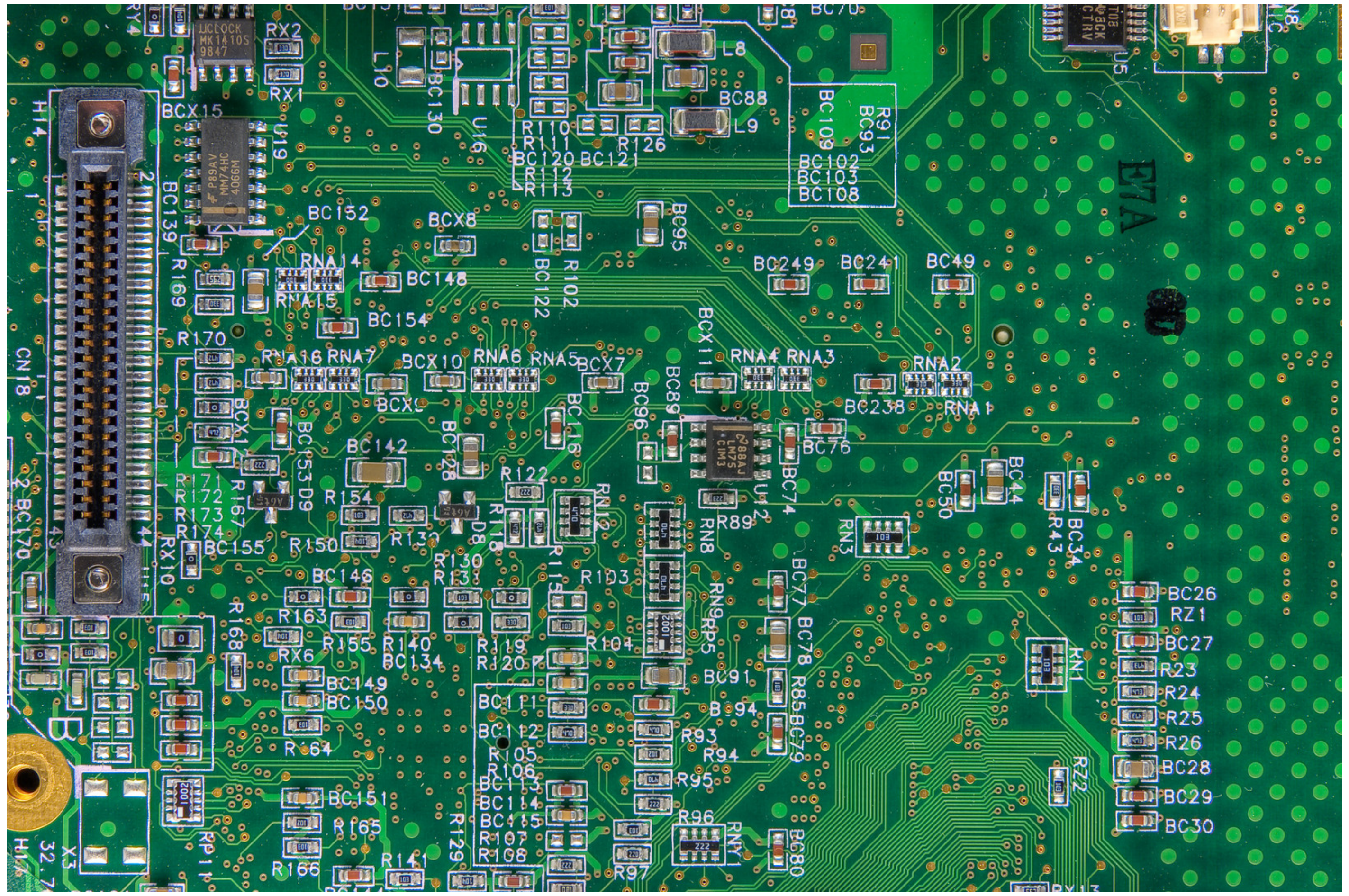
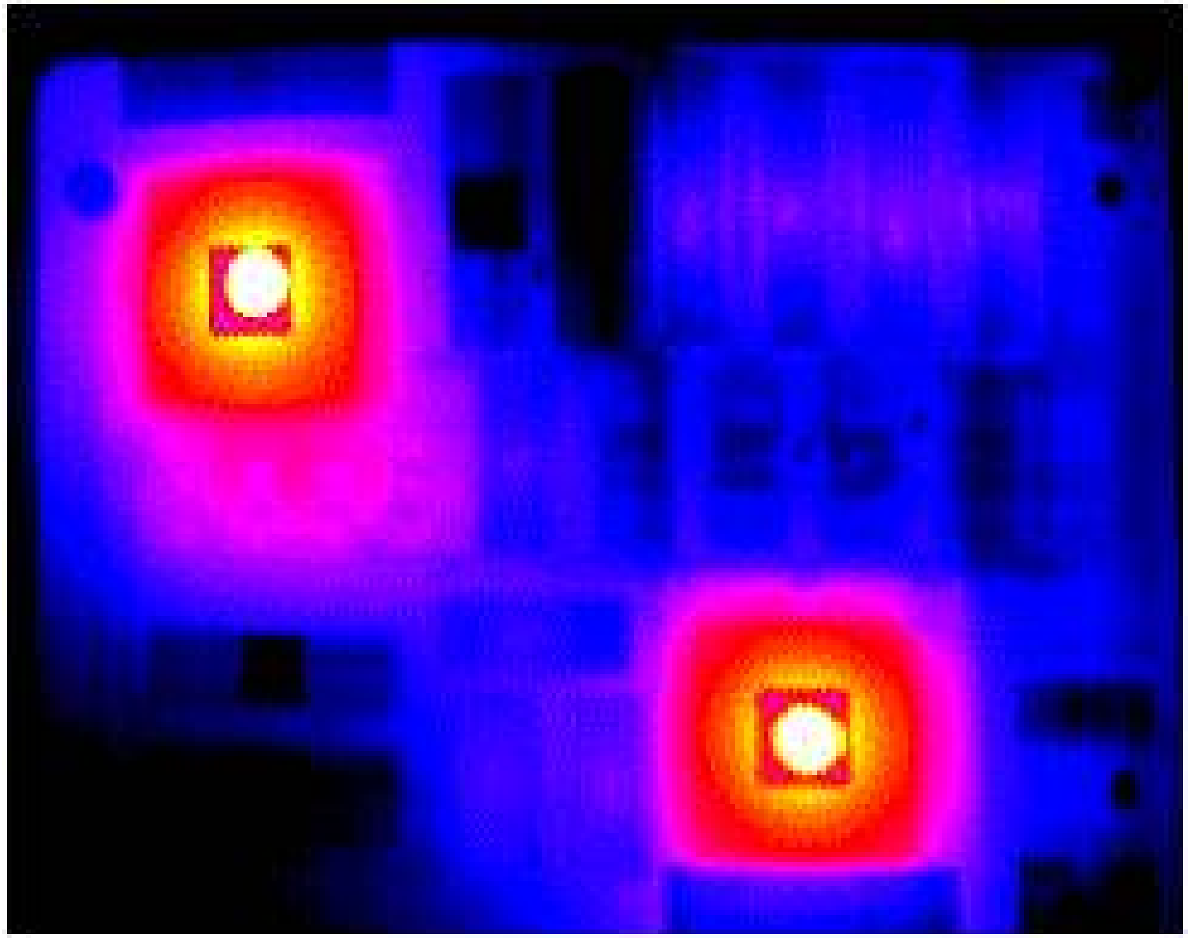
Quadtrees and Meshing

definition & properties
balanced quadtrees
meshing with quadtrees

Motivation: VLSI design

Simulation of heat emission on printed circuit boards





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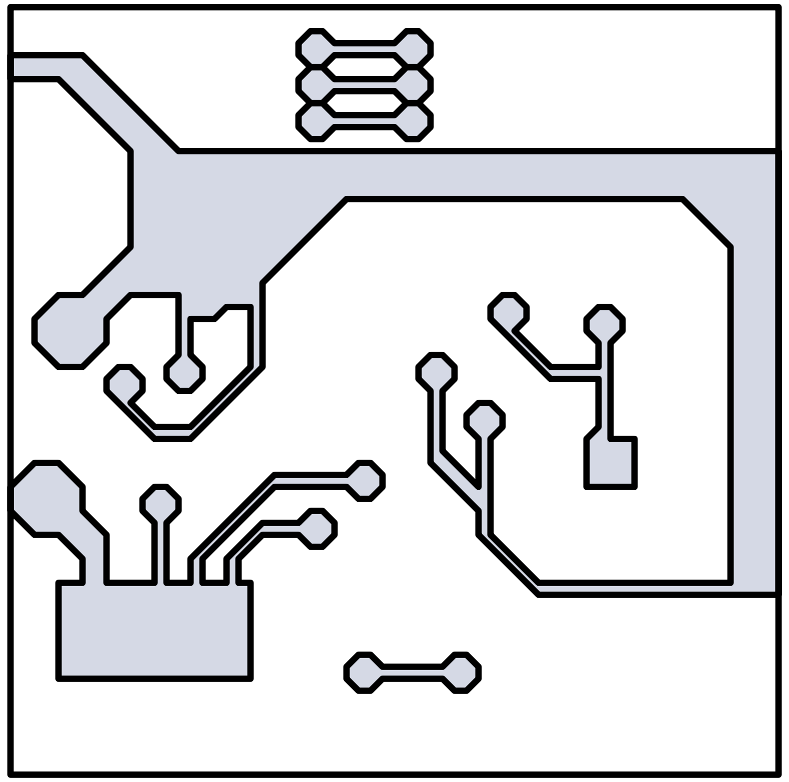

To simulate heat emission, use finite element method:

- partition board into small homogeneous elements (e.g. triangles) $\rightarrow$ mesh
- based on heat emission of each element and influence of neighbors numerically approximate the overall heat emission

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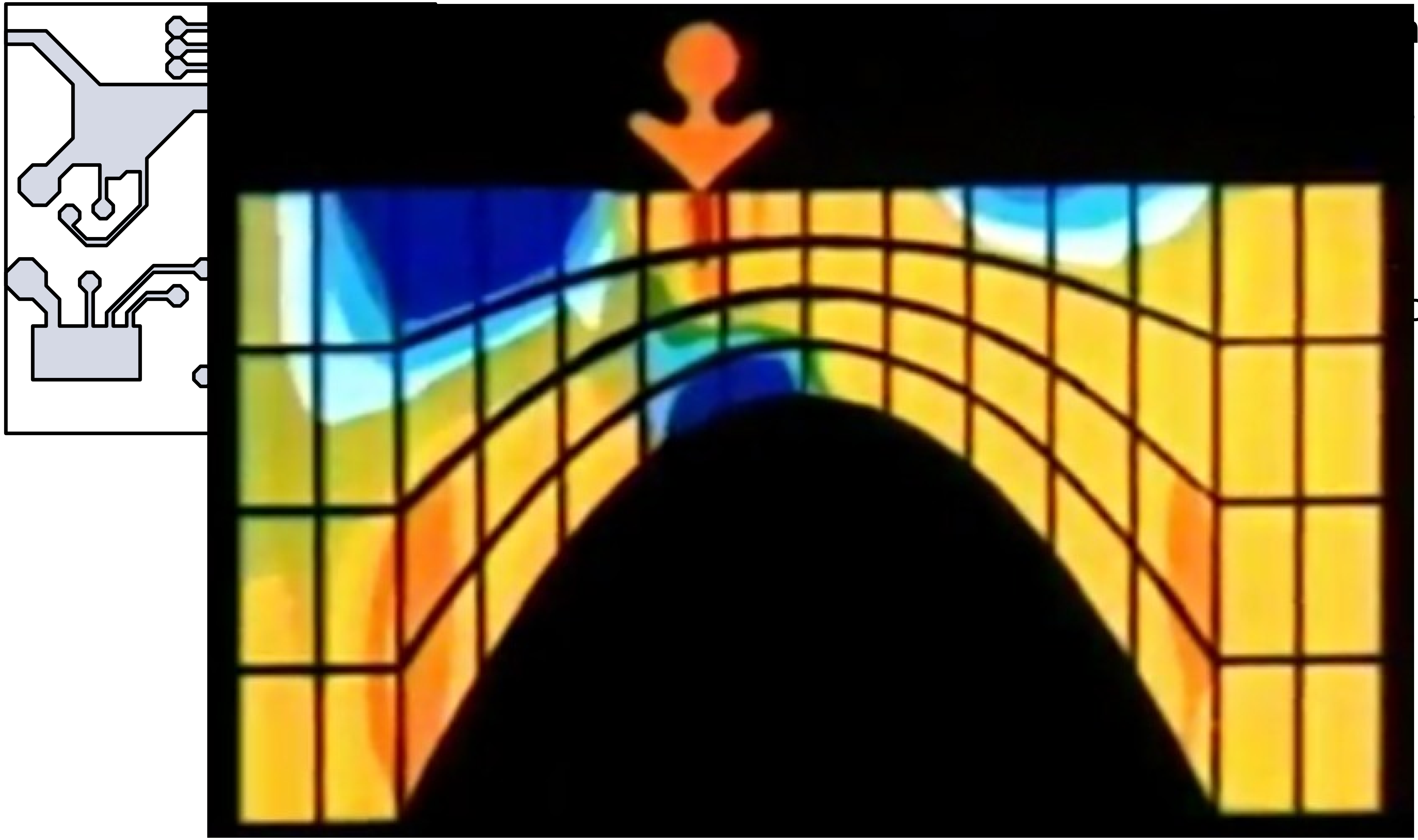

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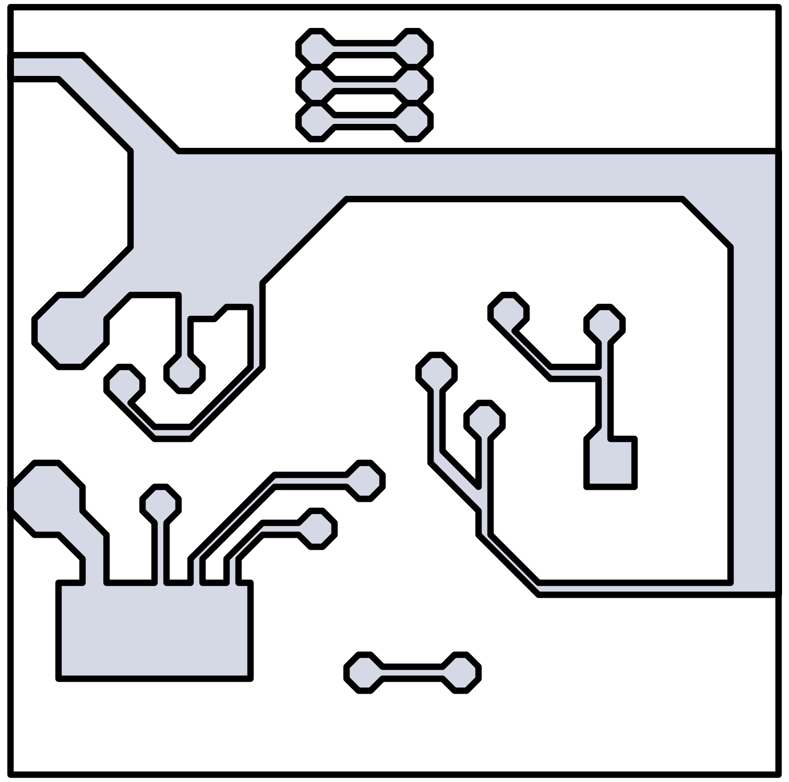

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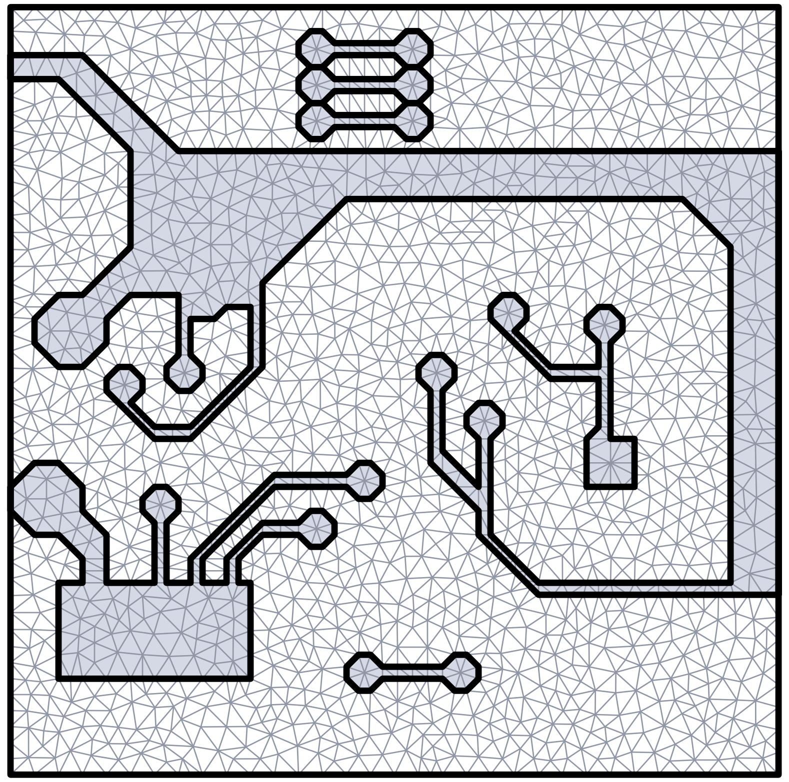

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quality criteria:

- finer mesh $\longrightarrow$ better approximation
- coarser mesh $\longrightarrow$ faster computation
- compact elements $\longrightarrow$ faster convergence

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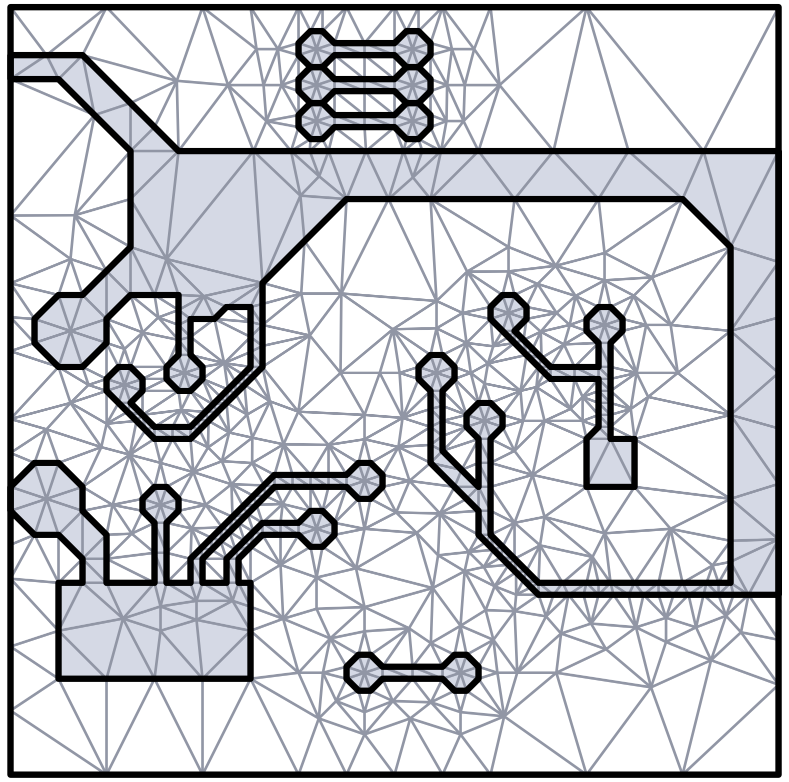

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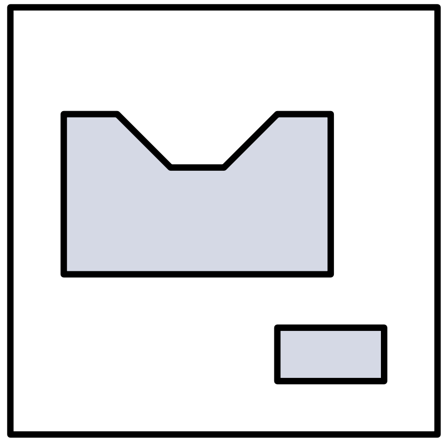
goal:
- non-uniform mesh $\longrightarrow$ small at boundaries, larger otherwise
- well-shaped triangles $\longrightarrow$ not too thin

Non-uniform meshes

Given: octilinear polygons with integer coordinates within a square $Q = [0, U] \times [0, U]$ with $U = 2^j$ a power of two.



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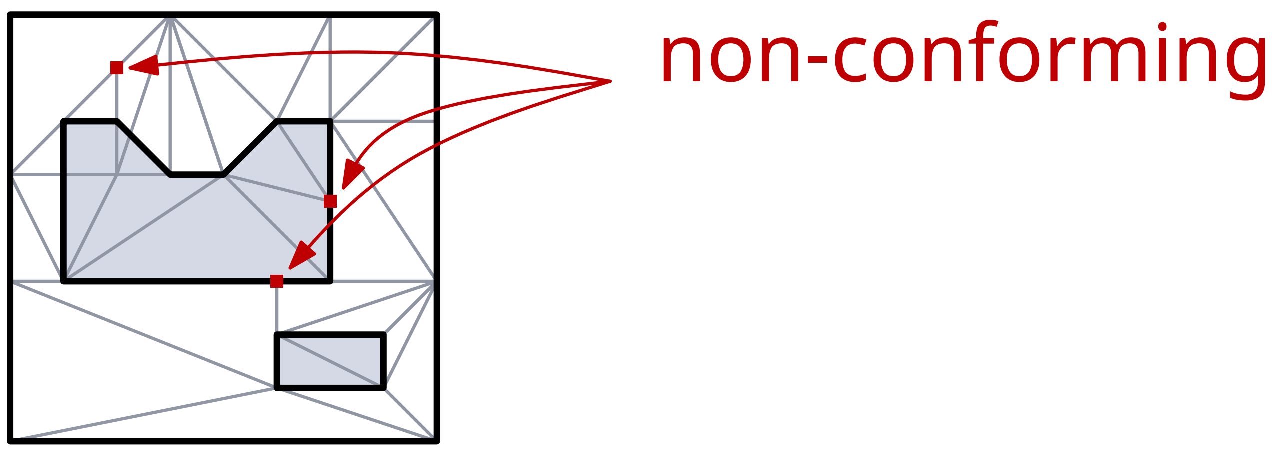

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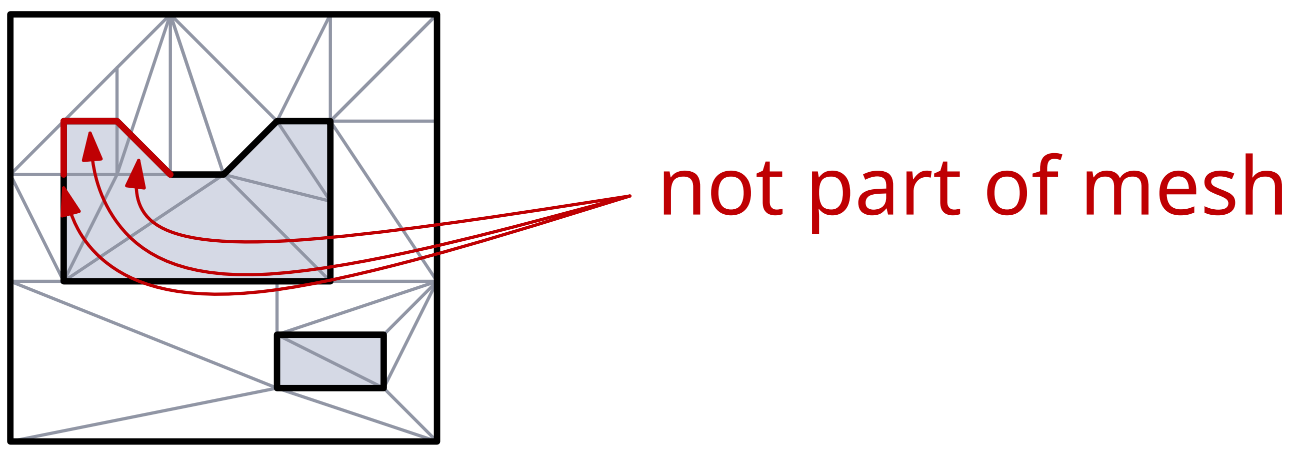

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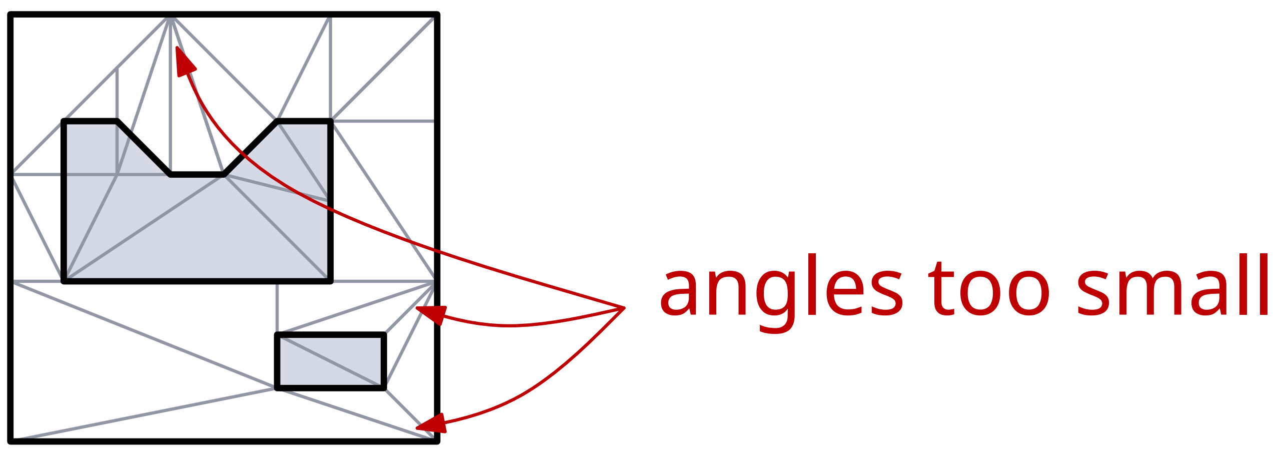

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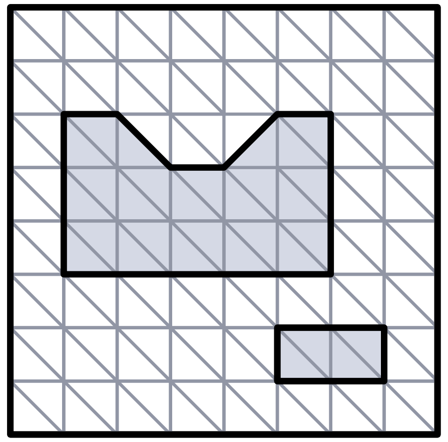

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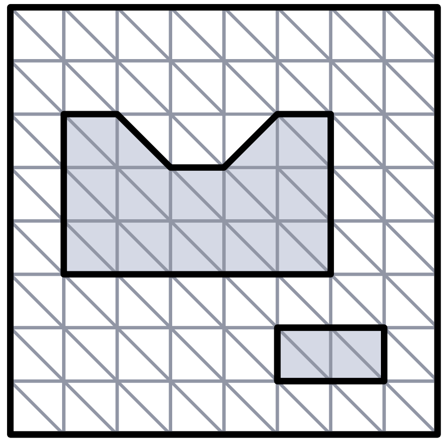

uniform mesh

Do we need Steiner points (i.e. non-input vertices)?

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Triangulation of subdivision?

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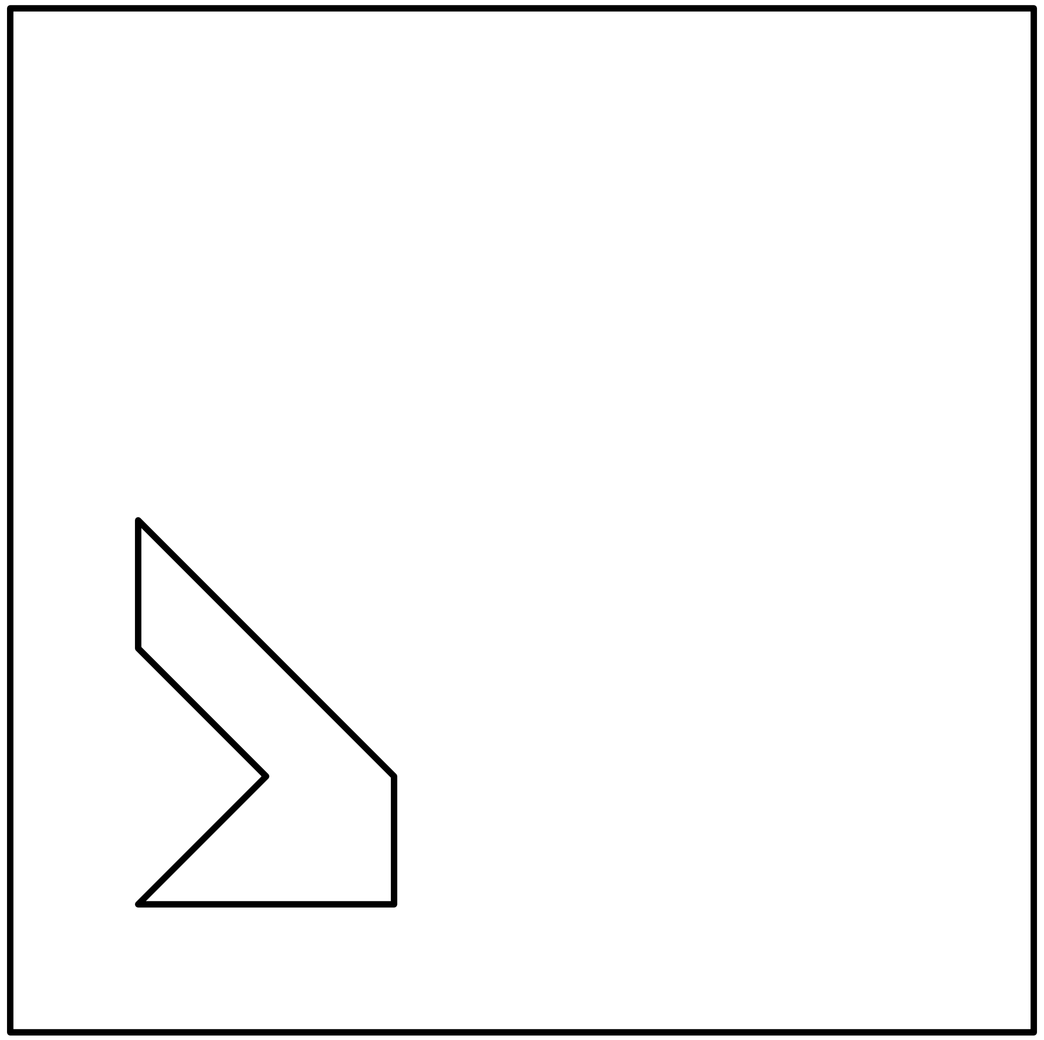

Triangulation of subdivision?

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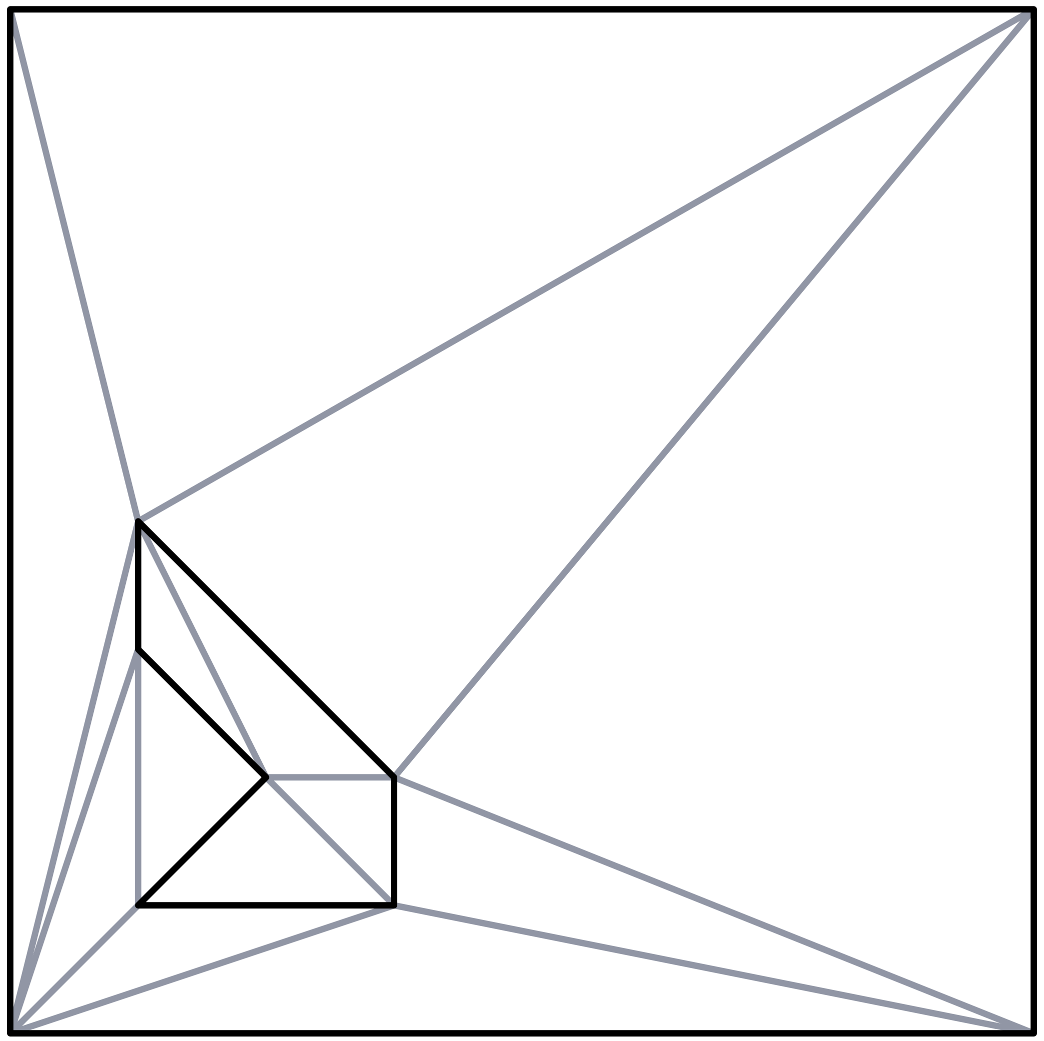

Triangulation of subdivision?

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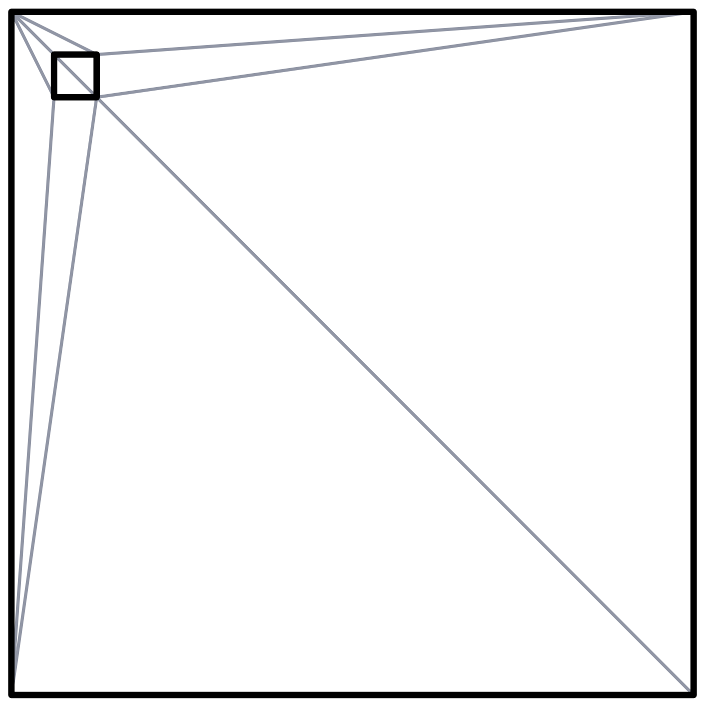
- without Steiner points: might have very small angles



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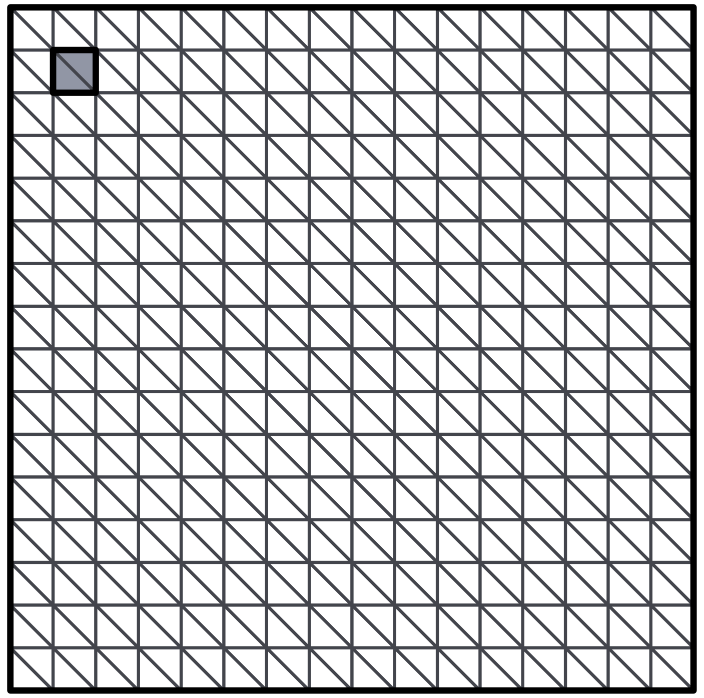

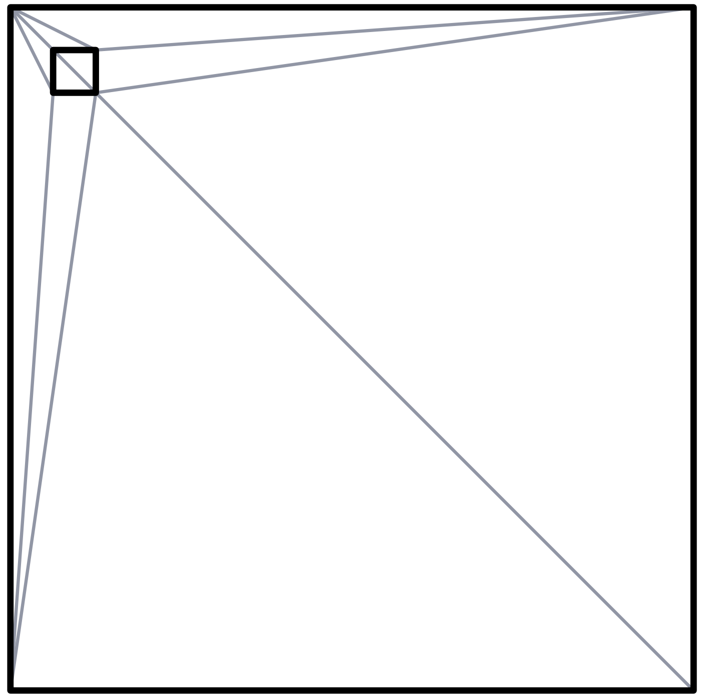

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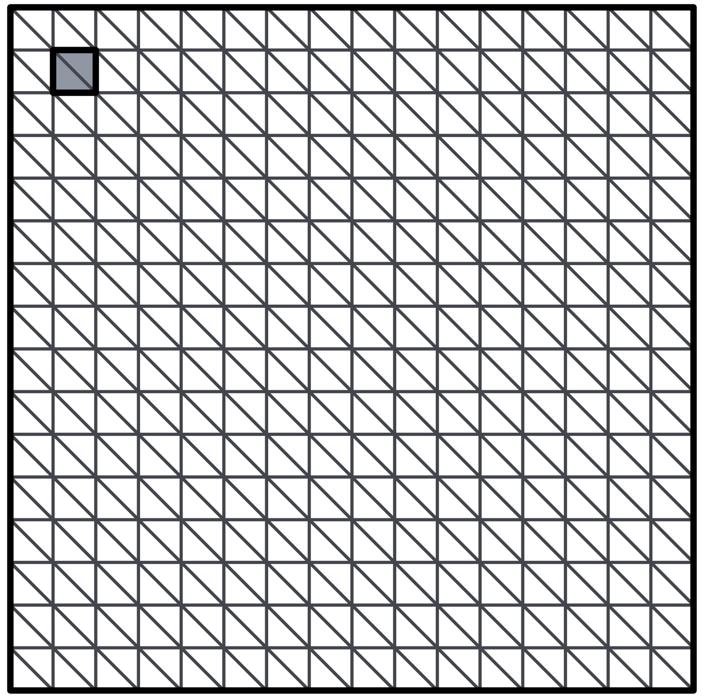
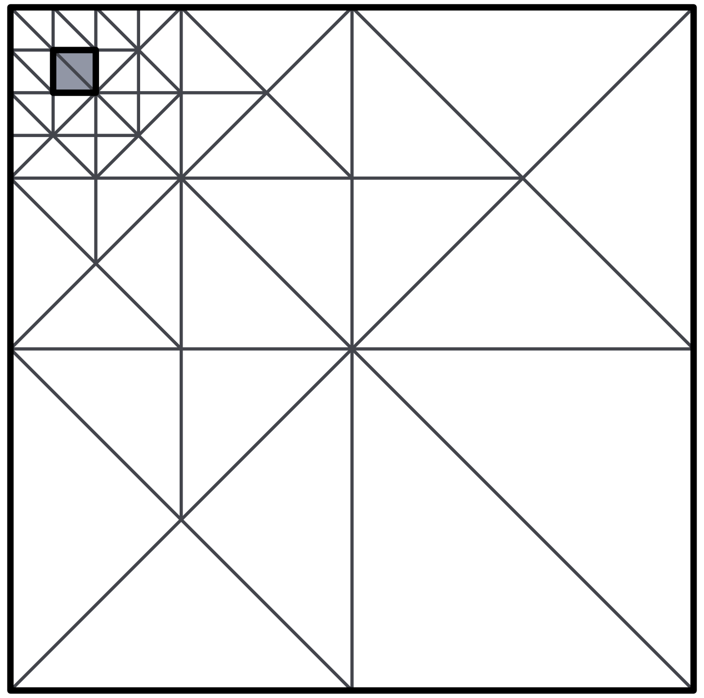
well-shaped, but uniform



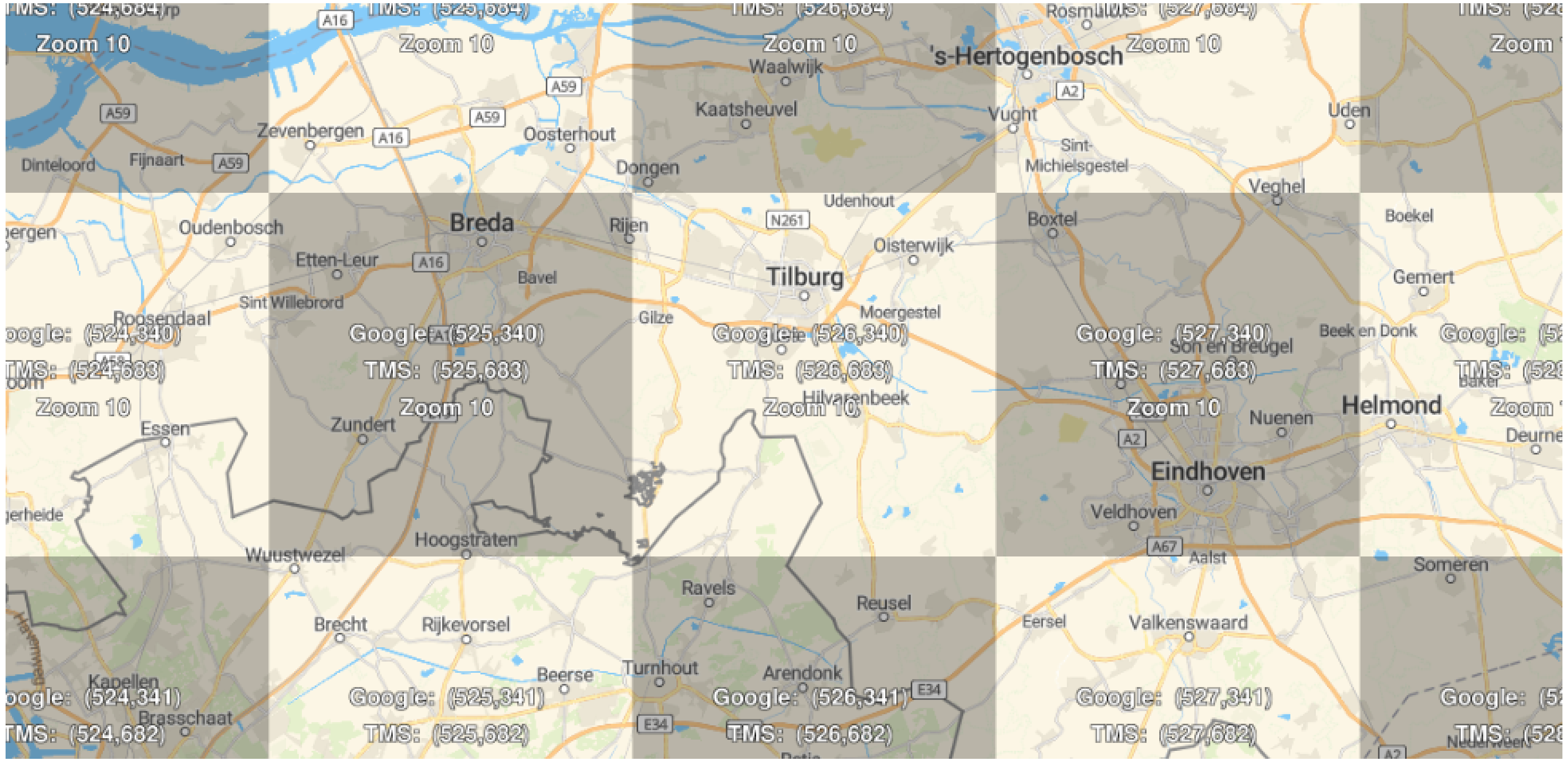
512 triangles

well-shaped, non-uniform



52 triangles

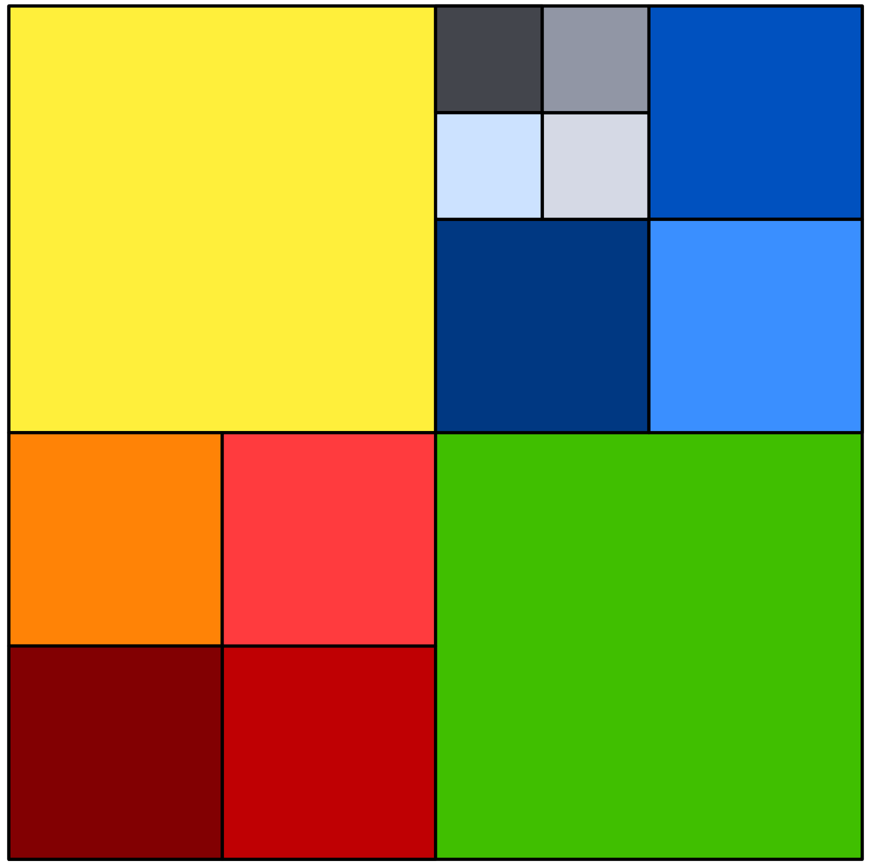
Quadtrees

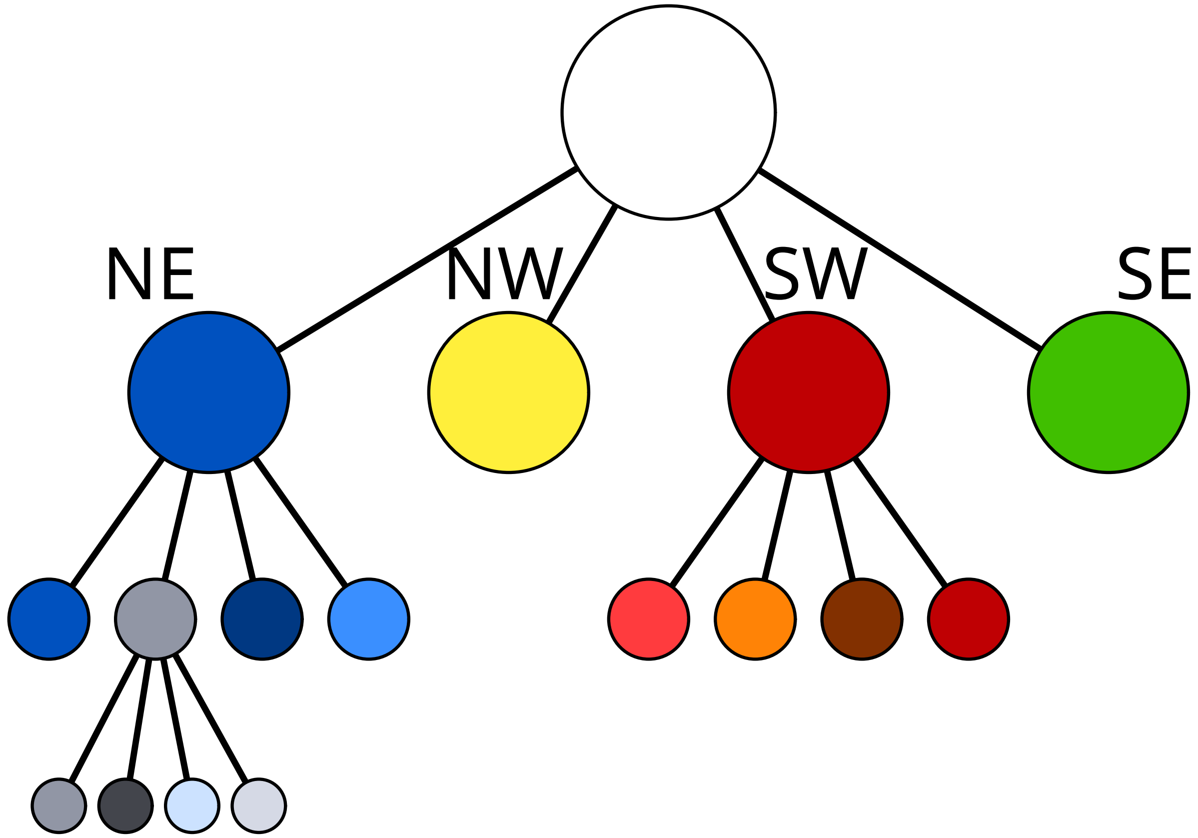


https://google.github.io/closure-library/source/closure/goog/demos/quadtree.html, http://www.maptiler.org/google-maps-coordinates-tile-bounds-projection

Quadtrees

Definition: A quadtree is a rooted tree, in which every interior node has 4 children. Every node corresponds to a square, and the squares of children are the quadrants of the parent's square.





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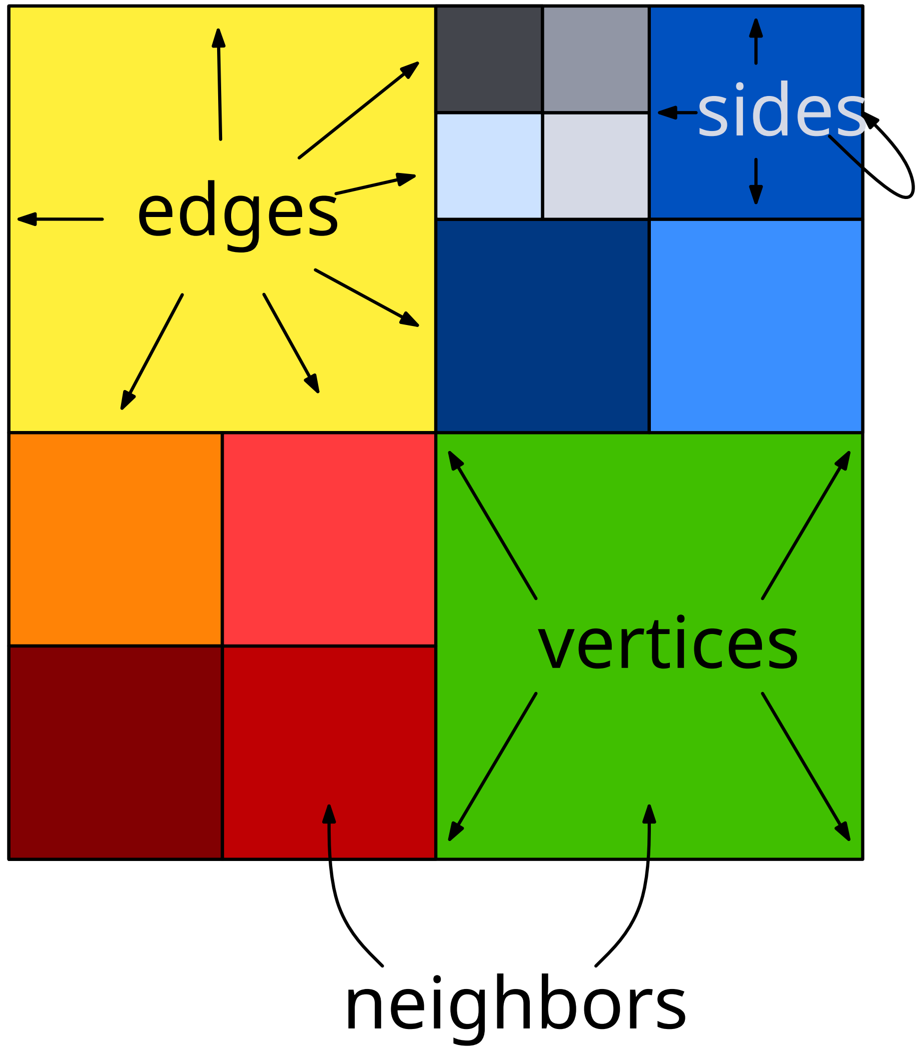
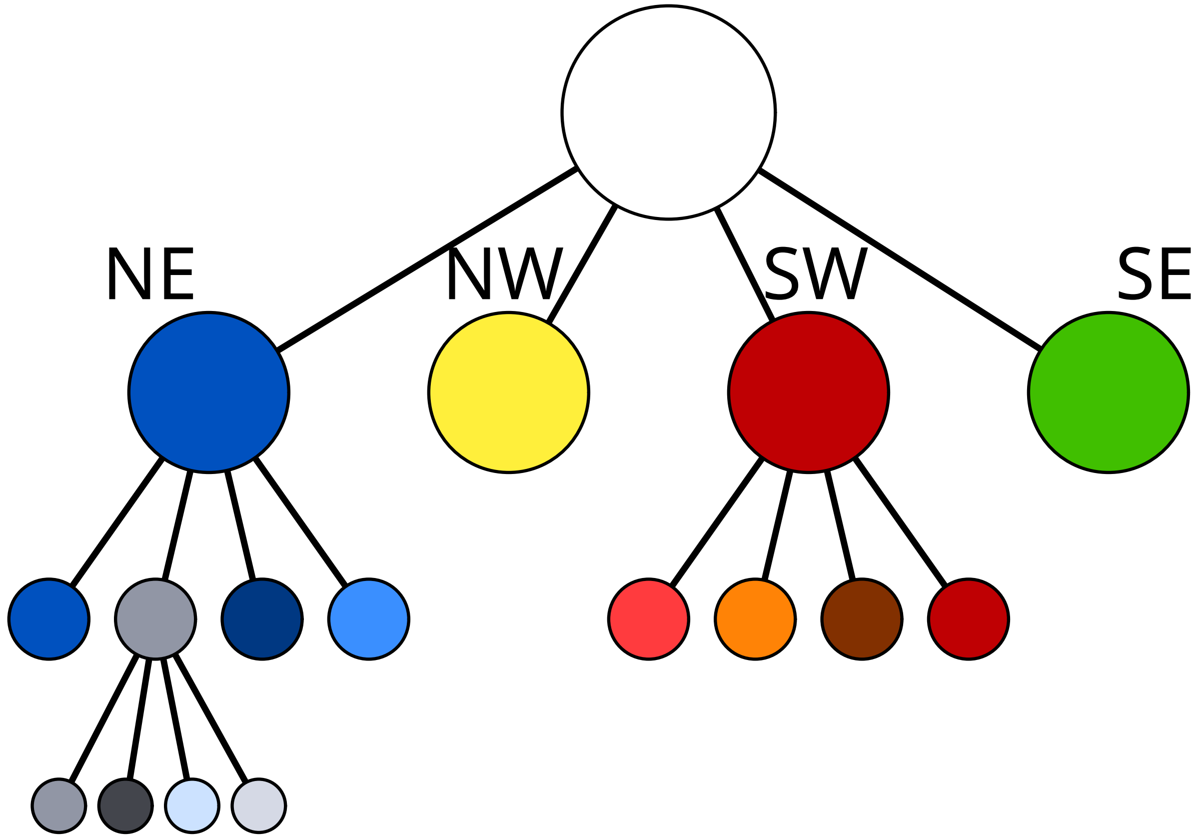




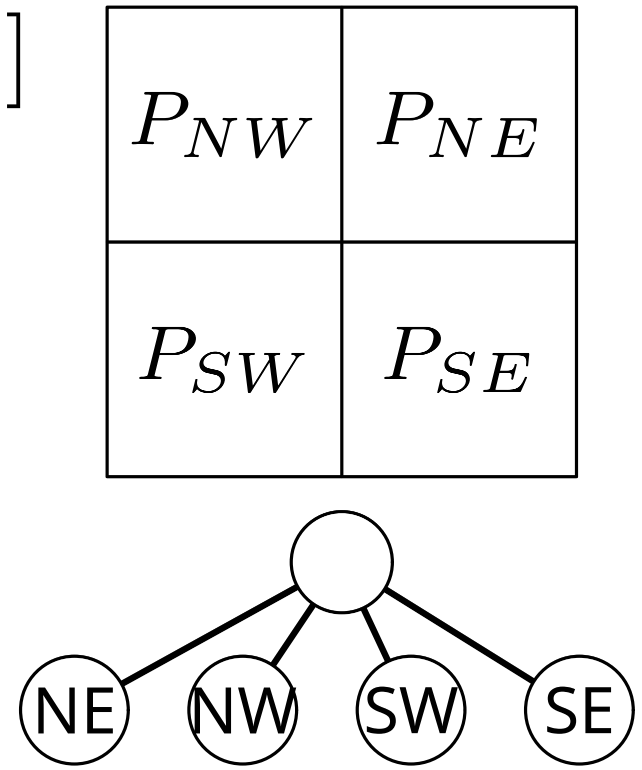
Quadtrees

Definition: For a point set P in a square $Q = [x_Q, x'_Q] \times [y_Q, y'_Q]$ the quadtree $\mathcal{T}(P)$ is:

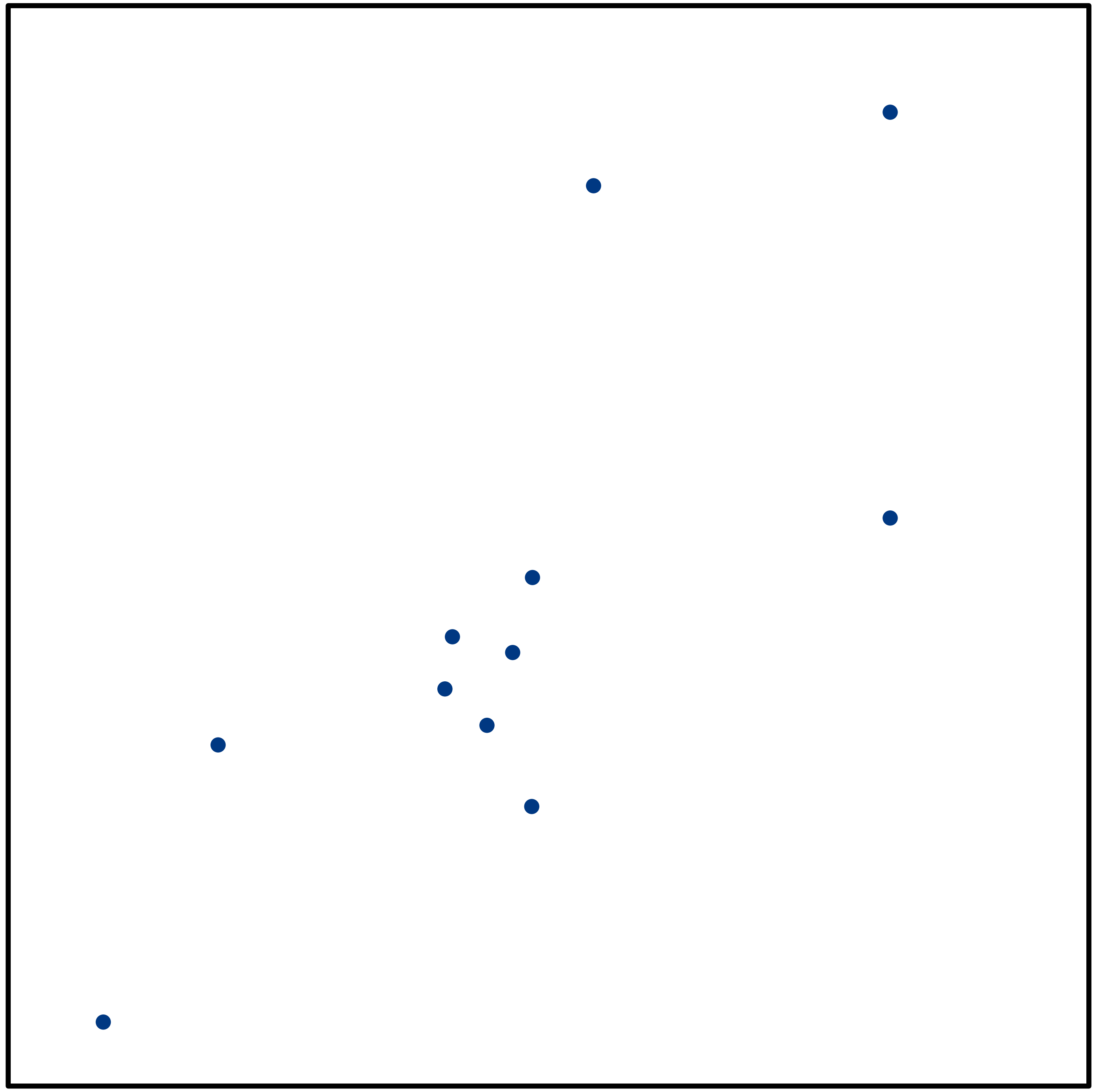
- if $|P| \leq 1$, is a leaf storing P and Q
- otherwise, is a node storing Q with four quadtrees as children in four quadrants for:

$$
\begin{aligned}
P_{NE} &:= \{p \in P \mid p_x > x_{\text{mid}} \text{ and } p_y > y_{\text{mid}}\}, \\
P_{NW} &:= \{p \in P \mid p_x \leq x_{\text{mid}} \text{ and } p_y > y_{\text{mid}}\}, \\
P_{SW} &:= \{p \in P \mid p_x \leq x_{\text{mid}} \text{ and } p_y \leq y_{\text{mid}}\}, \\
P_{SE} &:= \{p \in P \mid p_x > x_{\text{mid}} \text{ and } p_y \leq y_{\text{mid}}\},
\end{aligned}
$$

where $x_{\text{mid}} = \frac{x_Q + x'_Q}{2}$ and $y_{\text{mid}} = \frac{y_Q + y'_Q}{2}$.



Example



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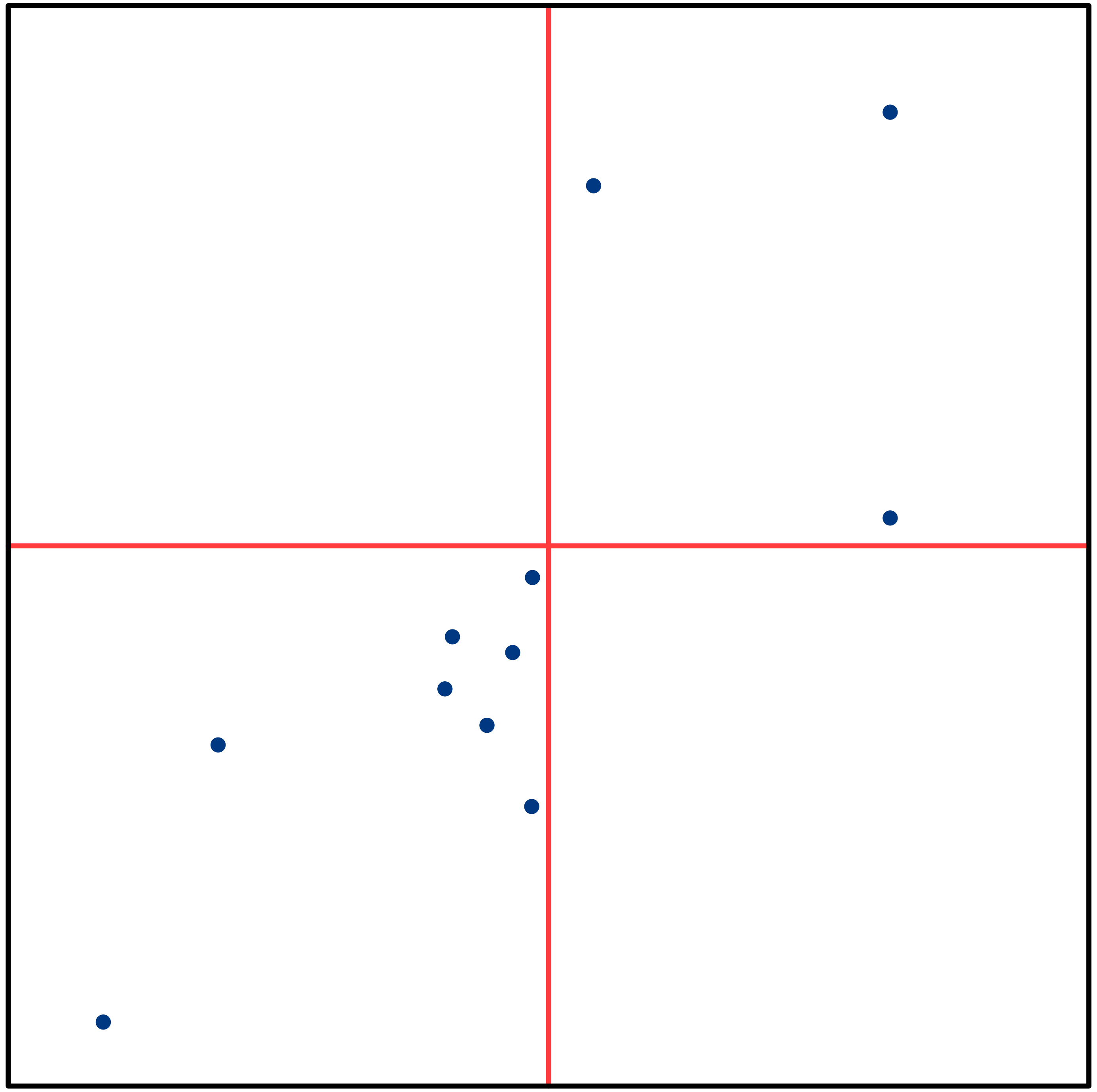

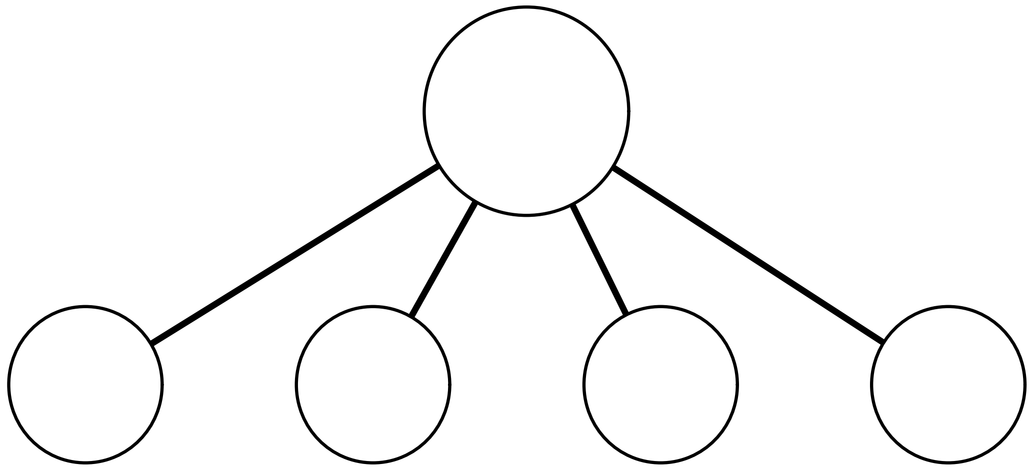

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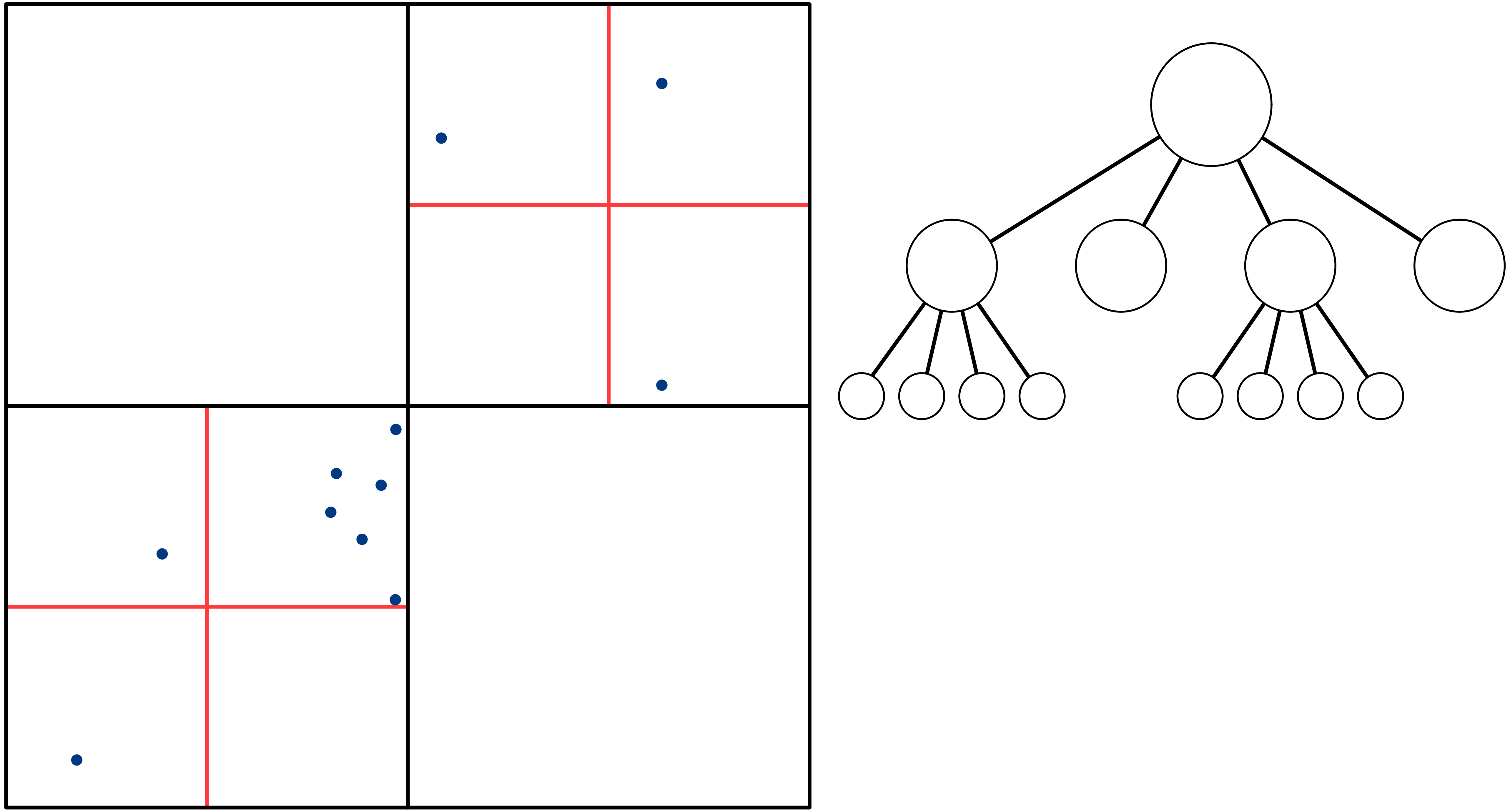

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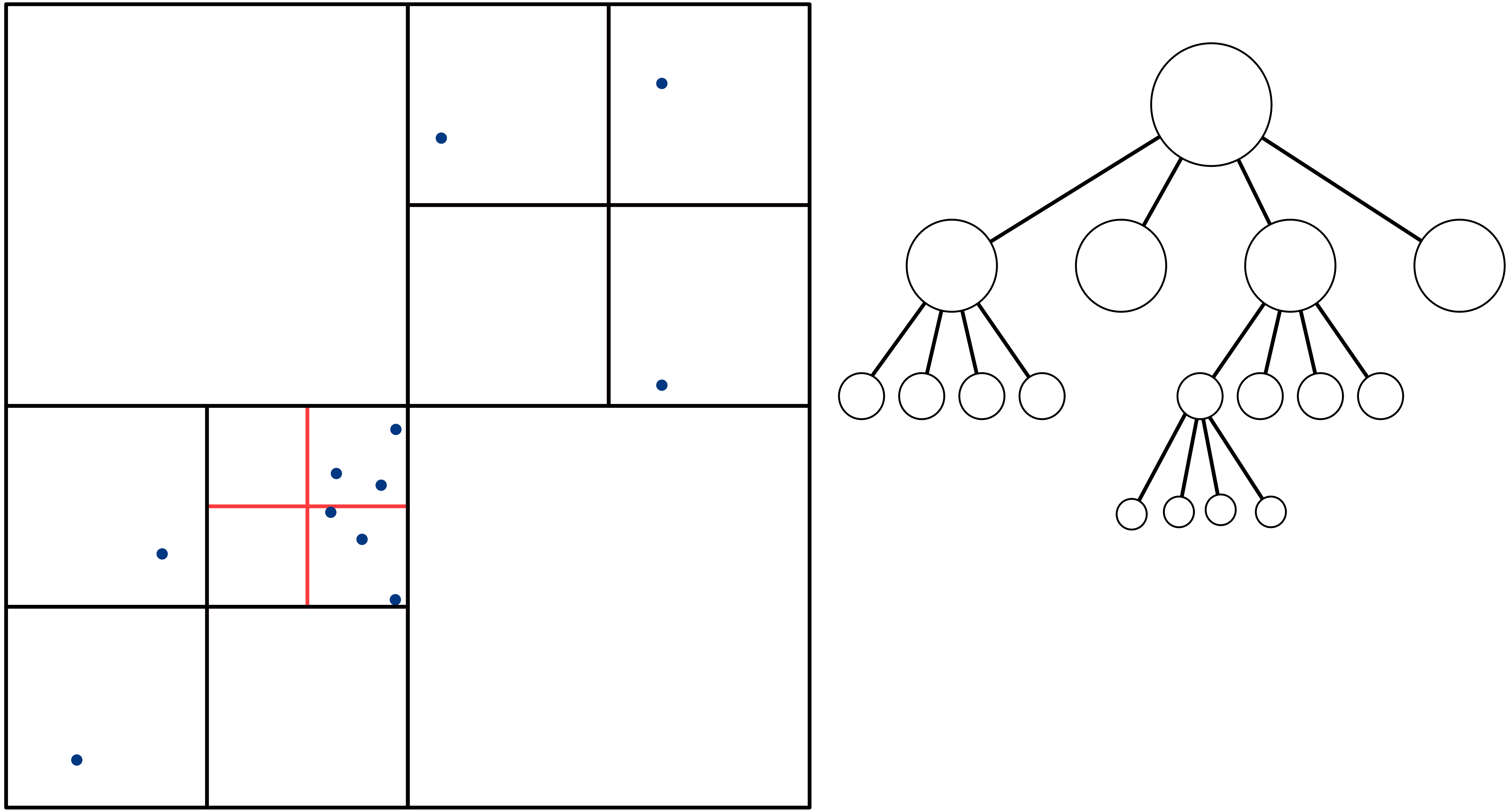

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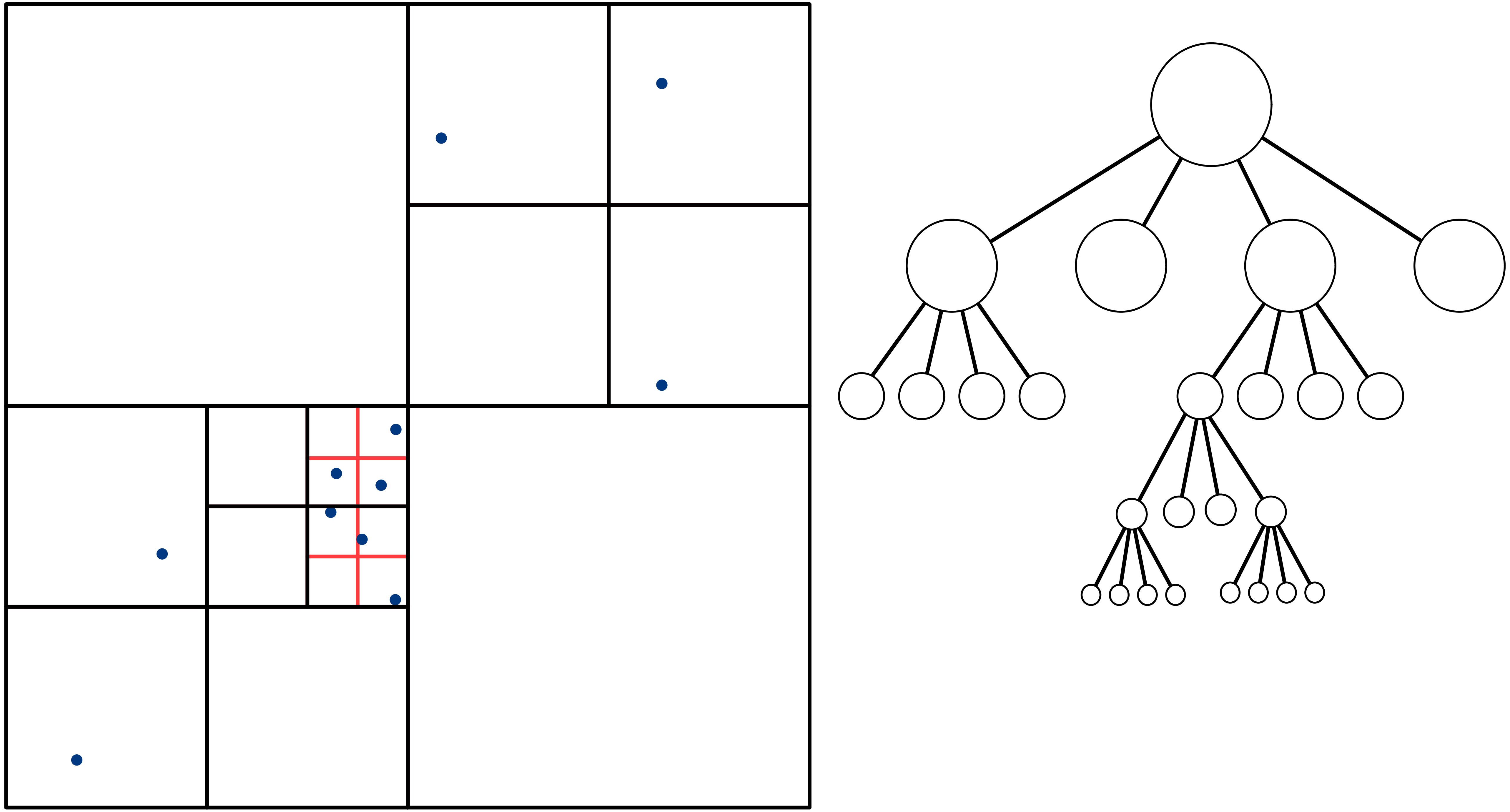

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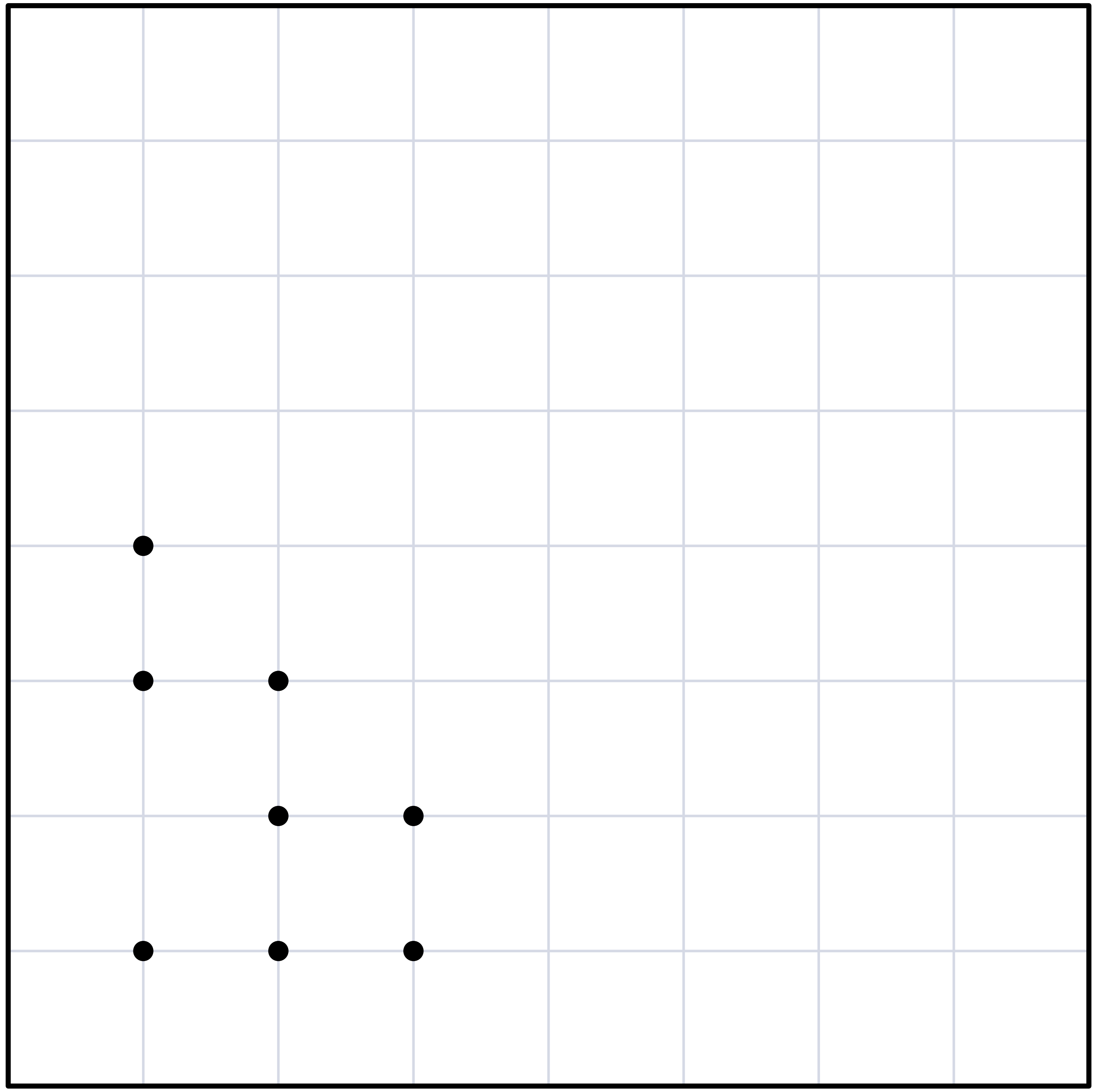

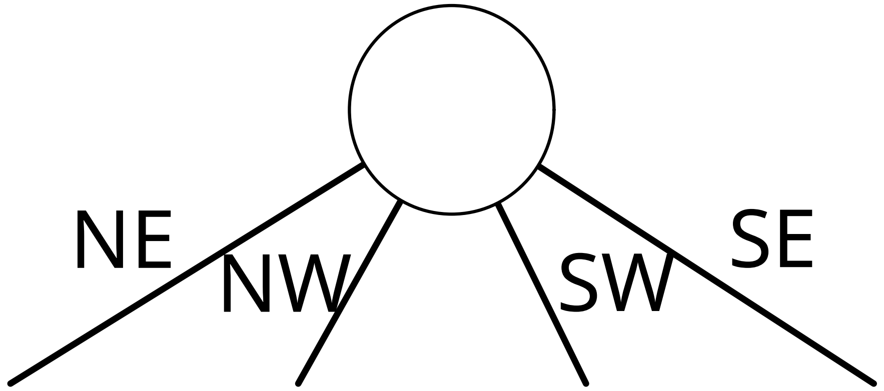

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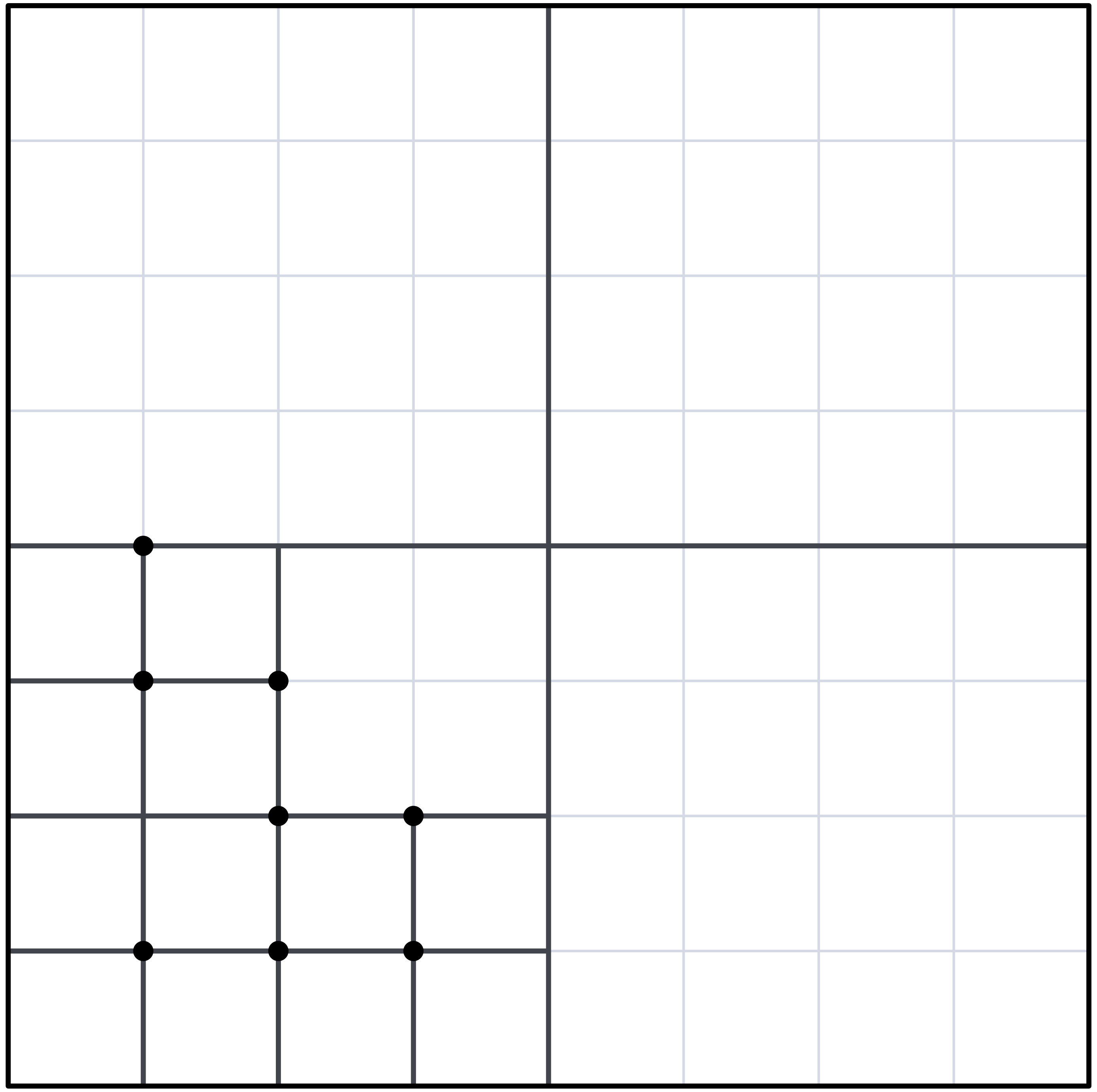

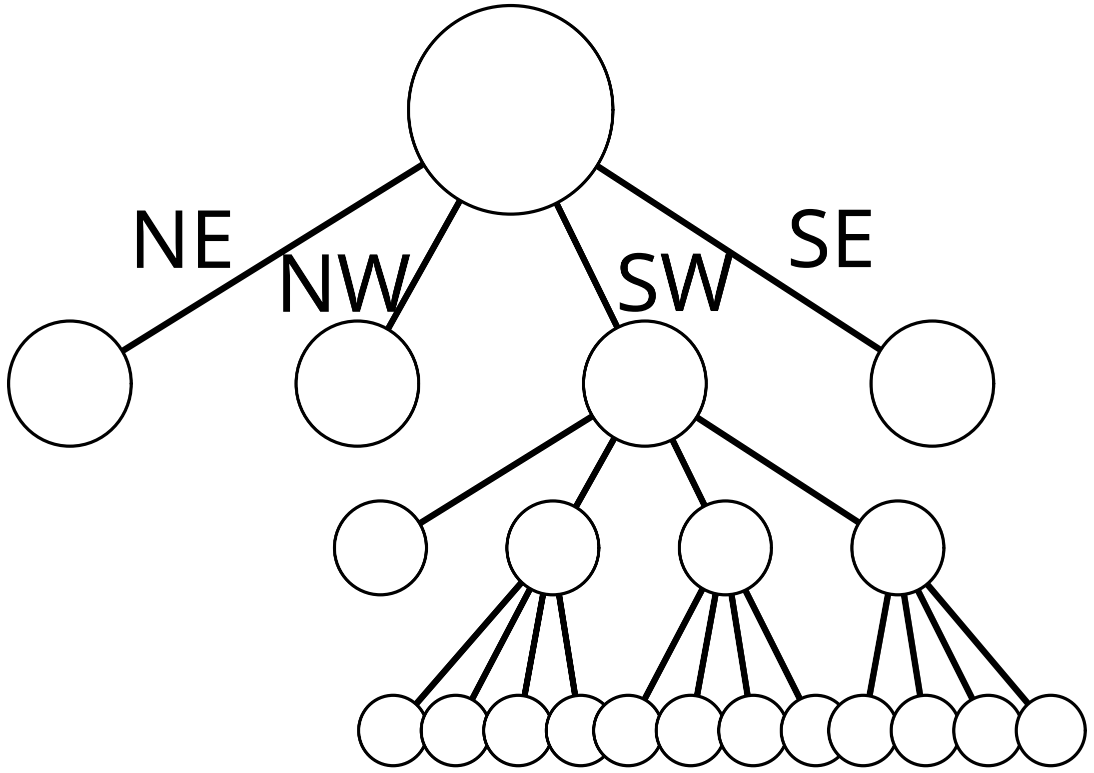

Quadtree properties

The recursive definition of a quadtrees immediately results in an algorithm

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Question: What is the depth of a quadtree with n nodes?

Quadtree properties

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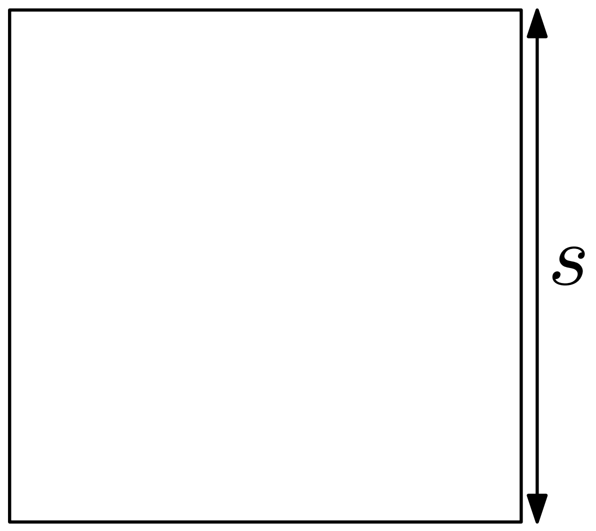
Lemma 1: Let c be the smallest distance between any two points in a point set P, and let s be the side length of the initial (biggest) square. Then the depth of a quadtree for P is at most $\log(s/c) + 3/2$.



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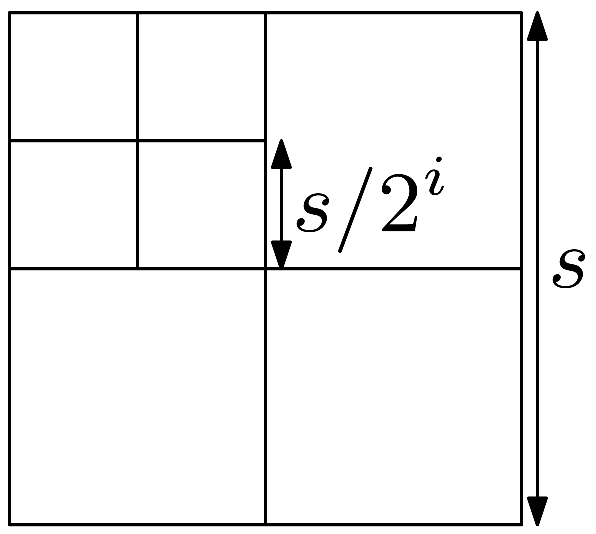

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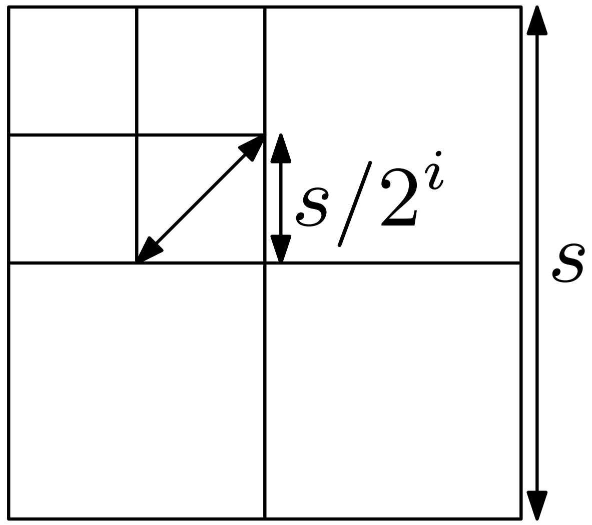

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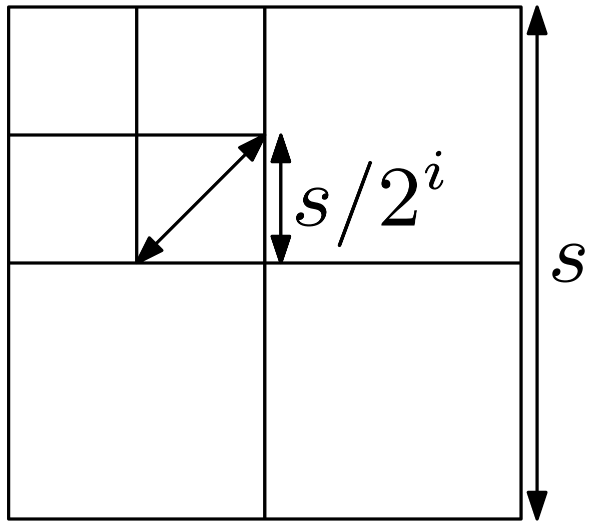

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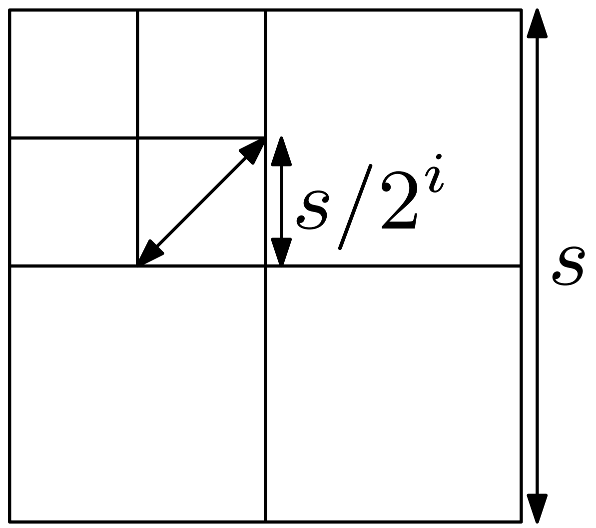

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$\Rightarrow i \leq \log(\sqrt{2}s/c) = \log(s/c) + 1/2$

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Theorem 1: A quadtree of depth d storing n points has $O((d+1)n)$ nodes and can be constructed in $O((d+1)n)$ time.

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$\Rightarrow$ for depth d overall $O((d+1)n)$ nodes.

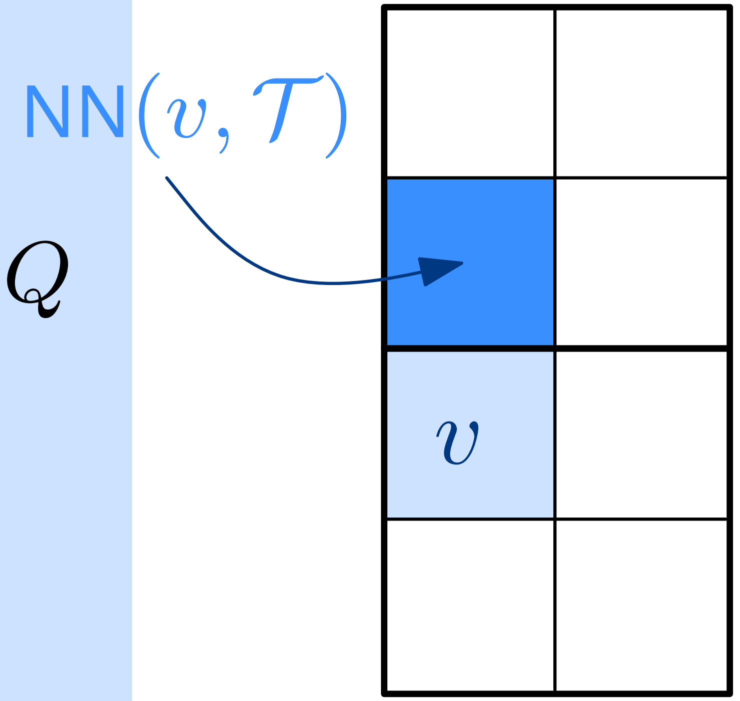
Finding neighbors

NORTHNEIGHBOR$(v, \mathcal{T})$

Input: node v in quadtree $\mathcal{T}$

Output: deepest v' not deeper than v, with $v'.Q$ north neighbor of $v.Q$

if $v =$ root$(\mathcal{T})$ **then return** NIL
$\pi \leftarrow$ parent(v)
if $v =$ SW(SE)-child of π **then return** NW(NE)-child of π
$\mu \leftarrow$ NORTHNEIGHBOR$(\pi, \mathcal{T})$
if $\mu =$ NIL or μ is a leaf **then return** μ
if $v =$ NW(NE)-child of π **then return** SW(SE)-child of μ



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```
if v = root(T) then return NIL
π ← parent(v)
if v = SW(SE)-child of π then return NW(NE)-child of π
μ ← NorthNeighbor(π, T)
if μ = NIL or μ is a leaf then return μ
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```

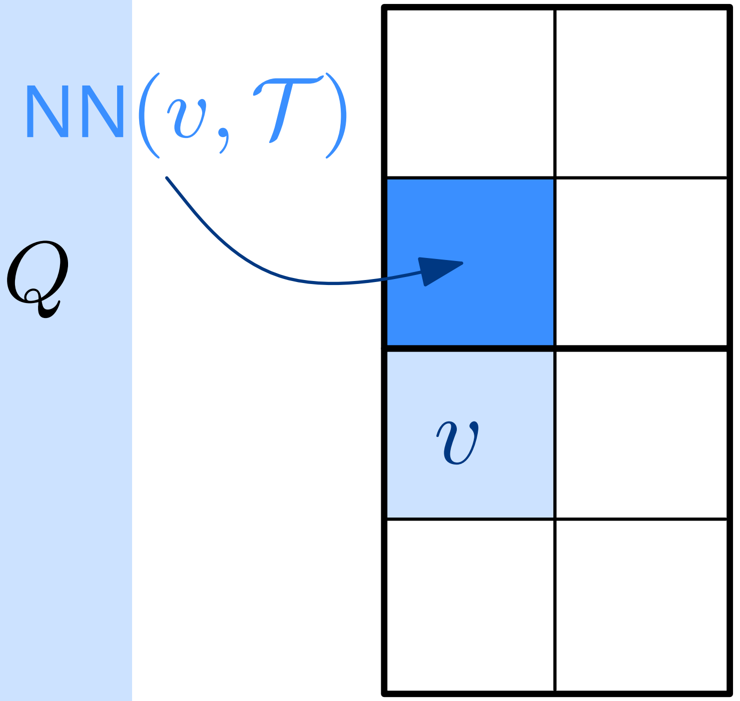

Theorem 2: Let $\mathcal{T}$ be a quadtree of depth d. The neighbors of a node v in any direction can be found in $O(d + 1)$ time.

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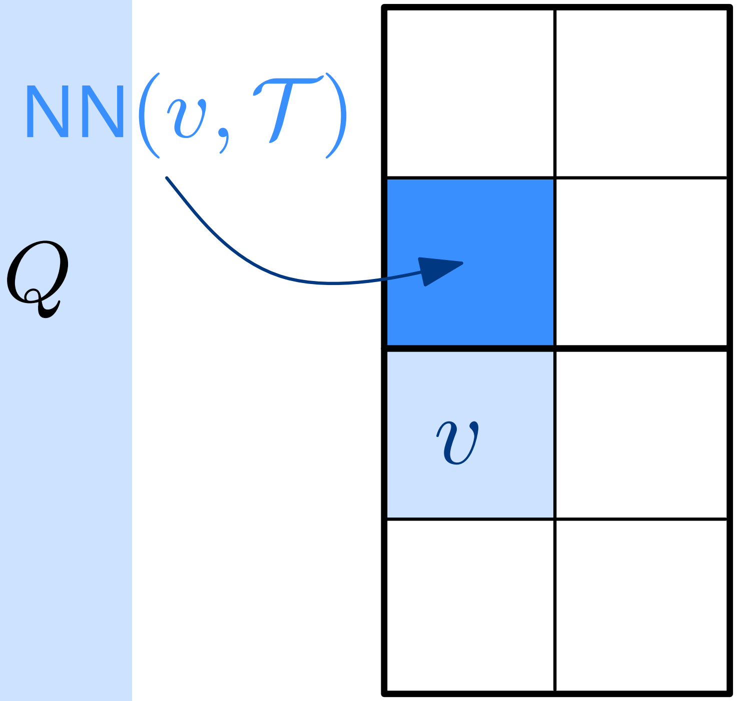
if $\mu =$ Nil or μ is a leaf **then return** μ

if $v =$ NW(NE)-child of π **then return** SW(SE)-child of μ



Theorem 2: Let $\mathcal{T}$ be a quadtree of depth d. The neighbors of a node v in any direction can be found in $O(d+1)$ time.

Proof:

- depth of recursion is $O(d+1)$
- cost per recursive step is $O(1)$

Balanced quadtrees

Definition: a quadtree is balanced if any two neighboring nodes differ by at most 1 in depth

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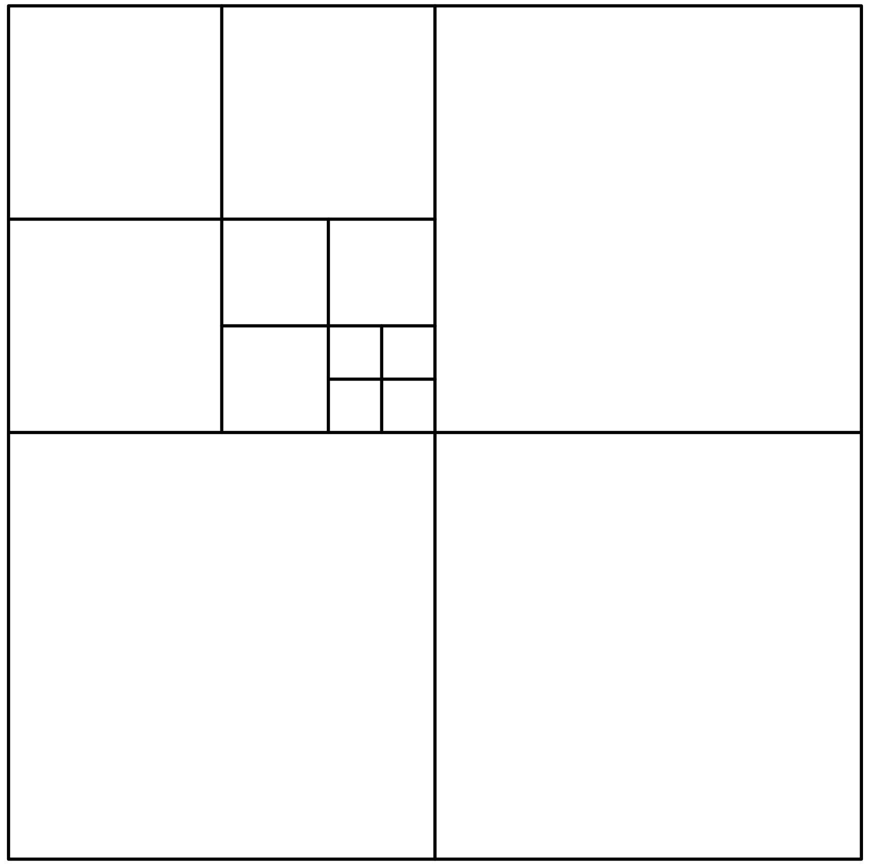

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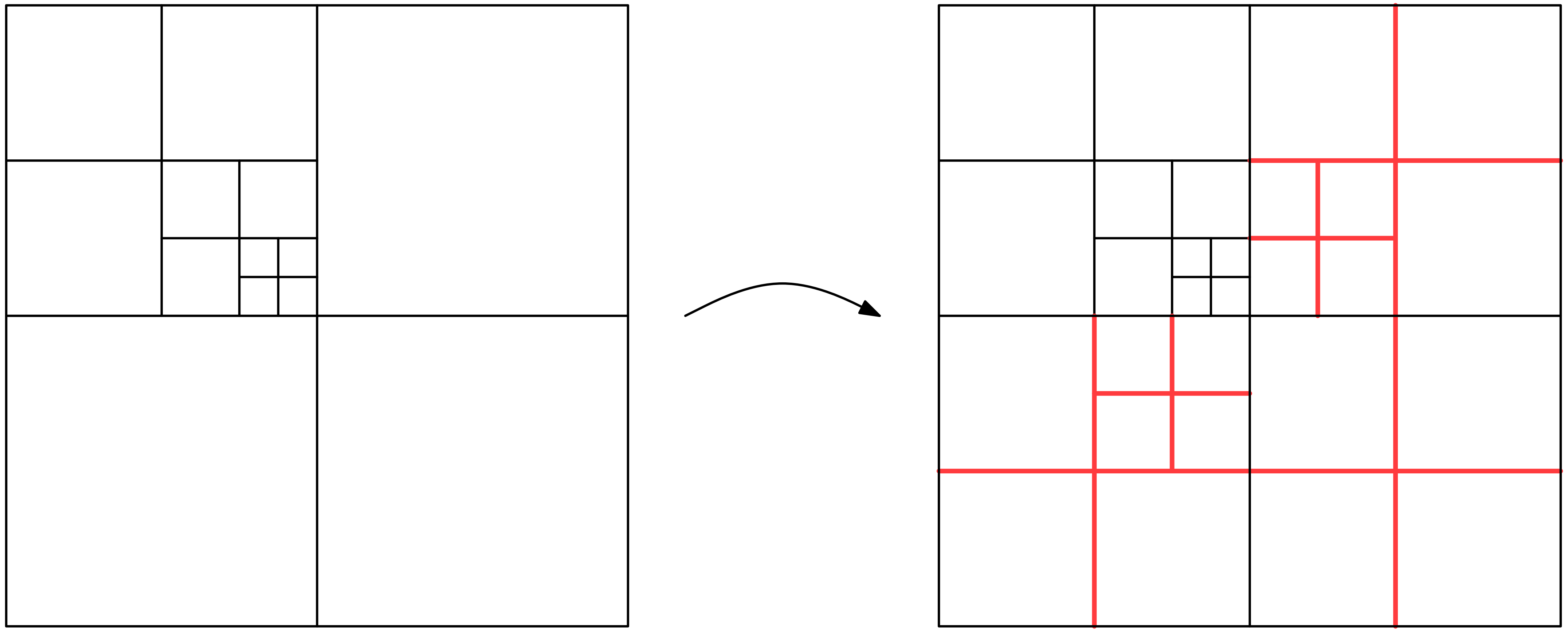

Balancing quadtrees

BALANCEQUADTREE($\mathcal{T}$)

Input: Quadtree $\mathcal{T}$

Output: Balanced quadtree $\mathcal{T}$

$L \leftarrow$ list of all leafs of $\mathcal{T}$
while L is not empty **do**
    $\mu \leftarrow$ extract a leaf from L
    if $\mu.Q$ is too large **then**
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        insert new leaves into L
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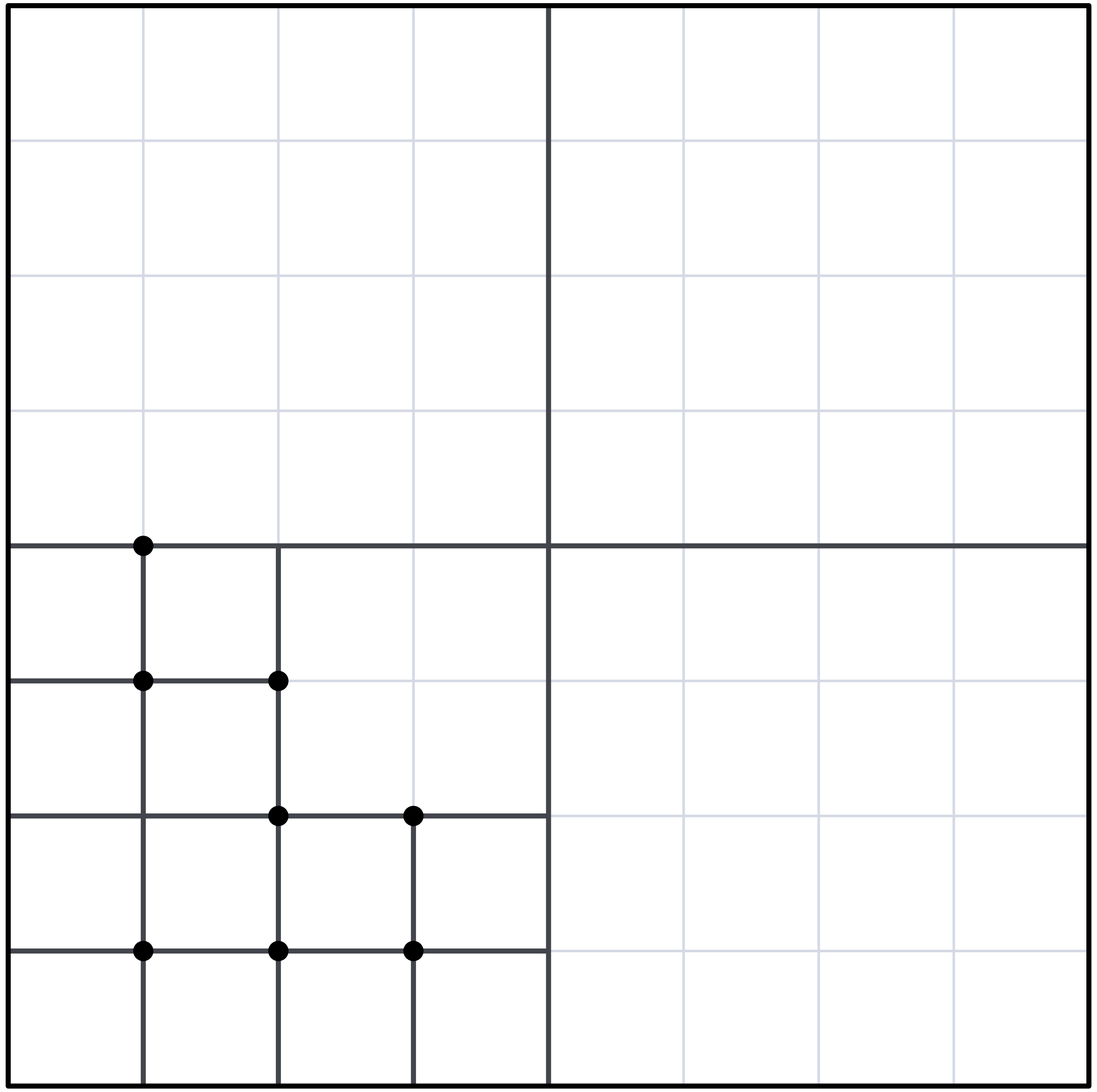
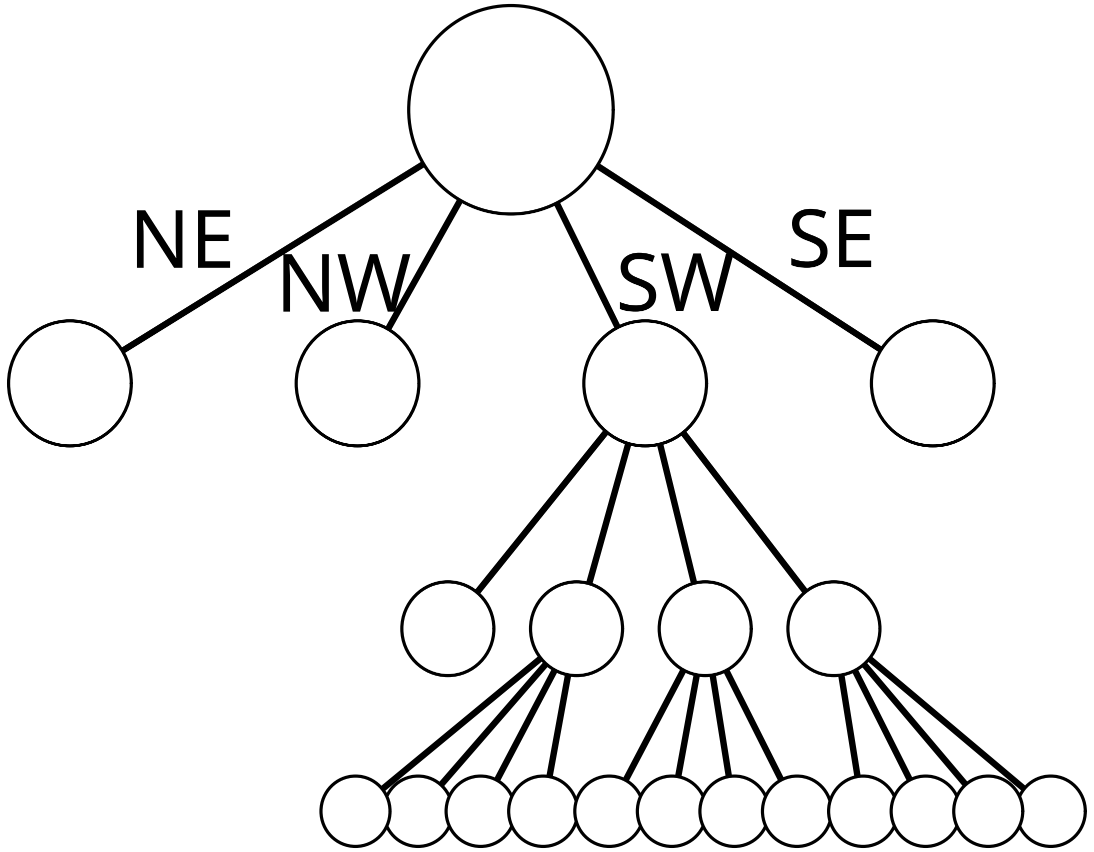
            insert them into L
return $\mathcal{T}$

Balancing quadtrees

Exercise:





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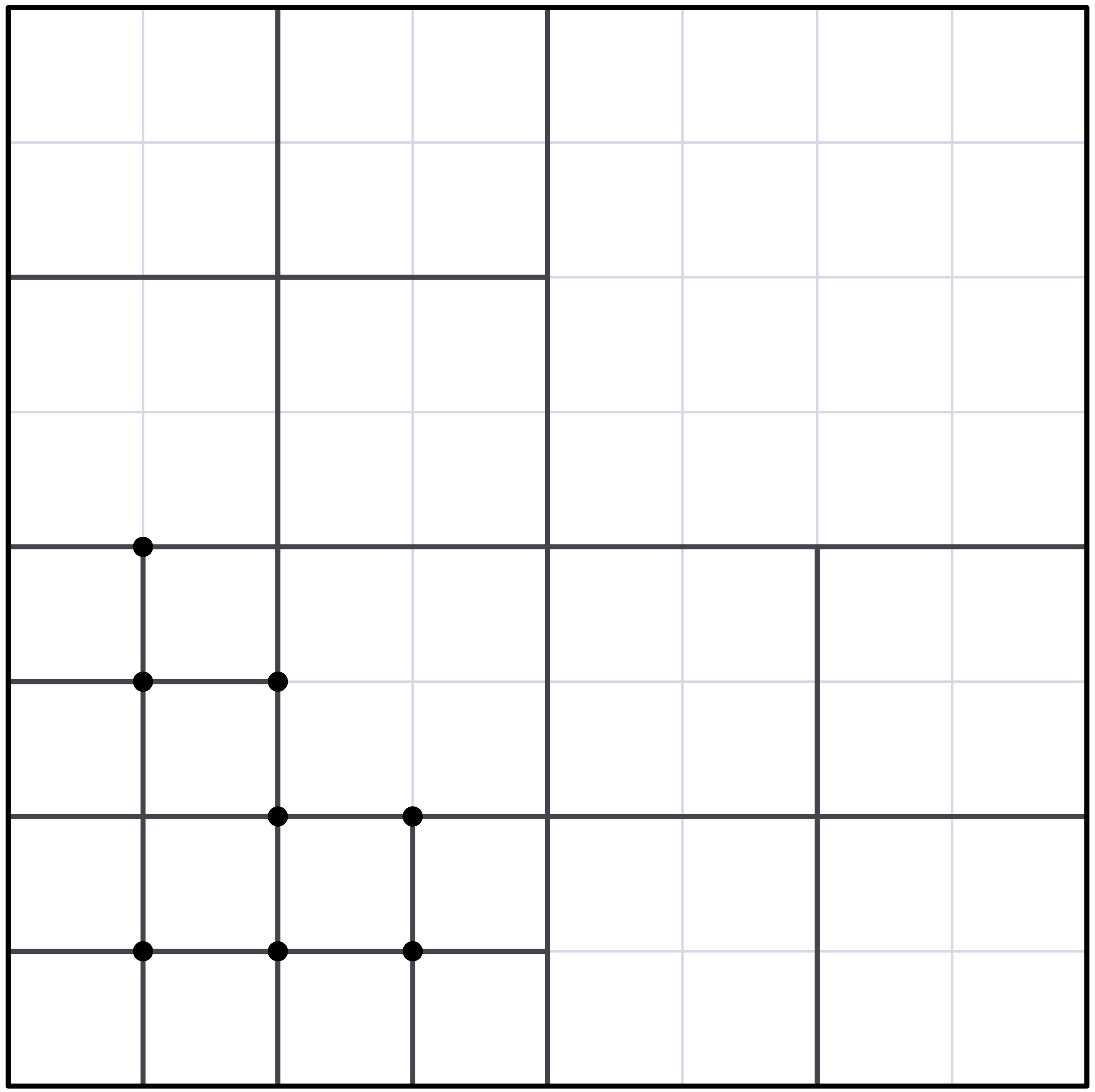

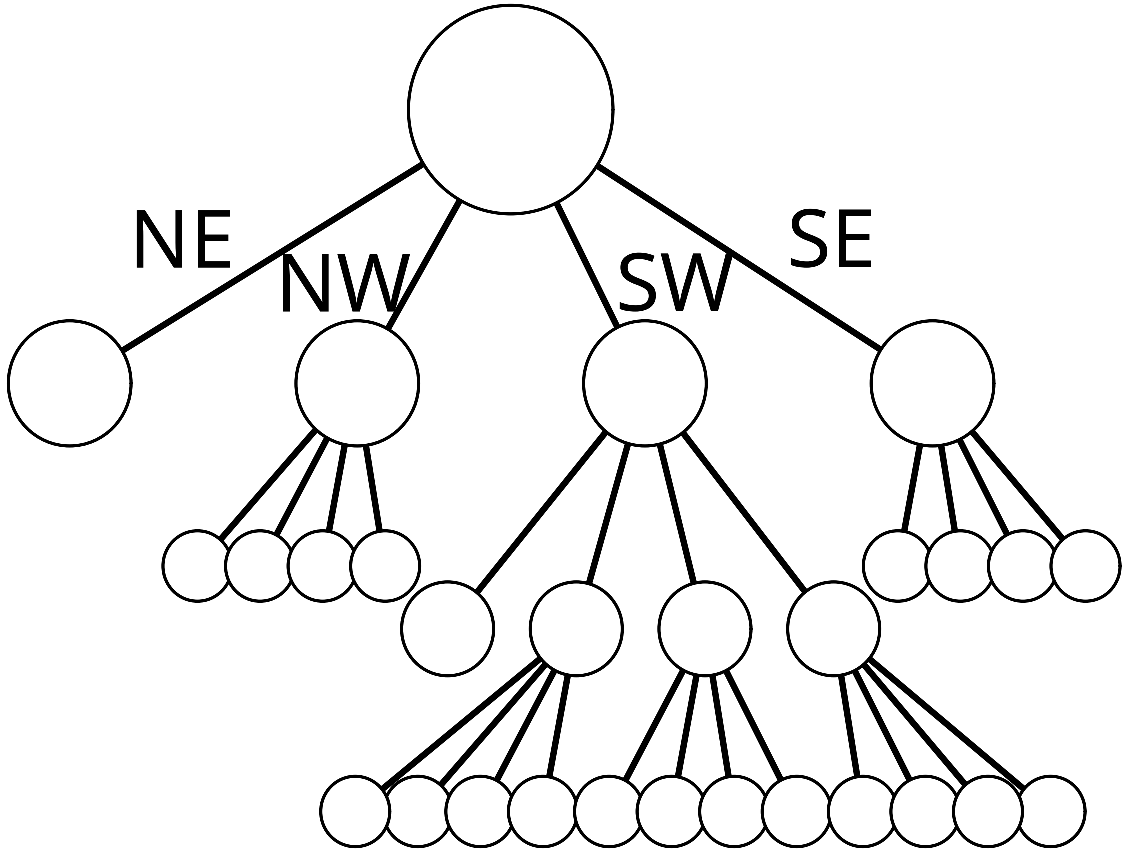

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How?

Balancing quadtrees

BalanceQuadtree$(\mathcal{T})$

Input: Quadtree $\mathcal{T}$

Output: Balanced quadtree $\mathcal{T}$

```
L ← list of all leafs of T
while L is not empty do
    μ ← extract a leaf from L          How?
    if μ.Q is too large then
        partition μ.Q into 4 quadrants and add 4 leaves to T
        insert new leaves into L
        if any neighbors of μ.Q are too large then
            insert them into L
return T                               How?
```

Question: How large can a balanced quadtree get?

Balancing quadtrees

Theorem 3: Let $\mathcal{T}$ be a quadtree with m nodes. Then the balanced version of $\mathcal{T}$ has $O(m)$ nodes and can be constructed in $O((d+1)m)$ time.

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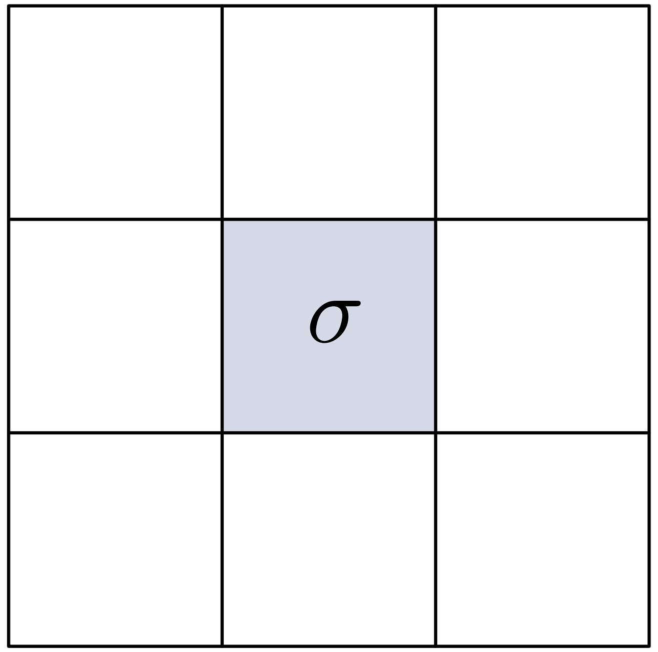

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- assume square σ is split $\Rightarrow$ σ has “old” nbr (among 8 nbrs)
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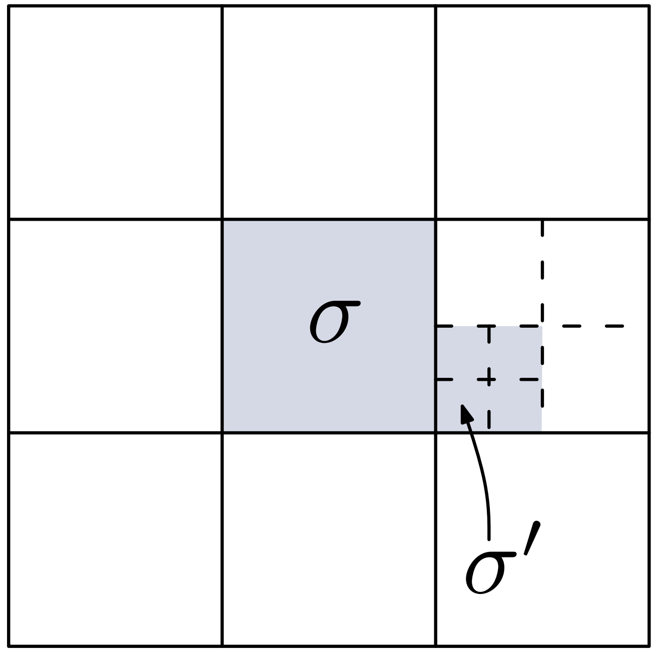

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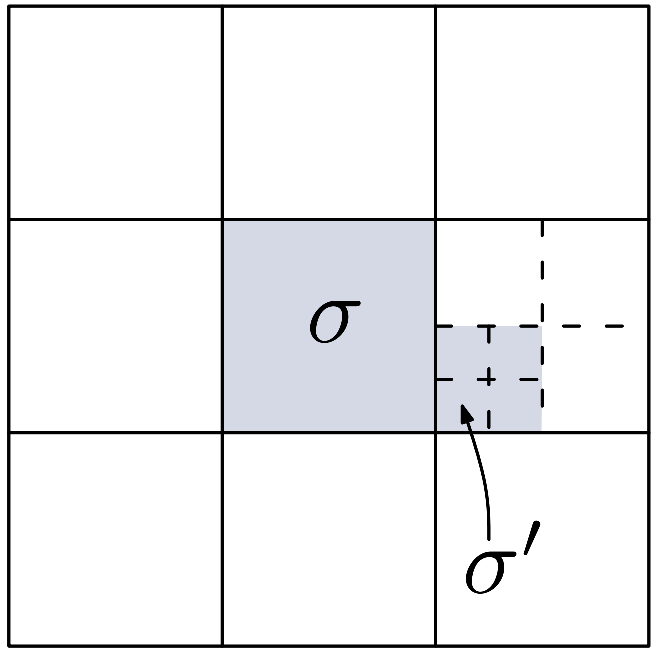

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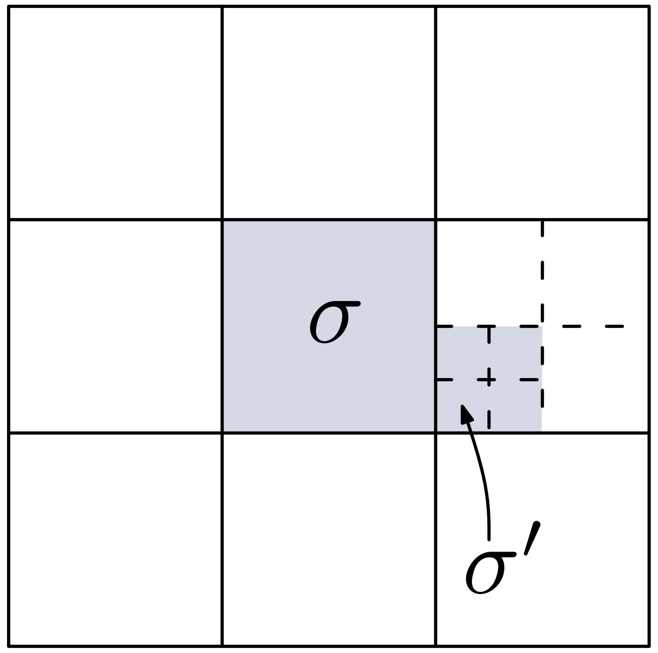

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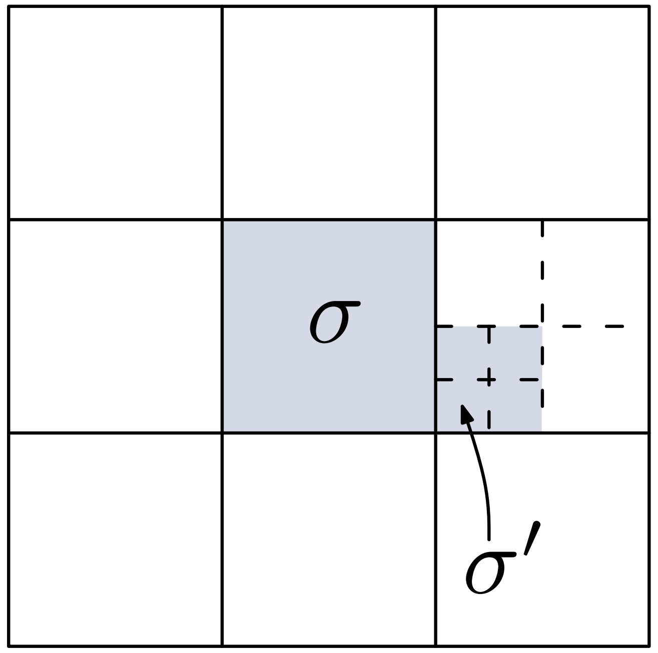

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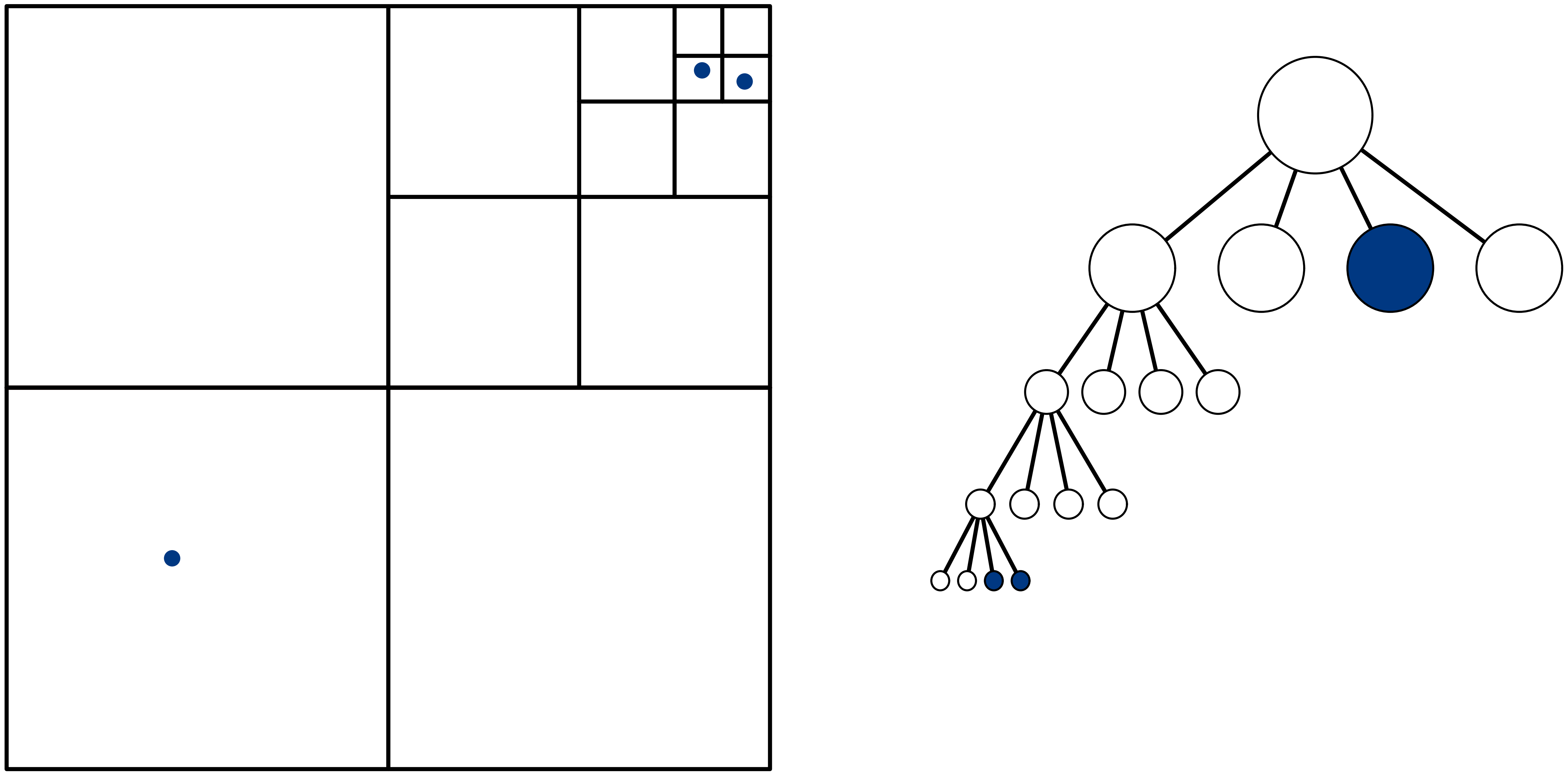
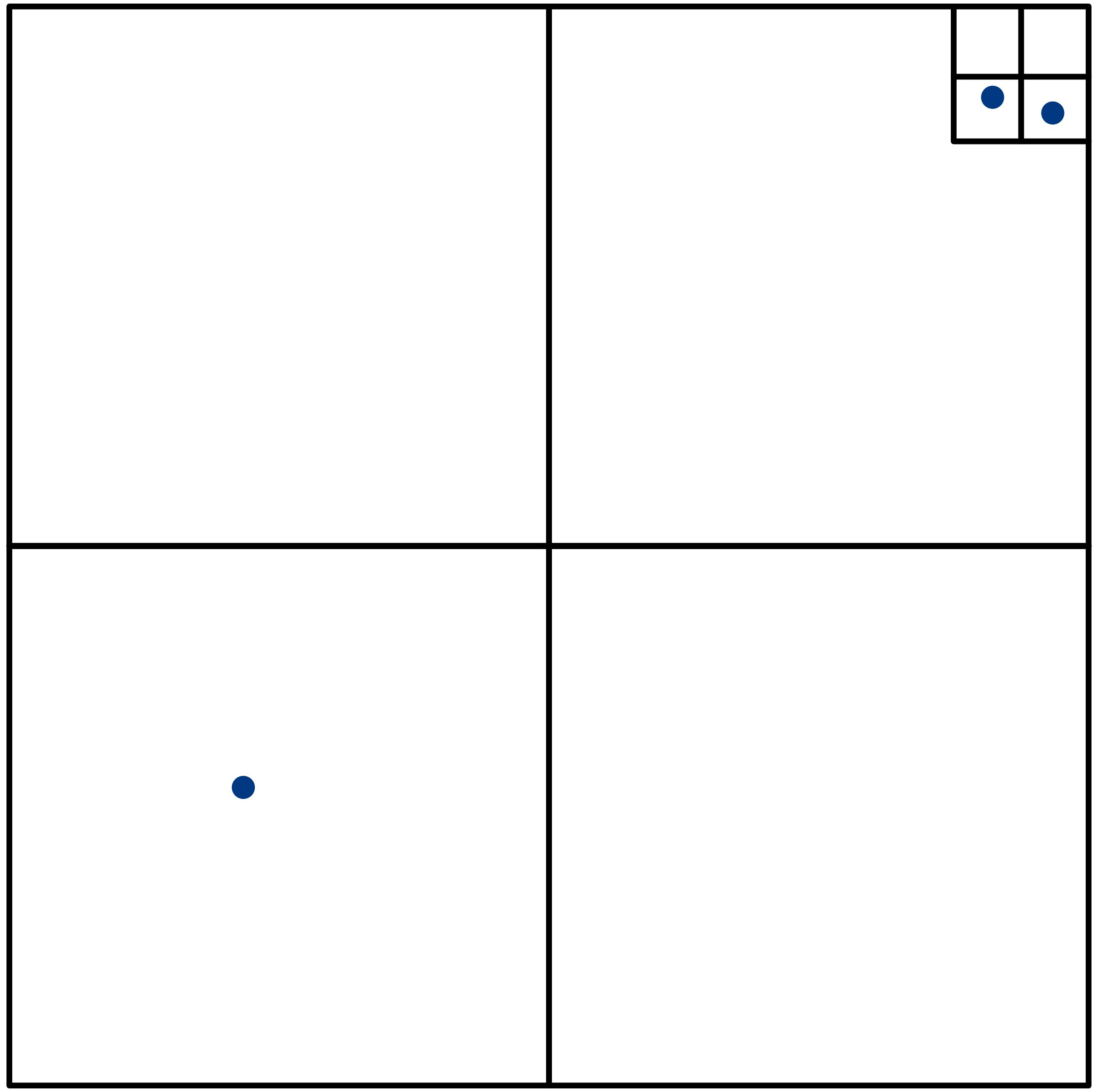
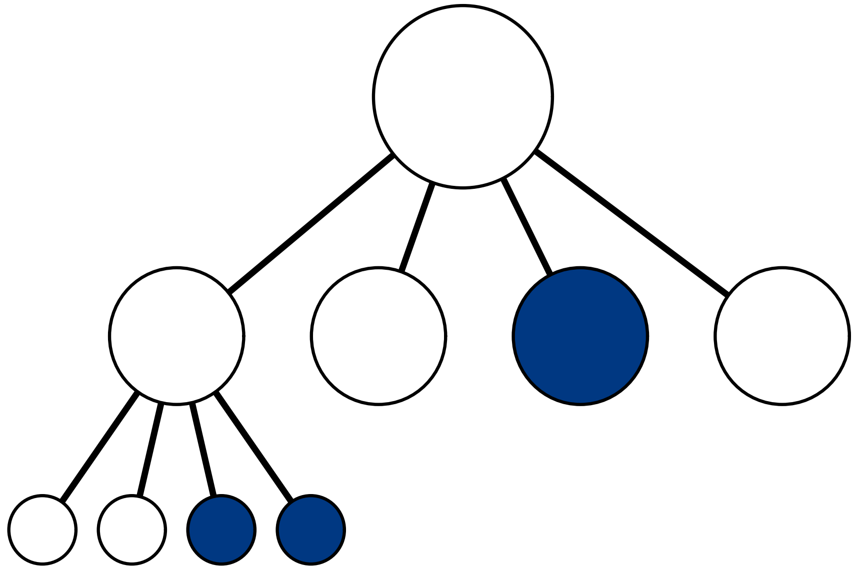
Compressed quadtrees



Paths of nodes with only one non-empty child can be compressed to an edge

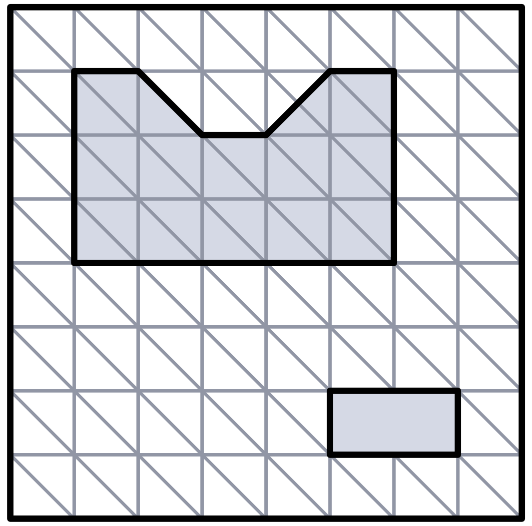
Compressed quadtrees





Paths of nodes with only one non-empty child can be compressed to an edge $\quad \rightarrow$ size $O(n)$

Quadtrees and non-uniform meshes

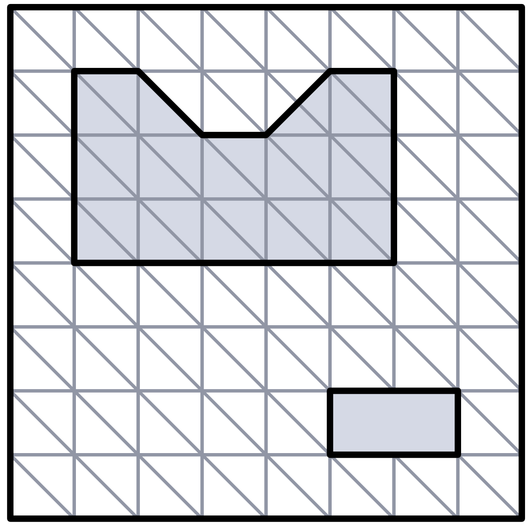


Given: octilinear polygons with integer coordinates within a square $Q = [0, U] \times [0, U]$ with $U = 2^j$ a power of two

Goal: triangular mesh of Q with the following properties

- conforming: exactly one triangle on each side of interior edges
- respect input: edges of input must be part of union of mesh edges
- well-shaped: angles between $45°$ and $90°$
- non-uniform: fine near boundaries, coarse otherwise

Quadtrees and non-uniform meshes



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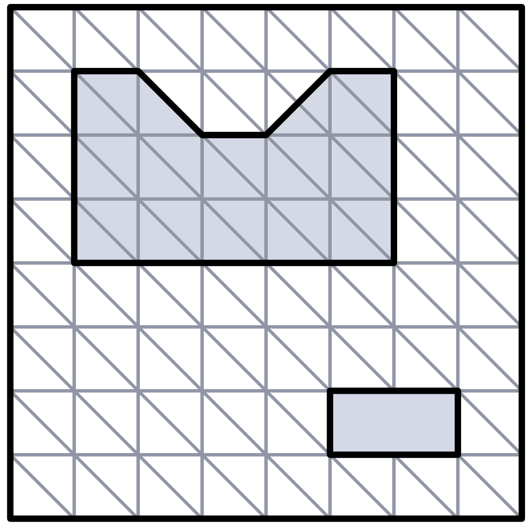

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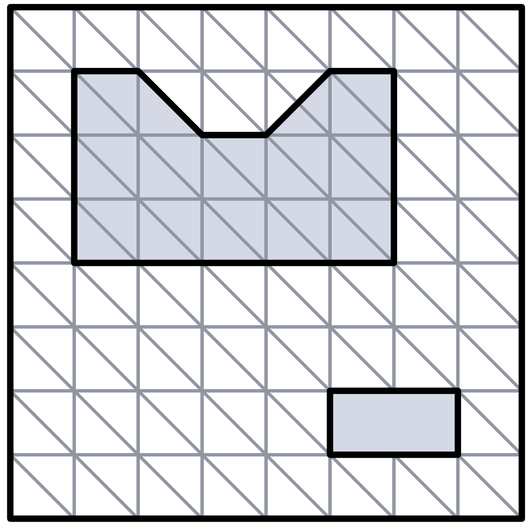

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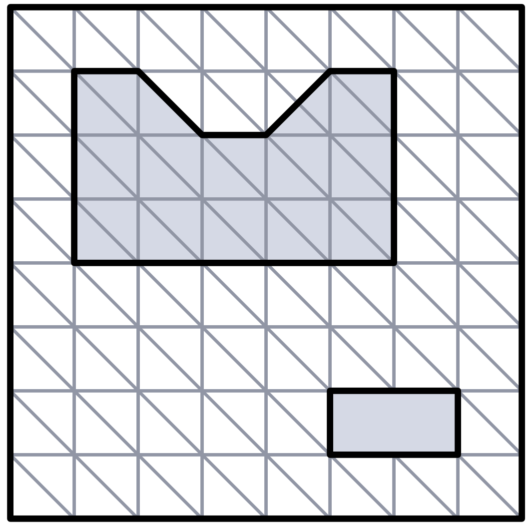

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- w … and 90°
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Idea …

Ques … t?

intersections also include:

- polygon boundary in square
- common edge, or even just common point

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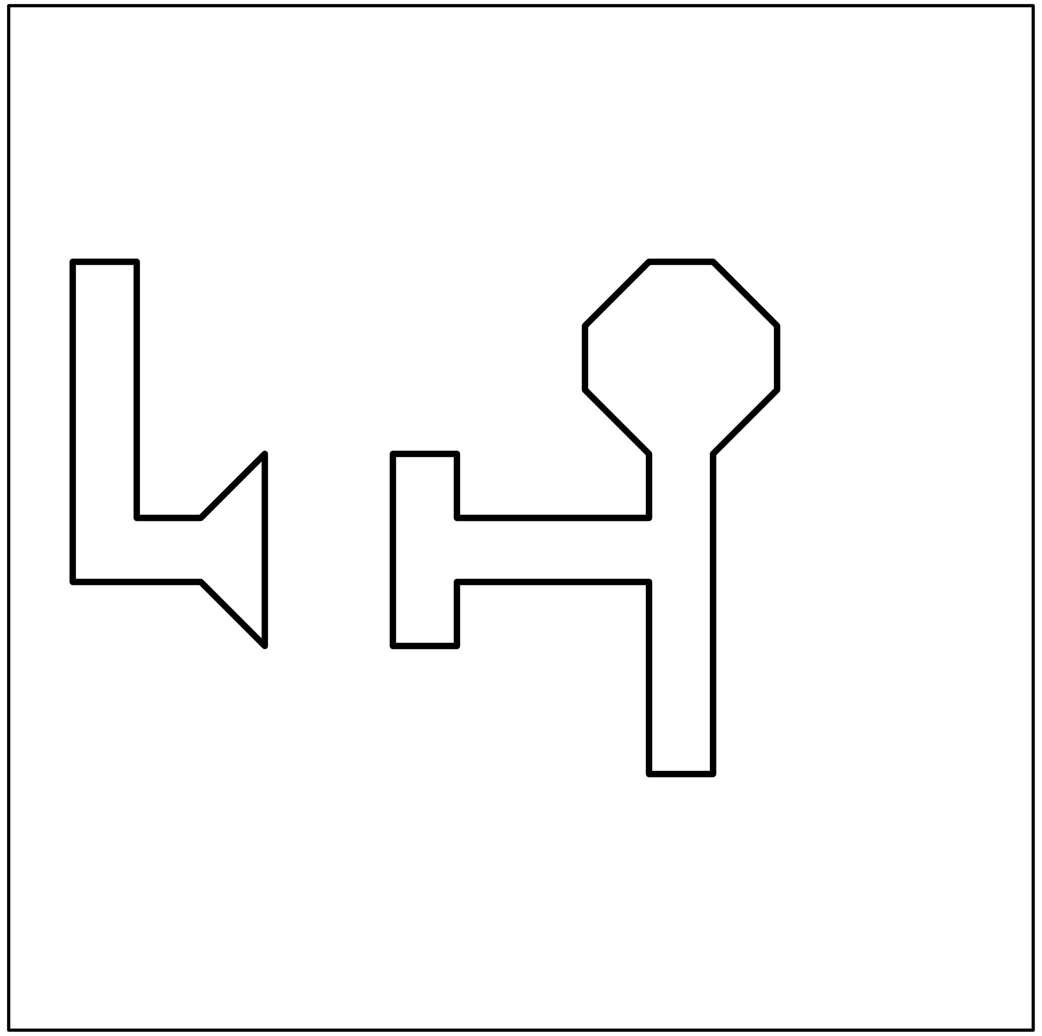
From quadtrees to meshes



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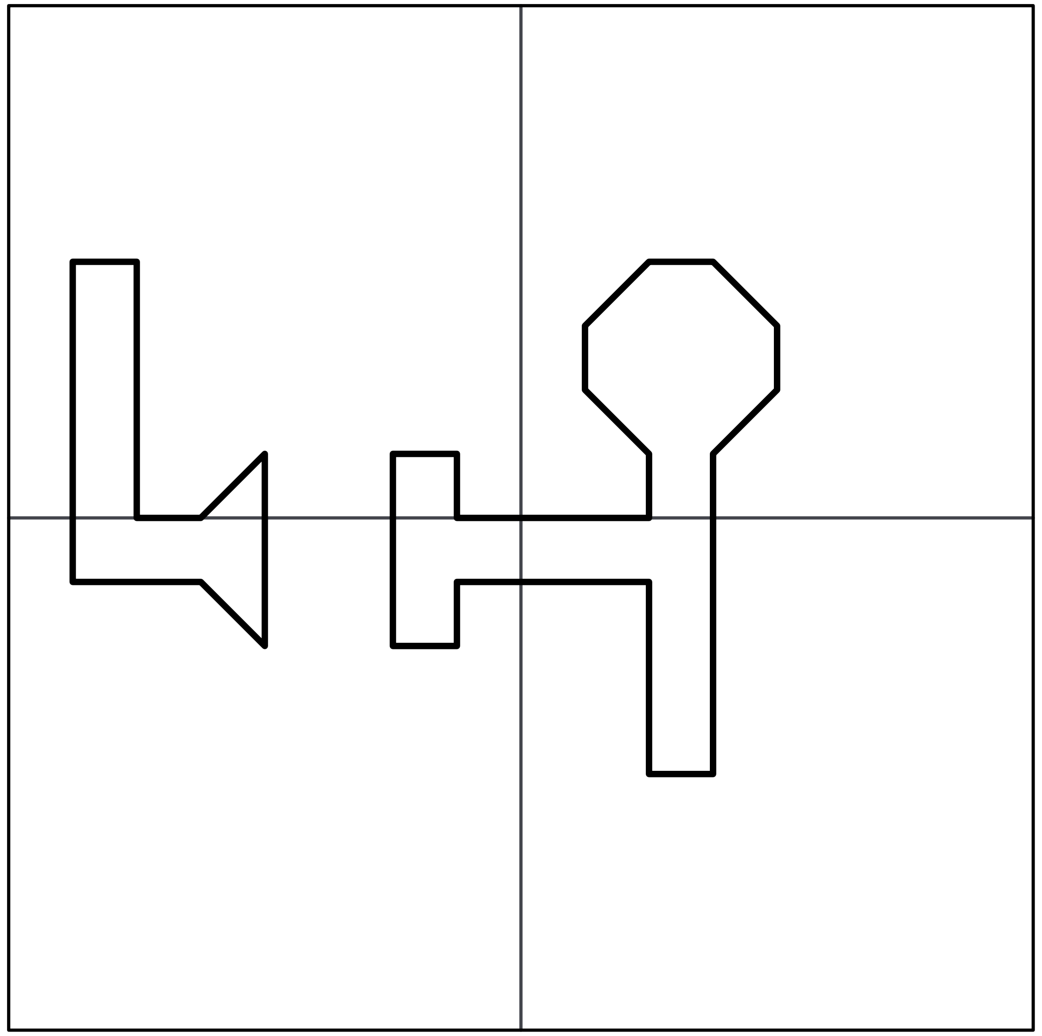

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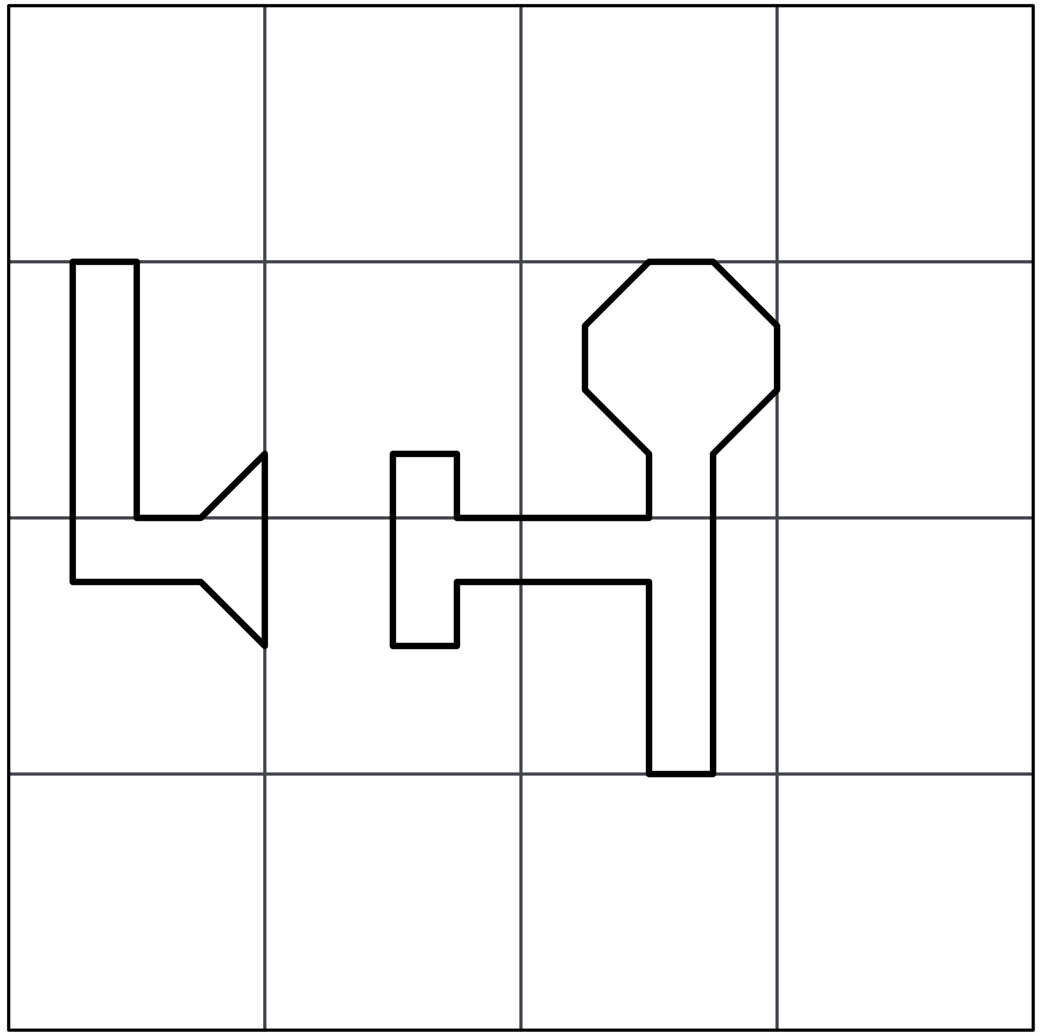

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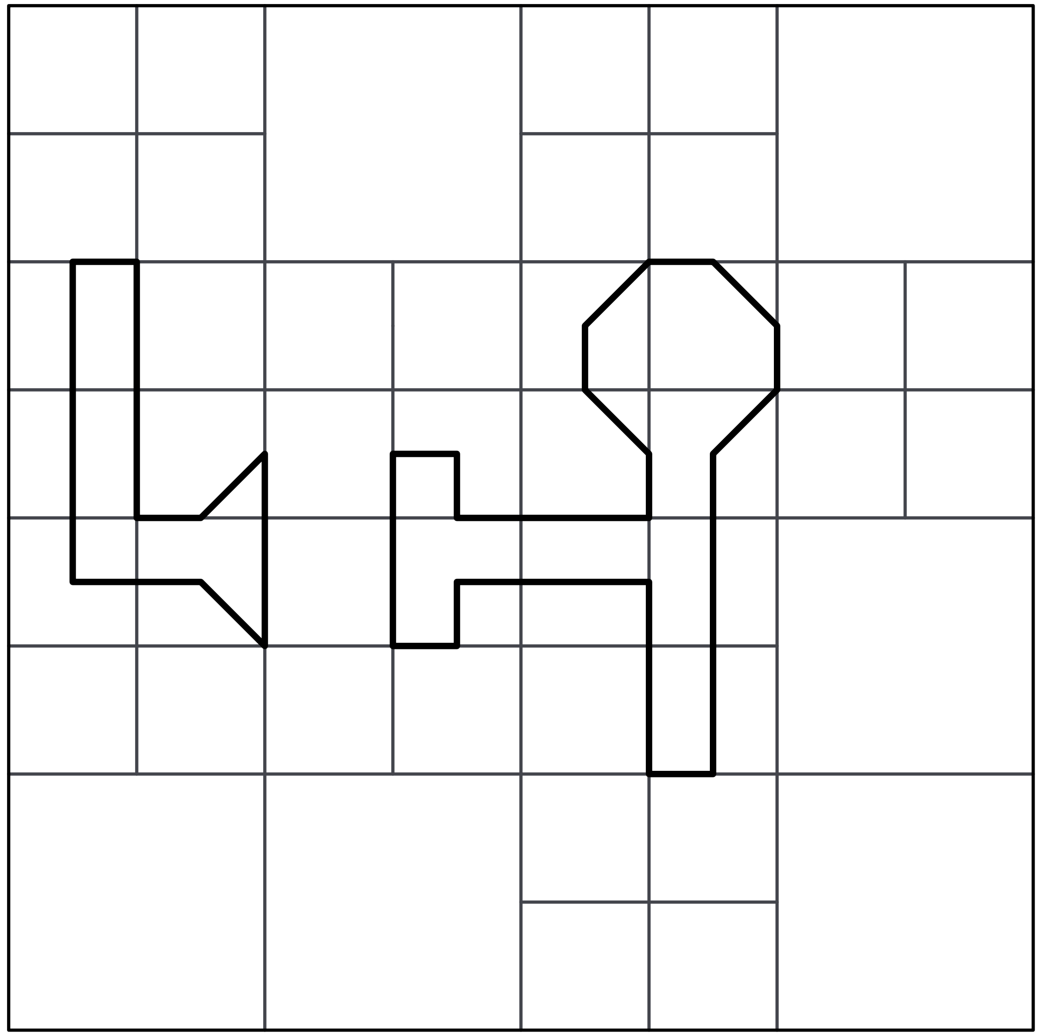

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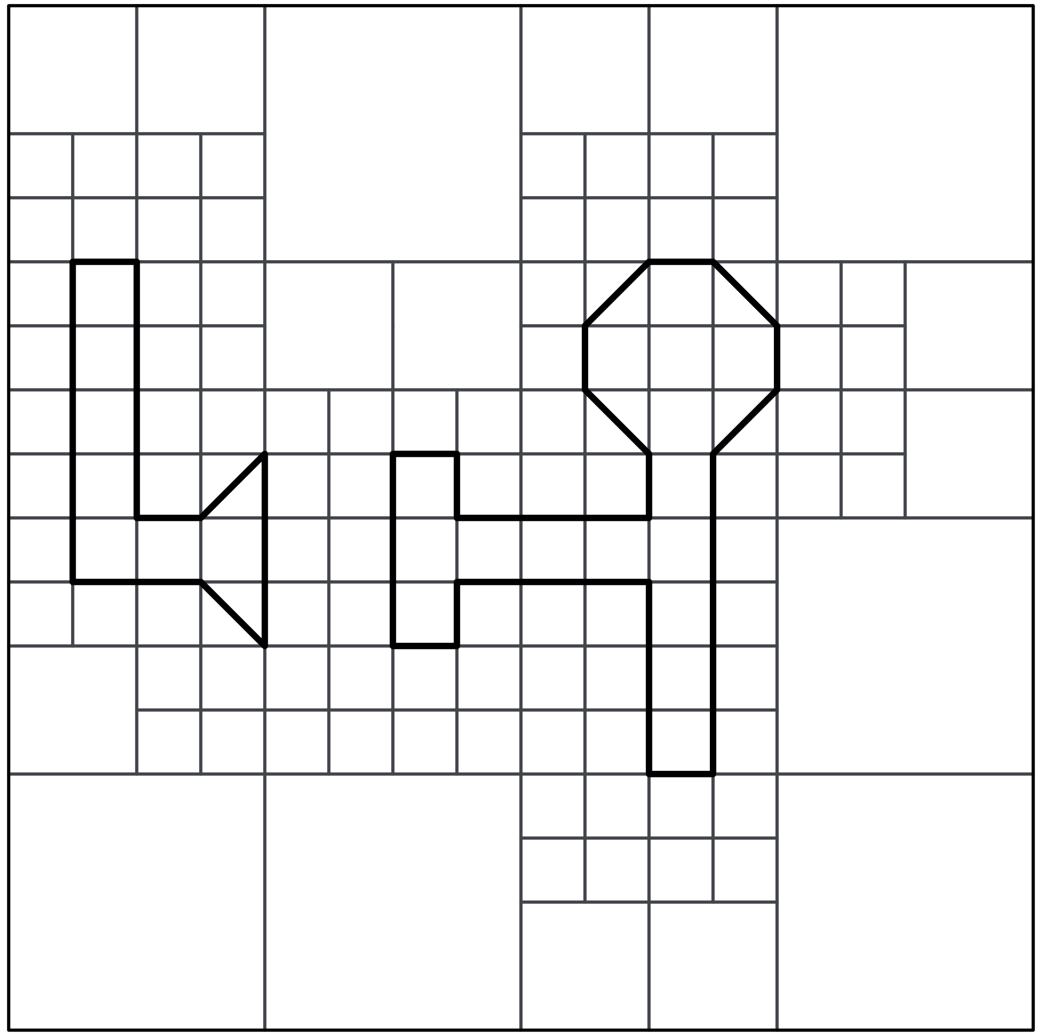

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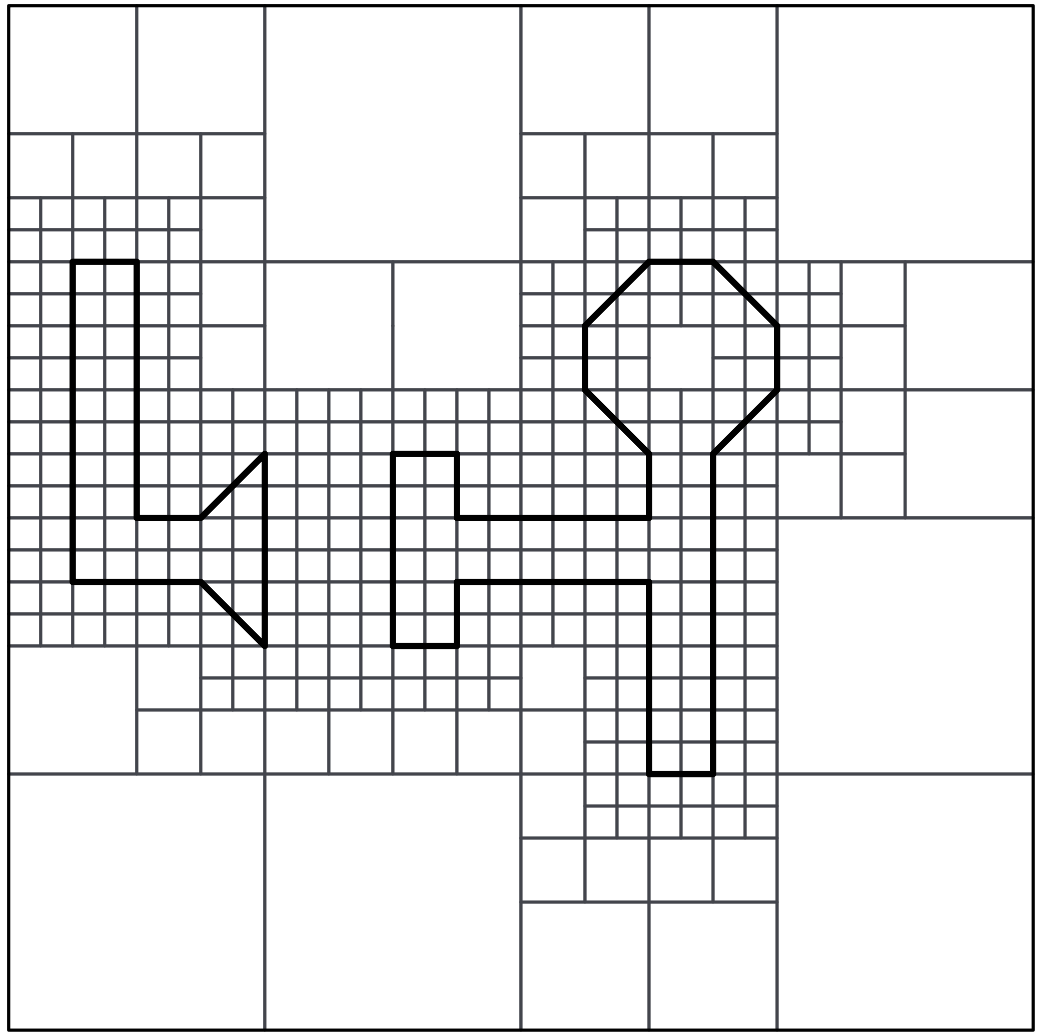

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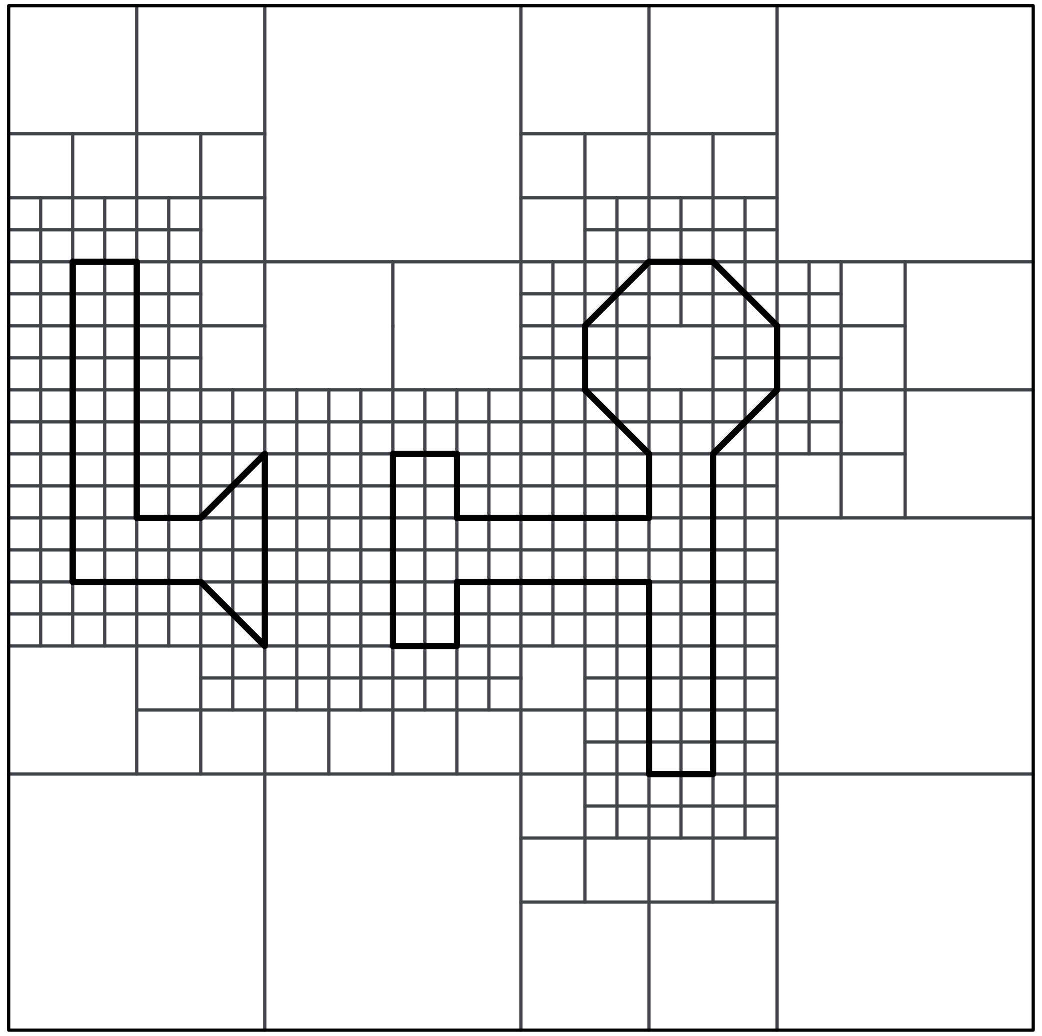

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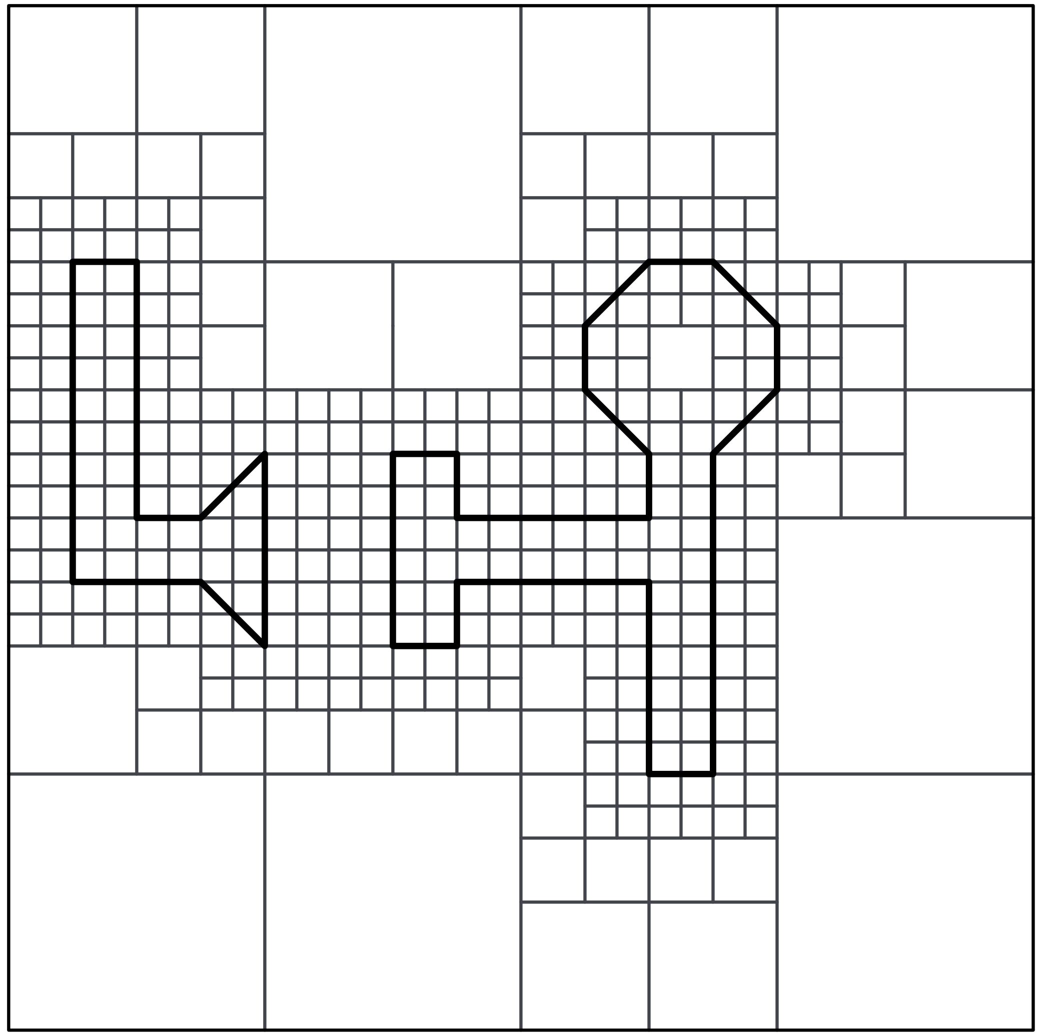

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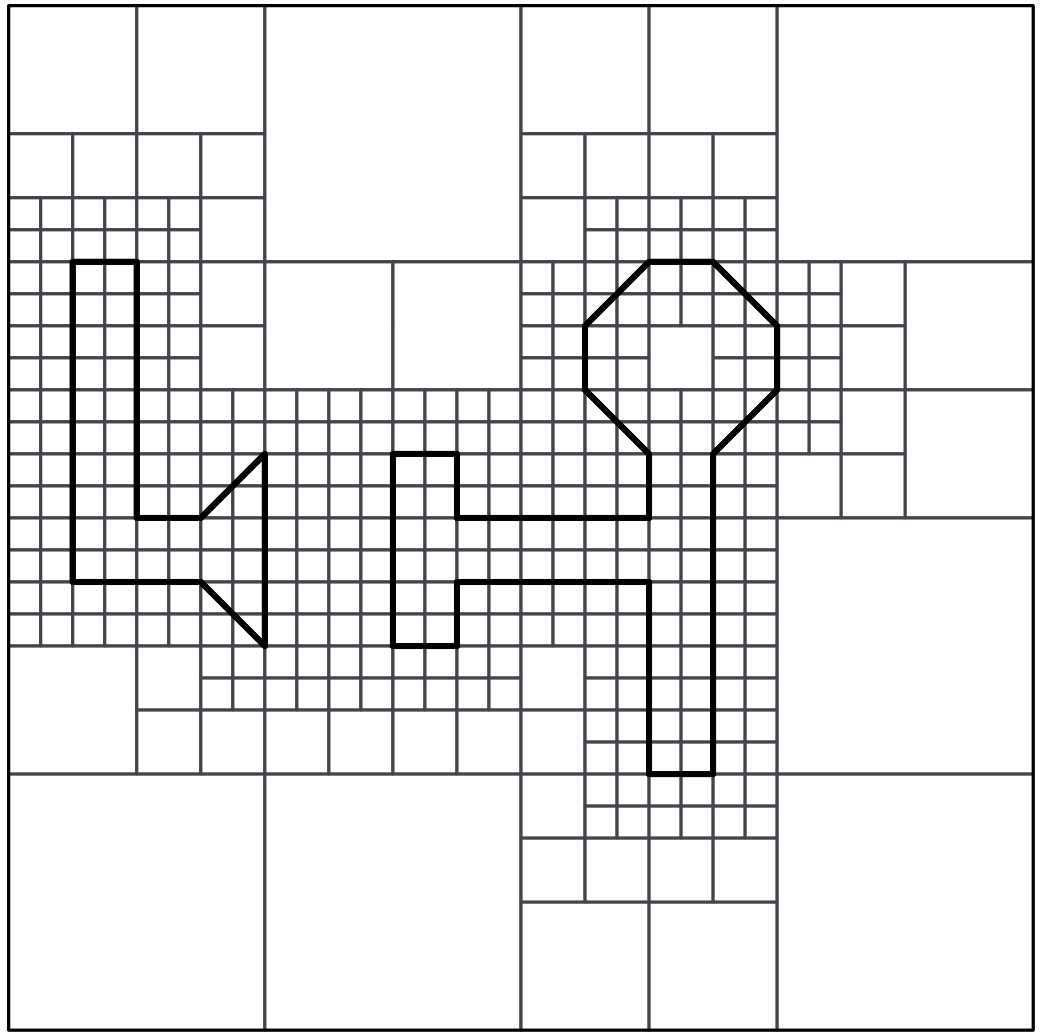

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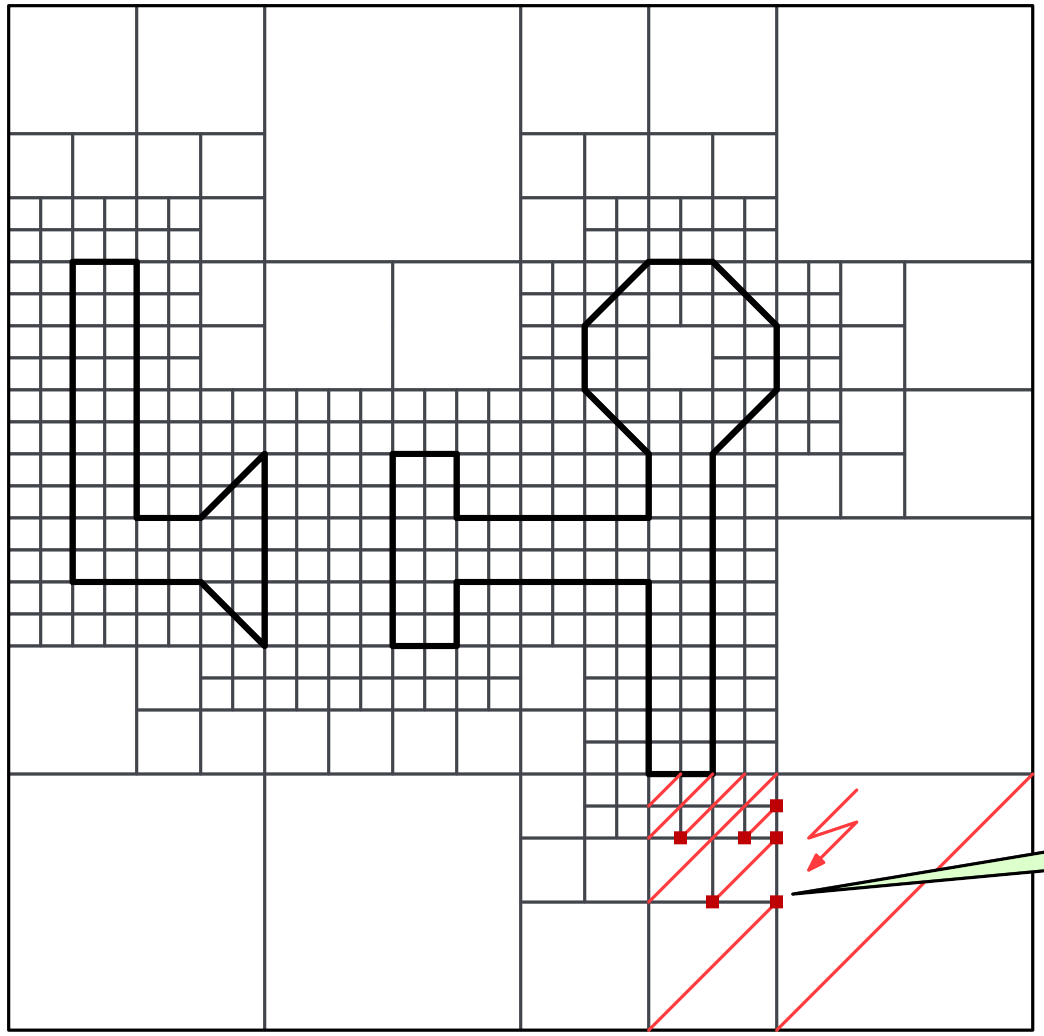
- Add diagonals for remaining squares?

From quadtrees to meshes



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non-conforming

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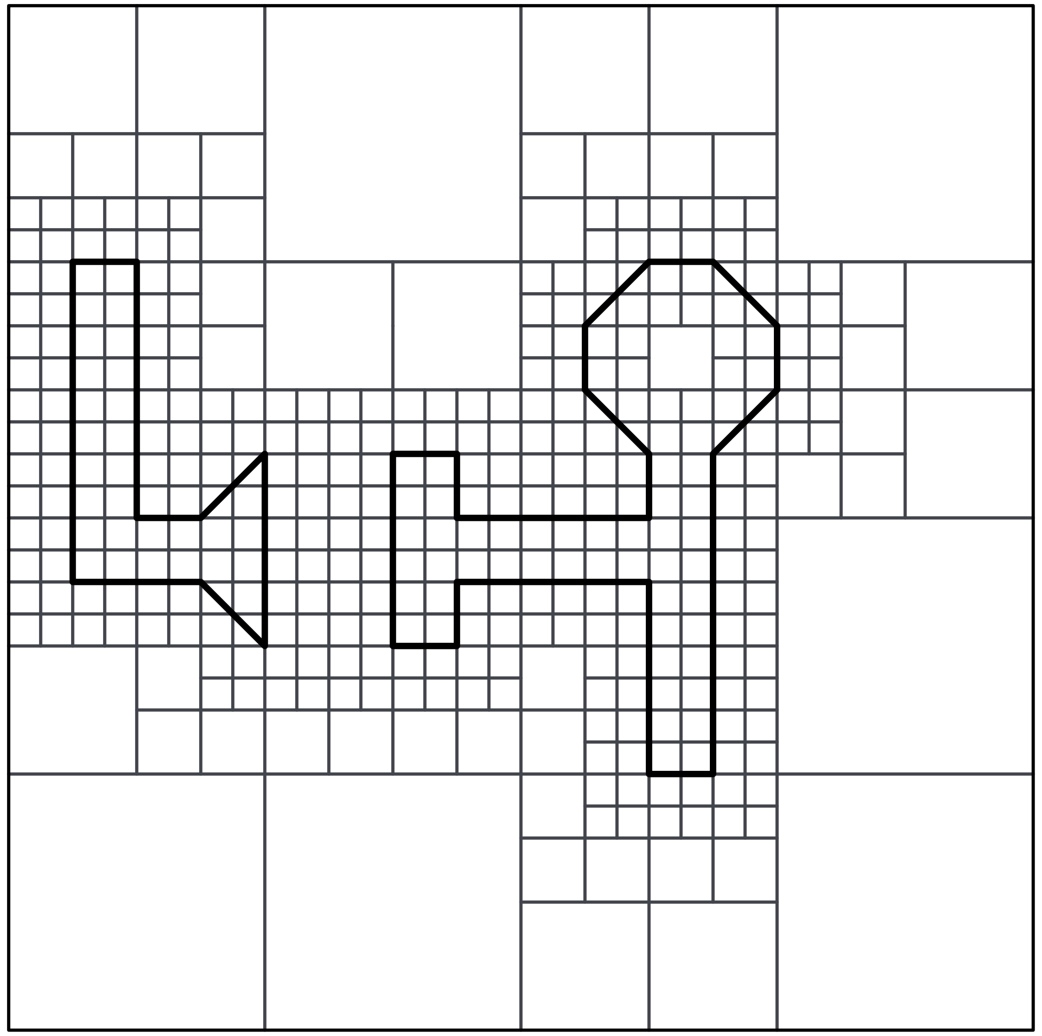

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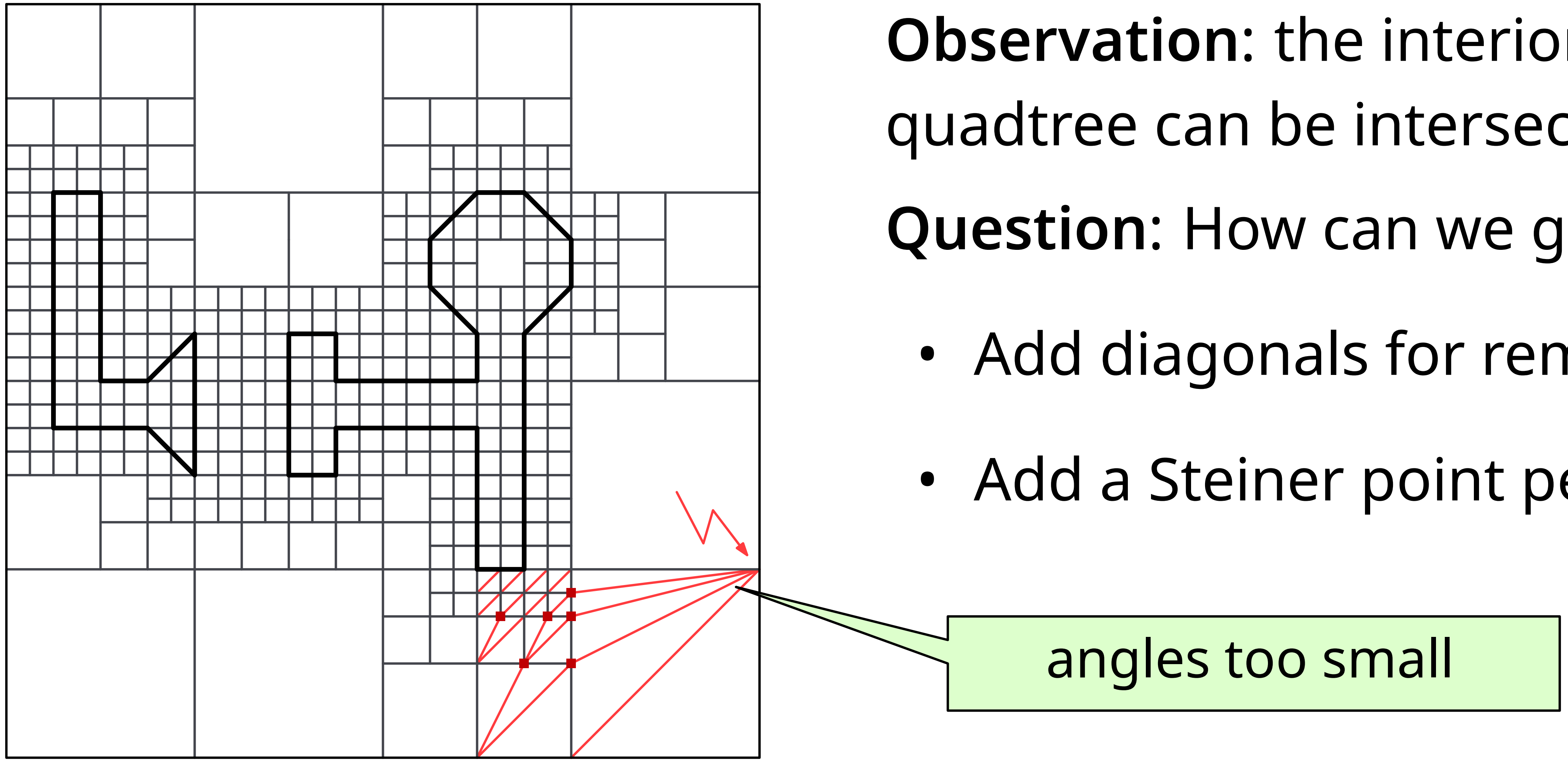
From quadtrees to meshes



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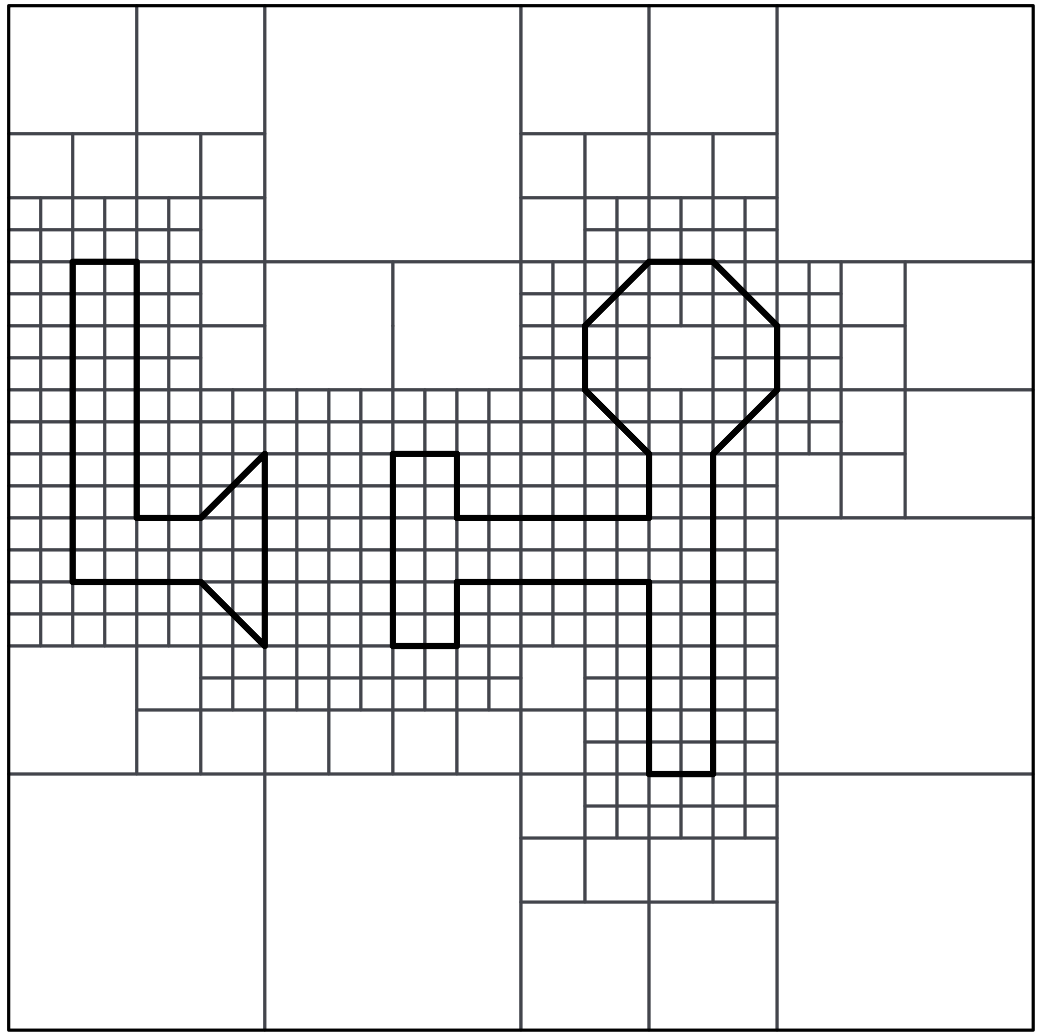

Observation: the interior of a square in the quadtree can be intersected only by a diagonal

Question: How can we get a valid mesh?

- Add diagonals for remaining squares? no!
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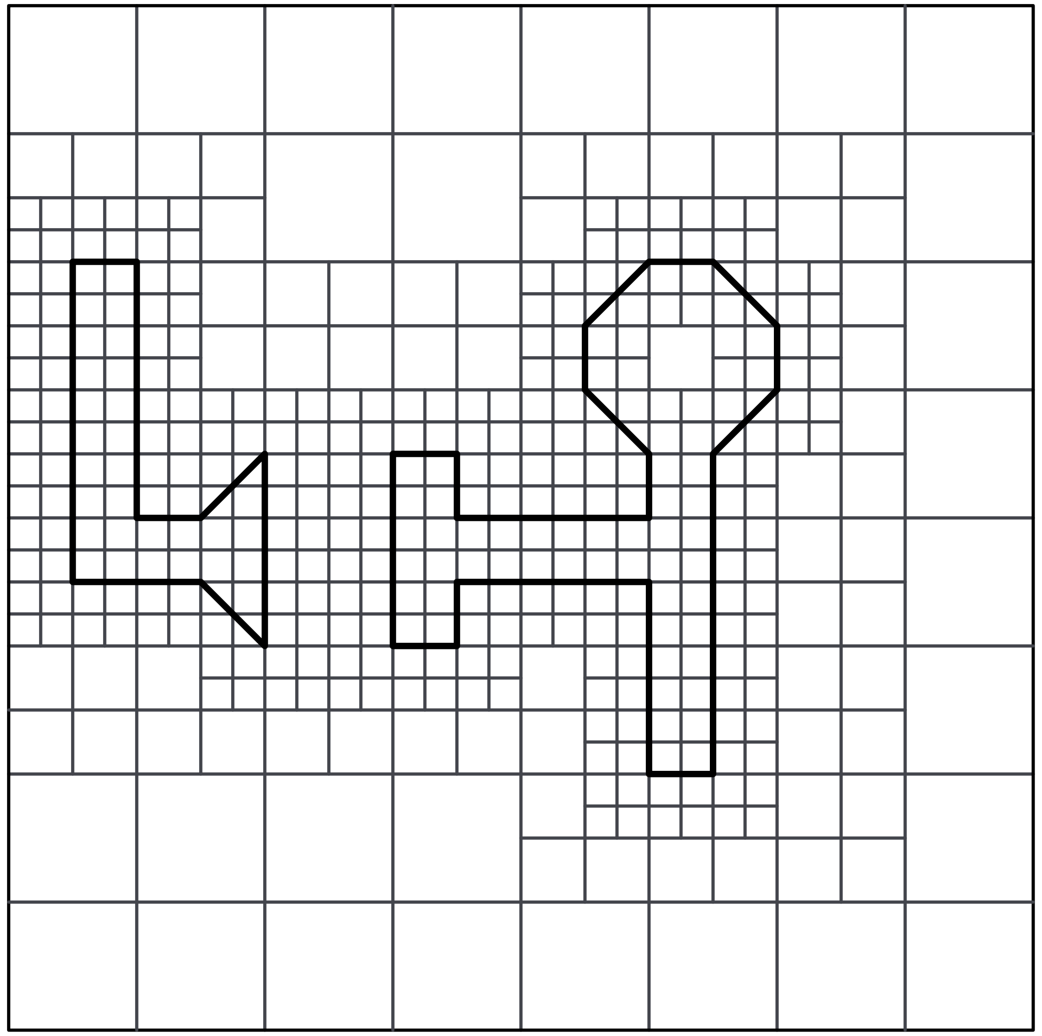
- Balanced quadtree and add Steiner points if necessary!

From quadtrees to meshes



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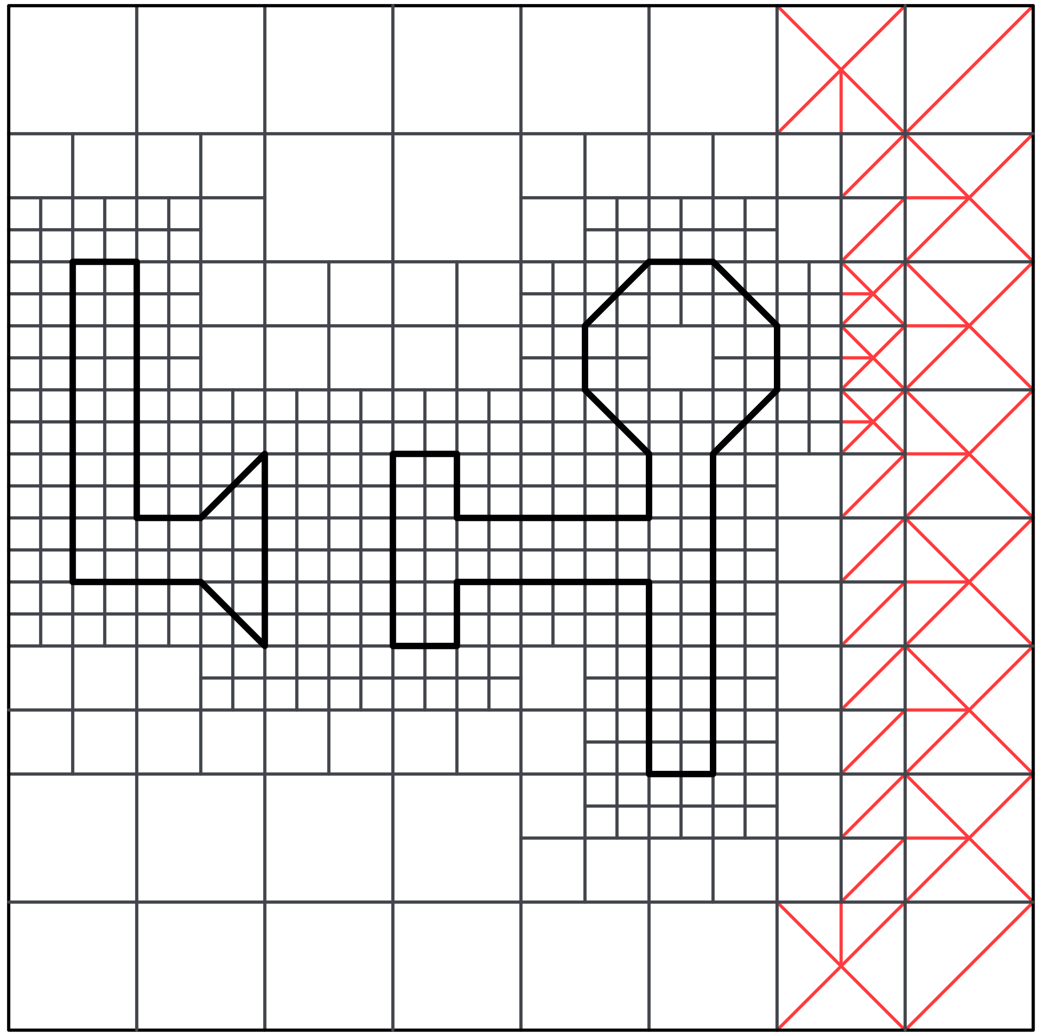

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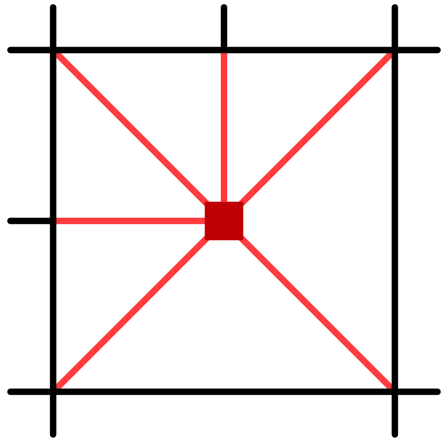


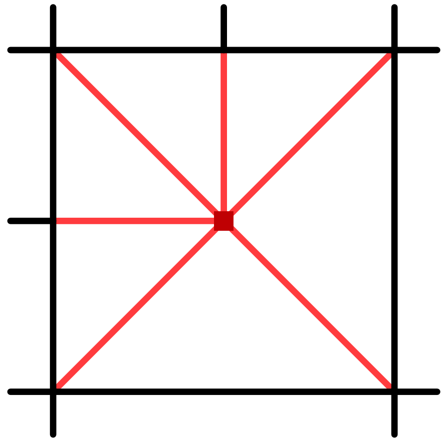
Triangulating quadtrees

TRIANGULATEQUADTREE($\mathcal{T}$)

Input: quadtree $\mathcal{T}$

Output: triangulation of $\mathcal{T}$

$\mathcal{D} \leftarrow$ DCEL for partition of Q by $\mathcal{T}$
for each facet f in $\mathcal{D}$ **do**
    if int(f) is intersected by a polygon **then**
        add corresponding diagonal in f to $\mathcal{D}$
    else
        if vertices only add corners of f **then**
            add a diagonal in f to $\mathcal{D}$
        else
            create Steiner point in the middle of f and
            connect in $\mathcal{D}$ to all vertices on ∂f
return $\mathcal{D}$



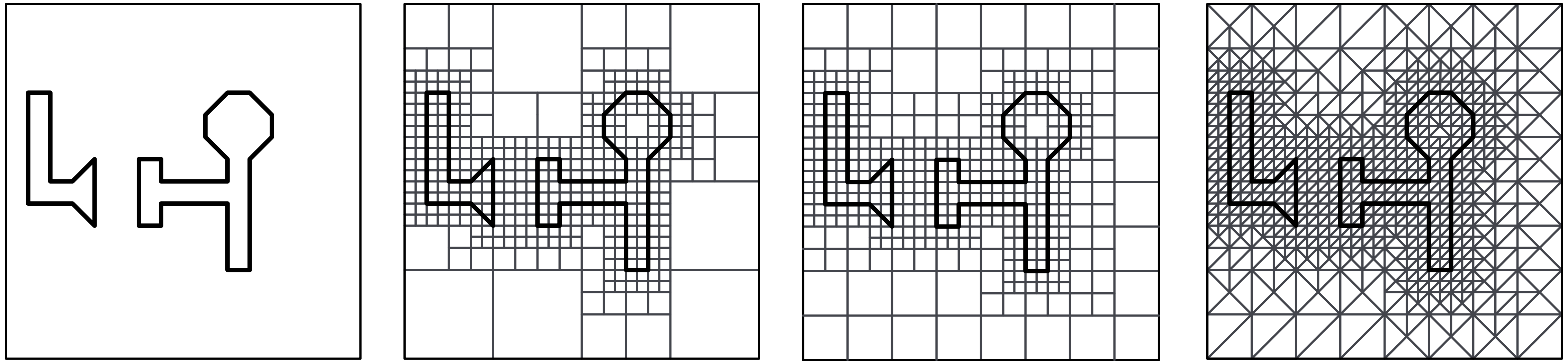
Algorithm

CREATEMESH(S)

Input: set S of octilinear polygons with integer coordinates in $Q = [0, 2^j] \times [0, 2^j]$

Output: valid, non-uniform triangular mesh S

1: $\mathcal{T} \leftarrow$ CREATEQUADTREE
2: $\mathcal{T} \leftarrow$ BALANCEQUADTREE$(\mathcal{T})$
3: $\mathcal{D} \leftarrow$ TRIANGULATEQUADTREE$(\mathcal{T})$
4: **return** $\mathcal{D}$



Exercise

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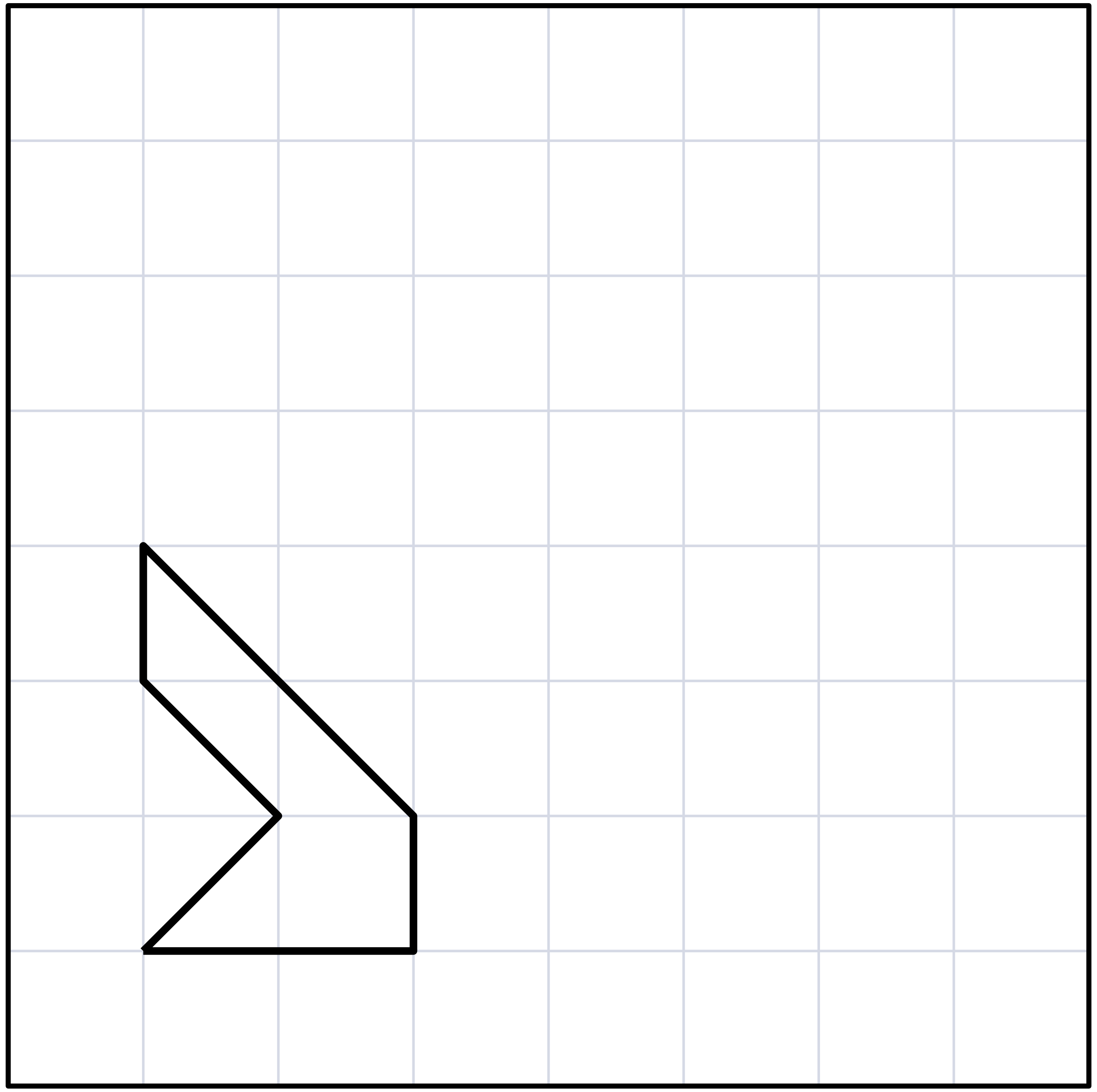

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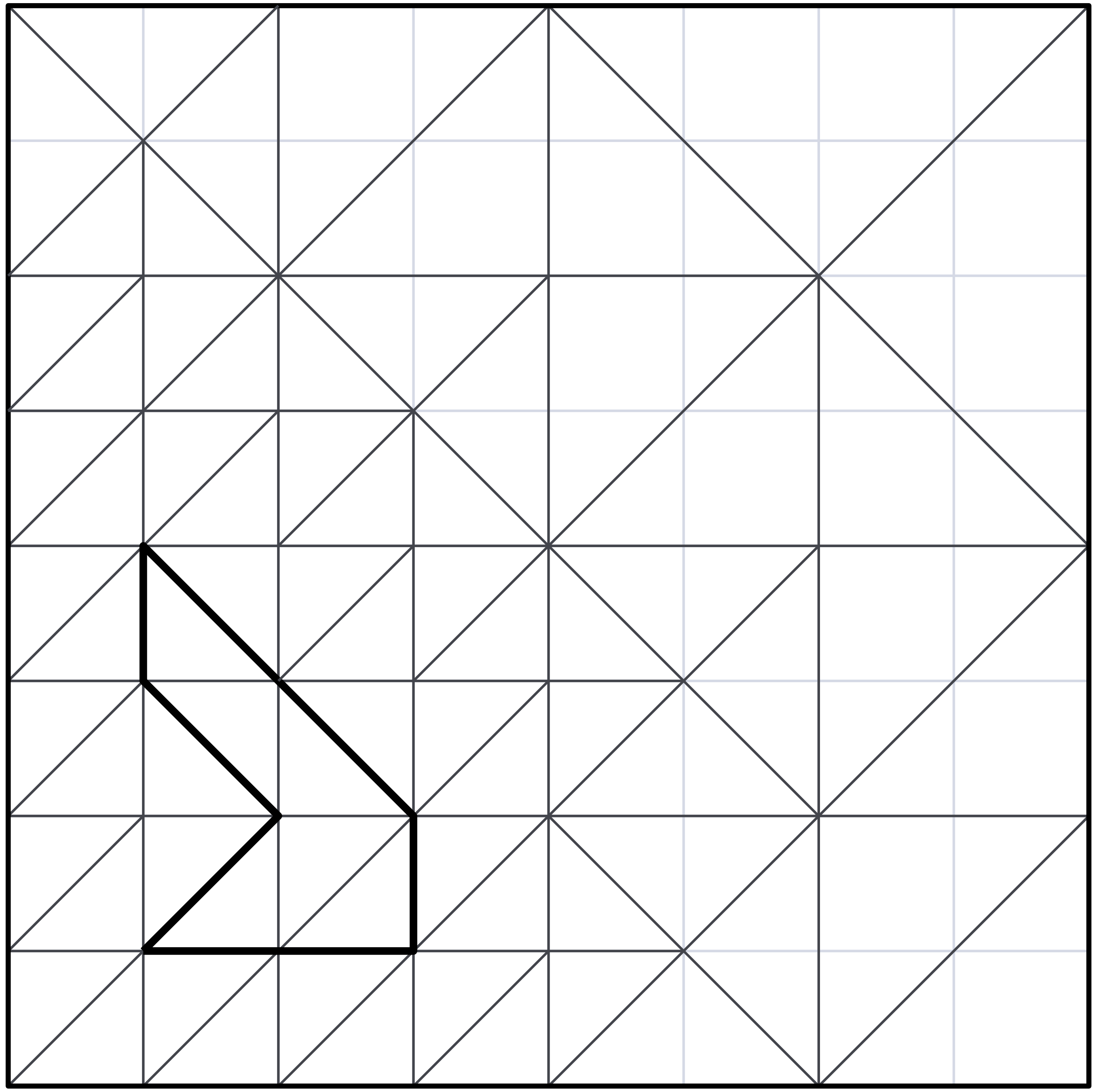

Summary

Theorem 4: Let S be a set of disjoint polygonal objects with vertices on a (integer) grid $[0, U] \times [0, U]$. Then

- there exists a non-uniform triangular mesh for S that is conforming, well-shaped and respects the input
- the number of triangles is $O(p(S) \log U)$, where $p(S)$ is the sum of (lengths of) perimeters of the objects
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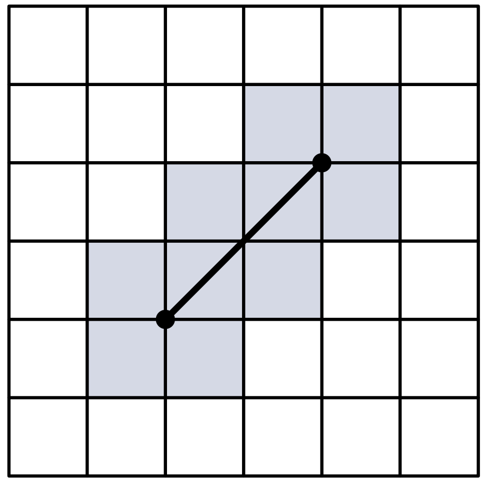
- the mesh can be constructed in $O(p(S) \log^2 U)$ time

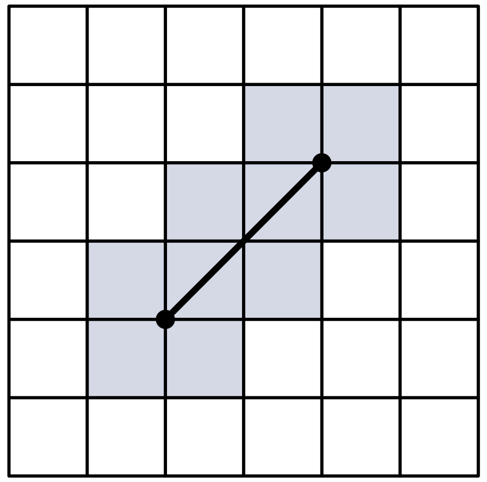
Proof: size of mesh: # squares intersected in a layer is $O(p(S))$, depth is $O(\log U)$



Summary

Theorem 4: Let S be a set of disjoint polygonal objects with vertices on a (integer) grid $[0, U] \times [0, U]$. Then

- there exists a non-uniform triangular mesh for S that is conforming, well-shaped and respects the input
- the number of triangles is $O(p(S) \log U)$, where $p(S)$ is the sum of (lengths of) perimeters of the objects
- the mesh can be constructed in $O(p(S) \log^2 U)$ time



Proof: size of mesh: # squares intersected in a layer is $O(p(S))$, depth is $O(\log U)$

construction time:

1. quadtree: linear in size
2. balancing: extra log-factor (by Thm 3)
3. triangulating: linear in size

Discussion

Can we compute/update compressed quadtrees efficiently?

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Yes, skip quadtrees have complexity $O(n)$ and we can insert, delete and search in $O(\log n)$ time in a suitable model of computation

[Eppstein et al., '05]

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Other applications?

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Other applications?

Quadtrees are used in many applications including computer graphics, image processing, GIS etc. In geometry used for approximation algorithms, but also connections to Delaunay triangulations.

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Directly generalize. In 3D quadtrees $\rightarrow$ octtrees