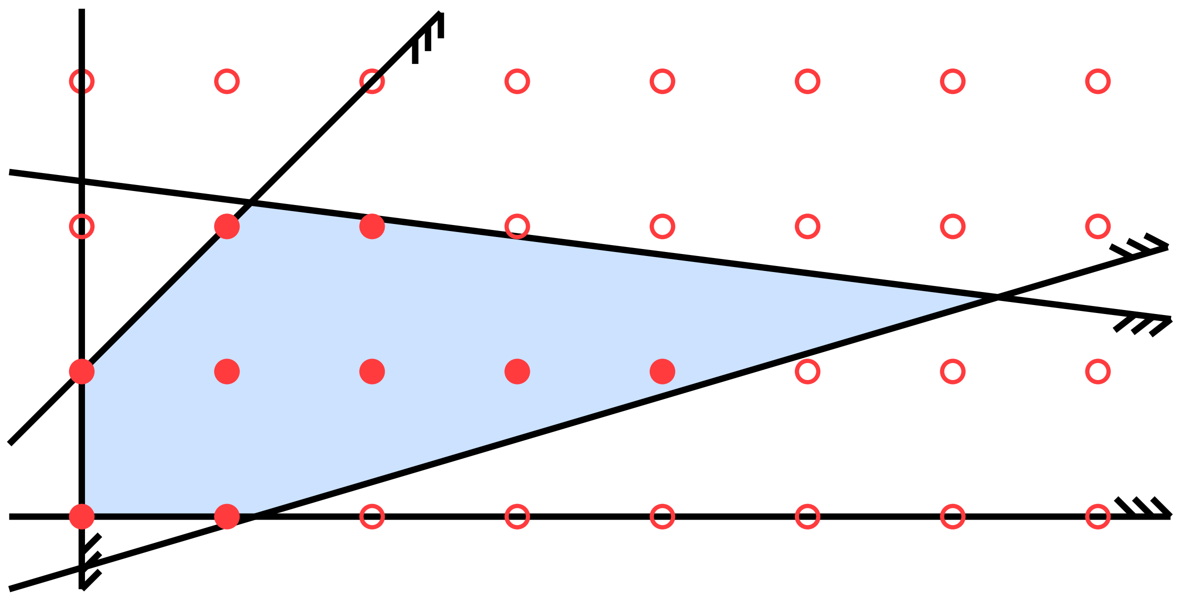
Linear Programming

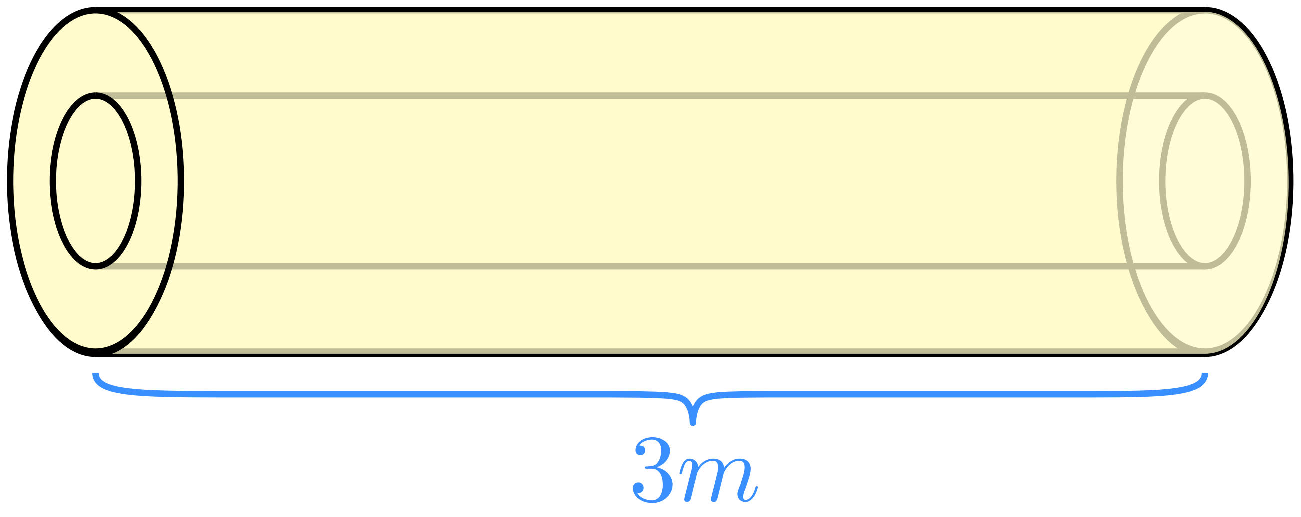
Integer Linear Programming



Recall Example: Cutting Paper Rolls

What’s the fewest number of rolls need to satisfy an order of:

- 97 rolls width 135cm
- 610 rolls width 108cm
- 395 rolls width 93cm
- 211 rolls width 42cm



Possible ways to cut roll with <42cm wasted:

P1: $2 \cdot 135$

P2: $135 + 108 + 42$

P3: $135 + 93 + 42$

P4: $135 + 3 \cdot 42$

P5: $2 \cdot 108 + 2 \cdot 42$

P6: $108 + 2 \cdot 93$

P7: $108 + 93 + 2 \cdot 42$

P8: $108 + 4 \cdot 42$

P9: $3 \cdot 93$

P10: $2 \cdot 93 + 2 \cdot 42$

P11: $93 + 4 \cdot 42$

P12: $7 \cdot 42$

Example: Cutting Paper Rolls

For each possibility P_j, add a variable $x_j \geq 0$ representing # rolls cut that way.

minimize $\displaystyle\sum_{j=1}^{12} x_j$ (total # of rolls cut)

subject to
$$2x_1 + x_2 + x_3 + x_4 \geq 97$$
$$x_2 + 2x_5 + x_6 + x_7 + x_8 \geq 610$$
$$x_3 + 2x_6 + x_7 + 3x_9 + 2x_{10} + x_{11} \geq 395$$
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Here, $48, 206, 197, 1$ (for x_9) is an optimal integral solution.

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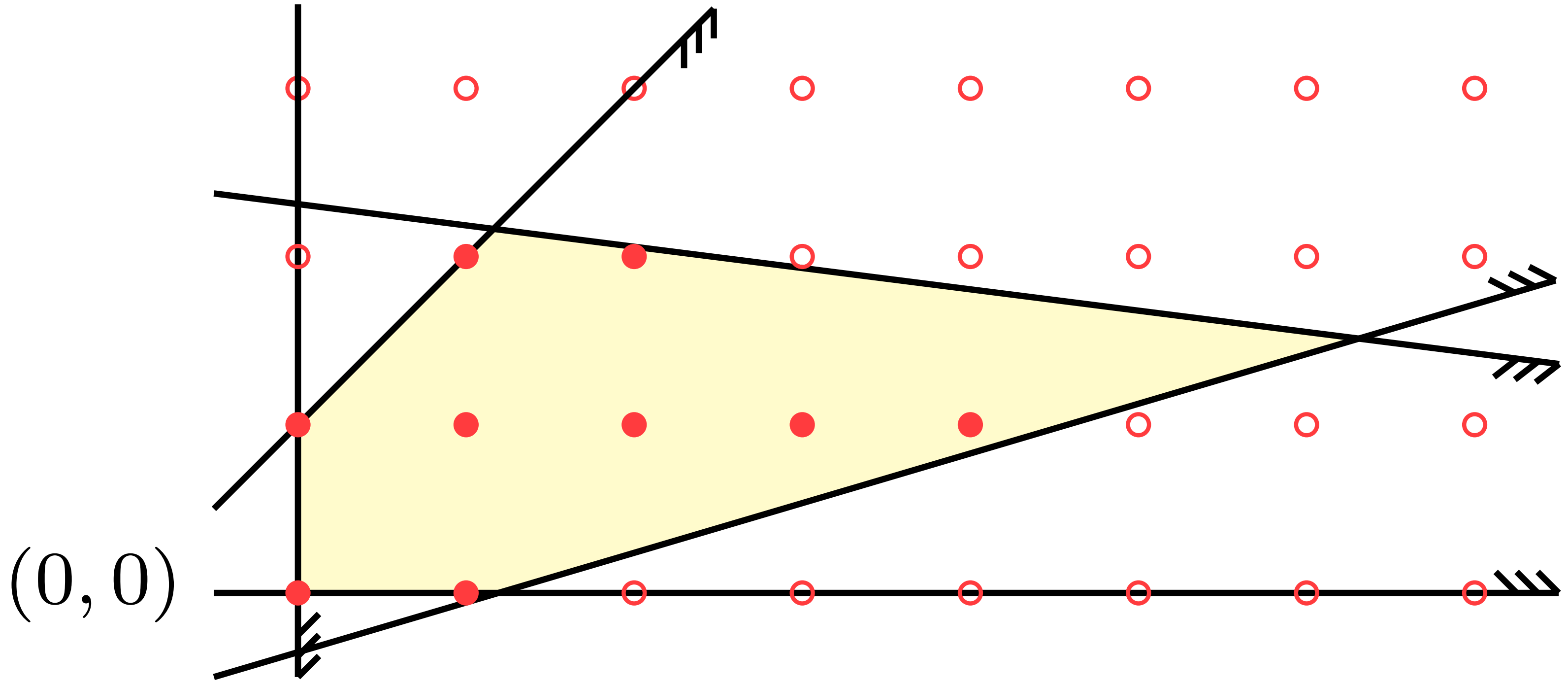

maximize $c^T x$

subject to $Ax \leq b$

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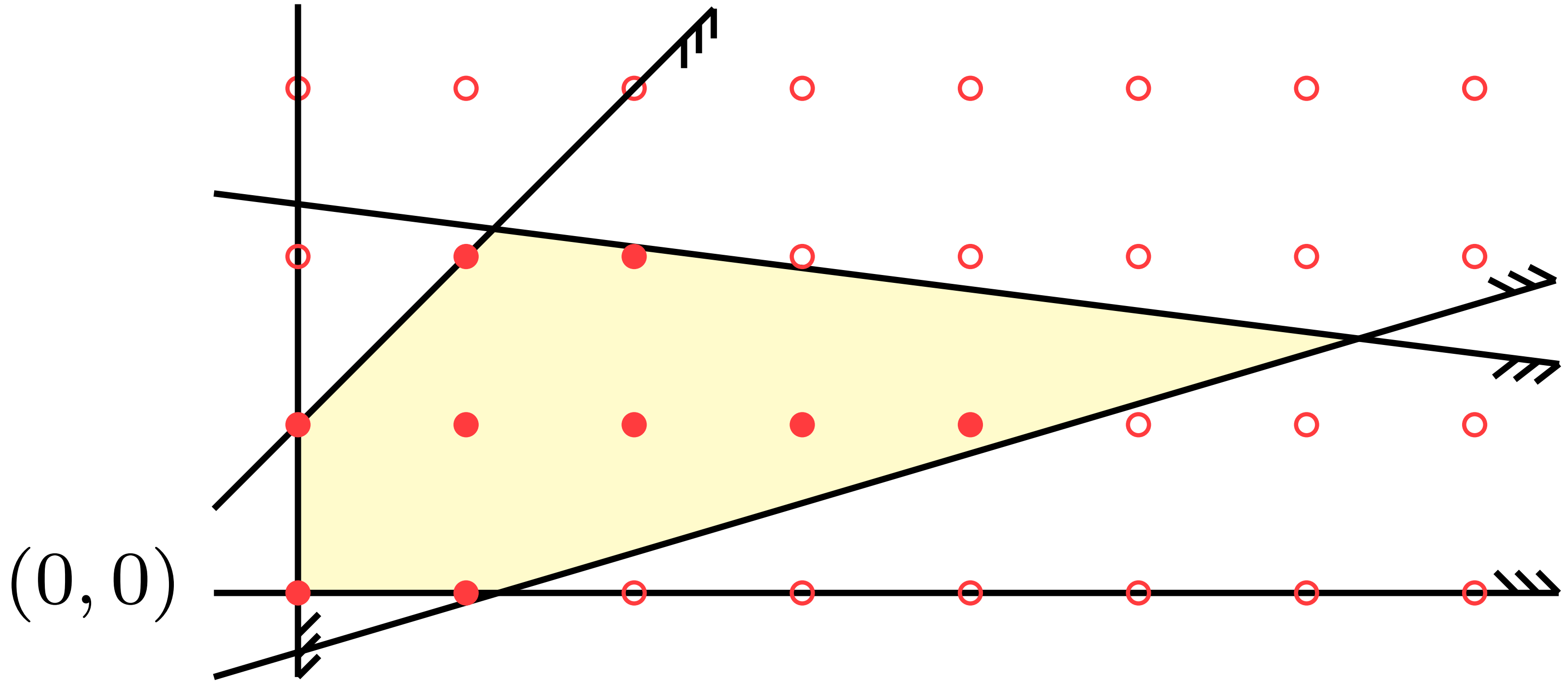

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Integer Linear Programming (ILP) is NP-hard.

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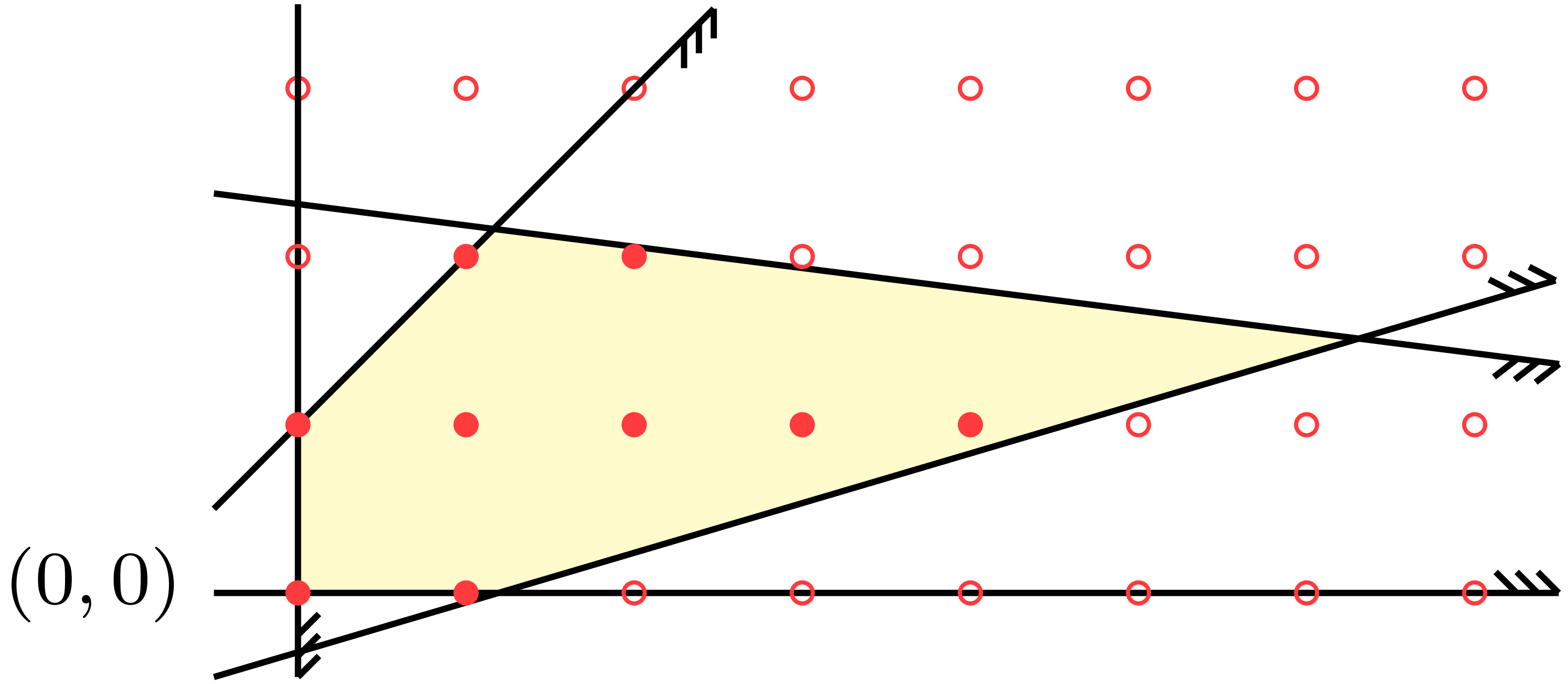

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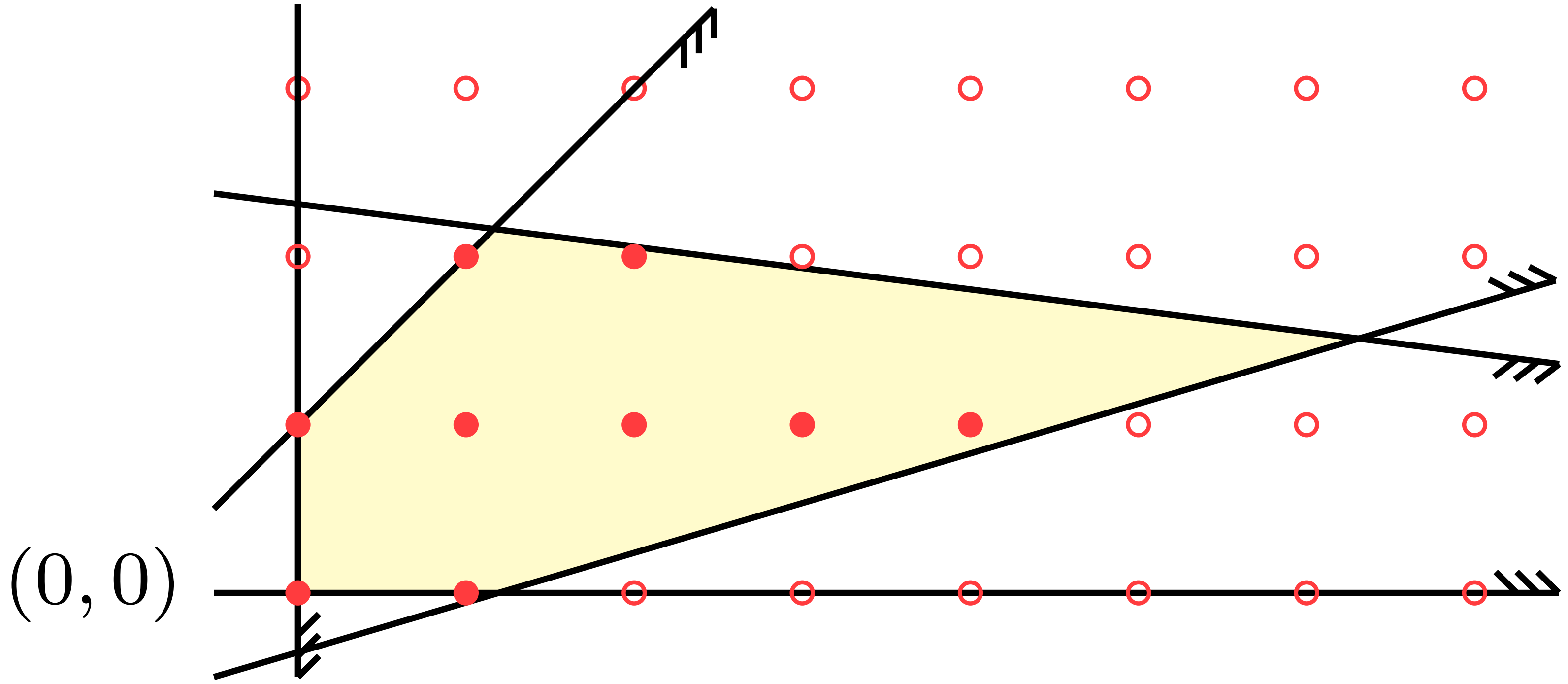

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ILP and variants most wide-spread use of LPs.

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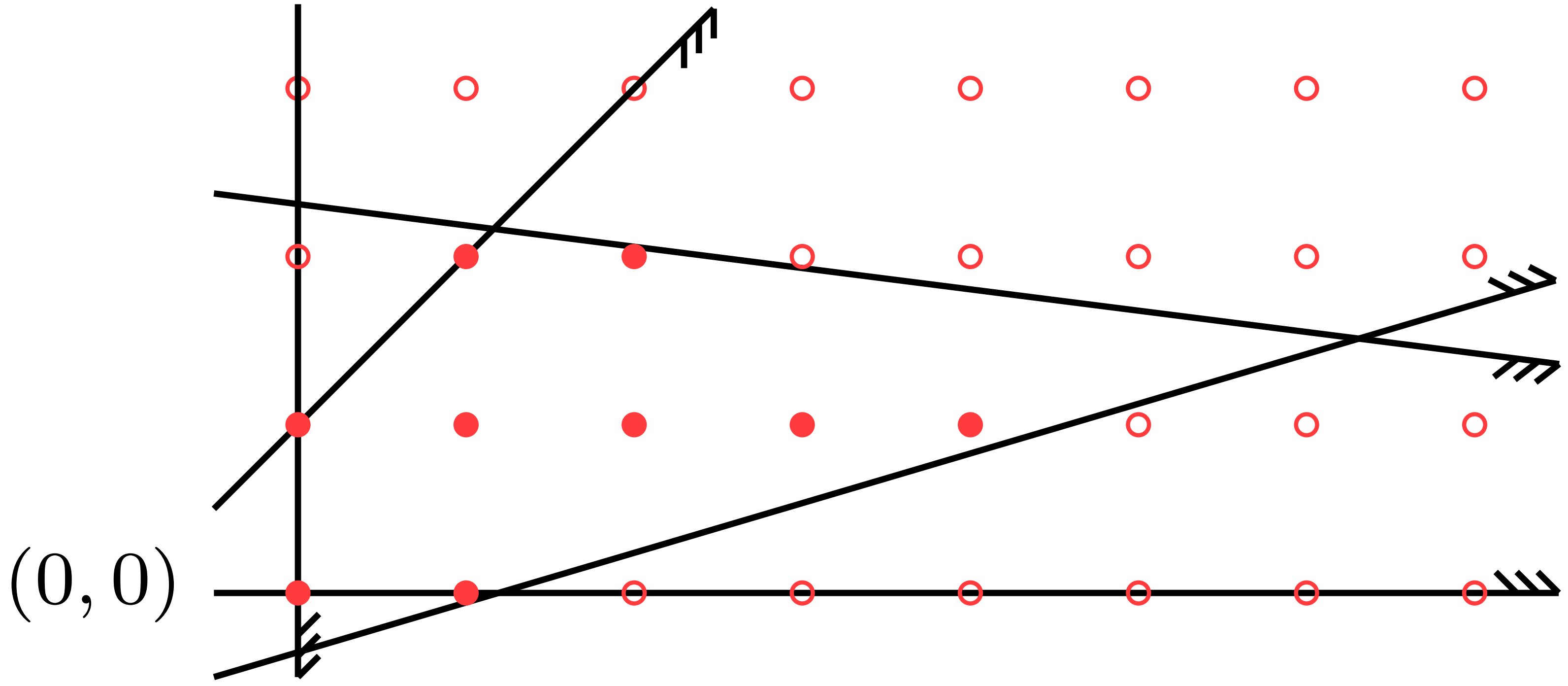

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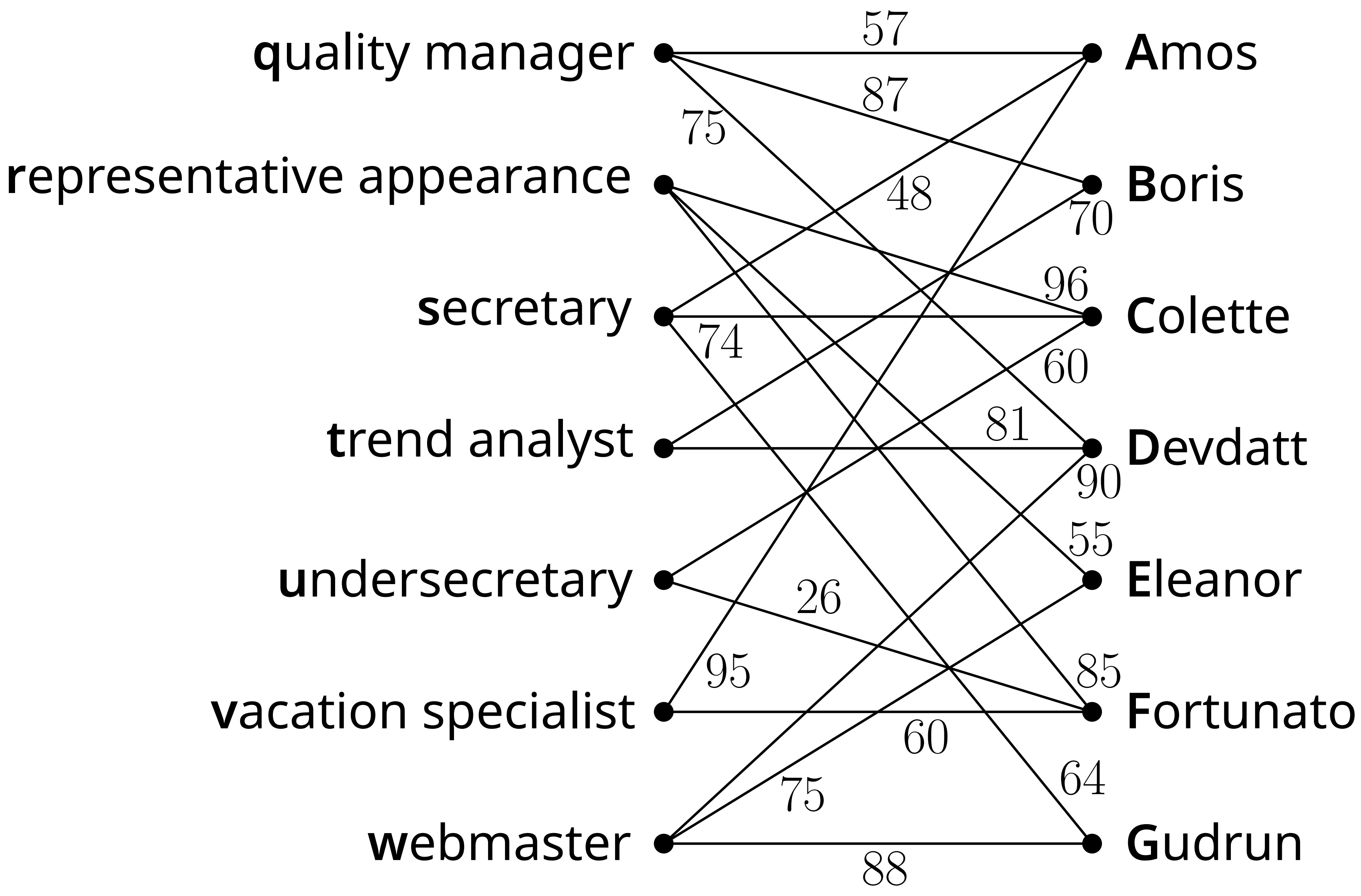
Next: Example “easy, medium, hard” integer programs.

- Maximum weight matching
- Minimum vertex cover
- Maximum independent set

Maximum weight perfect matching

A company is assigning workers to jobs. In the bipartite graph an edge connects a worker to a job they are willing to take, weighted by their suitability for the job.



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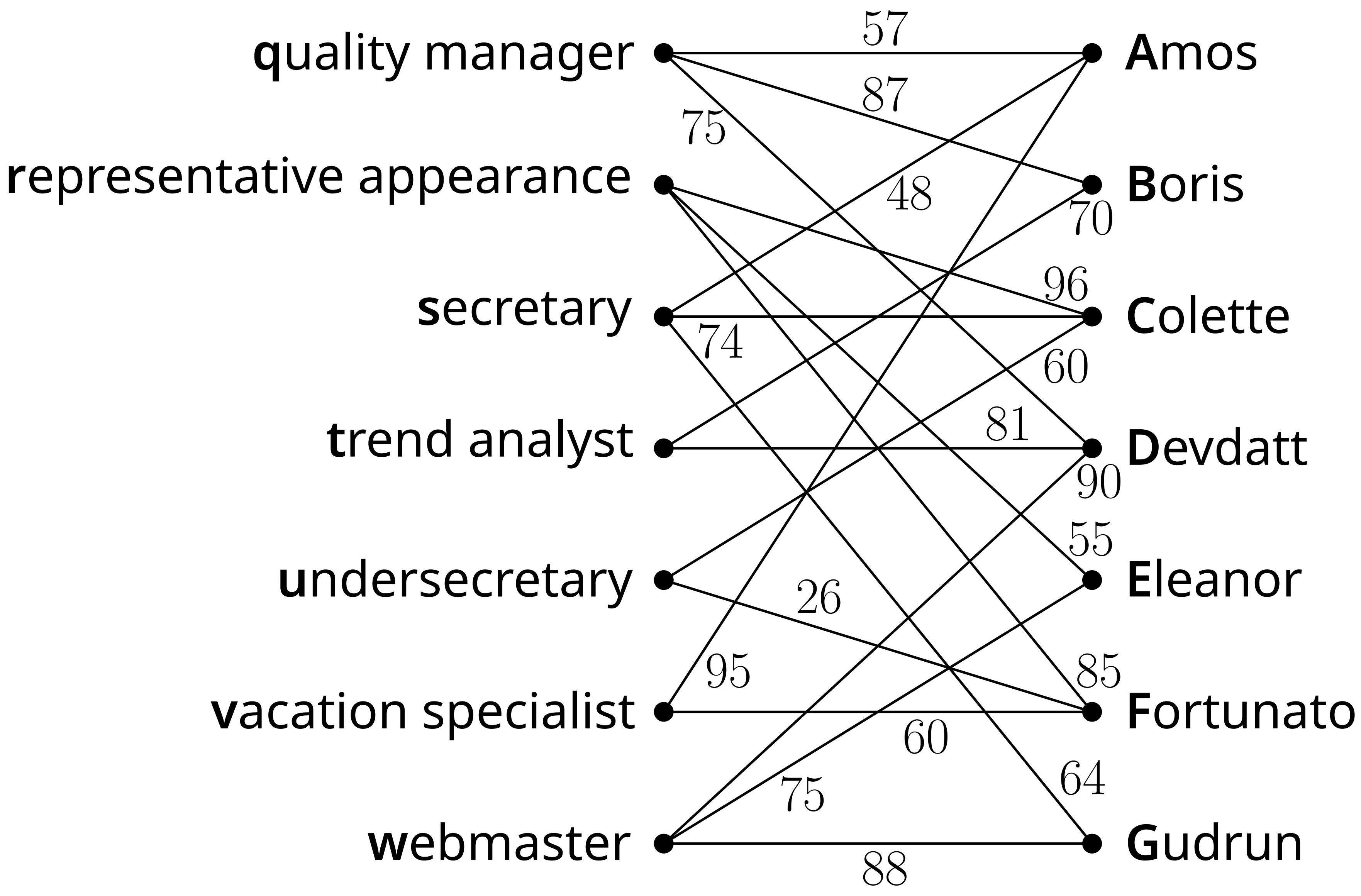

How to find an optimal assigment?

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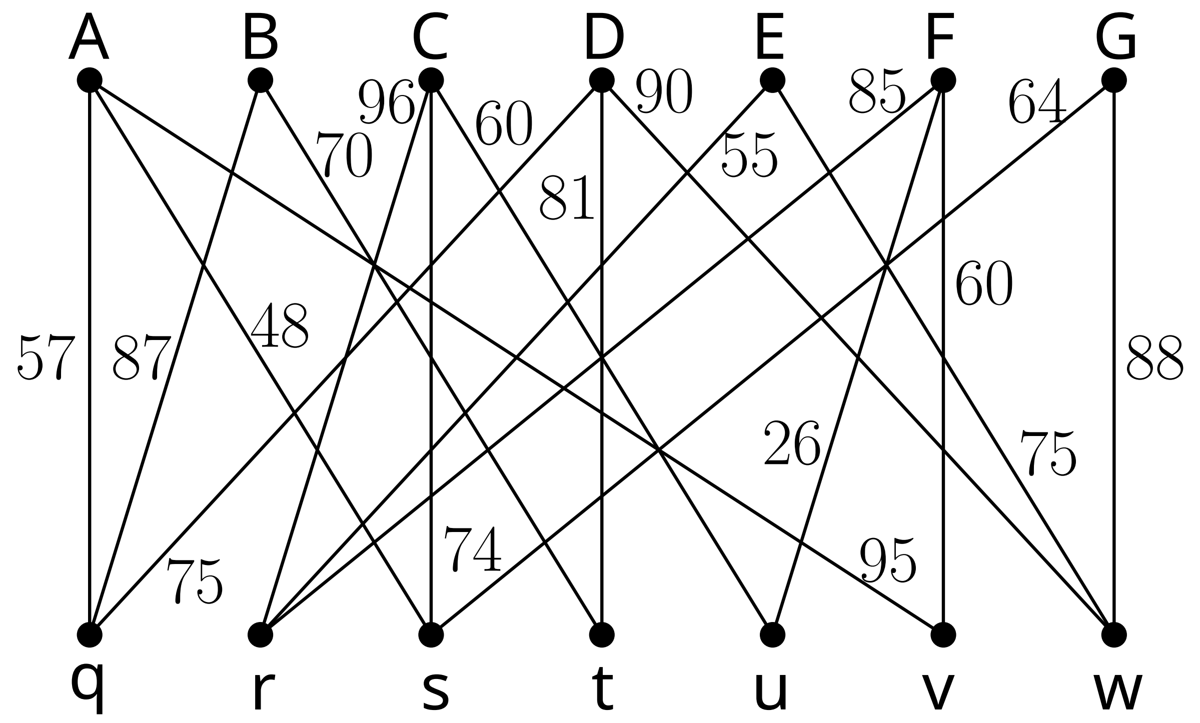
Greedy approach

always assign largest possible score



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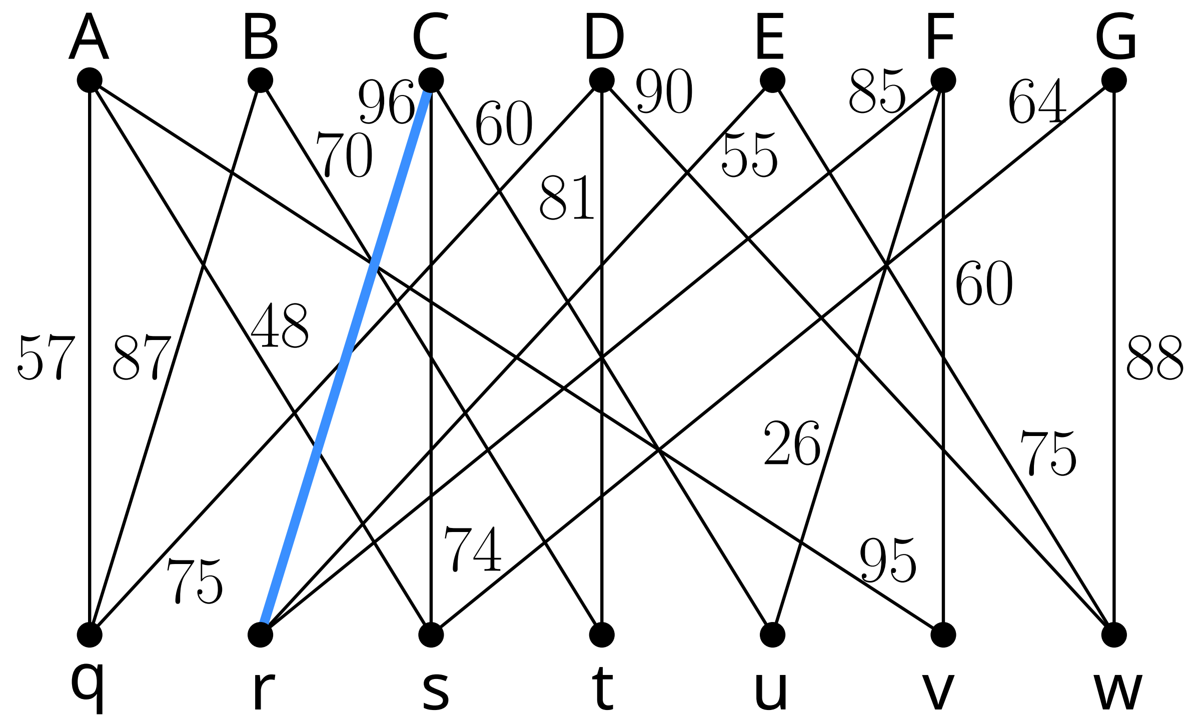

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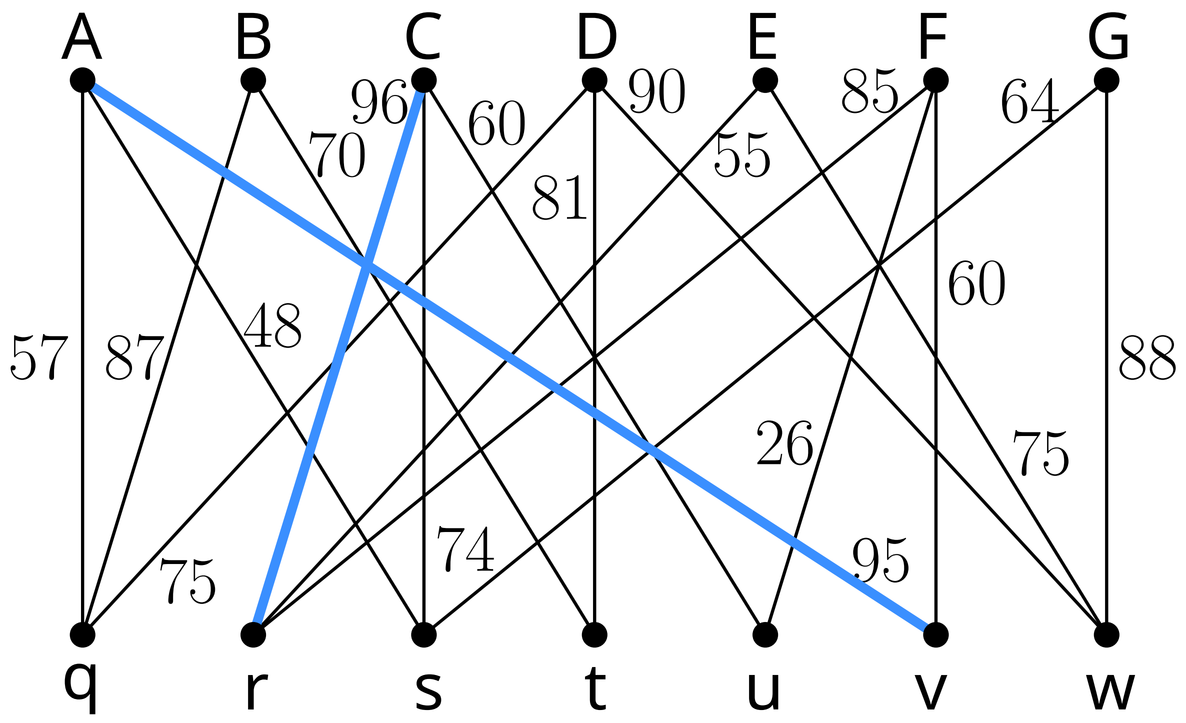

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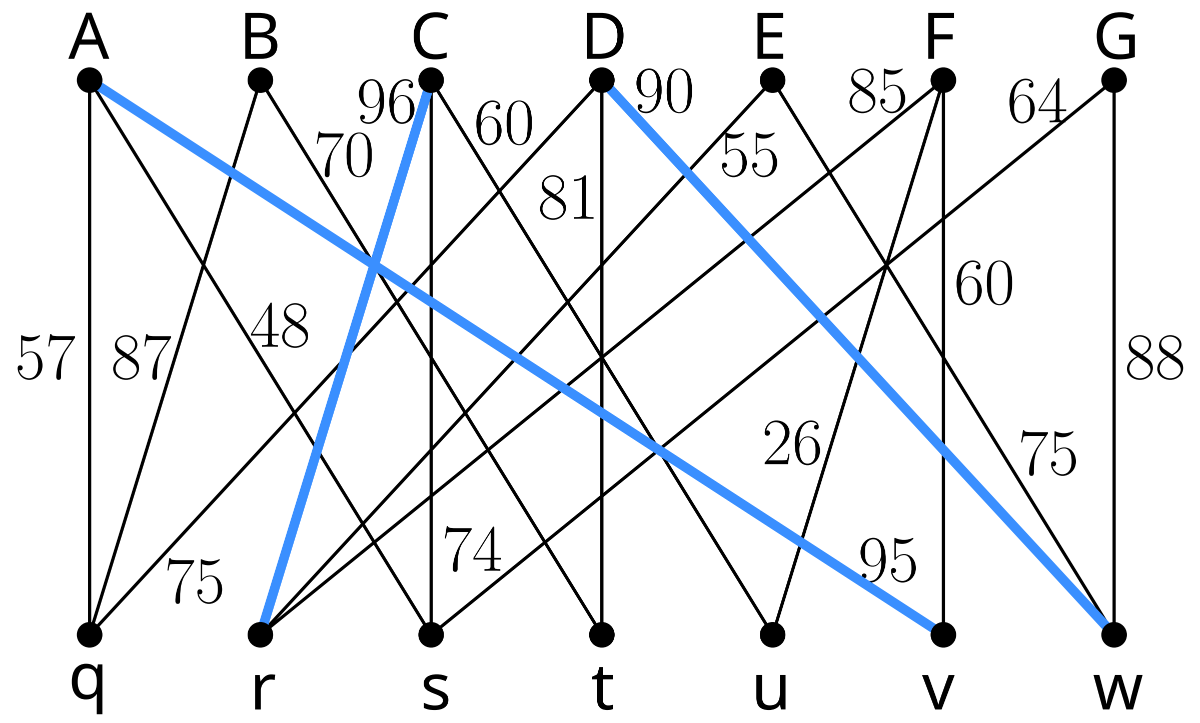

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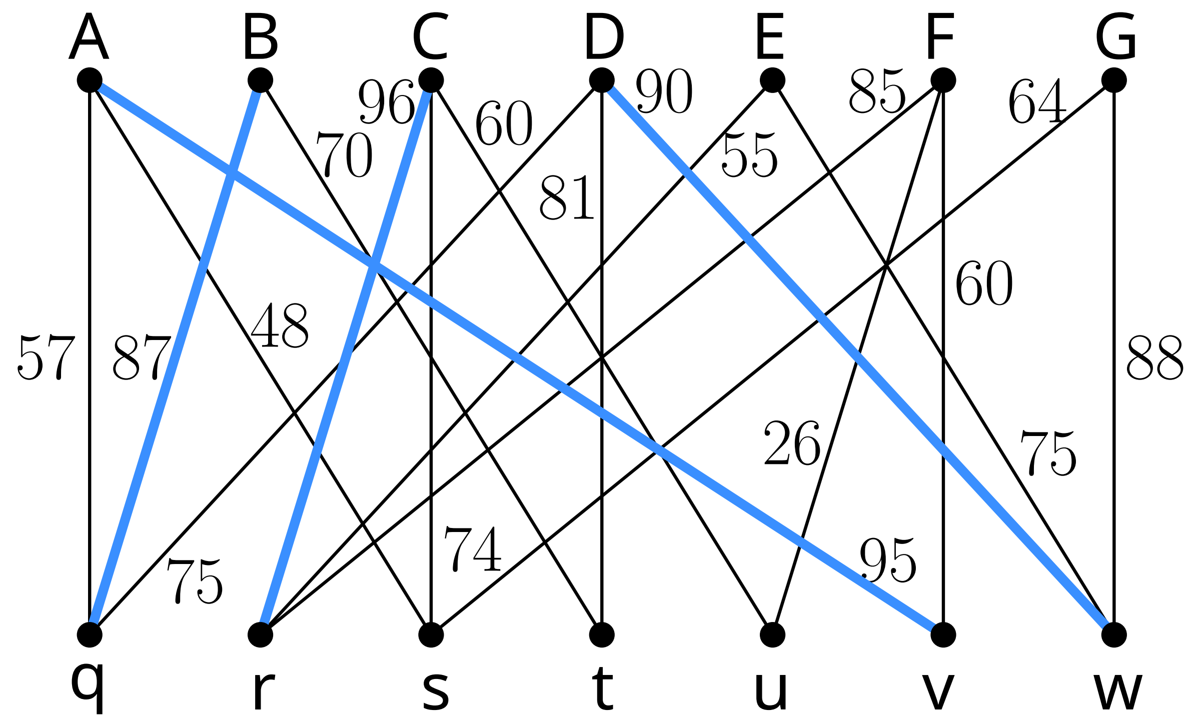

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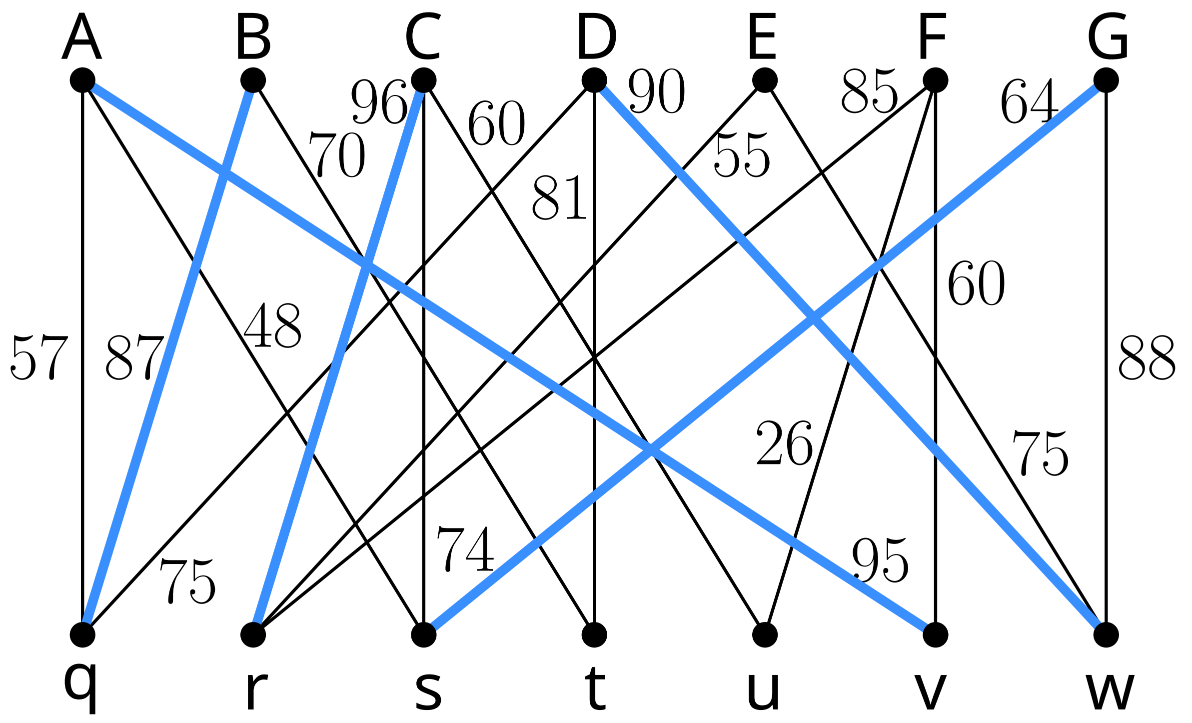

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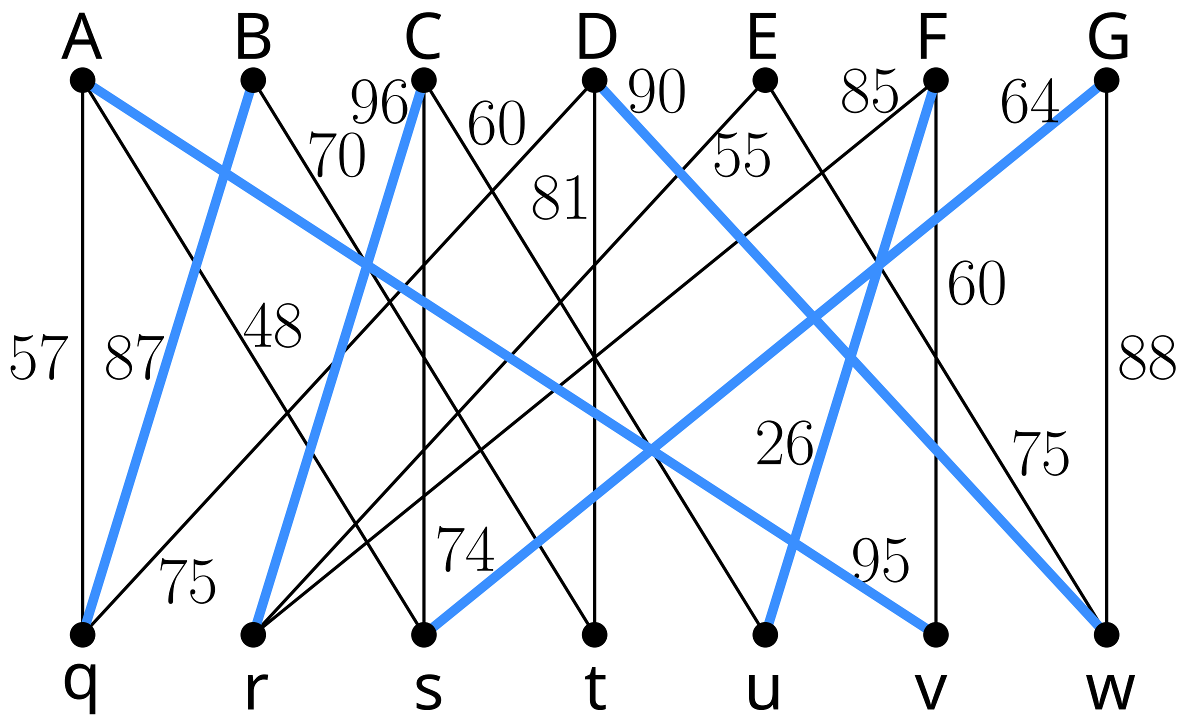

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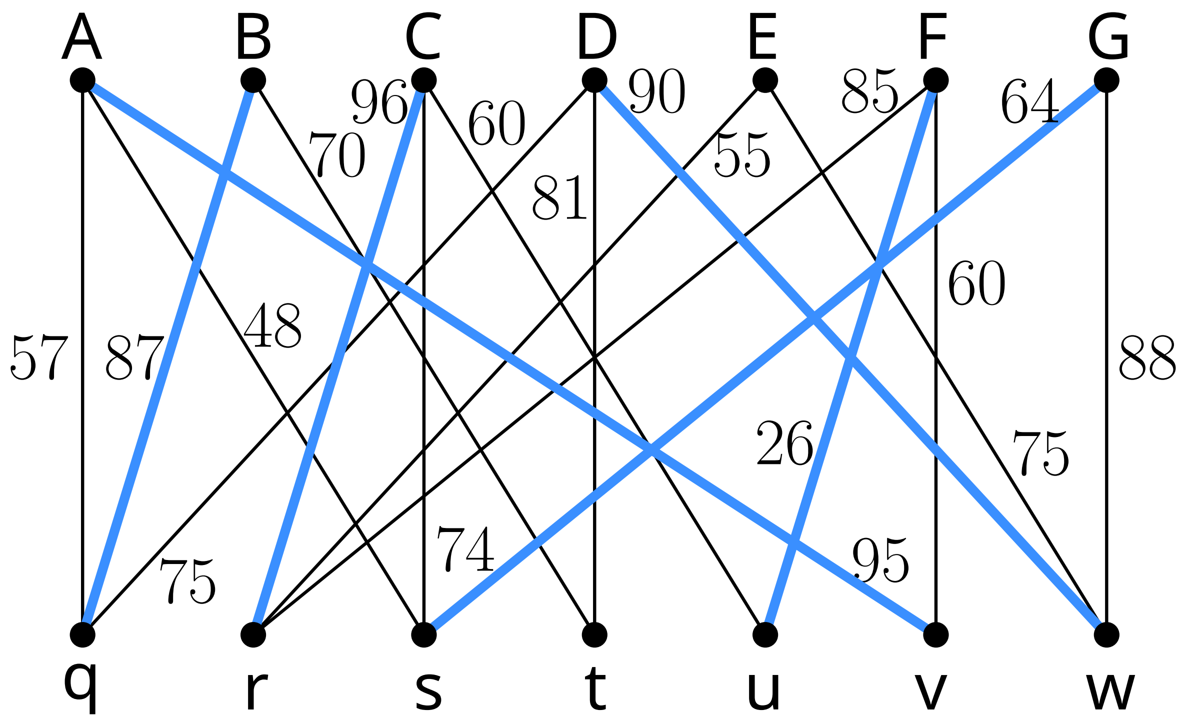

total score:

$$95 + 87 + 96 + 90 + 0 + 26 + 64 = 458$$

Eleanor is not assigned to any job!

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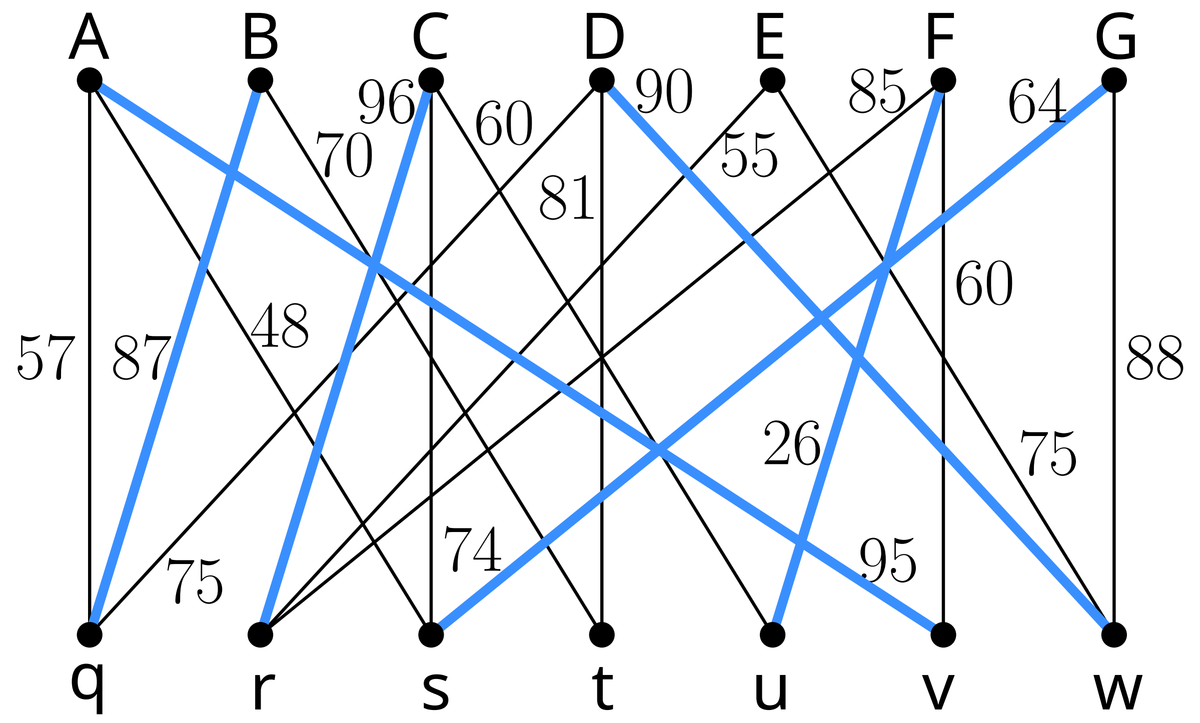

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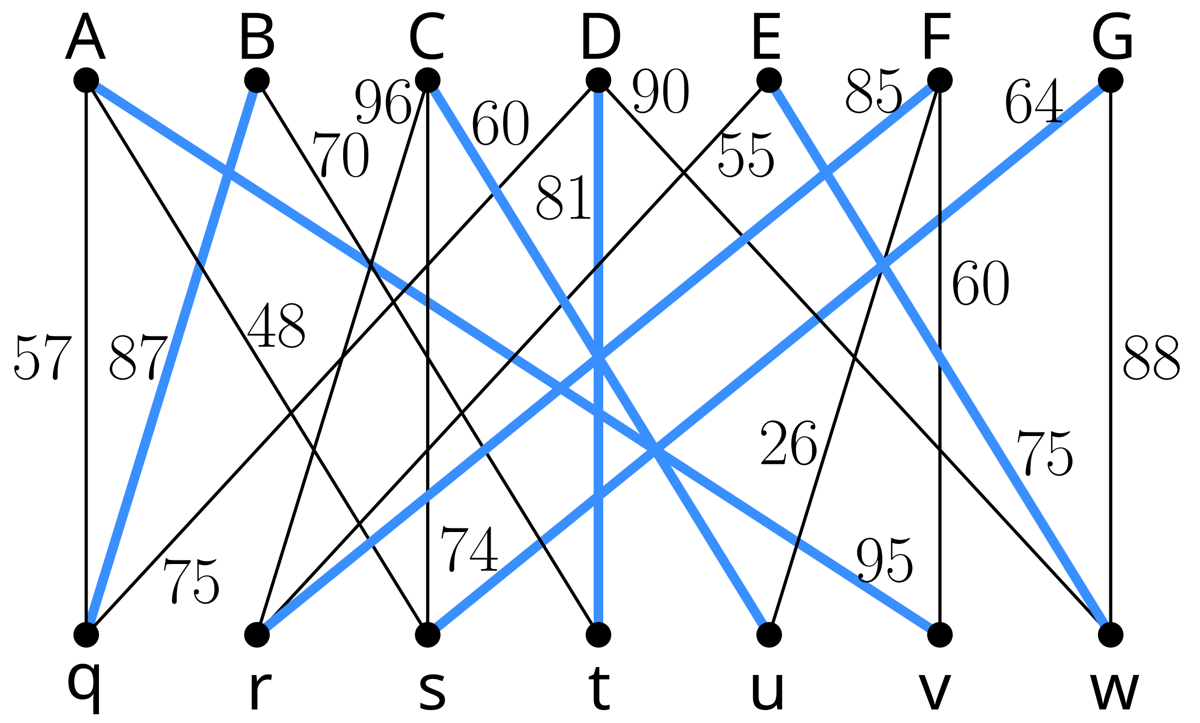

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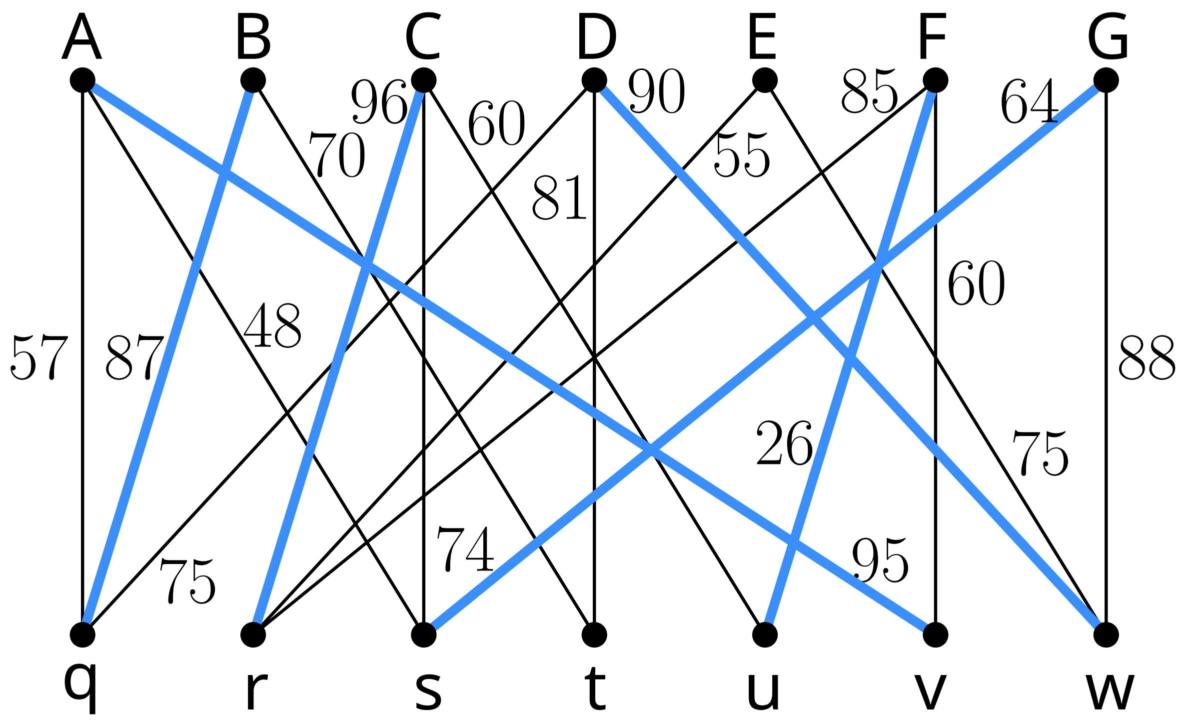

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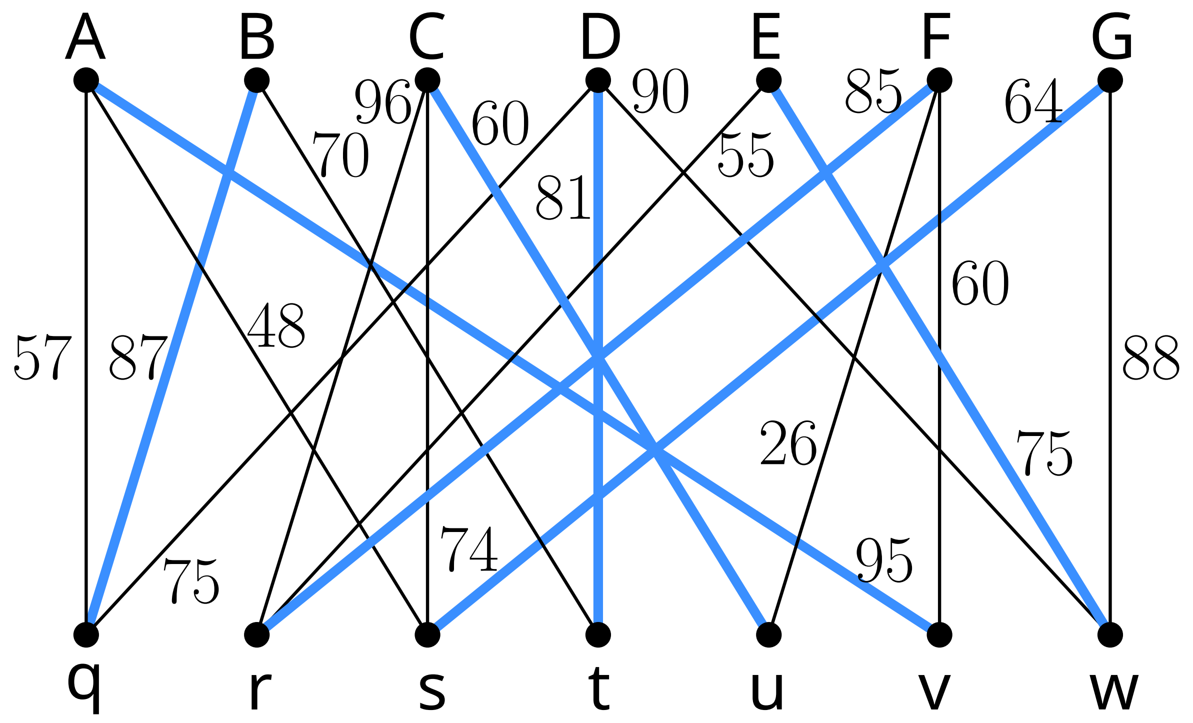

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How to model this as an ILP?

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Integer program

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subject to $\sum_{e \in E : v \in e} x_e = 1$ for each vertex $v \in V$, and

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variable x_e encodes whether edge e is contained in the matching

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if we "relax" the requirement that the x_e are integral we get . . .

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If the LP is infeasible so is the IP.

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Here this works out nicely.

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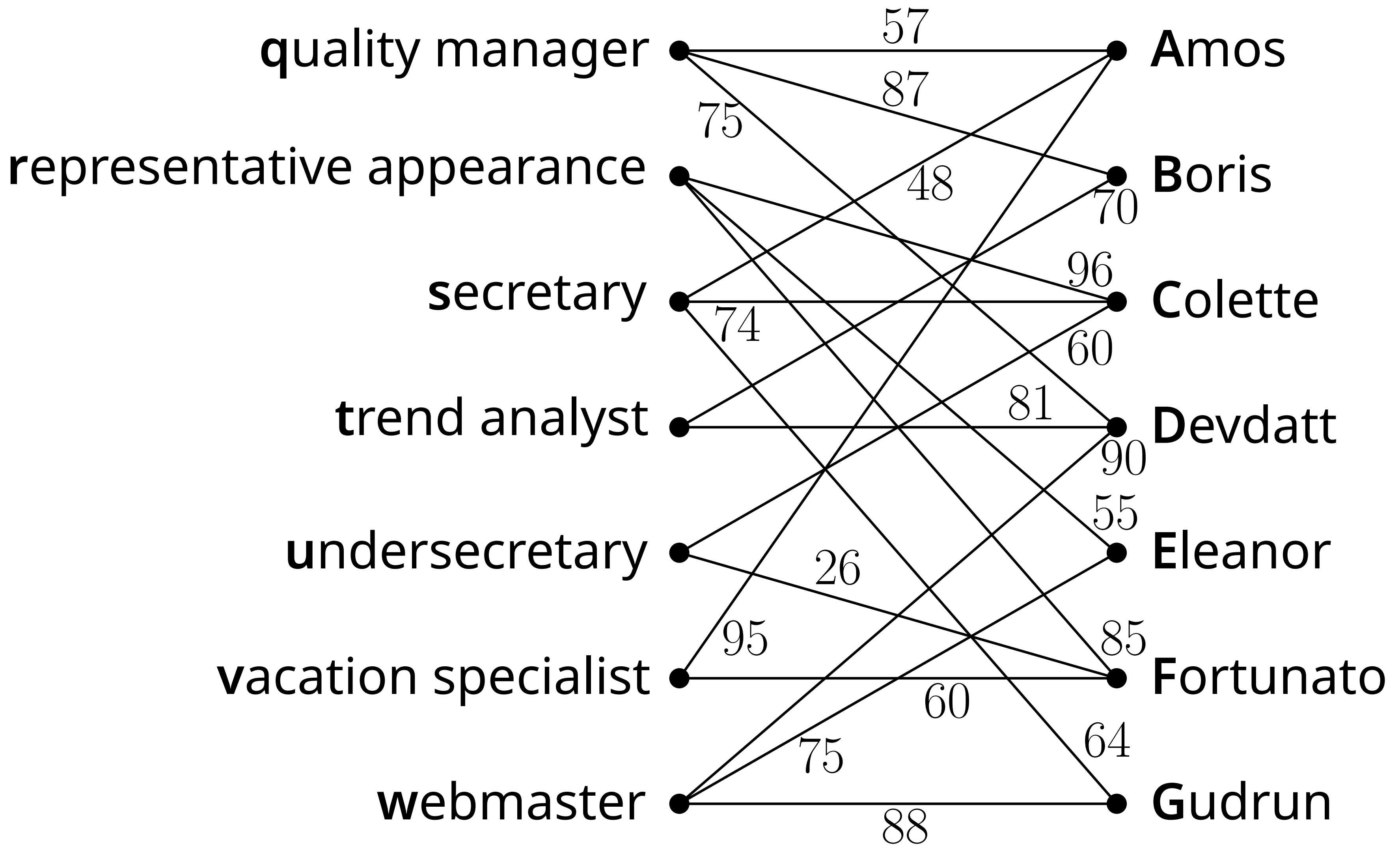
Theorem 3.2.1

Let G be an arbitrary weighted bipartite graph. The LP relaxation from before has an integer optimal solution (which also solves the integer program).



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If $k(x^*) = 0$, then we are done.

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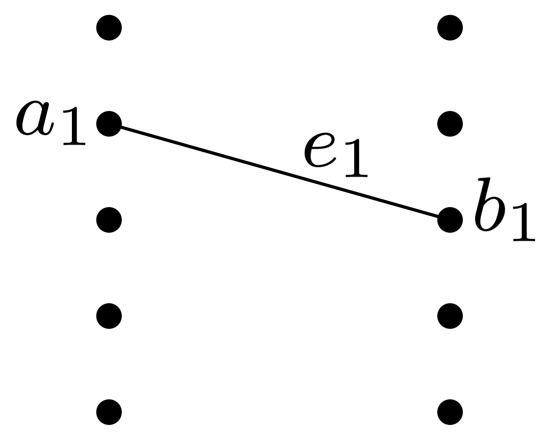

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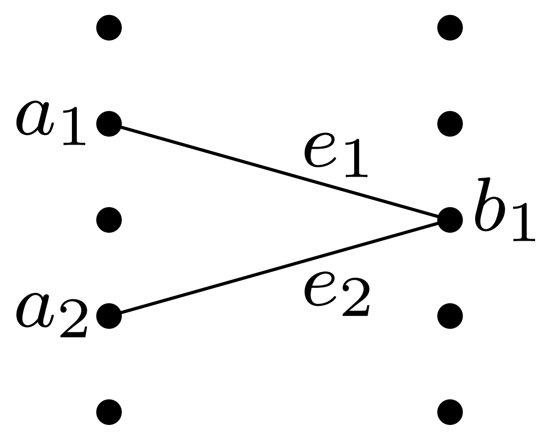

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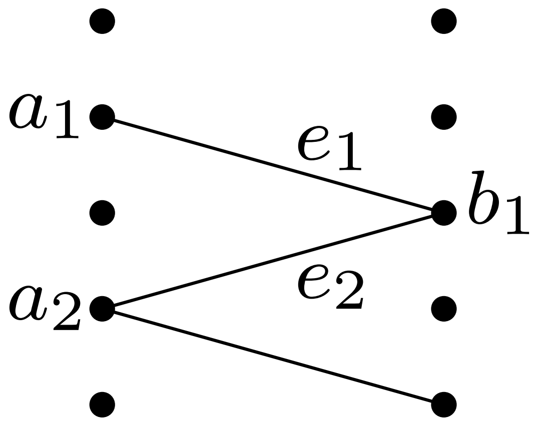

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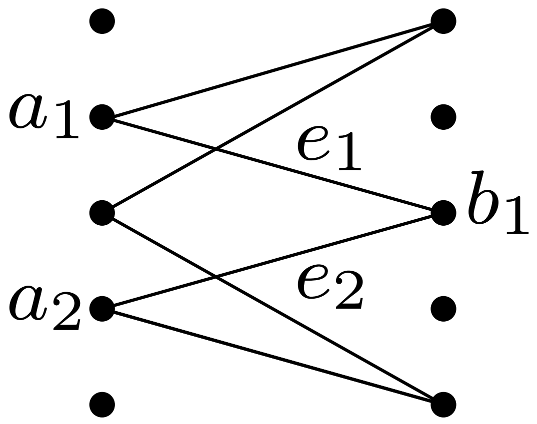

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Let $w(x^*) = \sum_{e \in E} w_e x^*_e$ its value and $k(x^*)$ its number of non-integral components.

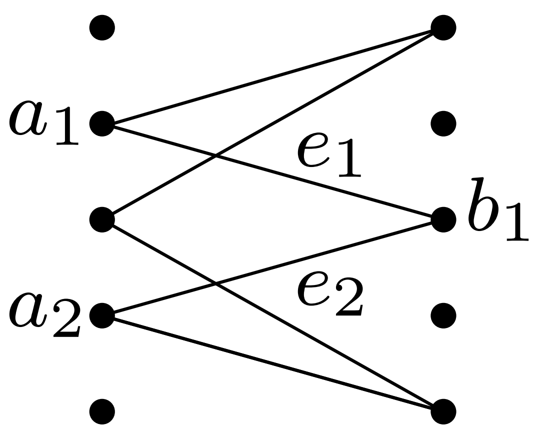
If $k(x^*) > 0$, we find an optimal solution $\tilde{x}$ with $k(x^*) > k(\tilde{x}^*)$.

Let $x^*_{e_1}$ be a non-integral component of x for edge $e_1 = (a_1, b_1)$.

Since $0 < x^*_{e_1} < 1$ and $\sum_{b_1 \in e} x^*_e = 1$ there exists $e_2 = (a_2, b_1) \neq e_1$ s.t. $x^*_{e_2}$ non-integral.

Similarly, there exists edge e_3 and so on, until we find a cycle.

Since G is bipartite, the cycle – say $e_1, \ldots, e_t$ – has even length t.



Maximum weight perfect matching

Theorem 3.2.1

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Proof: Let x^* be an optimal solution of the LP relaxation.

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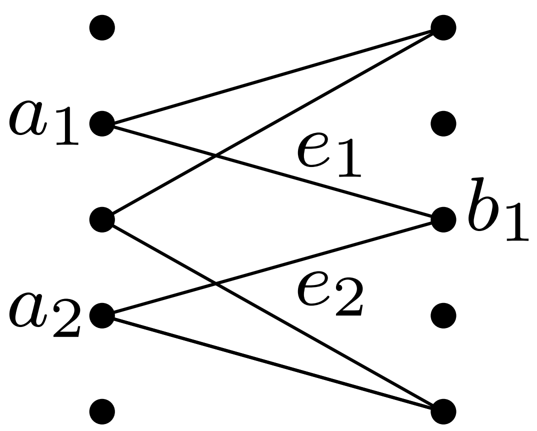
For small $\varepsilon > 0$ we define

$$\tilde{x}_e = \begin{cases} x_e^* - \varepsilon & \text{for } e \in \{e_1, \ldots, e_{t-1}\} \\ x_e^* + \varepsilon & \text{for } e \in \{e_2, \ldots, e_t\} \\ x_e^* & \text{else} \end{cases}$$



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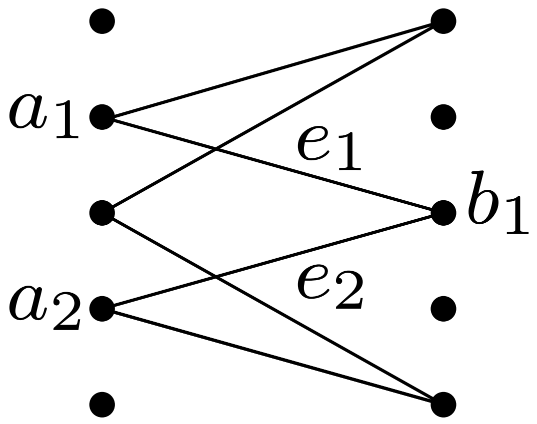

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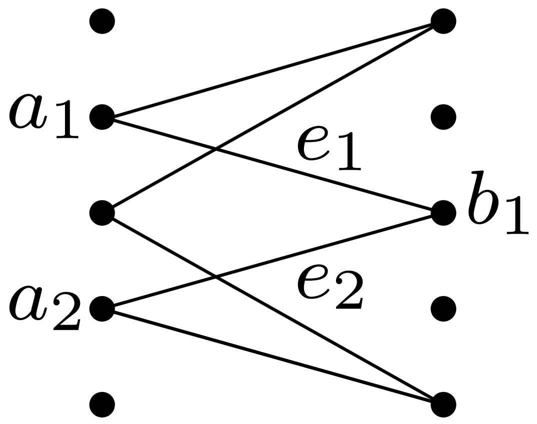

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What about $w(\tilde{x})$?

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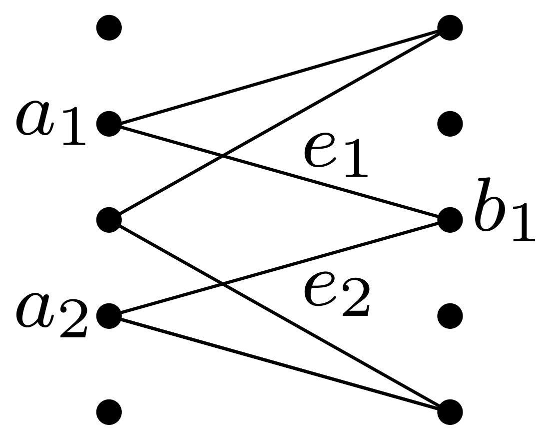

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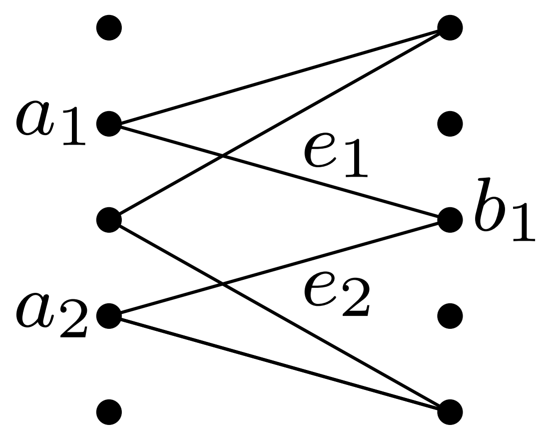

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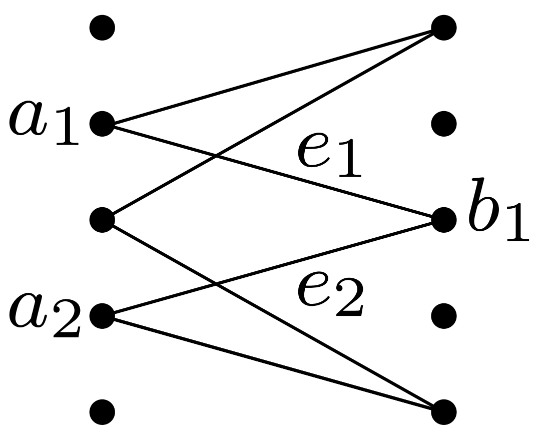

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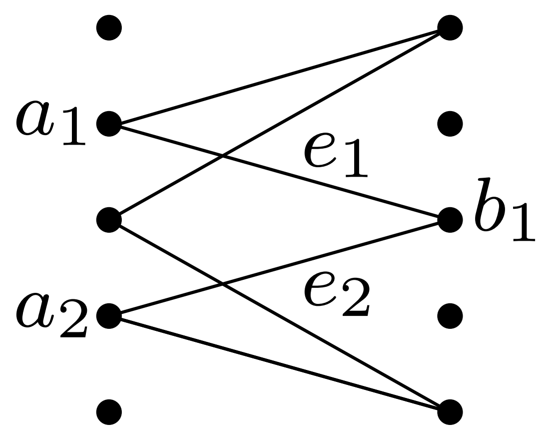

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How do we choose ε?

For small $\varepsilon > 0$ we define

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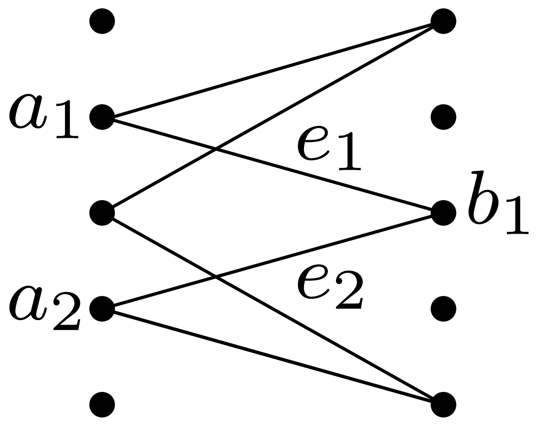

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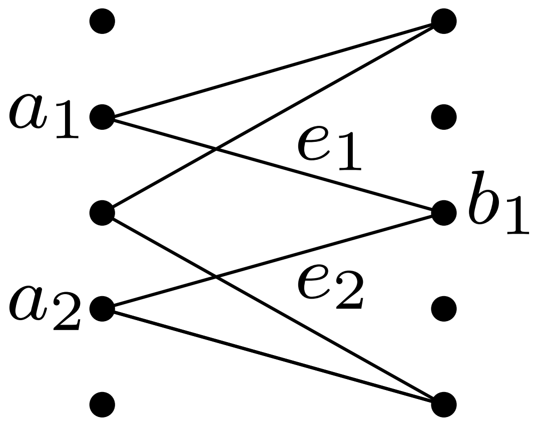

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We choose ε largest s.t. $\tilde{x}$ is still feasible. Then $k(\tilde{x}) < k(x^*)$.

We continue this procedure until all components are integral.

For small $\varepsilon > 0$ we define

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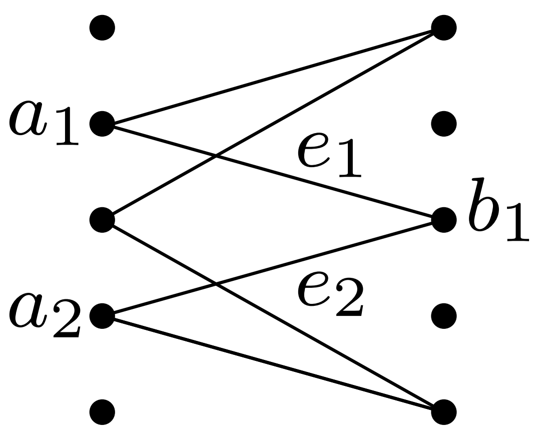


Maximum weight perfect matching

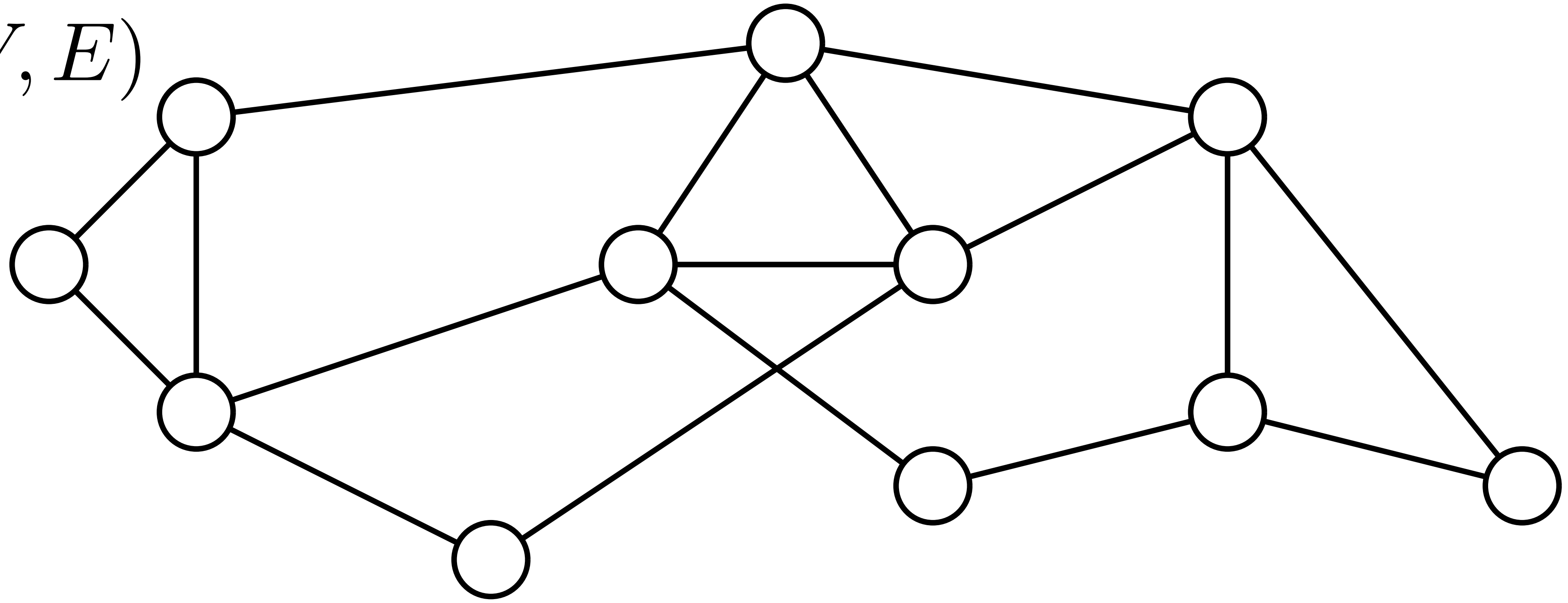
Theorem 3.2.1

Let G be an arbitrary weighted bipartite graph. The LP relaxation from before has an integer optimal solution (which also solves the integer program).

Moral: Sometimes solving an integer program is no harder than solving a linear program.

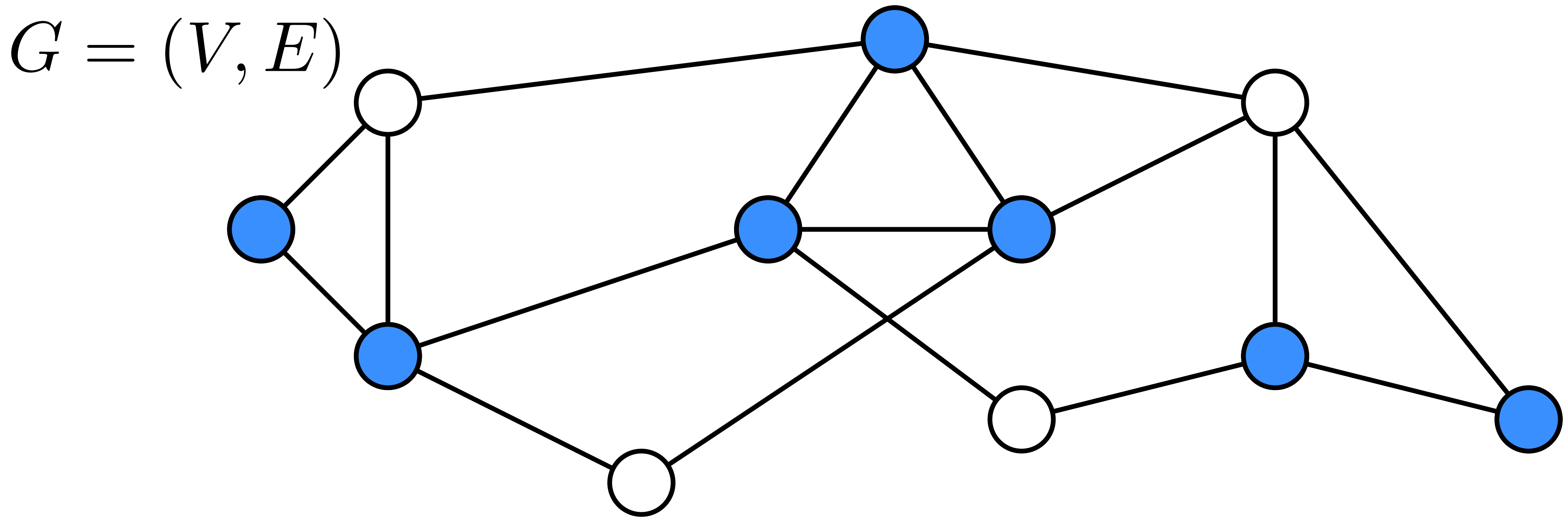
Minimum vertex cover

$G = (V, E)$



Recall: a minimum vertex cover, is a smallest possible subset $V' \subseteq V$ such that for every edge $uv \in E$, it holds that $u \in V'$ or $v \in V'$.

Minimum vertex cover



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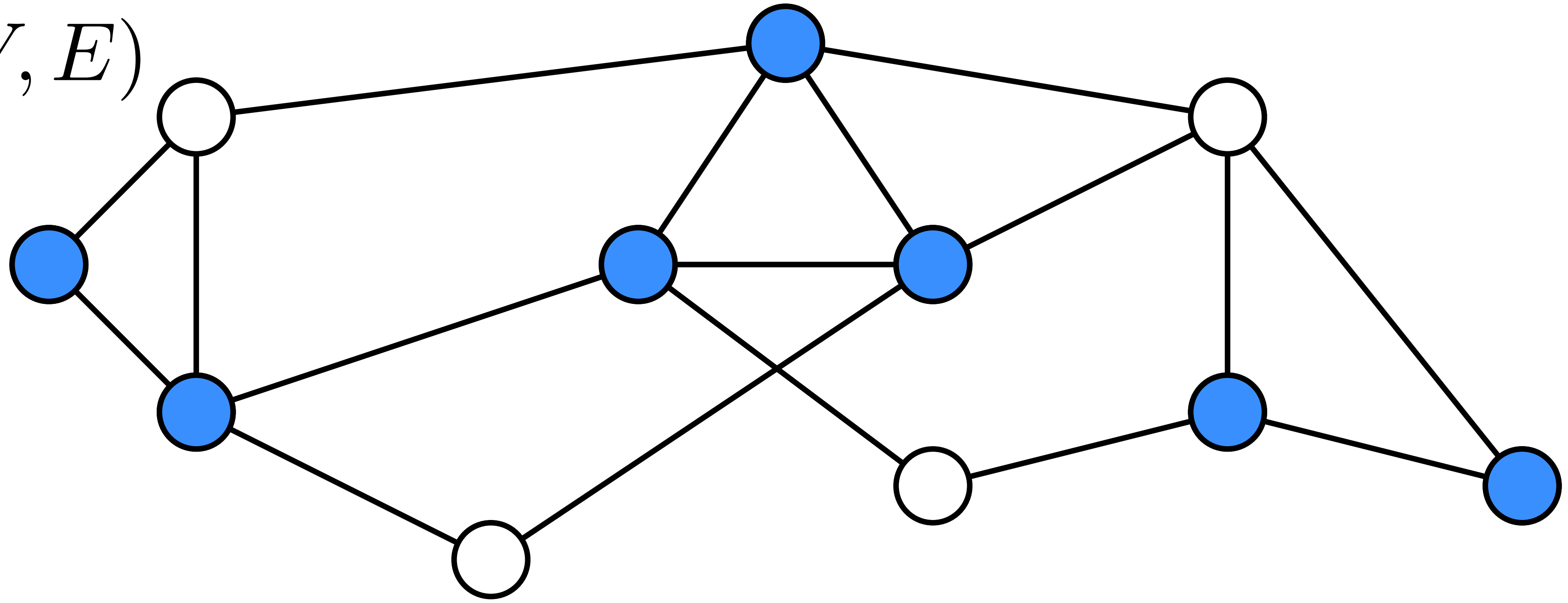
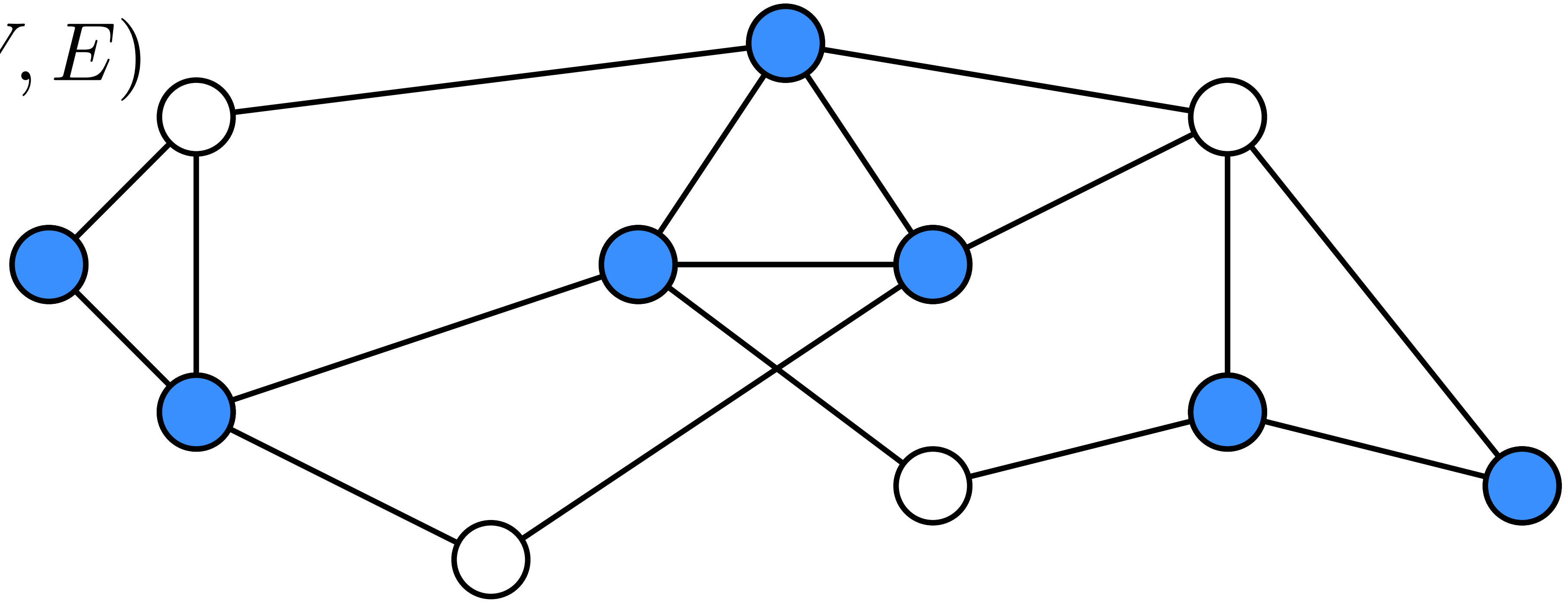


Recall: a minimum vertex cover, is a smallest possible subset $V' \subseteq V$ such that for every edge $uv \in E$, it holds that $u \in V'$ or $v \in V'$.

How can we formulate this as an ILP?

Minimum vertex cover

$G = (V, E)$



minimize $\sum_{v \in V} x_v$

subject to $x_u + x_v \geq 1$ for every edge $\{u, v\} \in E$

$x_v \in \{0, 1\}$ for all $v \in V$.

variable x_v encodes whether vertex v is contained in the cover

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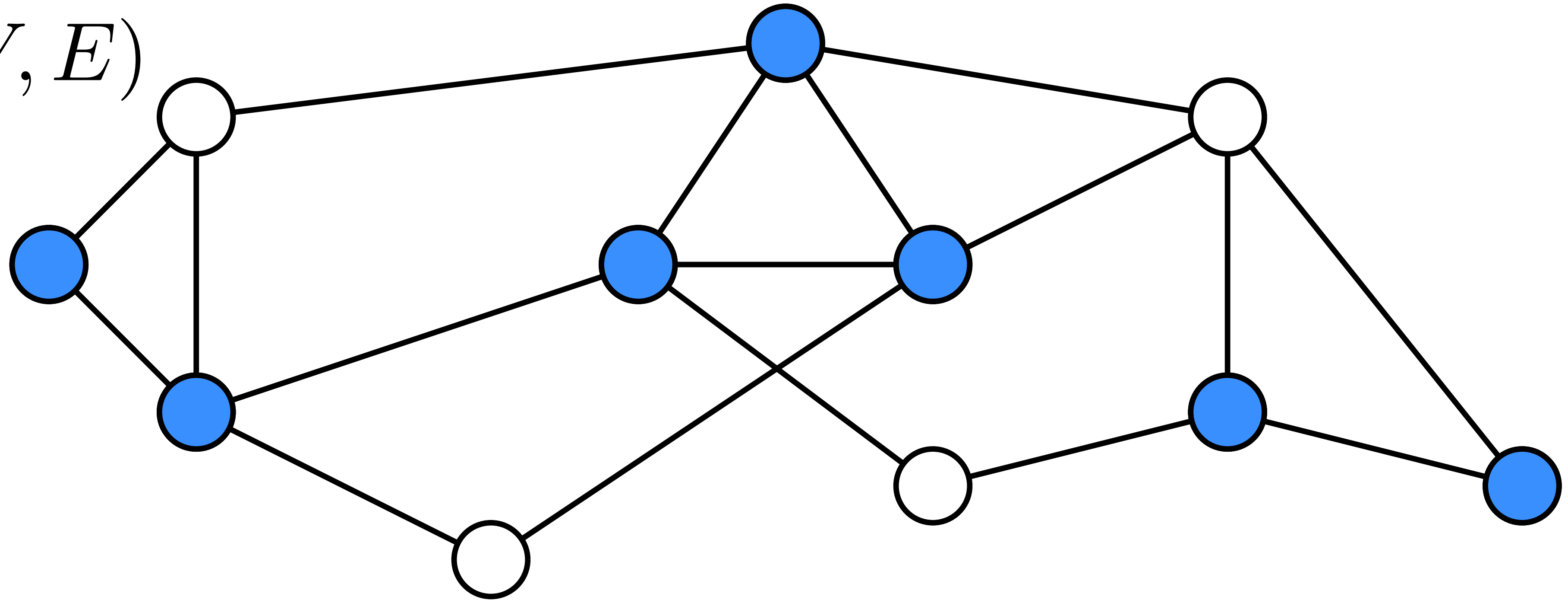

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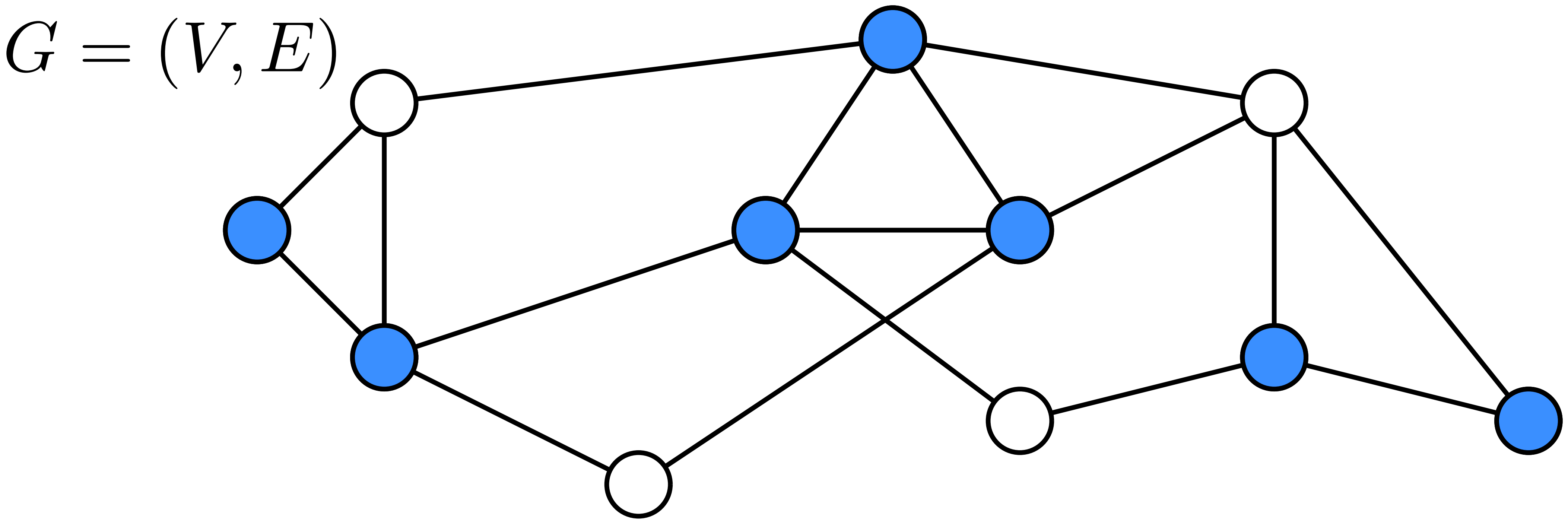
LP relaxation: $0 \leq x_v \leq 1$

Minimum vertex cover



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LP relaxation: $0 \leq x_v \leq 1$ How does this help?

Minimum vertex cover

Rounding:

Let $S_{\text{IP}} \subseteq V$ be a vertex cover solving IP.

Let $S_{\text{LP}} \subseteq V$ be a vertex cover solving LP,
obtained by $S_{\text{LP}} = \{v \in V \mid x_v \geq \frac{1}{2}\}$.

minimize $\sum_{v \in V} x_v$

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What solution do we get here?

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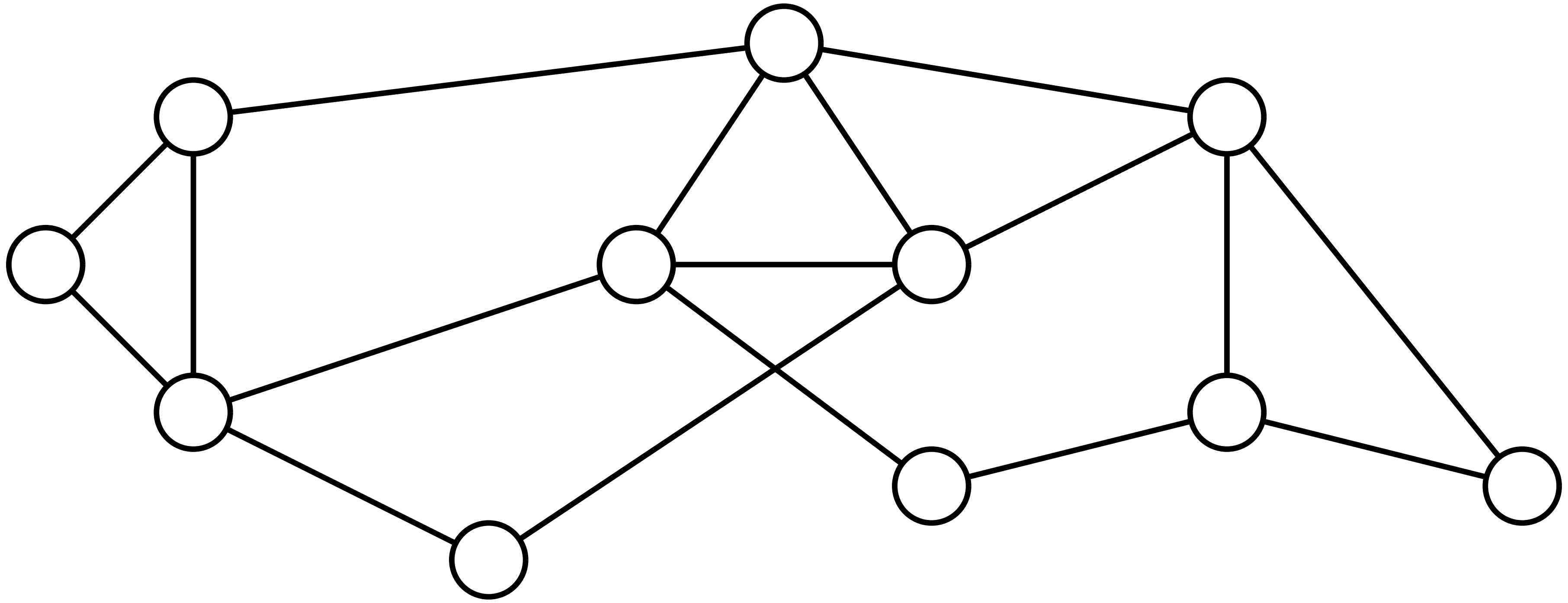

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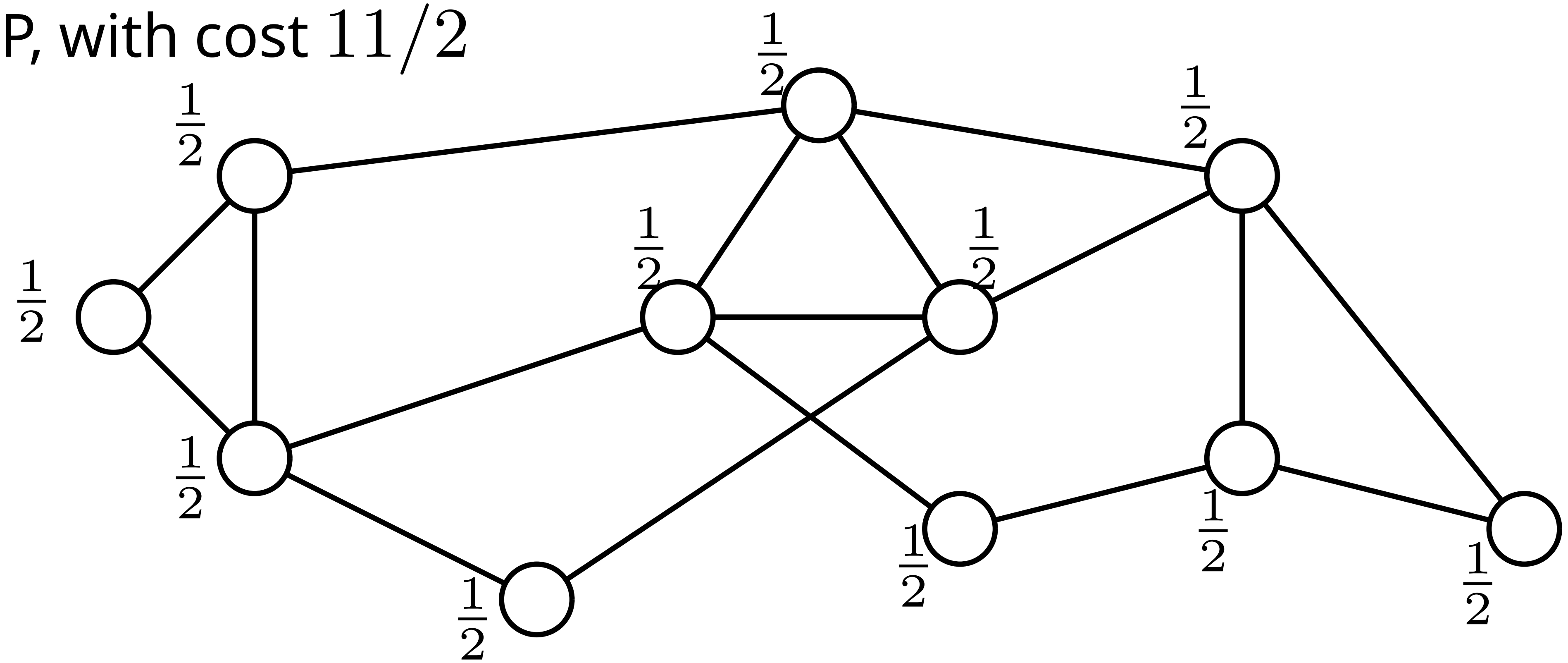
solution to LP, with cost $11/2$



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$$\text{minimize} \sum_{v \in V} x_v$$

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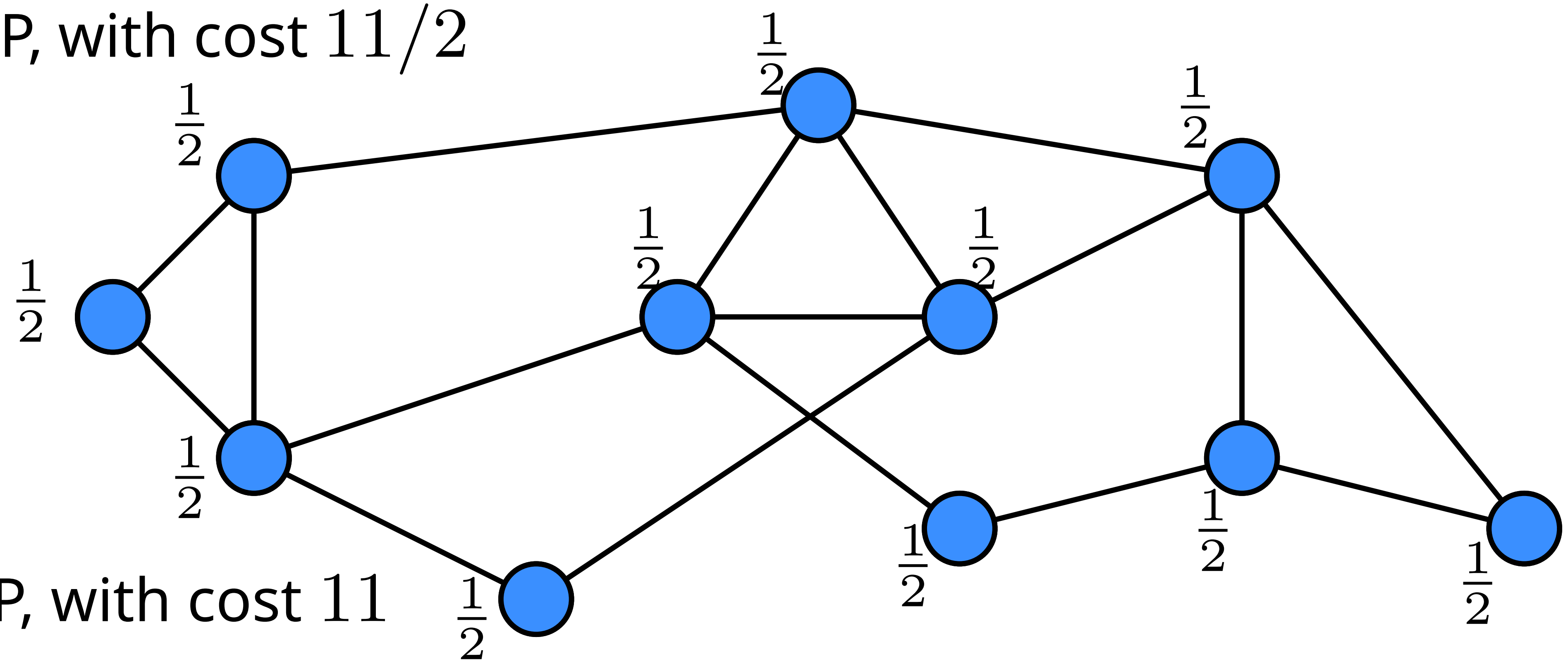
What solution do we get here?

solution to LP, with cost $11/2$



solution to IP, with cost 11

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What solution do we get here?

minimize $\sum_{v \in V} x_v$

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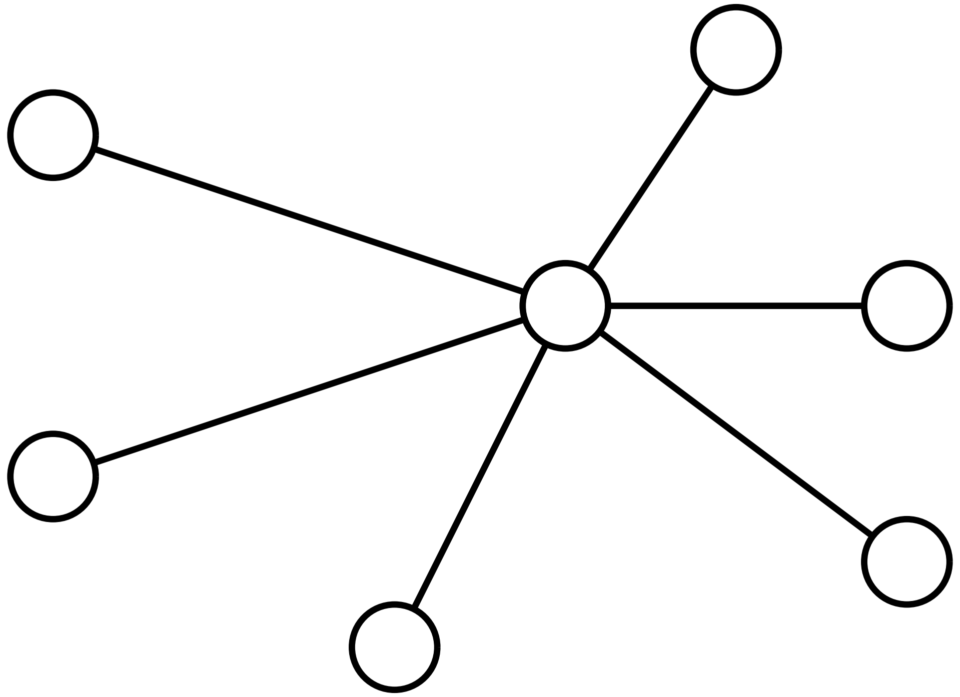

Minimum vertex cover

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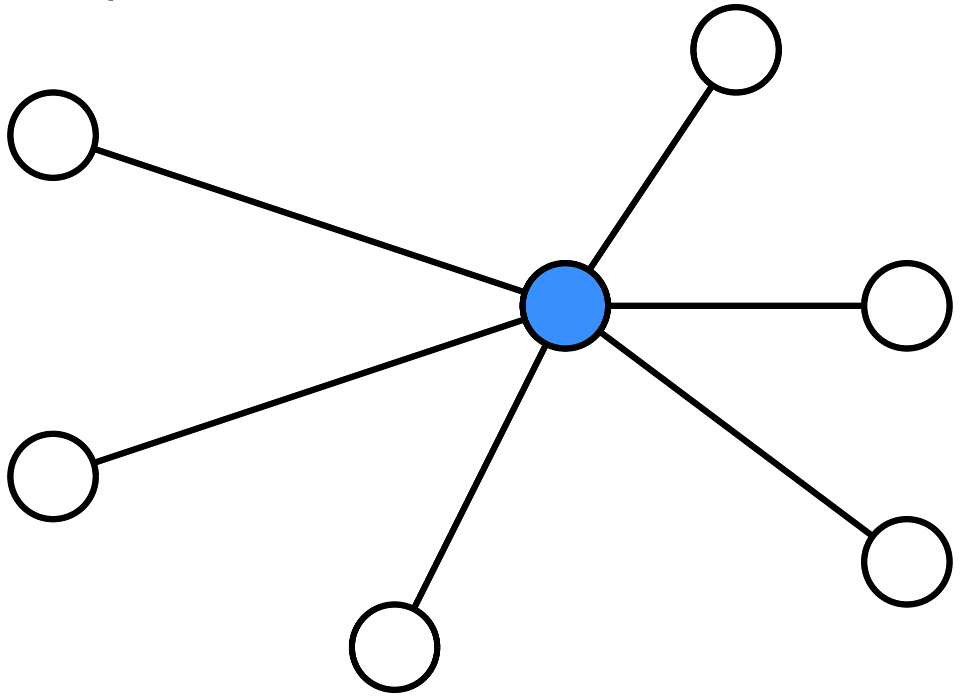
What solution do we get here?

solution to LP also to IP, with cost 1



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Fact: $|S_{\text{IP}}| \leq |S_{\text{LP}}| \leq 2|S_{\text{IP}}|$

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Fact: $|S_{\text{IP}}| \leq |S_{\text{LP}}| \leq 2|S_{\text{IP}}|$

x_v^*: optimal LP solution,
$\tilde{x}_v$: optimal ILP solution

Proof of second inequality

$$|S_{\text{LP}}| \leq 2\sum_{v \in V} x_v^* \leq 2\sum_{v \in V} \tilde{x}_v = 2 \cdot |S_{\text{IP}}|$$

by definition of S_{LP}

since any solution to IP is also a feasible solution for the LP relaxation.

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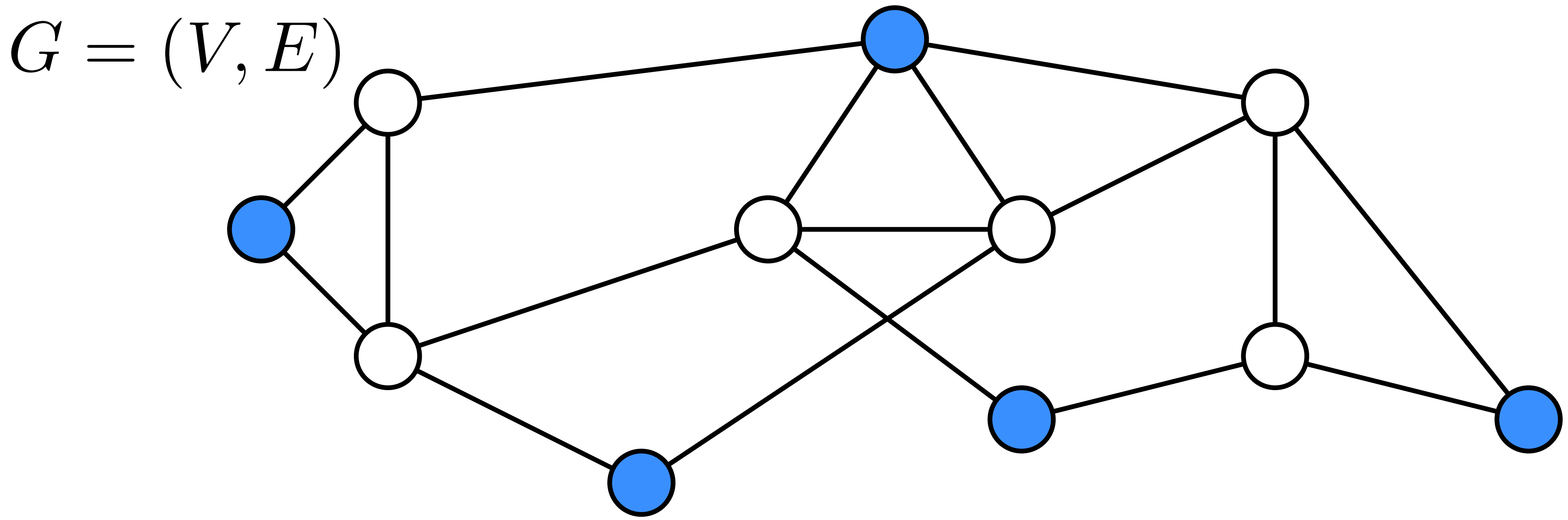
Moral: Sometimes solving an LP relaxation gives an approximate solution to an **NP**-hard integer program.

Maximum independent set

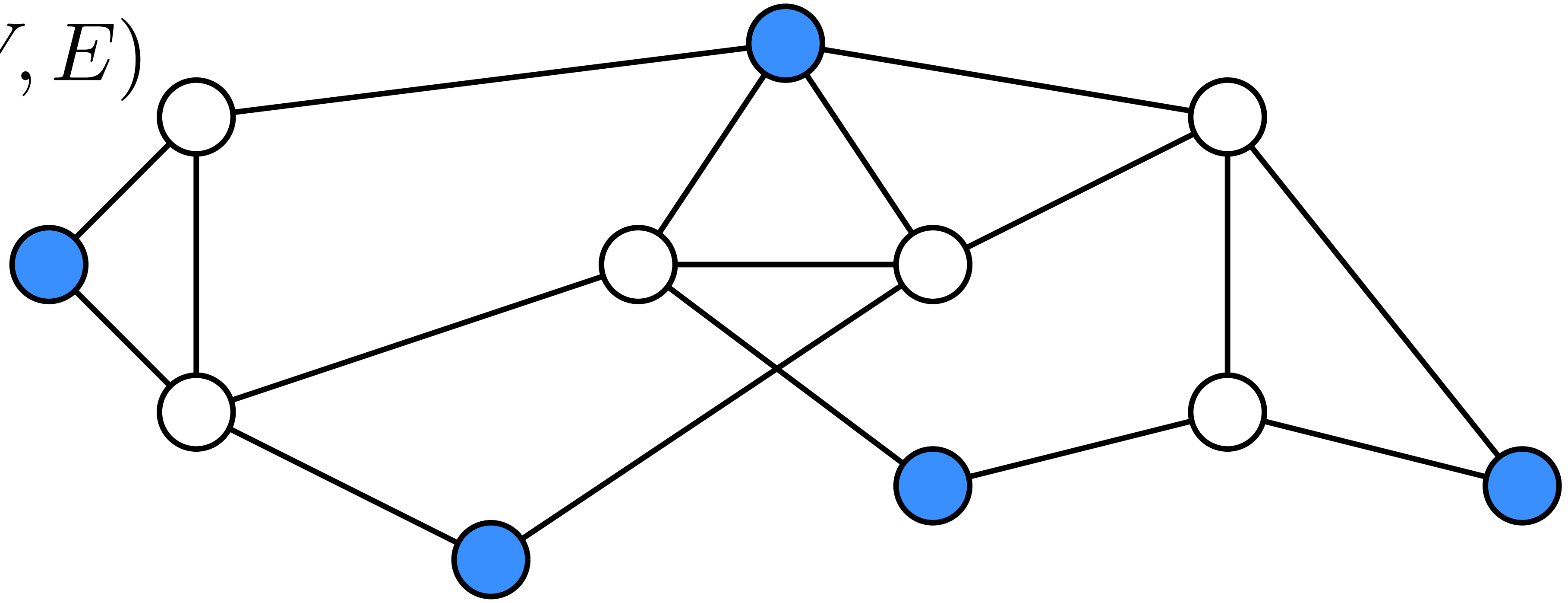


Recall: a maximum independent, is a largest possible subset $V' \subseteq V$ such that for any $u, v \in V'$, it holds that $(u, v) \notin E$.

How can we formulate this as an ILP?

Maximum independent set

$G = (V, E)$



maximize $\sum_{v \in V} x_v$

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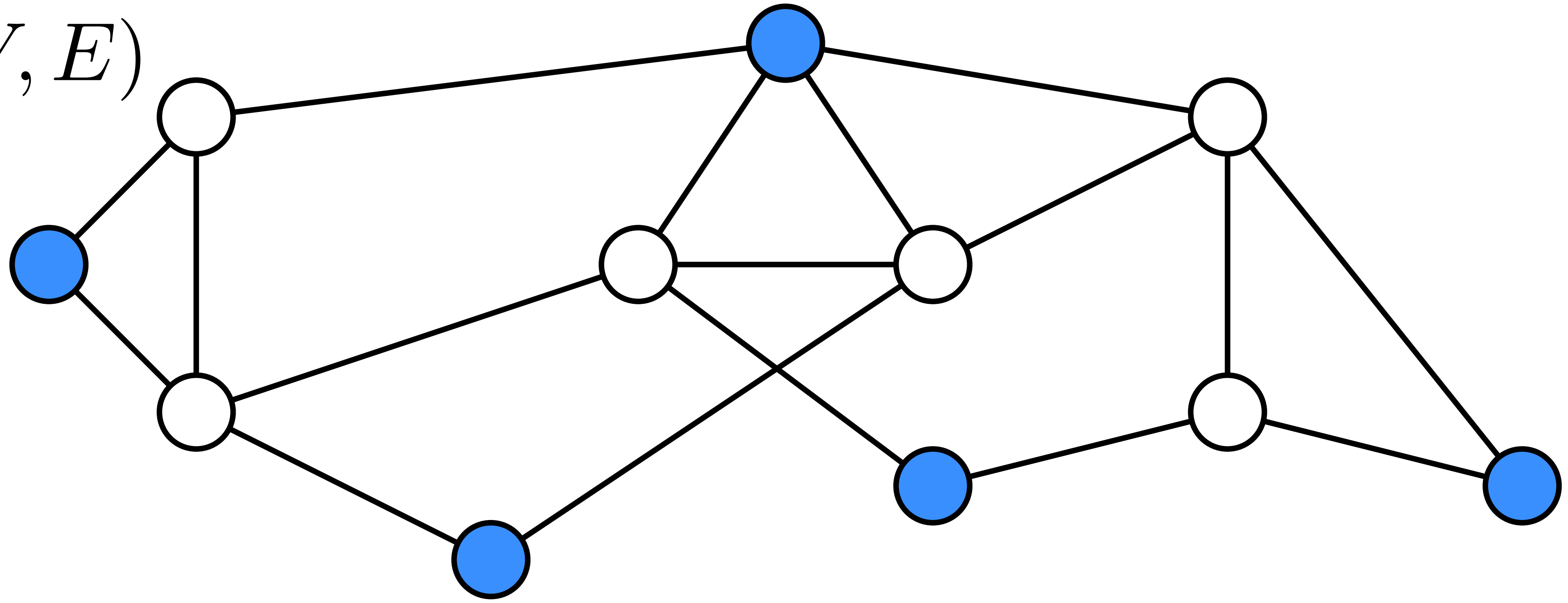

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LP relaxation: $0 \leq x_v \leq 1$

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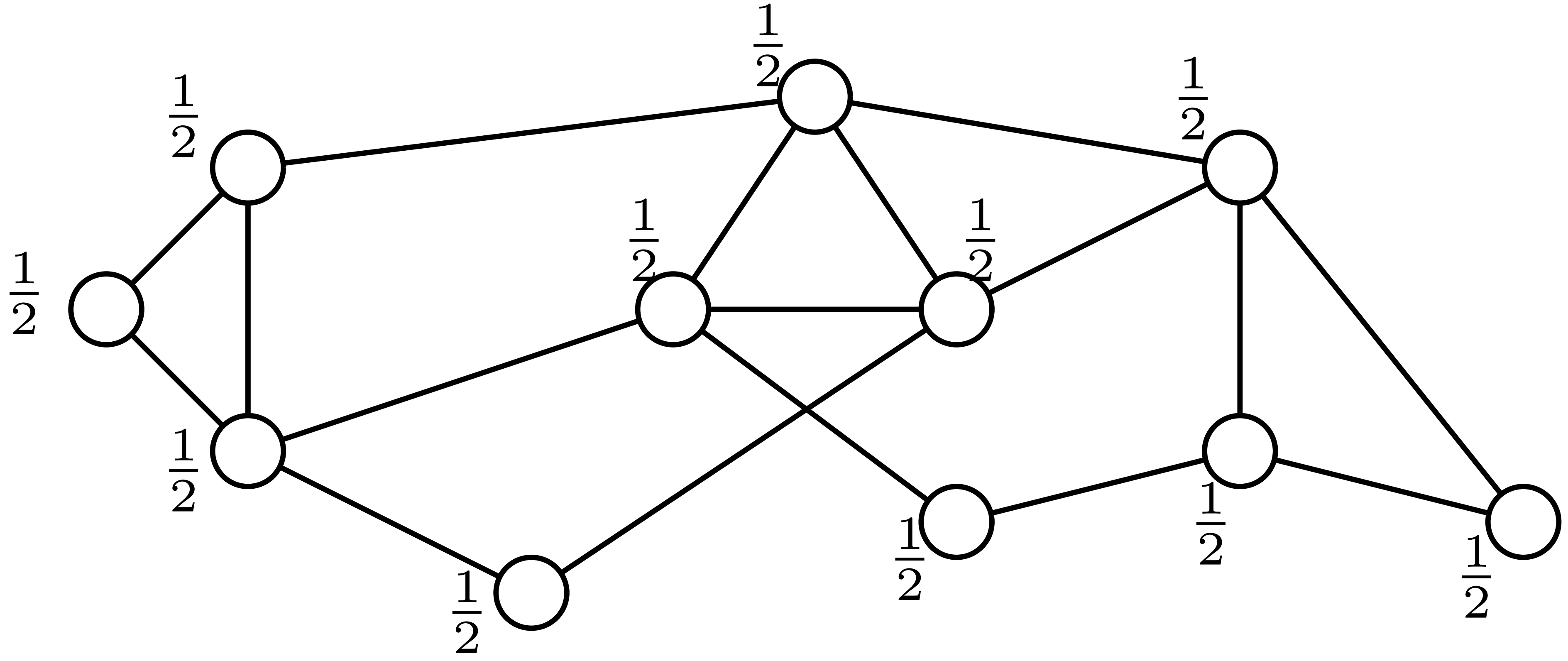

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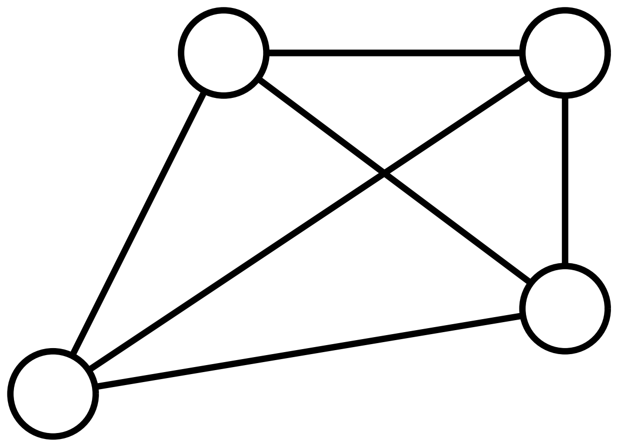
In a complete graph, what are the optimum LP and IP solution?



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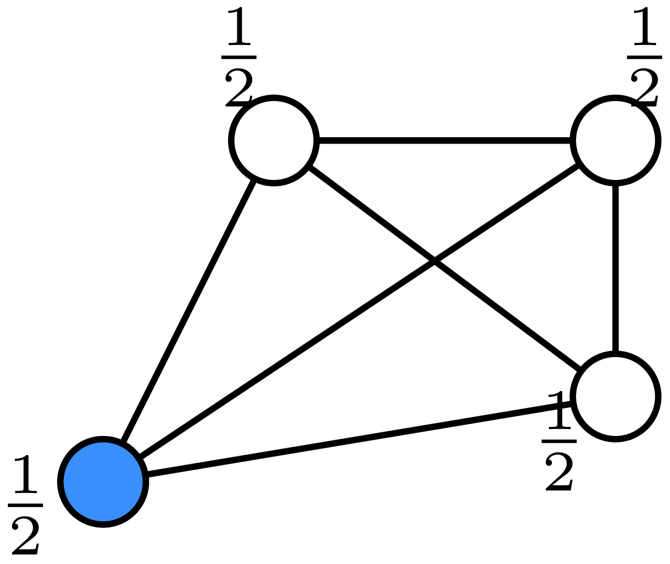

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Approximation for this problem is known to be hard:

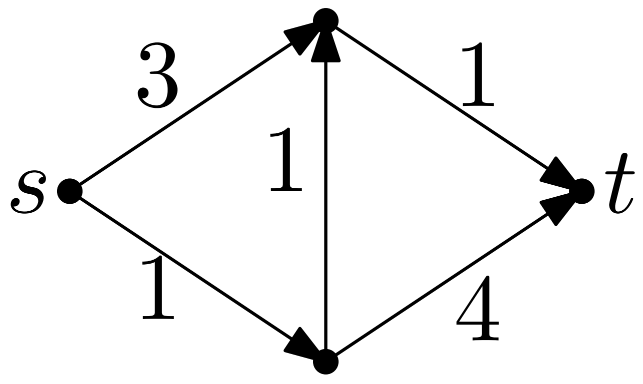
J. Håstad: Clique is hard to approximate within $n^{1-\epsilon}$,
Acta Math. 182(1999): 105-142

Bonus Examples

Shortest Path and TSP

Shortest Path Problem

Given a directed graph $G = (V, E)$ with edge weights w, we are looking for a shortest path from s to t.



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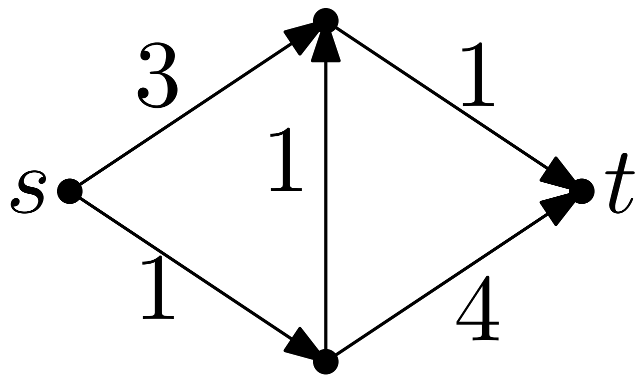

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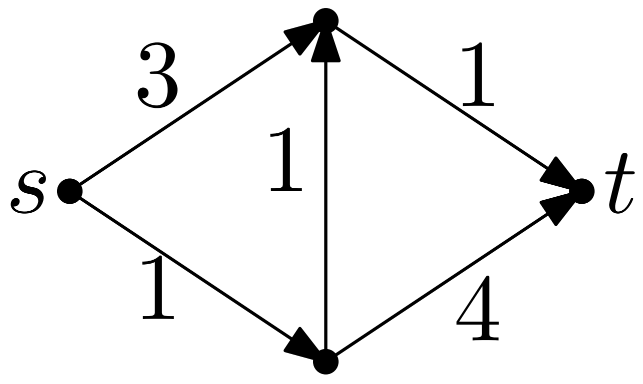

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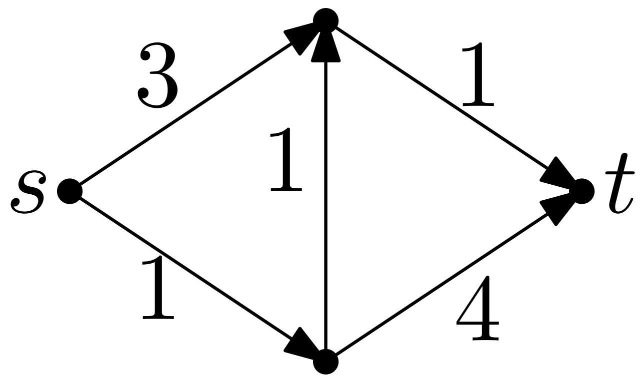

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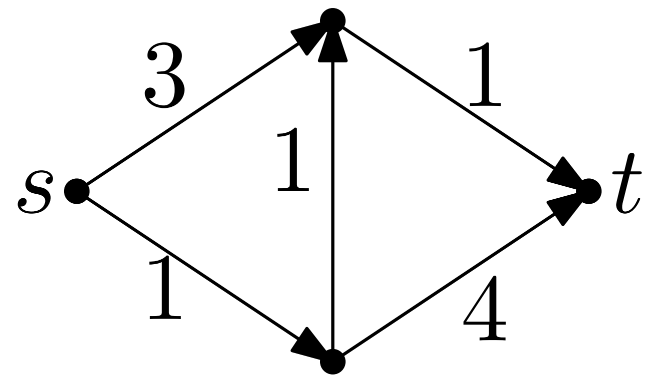

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LP-relaxation has $\{0, 1\}$-solution, like MinWeight Perfect Matching

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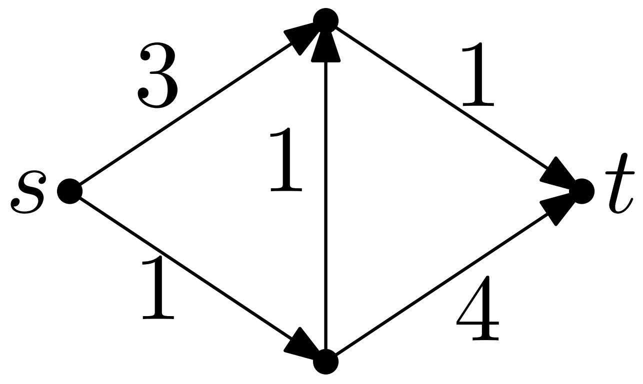

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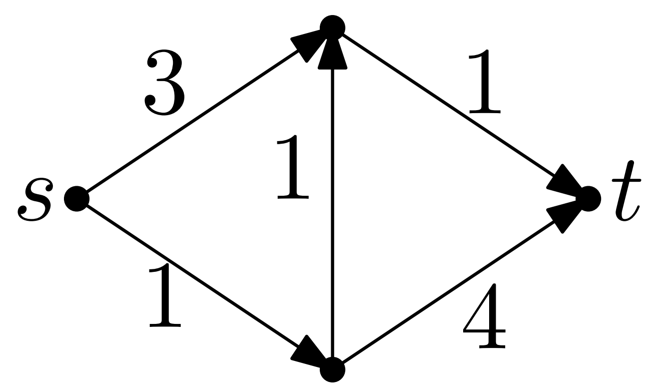

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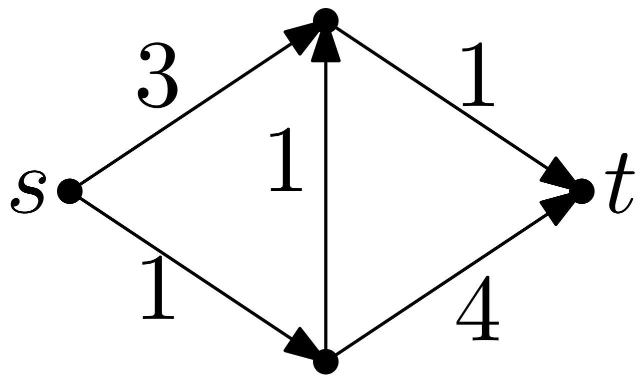

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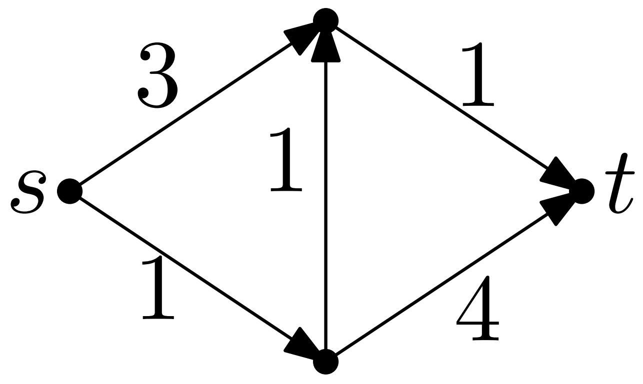

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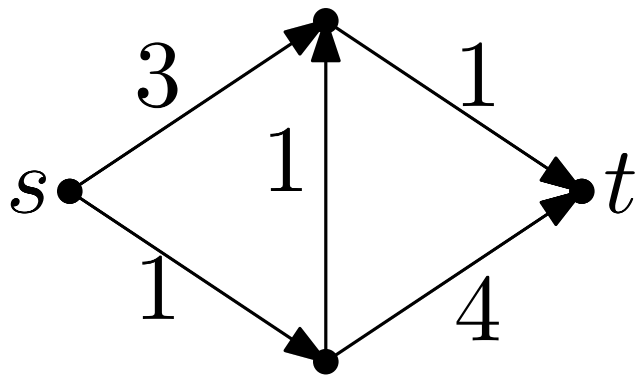

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simulating a $\min(\cdot)$ by $\leq$ and maximize

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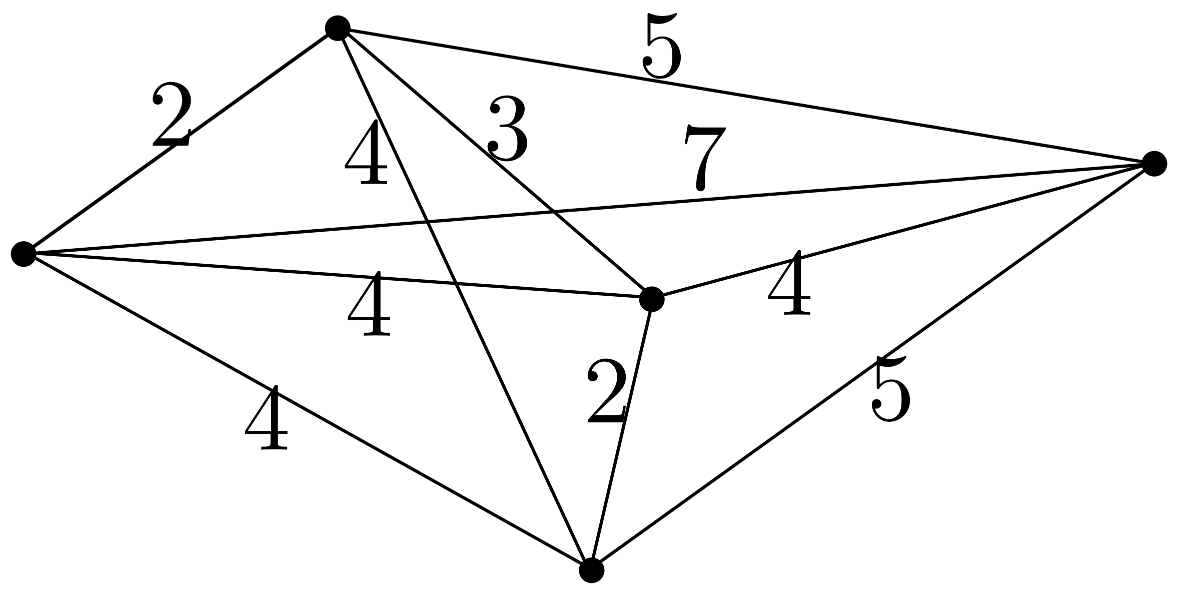
Traveling Salesperson Tour

Given an undirected complete graph $G = (V, E)$ with edge weights $c\colon E \to \mathbb{R}$, find a shortest Hamilton circuit in G.



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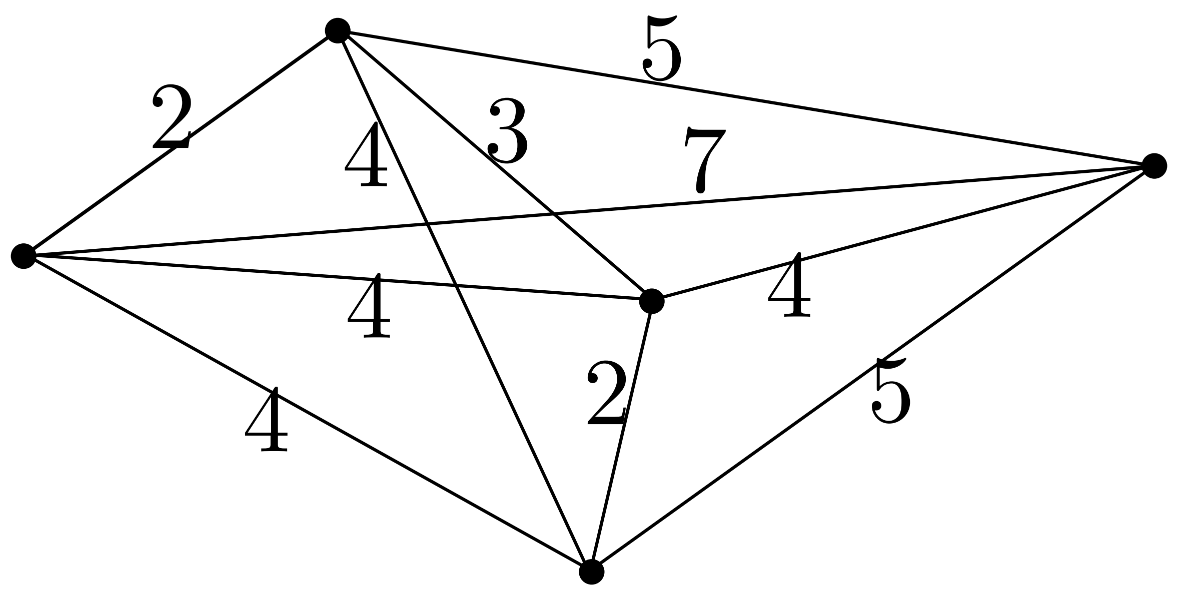

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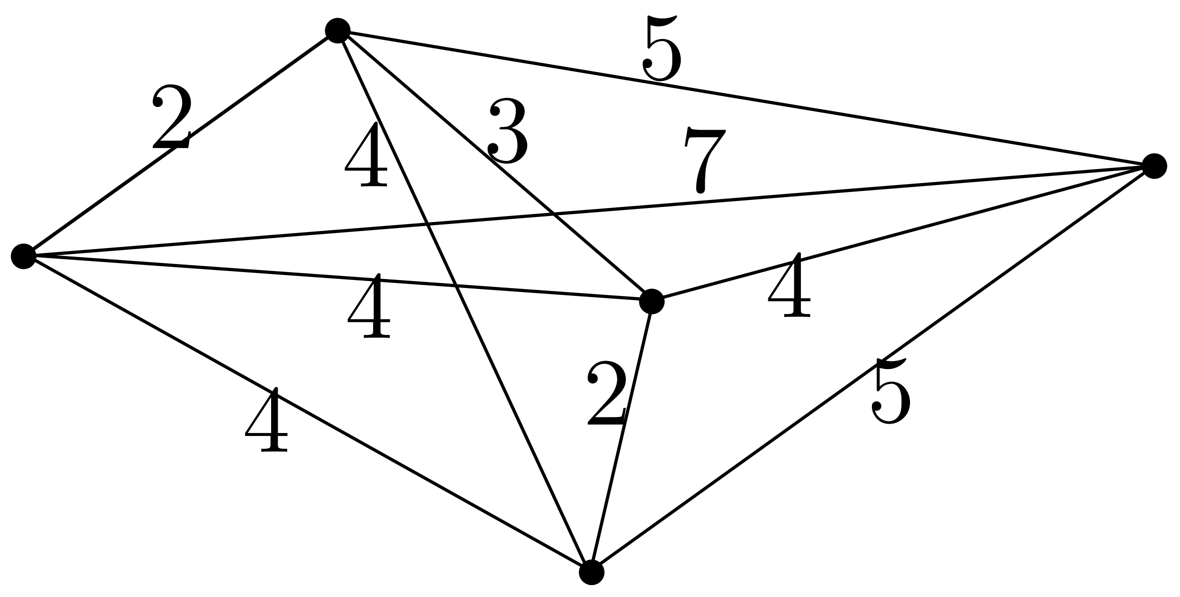

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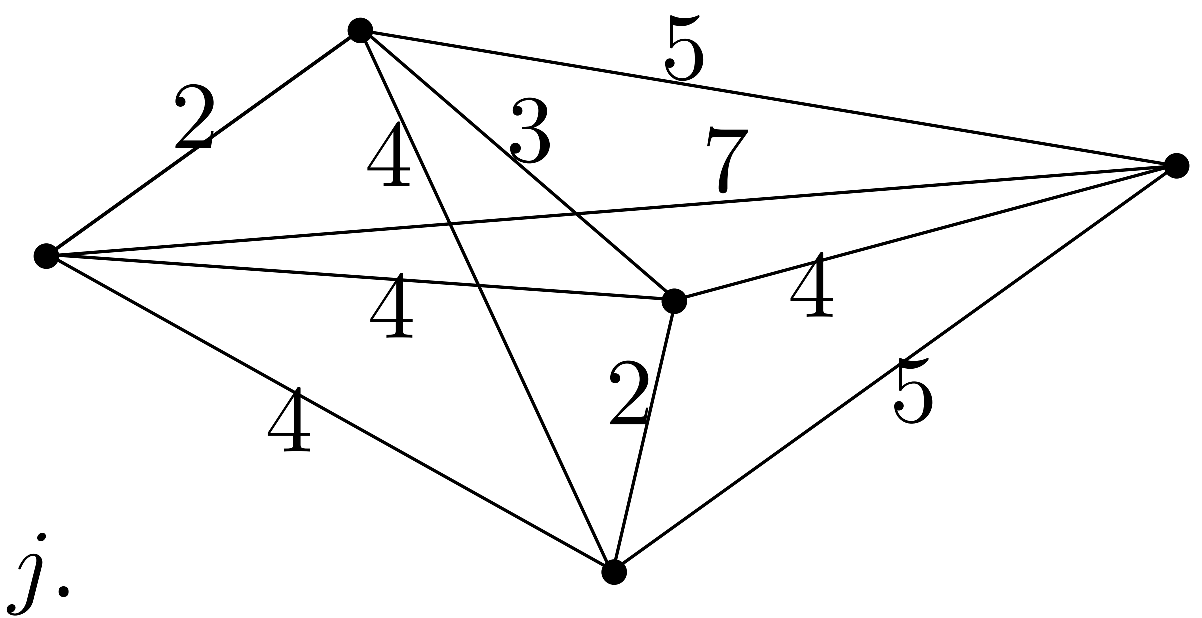

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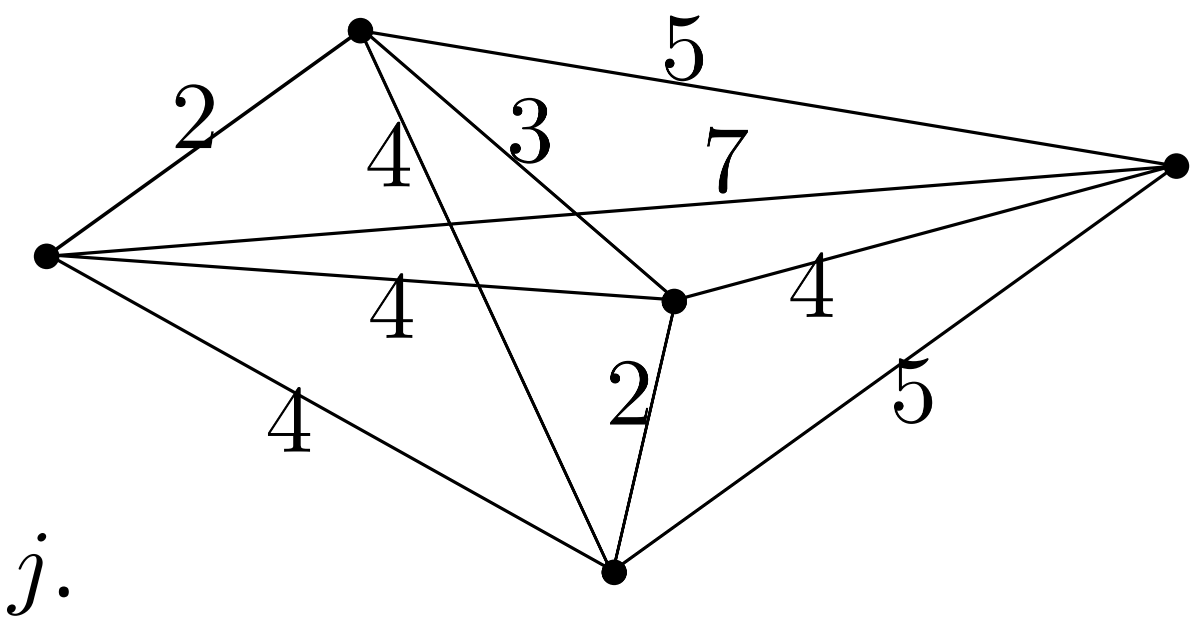

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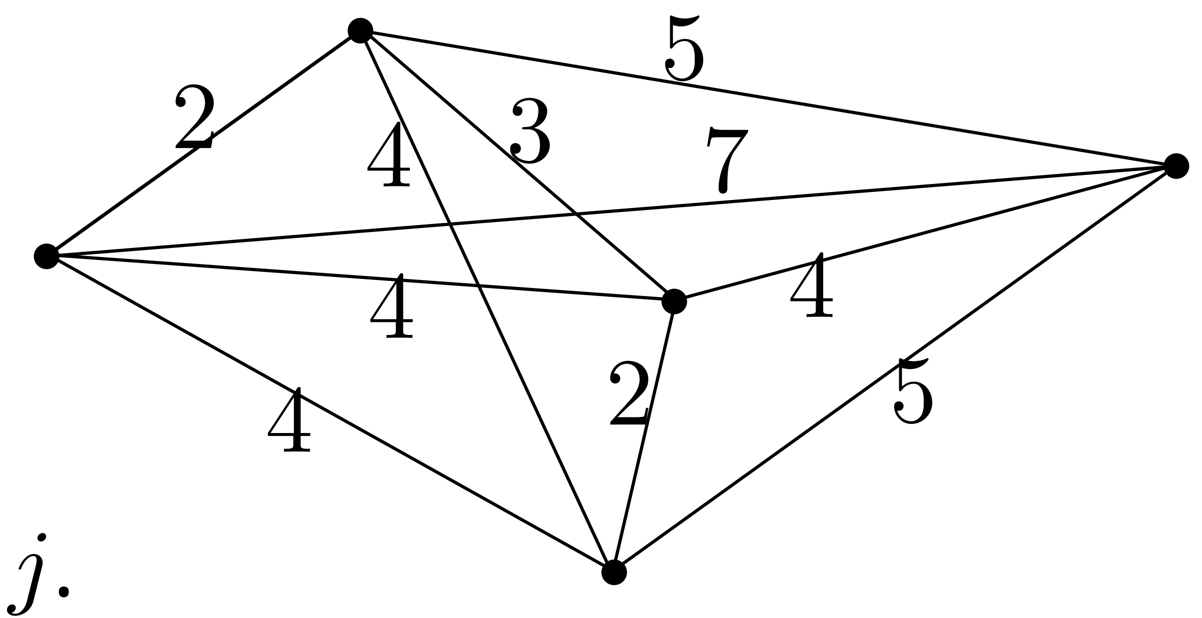

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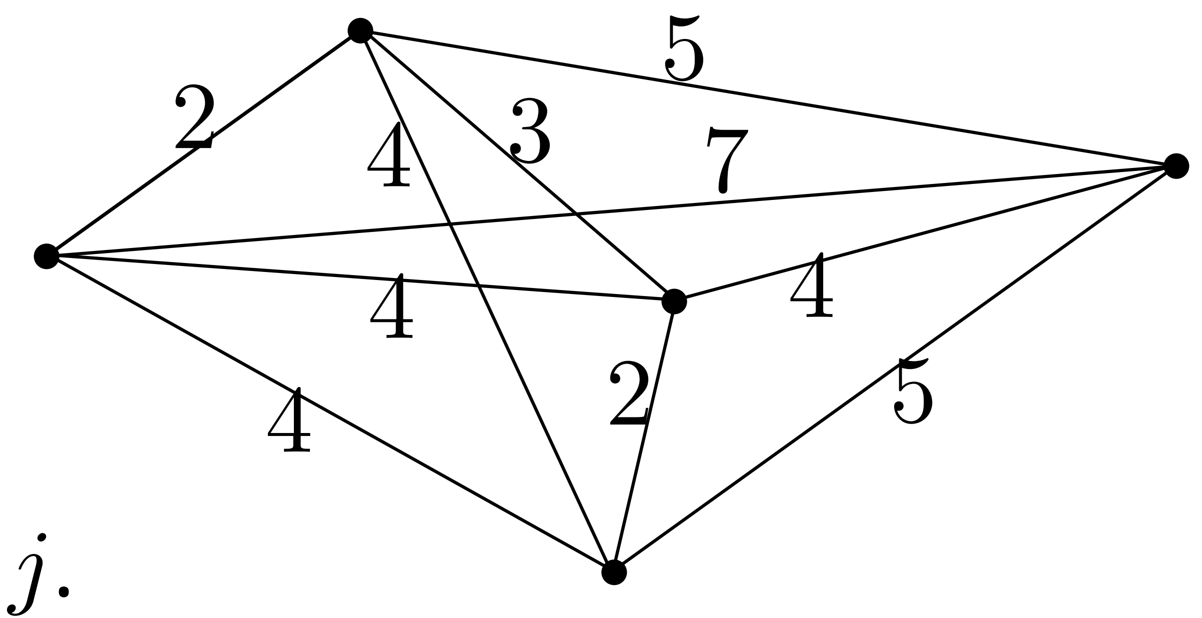

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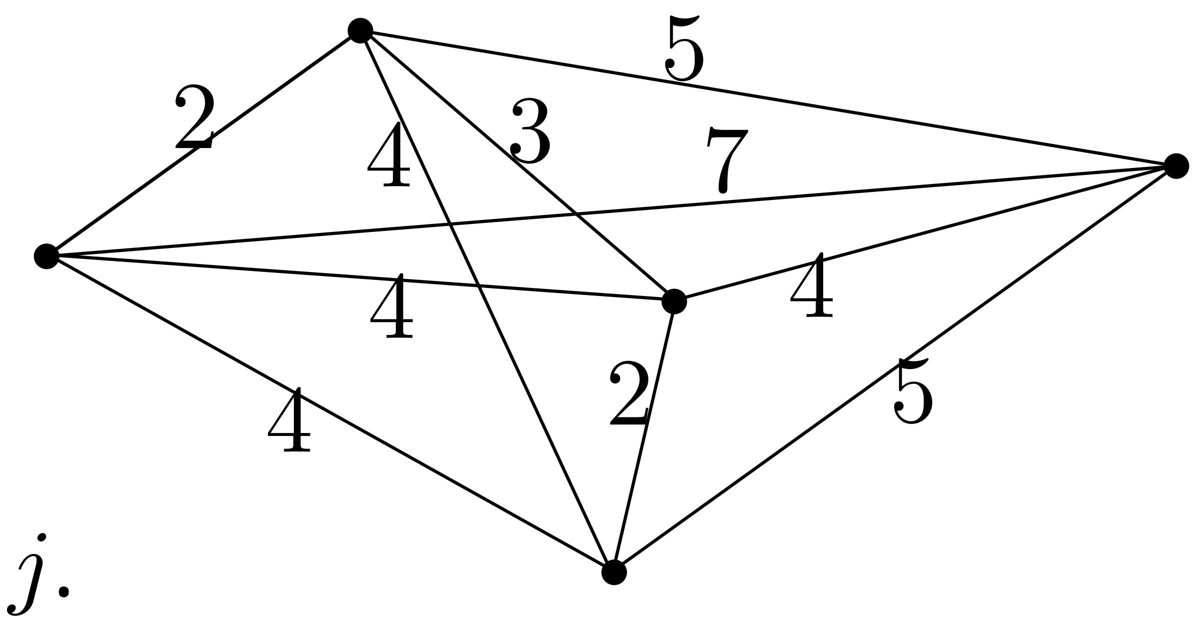

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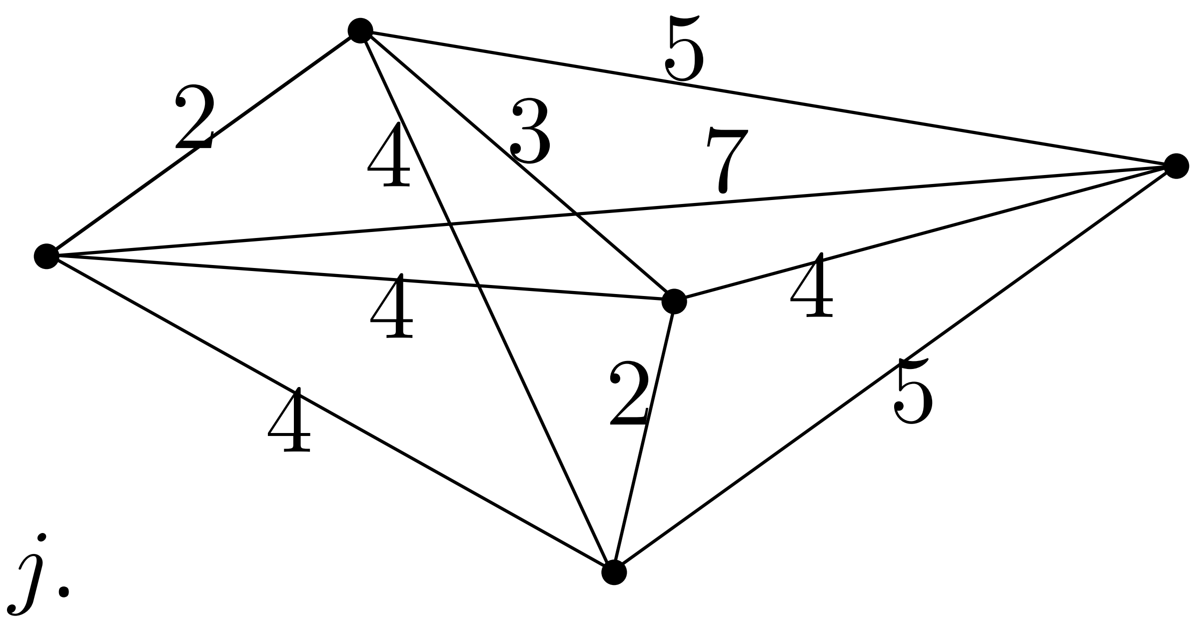

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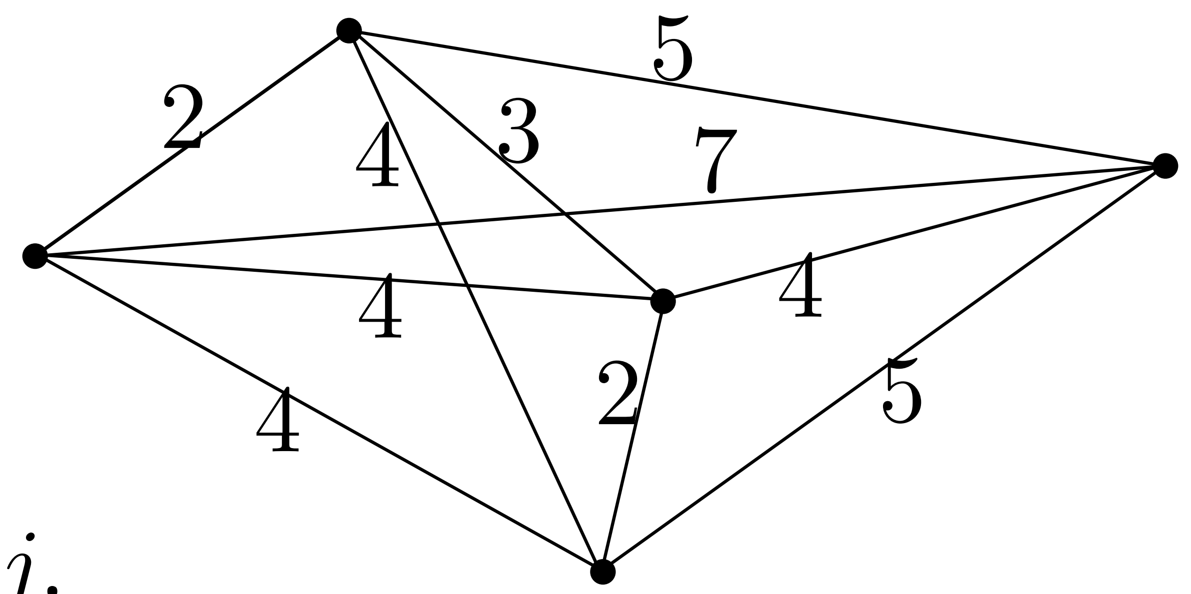

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by $u_j + (n-1) \ge u_i + n x_{ij}$
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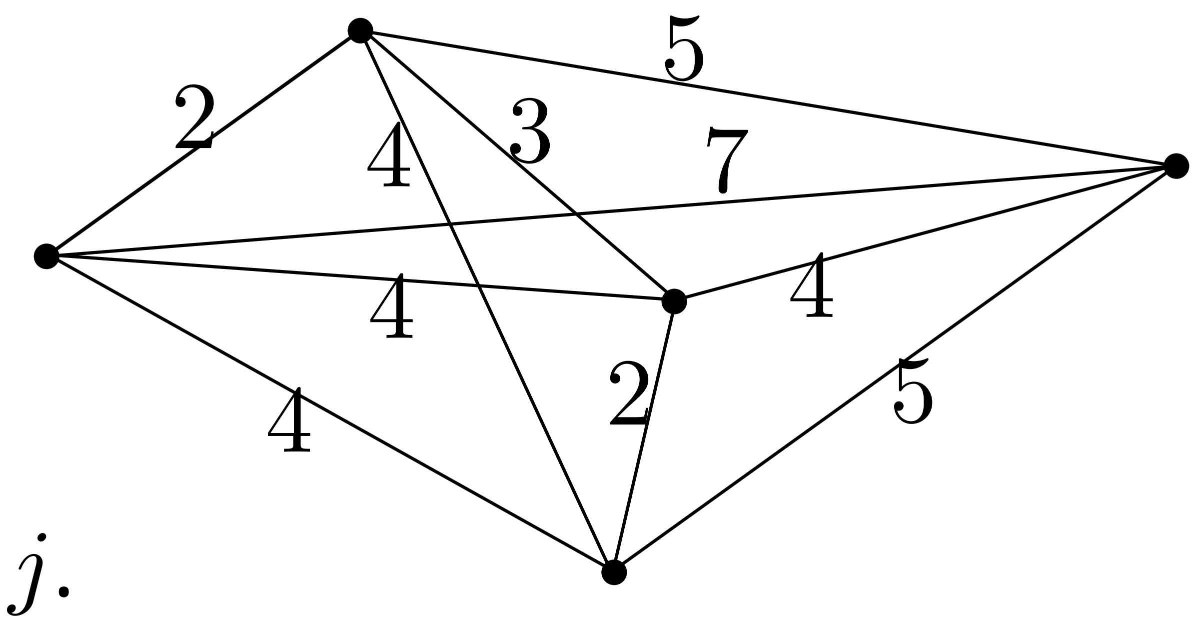

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Miller-Tucker-Zemlin formulation

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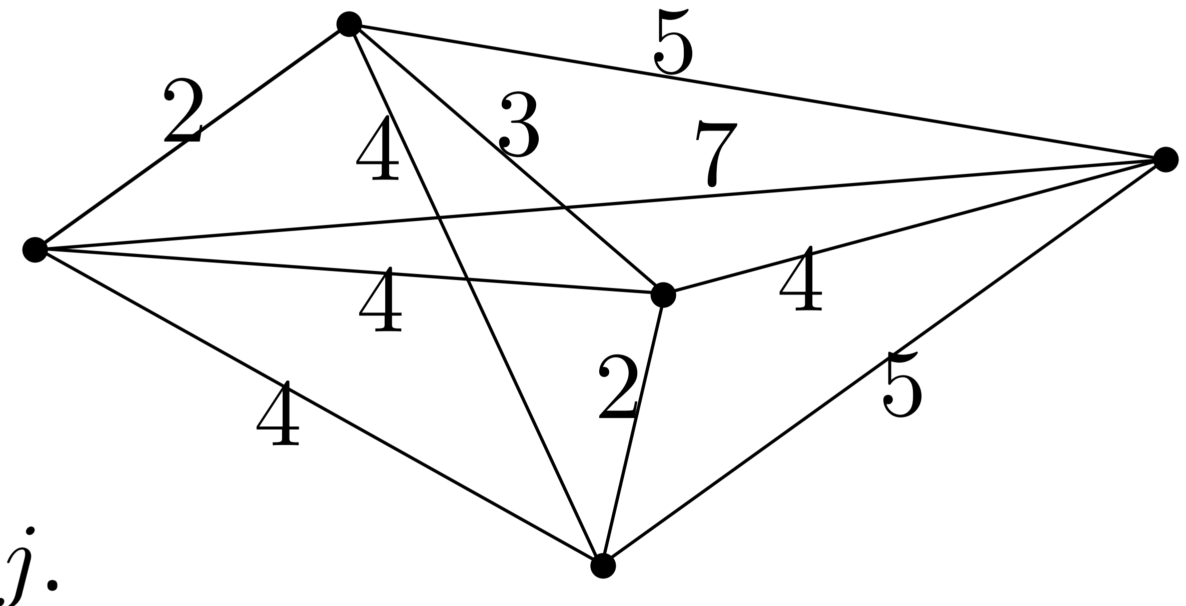

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$$\sum_{i\neq j\,:\,i,j\in Q} x_{ij} \le |Q| - 1$$

$$\forall Q \subsetneq \{1, \dots, n\}, |Q| \ge 2$$

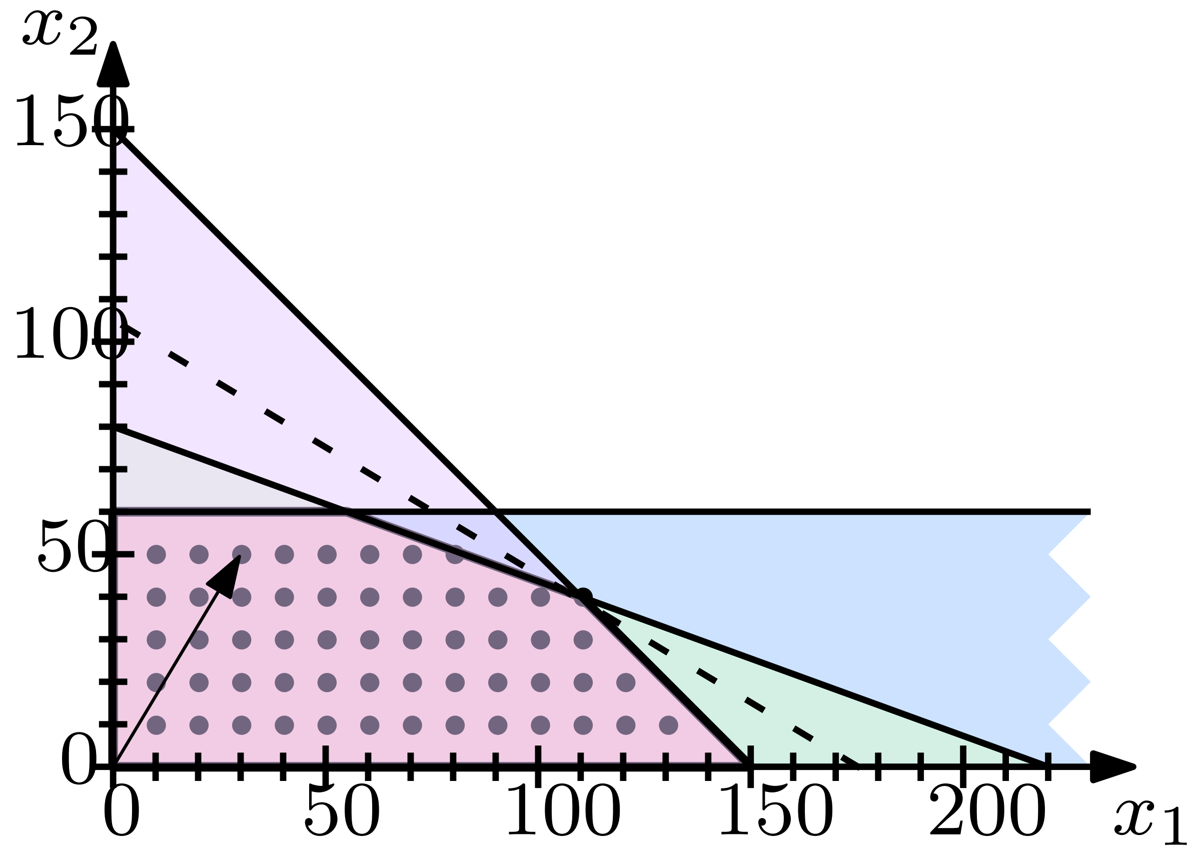
Miller-Tucker-Zemlin formulation

alternative:

Dantzig–Fulkerson–Johnson formulation with subtour-elimination constraint

Solving Integer Linear Programs

How can we solve an ILP?



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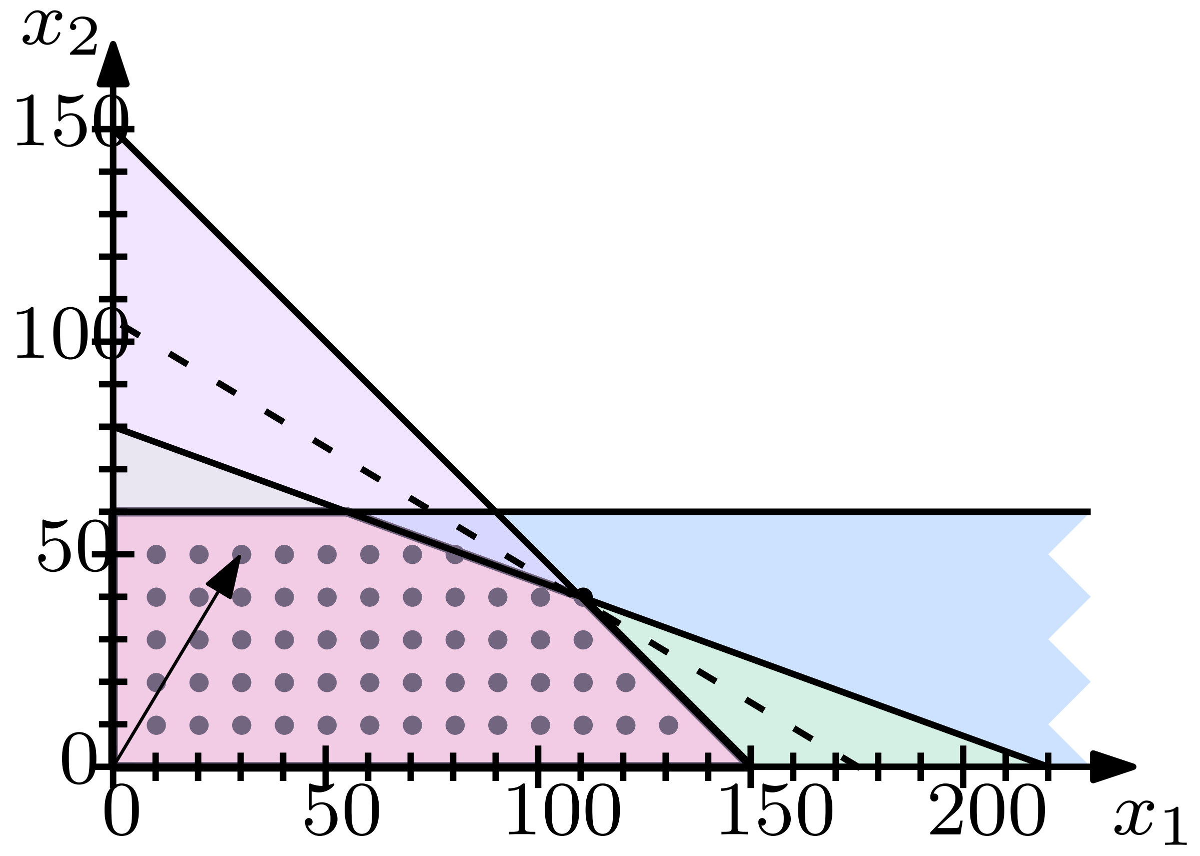

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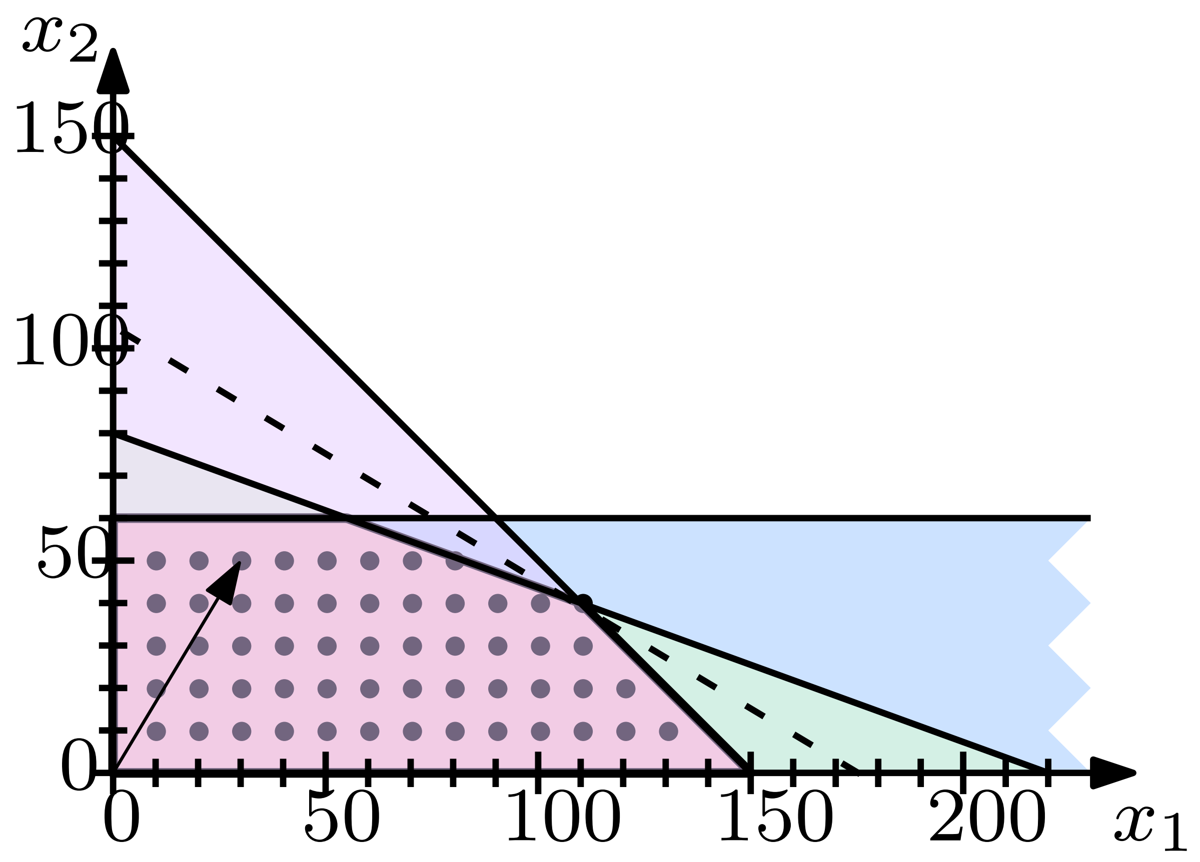
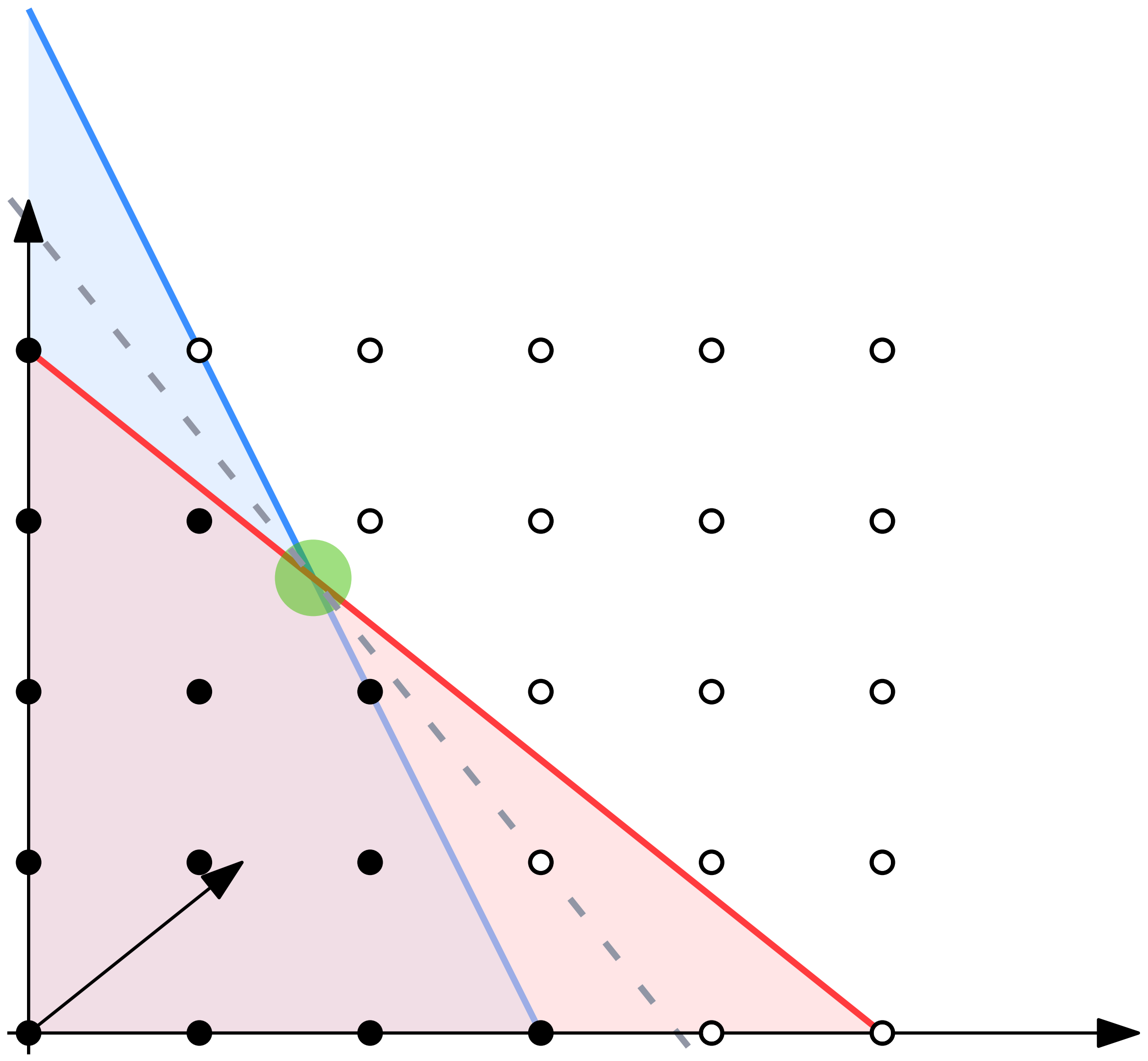
Techniques

- Branch-and-Bound
- Cutting Planes
- Branch-and-Cut



Branch and Bound for Solving ILP

Idea: branch: decompose in two subproblems
bound: discard if possible subproblems



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Sketch

- solve relaxed problem
- if solution non integer, choose variable x_i with non-integer value α_i and split into

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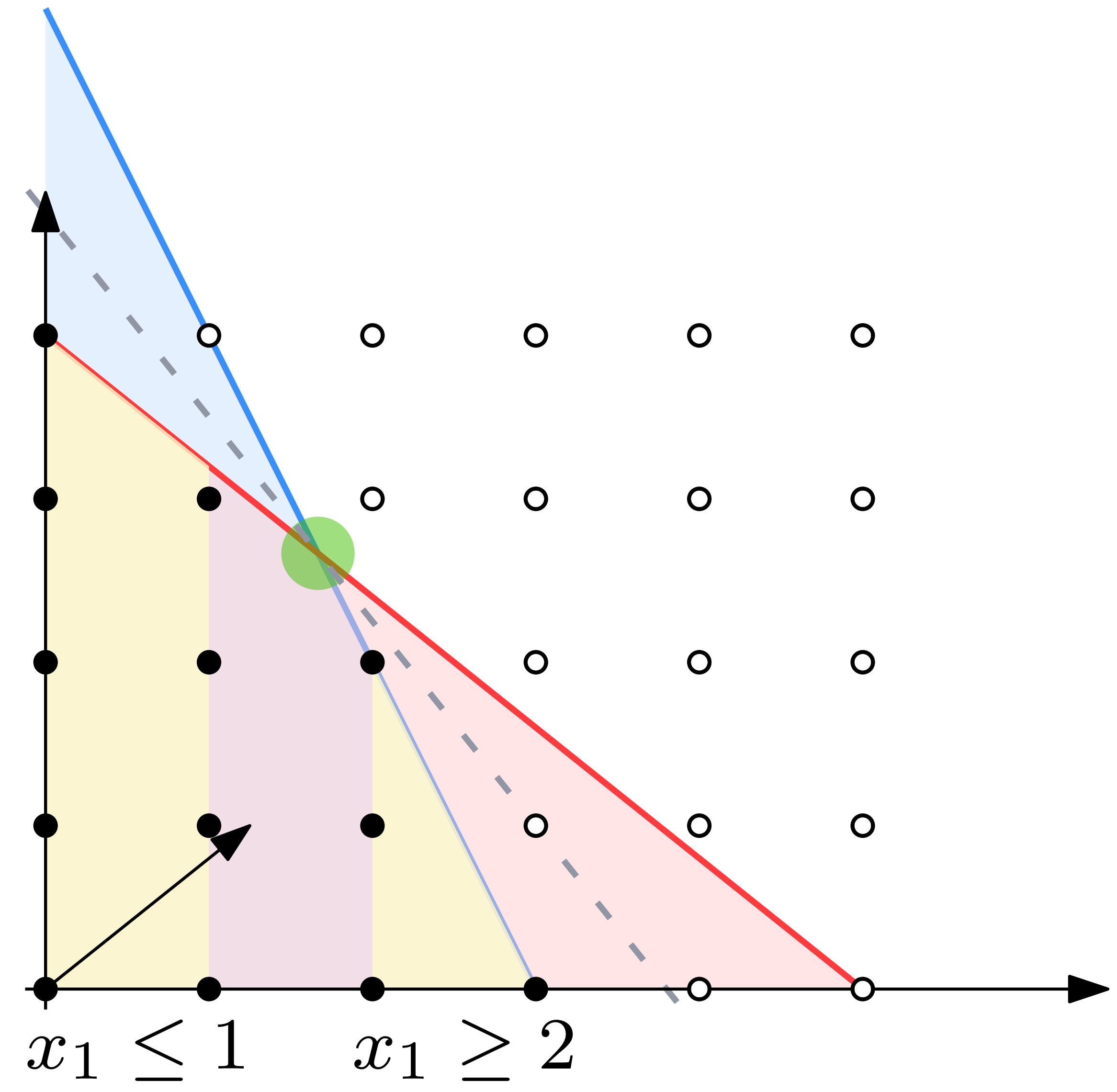

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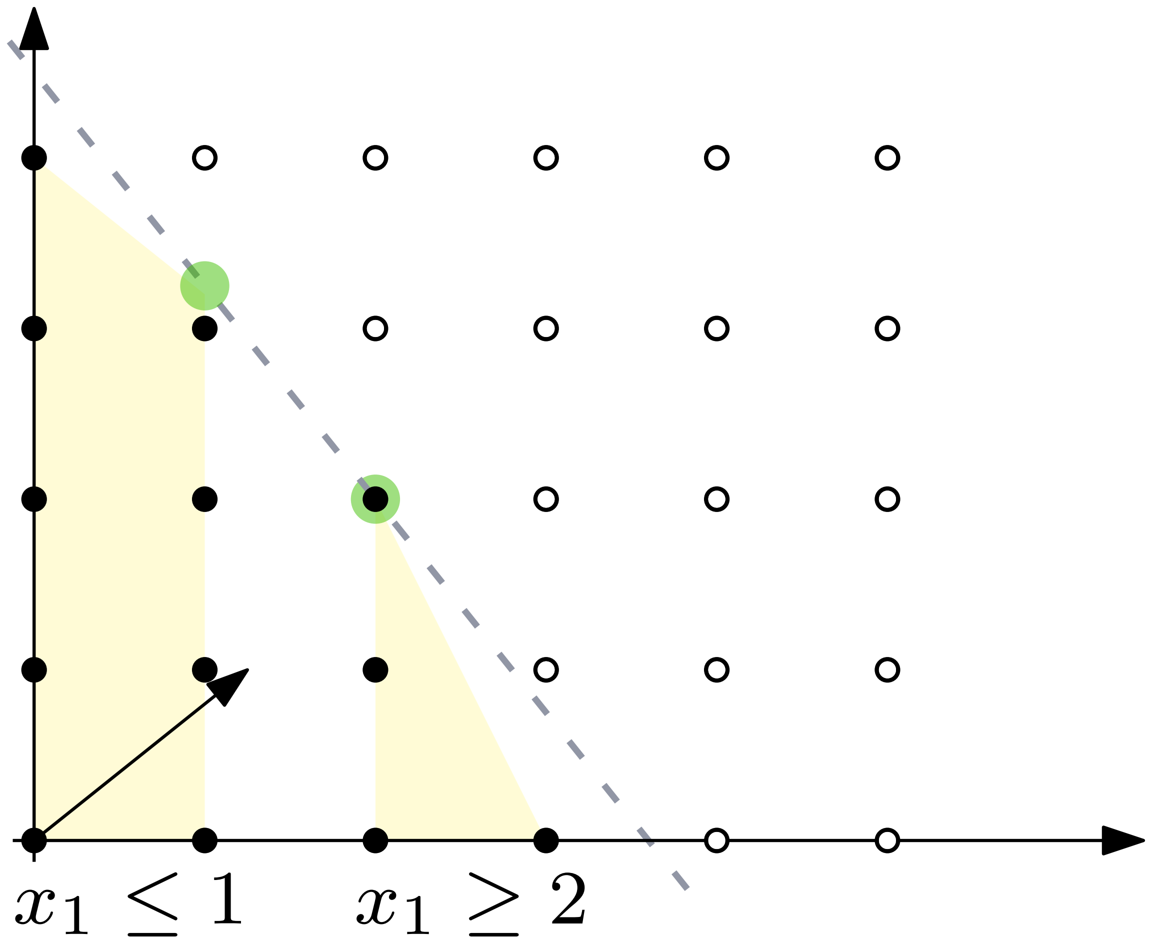

Branch and Bound for Solving ILP

Idea: branch: decompose in two subproblems
bound: discard if possible subproblems

Sketch

- solve relaxed problem
- if solution non integer, choose variable x_i with non-integer value α_i and split into

problem P_1: P with $x_i \leq \lfloor \alpha_i \rfloor$

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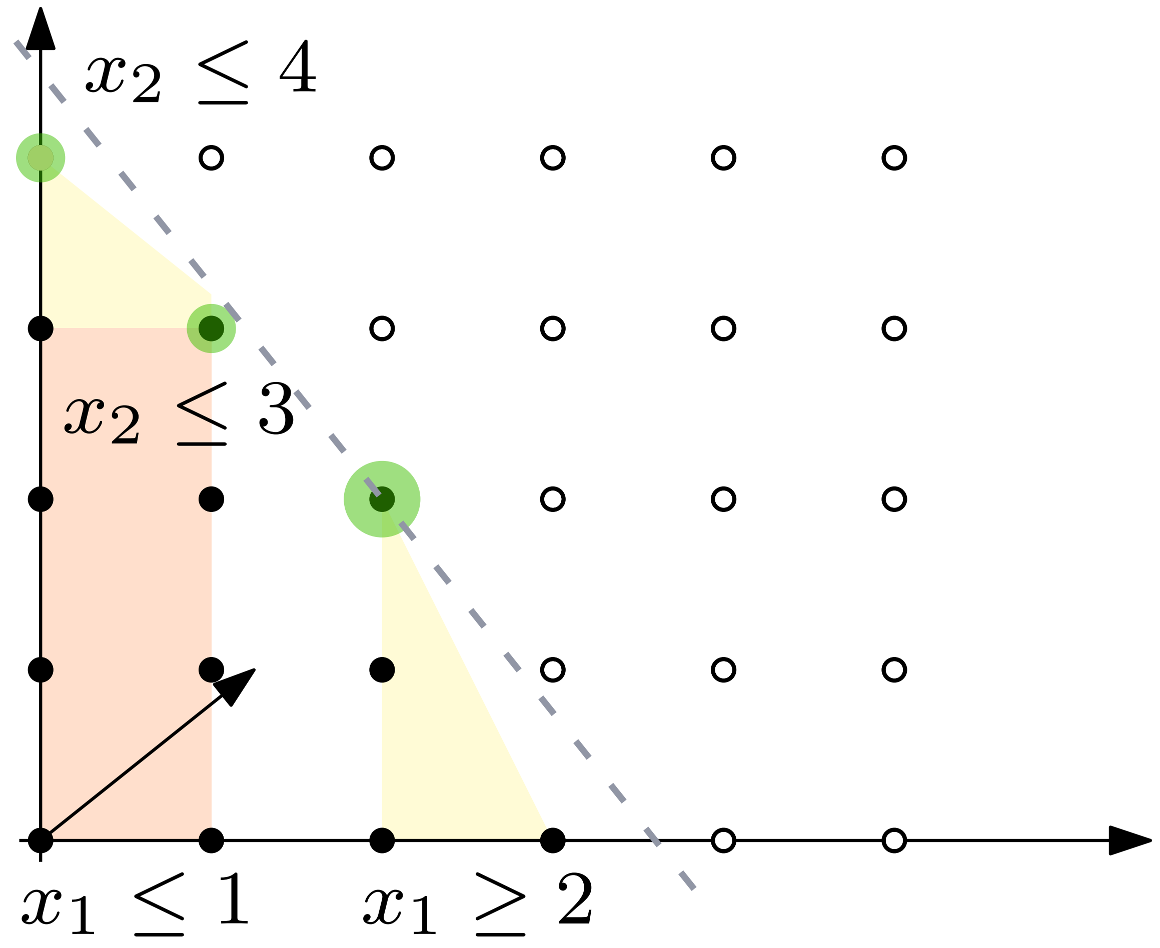

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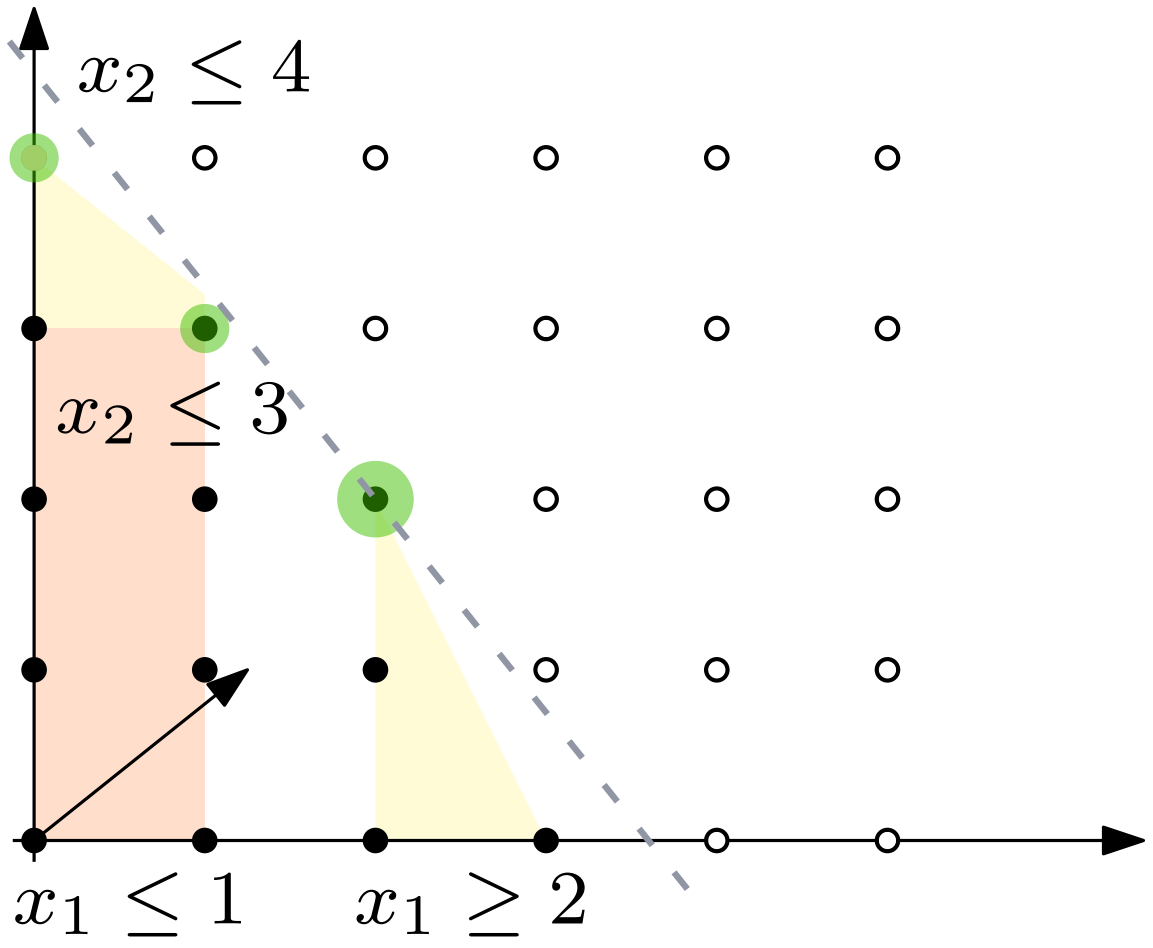

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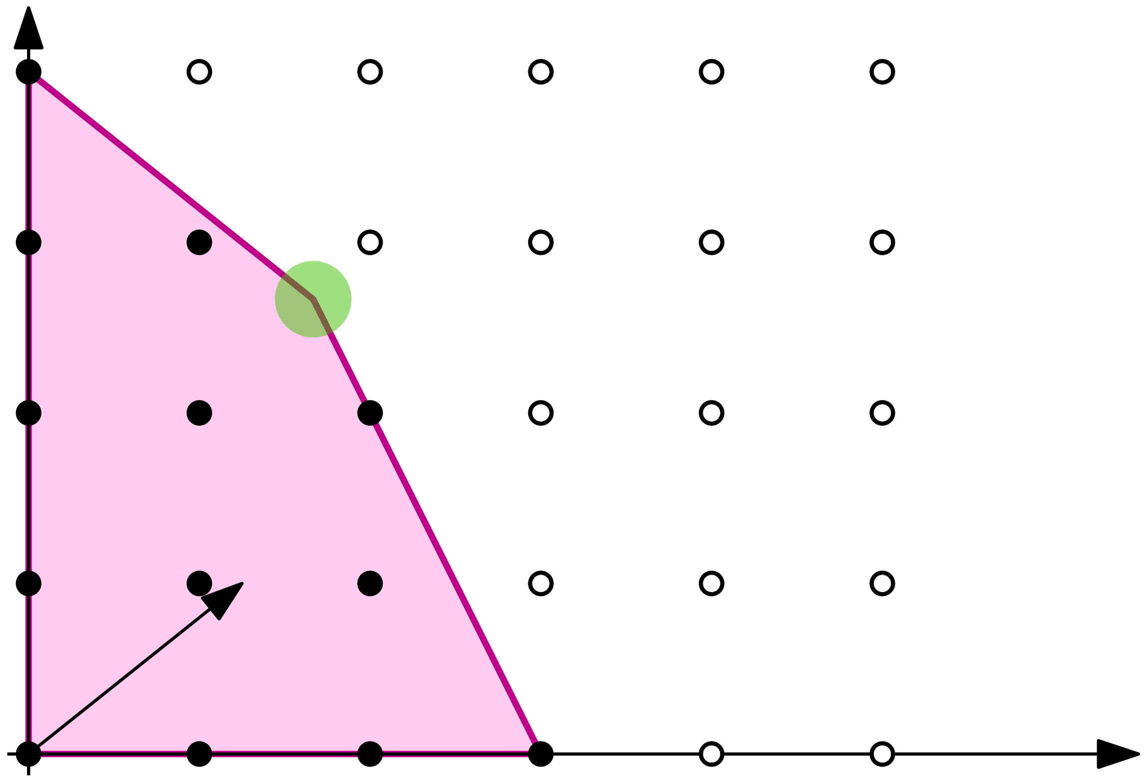
three possibilities for subproblems:

- LP infeasible $\rightarrow$ discard branch
- solution integer $\rightarrow$ update OPT and discard branch
- solution non integer $\rightarrow$ stop if solution worse than OPT, else continue branching

Cutting Planes for Solving ILPs

Idea: solve relaxed LP and cut off non-integer solutions found



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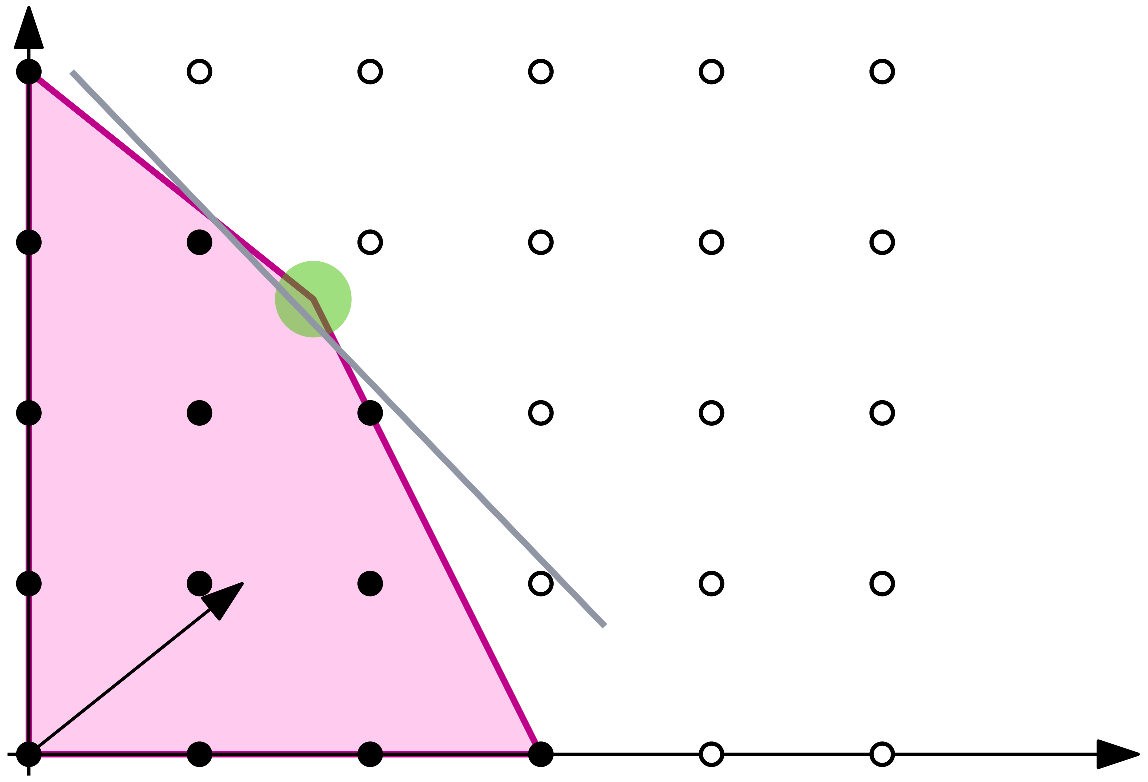

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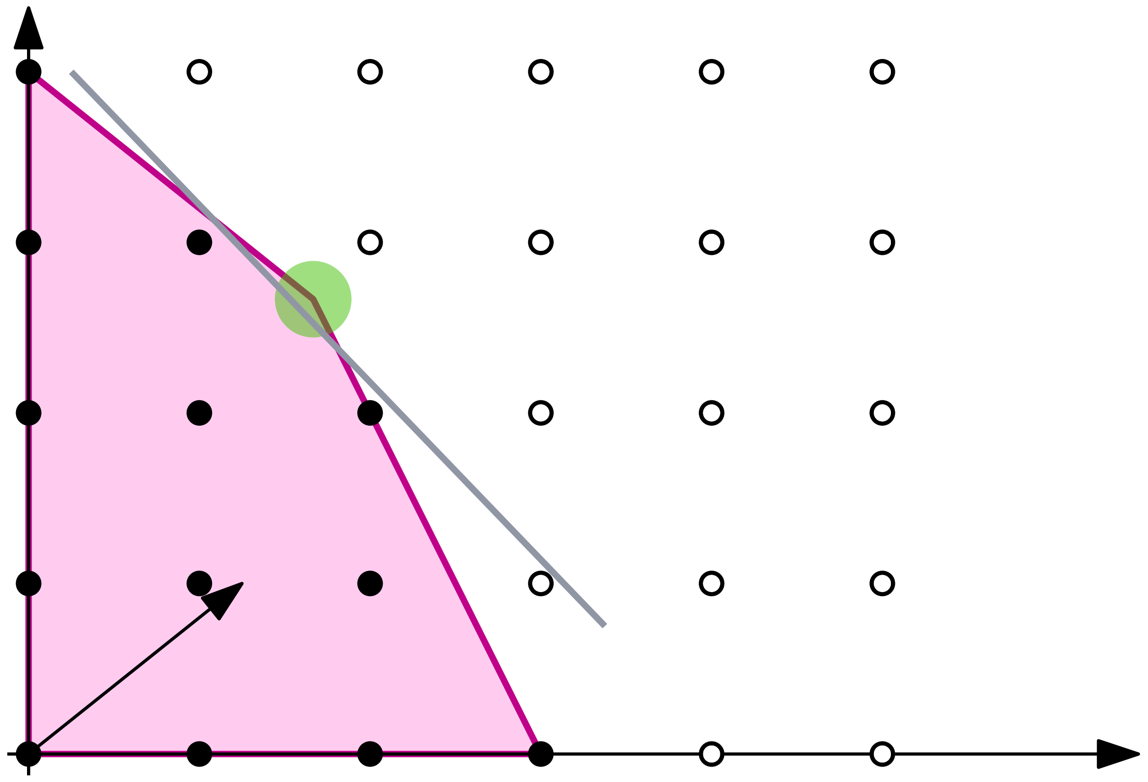

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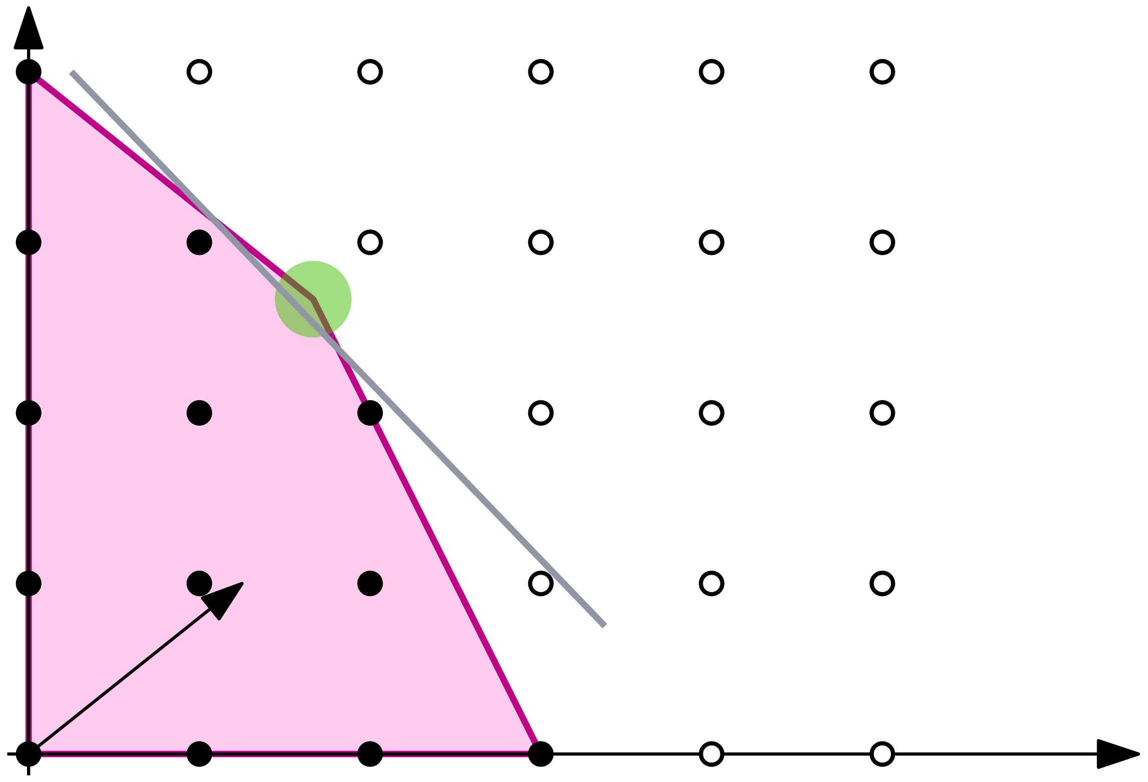

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general-purpose cuts, e.g. Gomory Cuts
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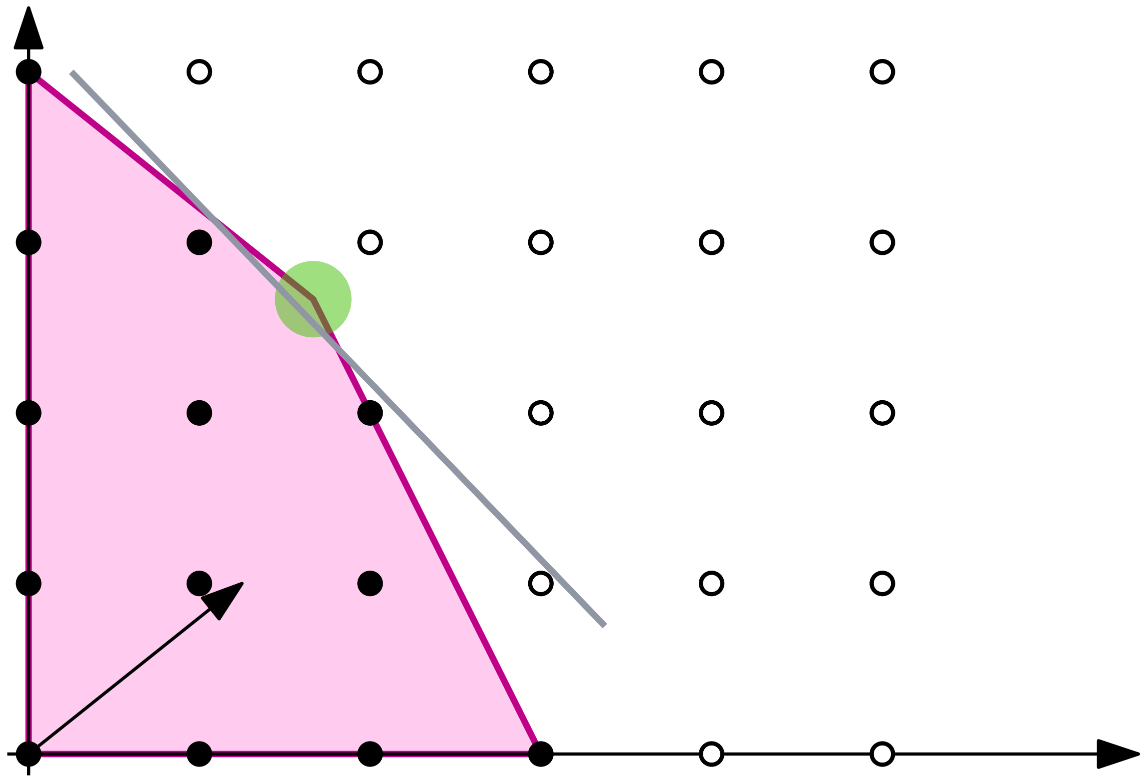

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Even better is to combine both techniques:

Branch & Cut

Summary

An integer linear programm (IP) is of the form

$$\begin{aligned} \text{maximize } & c^T x \\ \text{subject to } & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Solving IPs in general is NP-hard

We formulated as IP

- Maximum weight matching
- Minimum vertex cover
- Maximum independent set
- Shortest path
- Traveling Salesperson Tour

Techniques: branch&bound, cutting planes, branch&cut

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Next steps:

- basics for solving LPs
- simplex algorithm