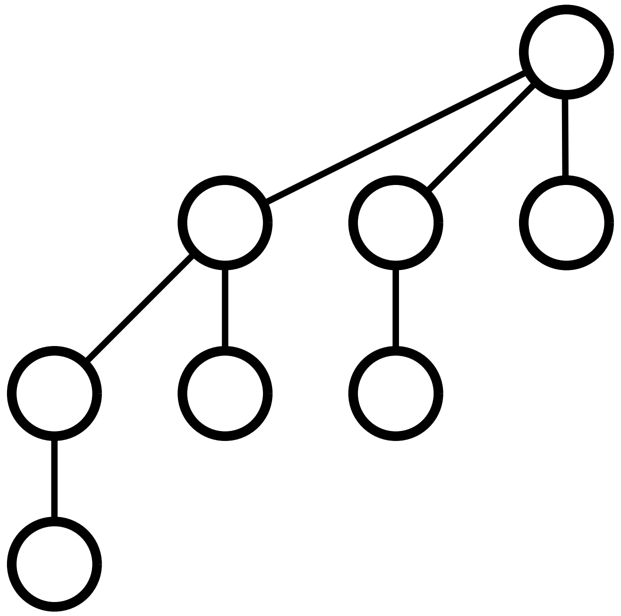
Binomial Heaps

Introduction and Worst-Case Analysis

Amortized Analysis of Insert

Amortized Analysis of Lazy-Union



Priority Queues

Abstract data typ: manage a set of elements with keys (their priority) under the operations *insert, minimum, deleteMinimum*, and optionally *decreaseKey, remove* and *merge*.

Implementations:

	deleteMin	decreaseKey	insert	build
Binary heaps	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(n)$
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Runtime for Dijkstras Algorithm: $O(m + n \log n)$ using Fibonacci heaps

Mergeable Priority Queues

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Goal today: Efficiently mergeable heaps

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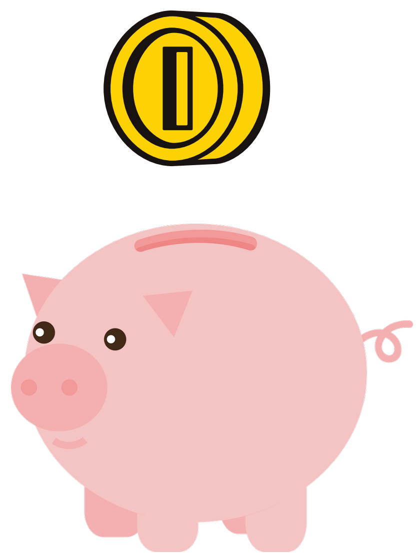
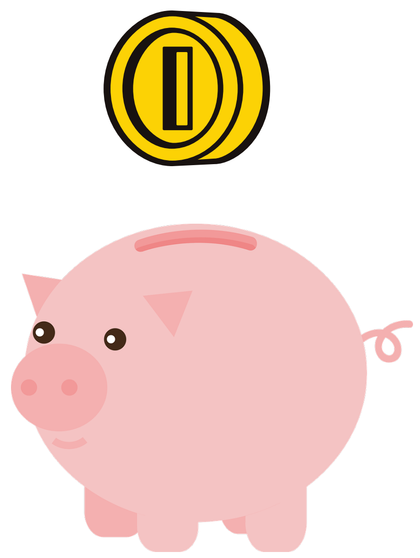
Short Recap: Amortized Analysis



Short Recap: Amortized Analysis

- $D_i =$ data structure after ith operation
- $c_i =$ actual cost of ith operation
- $\hat{c}_i =$ amortized cost of ith operation = coins



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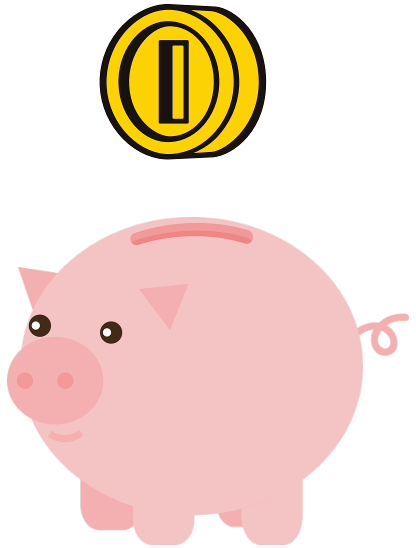

accounting method:

- assign amortized cost ("coins") to every operation
- check whether enough coins are saved to pay for all operations:

for all j : $\sum_{i=1}^{j} \hat{c}_i - \sum_{i=1}^{j} c_i \geq 0$ (coins saved in step i: $\hat{c}_i - c_i$)

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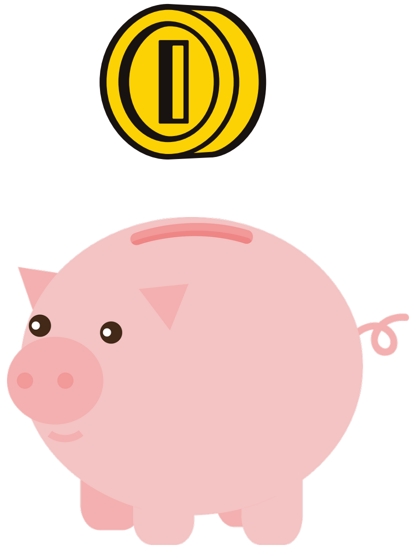

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potential method:

- say how many "coins" are saved with data structure D_i: $\Phi(D_i)$ ("potential")
- calculate amortized costs from potential:

$\hat{c}_i = c_i$ + coins saved in step i $= c_i + \Phi(D_i) - \Phi(D_{i-1})$

$= c_i$ + Δ_i (change in potential)

Short Recap: Amortized Analysis (Binary Counter)

Algorithm. increment a k- bit binary counter

Representation as array. $A[j]$: jth least-significant bit

example: $k = 6$

value	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
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costs. number of bits flipped. How many for n increments?

Short Recap: Amortized Analysis (Accounting Method)

actual cost per operation: 1 coin per bit flipped

invariant: ???

accounting/amortized cost ($\hat{c}_i$): ???

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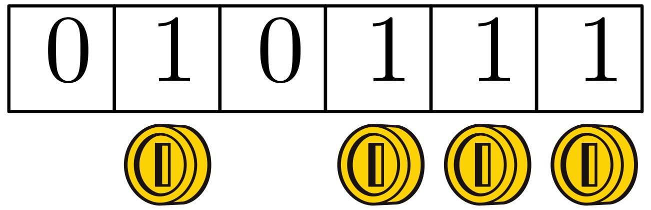

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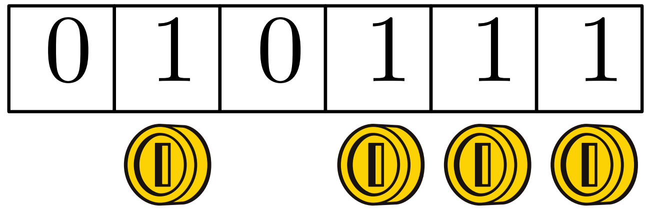

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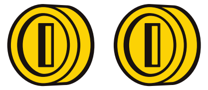
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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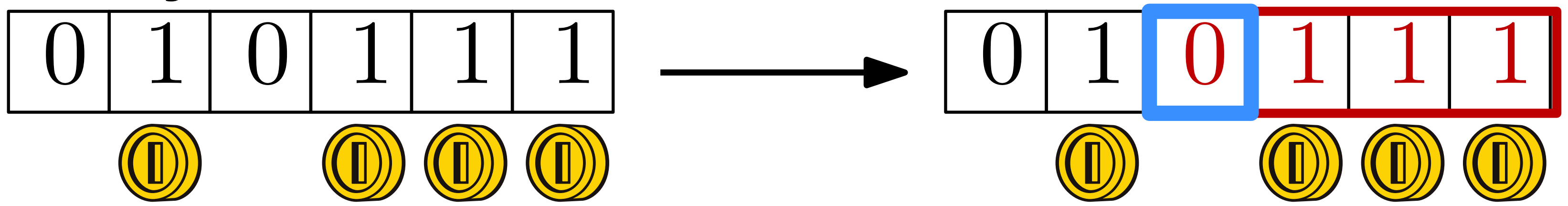

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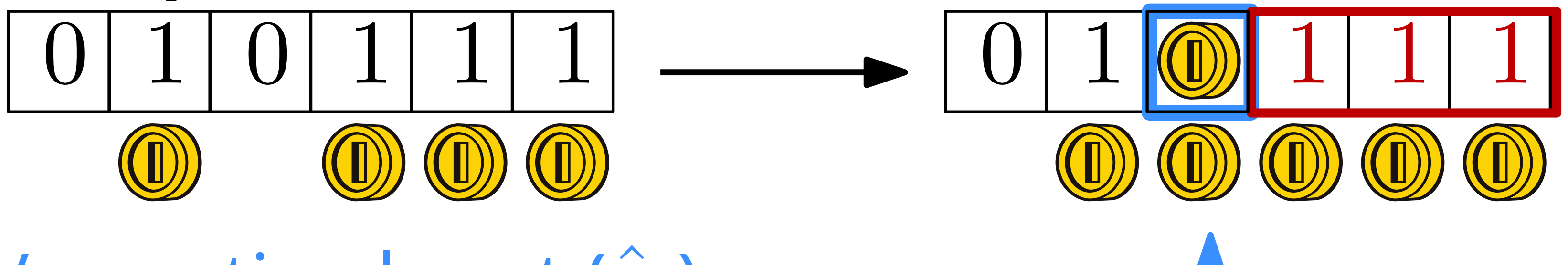

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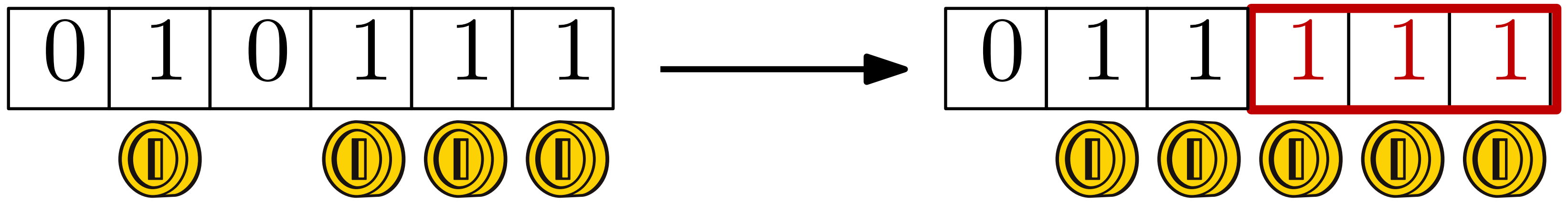

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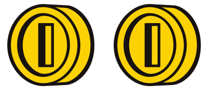
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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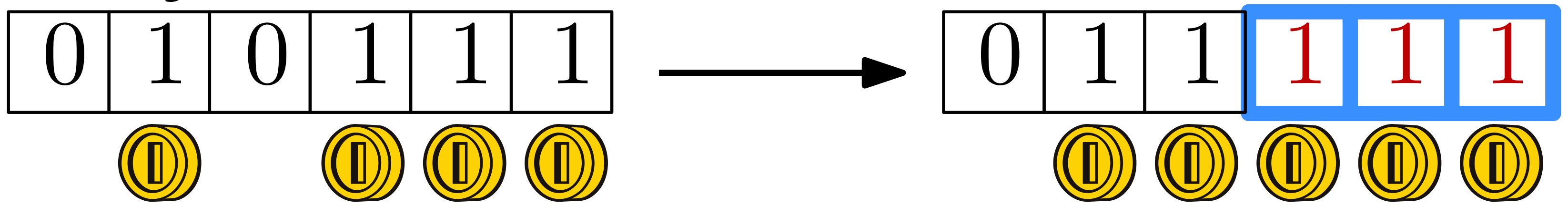

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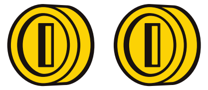
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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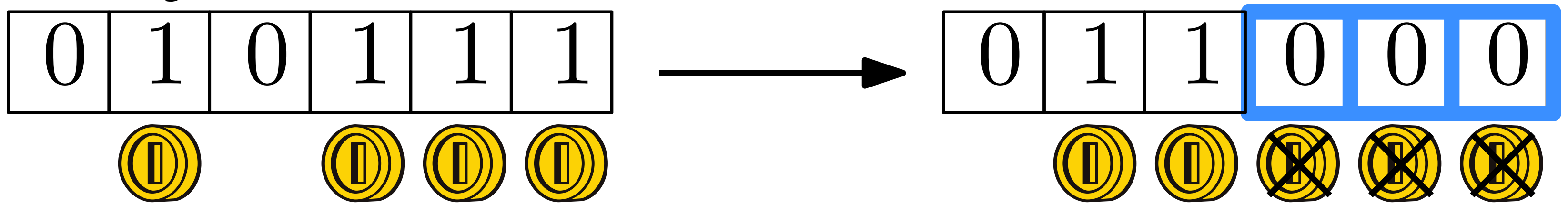

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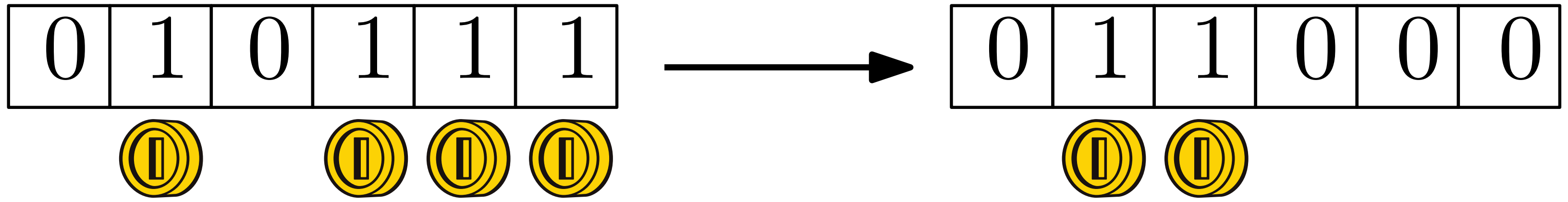

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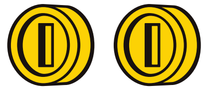
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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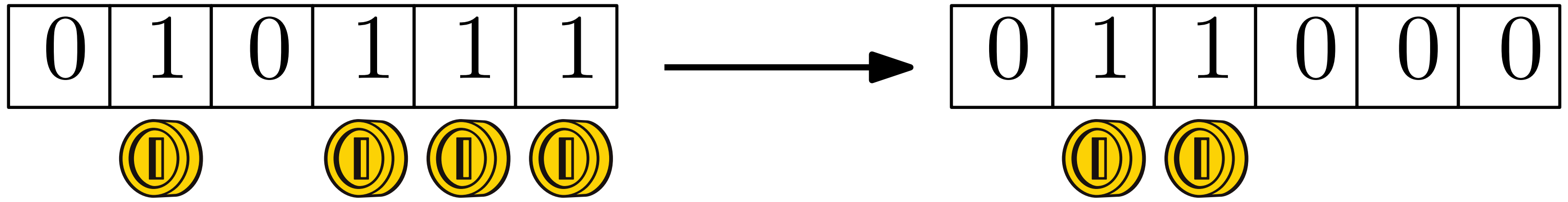

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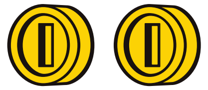
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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- invariant $\Rightarrow$ number of coins in data structure $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$
- amortized cost per operation $\hat{c}_i \leq 2$
- actual total costs $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \leq \sum_{i=1}^{n} 2 = 2n$

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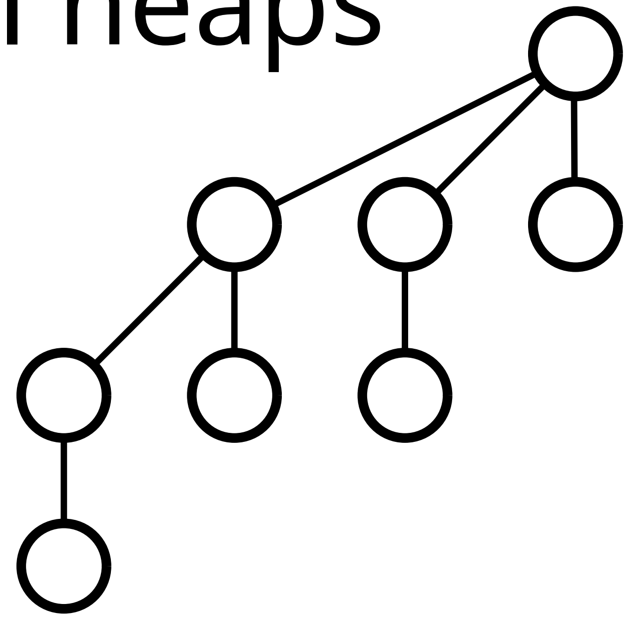
- c_i = number of bits flipped
  = number of bits flipped from 0 to 1 + number of bits flipped from 1 to 0

- change in potential: Δ_i = increase in number of 1s
  = number of bits flipped from 0 to 1 - number of bits flipped from 1 to 0
  = 1 $-$ number of bits flipped from 1 to 0

The worst-case running time of a sequence of n increments (starting from 0) is $O(n)$. The amortized running time of one increments is $O(\hat{c}_i) = O(1)$

Binomial trees and Binomial heaps

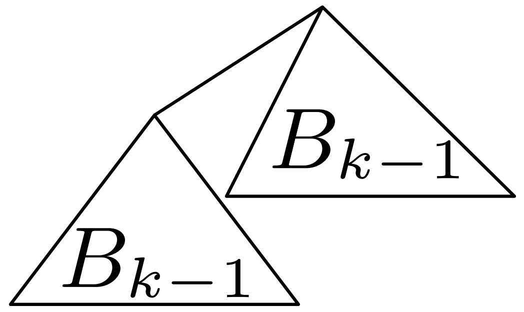


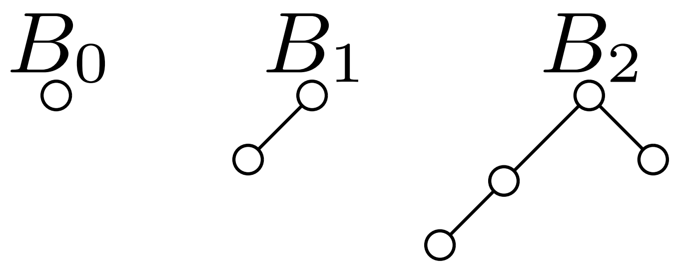
Binomial Trees

Binomial tree: recursively defined

B_0 single node

B_k two B_{k-1} joined by making one a child of the root of the other





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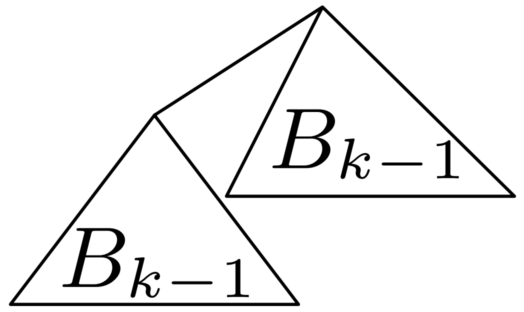

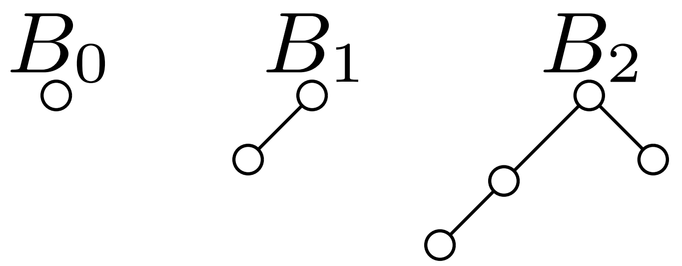

Question: What does B_3 look like?

Binomial Trees

Binomial tree: recursively defined

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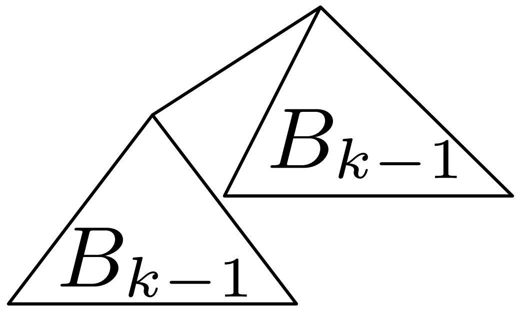
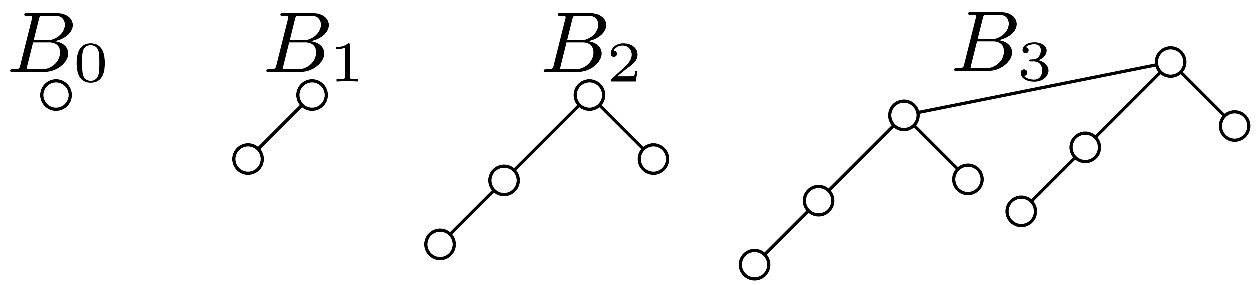
B_k two B_{k-1} joined by making one a child of the root of the other





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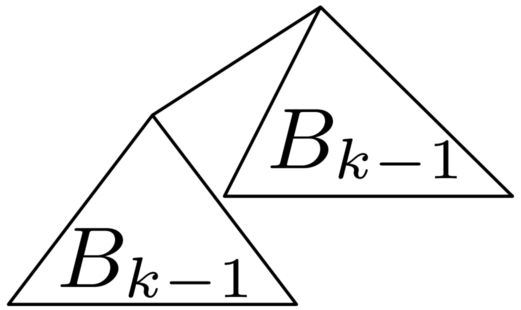

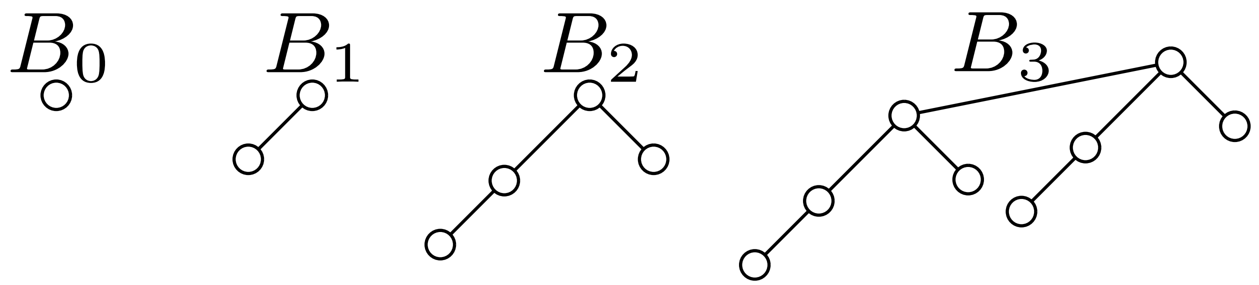

Lemma: A binomial tree B_k has

- 2^k nodes
- height k
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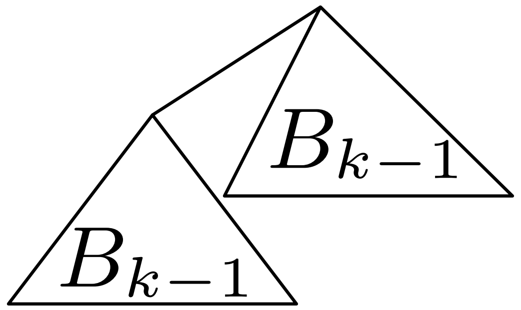

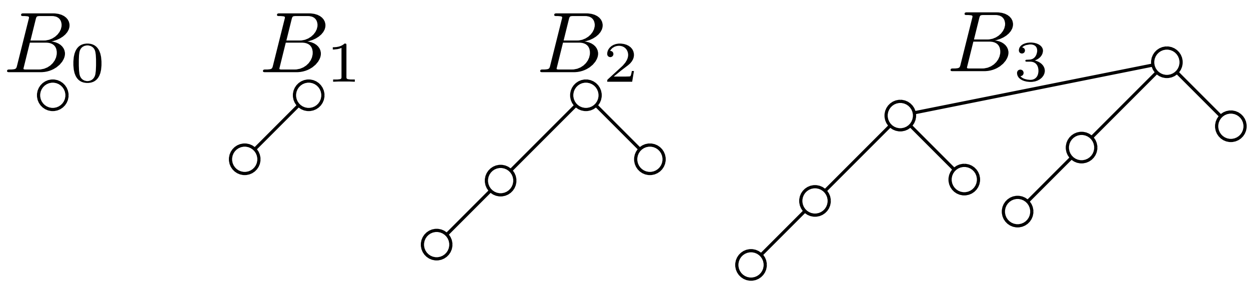

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Proof: Induction(k)

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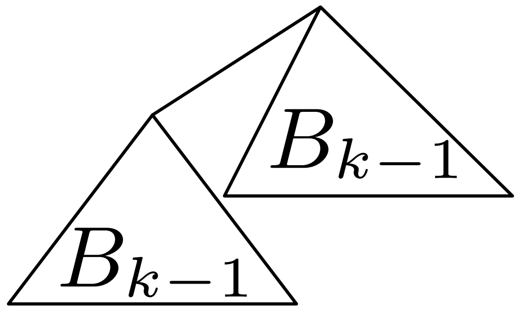

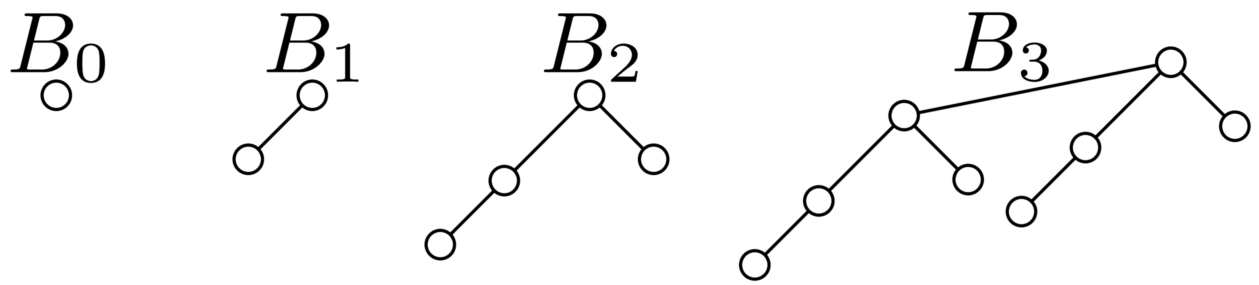

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Corollary: Maximal degree in a binomial tree on n nodes is $\log n$.

Binomial Trees – Properties

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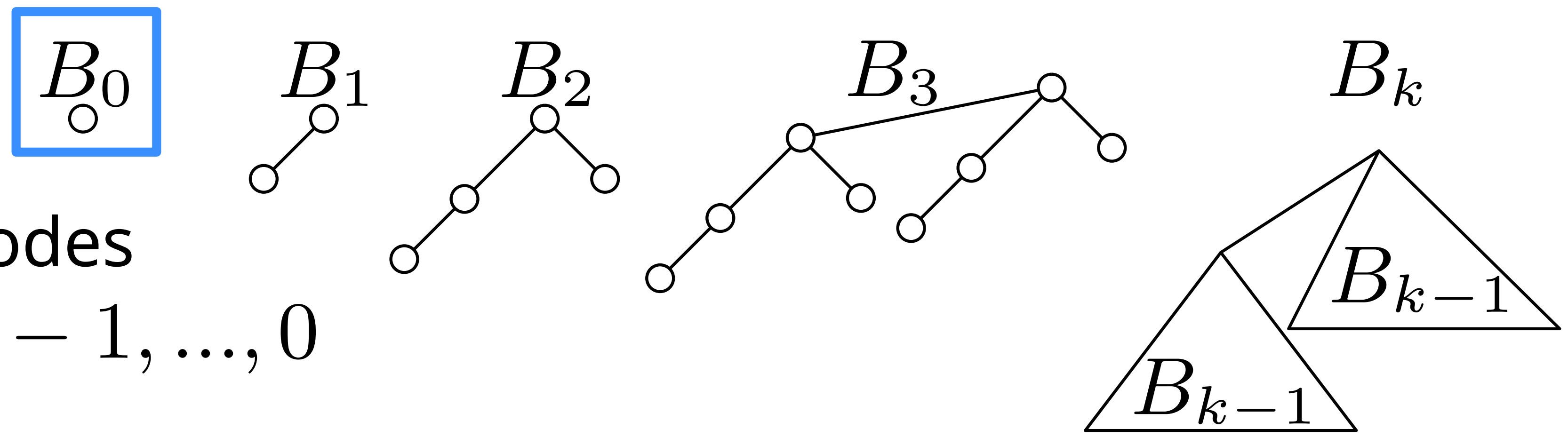

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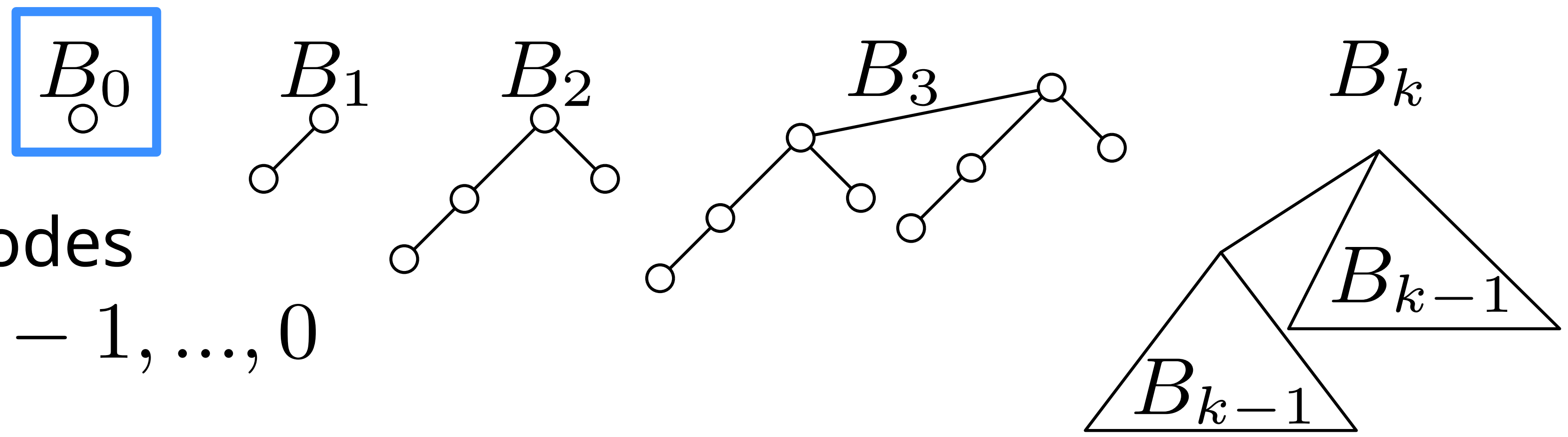

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base: $k = 0$

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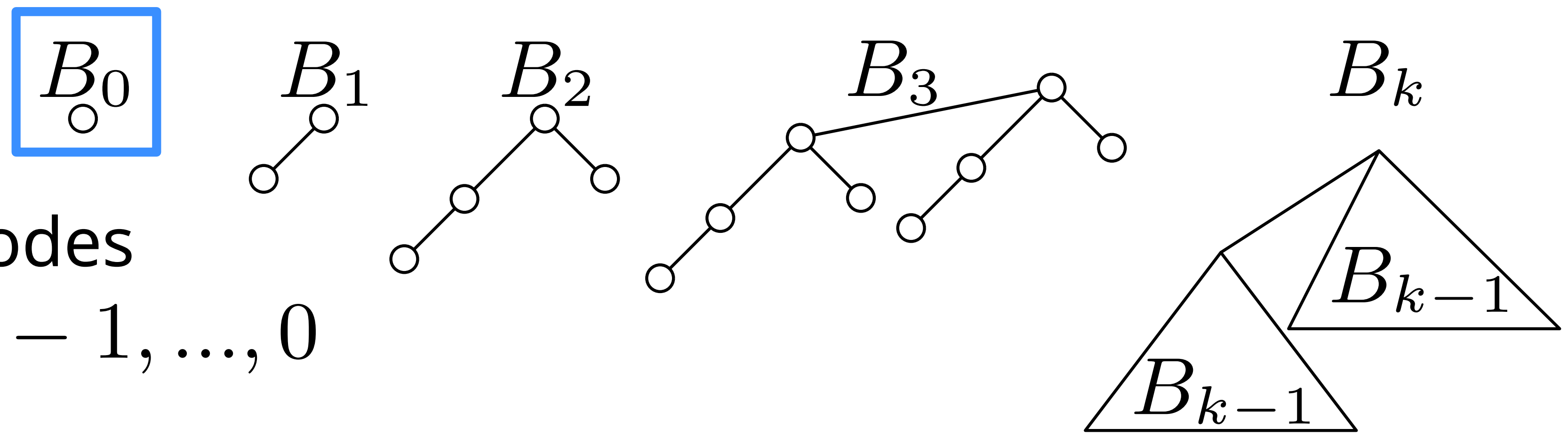

Proof: Induction(k)

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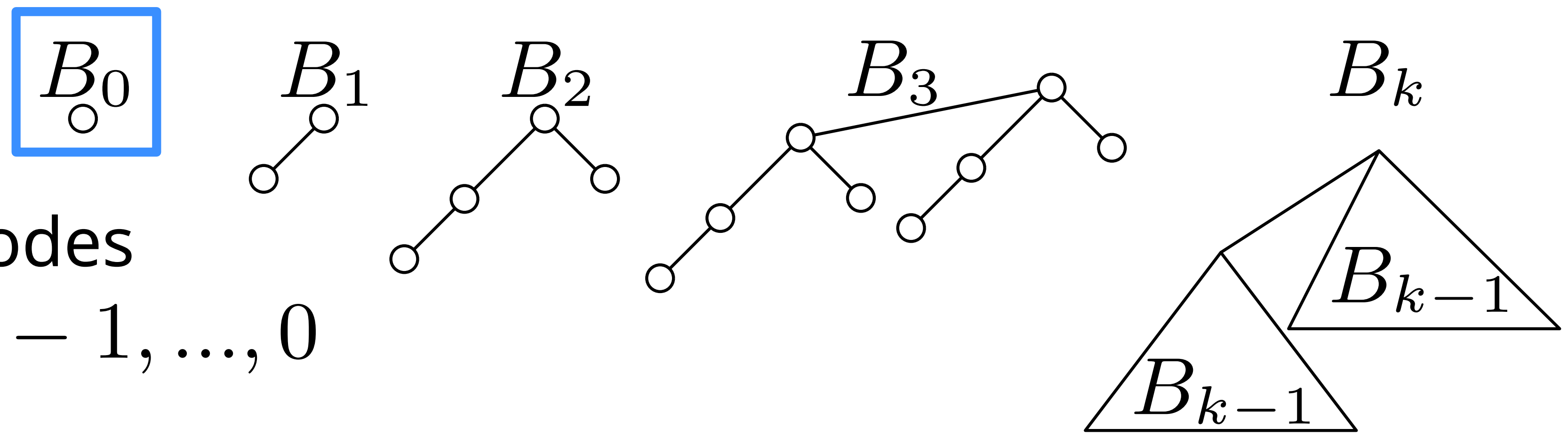

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- root of degree 0: no children

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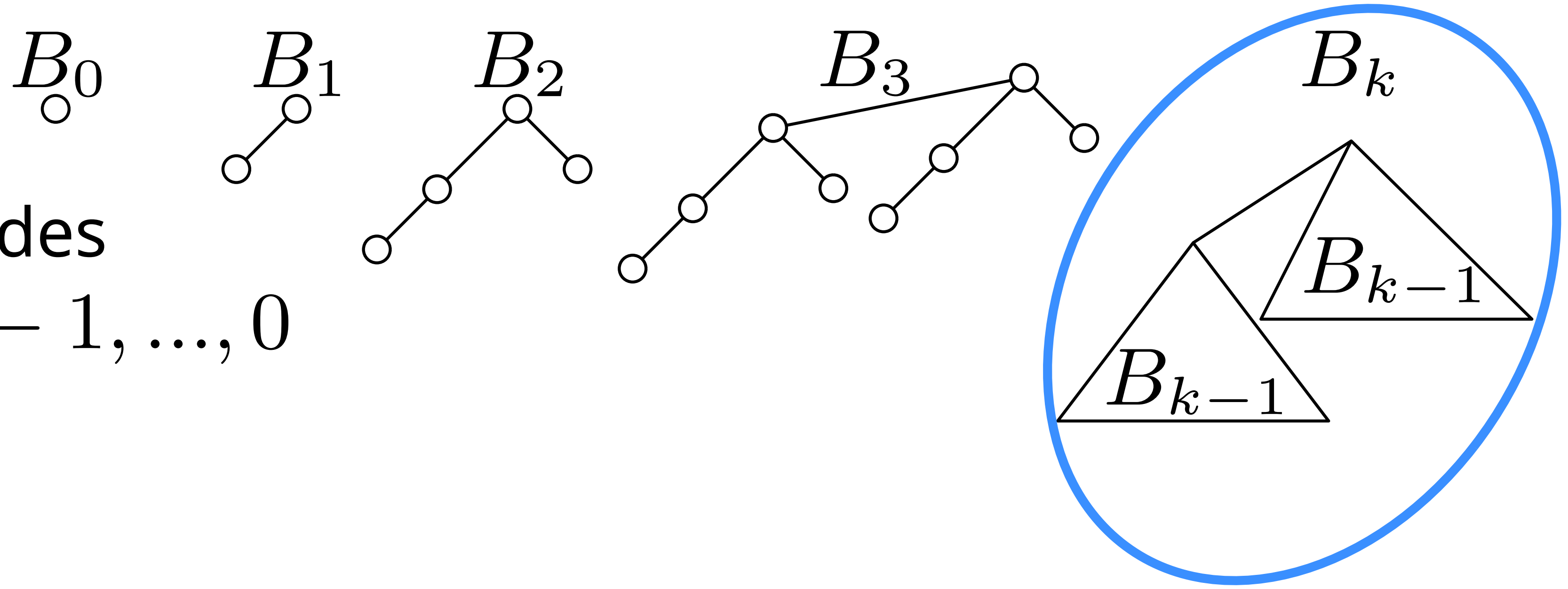

Proof: Induction(k)

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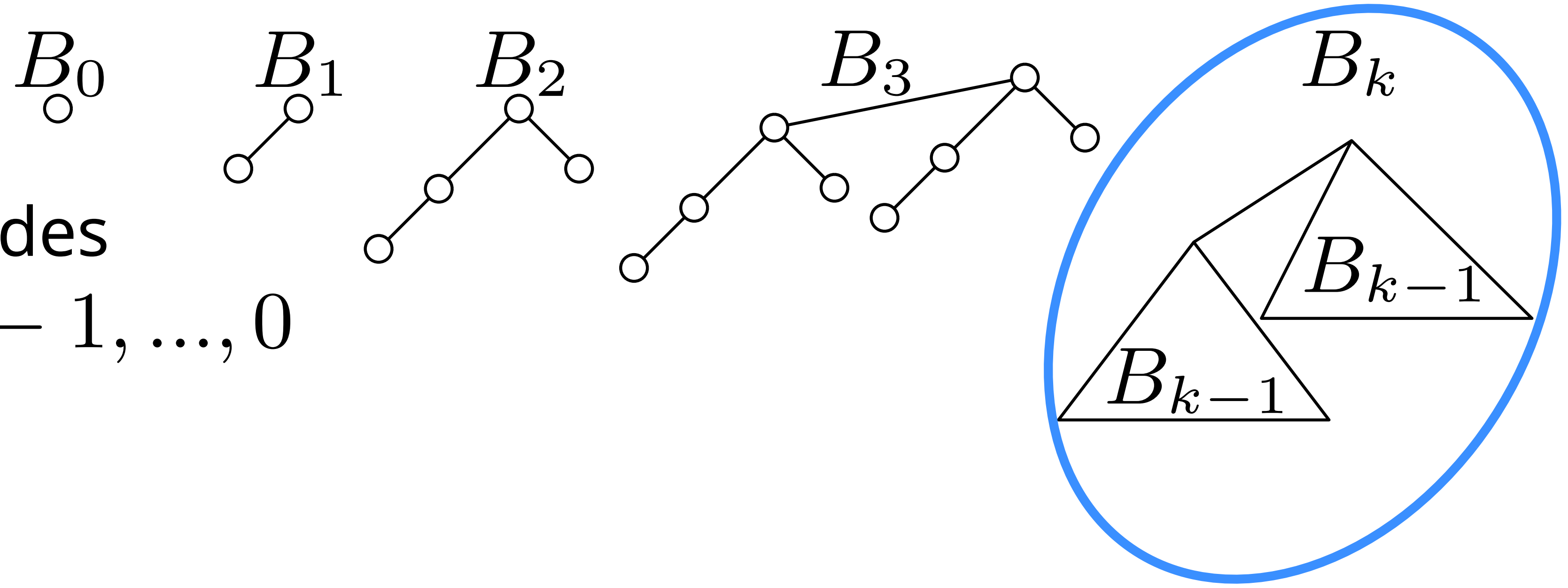

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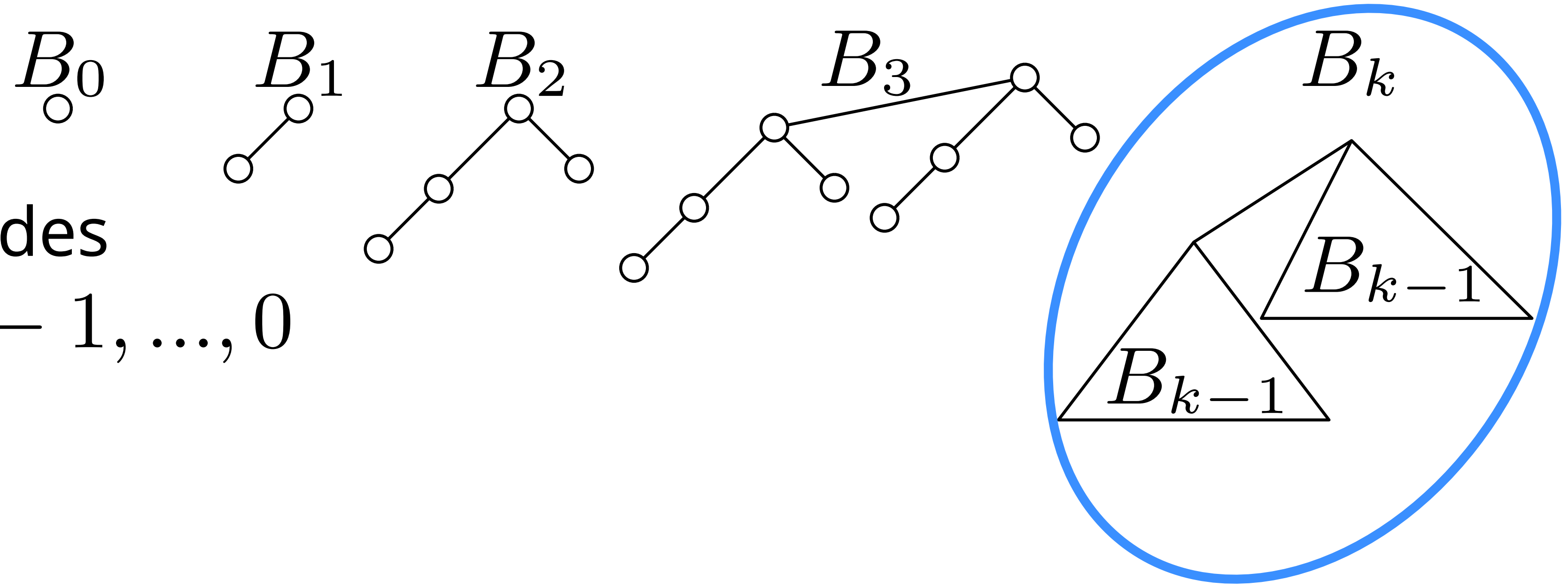

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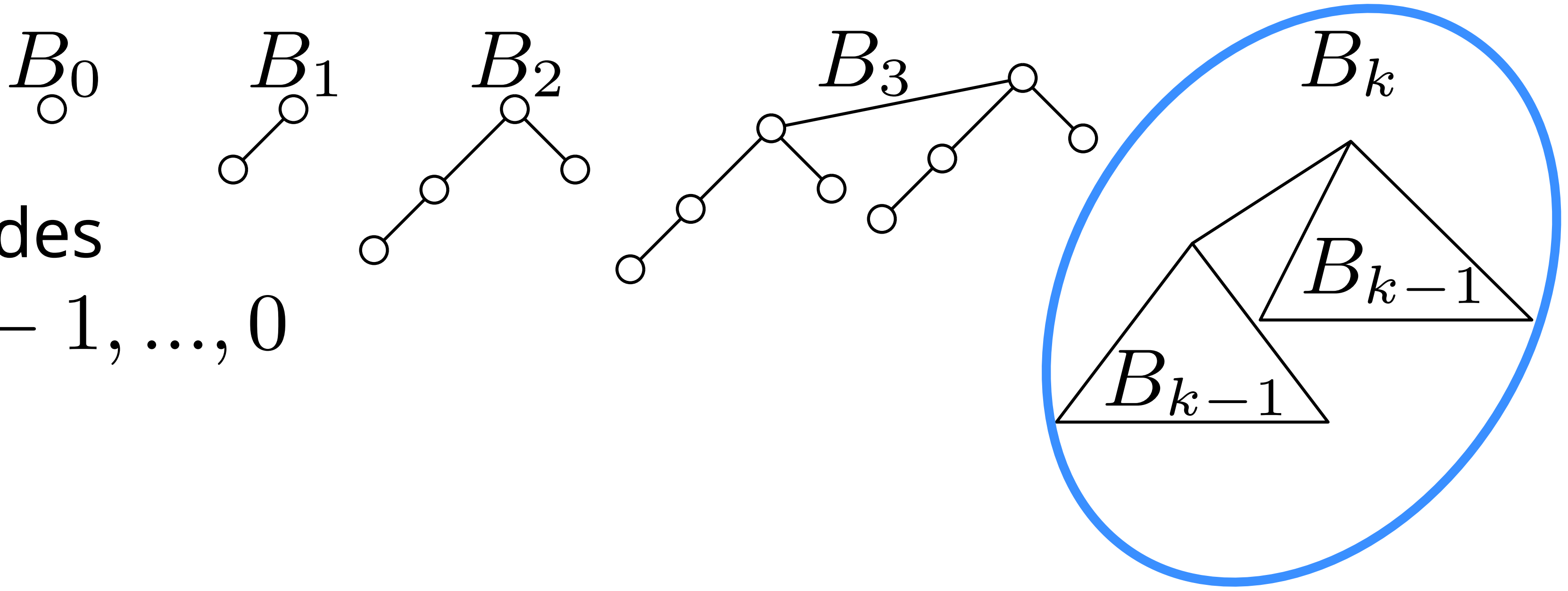

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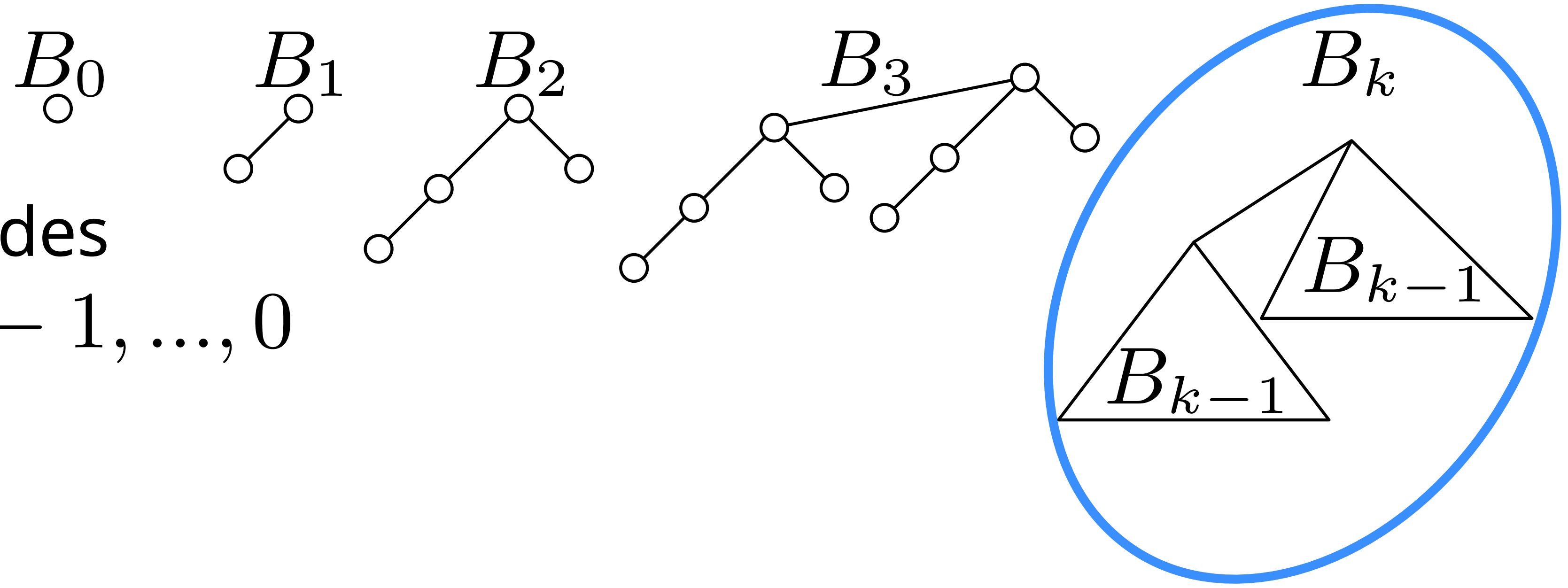

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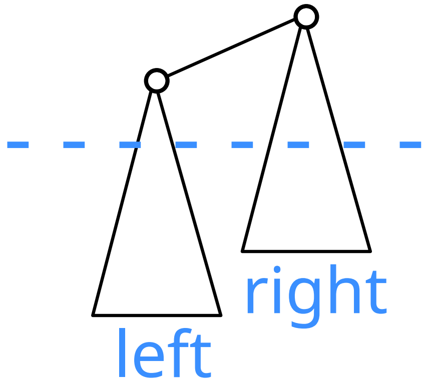

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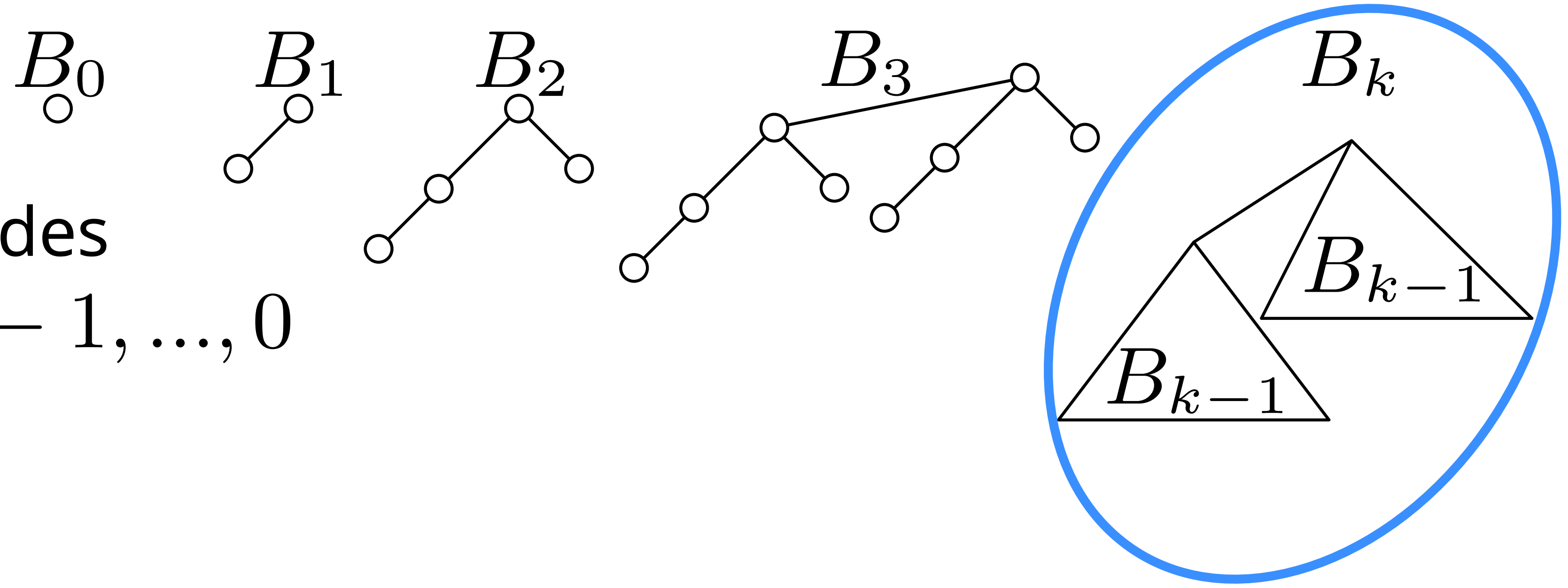

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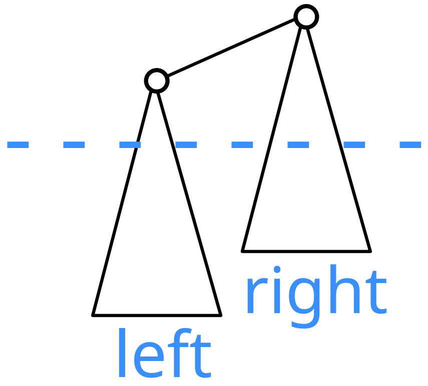

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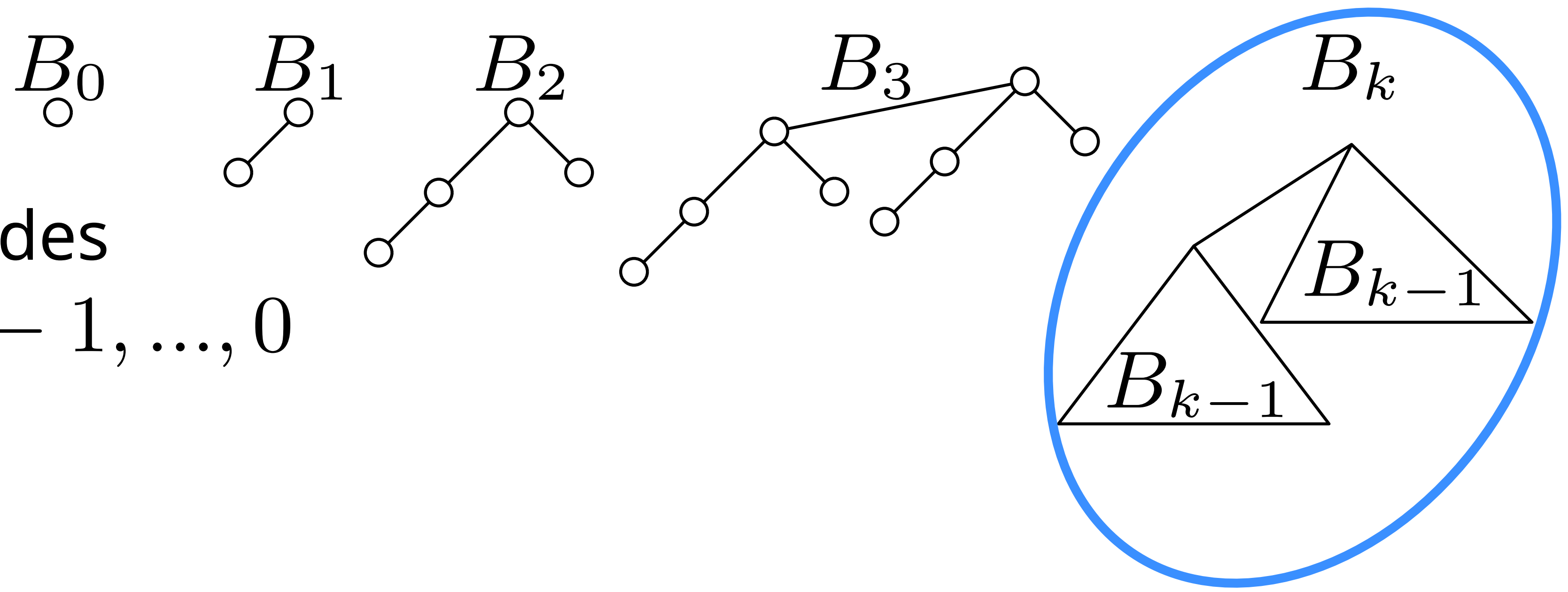

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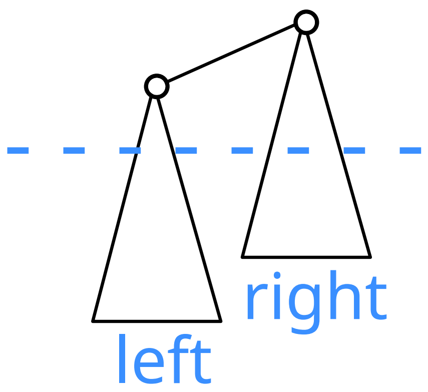

- degree of root: $(k-1)+1 = k$

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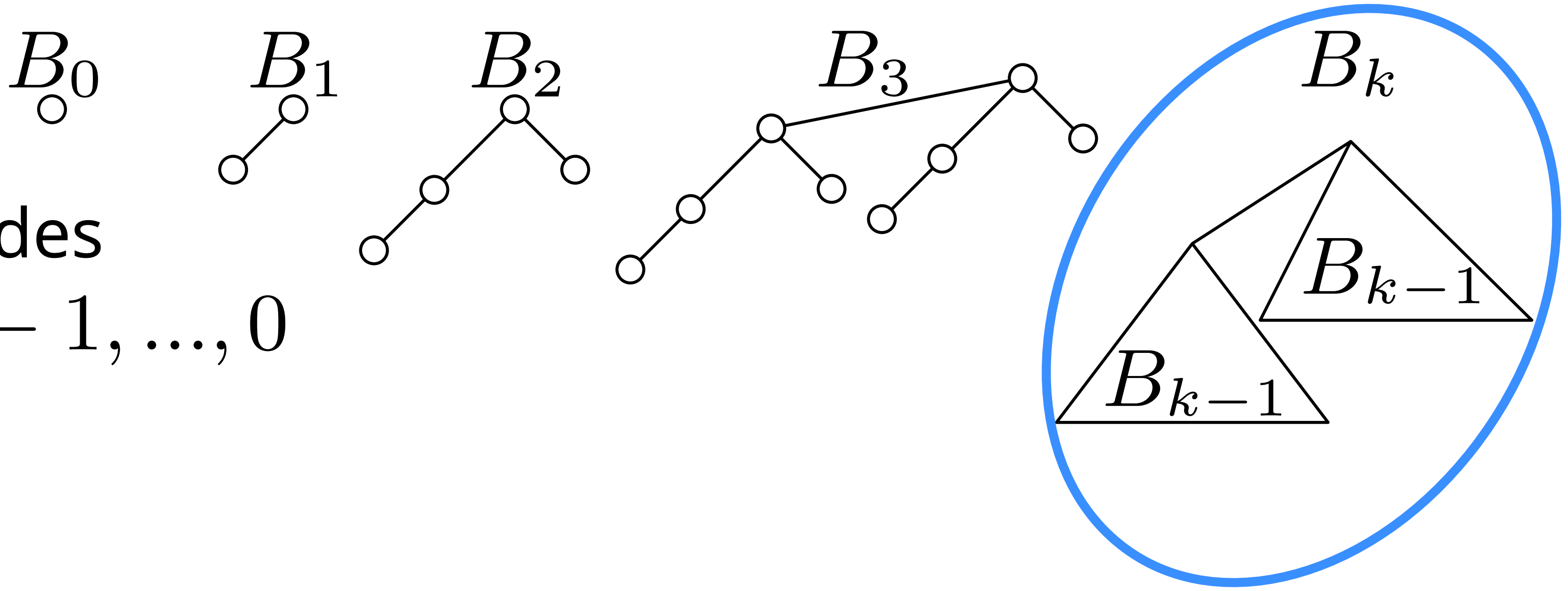

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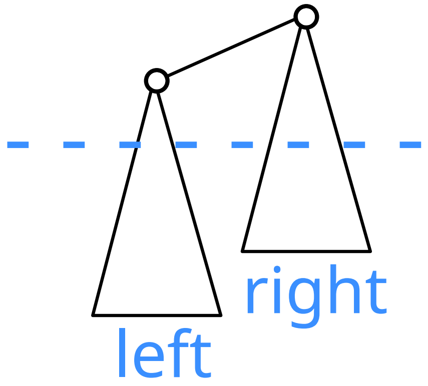
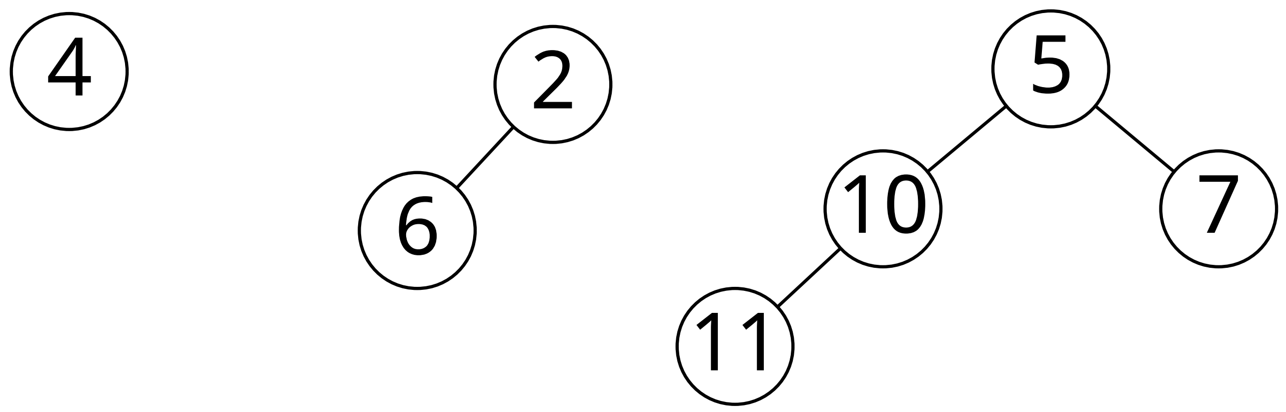


- degree of root: $(k-1)+1 = k$
- degree of children: $\underbrace{k-1}_{\text{left}},\ \underbrace{k-2, \ldots, 0}_{\text{right}}$

Binomial Heaps

Binomial Heap is a set of binomial trees where each node stores a key with the binomial heap property:

- each binomial tree fulfills the MinHeap-property
- for all $k \geq 0$ there is at most one binomial tree B_k



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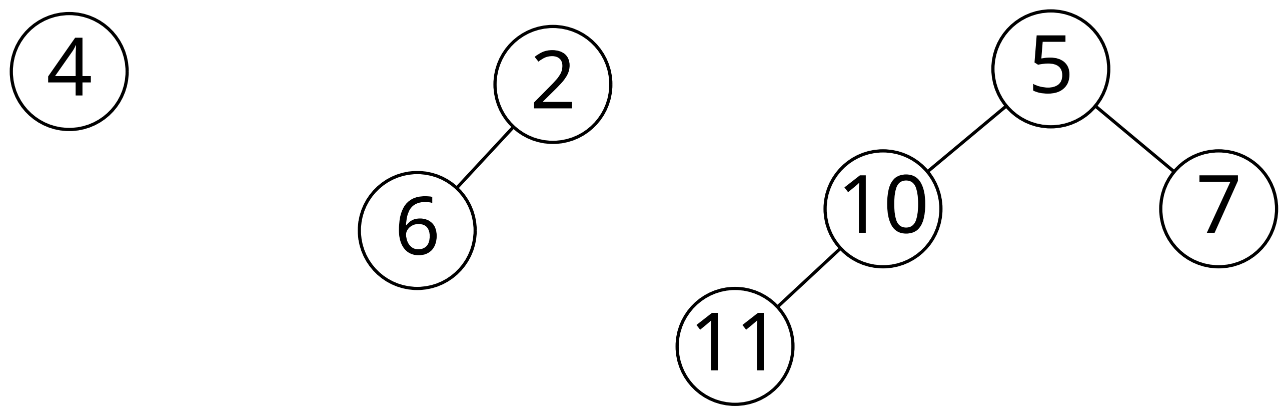
$\Rightarrow$ a binomial heap on n nodes consists of at most $\lfloor \log n \rfloor + 1$ binomial trees

and these correspond to the binary representation of $n = \sum_{i=0}^{\lfloor \log n \rfloor} b_i 2^i$



here $n = 7$, which is 111 in binary representation.

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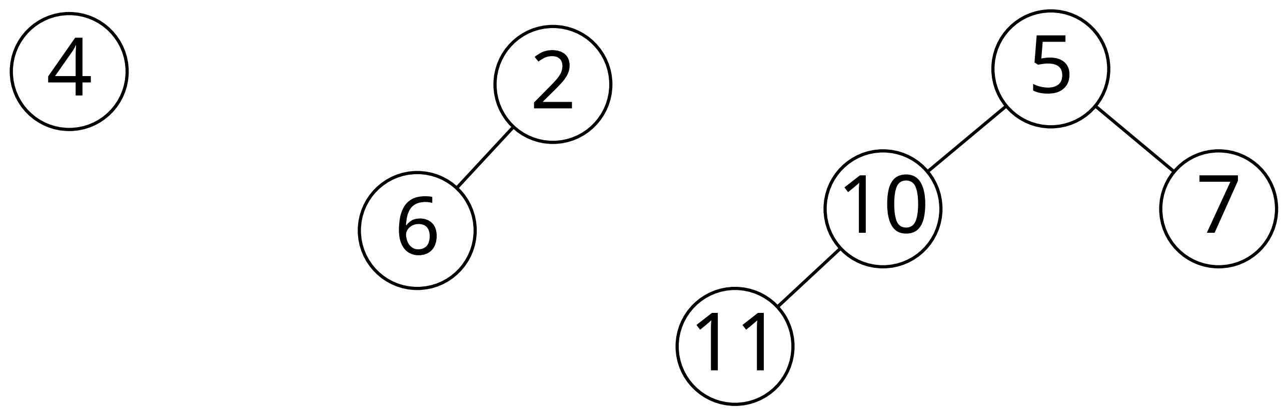

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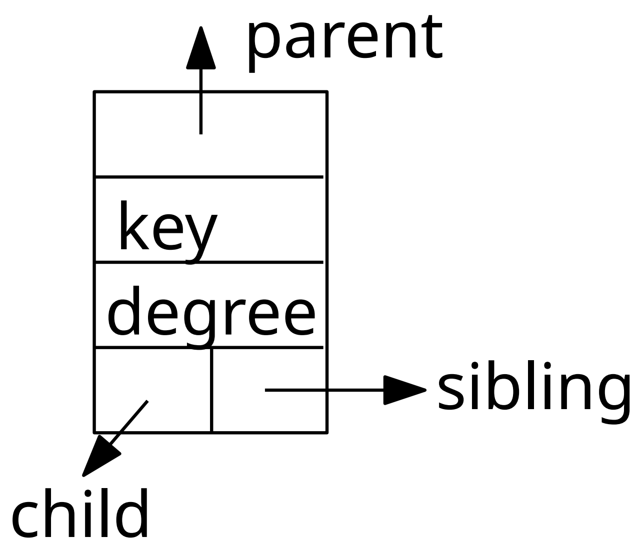
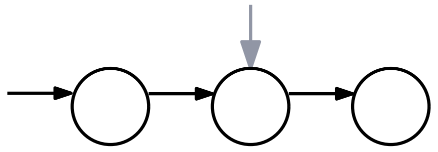
What does a binomial heap on $n = 6$ or $n = 8$ nodes look like?

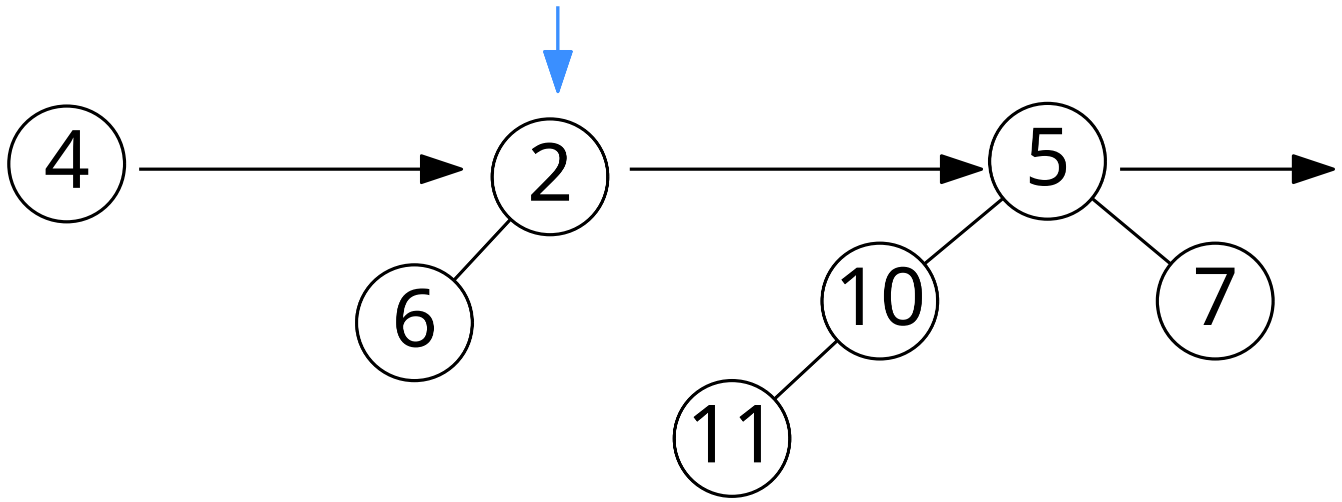
Binomial Heaps



Implementation:

- each node x stores key$[x]$, degree$[x]$, and three pointers to its parent, leftmost child, right sibling
- root of binomial heaps are stored in a linked list (of heaps of increasing size)





Operations

make-0: generate empty heap (i.e. empty list)

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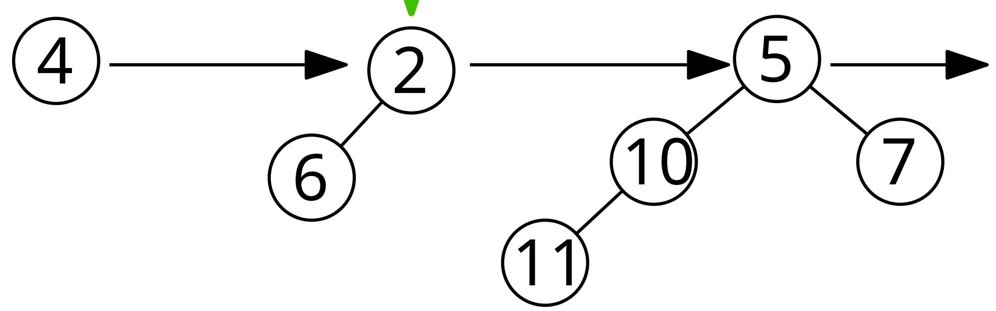

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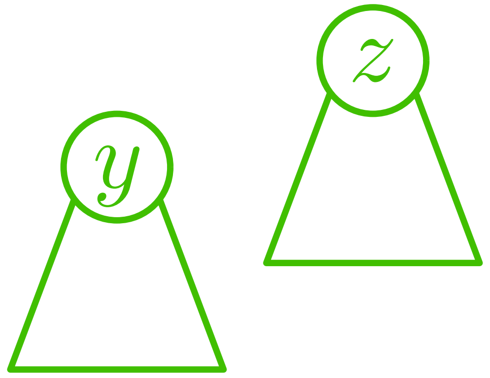
union(H_1, H_2):

- merge the lists of roots (by increasing degree/size)
- iterate through merged list and link binomial trees of equal k

link(y, z) (assuming key[z] $\leq$ key[y])



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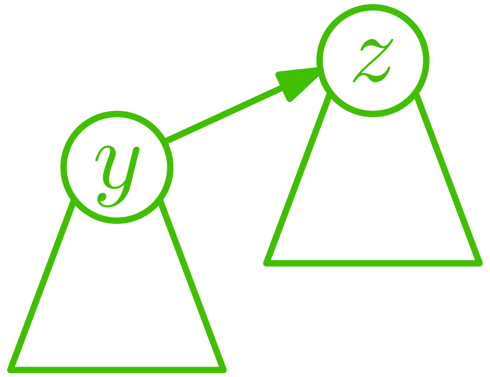
parent[y] $= z$



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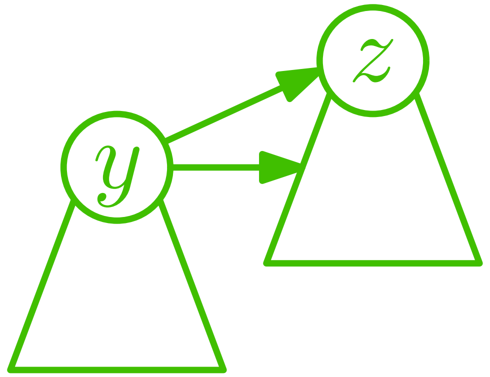
  parent[y] $= z$
  sibling[y] $=$ child[z]



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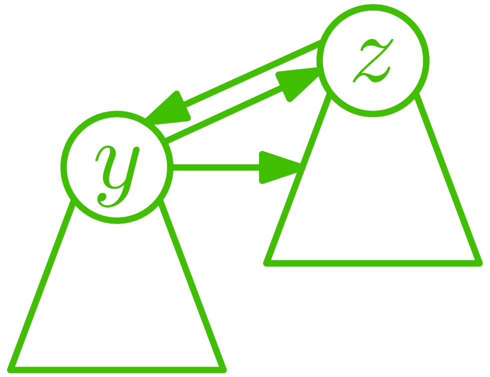

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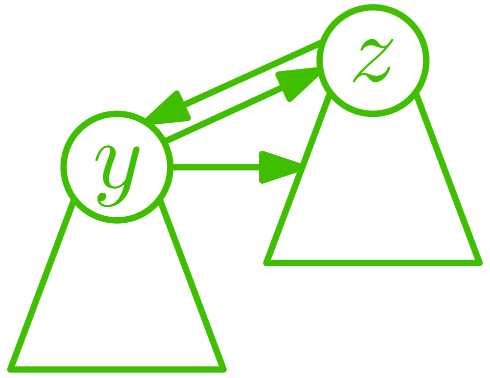
link(y, z) (assuming key[z] $\leq$ key[y])

parent[y] $= z$
sibling[y] $=$ child[z]
child[z] $= y$

degree[z] = degree[z] + 1



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- make a 1-binomial heap (make-1)
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- make a 1-binomial heap (make-1) How to make a 1-binomial heap?
- union with existing heap

Operations

make-0: generate empty heap (i.e. empty list)

min: iterate through list of roots (or store extra pointer)

union(H_1, H_2):

- merge the lists of roots (by increasing degree/size)
- iterate through merged list and link binomial trees of equal k

insert(H, a):

- make a 1-binomial heap (make-1)
- union with existing heap

How to make a 1-binomial heap?

parent[x] = NIL
child[x] = NIL
sibling[x] = NIL
key[x] = a
degree[x] = 0
Head[H'] = x

Operations

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decKey: Q: how to realize decreaseKey(x,k)?

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delete(x): Q: how to realize delete(x)?

Operations

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union(H_1, H_2):

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- union with existing heap

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- union with the list of children of x (in opposite direction)

decKey: siftUp/bubbleUp in corresponding tree like in ordinary heap

delete(x): decKey($x, -\infty$); delete-min

Operations – Examples

make-0

Operations – Examples

make-0 head $\rightarrow\!\!\blacktriangleright\!\!|$ $\emptyset$

Operations – Examples

make-0	head $\rightarrow\!\blacktriangleright\!|$	$\emptyset$
insert(5)	???	???

Operations – Examples

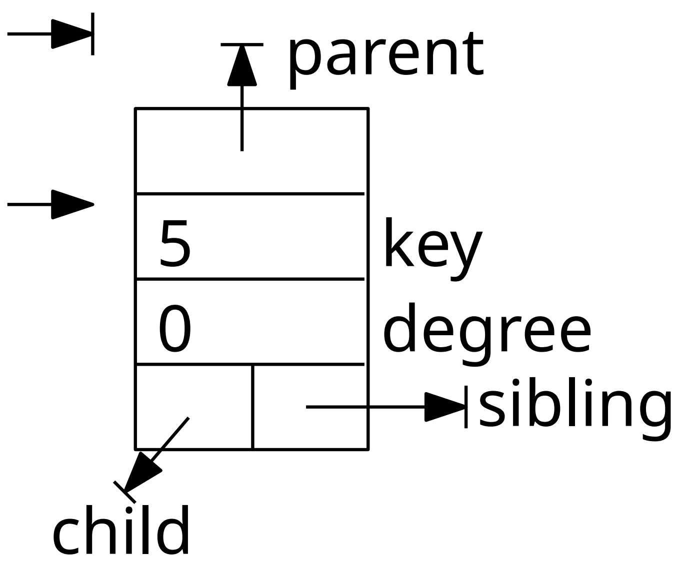
make-0 head $\rightarrow\!|$ $\emptyset$

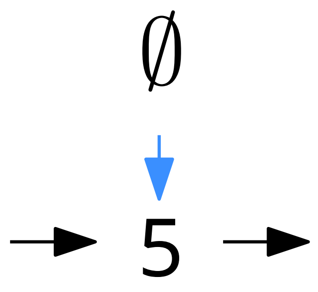
insert(5) $\rightarrow 5 \rightarrow$ (make-1)

Operations – Examples

make-0 head

insert(5) head



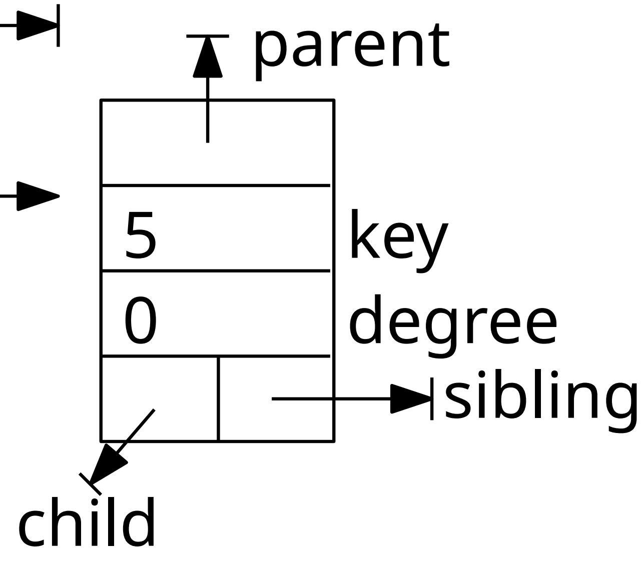


(make-1)

Operations – Examples

make-0 head → $\emptyset$

insert(5) head →



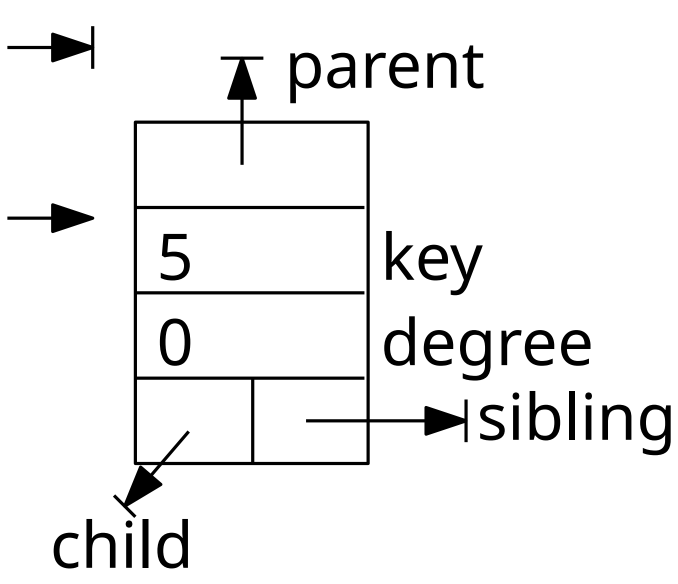
→ 5 → (make-1)

insert(7) ??? ???

Operations – Examples

make-0 head → $\emptyset$

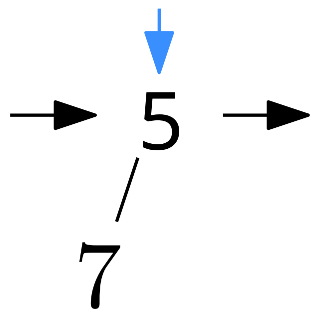
insert(5) head →



→ 5 →

(make-1)

insert(7)

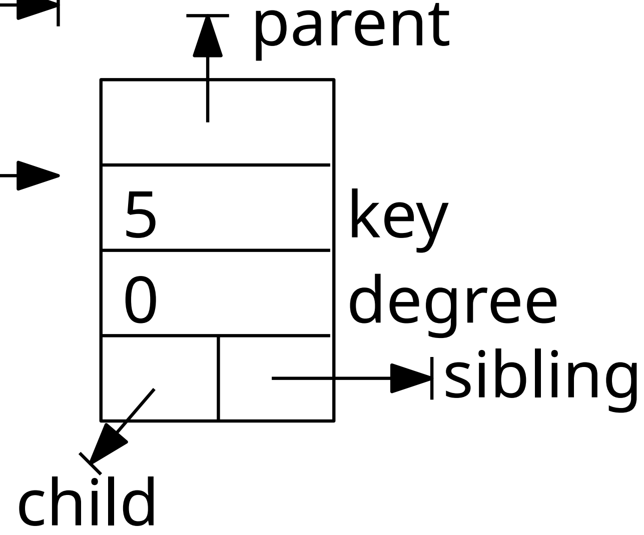


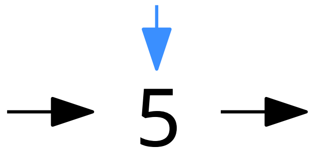
(make-1 +
union $\rightarrow$ link)

Operations – Examples

make-0 head $\rightarrow\!\mid$ $\emptyset$

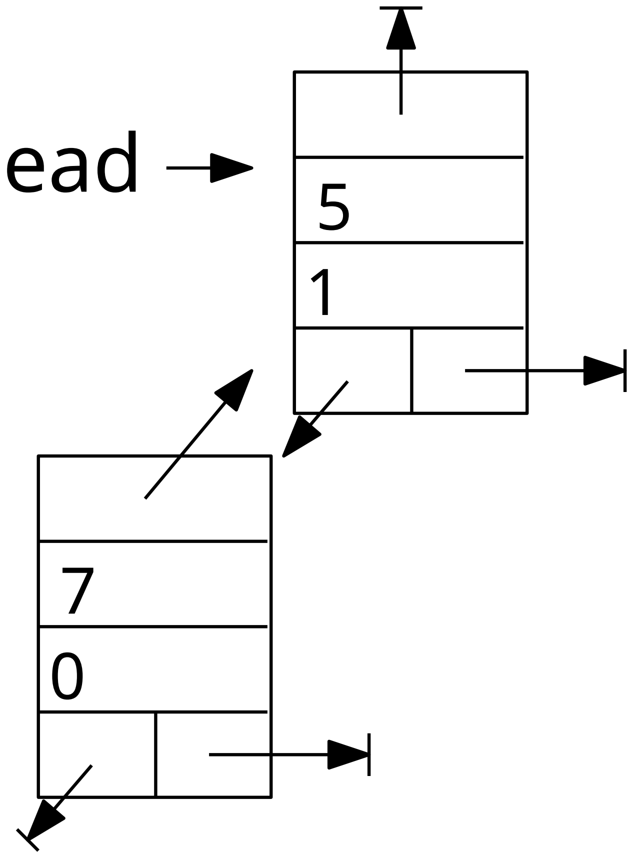
insert(5) head $\rightarrow$

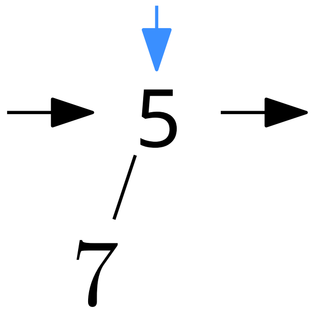




(make-1)

insert(7) head $\rightarrow$





(make-1 +
union $\rightarrow$ link)

Operations – Examples

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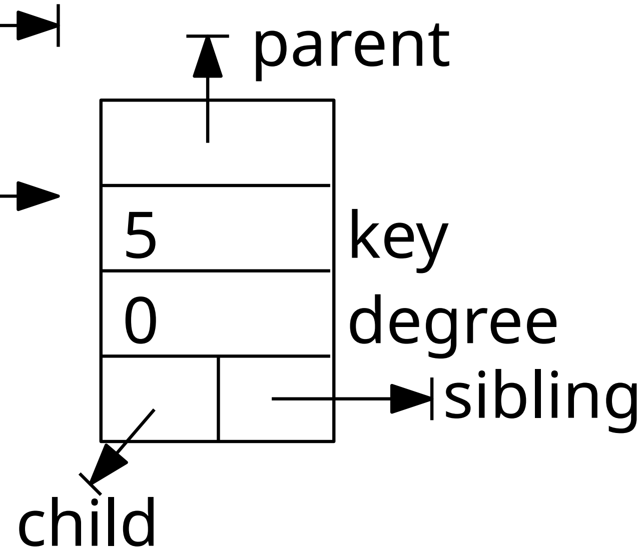
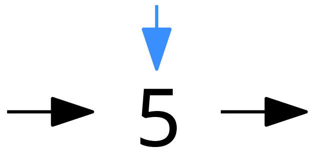
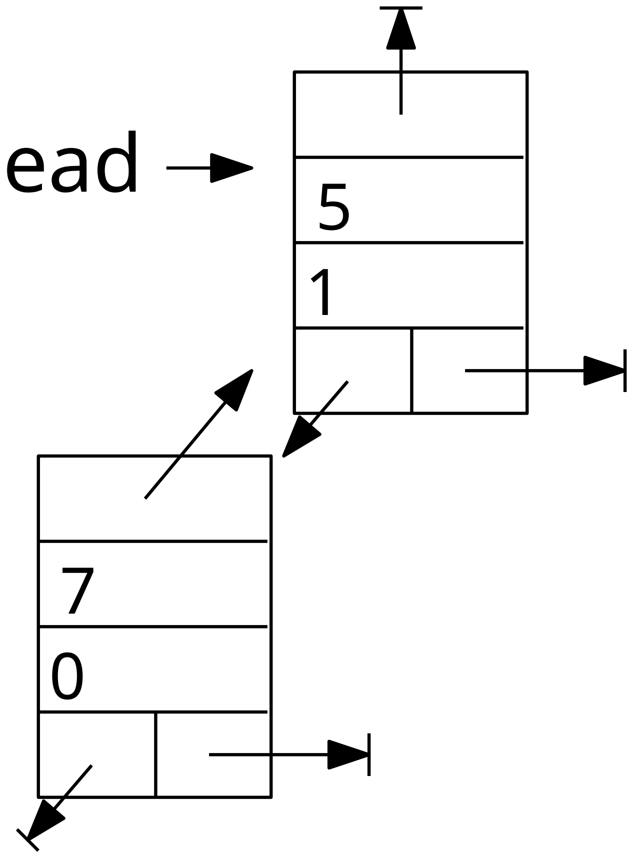
insert(5) head →

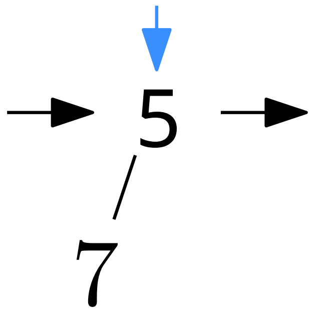




(make-1)

insert(7) head →



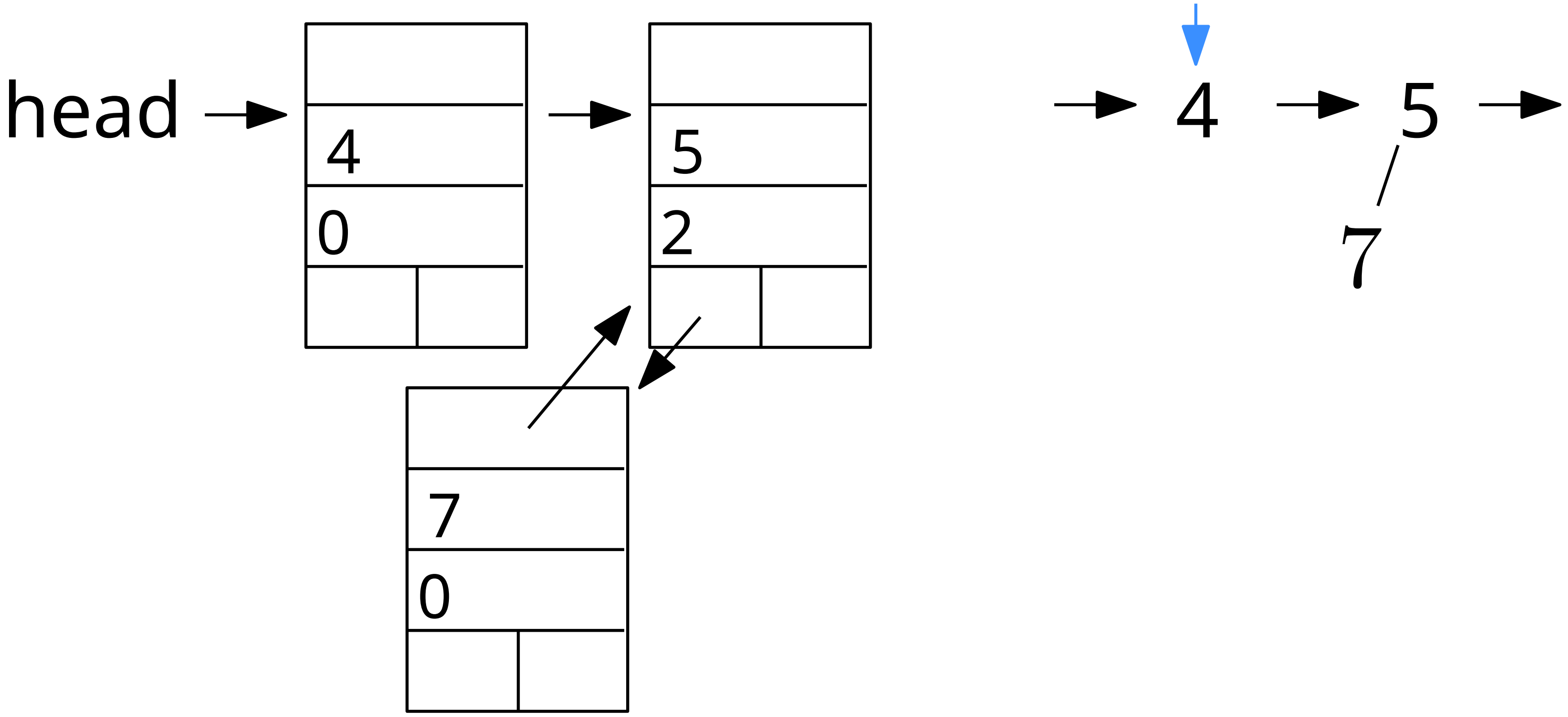


(make-1 + union → link)

insert(4) ???

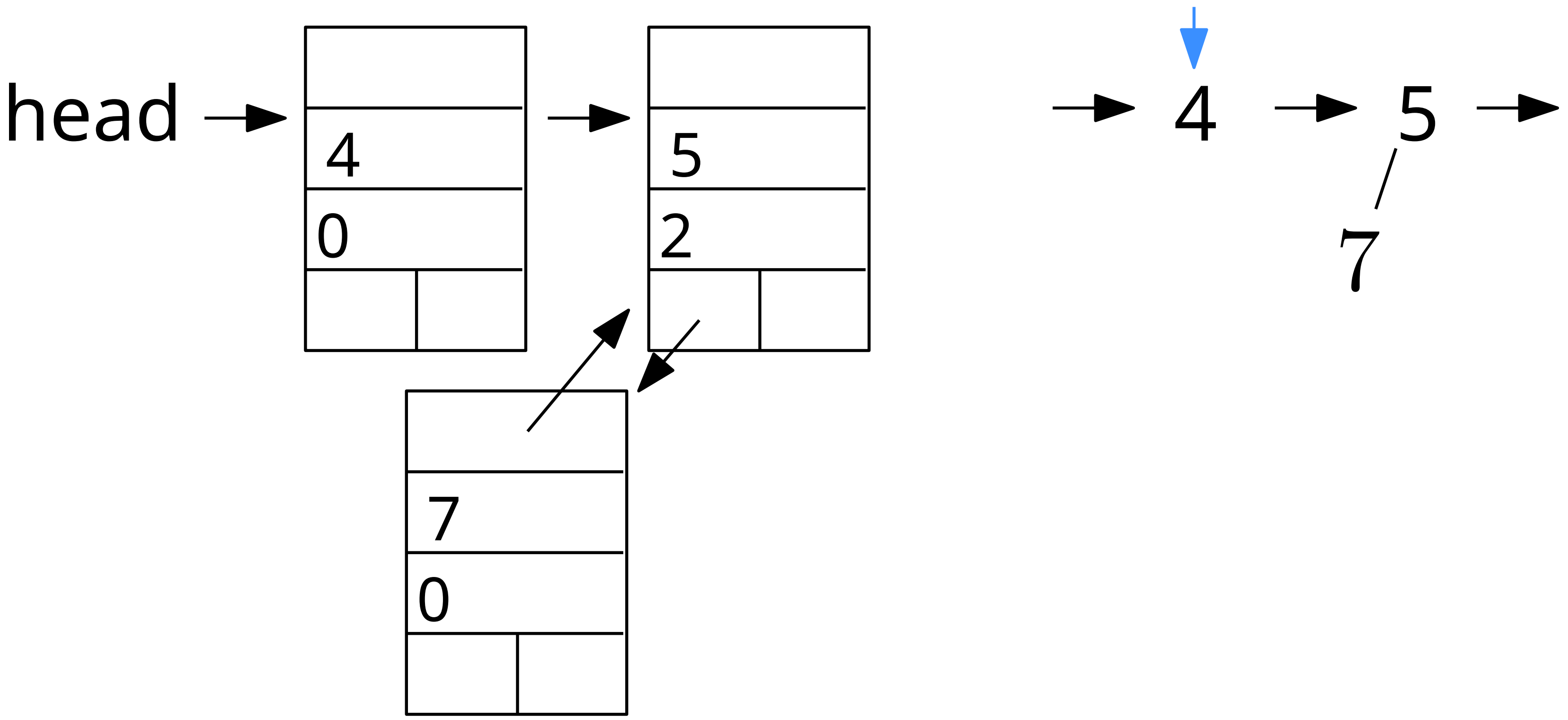
Operations – Examples

insert(4)



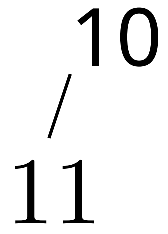
Operations – Examples

insert(4)



union(H, H')

H' :



Operations – Examples

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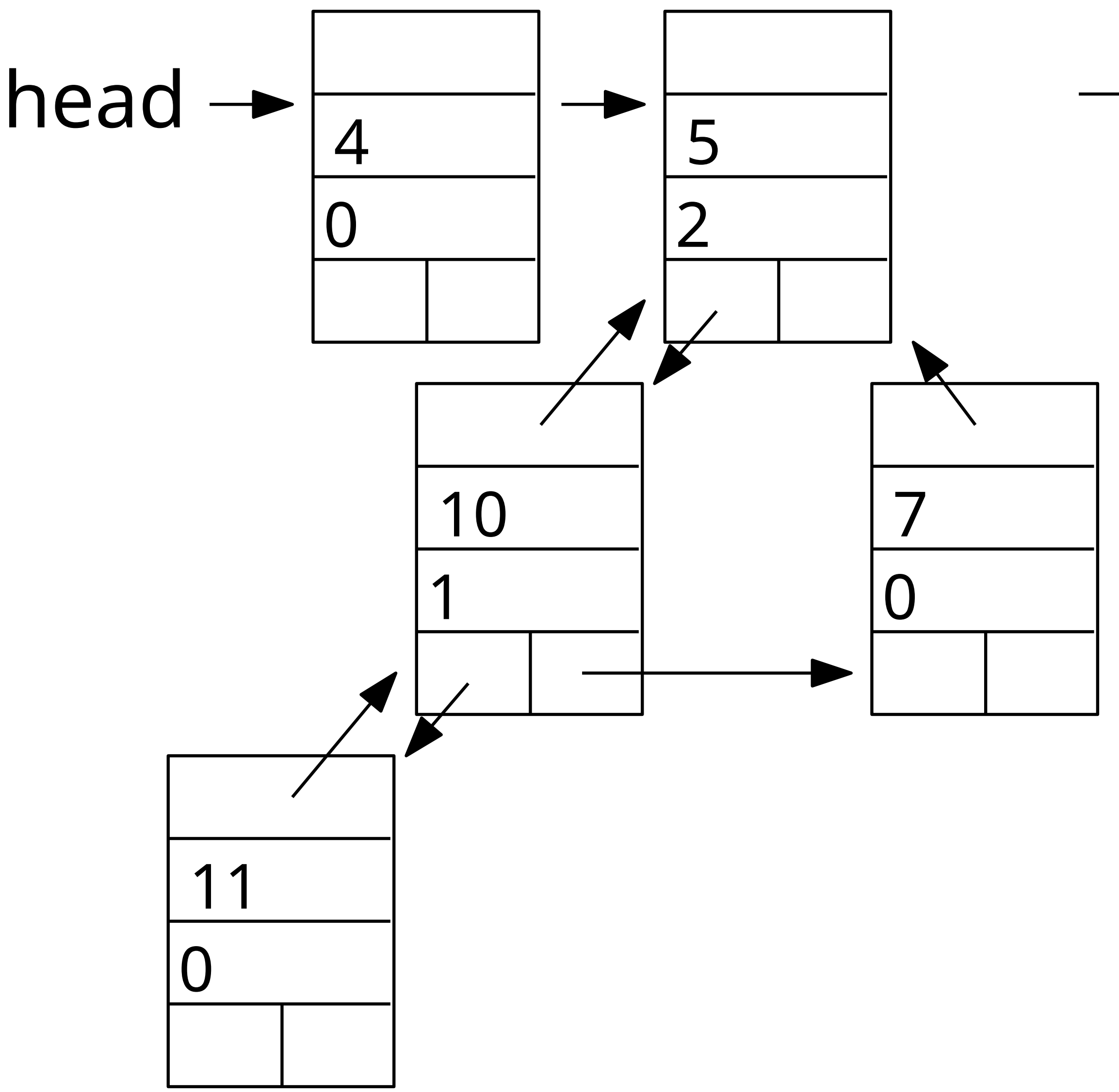

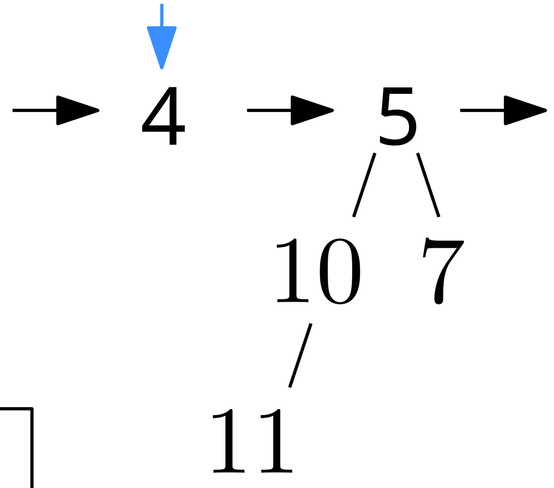

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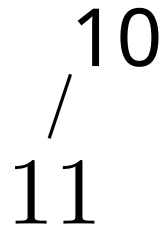
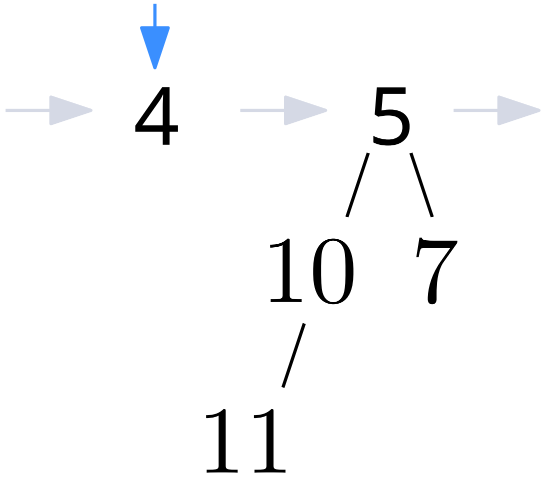
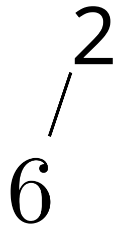


Operations – Examples



union(H, H')

H'' :

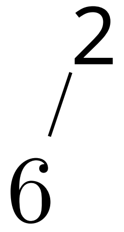


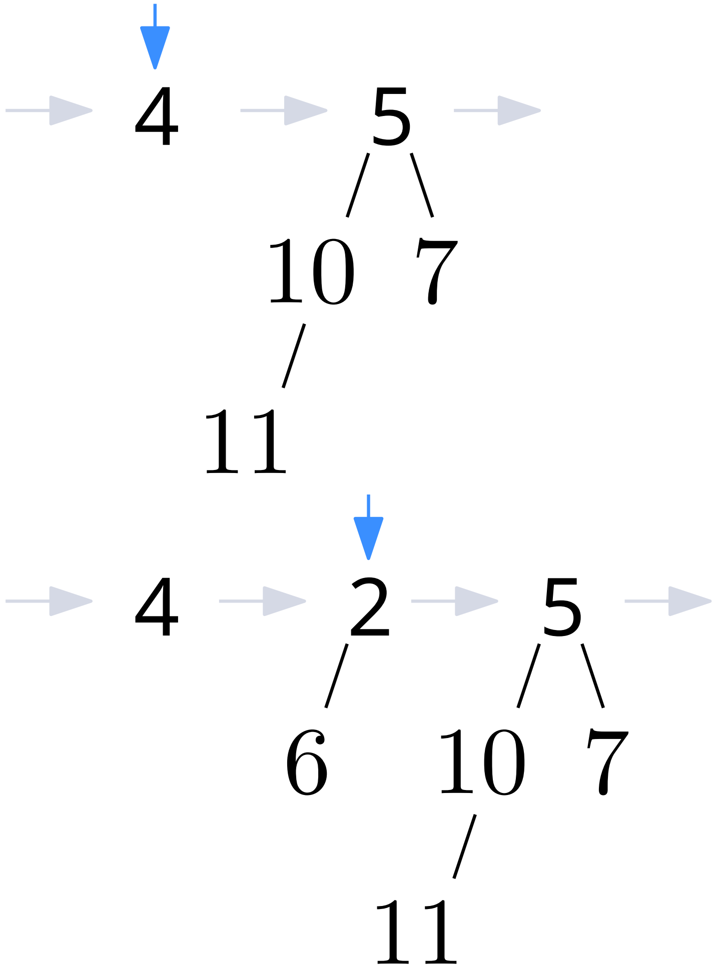
???

Operations – Examples

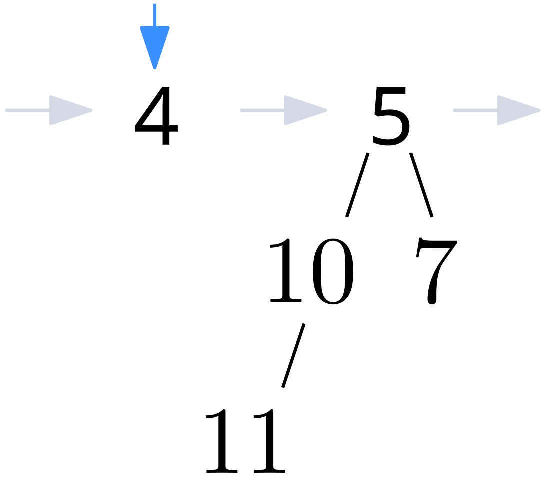
union(H, H')

H'' :

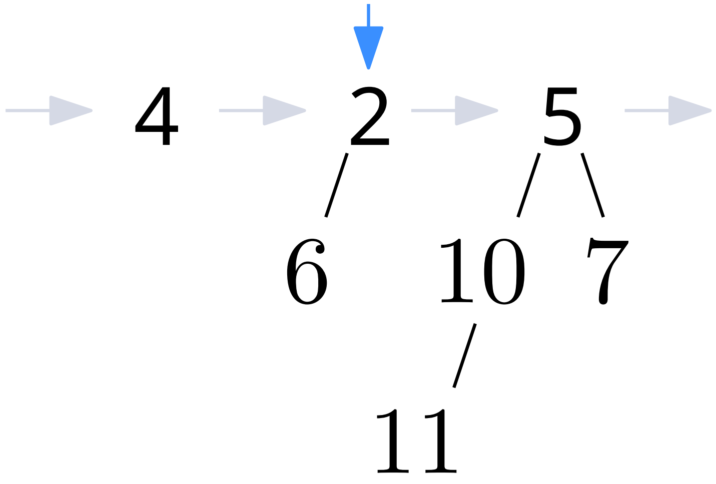




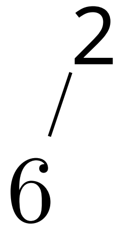
Operations – Examples



union(H, H')



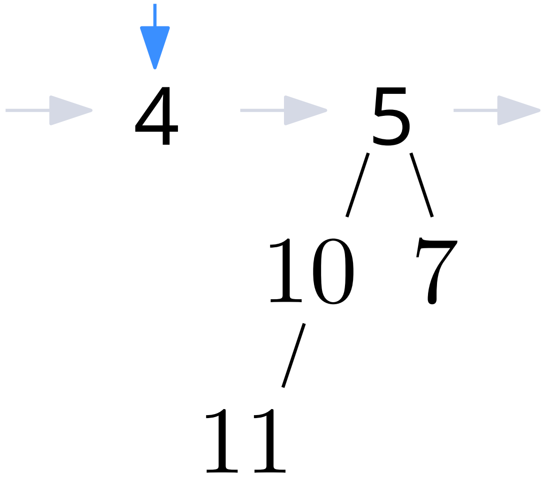
H'' :



deleteMin

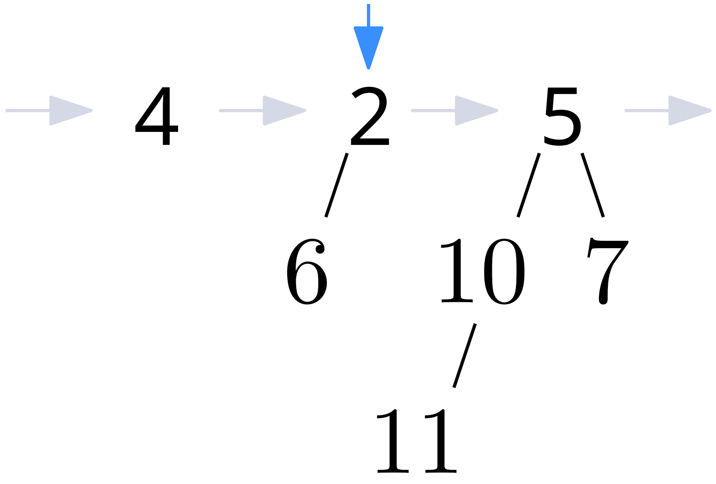
???

Operations – Examples

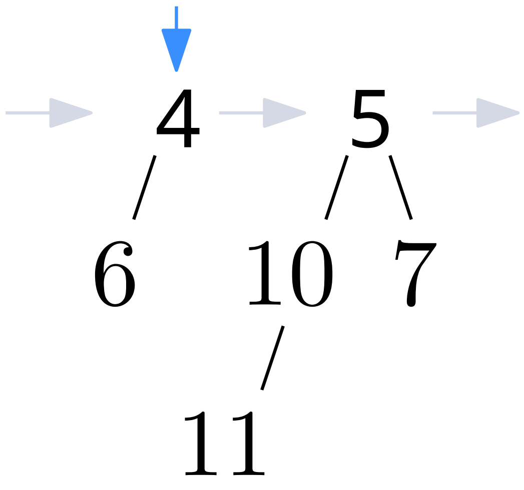


union(H, H')

H'' : 2 / 6



deleteMin



Operations – Runtime

Runtimes: ???

make-0: generate empty heap (i.e. empty list)

min: iterate through list of roots (or store extra pointer)

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- merge the lists of roots
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insert(H, a):

- make a 1-binomial heap (make-1)
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decKey: siftUp/bubbleUp in corresponding tree like in ordinary heap

delete(x): decKey($x, -\infty$); delete-min

Operations – Runtime

Runtimes: ???

make-0: generate empty heap (i.e. empty list) $O(1)$

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Runtimes

	worst-case	
make		
min		
insert		
delete-min		
union		

Runtimes

	worst-case	
make	$O(1)$	
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Runtimes

	worst-case	
make	$O(1)$	
min	$O(1)$	
insert	$O(\log n)$	
delete-min	$O(\log n)$	
union	$O(\log n)$	

Which operation has a lower amortized runtime?

Runtimes

	worst-case	amortised
make	$O(1)$	$O(1)$
min	$O(1)$	$O(1)$
insert	$O(\log n)$	$O(1)$
delete-min	$O(\log n)$	$O(\log n)$
union	$O(\log n)$	$O(\log n)$

Amortised Analysis with accounting or potential method

Binomial Heap – Accounting Method

idea: save 1 coin per tree

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make-0: $\hat{c} = 1$ for creating empty heap

Binomial Heap – Accounting Method

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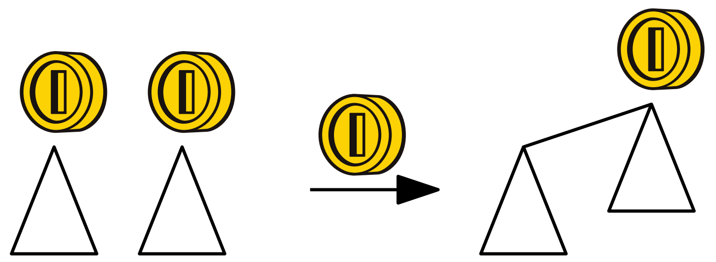
link: $\hat{c} = 0$ (pay for link with existing coin)



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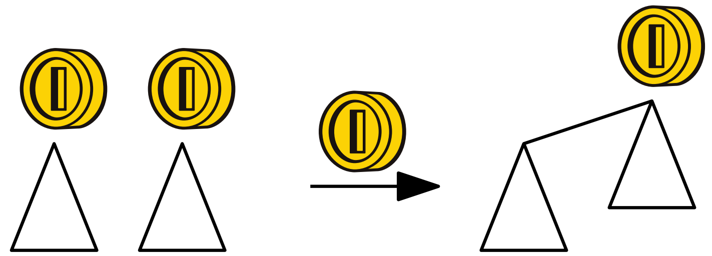

insert: $\hat{c} = 3$ (2 for make-1 + 1 for calling union + 0 for linking)

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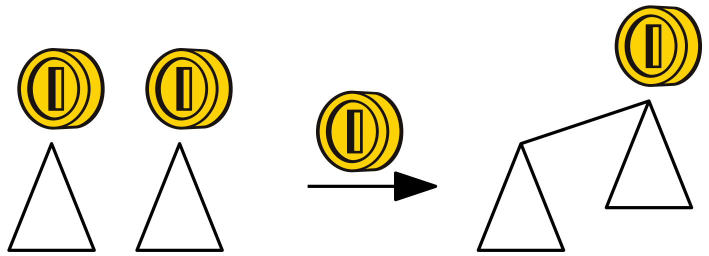

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merge lists: $\Theta(m)$, where $m = $ #roots$= O(\log n)$

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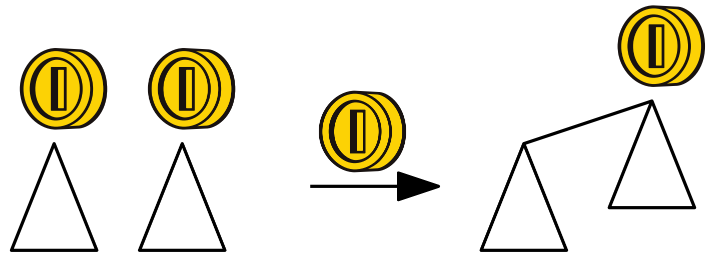

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link trees of same degree: pay with coins

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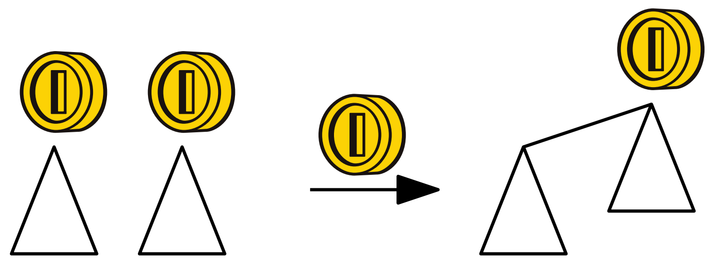

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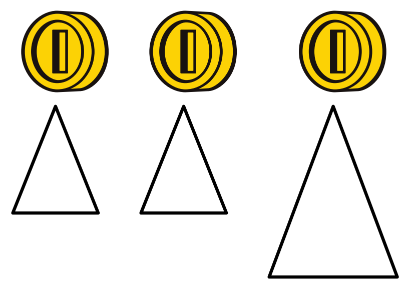

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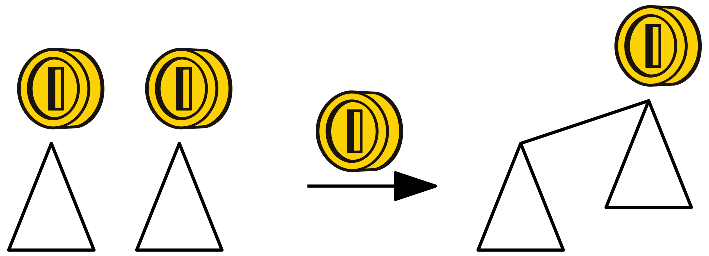

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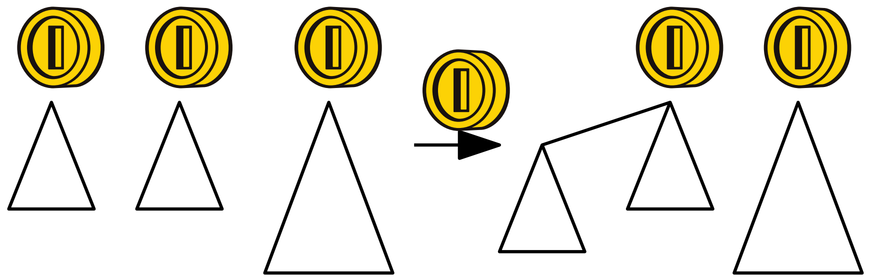

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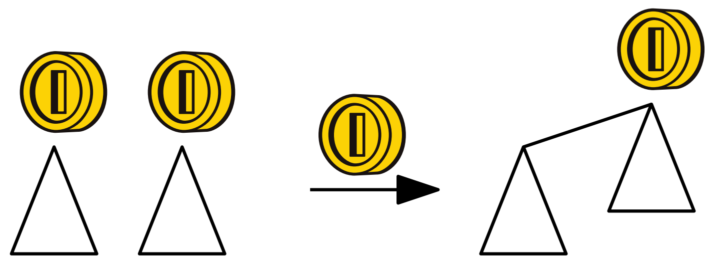

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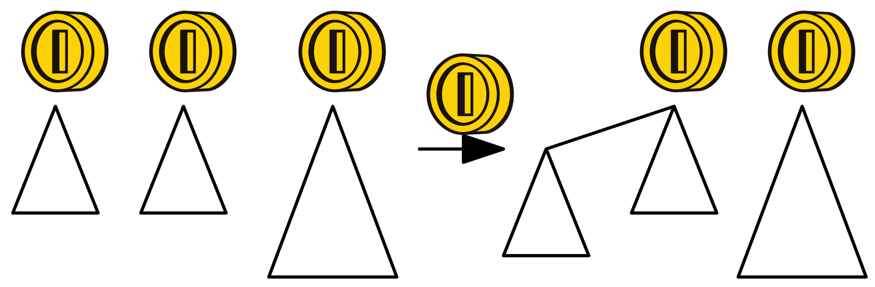
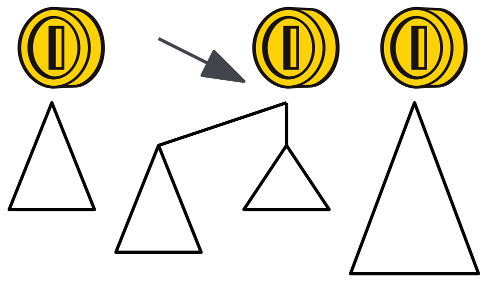
link trees of same degree: pay with coins



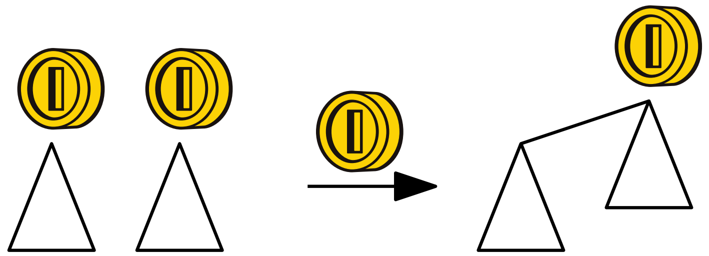
deleteMin:



Binomial Heap – Accounting Method

idea: save 1 coin per tree

make-0: $\hat{c} = 1$ for creating empty heap
link: $\hat{c} = 0$ (pay for link with existing coin)



insert: $\hat{c} = 3$ (2 for make-1 + 1 for calling union + 0 for linking)

union:
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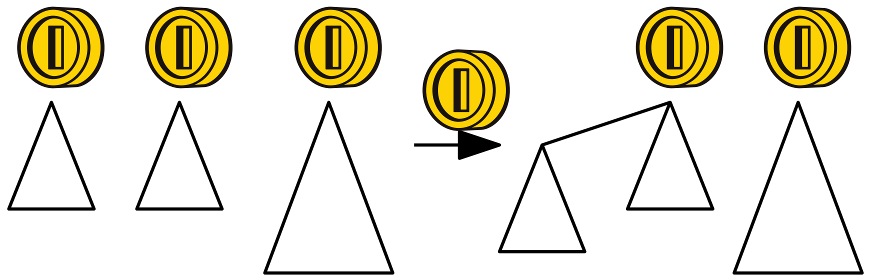
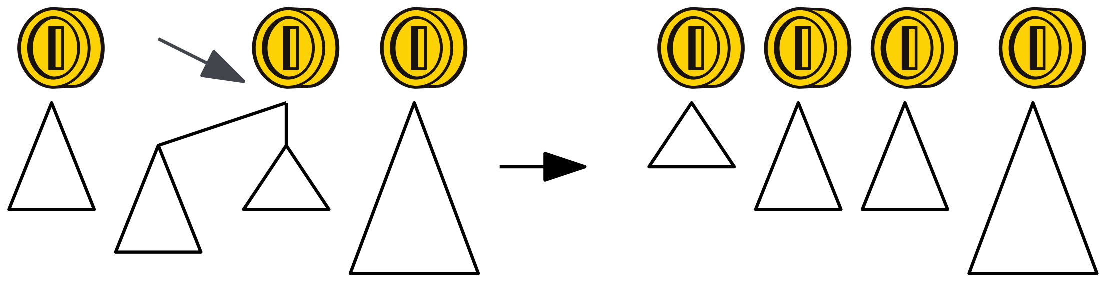


deleteMin: $\leq \log n$ children of min-node (need a coin each)



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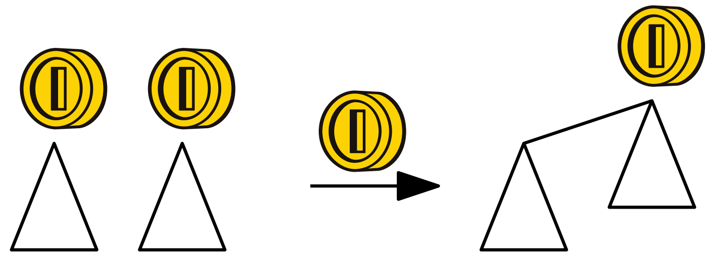

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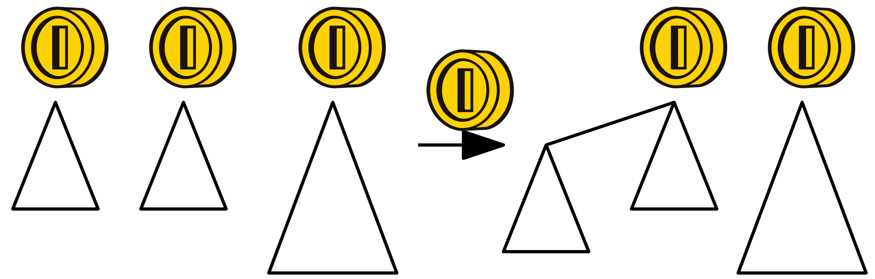

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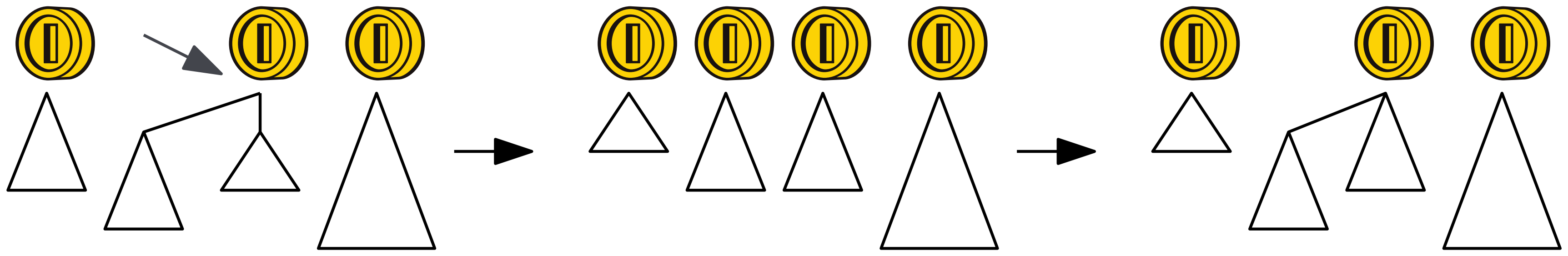

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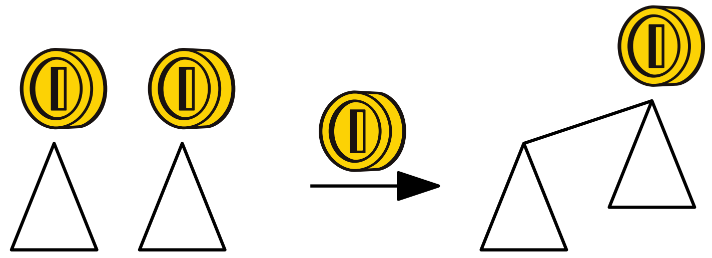

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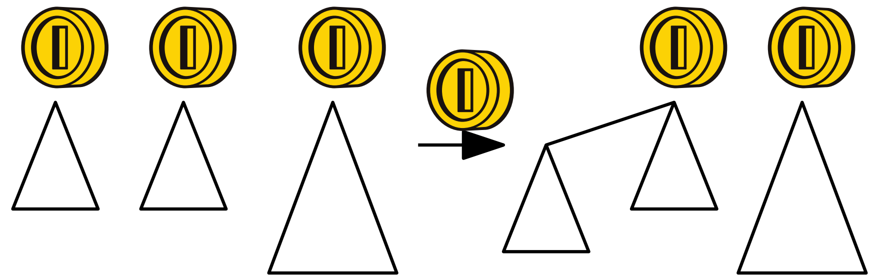

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decreaseKey: does not change number of trees,
$\hat{c} = $ actual cost $= O(\log n)$

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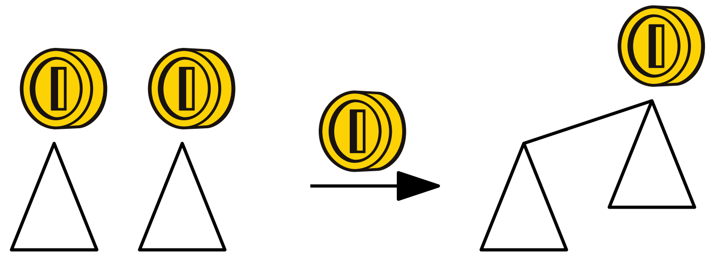

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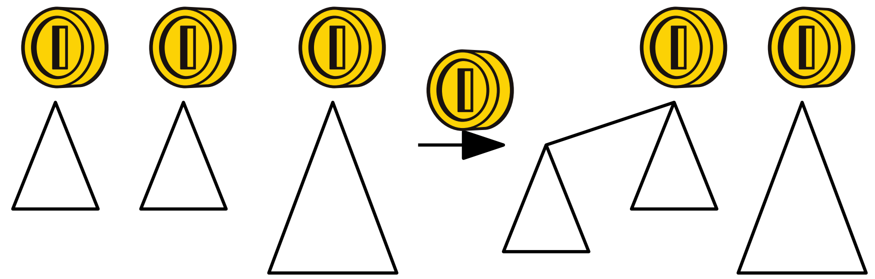

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all amortized costs as stated
+ always enough coins to pay
for actual costs

Binomial Heap – Potential Method

$\Phi(D_i) = c \cdot \#\text{trees in } D_i$

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$$\hat{c}_i = c_i + \Delta_i = 1 + k \; + c - c(k-1)$$

$$= 2 + c \; - (c-1)(k-1) \leq 2 + c = \Theta(1)$$

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deleteMin: $c_i = \underbrace{1}_{\text{find and remove min}} + \underbrace{(r + k) + \ell}_{\text{union}} + \underbrace{\log n}_{\text{update min pointer}}$, where $r =$ degree of min

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$\Delta_i = c \cdot (r - 1) - c \cdot \ell = c \cdot (r - \ell - 1)$

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$\rightarrow$ all amortized costs as claimed

Runtimes

	worst-case	amortised
make	$O(1)$	$O(1)$
min	$O(1)$	$O(1)$
insert	$O(\log n)$	$O(1)$
delete-min	$O(\log n)$	$O(\log n)$
union	$O(\log n)$	$O(\log n)$

Amortised Analysis with accounting or potential method

Lazy Union:
only concatenate lists and link only for a delete-min

Runtimes

	worst-case	amortised	amortised lazy union
make	$O(1)$	$O(1)$	$O(1)$
min	$O(1)$	$O(1)$	$O(1)$
insert	$O(\log n)$	$O(1)$	$O(1)$
delete-min	$O(\log n)$	$O(\log n)$	$O(\log n)$
union	$O(\log n)$	$O(\log n)$	$O(1)$

Amortised Analysis with accounting or potential method

Lazy Union:

only concatenate lists and link only for a delete-min

Lazy Union

store: roots in doubly-linked list and maintain min-pointer

operations:
make-0 and link: as before

Lazy Union

store: roots in doubly-linked list and maintain min-pointer

operations:

make-0 and link: as before

union: only concatenate lists (!) + update min-pointer

Lazy Union

store: roots in doubly-linked list and maintain min-pointer

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make-0 and link: as before
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example:

insert(5) $\overset{\downarrow}{5}$

Lazy Union

store: roots in doubly-linked list and maintain min-pointer

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union: only concatenate lists (!) + update min-pointer

insert: make-1 + union

example:

insert(5) 5

insert(7) $7 \rightarrow 5$

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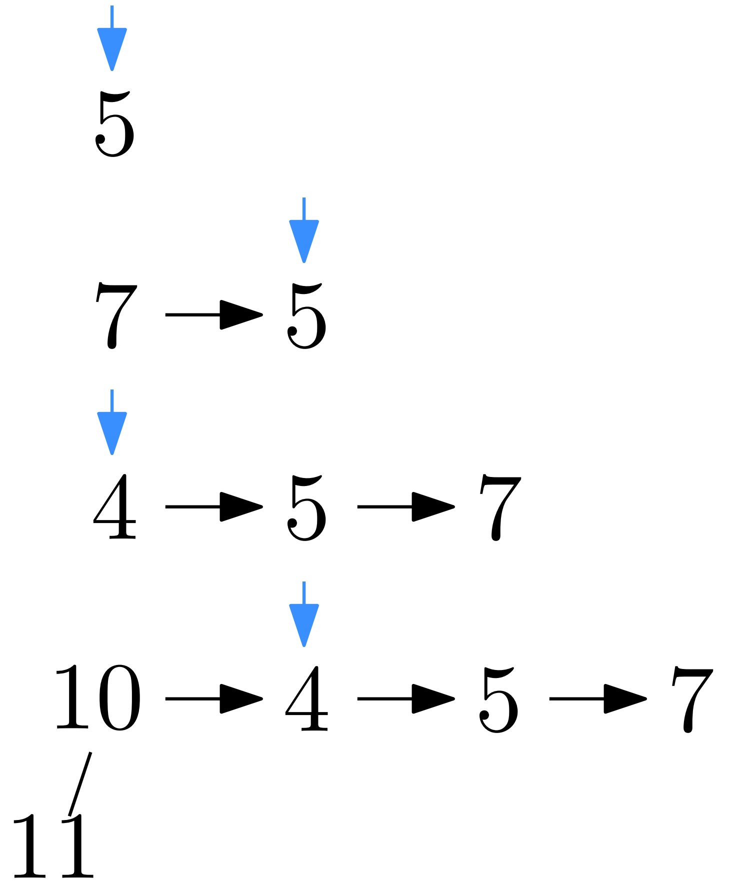

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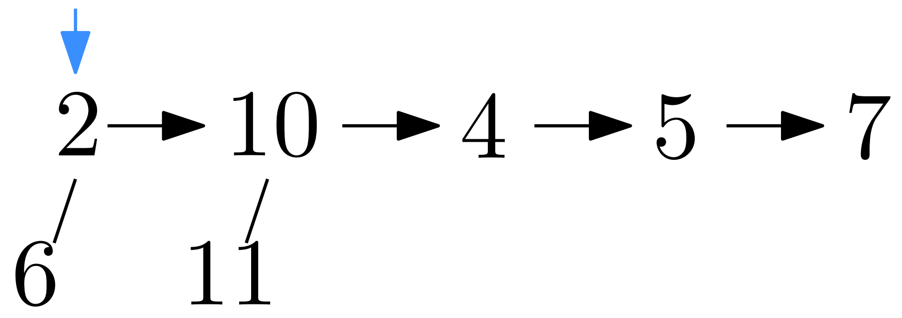

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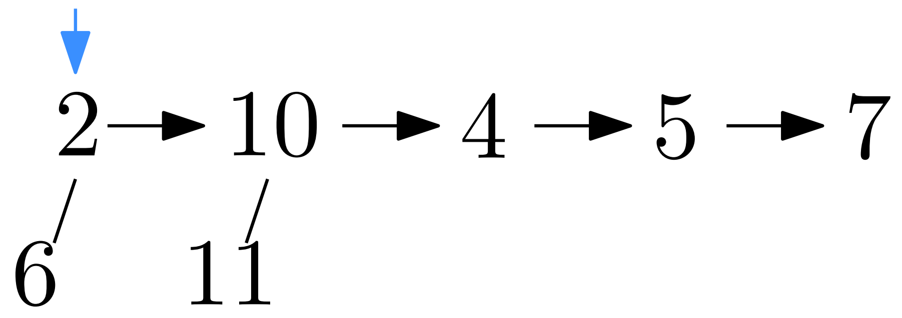

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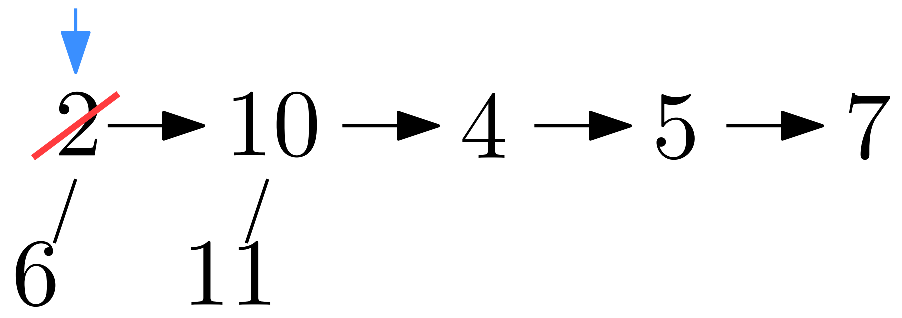

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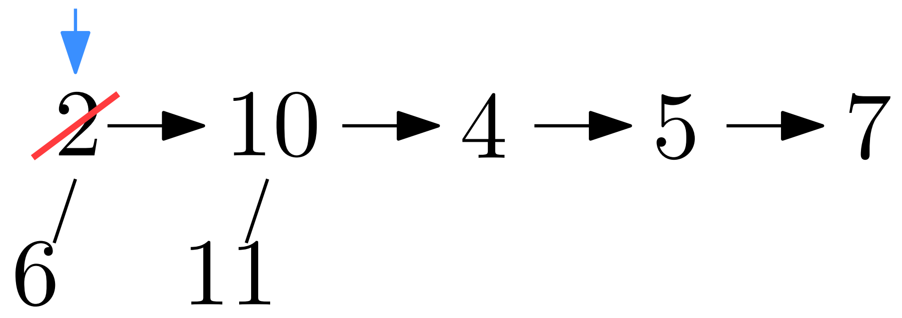

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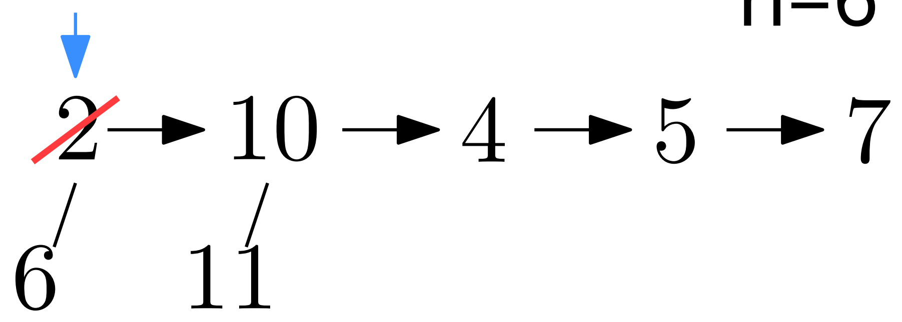

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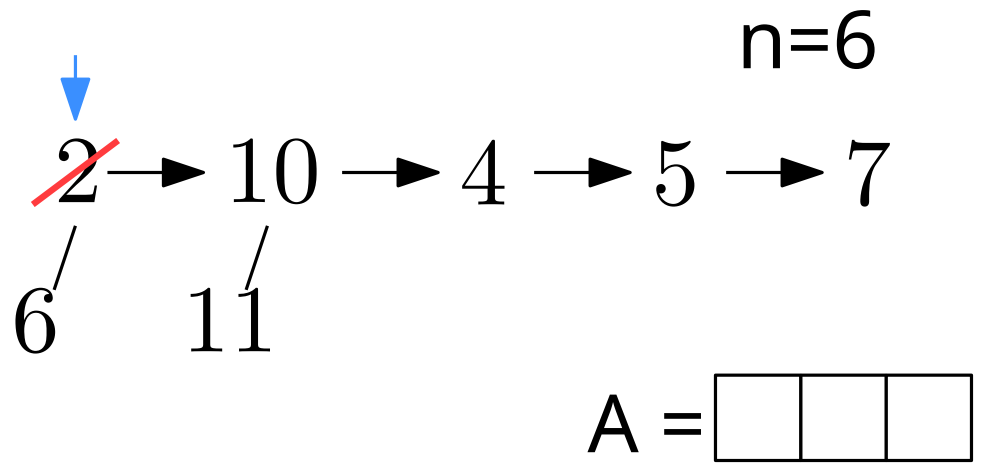

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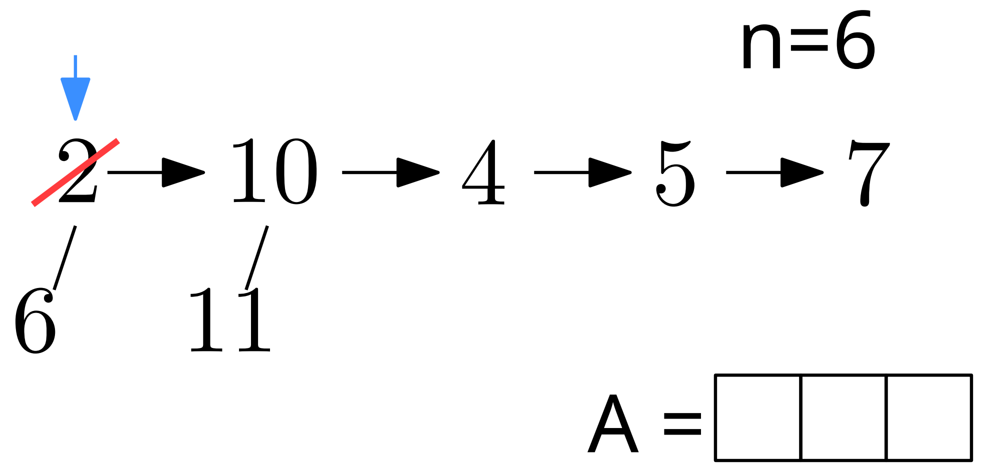

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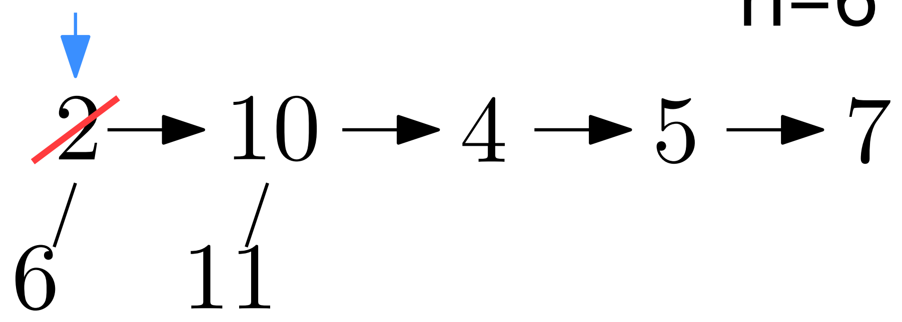

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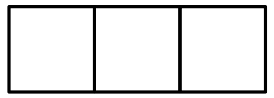
A = 

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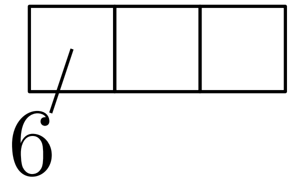

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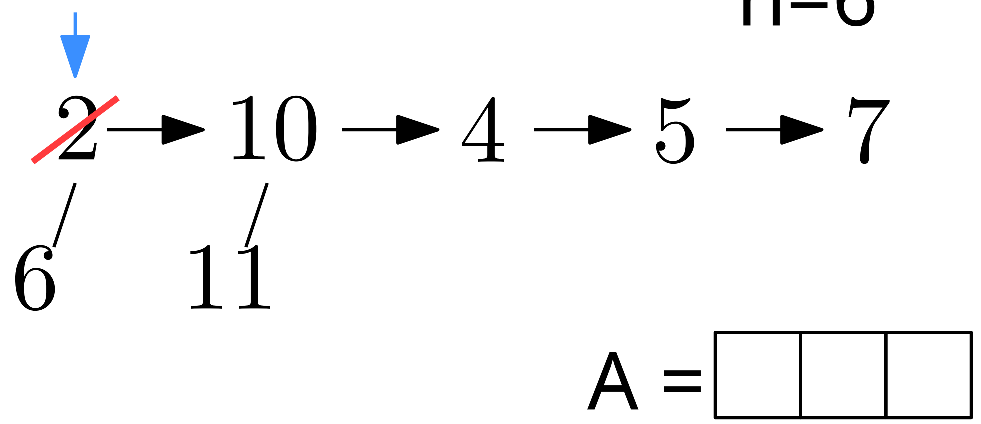

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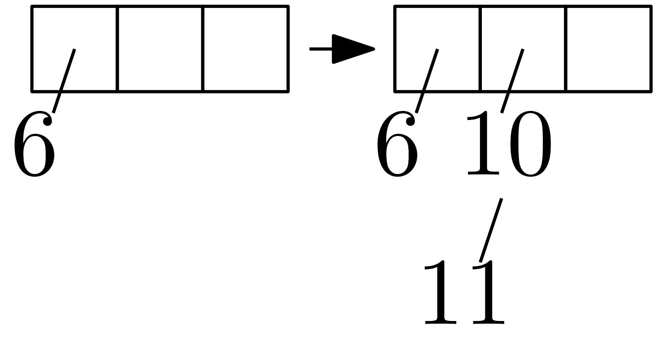

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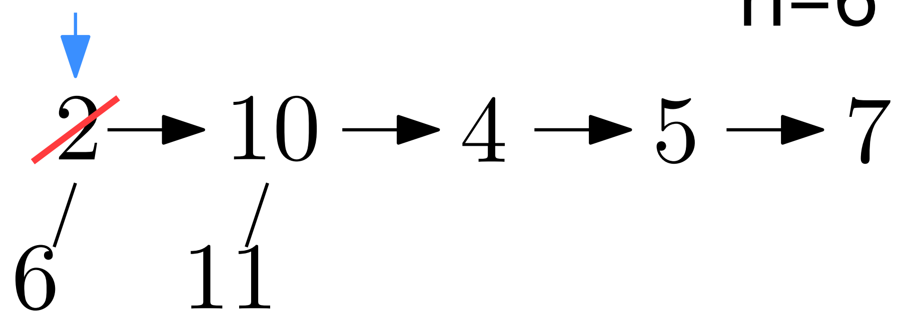

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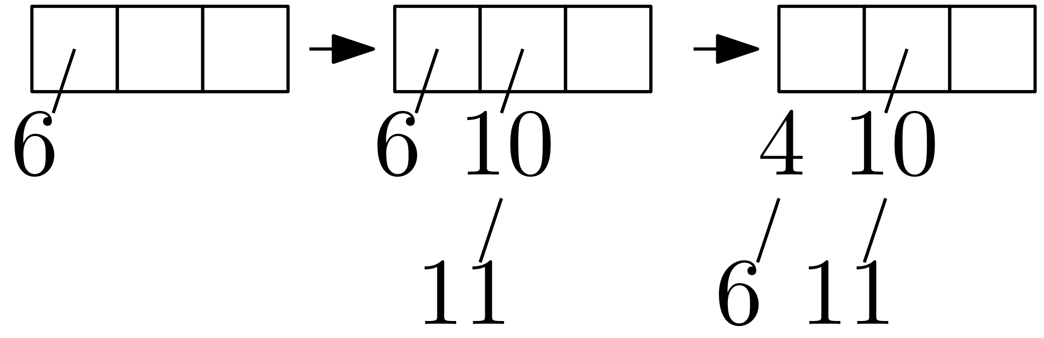

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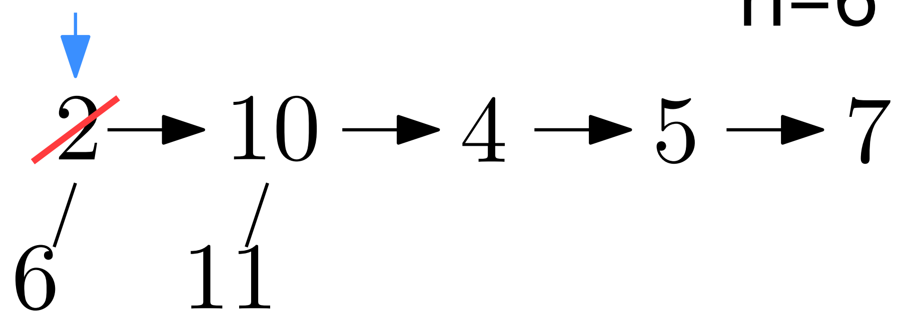

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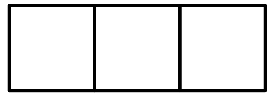
A = 

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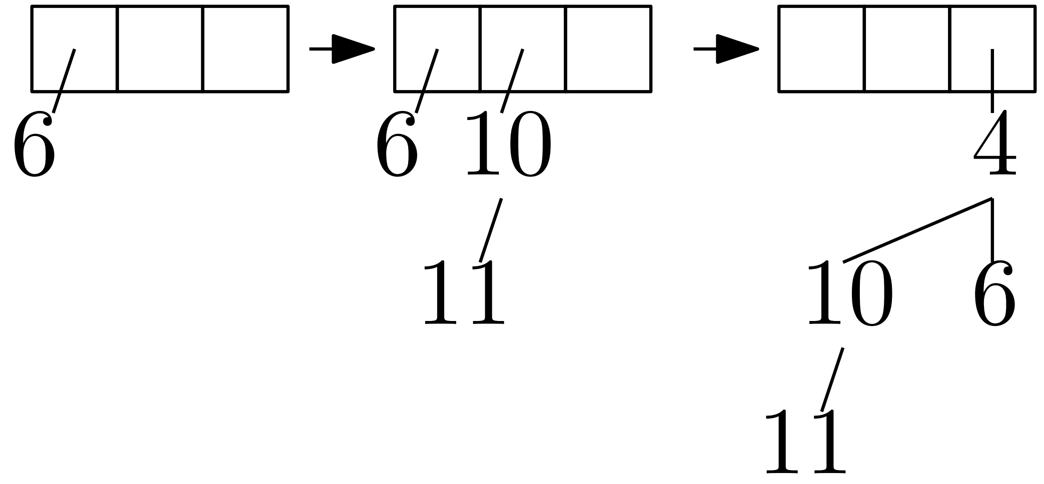

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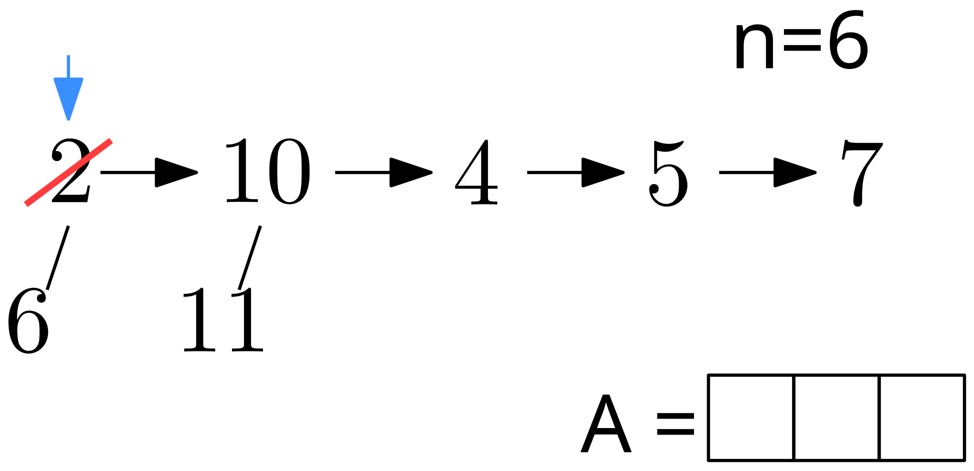

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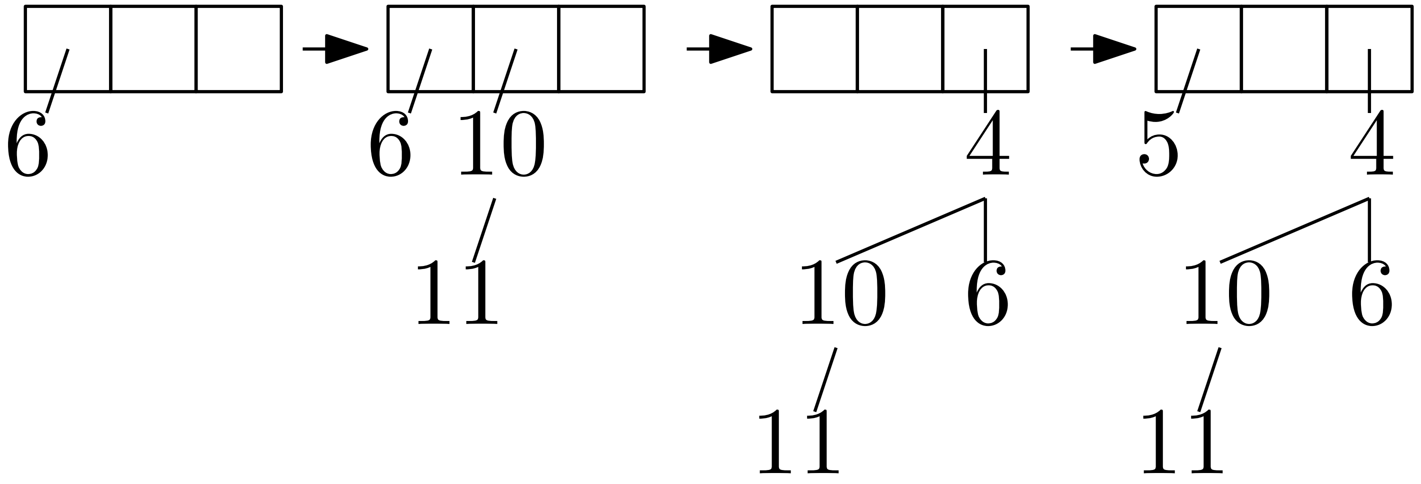

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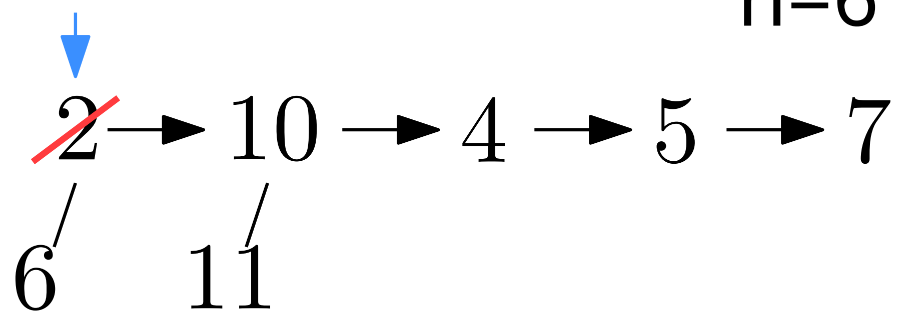

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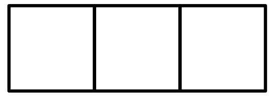
A = 

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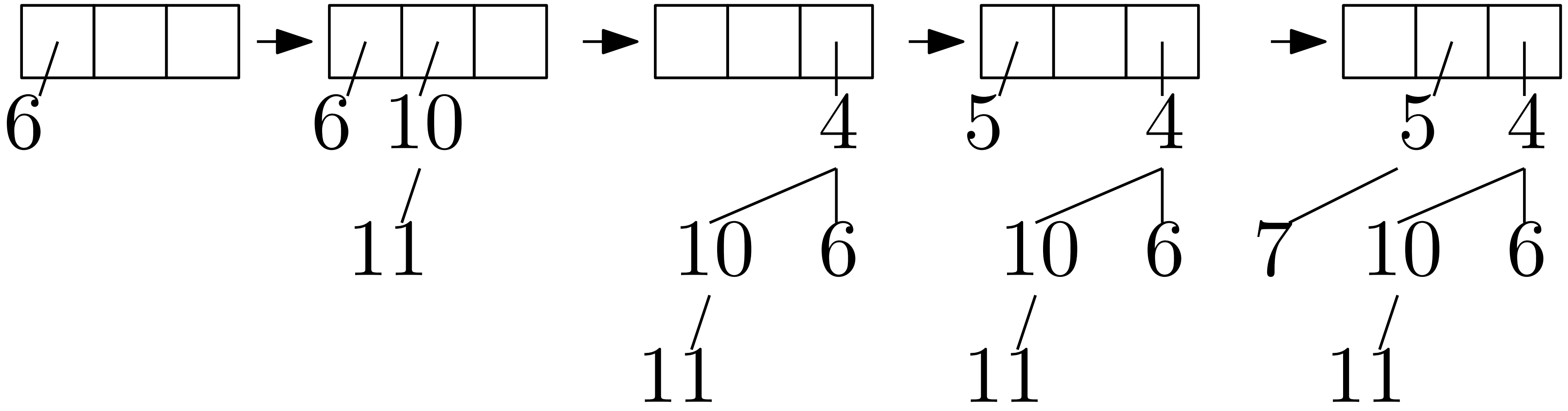

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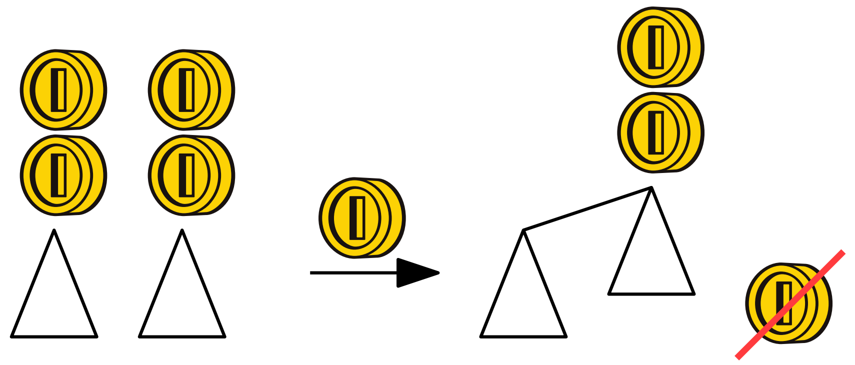

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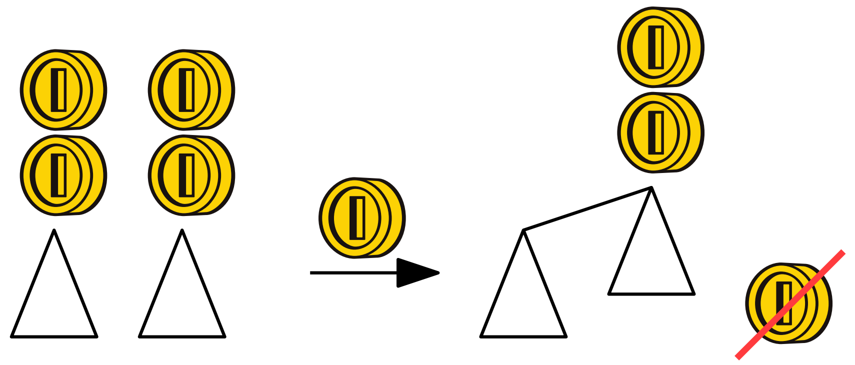

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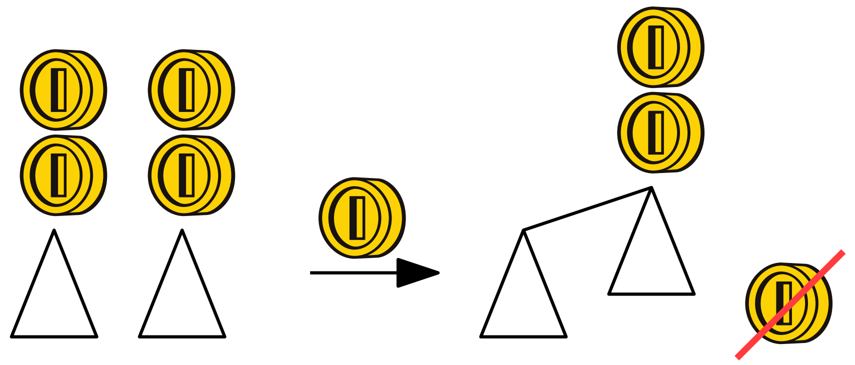

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add to array
+ link

everything else

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$$\Delta_i \leq c \log n - c \cdot \ell$$

new trees: children of min

trees removed by linking

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$$\begin{aligned} \hat{c}_i = c_i + \Delta_i &= t + \ell - c \cdot \ell + O(\log n) \\ &\leq t - \ell + O(\log n) \end{aligned}$$

Lazy Union – Potential Method

$\Phi(D_i) = c \cdot \#$trees in D_i , where $c \geq 2$ is a constant. *(here: $c = 2$ works)*

$\Phi(D_0) = 0$, $\Phi(D_i) \geq 0 = \Phi(D_0)$ ✓

$\hat{c}_i = c_i + \Delta_i$, where $\Delta_i = \Phi(D_i) - \Phi(D_{i-1})$

make, link, insert: as for regular union

union: $c_i = 1, \Delta_i = 0 \rightarrow \hat{c}_i = 1$

deleteMin: $c_i = t + \ell + O(\log n)$, where $t = \#$trees to start (after removing min)
and $\ell = \#$links

$$\Delta_i \leq c \log n - c \cdot \ell$$

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since $t - \ell = \#$trees at end $= O(\log n)$

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all amortized costs as claimed

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Runtimes

	worst-case	amortised	amortised lazy union
make	$O(1)$	$O(1)$	$O(1)$
min	$O(1)$	$O(1)$	$O(1)$
insert	$O(\log n)$	$O(1)$	$O(1)$
delete-min	$O(\log n)$	$O(\log n)$	$O(\log n)$
union	$O(\log n)$	$O(\log n)$	$O(1)$

Amortised Analysis with accounting or potential method

Lazy Union:
only concatenate lists and link only for a delete-min

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Amortised Analysis with accounting or potential method

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but decreaseKey still costs $O(\log n)$ time! $\longrightarrow$ Fibonacci Heaps !