MaxSat via Randomized Rounding

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Is a given formula in conjunctive normal form satisfiable?

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Problem is NP-hard since SATISFIABILITY (SAT) is NP-hard:
Is a given formula in conjunctive normal form satisfiable?
E.g. $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

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We need to show:

$$E[W] \geq \mathrm{OPT}/2.$$

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$E[Y] = \Pr[Y = 1]$ holds
for 0/1 random variables

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$$l_j := \text{length}(C_j) \;\Rightarrow\; \Pr[C_j \text{ satisfied}] = 1 - (1/2)^{l_j} \geq$$

1 out of 2^{l_j} assignments
not satisfying

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Thus, $E[W] \geq 1/2 \sum_{j=1}^{m} w_j \geq \mathrm{OPT}/2.$ $\square$

Derandomization by Conditional Expectation

Theorem. The previous algorithm can be derandomized, i.e., there is a deterministic $1/2$-approximation algorithm for MaxSat.

Derandomization: process of taking a randomized algorithm and turning it into a deterministic algorithm.

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Idea: Incrementally replace the random choices by deterministic choices, maximizing the conditional expectation in every step

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Questions to answer:

1. How can we calculate $E[W|x_1 = 1]$ and $E[W|x_1 = 0]$?
2. Why is this still a $1/2$-approximation?

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Computing $E[W|x_1 = 1]$ and $E[W|x_1 = 0]$:

$E[W] = \sum_{j=1}^{m} w_j \Pr[C_j \text{ satisfied}]$ is a polynomial in $p_1, \dots p_n$,
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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly w_j to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

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If C_j is not yet satisfied and contains k unassigned variables, then it contributes exactly $w_j(1 - (1/2)^k)$ to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

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The conditional expectation is simply the sum of the contributions from each clause.

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Approximation factor:

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[because of original random choice of x_1]

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$E[W] = \left(E[W|x_1 = 0] + E[W|x_1 = 1]\right)/2.$ [because of original random choice of x_1]

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[because of original random choice of x_1]

If x_1 was set to $b_1 \in \{0, 1\}$,
then $E[W|x_1 = b_1] \geq E[W] \geq \mathrm{OPT}/2$.

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Approximation factor:

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Then (similar to the base case):

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Global optimization?

Randomized Rounding

An ILP for MAXSAT

maximize

?

subject to

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

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& y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\
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$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\ \textbf{subject to} & \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \text{for } j = 1, \ldots, m \\ & y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\ & z_j \in \{0, 1\}, & \text{for } j = 1, \ldots, m \end{array}$$

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… and its Relaxation

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\begin{array}{lll}
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 & 0 \leq y_i \leq 1, & \text{for } i = 1, \ldots, n \\
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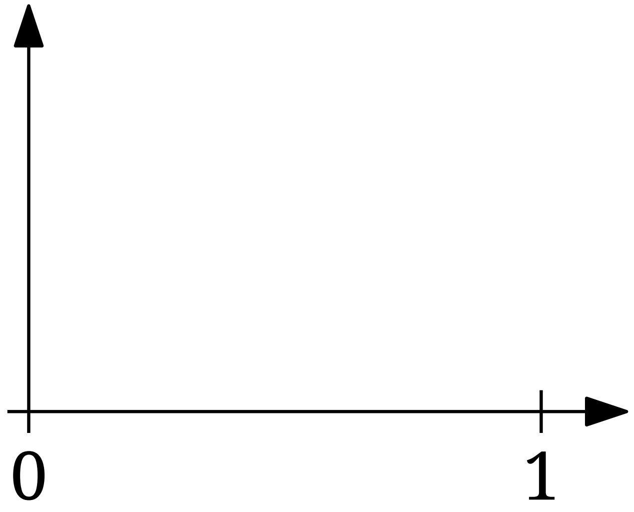

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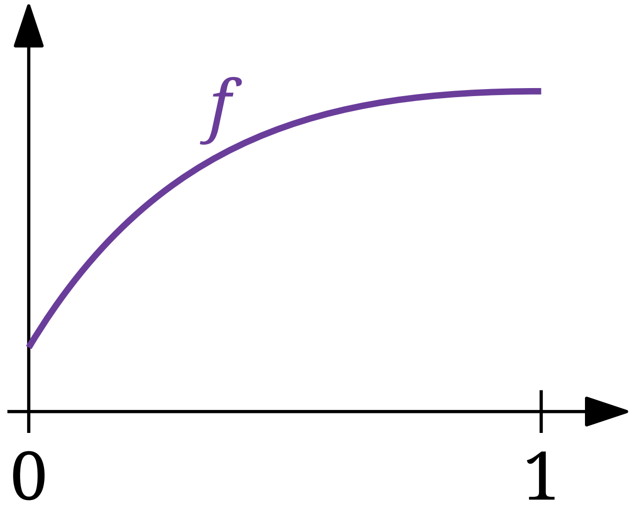

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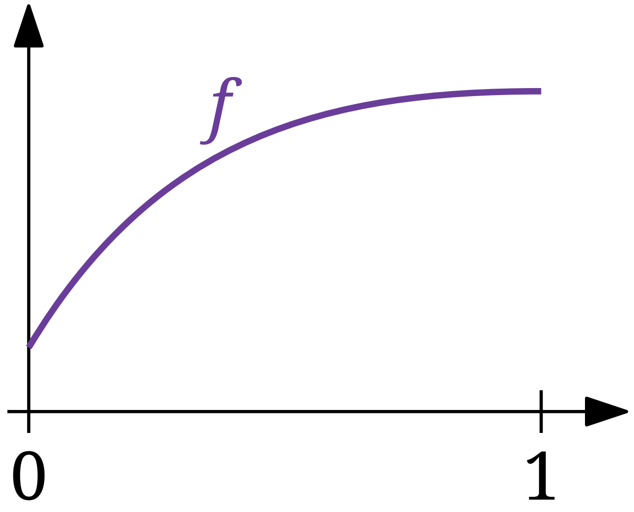

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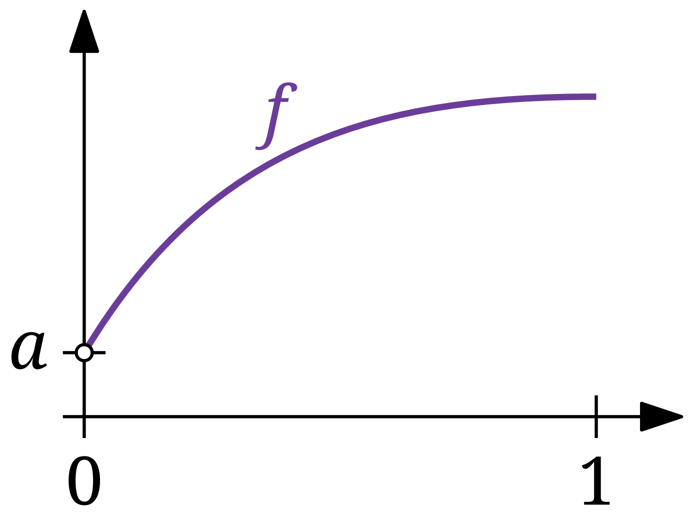

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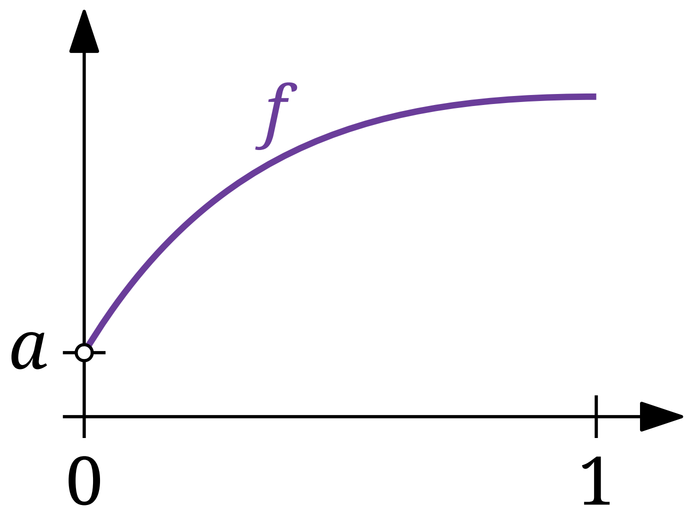

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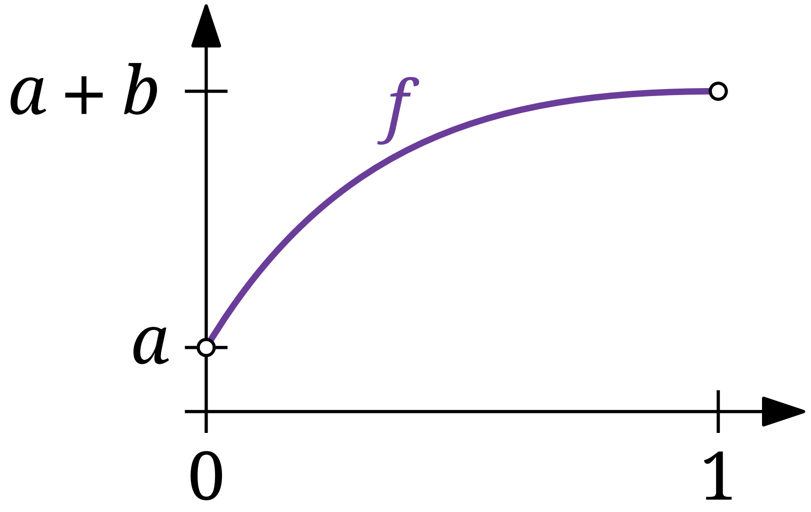

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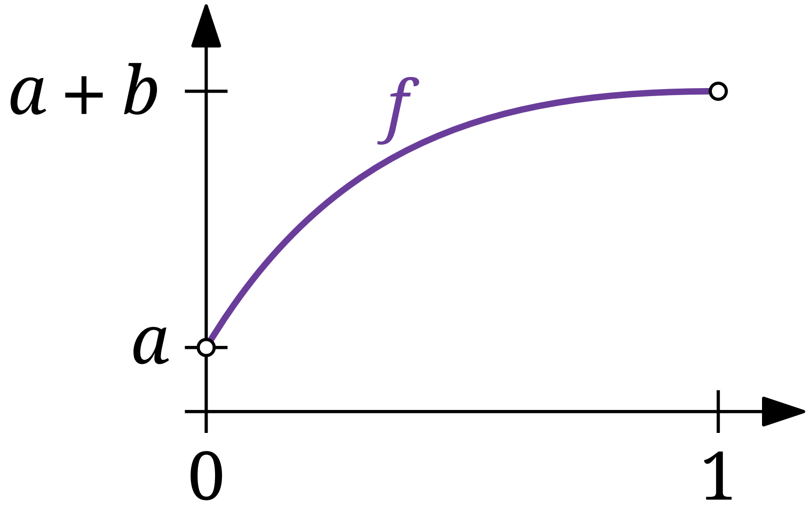

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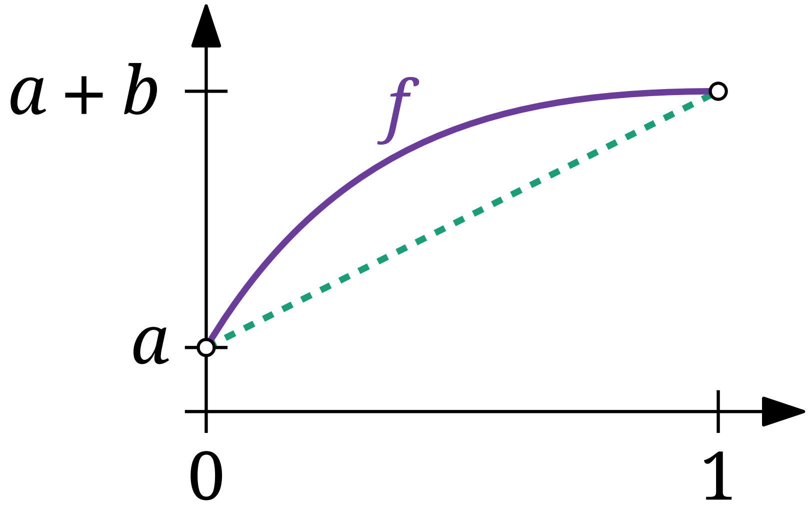

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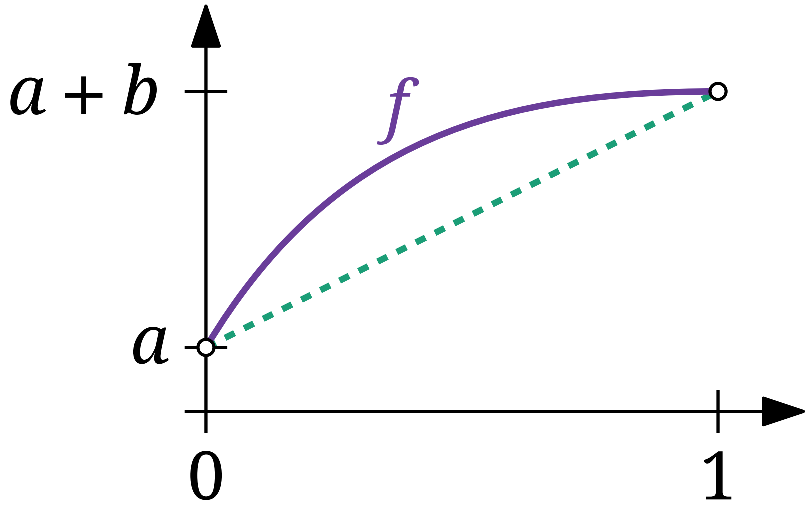

Arithmetic–Geometric Mean Inequality (AGMI):

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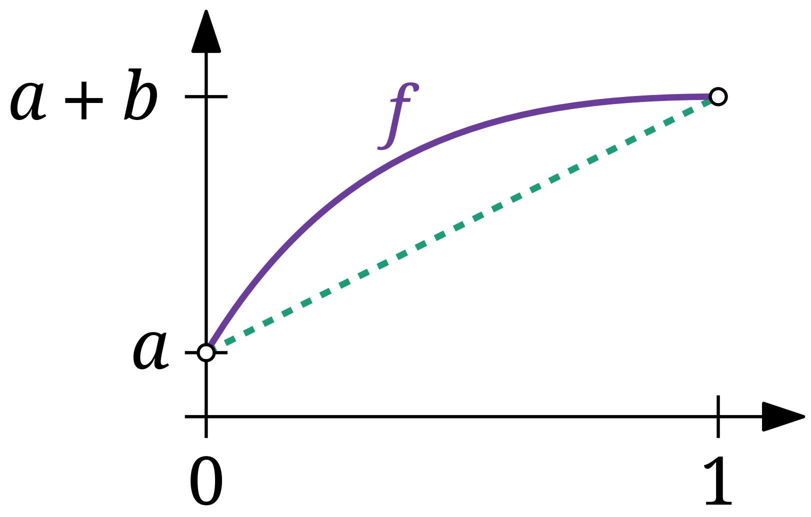

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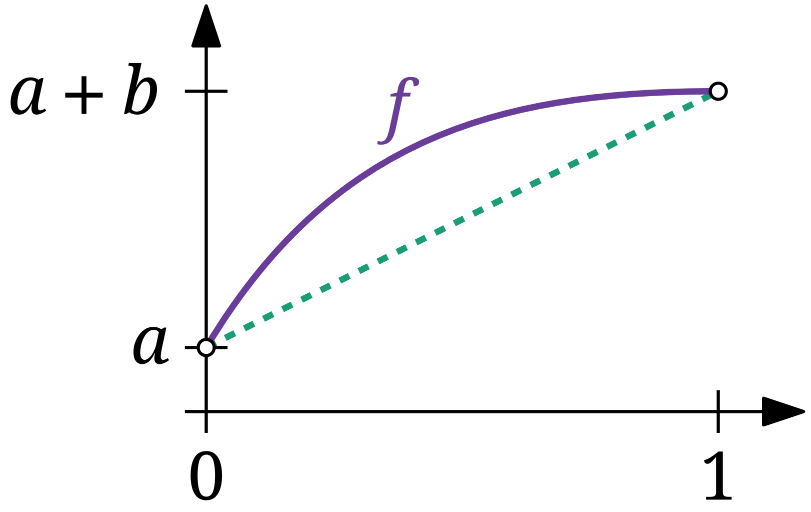

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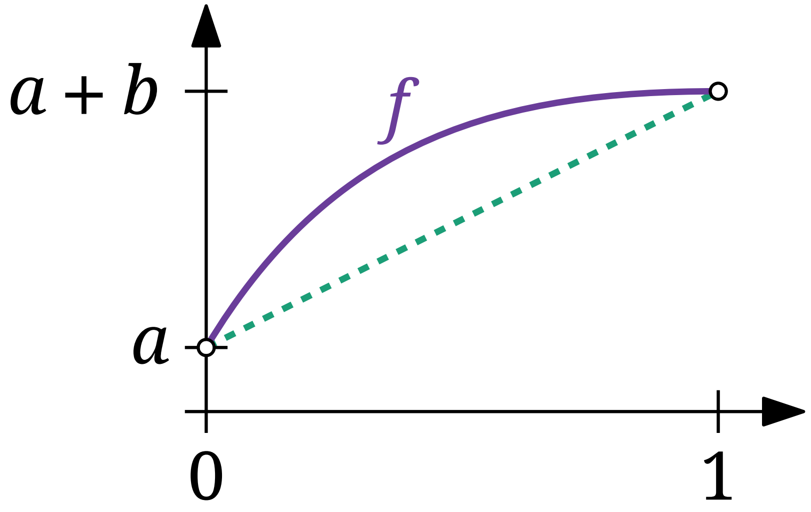

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&\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^* \\
&\geq
\end{aligned}
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Randomized Rounding (Proof)

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LP objective function

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LP objective function

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Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a   -approximation for MaxSat.

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We use another probabilistic argument.
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The better solution is at least as good as the expectation of the above algorithm.

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The probability that clause C_j is satisfied is at least:

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Visualization and Derandomization

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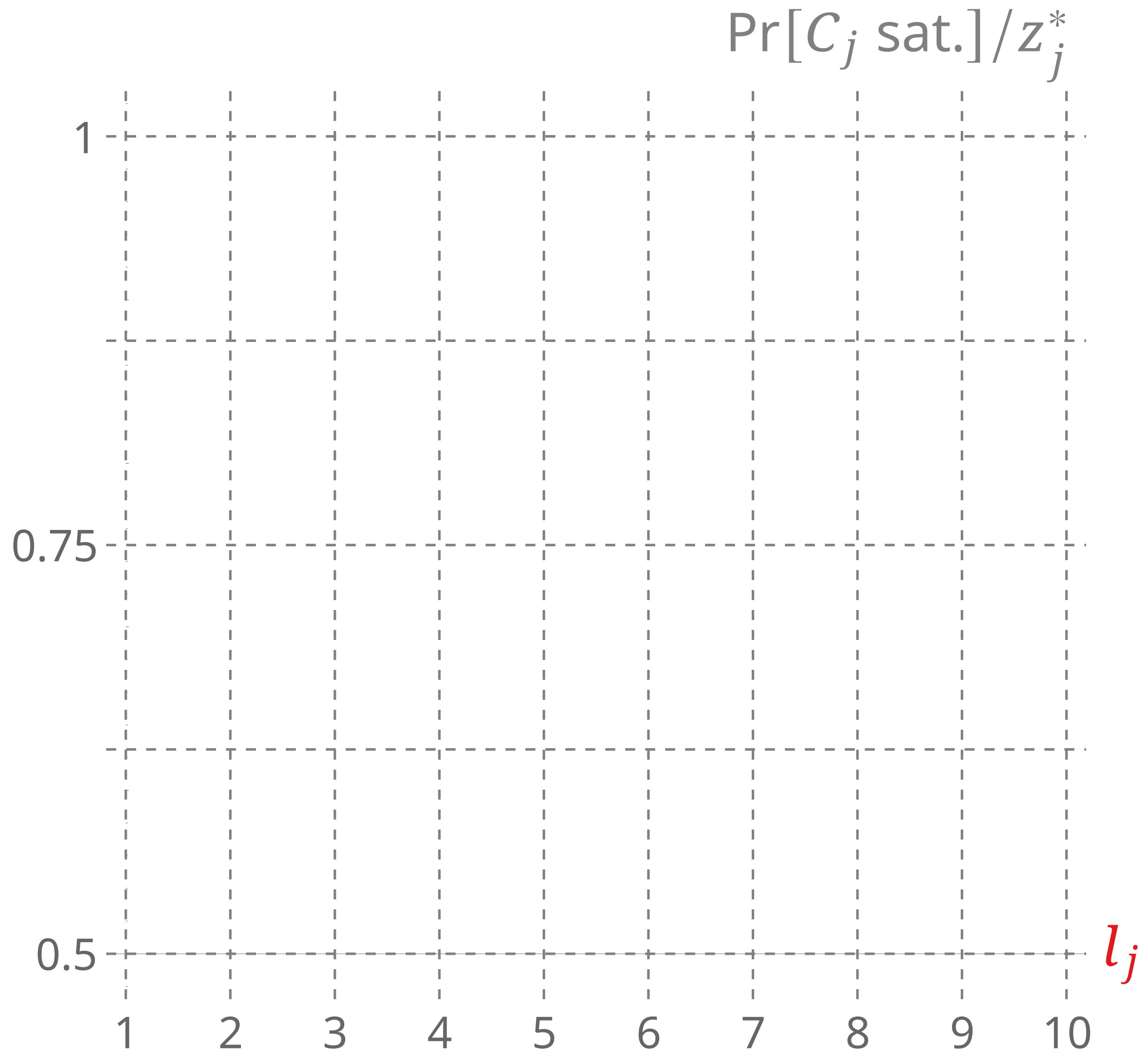
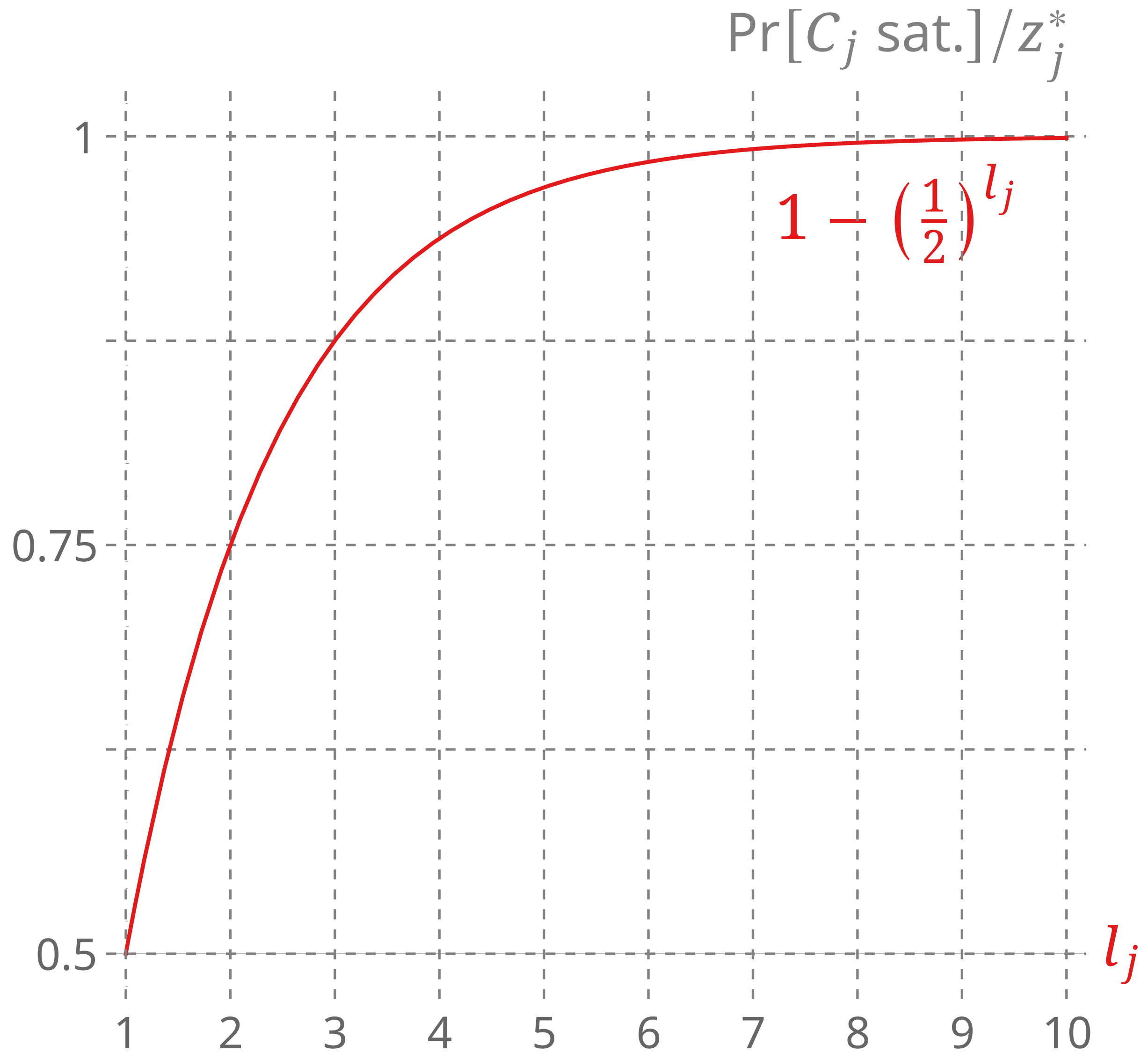


Visualization and Derandomization



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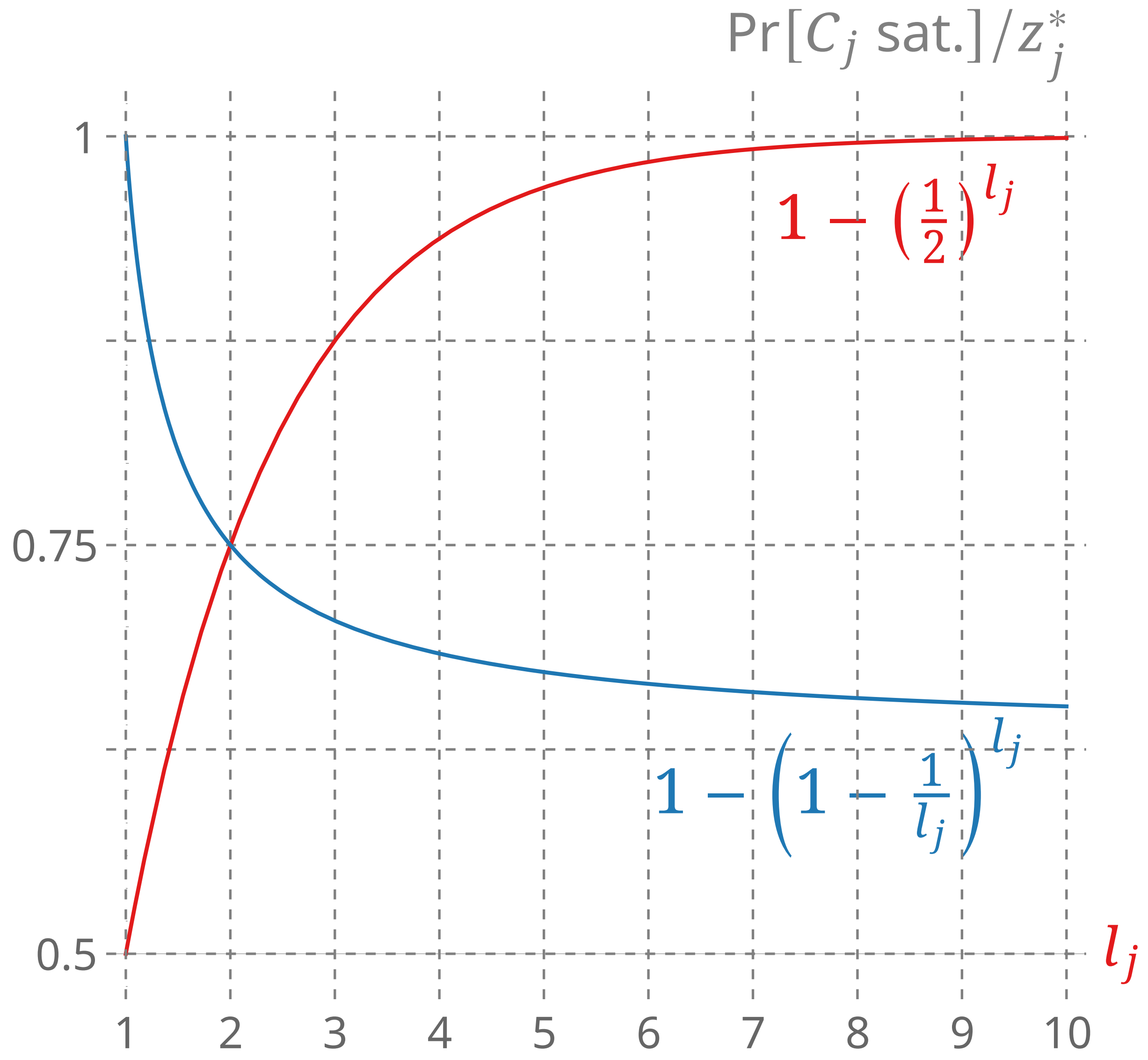
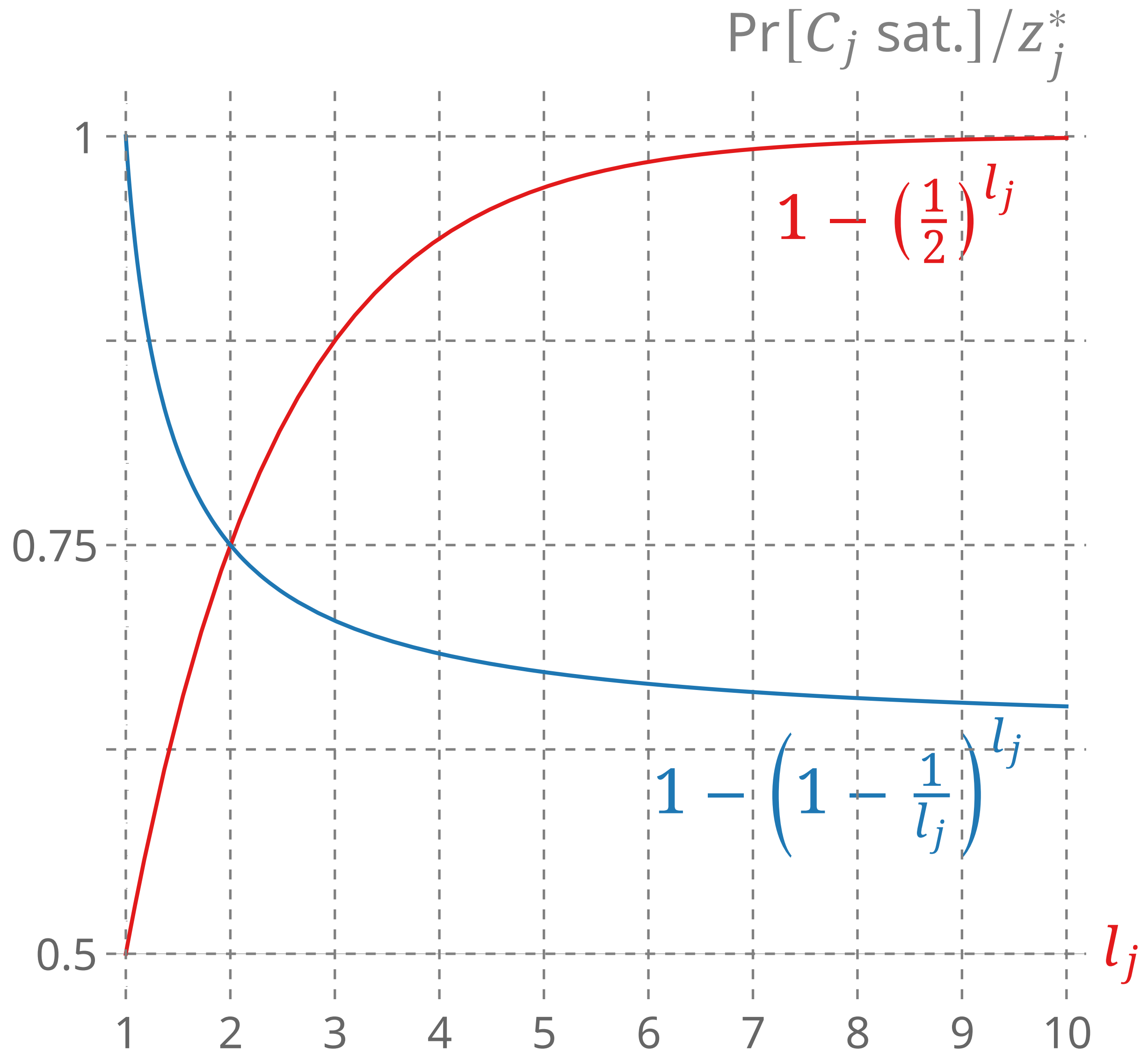


Visualization and Derandomization

- Randomized alg. is better for large values of l_j.



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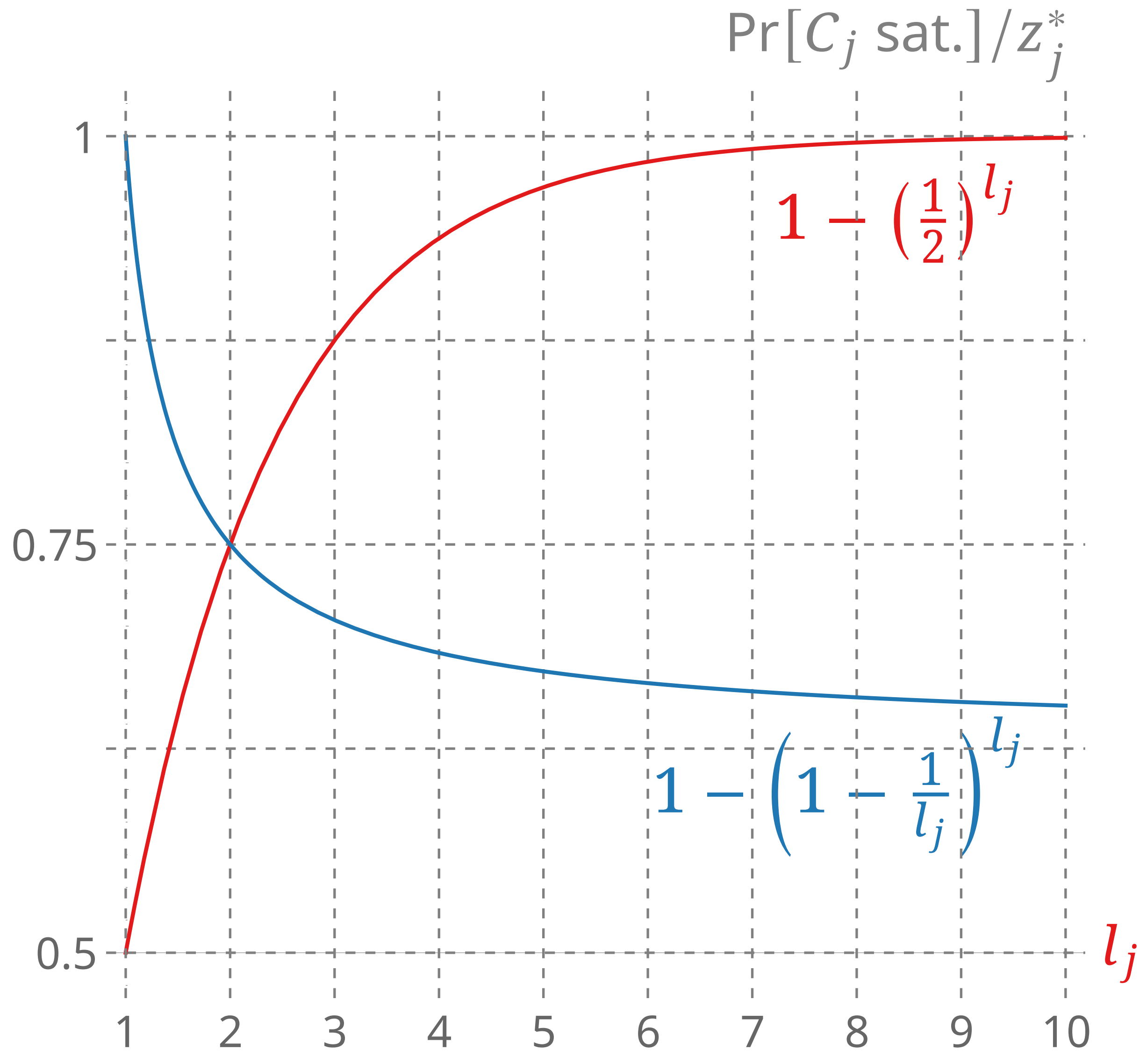

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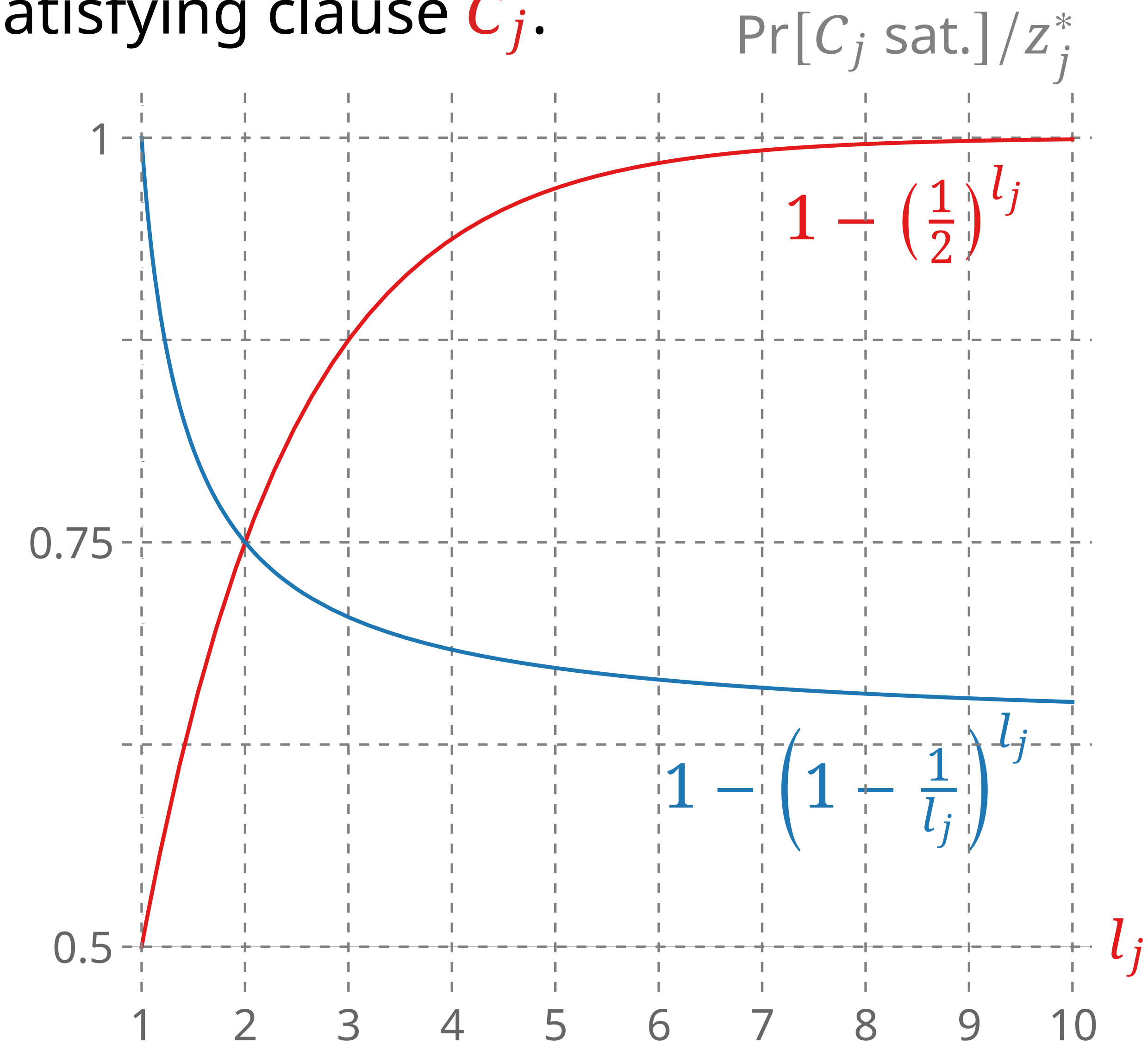
$\Rightarrow$ higher probability of satisfying clause C_j.



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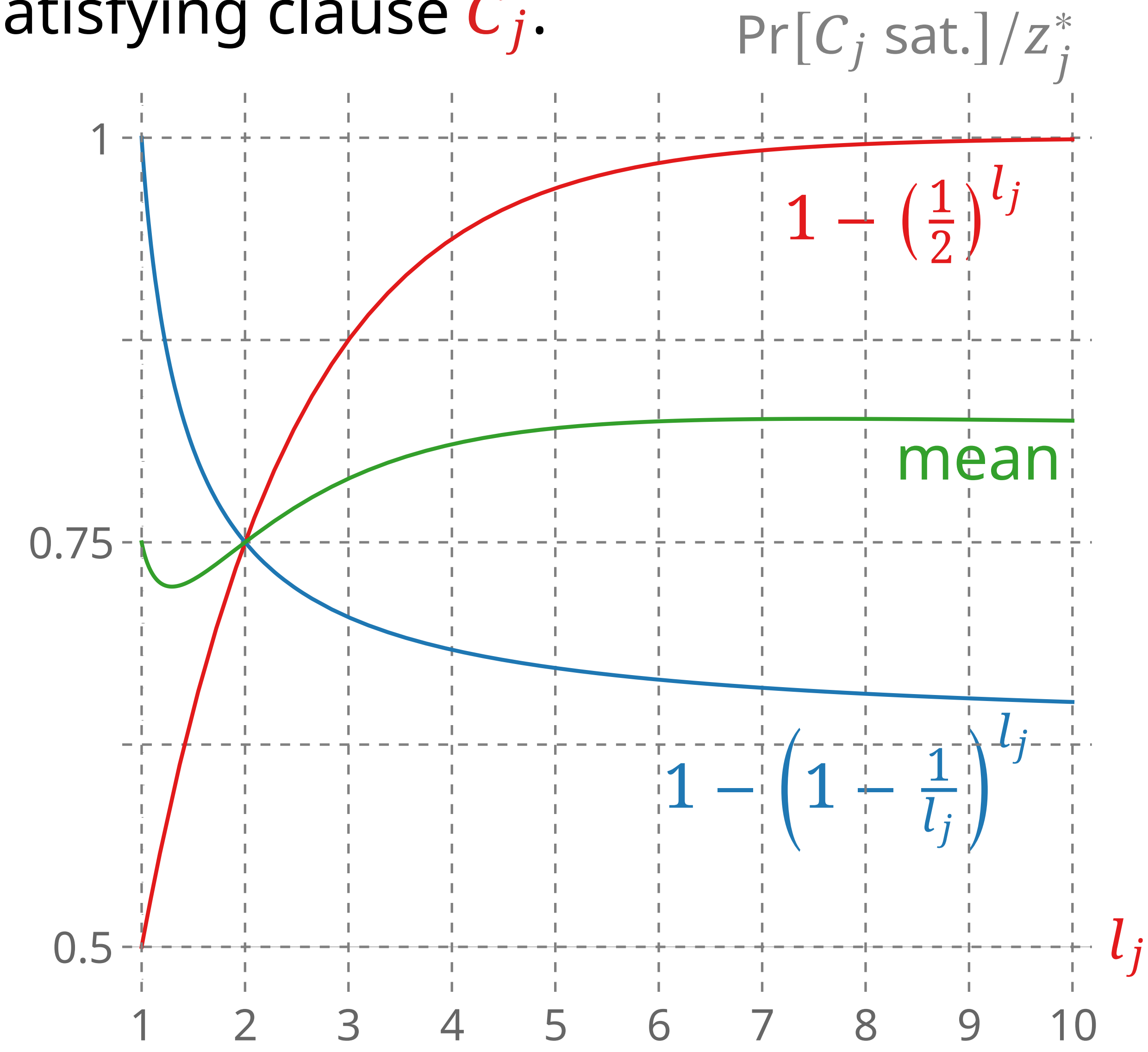

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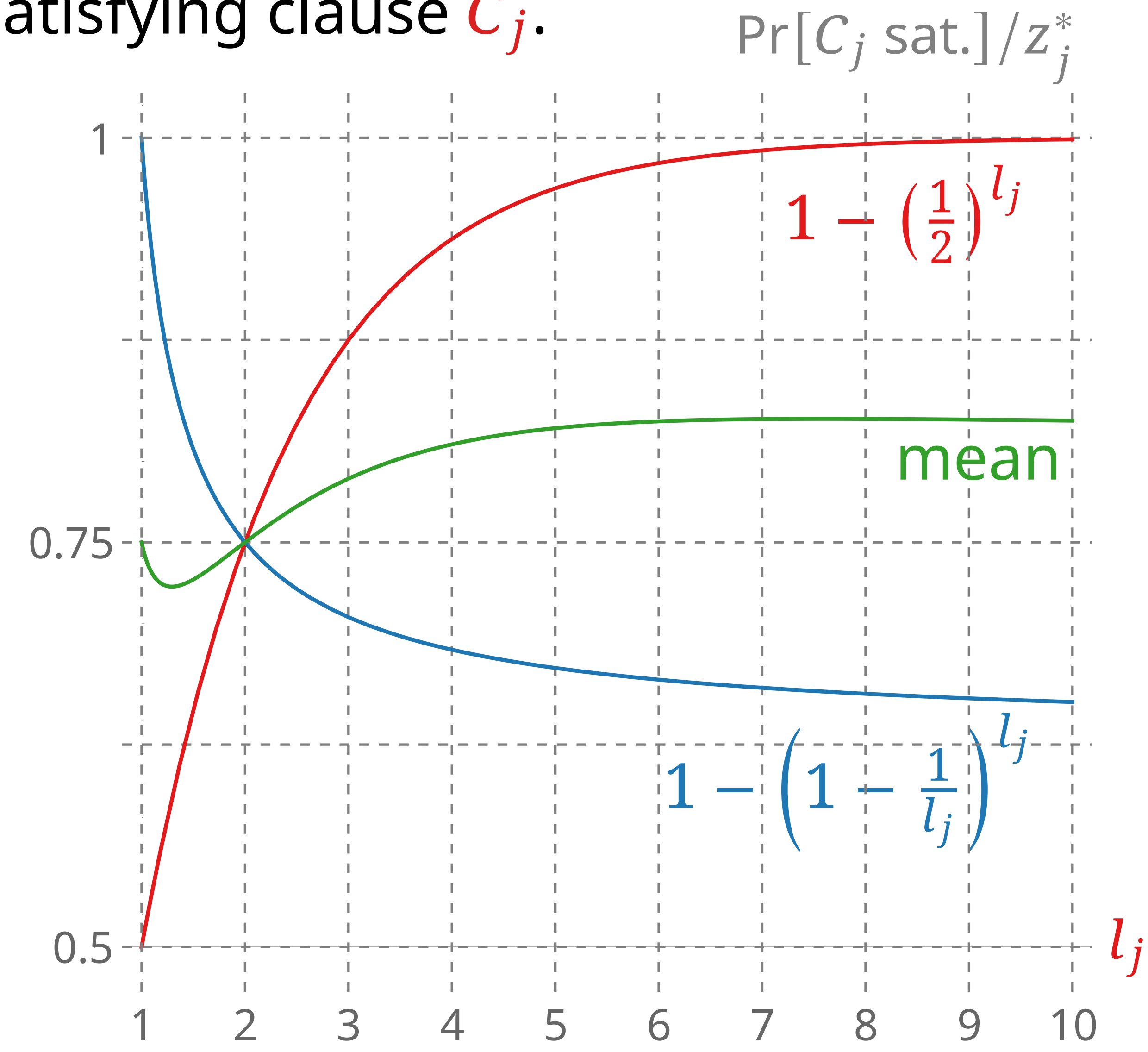

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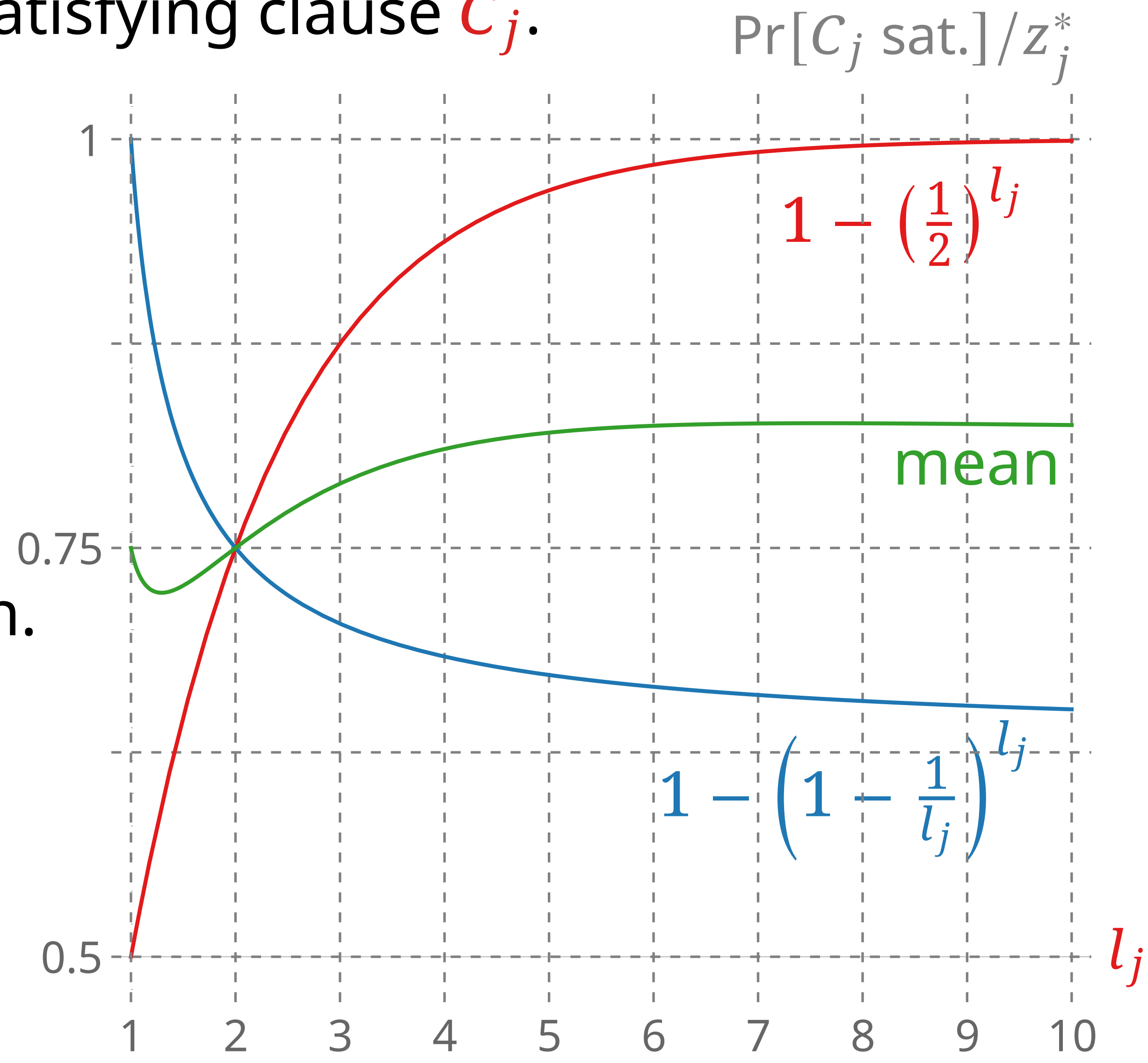

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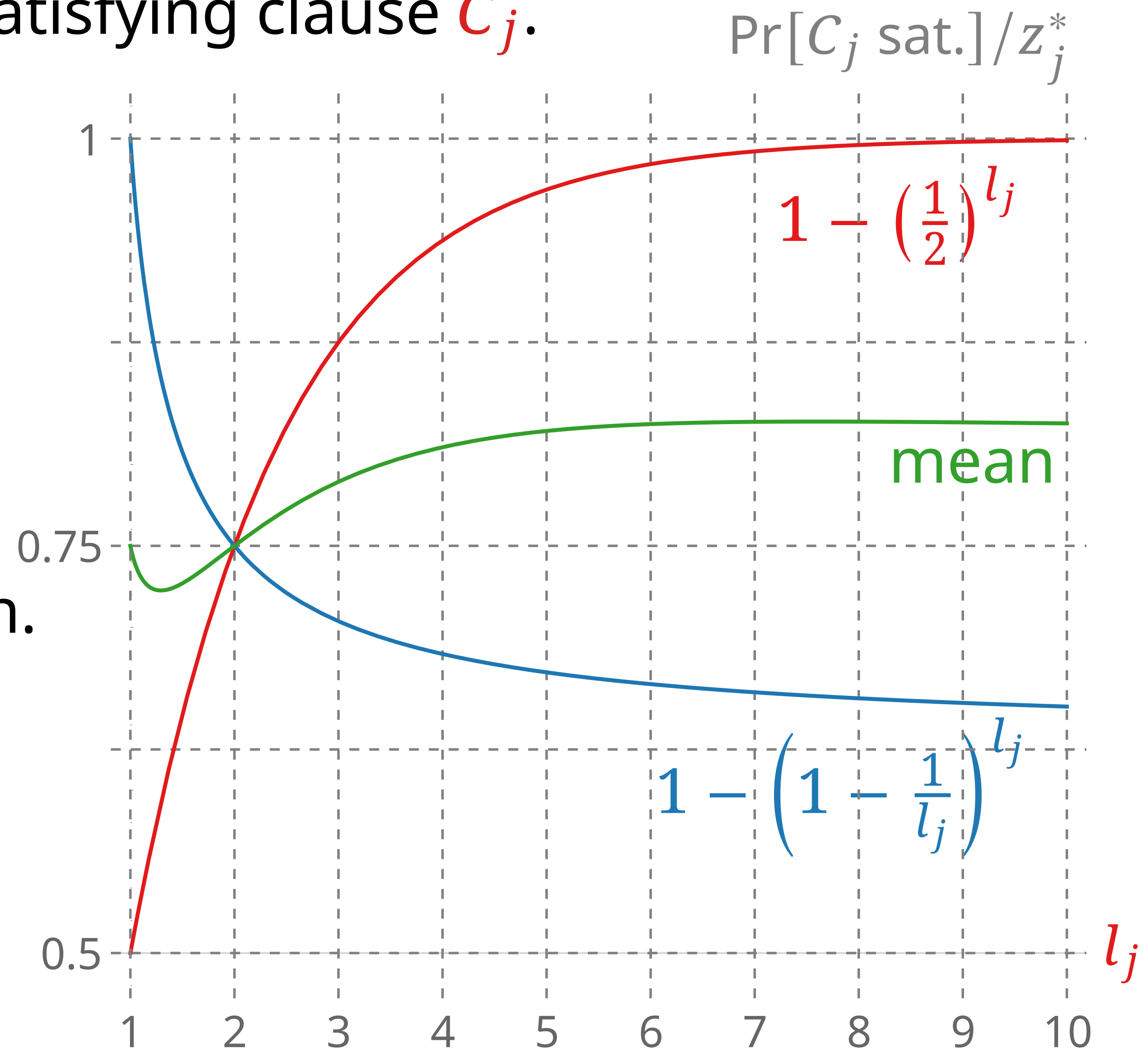
This algorithm, too, can be derandomized by conditional expectation.



Summary

Simple randomized algorithm: expected 1/2-approximation

Randomized LP-rounding: expected $(1 - 1/e)$-approximation

combination: 3/4-approximation

Derandomization by Conditional Expectation

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Derandomization by Conditional Expectation

Can we do better?

Best approximation factor known: 0.7968 [Avidor, Berkovitch, and Zwick 2006]

Max-E3-Sat: Every clause contains exactly 3 different literals

- Cannot be approximated within a factor $7/8 + \varepsilon$ for $\varepsilon > 0$ unless P=NP [Hastad 2001]
- simple randomized algorithm achieves $7/8$ for Max-E3-Sat: optimal!