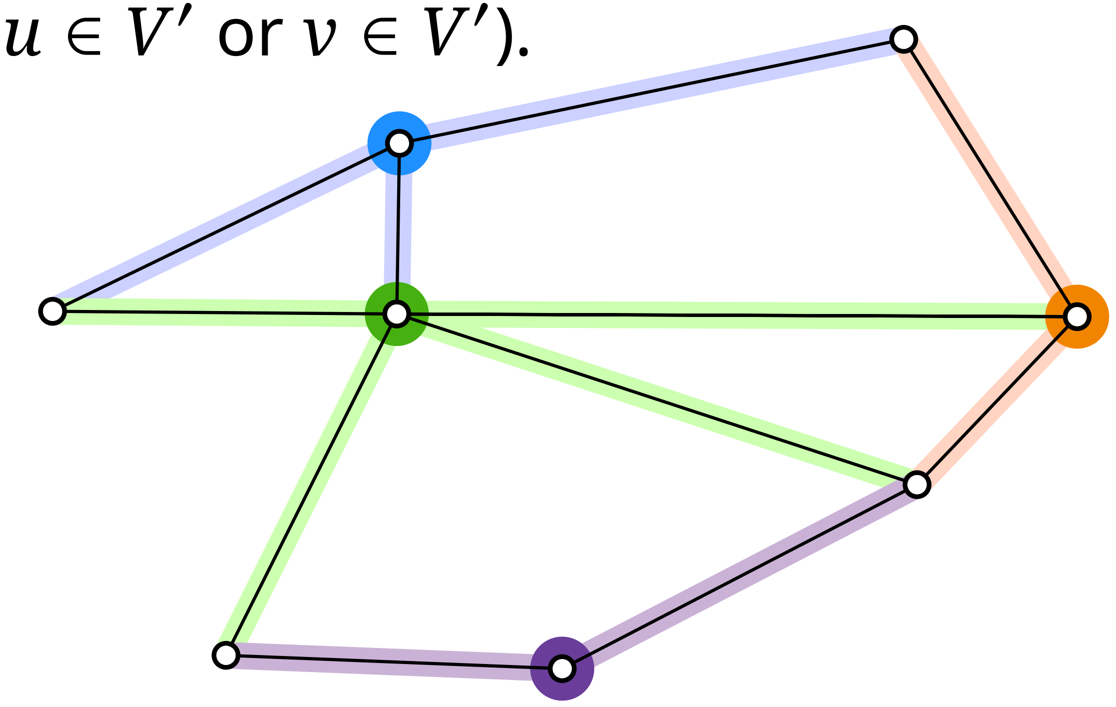

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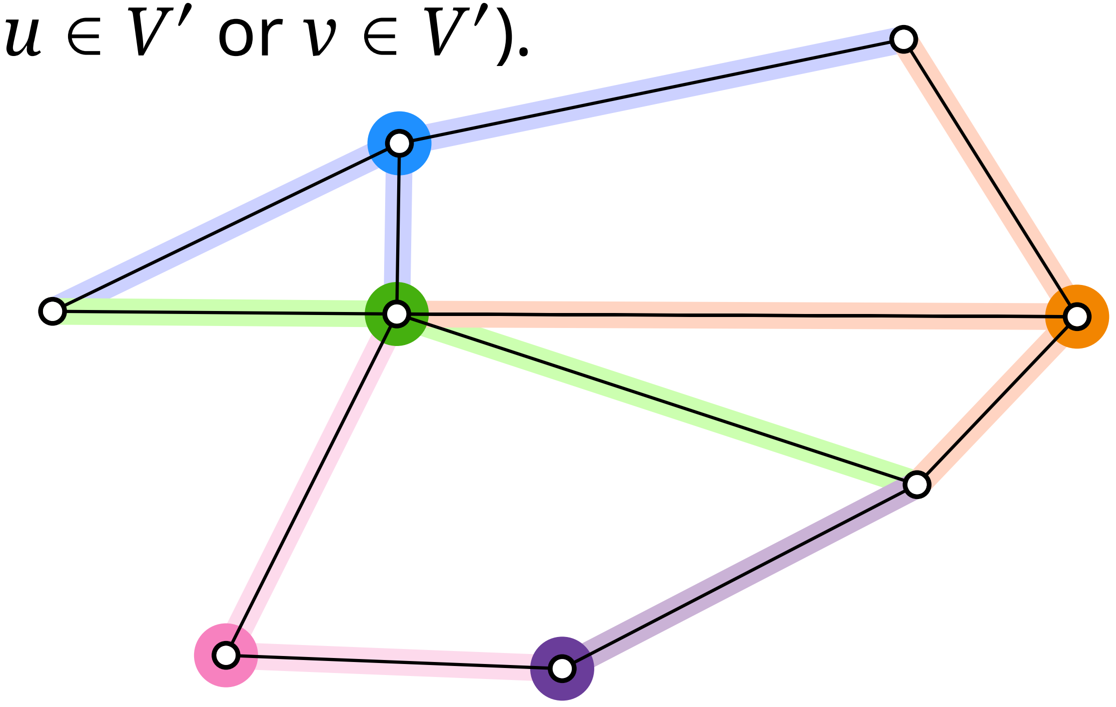

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"good" ($5/4$-) approximate solution

Key Concepts: NP-Optimization Problems and Approximation Algorithms

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- Π is either a minimization or maximization problem.

VERTEXCOVER: NP-Optimization Problem

Task: Fill in the gaps for the problem Π = VERTEX COVER.

D_Π = ???

For $I \in D_\Pi$: ??? $|I|$ = ???

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The optimal value $\mathrm{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\mathrm{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

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A factor-α approximation algorithm for Π is an efficient algorithm that provides, for **any** instance $I \in D_{\Pi}$, a feasible solution $s \in S_{\Pi}(I)$ such that

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A factor-α approximation algorithm for Π is an efficient algorithm that provides, for **any** instance $I \in D_\Pi$, a feasible solution $s \in S_\Pi(I)$ such that

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Approximation Algorithm for VertexCover

Approximation Algorithm for VERTEXCOVER

Ideas?

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- Edge-Greedy

Approximation Algorithm for VERTEXCOVER

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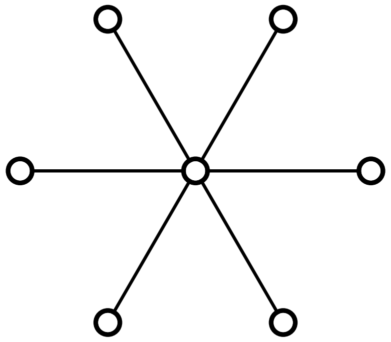

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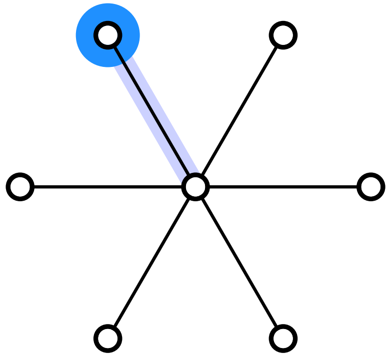

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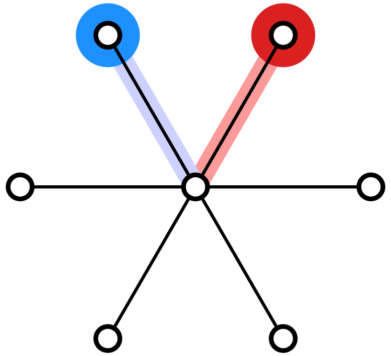

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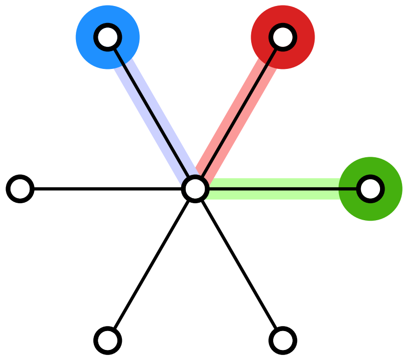

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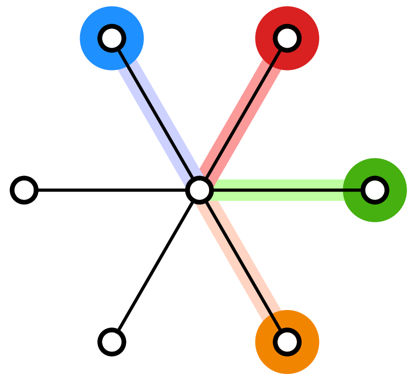

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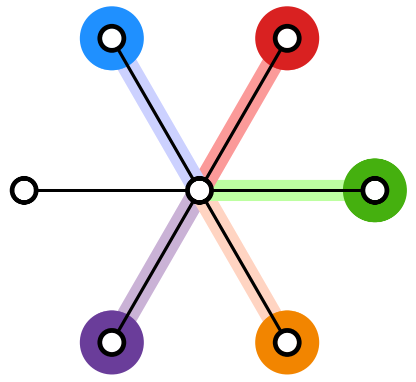

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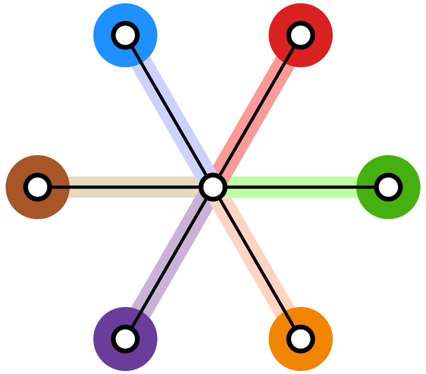

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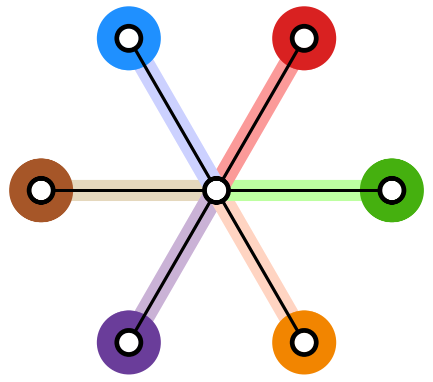

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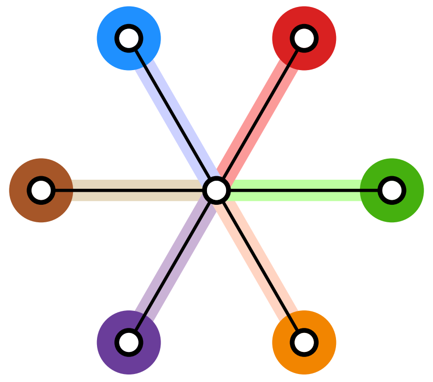

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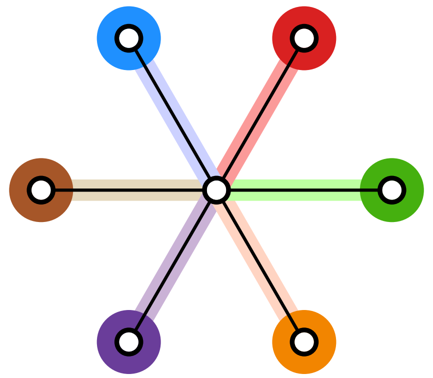

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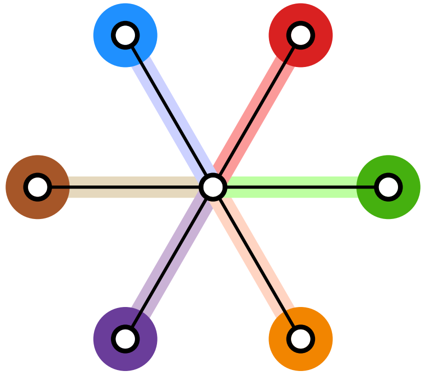

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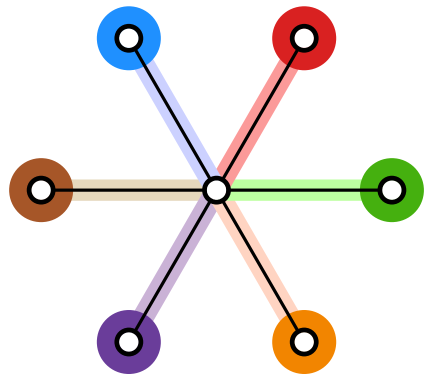

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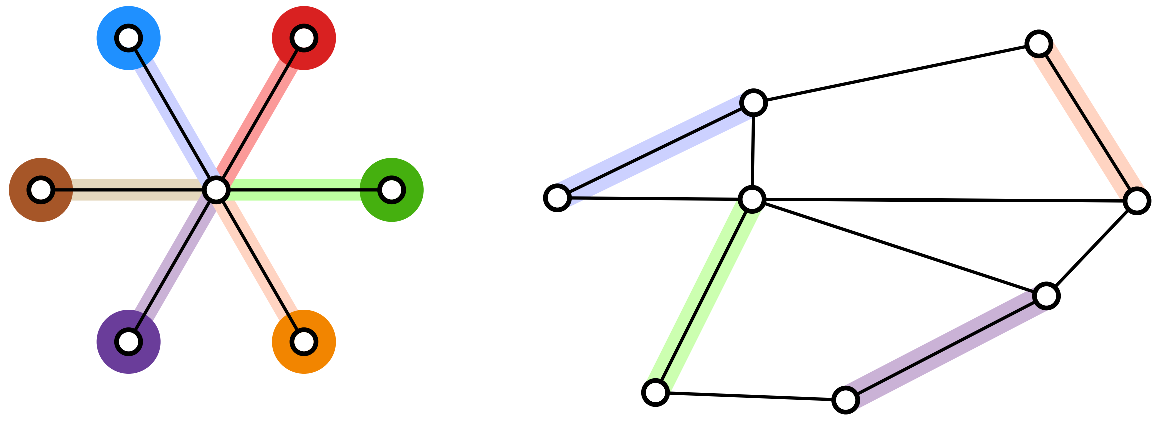

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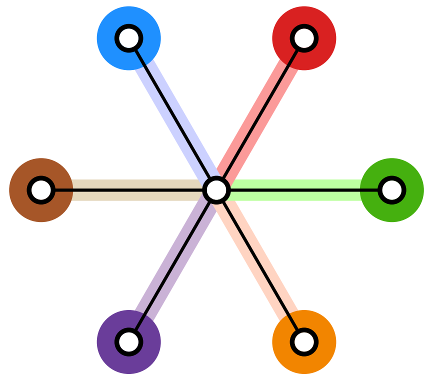

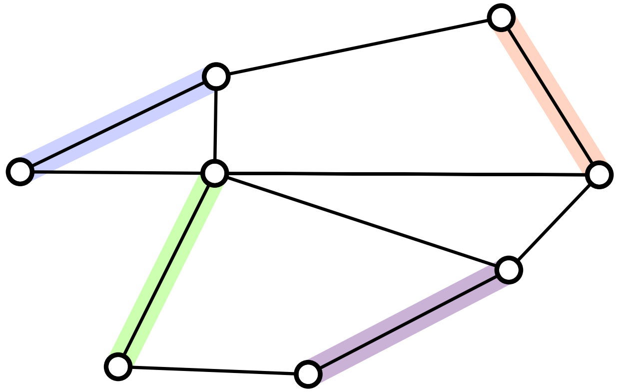

How did we argue that $OPT = 4$ for this instance?

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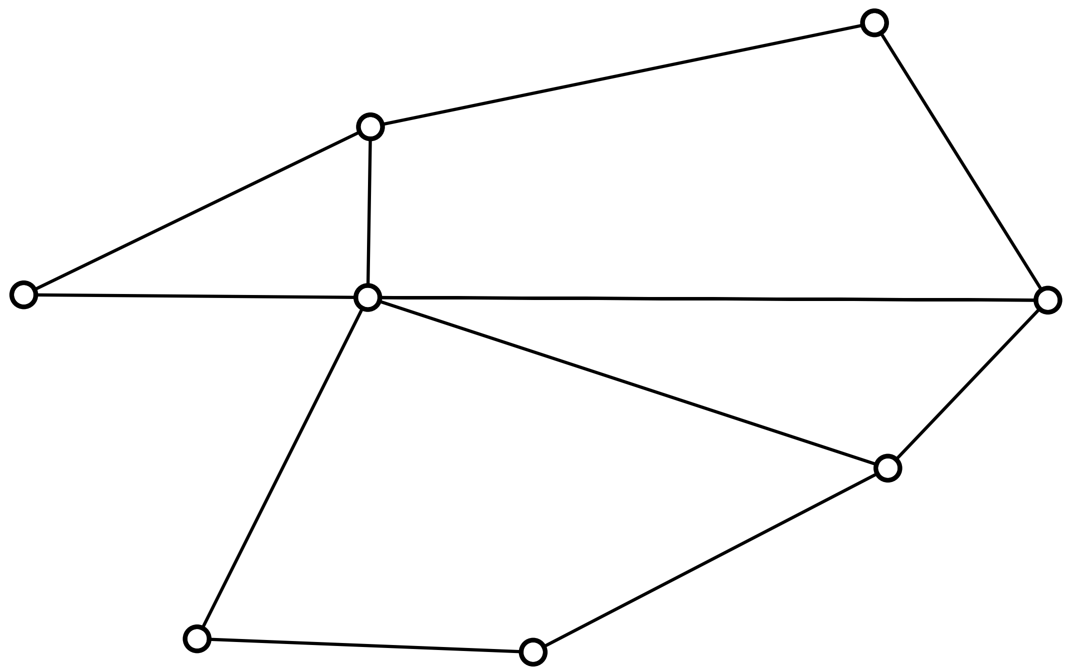
need at least one vertex for each edge in a vertex-disjoint set of edges

Lower Bound by Matchings



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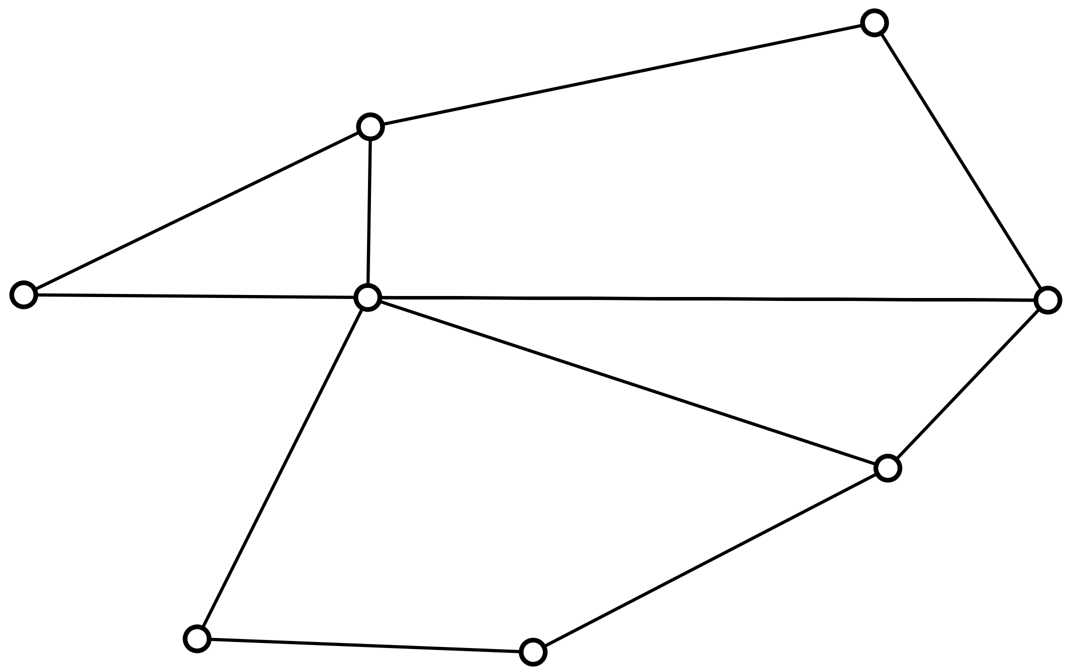

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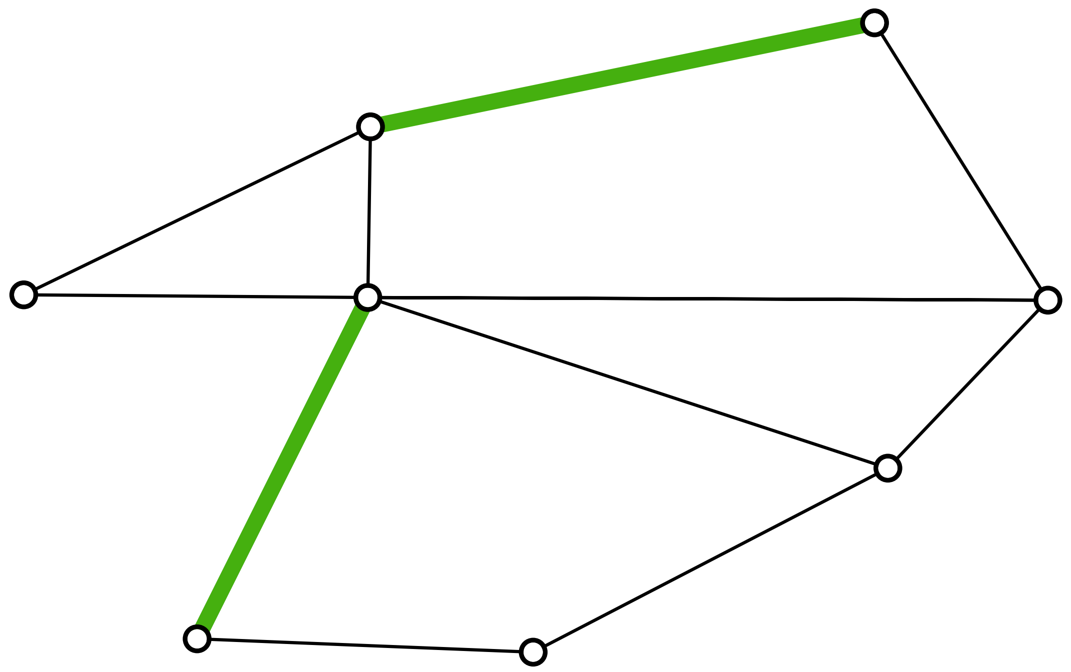

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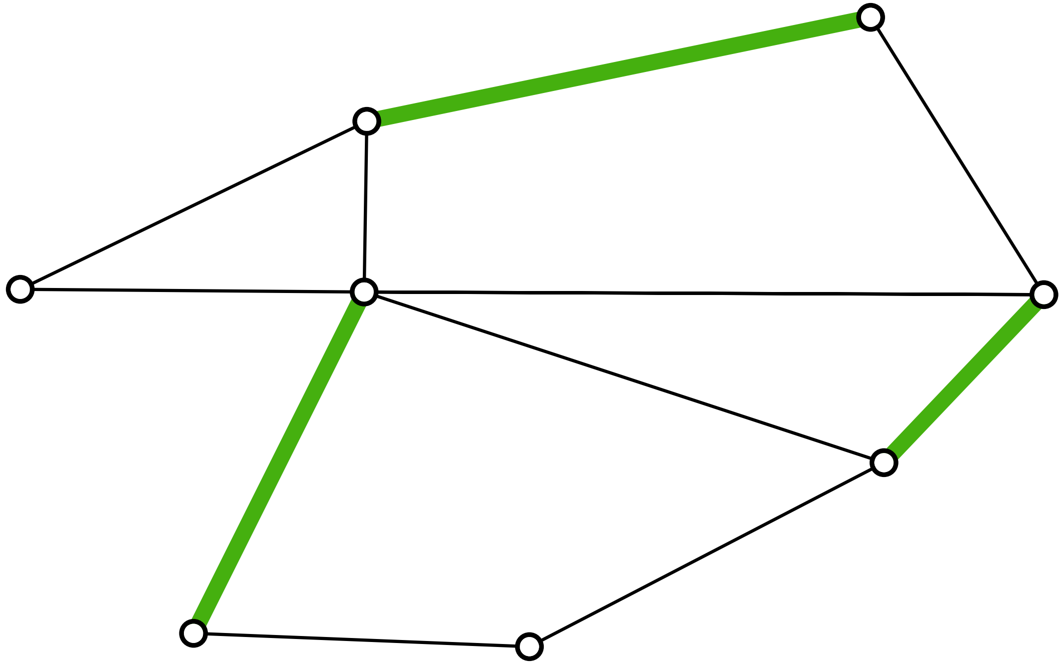

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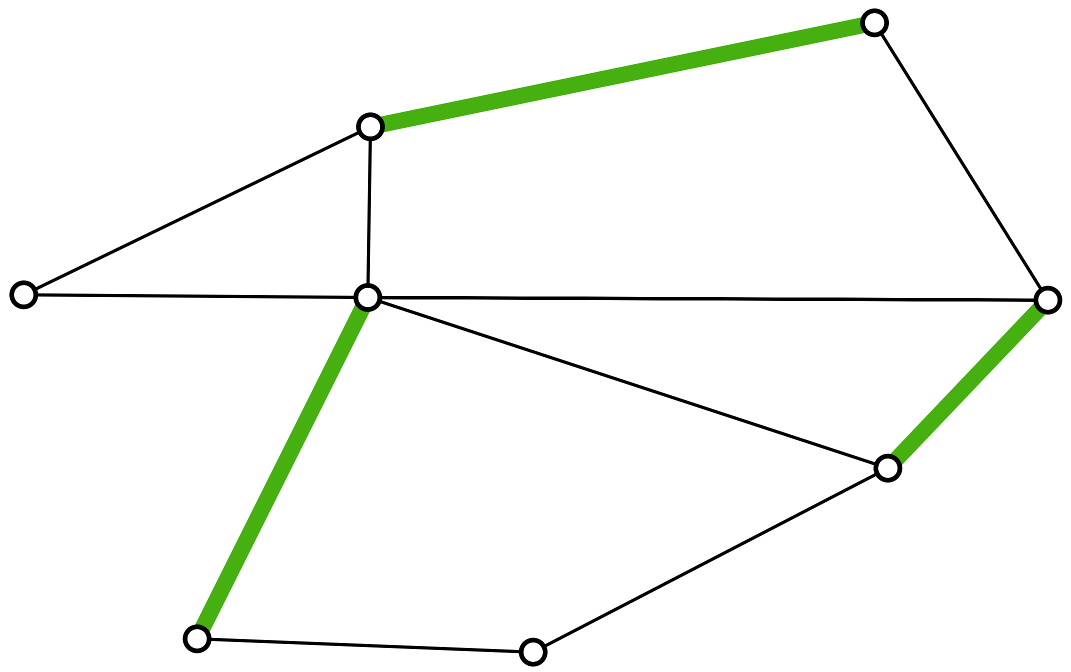

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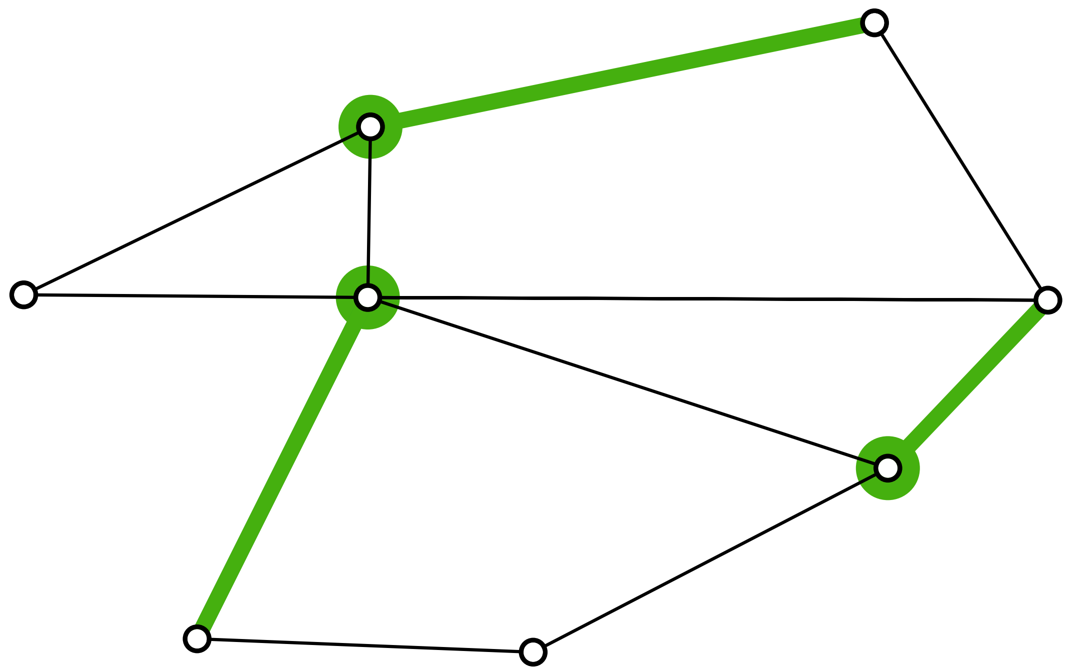
Vertex cover of M



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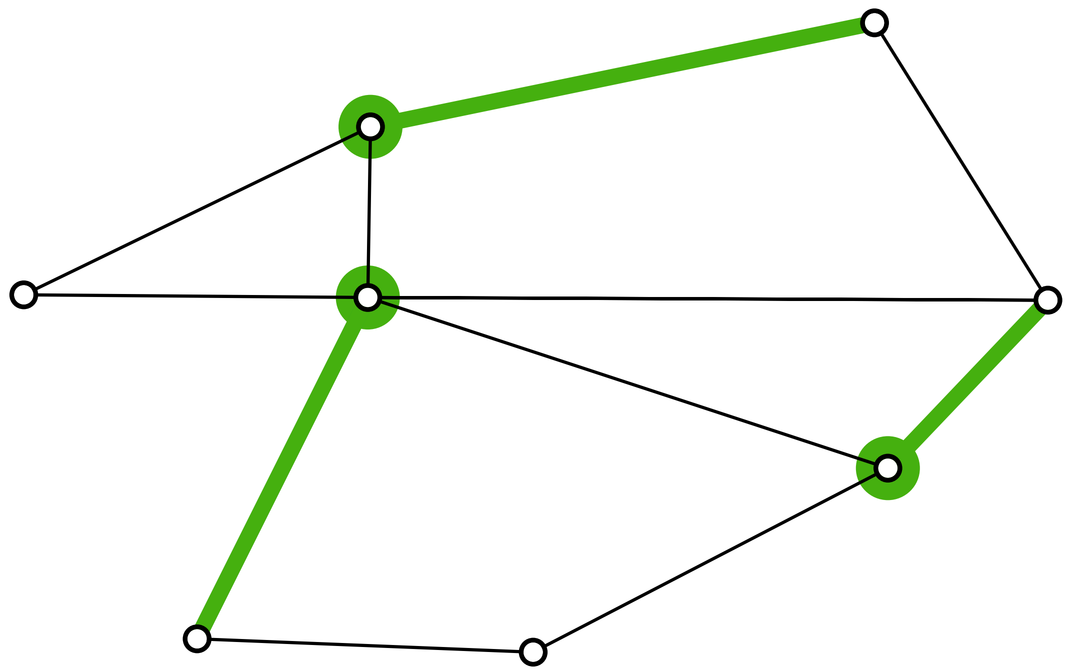

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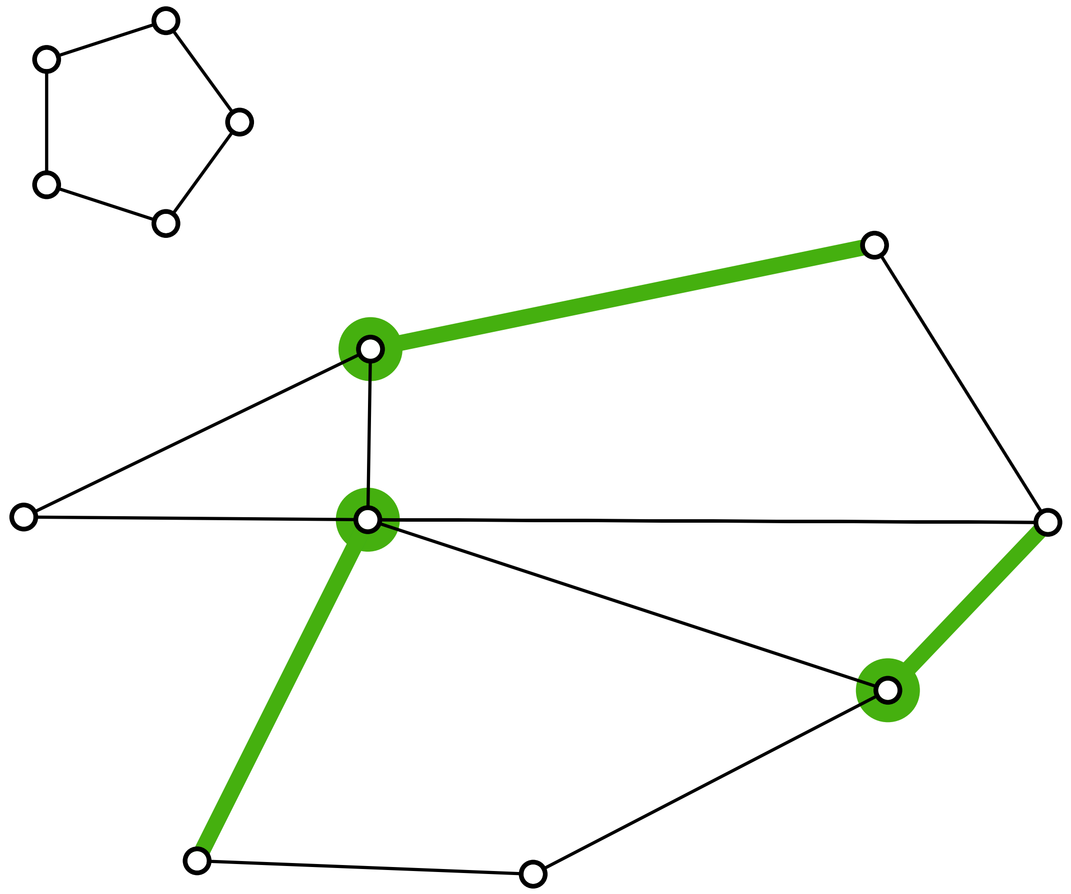

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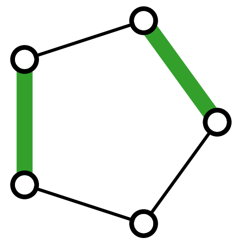

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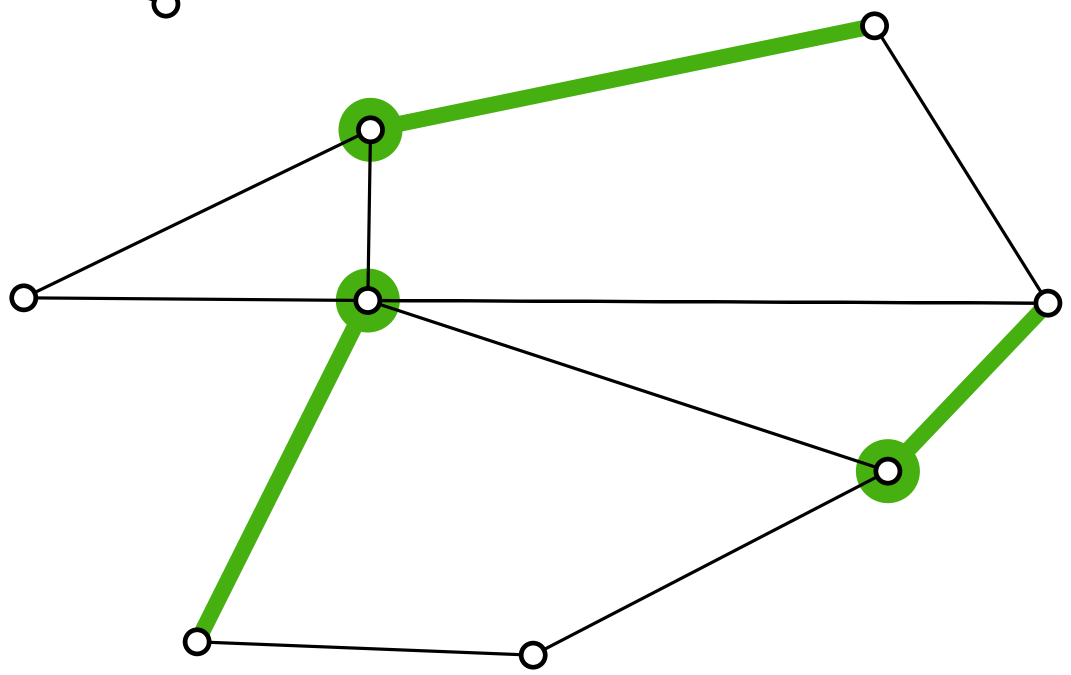

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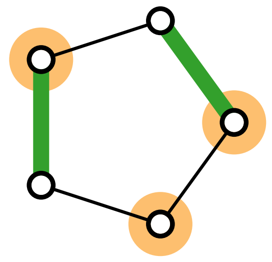

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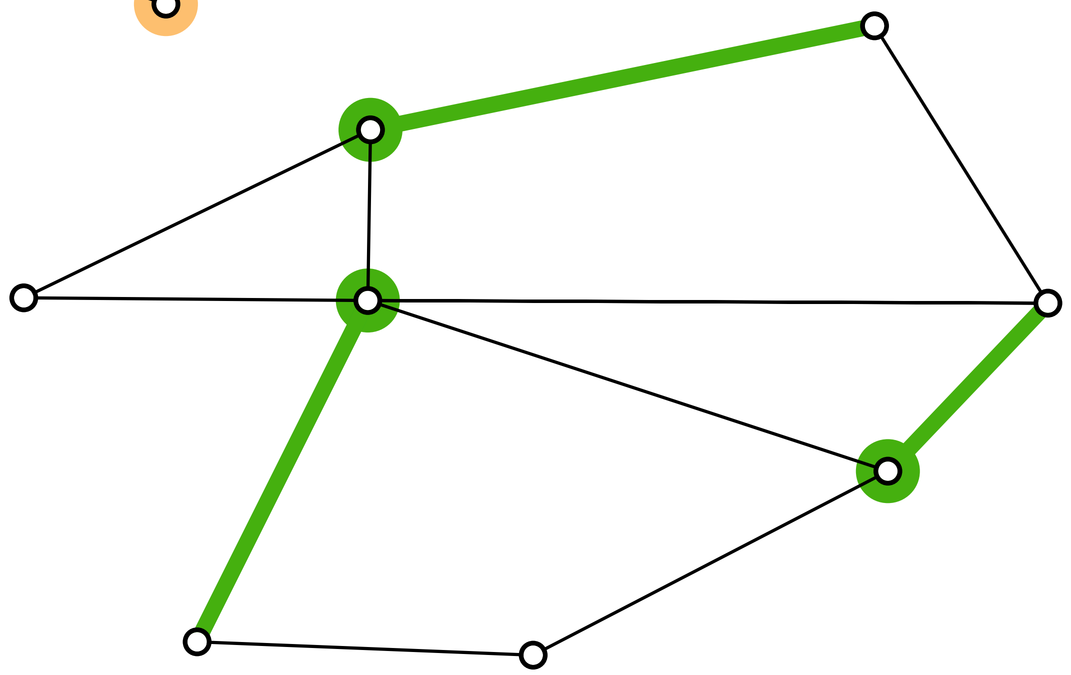

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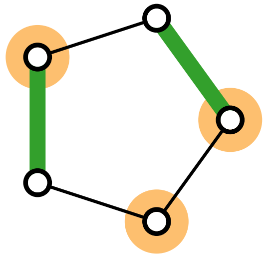

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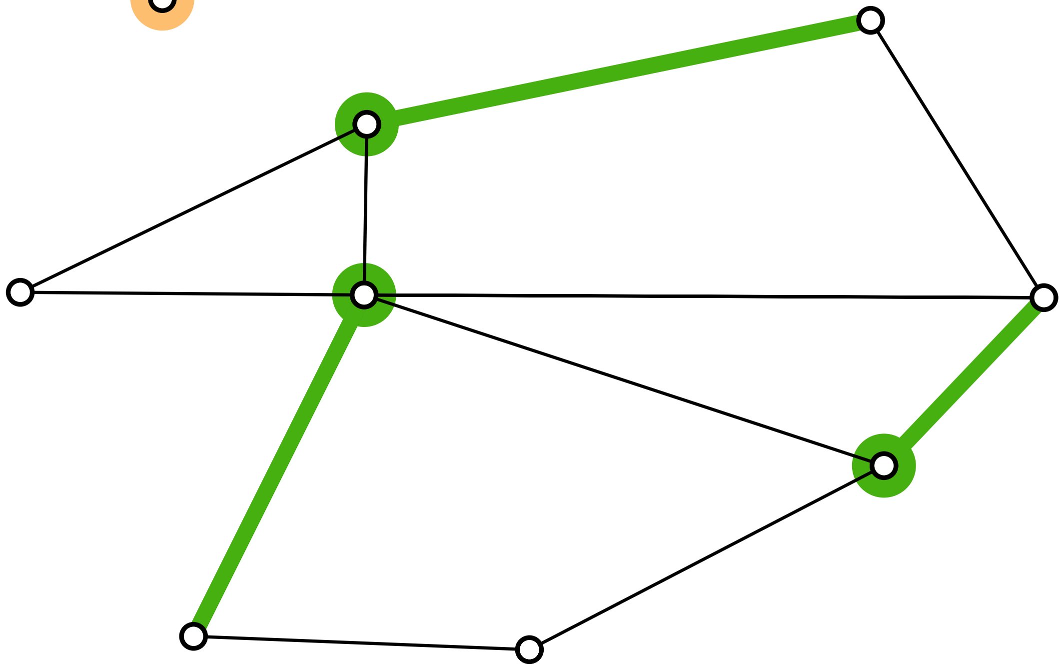

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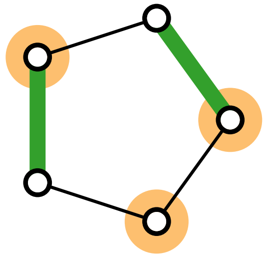

Can you come up with a vertex cover based on M?

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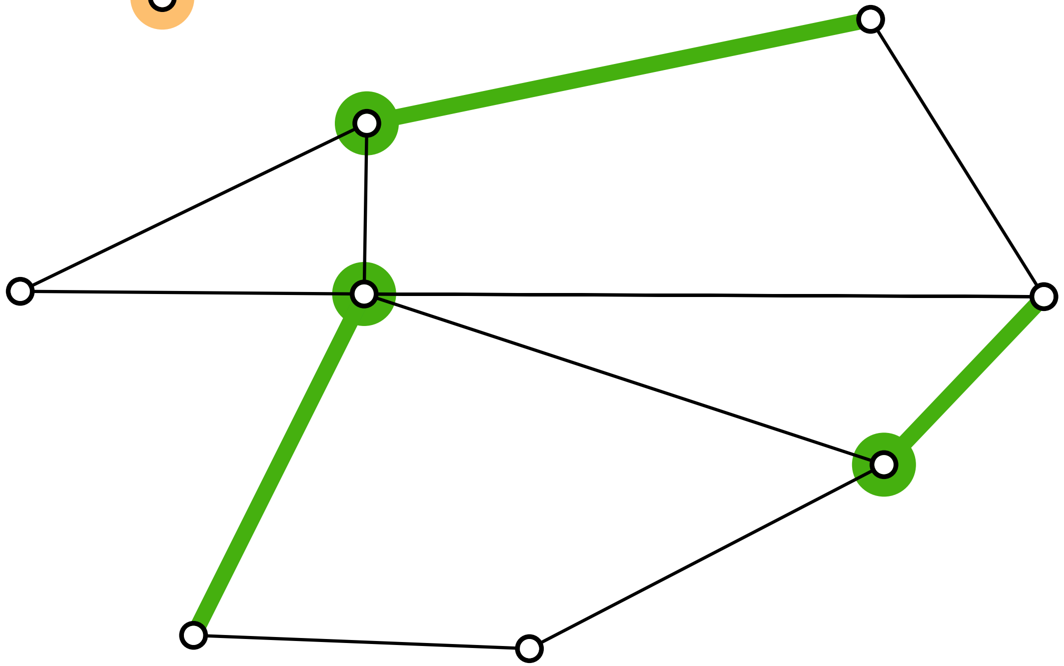
Vertex cover of M



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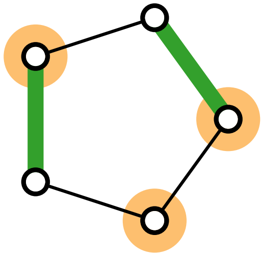

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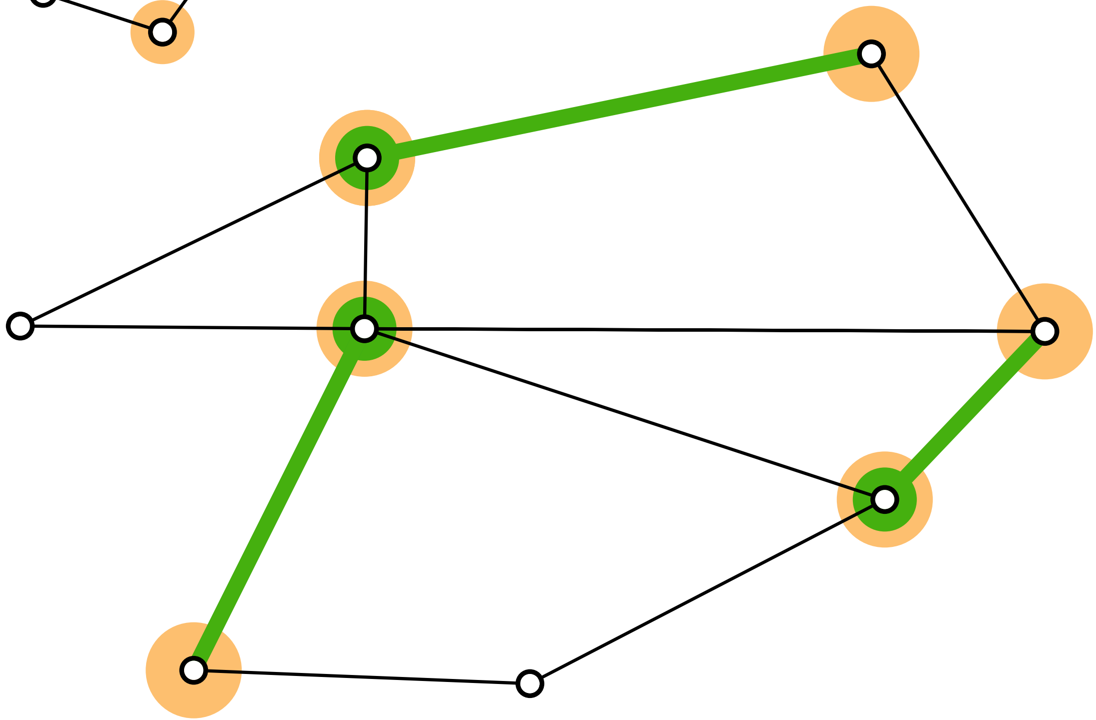

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Vertex cover of M

Vertex cover of E

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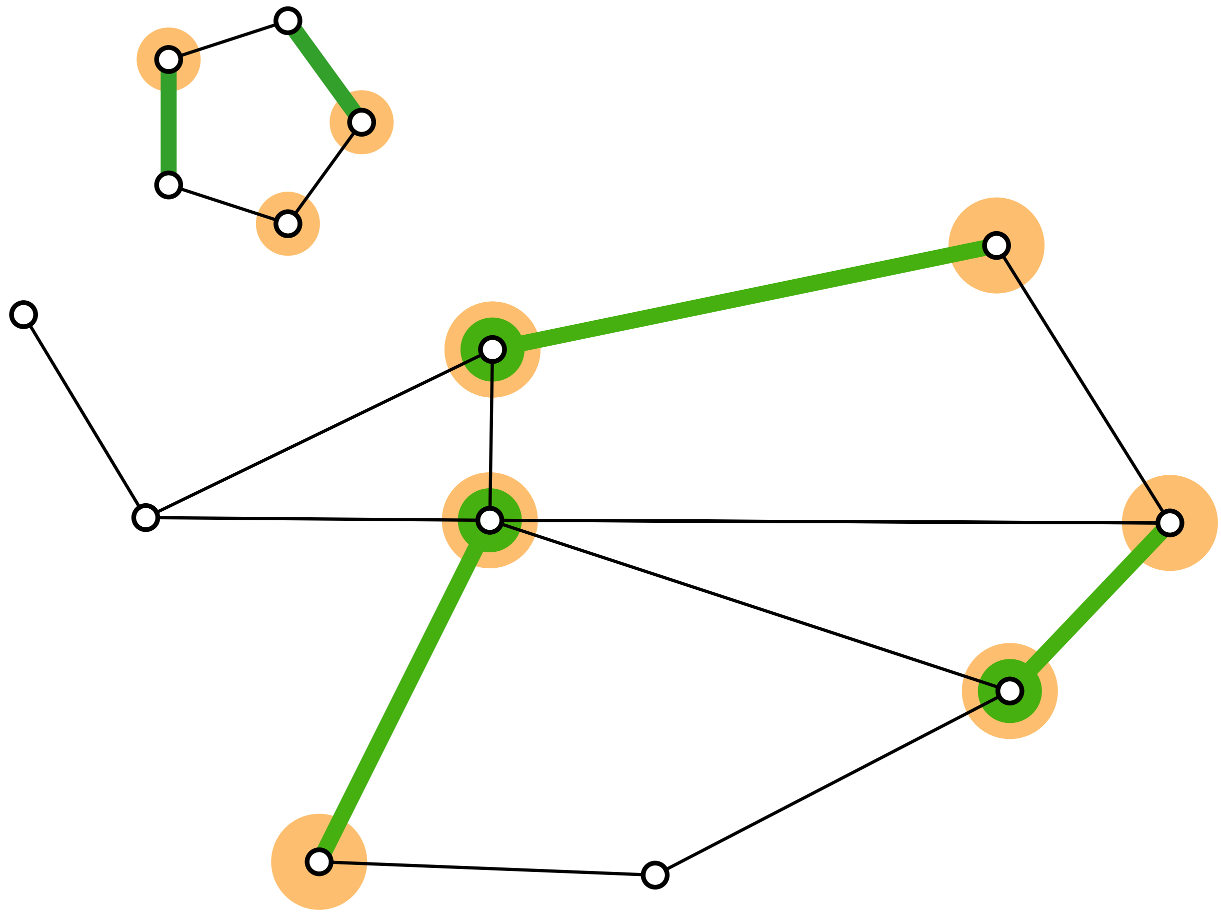

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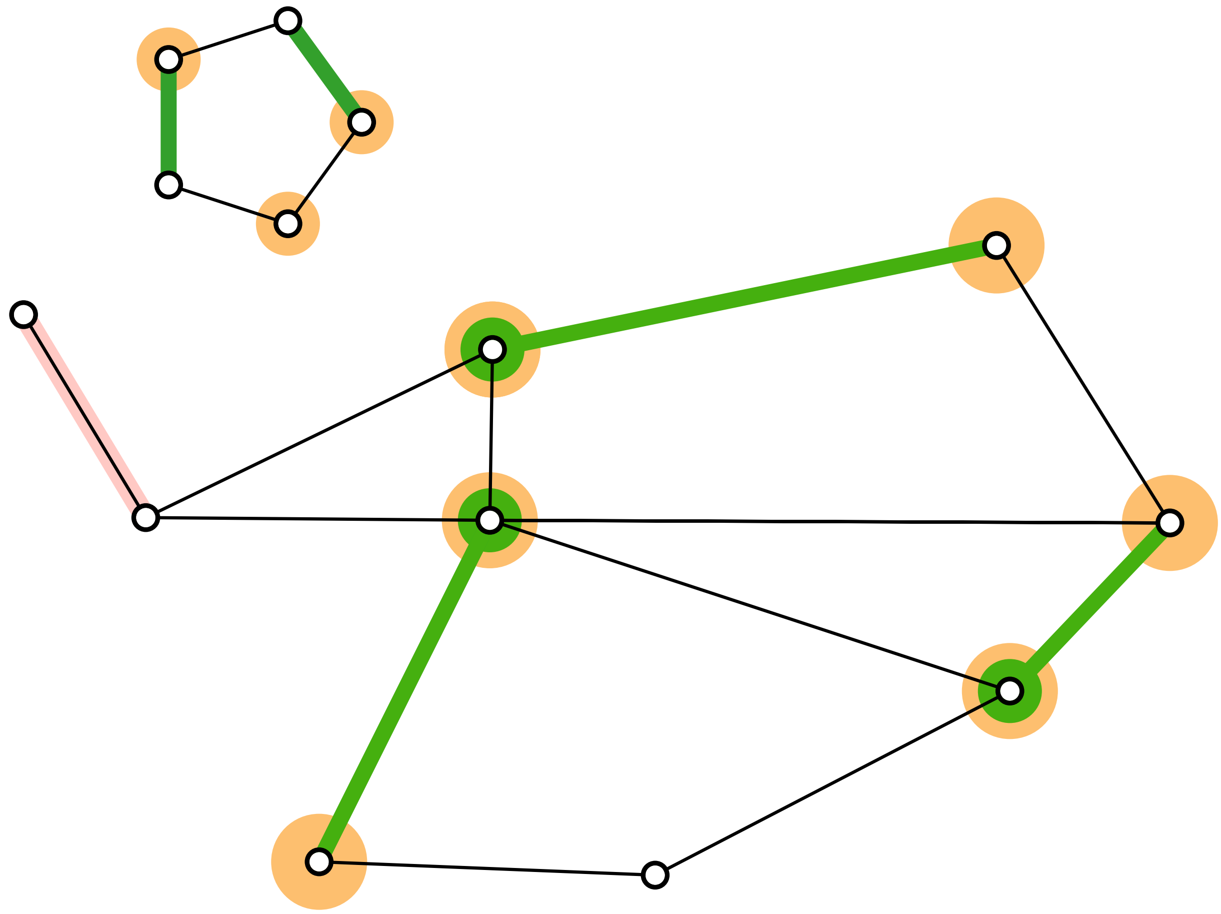

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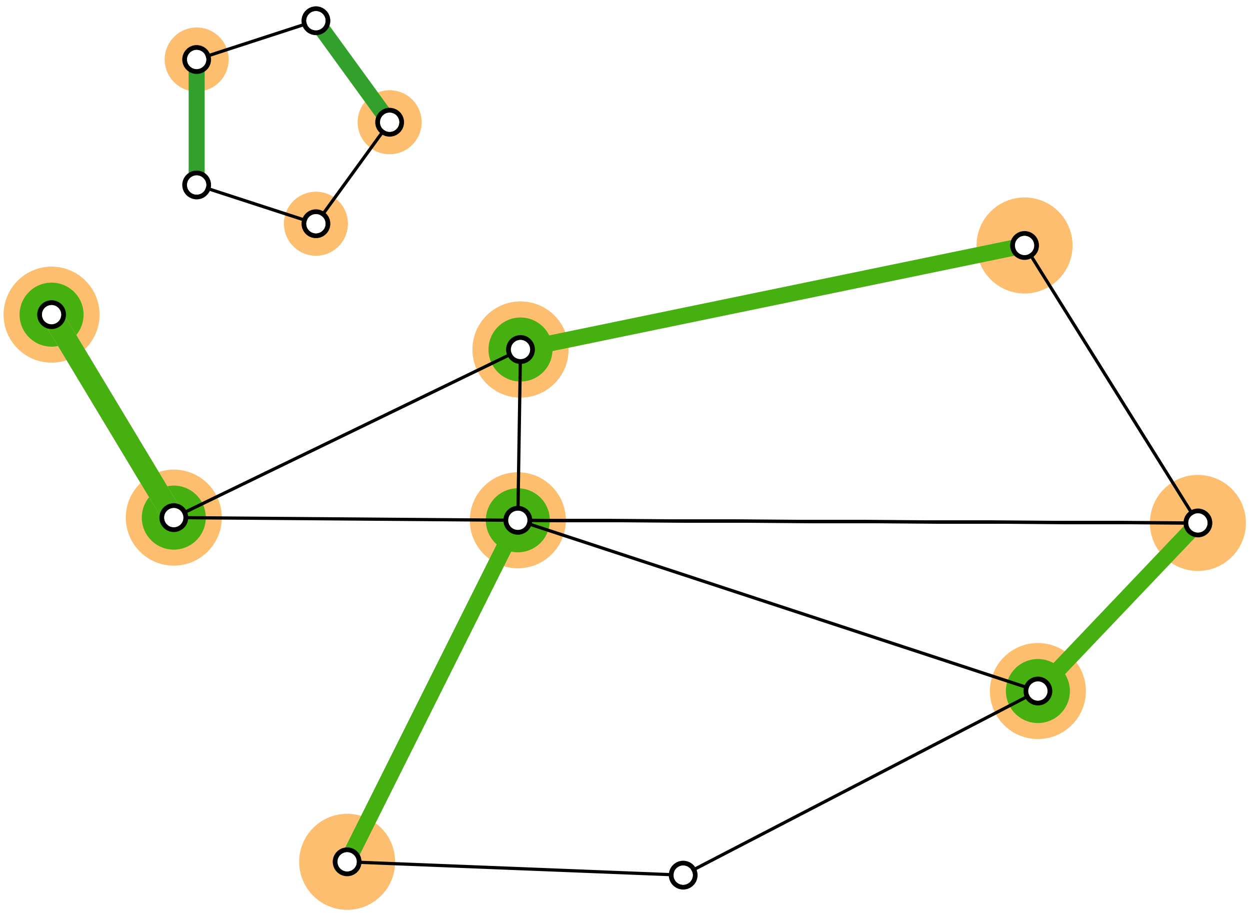

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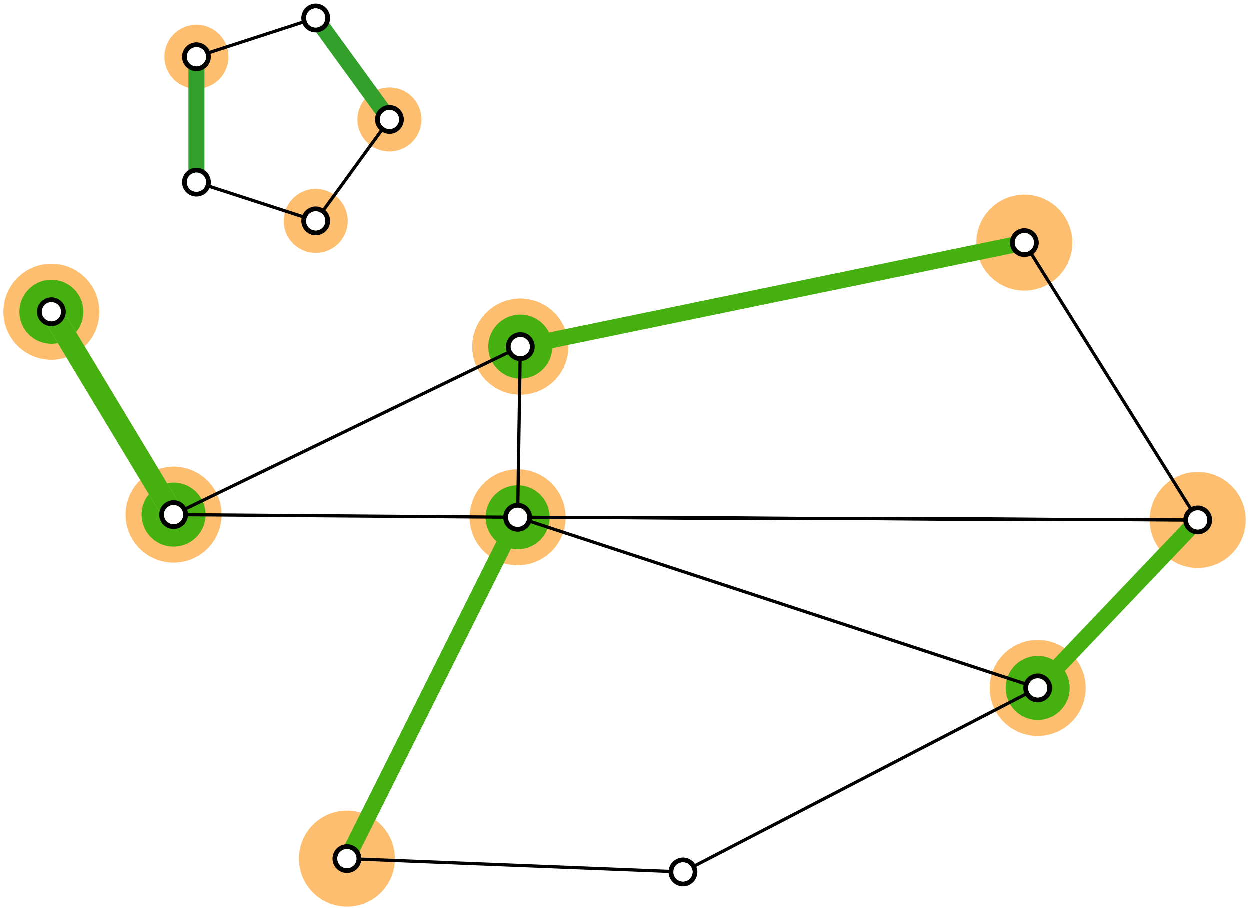

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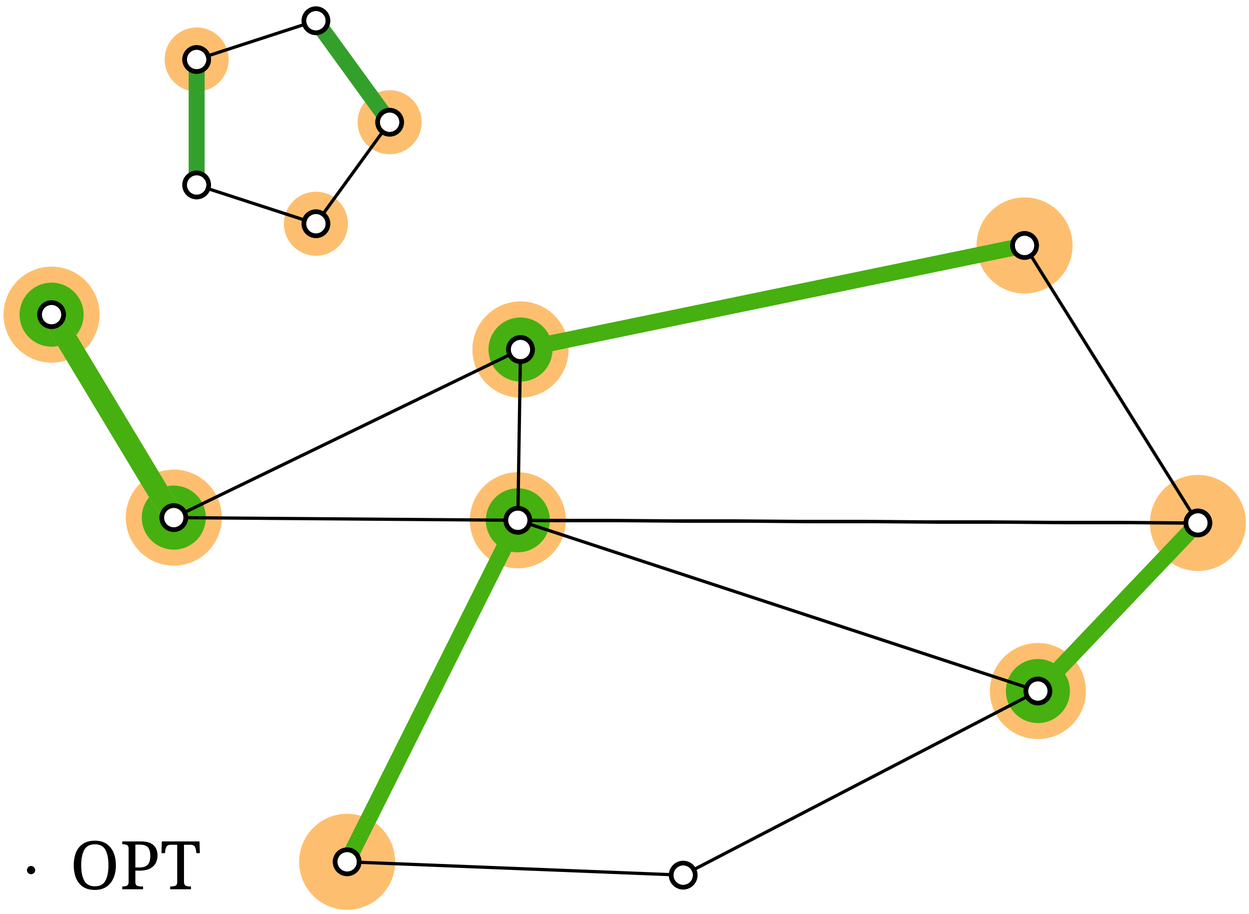

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Approximation Algorithm for VERTEXCOVER

Algorithm VertexCover(G)

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Theorem. The above algorithm is a factor-2 approximation algorithm for VertexCover.

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Questions:
- better analysis of this algorithm?
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Approximation Alg. for Vertex Cover

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tight example: graph with $ALG = 2\ \mathrm{OPT}$?

Approximation Alg. for Vertex Cover

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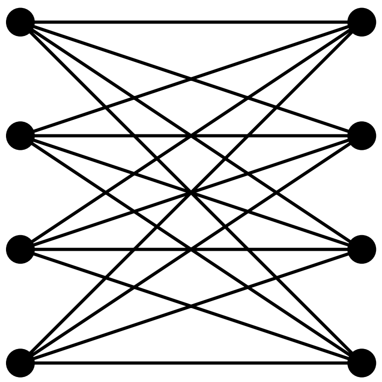

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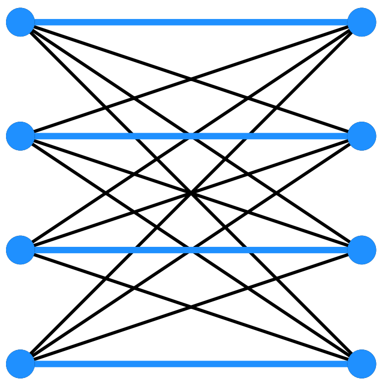
because any maximal matching has size n



Approximation Alg. for Vertex Cover

Questions:
- better analysis of this algorithm?
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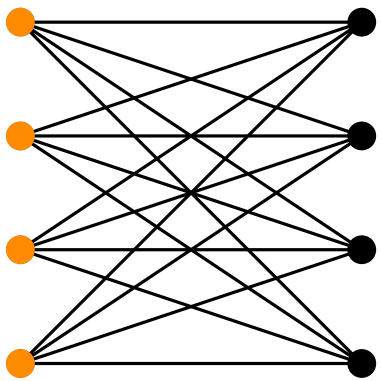

but an optimal vertex cover has size n

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tight example: graph with $|M| = \mathrm{OPT}/2$?

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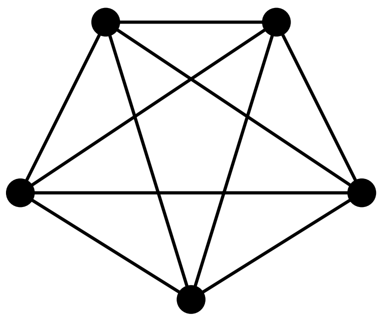

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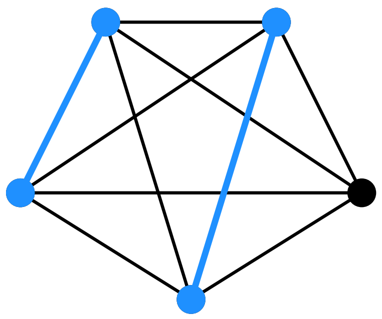

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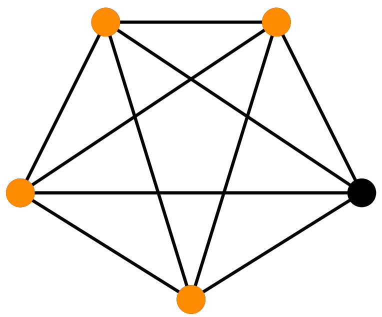

of a maximal matching is $(n-1)/2$

of a minimal vertex cover is $n-1$

Approximation Alg. for Vertex Cover

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Still open!

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The best known approximation factor for VertexCover is

$$2 - \Theta(1/\sqrt{\log n}).$$

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If P $\neq$ NP, VertexCover cannot be approximated within a factor of 1.3606.

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The best known approximation factor for VertexCover is

$$2 - \Theta(1/\sqrt{\log n}).$$

If P ≠ NP, VertexCover cannot be approximated within a factor of 1.3606.

VertexCover cannot be approximated within a factor of $2 - \Theta(1)$
– if the Unique Games Conjecture holds.

assuming that a certain problem
is NP-hard to approximate

Matching and Vertex Cover

we used the lower bound *max |matching| ≤ min |vertex cover|*

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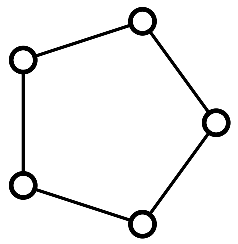

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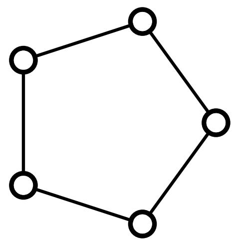

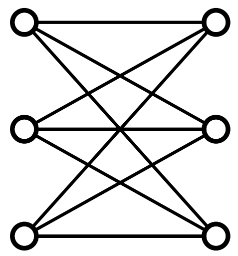

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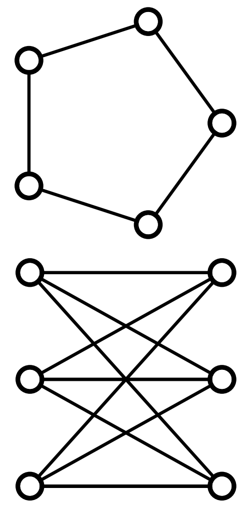
in fact, this holds for all bipartite graphs!



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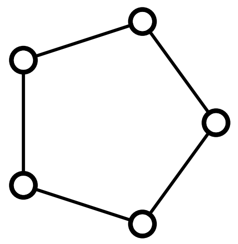

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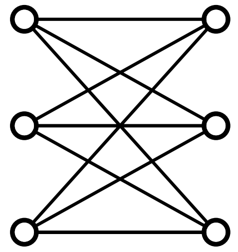

Theorem [König-Egerváry, 1931]
In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

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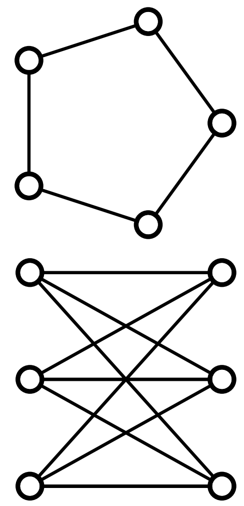

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Hence, (only) for bipartite graphs, these problems are equal!

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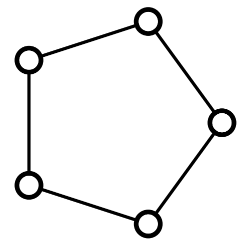

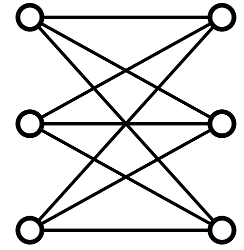

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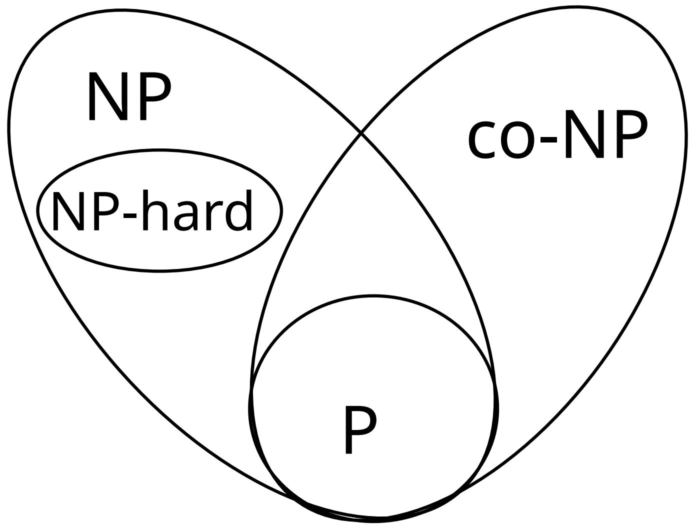
A little complexity theory

Q: Where are vertex cover and matching in general?



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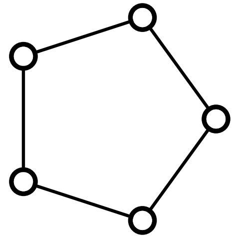

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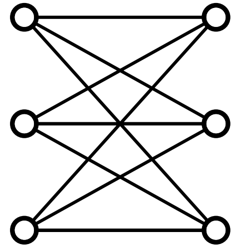

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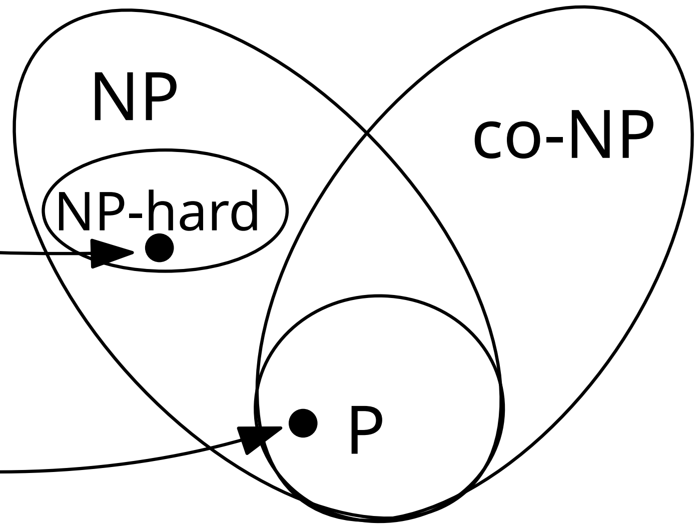

Summary

Theorem. Minimum-cardinality VERTEXCOVER can be approximated within a factor-2 by greedily computing a maximal matching M, and taking as vertex cover the endpoints of the edges in M.

- key ingredient: lower bound $|M|$ on OPT
- giving tight examples provides crucial insight into the functioning and the algorithm and can provide ideas what to improve

Acknowledgements

Slides for approximation algorithms are mostly due to colleagues in Würzburg,
in particular Joachim Spoerhase and Alexander Wolff. **Thanks!**