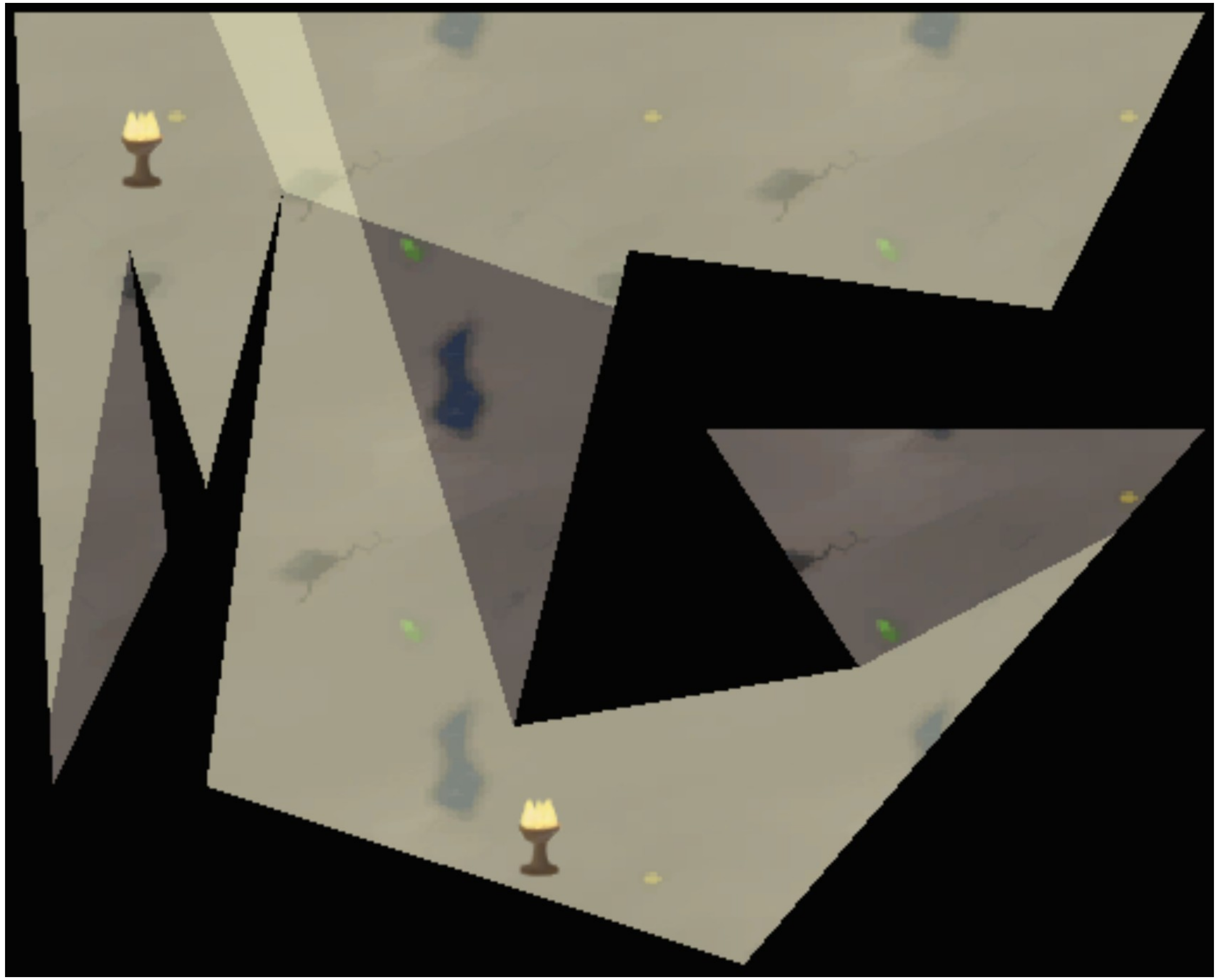
Polygon Triangulation

Geometric Algorithms

The Art Gallery Problem



https://kbuchin.github.io/ruler/art/

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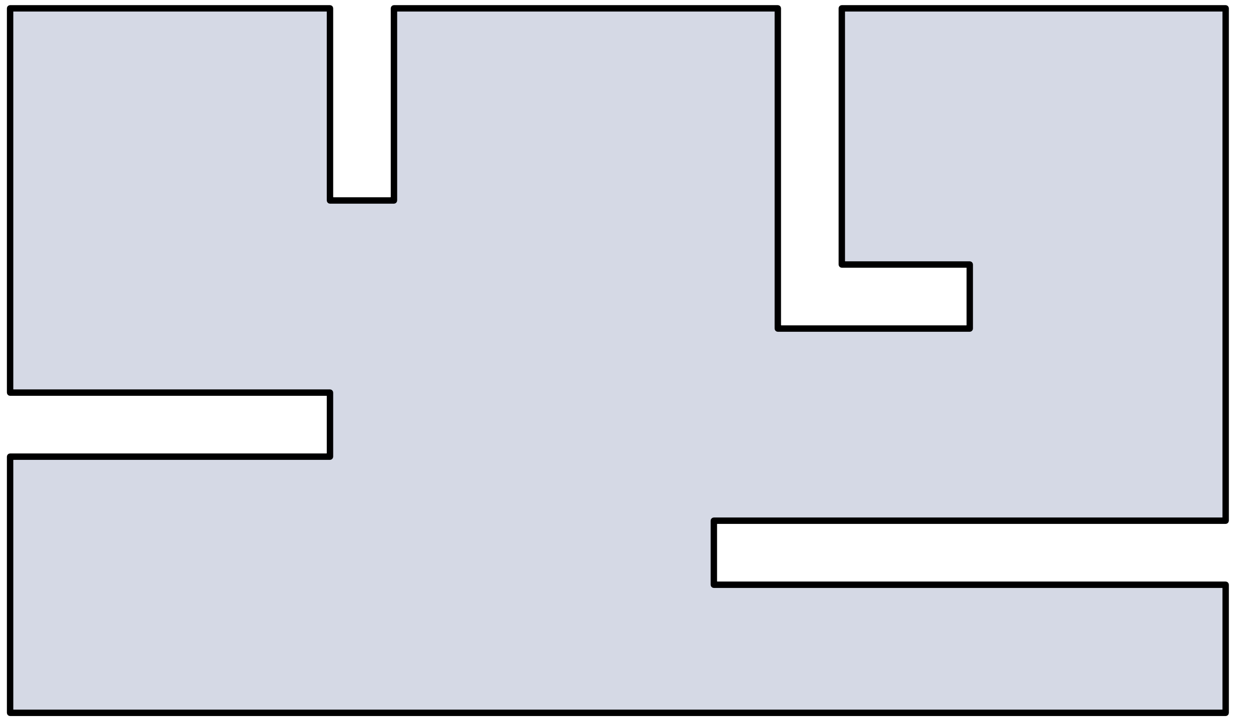
Problem: Install 360°-cameras for the surveillance of an art gallery such that the whole gallery is seen.



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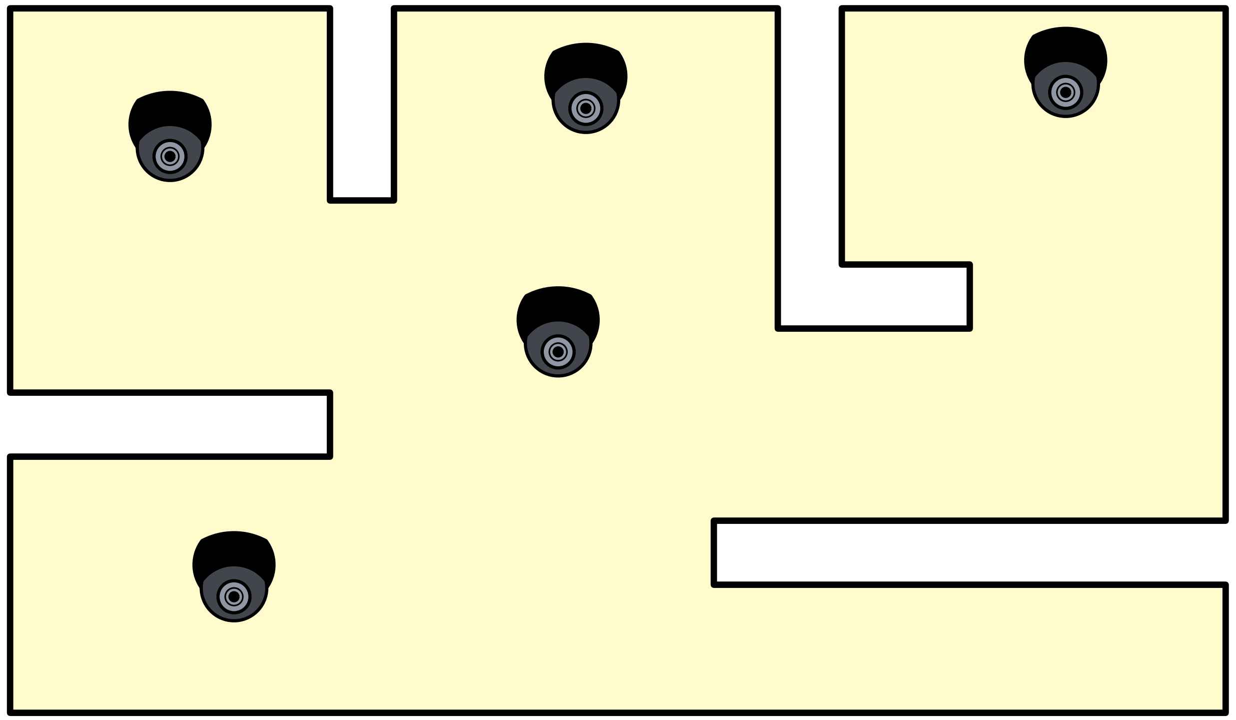

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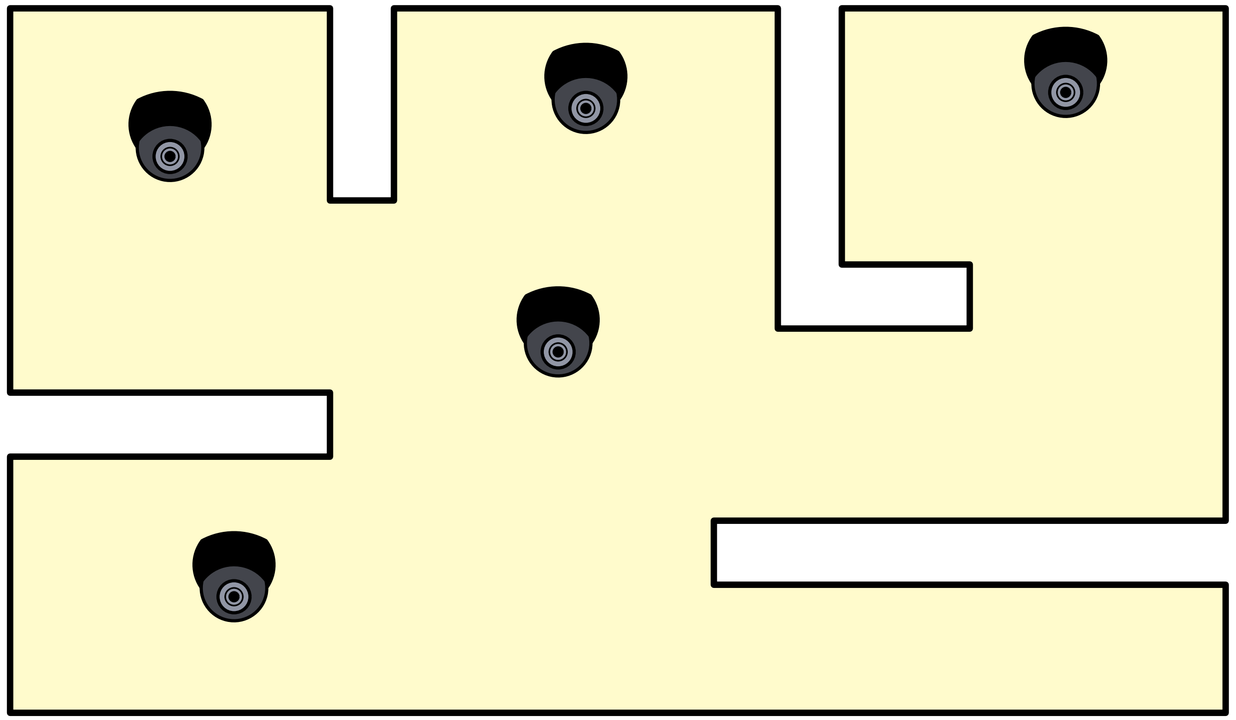

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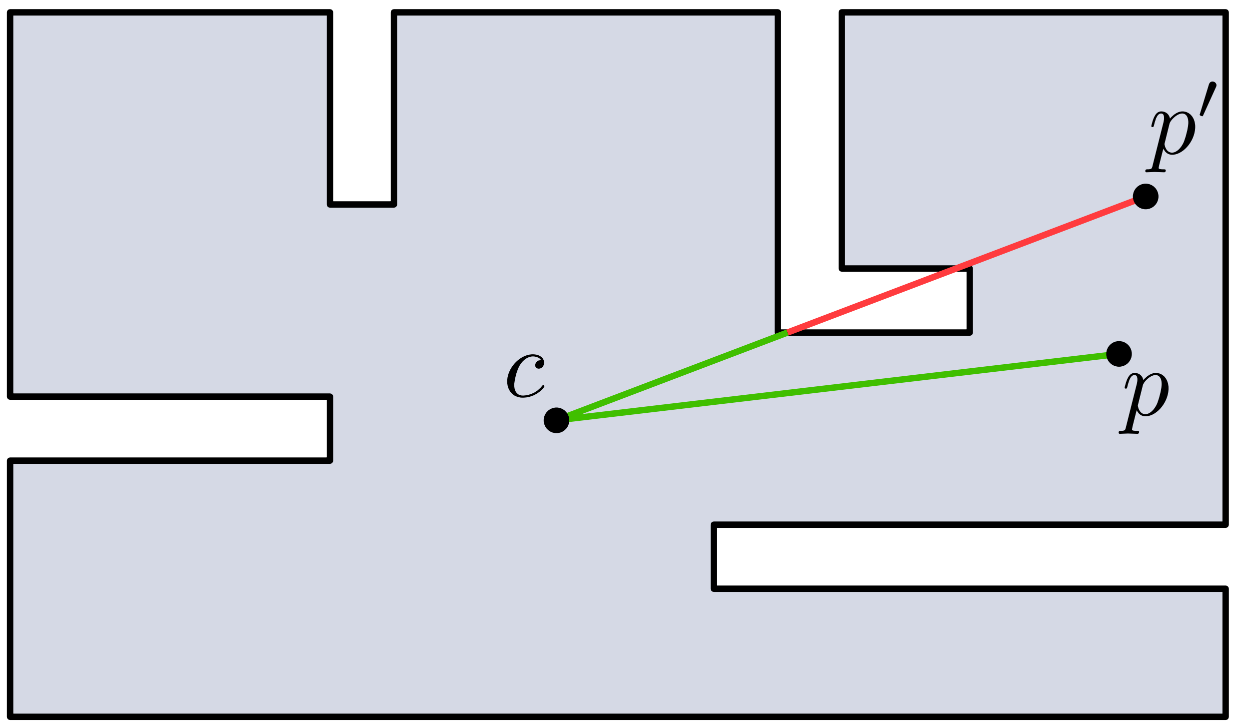

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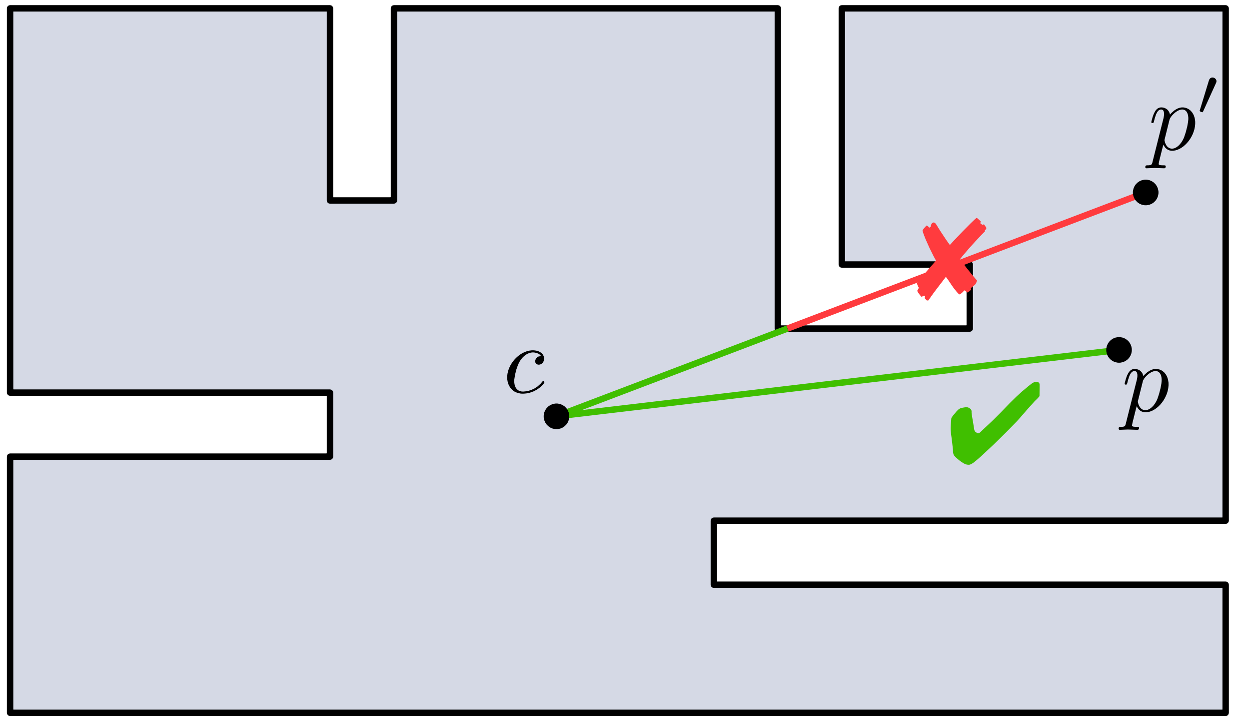

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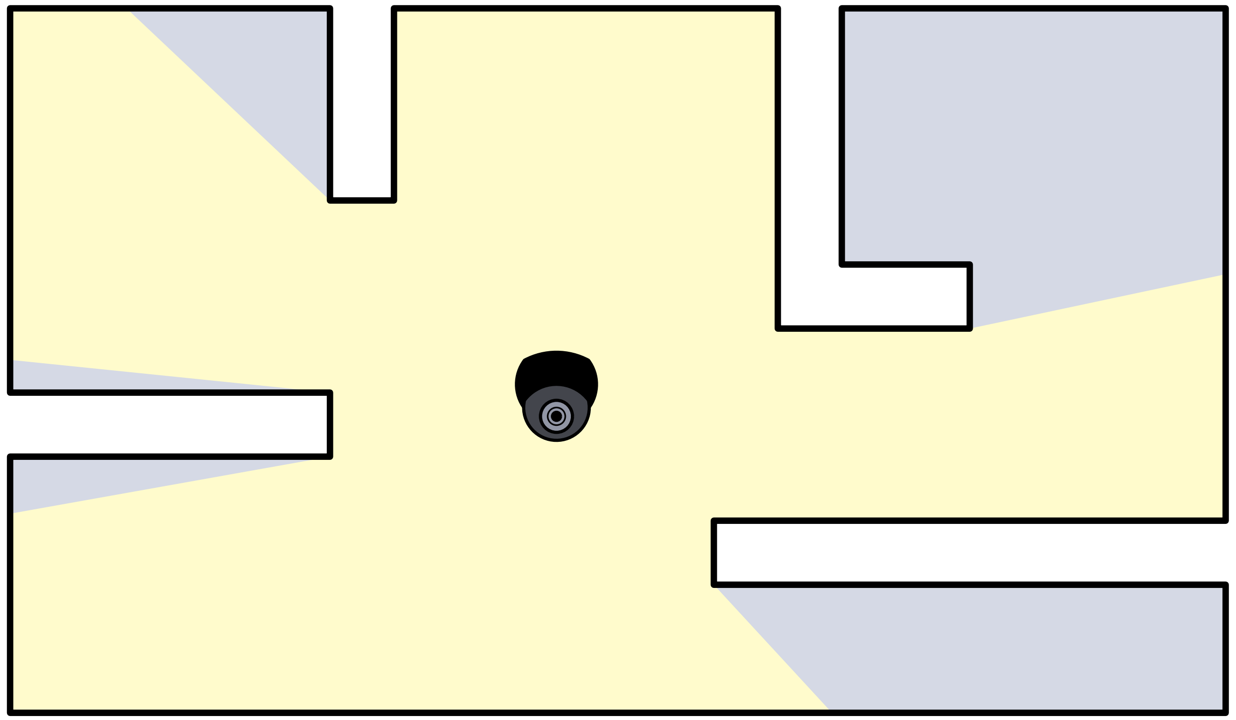

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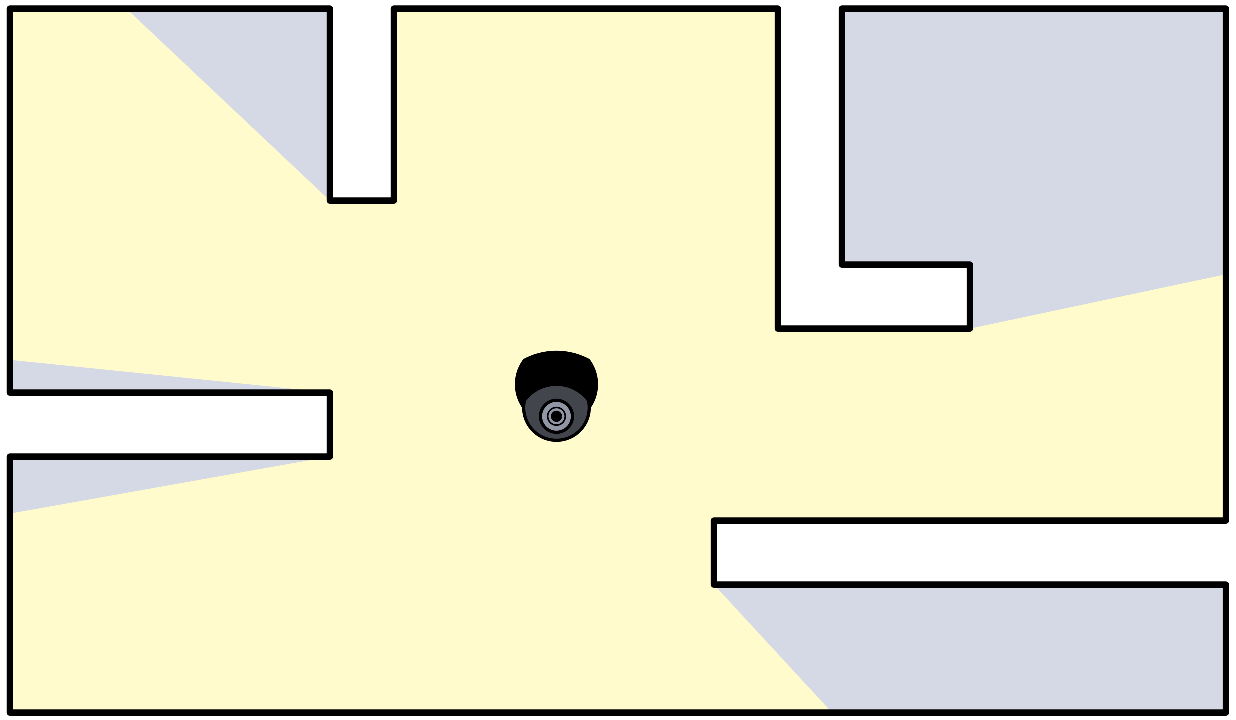

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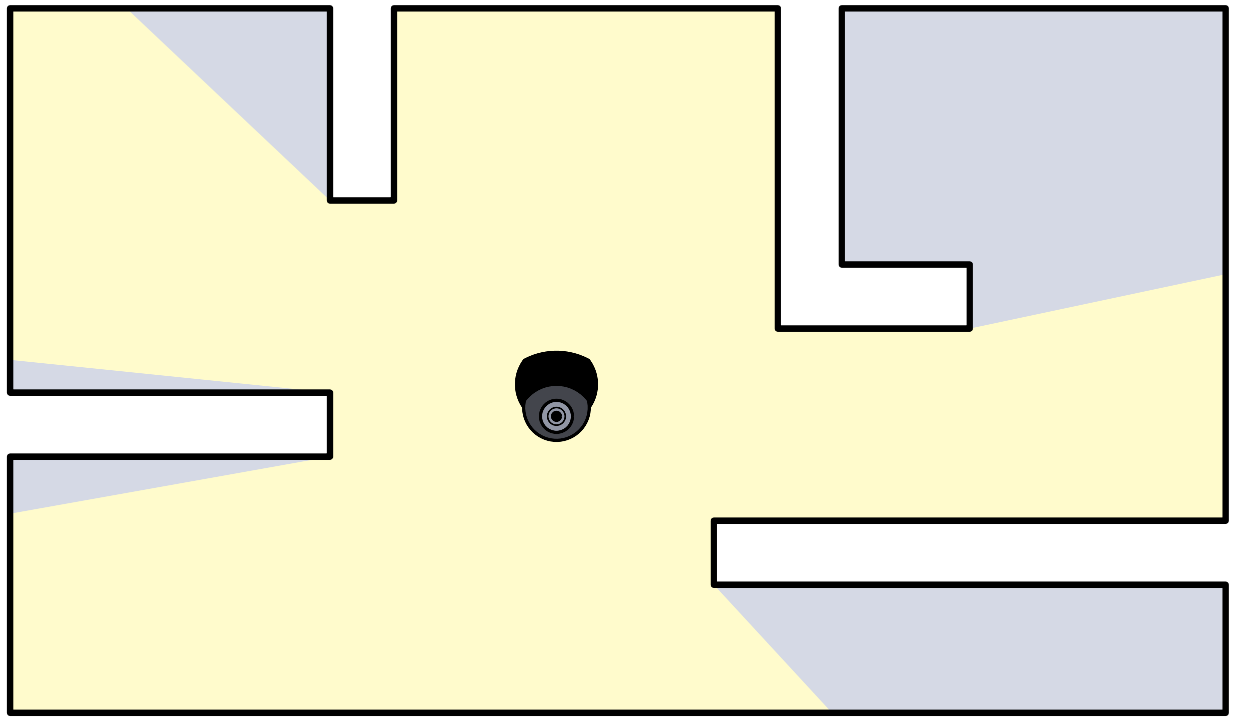

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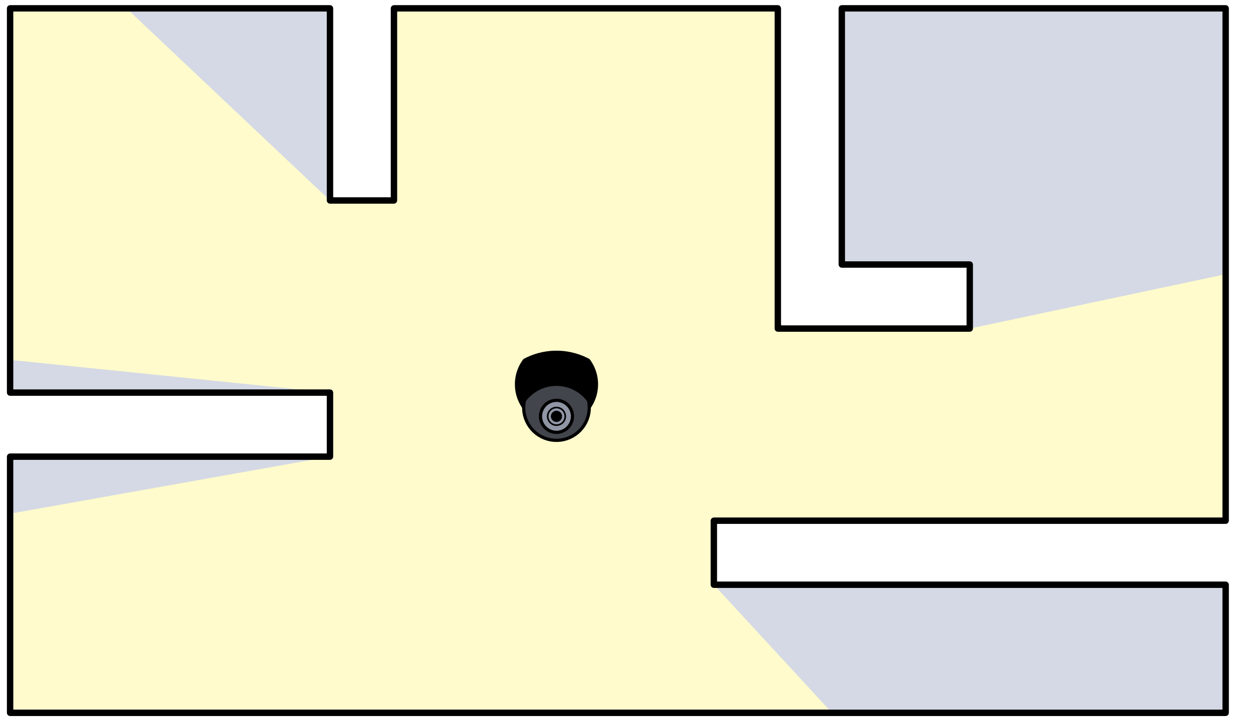

Goal: Use as few cameras as possible! NP-hard!

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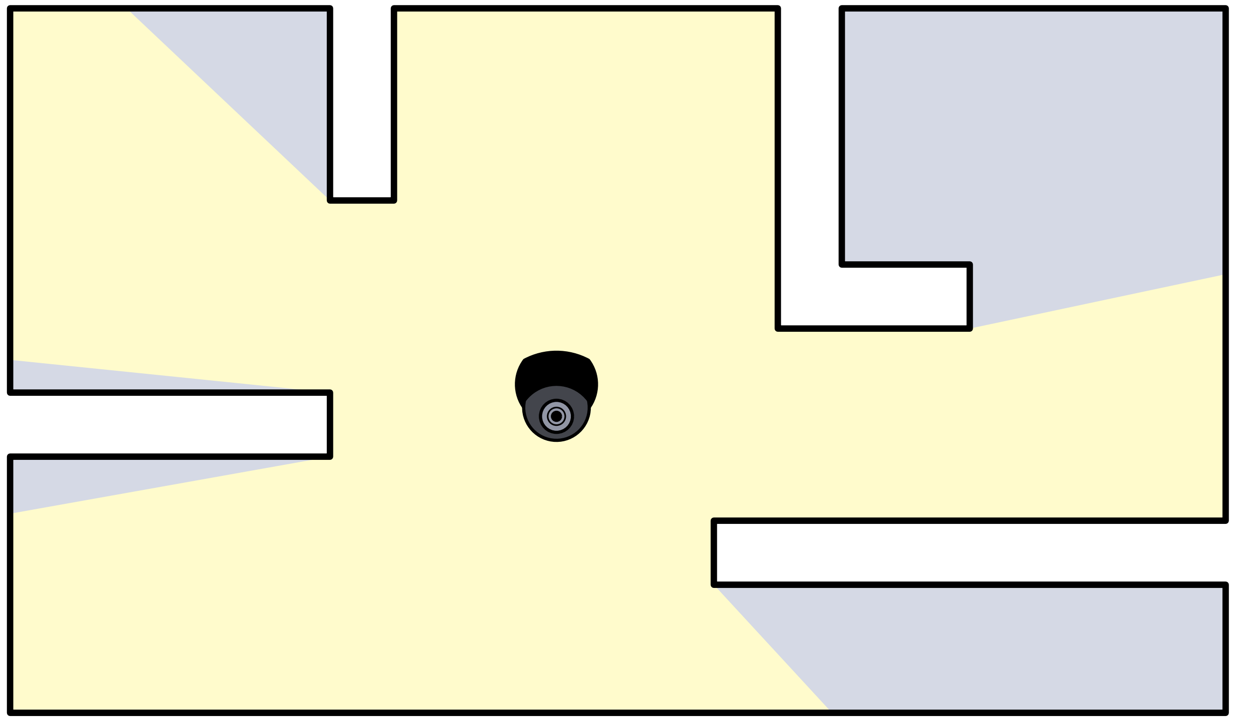

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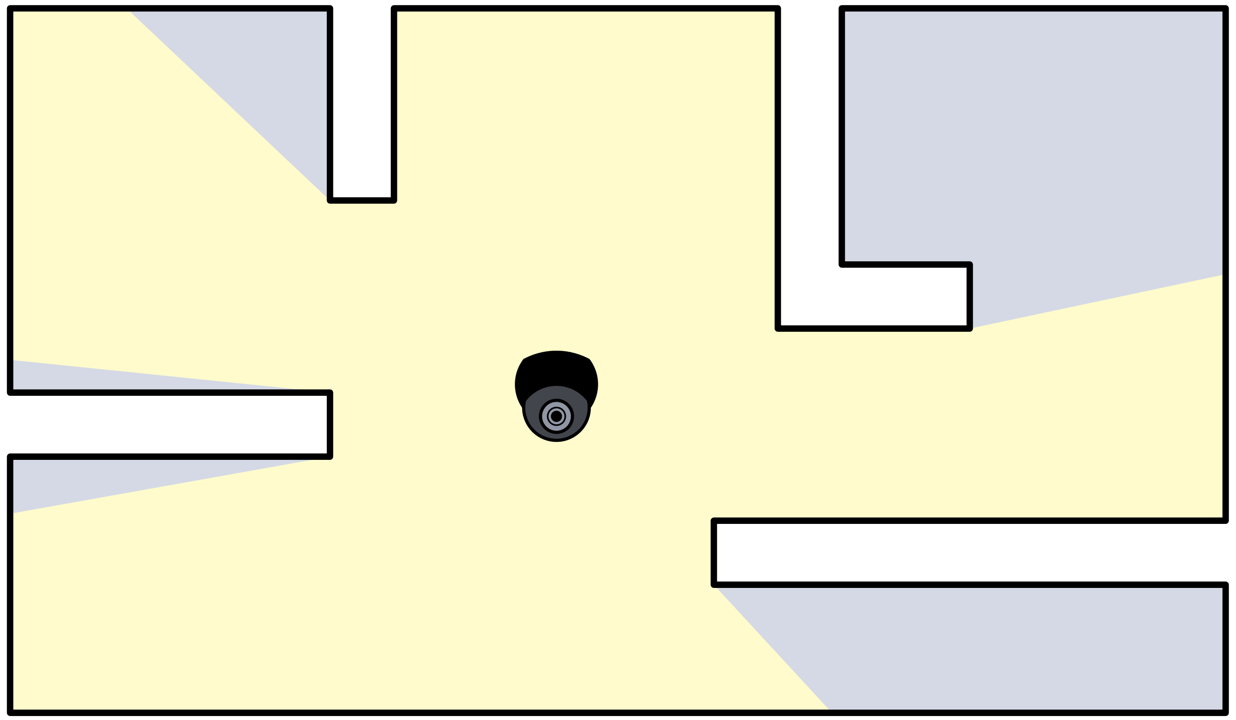

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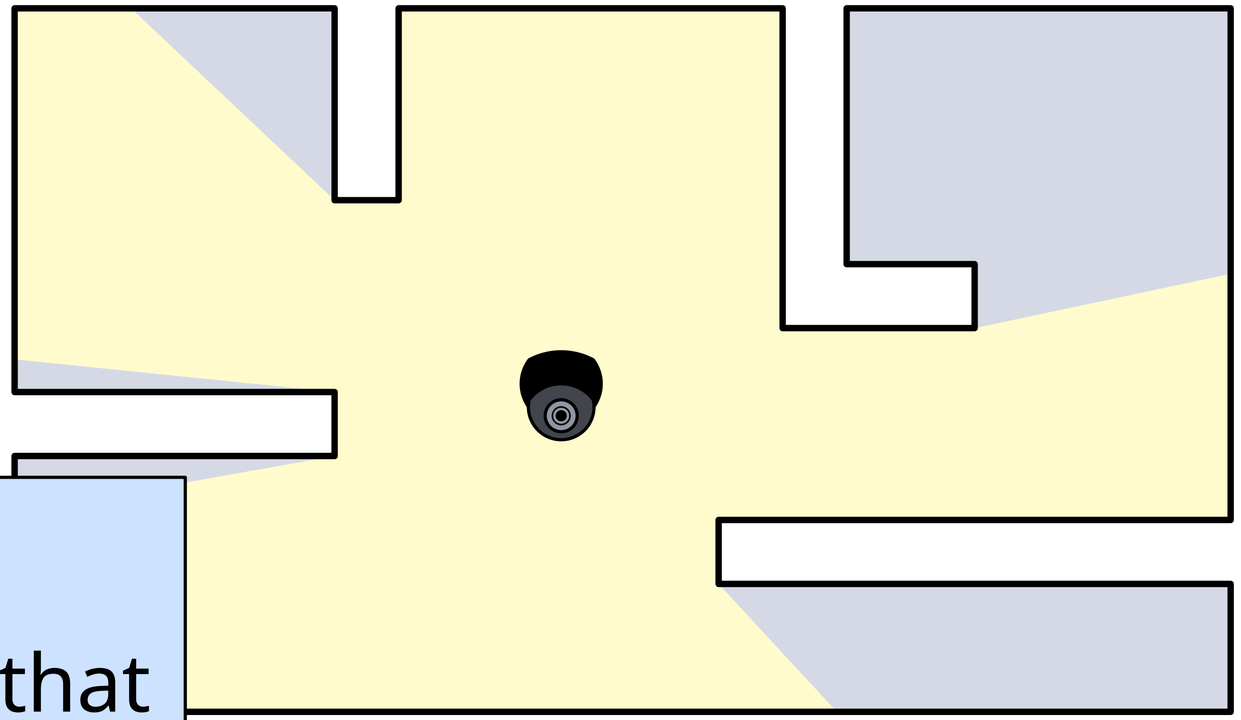

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Definitio

Point p

Observati

Every c

Goal: Use

Try to find bounds.

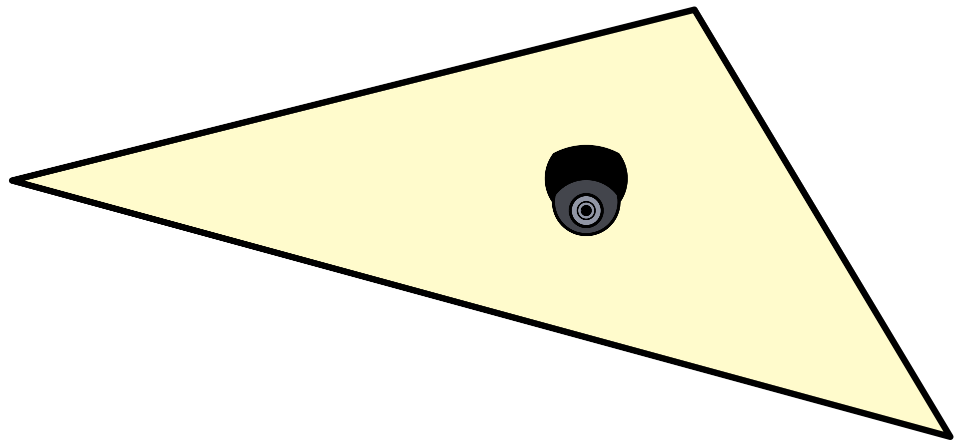
Upper bound: prove, for any polygon that that many cameras suffices.

Lower bound: construct family of polygons that needs many cameras.

Simple upper and lower bounds? between 1 and n

Simplifying the Problem

Observation: Triangles are easy to guard.



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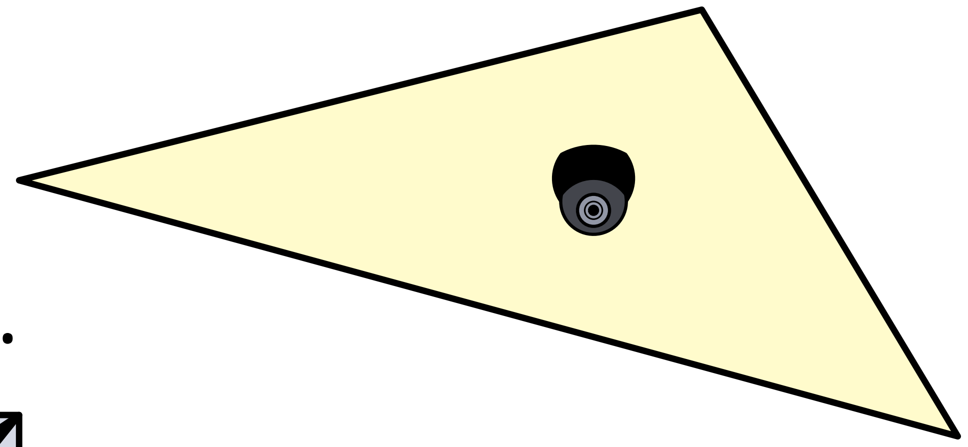

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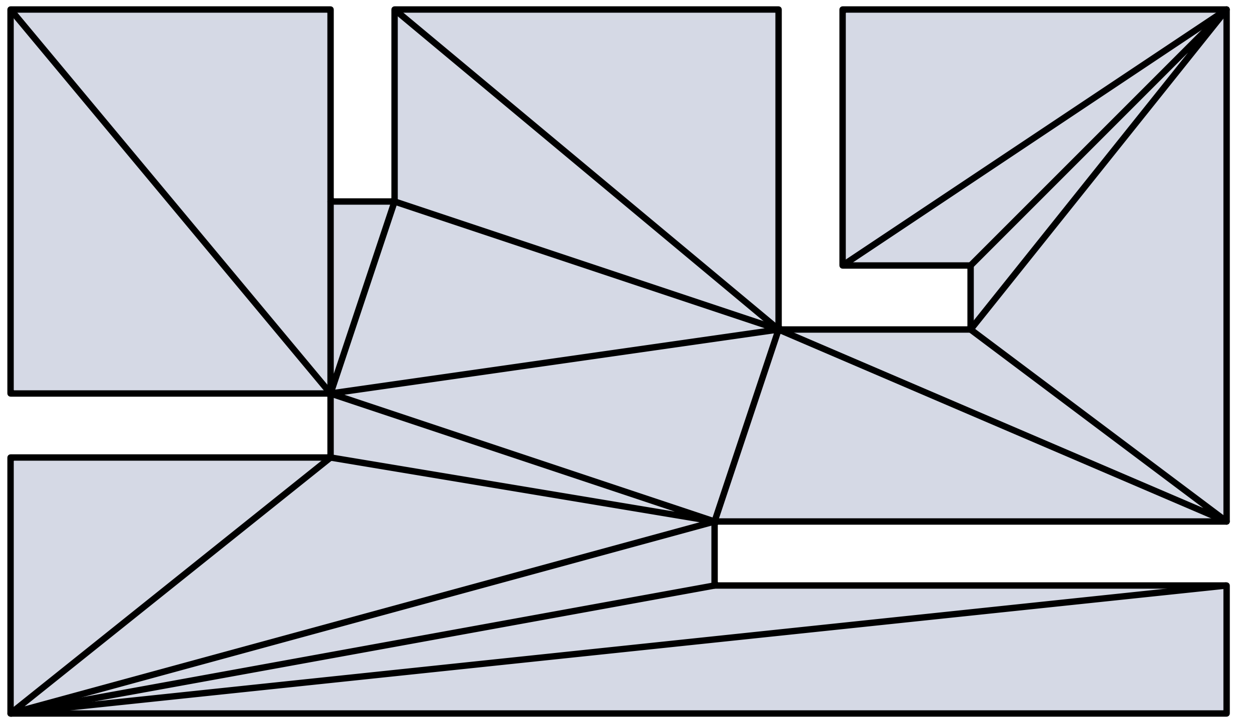

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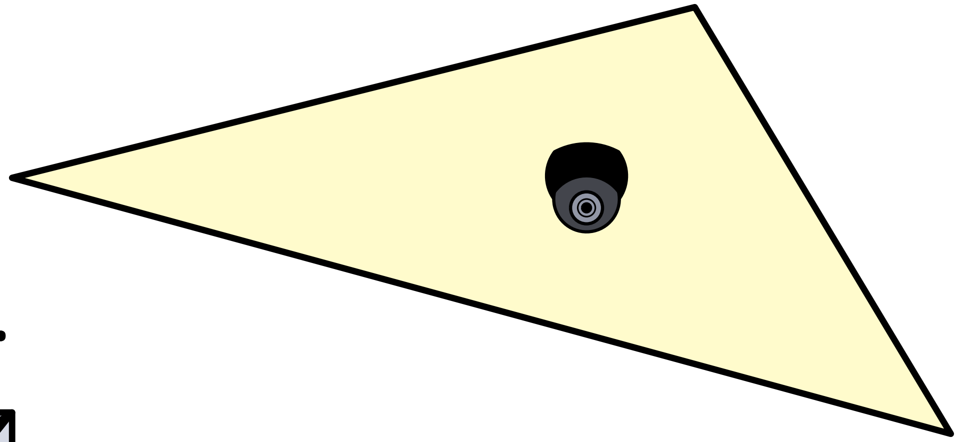

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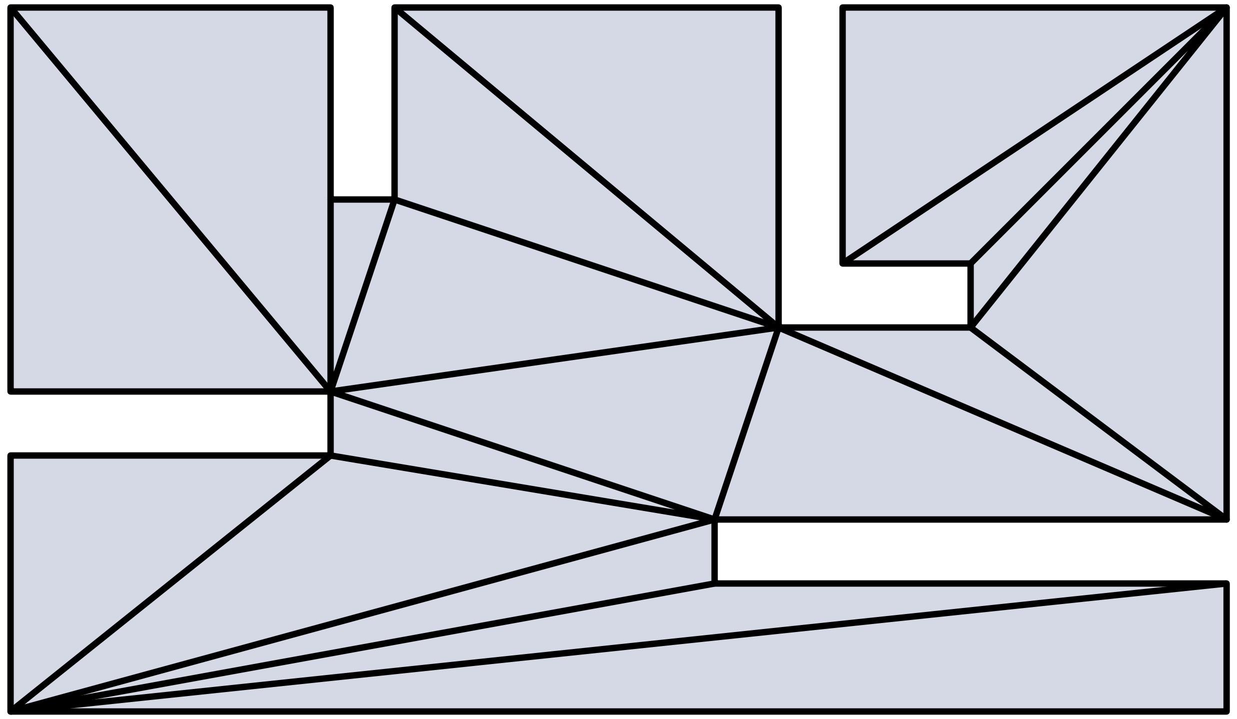
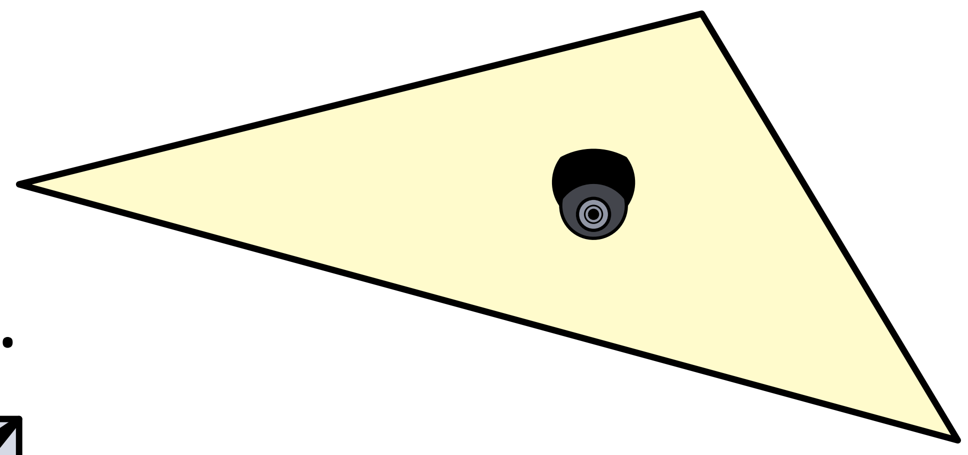


Does a triangulation always exist?

Is it unique?

How many triangles does a triangulation have?

Simplifying the Problem



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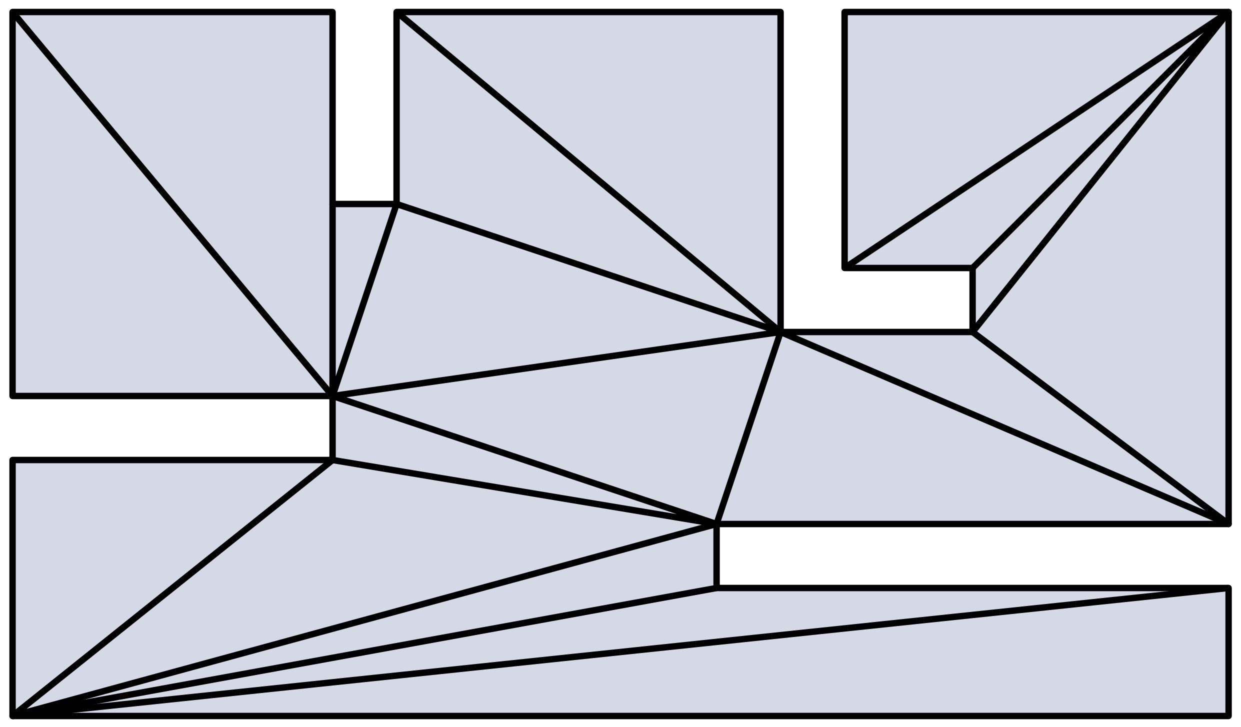

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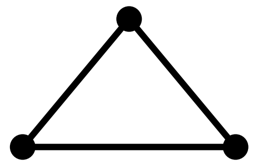
$n = 3:$  trivial

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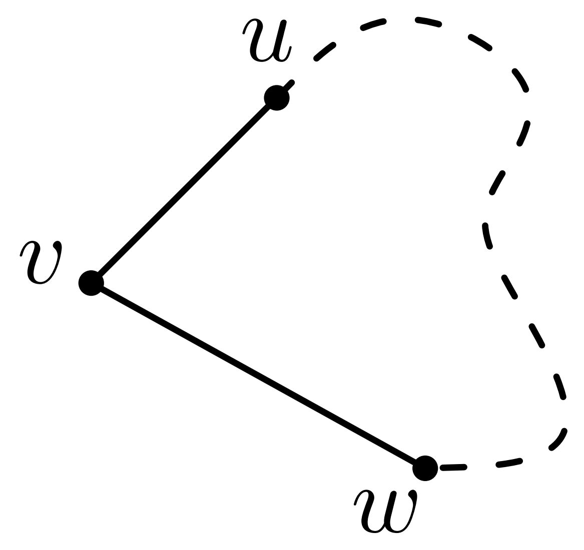

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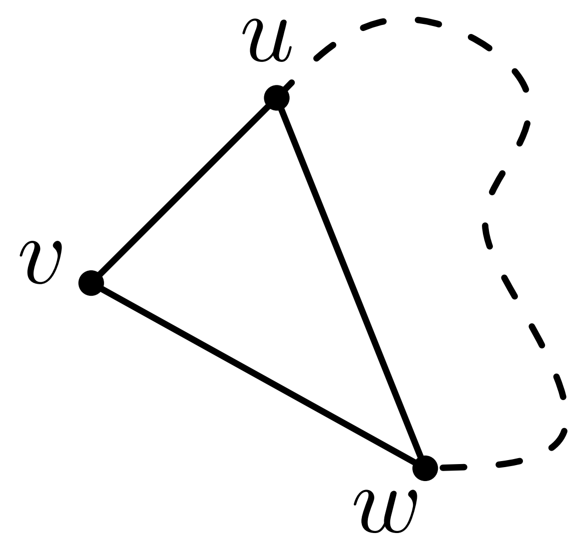

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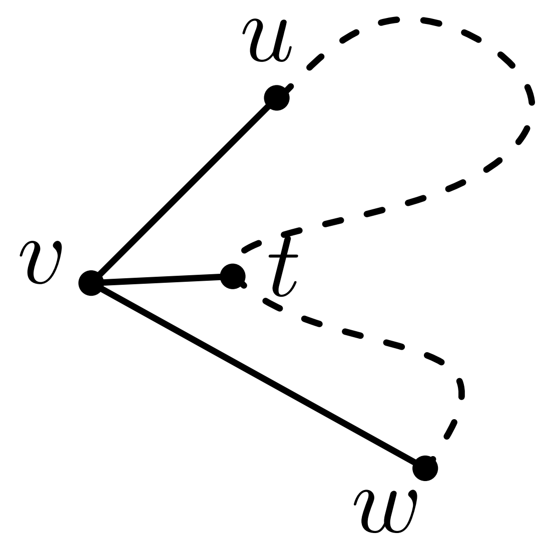

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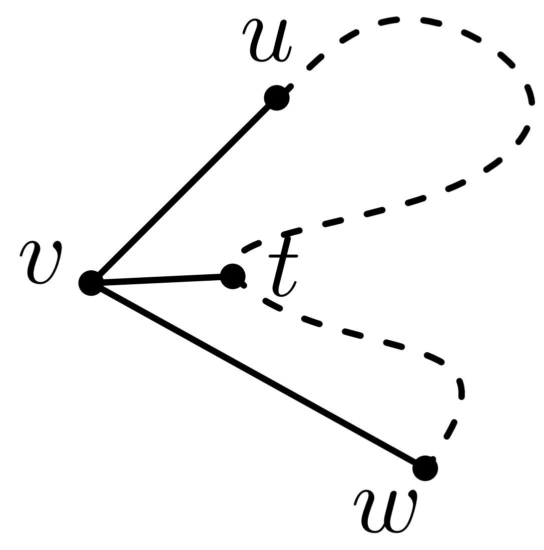

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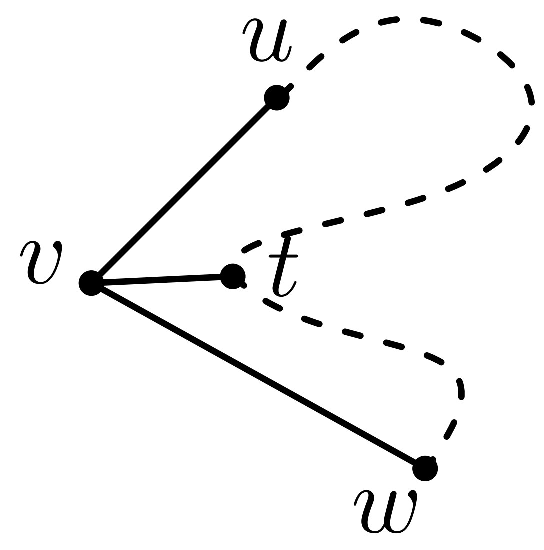

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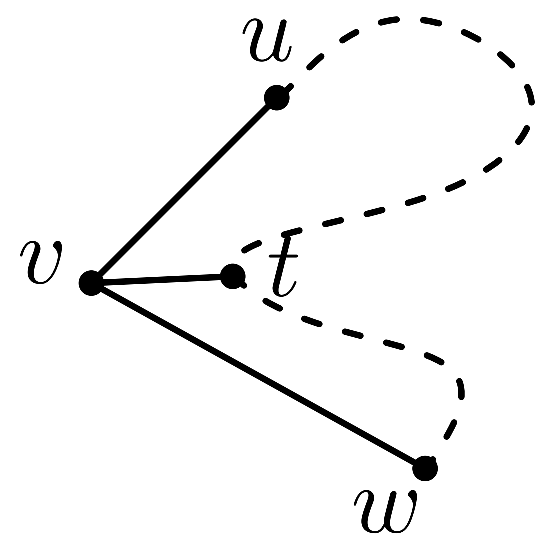

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Does the proof provide an algorithm?
Running time?

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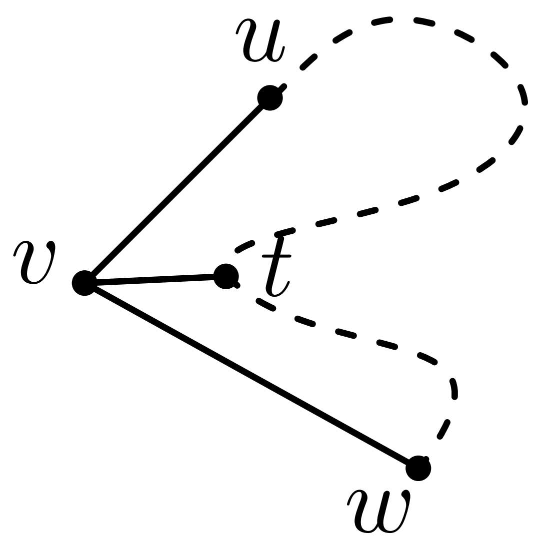

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In both cases: partition into polygons of size m and $n-m+2$,

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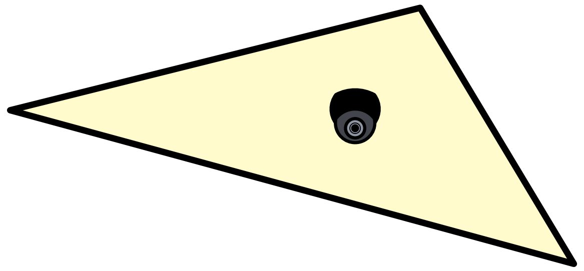
Proof results in recursive $O(n^2)$-algorithm!

Simplifying the Problem



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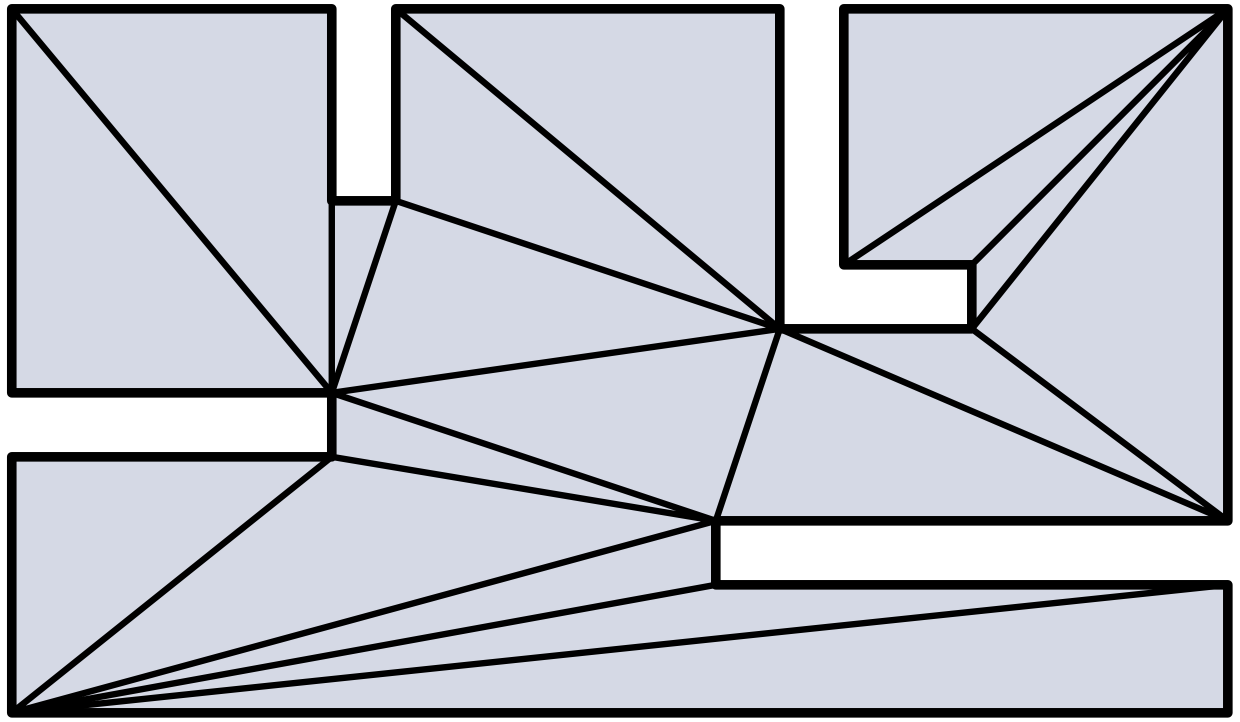

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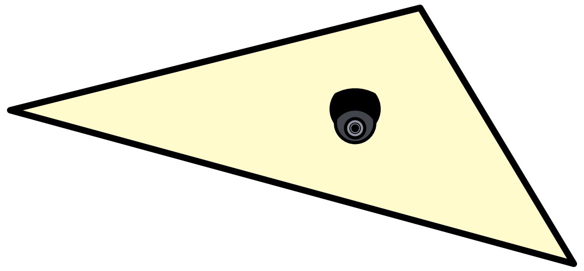
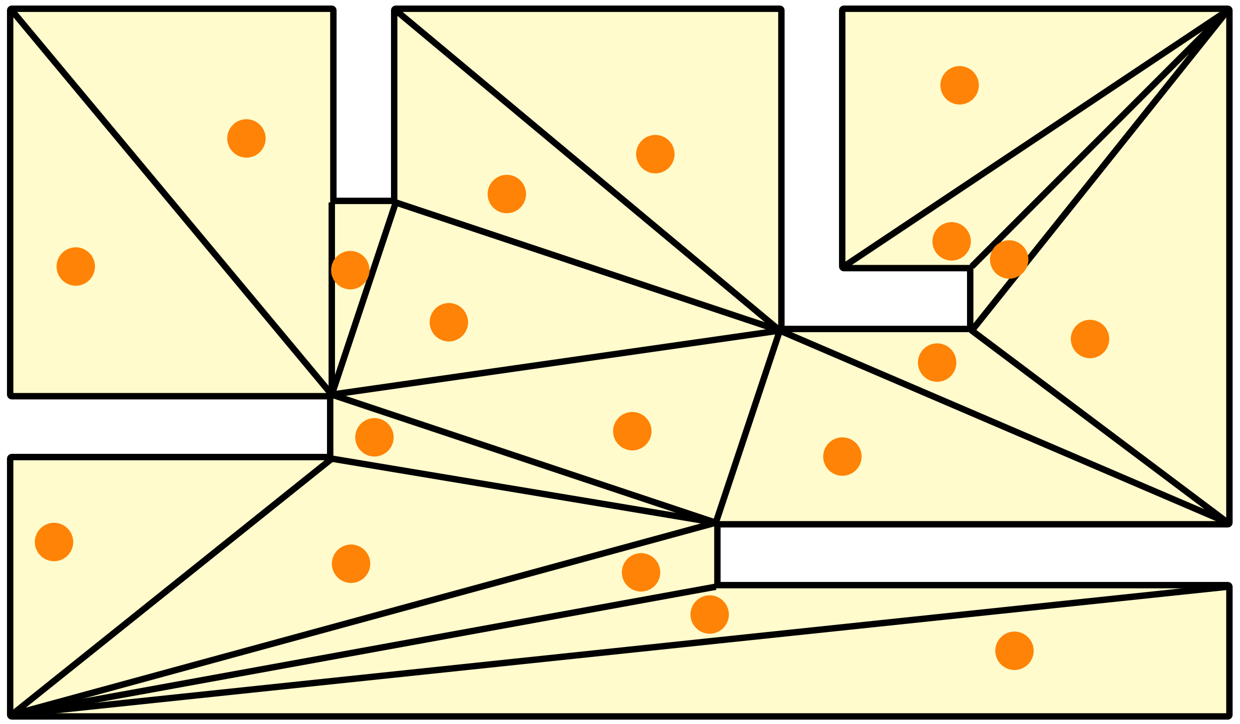


Idea: Partition P into triangles and guard every triangle.

- P can be guarded with $n-2$ cameras



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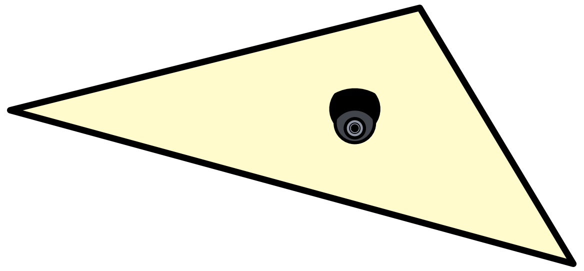

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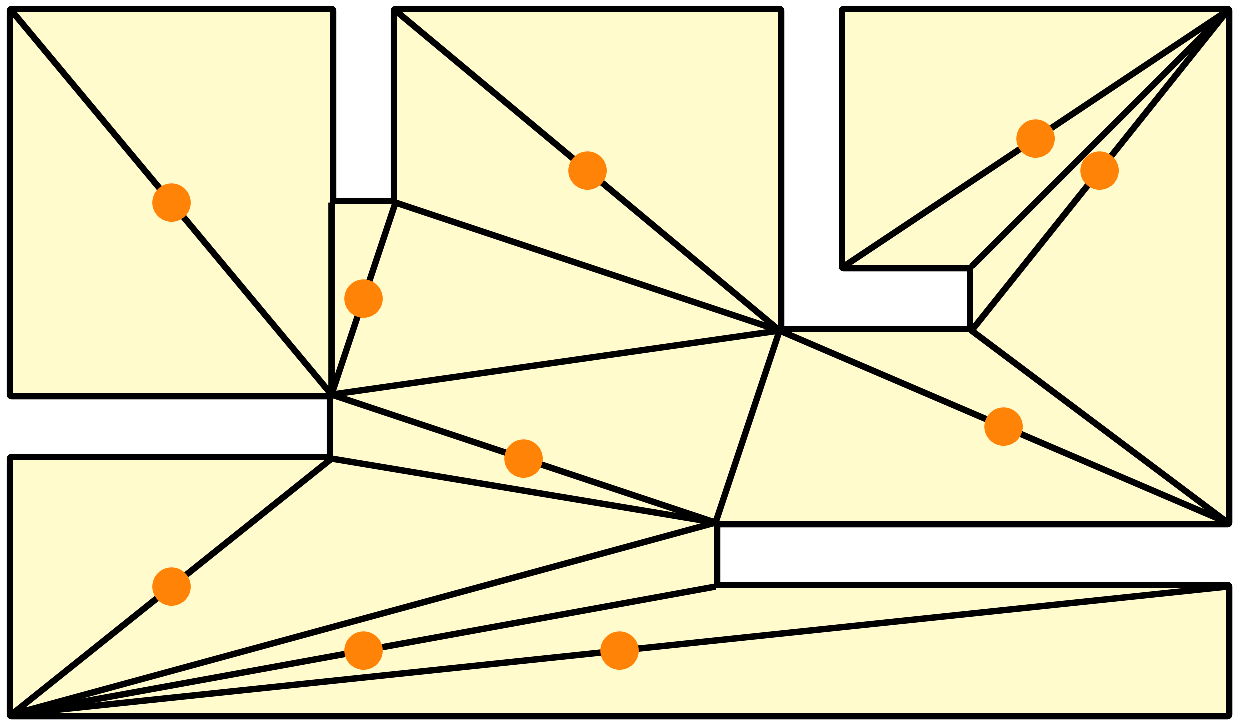

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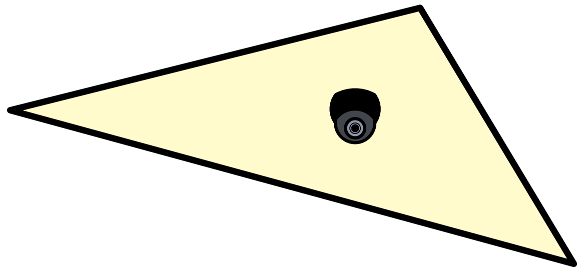

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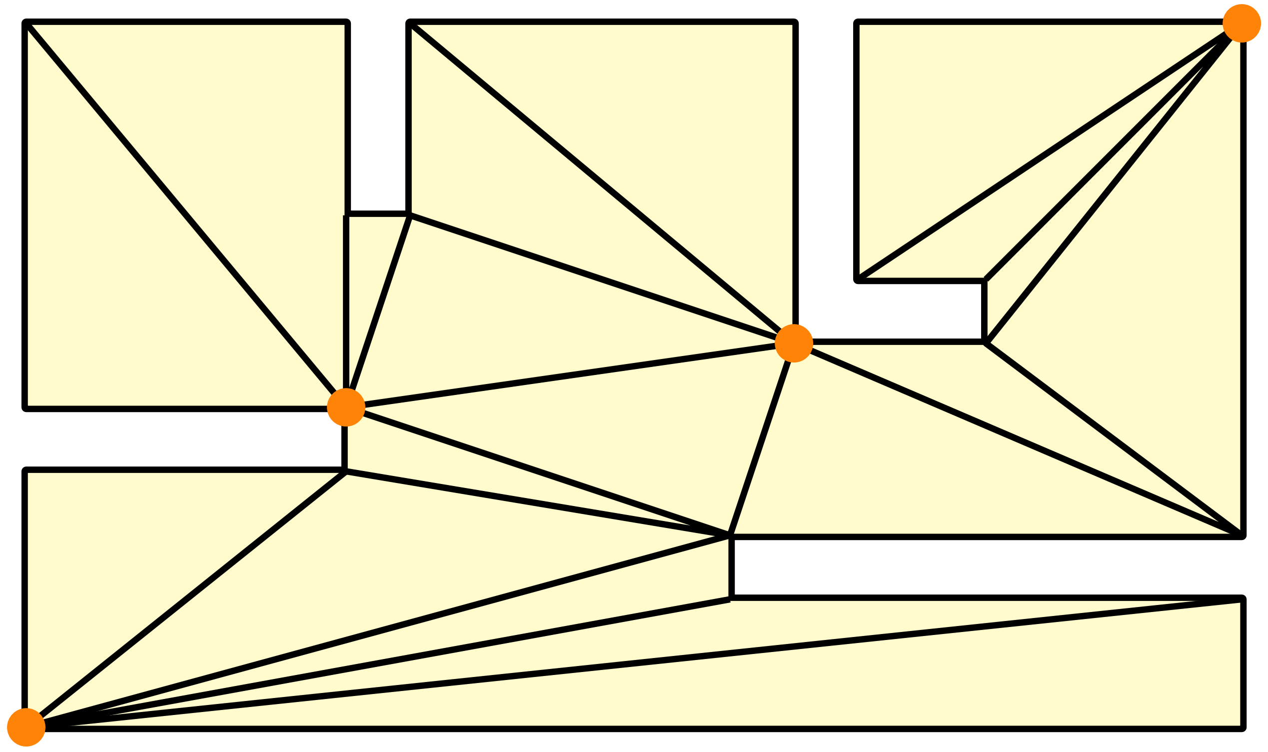

- P can be guarded with $n-2$ cameras
- P can be guarded with $\approx n/2$ cameras
- P can be guarded with even fewer vertex-guards (guards on vertices)

Theorem 1: Every simple polygon with n vertices has a triangulation; every such triangulation consists of $n-2$ triangles.

The Art Gallery Theorem [Chvátal ’75]

Theorem 2: $\lfloor n/3 \rfloor$ guards are sometimes necessary and always sufficient to guard a simple polygon with n vertices.

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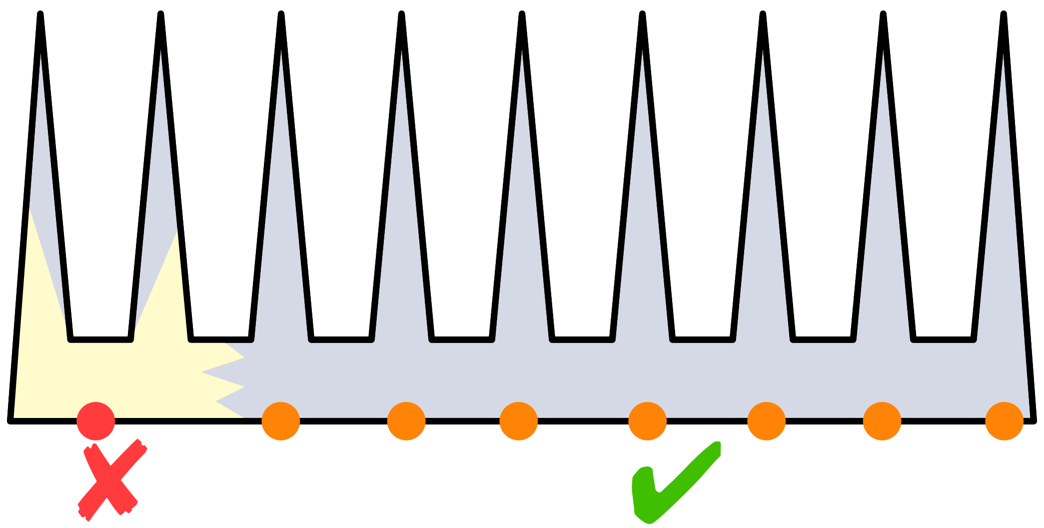

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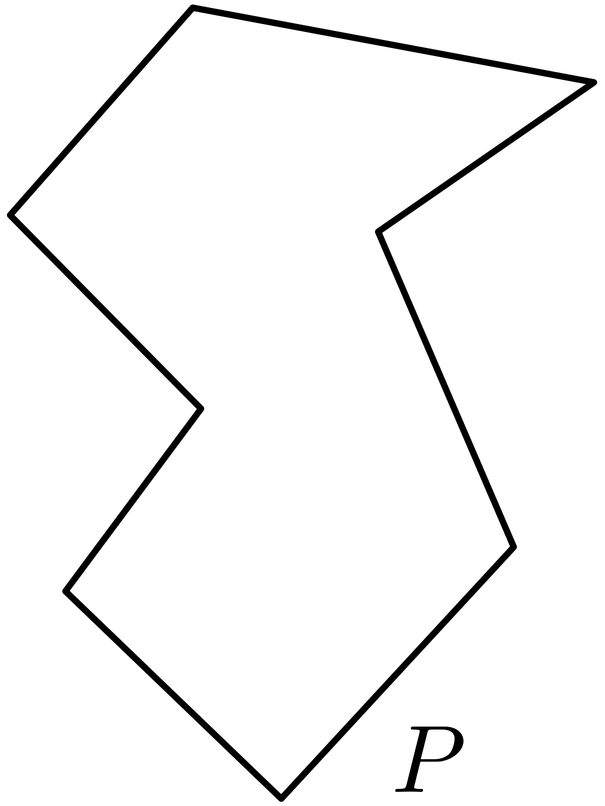

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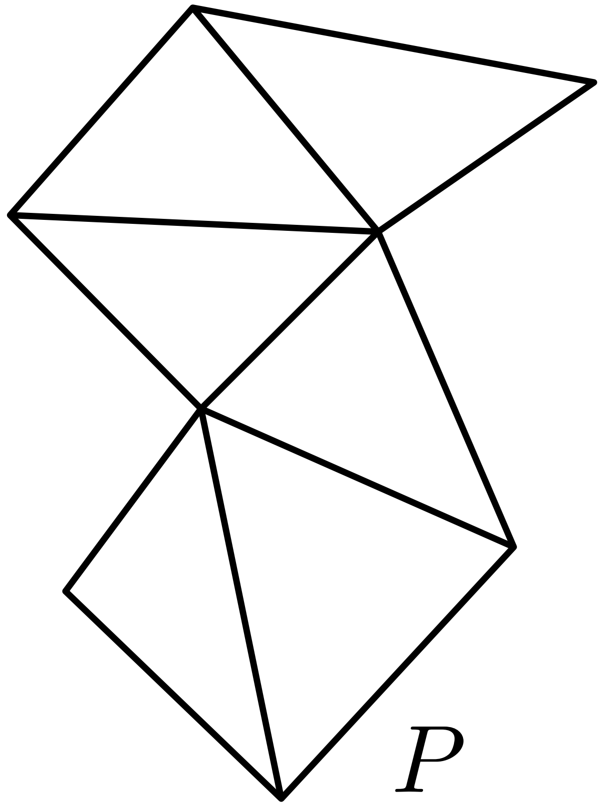

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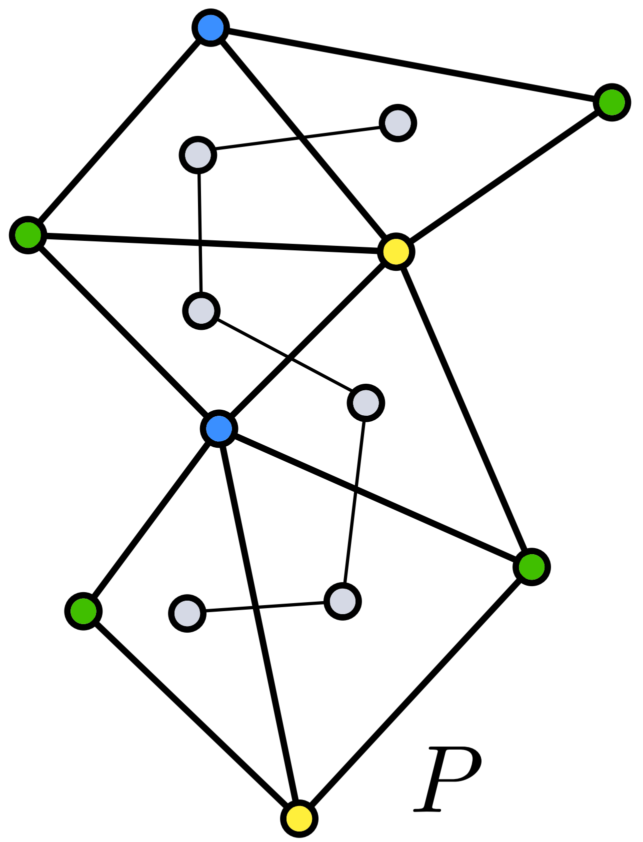
- Triangulation can be 3-colored (induction or consider dual graph)



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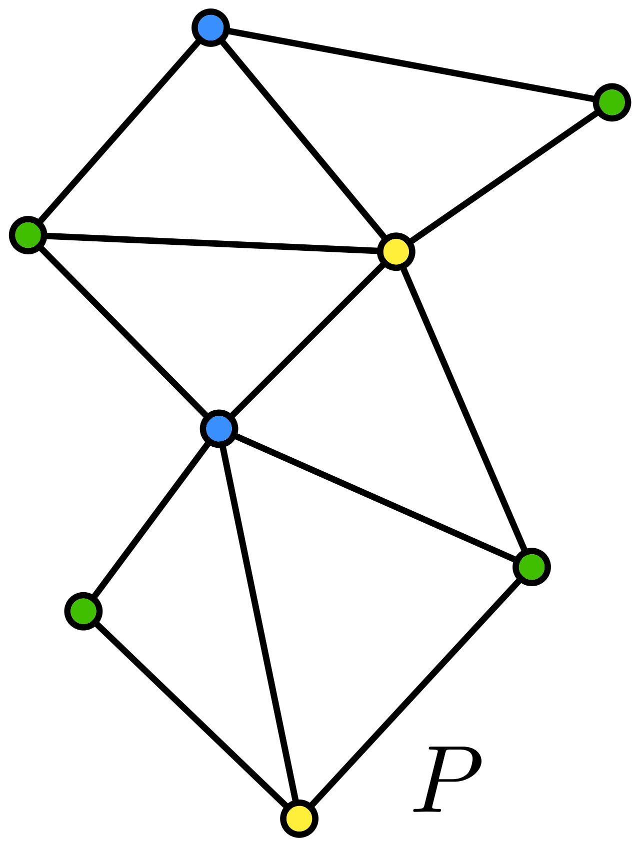
- Smallest color class has $\lfloor \frac{n}{3} \rfloor$ vertices (pigeon-hole principle)



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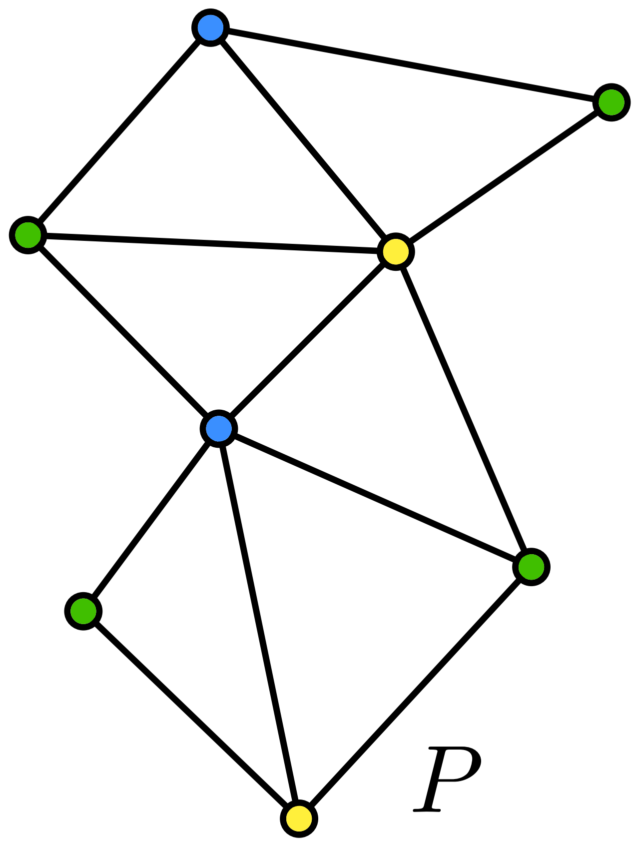
Algorithm:

- compute triangulation
- compute dual graph
- color triangulation
- select smallest color class



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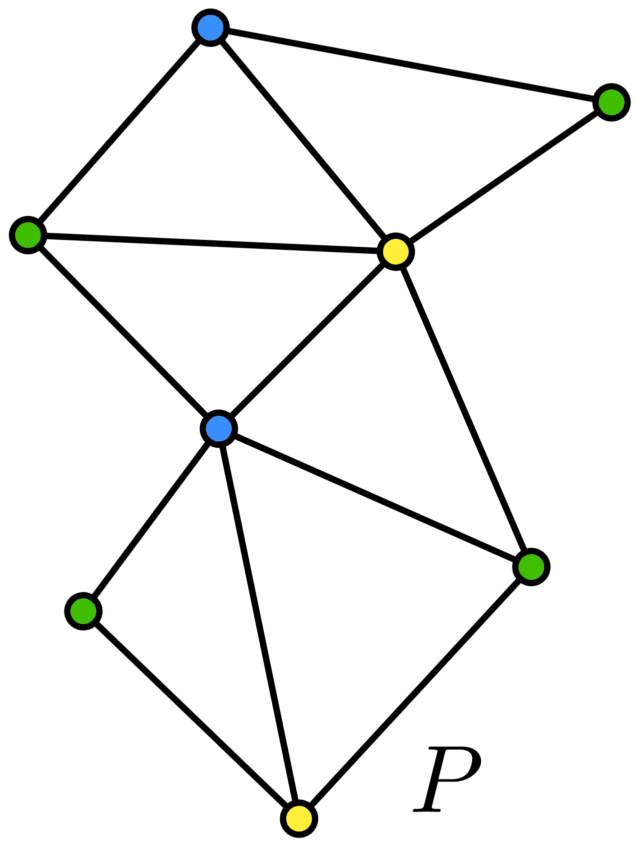

Running time?

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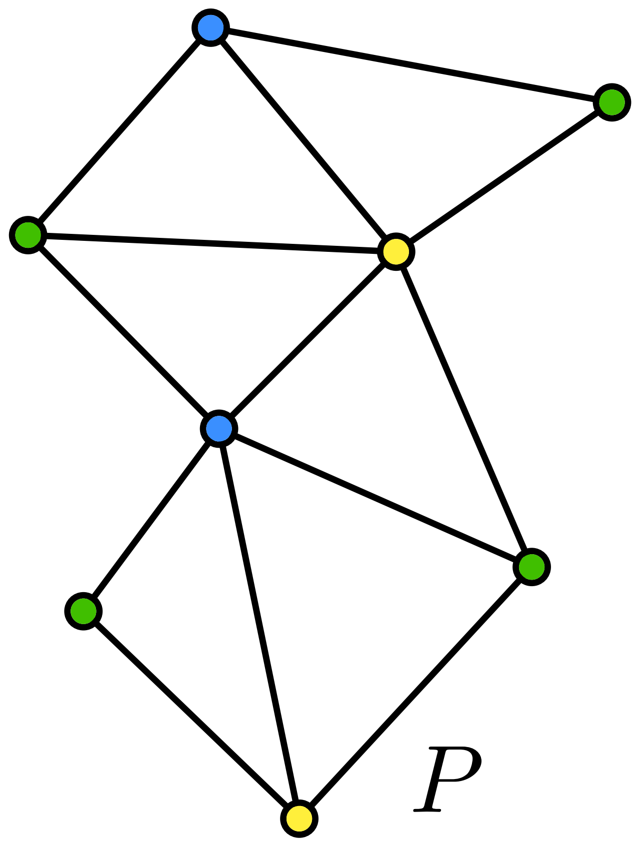
Algorithm:

- compute triangulation $O(n^2)$
- compute dual graph $O(n)$
- color triangulation
- select smallest color class

Running time? $O(n^2)$



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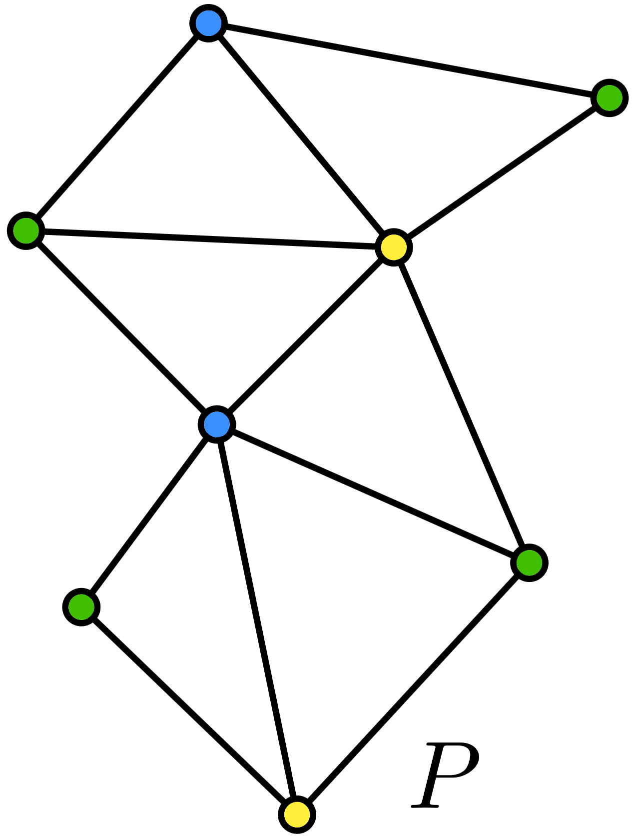
Now: faster triangulation algorithm



Triangulation: Overview

Idea: Partition into simpler parts and triangulate those.

Triangulation: Overview

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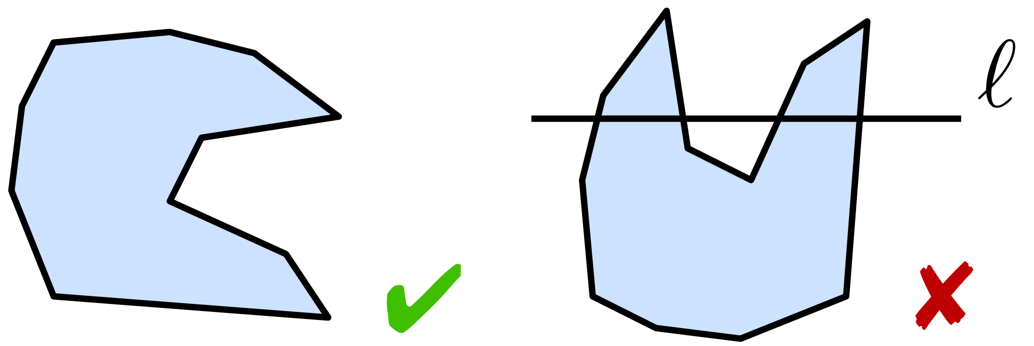
Which polygons are easy to triangulate?

Triangulation: Overview

2-step procedure:

- step 1: partition P into y-monotone subpolygons

Definition: A polygon P is y-monotone if, for every horizontal line ℓ, the intersection $\ell \cap P$ is connected.



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2-step procedure:

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boundary from top to bottom never goes up

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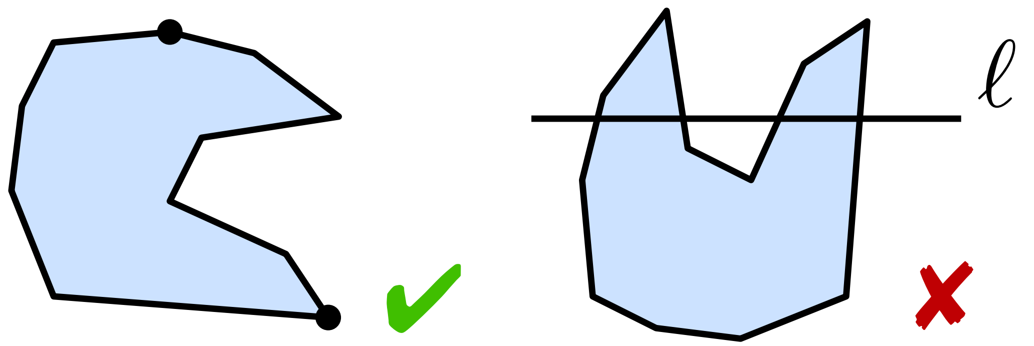

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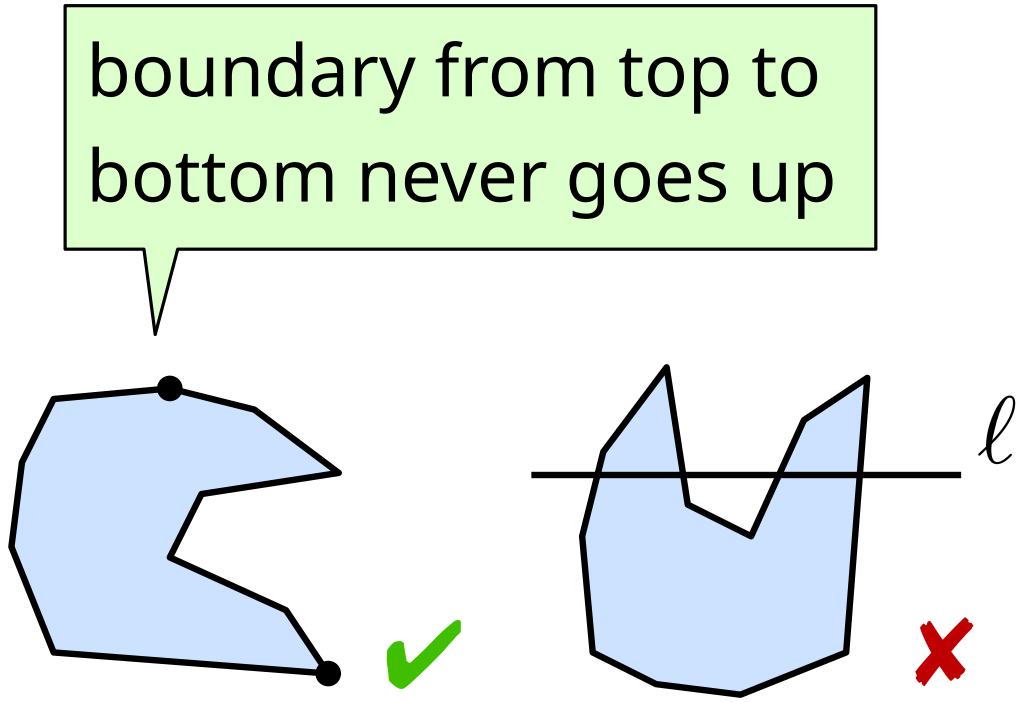

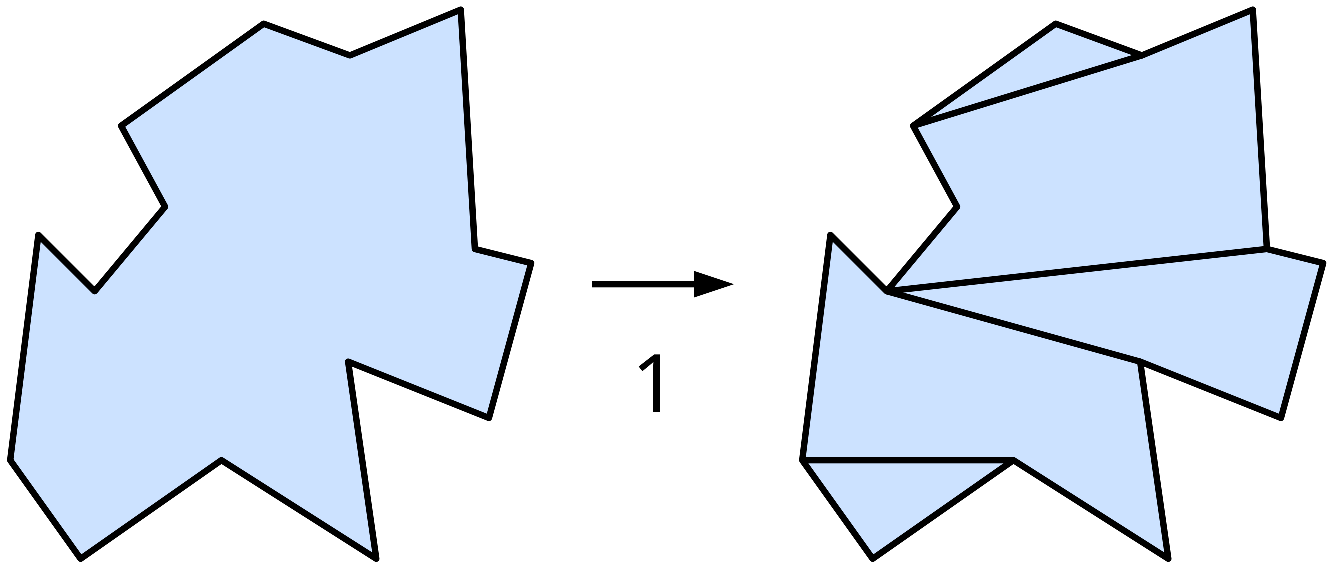

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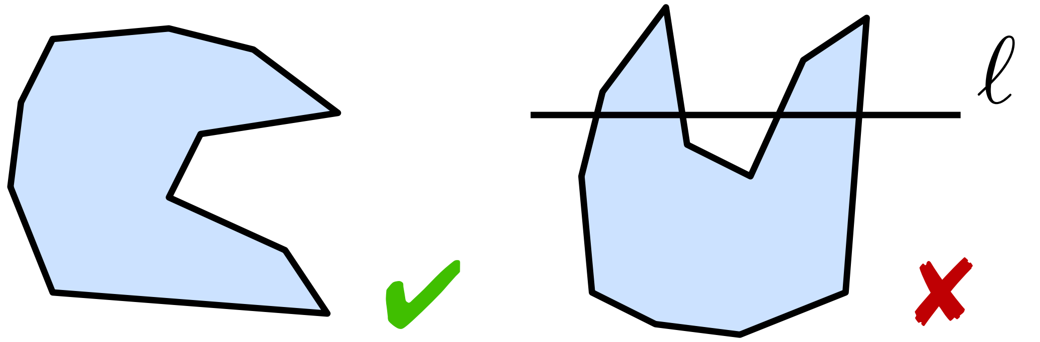
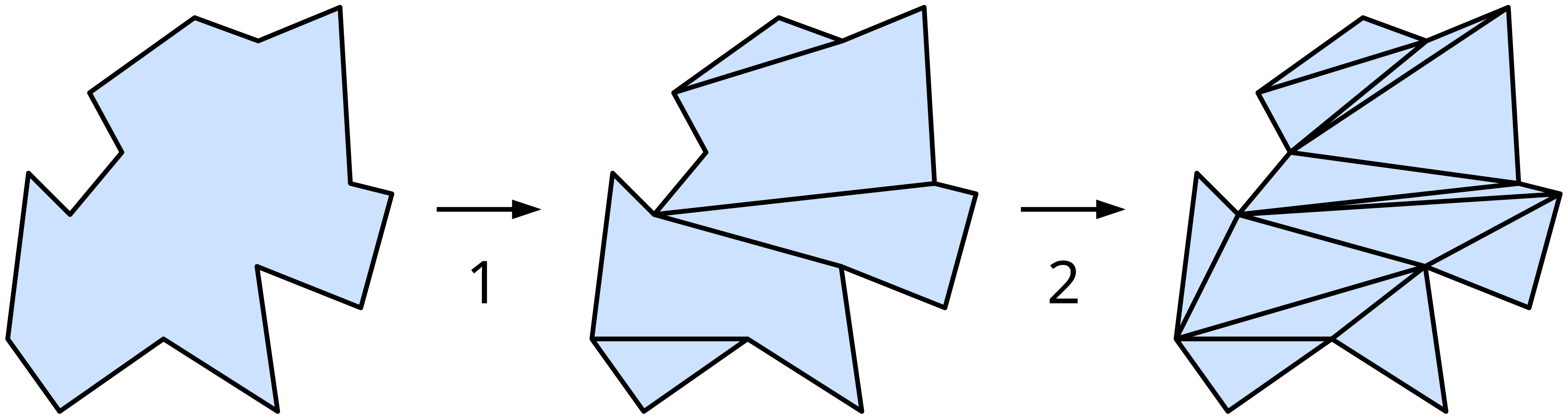
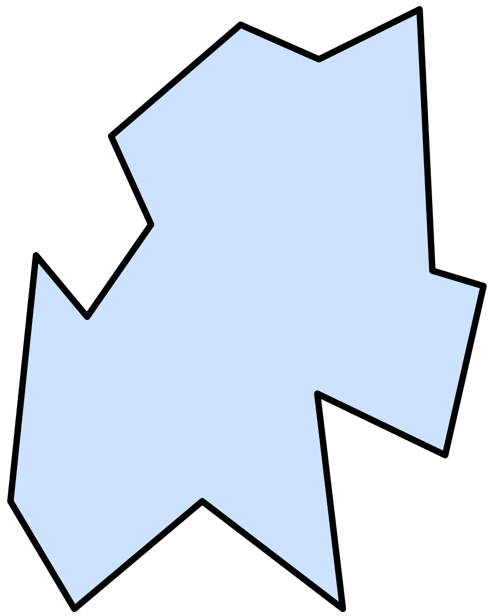


- step 2: triangulate y-monotone subpolyons



Partition into y-monotone Pieces

Idea: Distinguish 5 types of vertices



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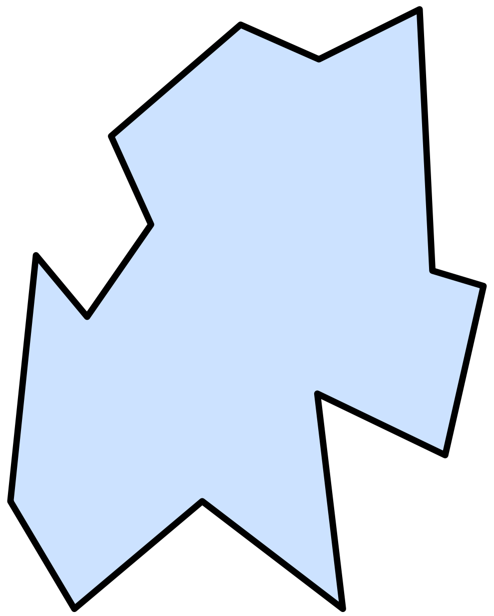

– regular vertex

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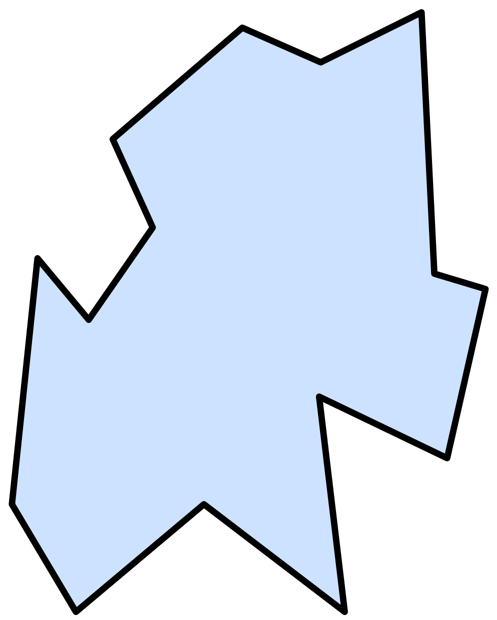

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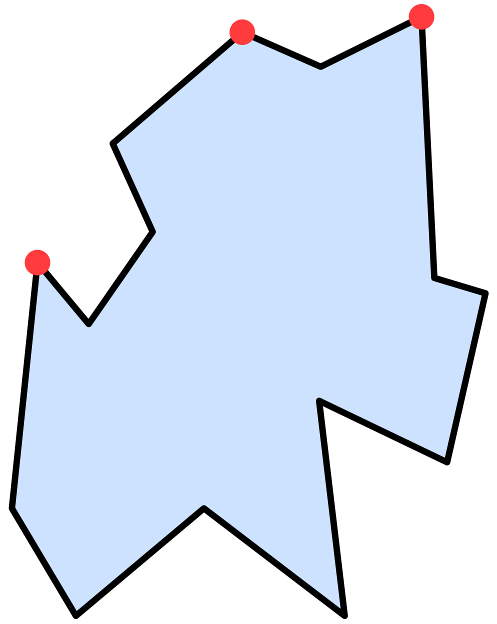
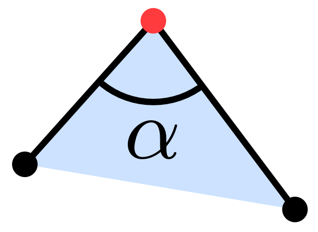


- start vertex



if $\alpha < 180°$

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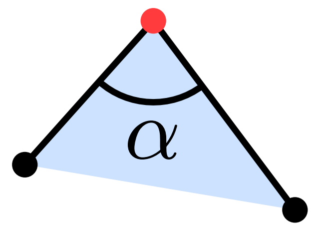
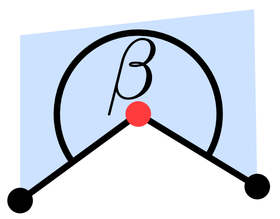
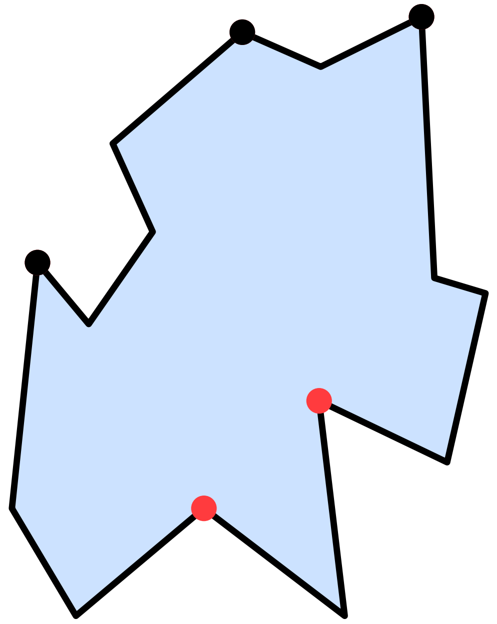
- start vertex  if $\alpha < 180°$

- split vertex  if $\beta > 180°$



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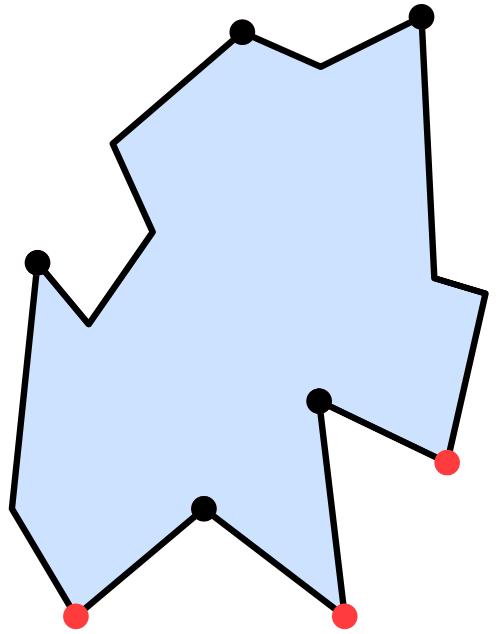

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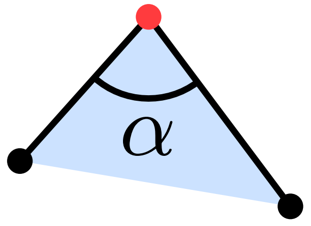

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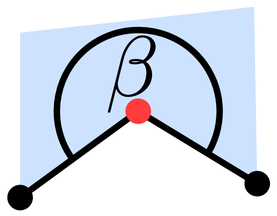

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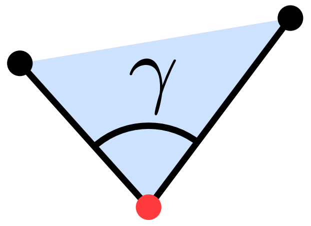

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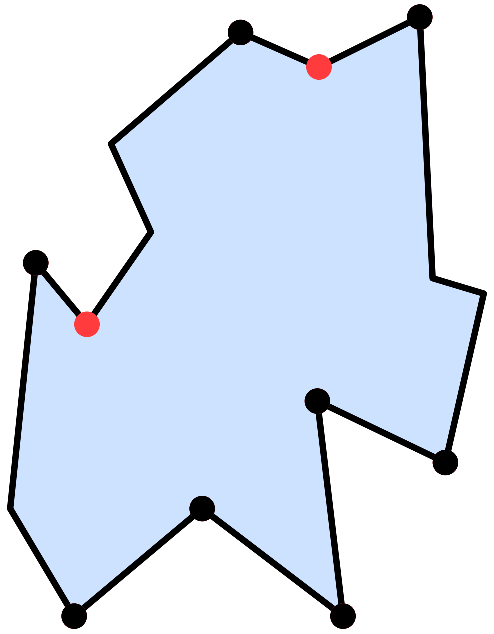

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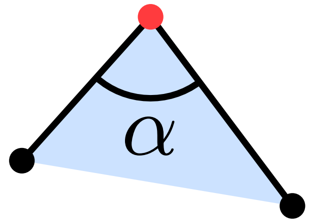
- start vertex 

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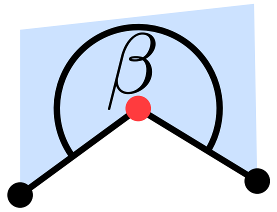
- split vertex 

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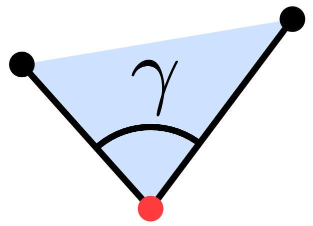
- end vertex 

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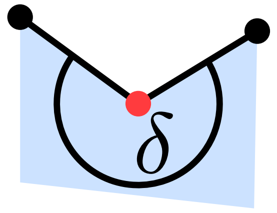
- merge vertex 

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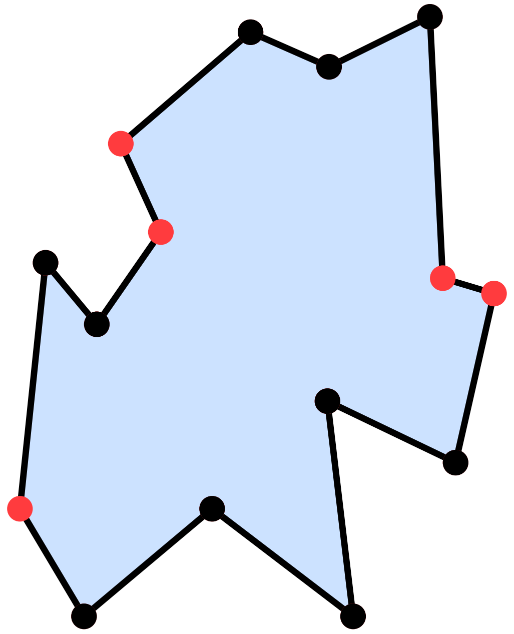

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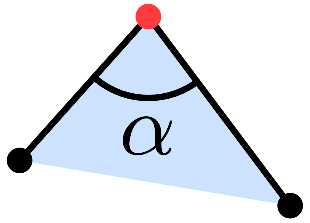
- start vertex 

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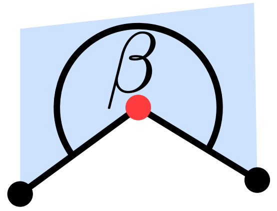
- split vertex 

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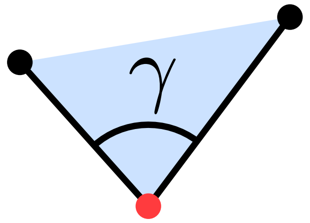
- end vertex 

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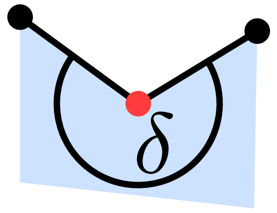
- merge vertex 

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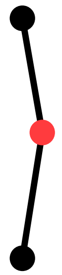
– regular vertex 

Characterization

Lemma 1: If a polygon does not contain split and merge vertices then it is y-monotone.

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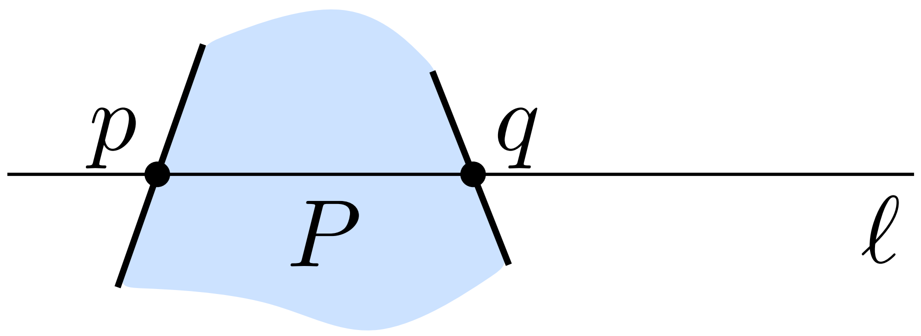
Let ℓ be a horizonal line intersecting several components of P.

Cases:



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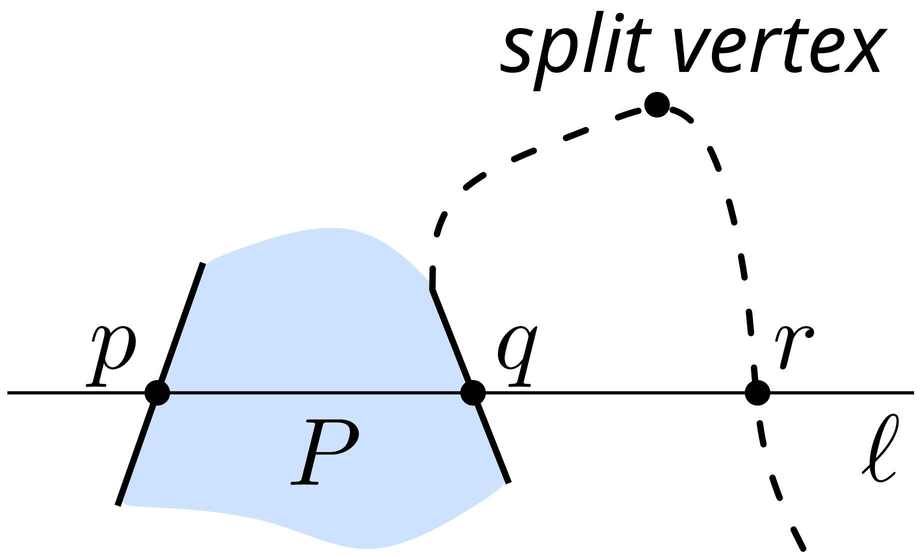

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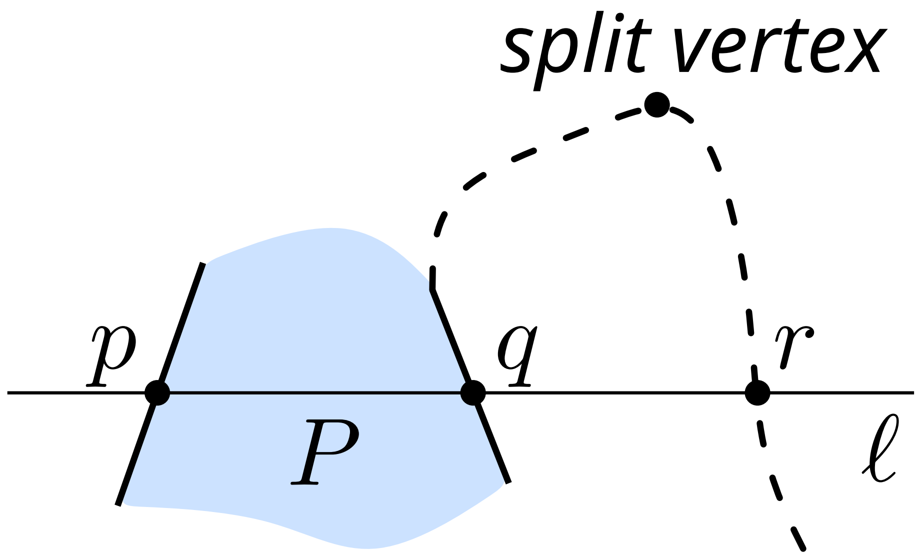

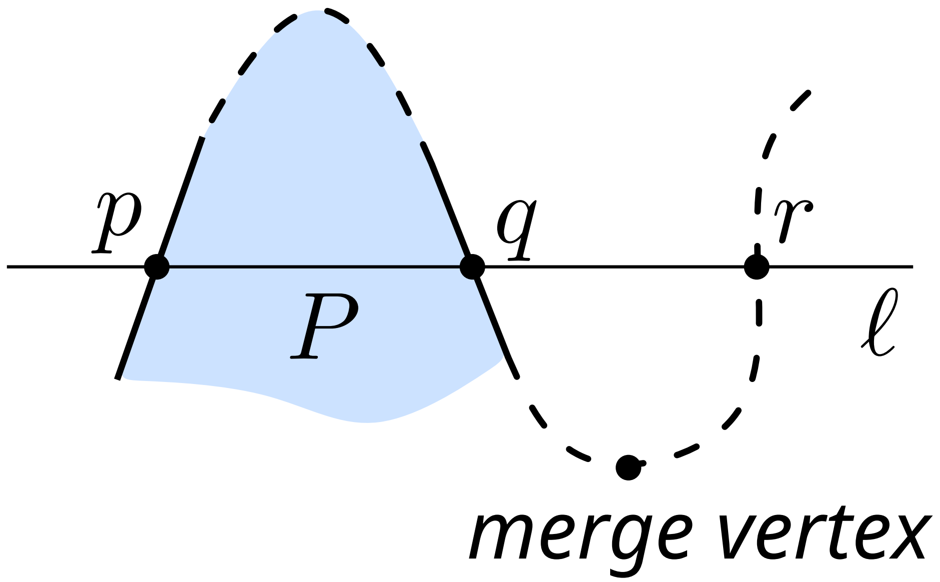

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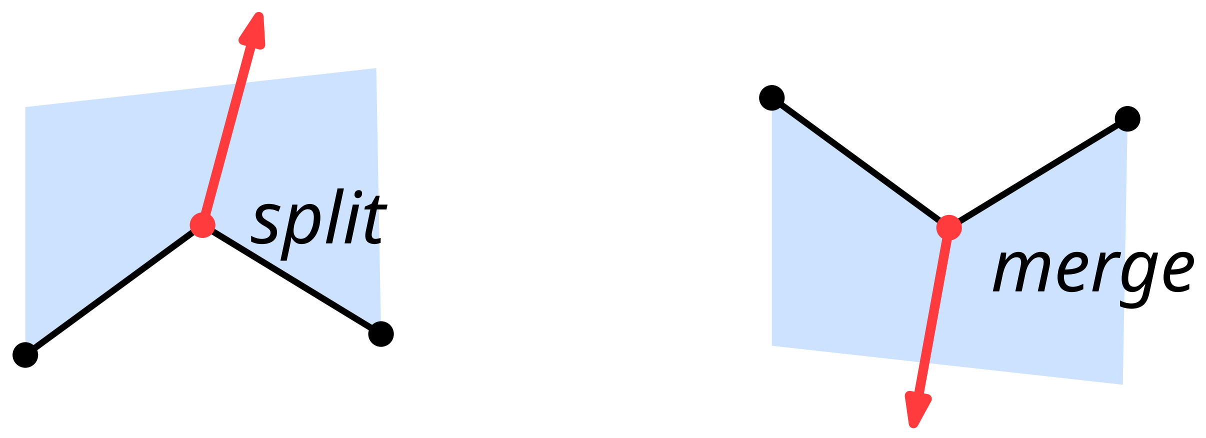
$\Rightarrow$ We need to remove all split and merge vertices by adding diagonals



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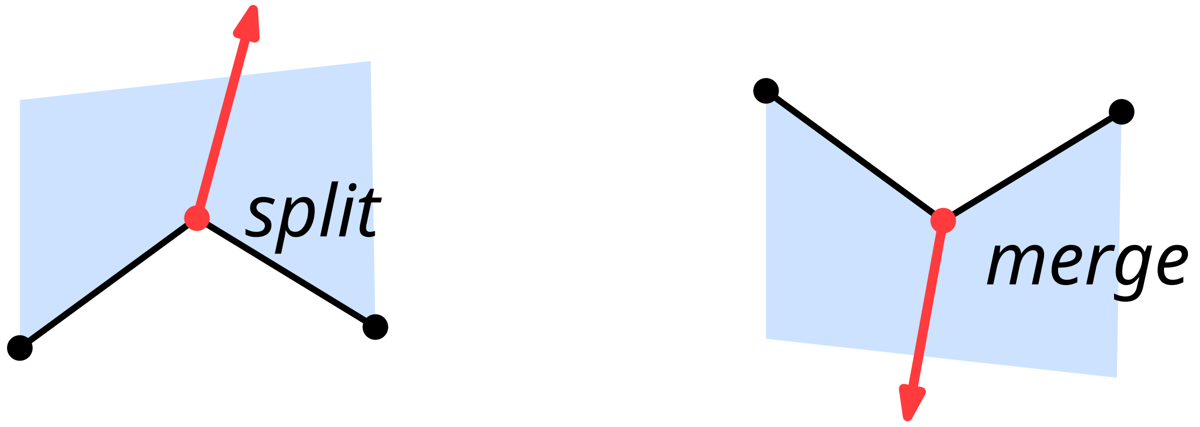
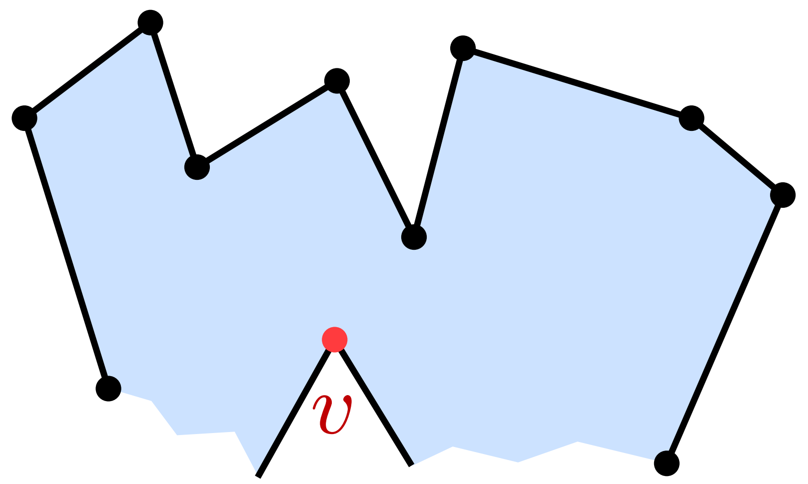


Careful: Diagonals shouldn't intersect edges of P or other diagonals

Towards a Sweepline Algorithm

1) Diagonals for split vertices



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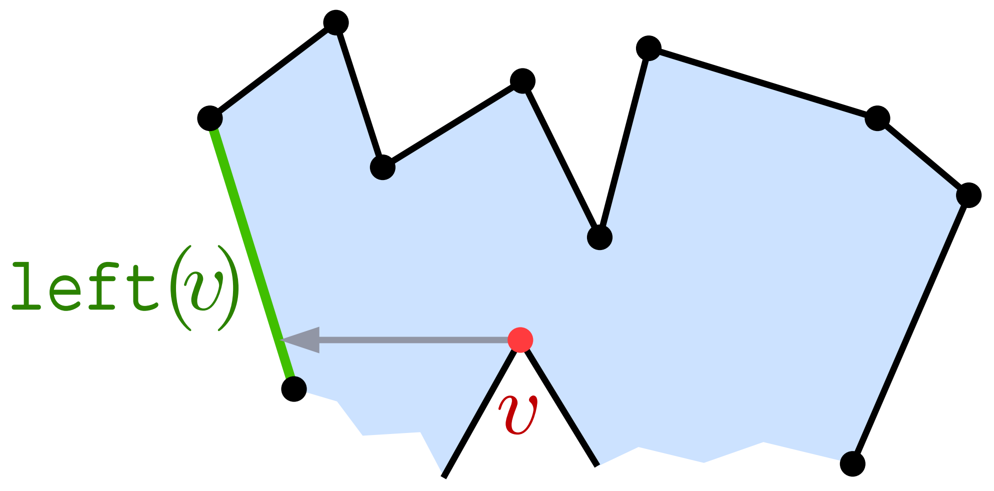

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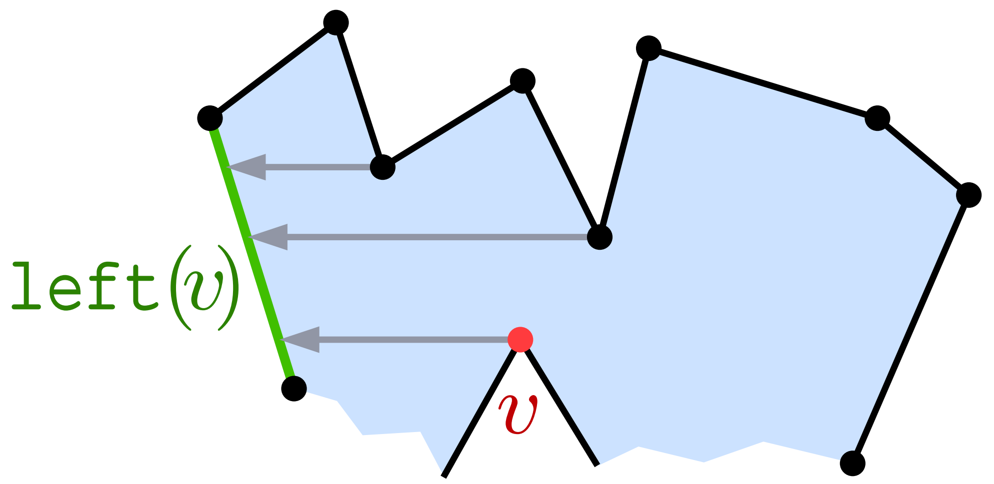

Towards a Sweepline Algorithm

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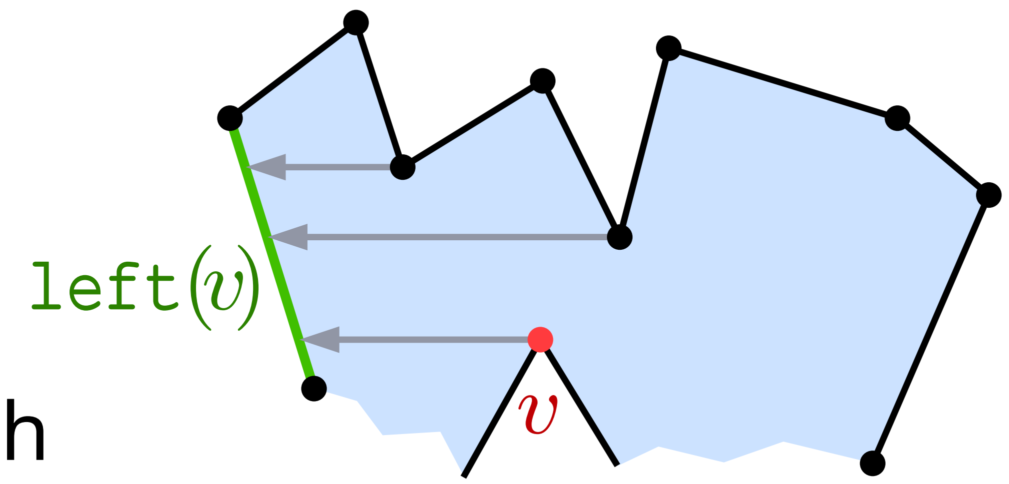

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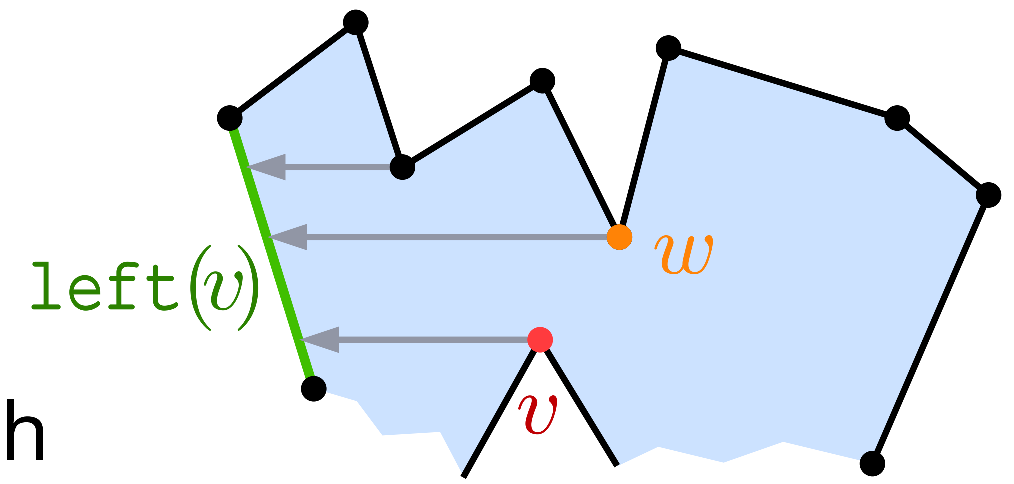

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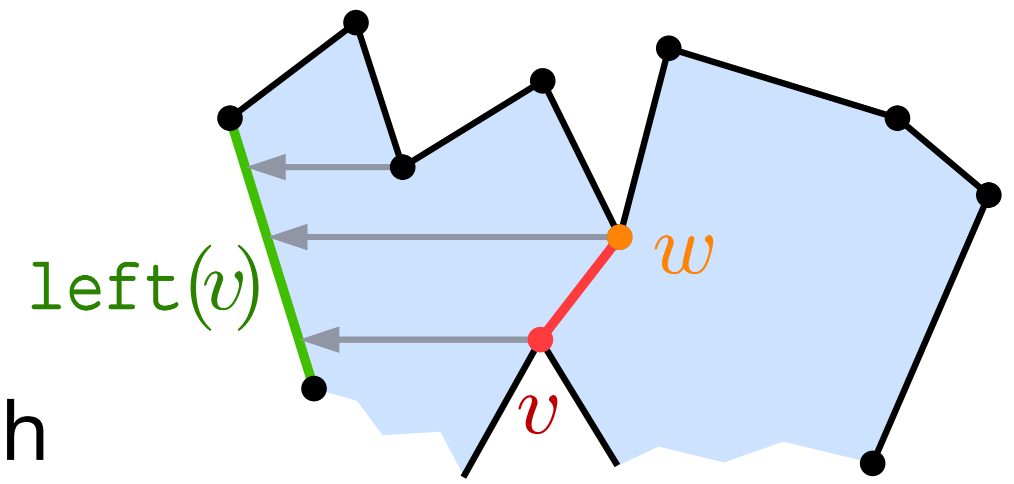

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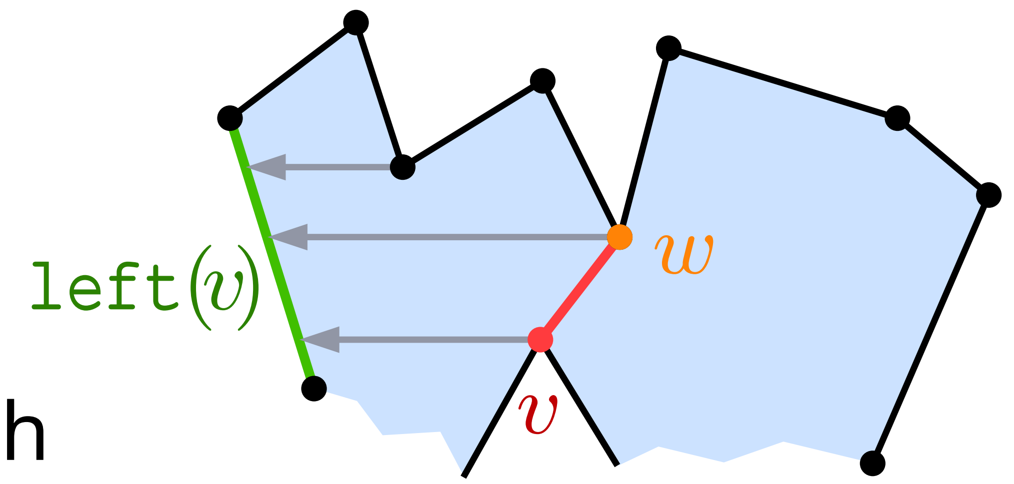

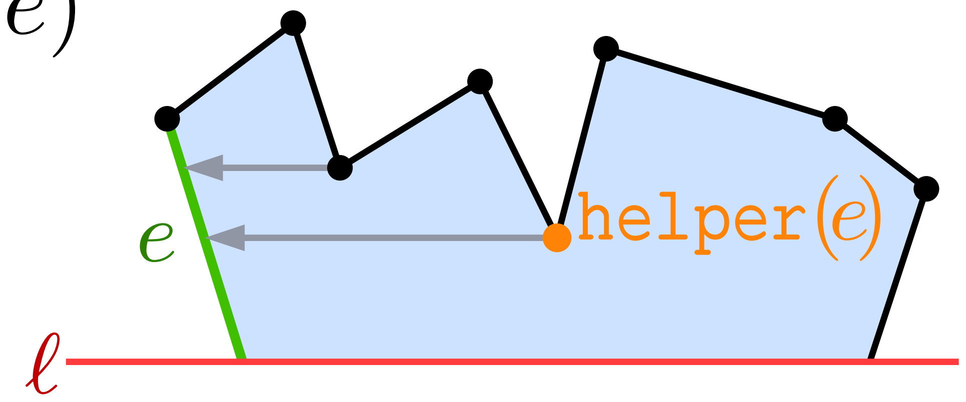

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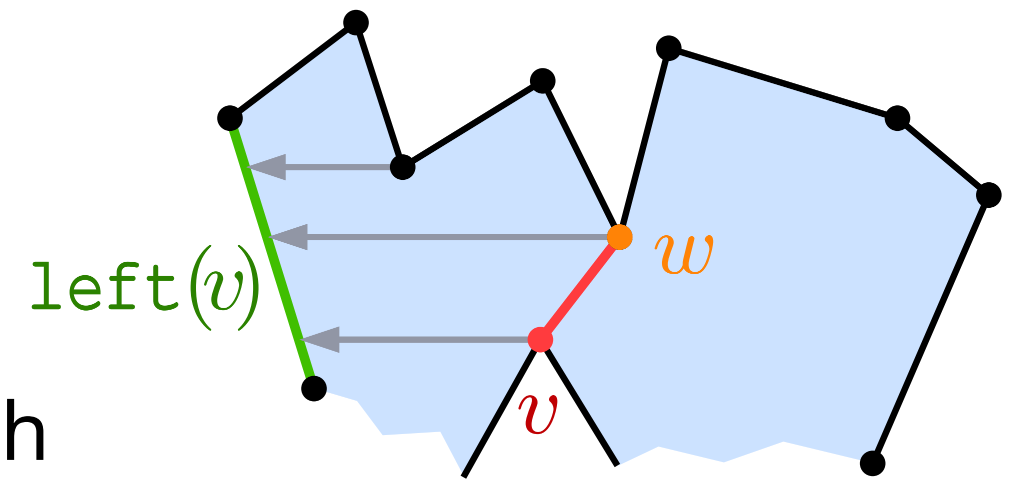

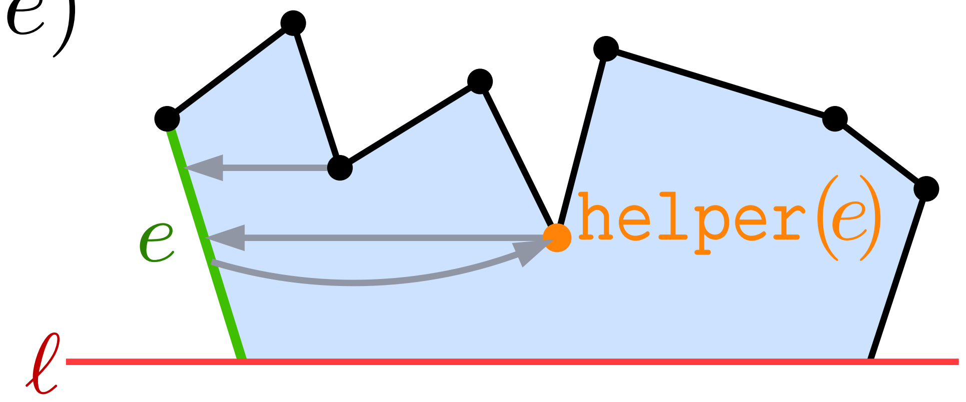

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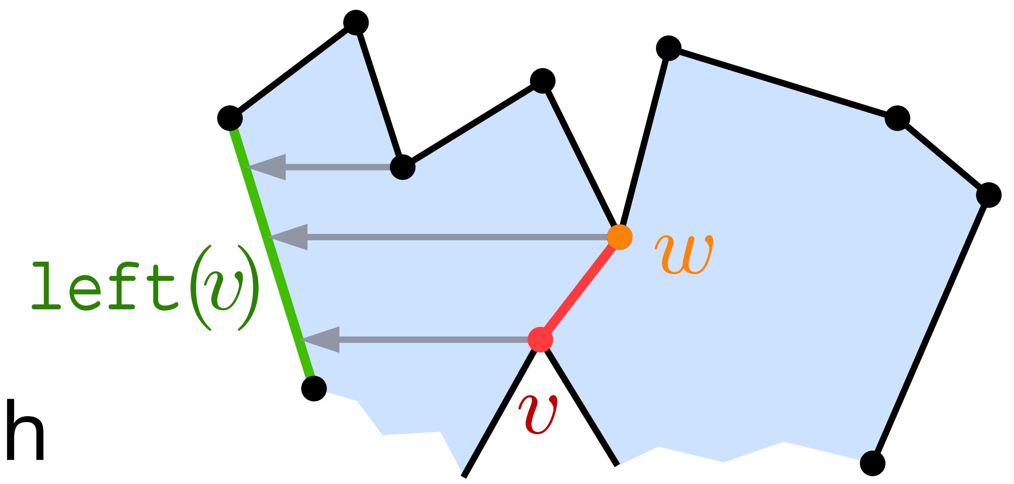
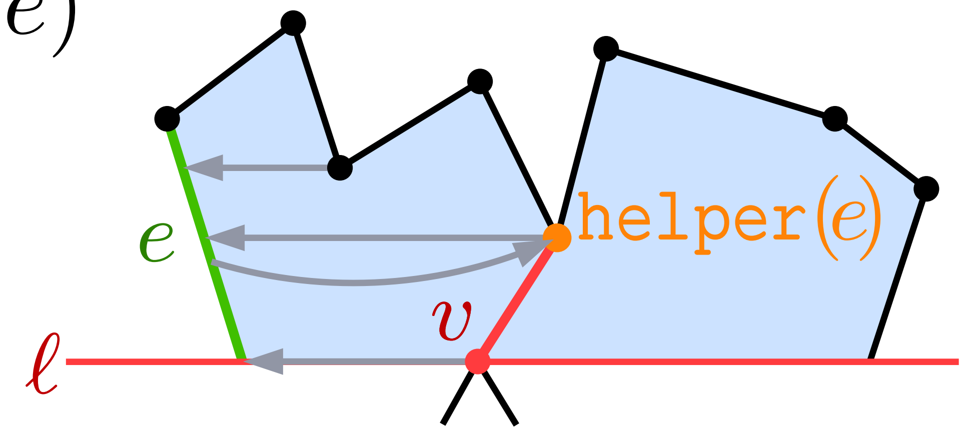
- when ℓ reaches split node v: connect v to $\texttt{helper}(\texttt{left}(v))$





Towards a Sweepline Algorithm

2) Diagonals for merge vertices

- merge vertex v reached:
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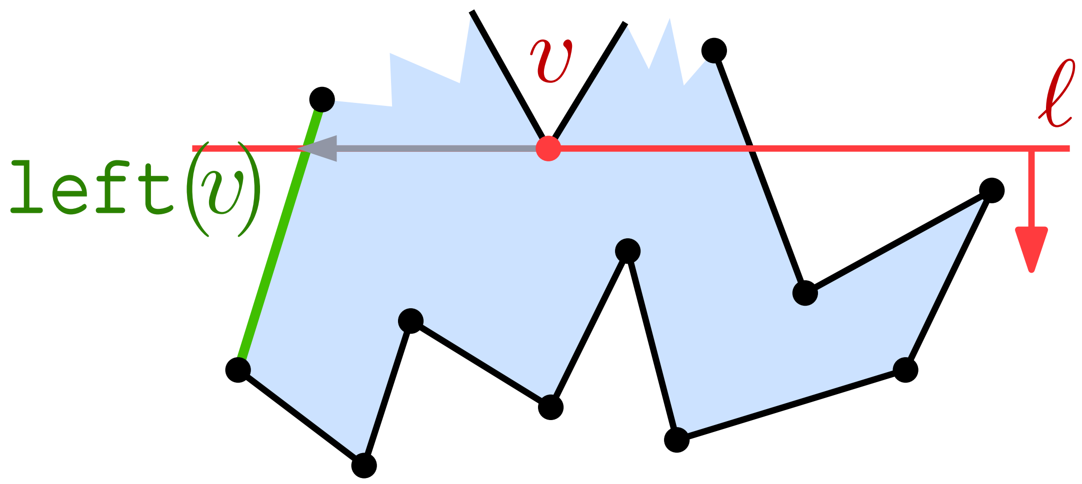

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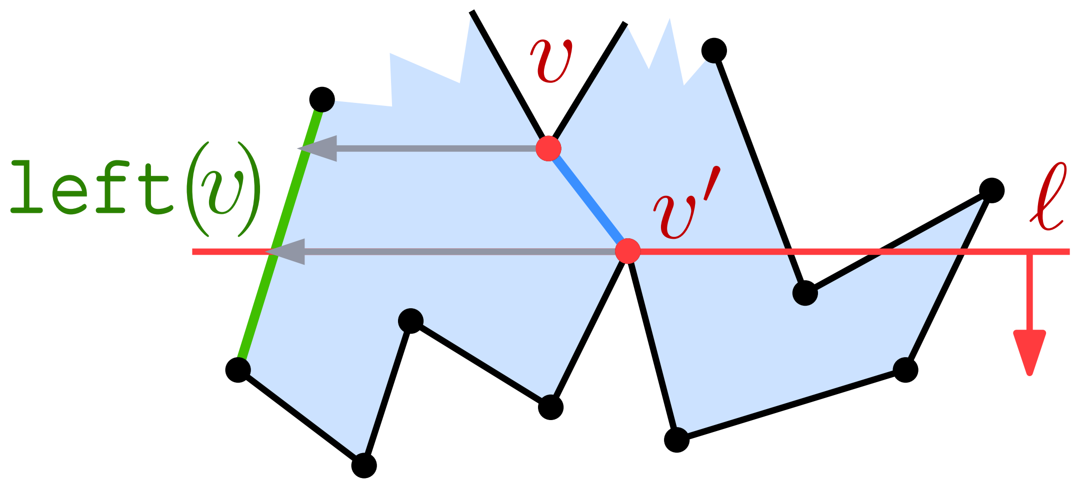

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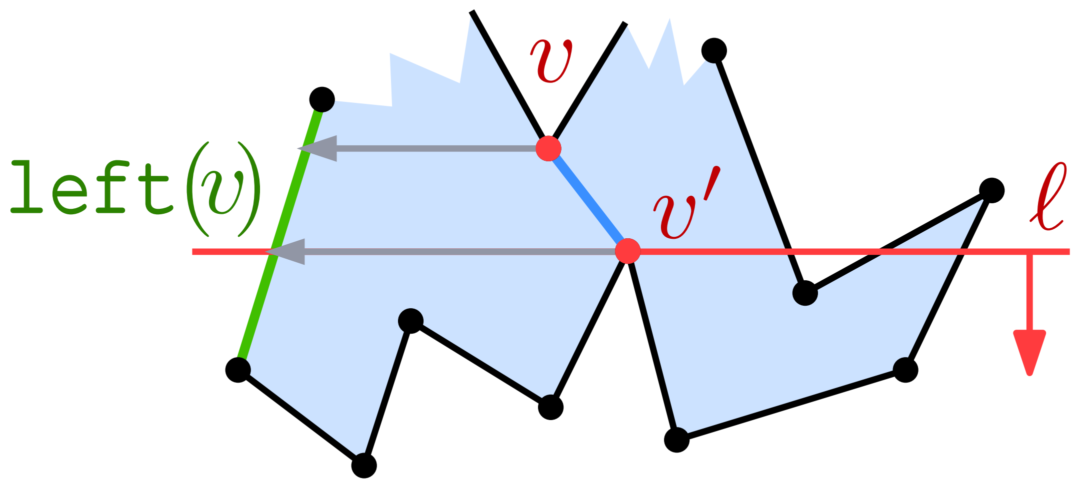

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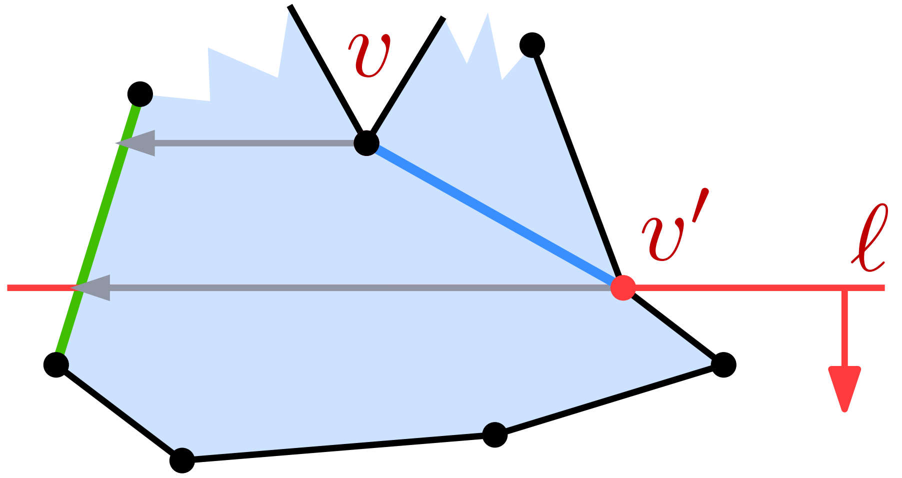

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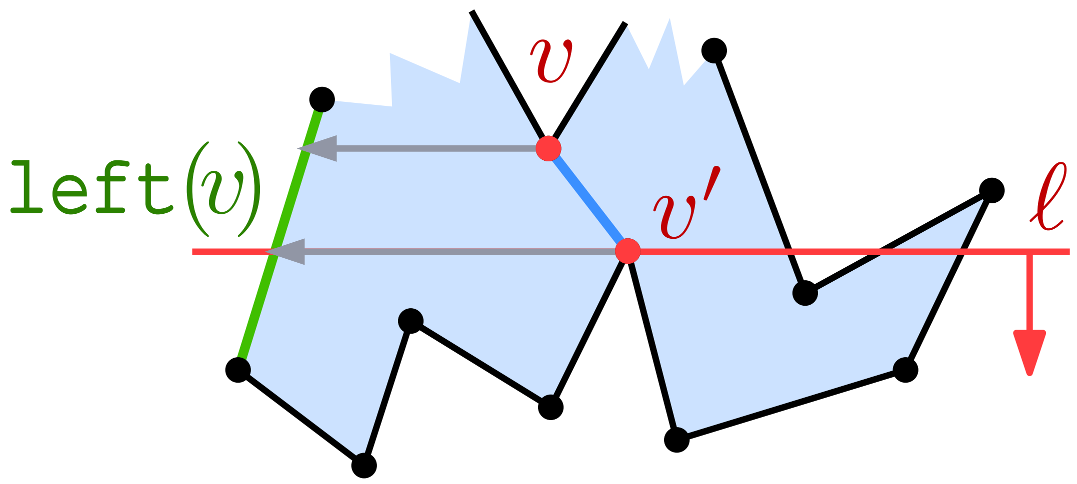

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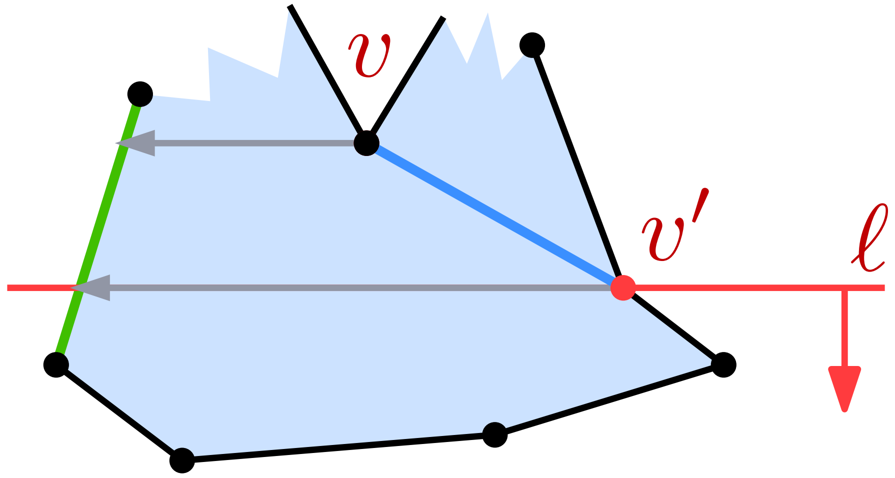

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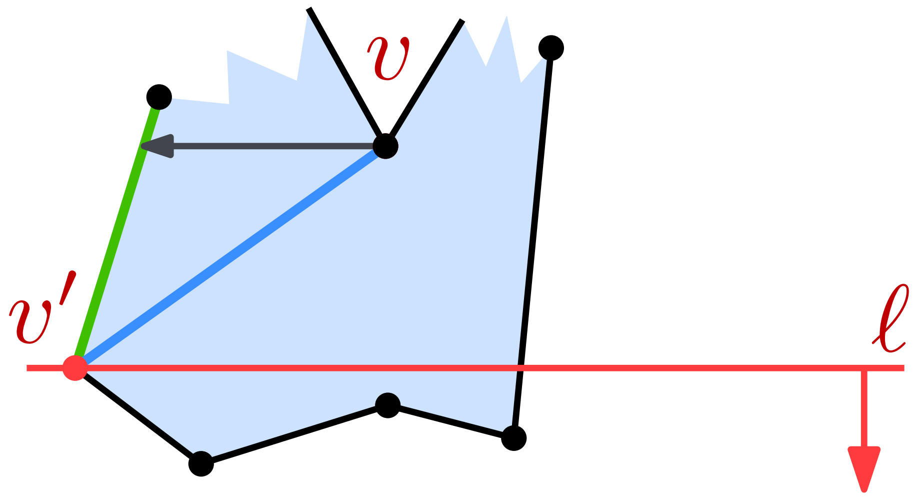

Towards a Sweepline Algorithm

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- merge ver
  update he
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Alternative:

Handle merge vertices in separate sweep.

merge = upside-down split

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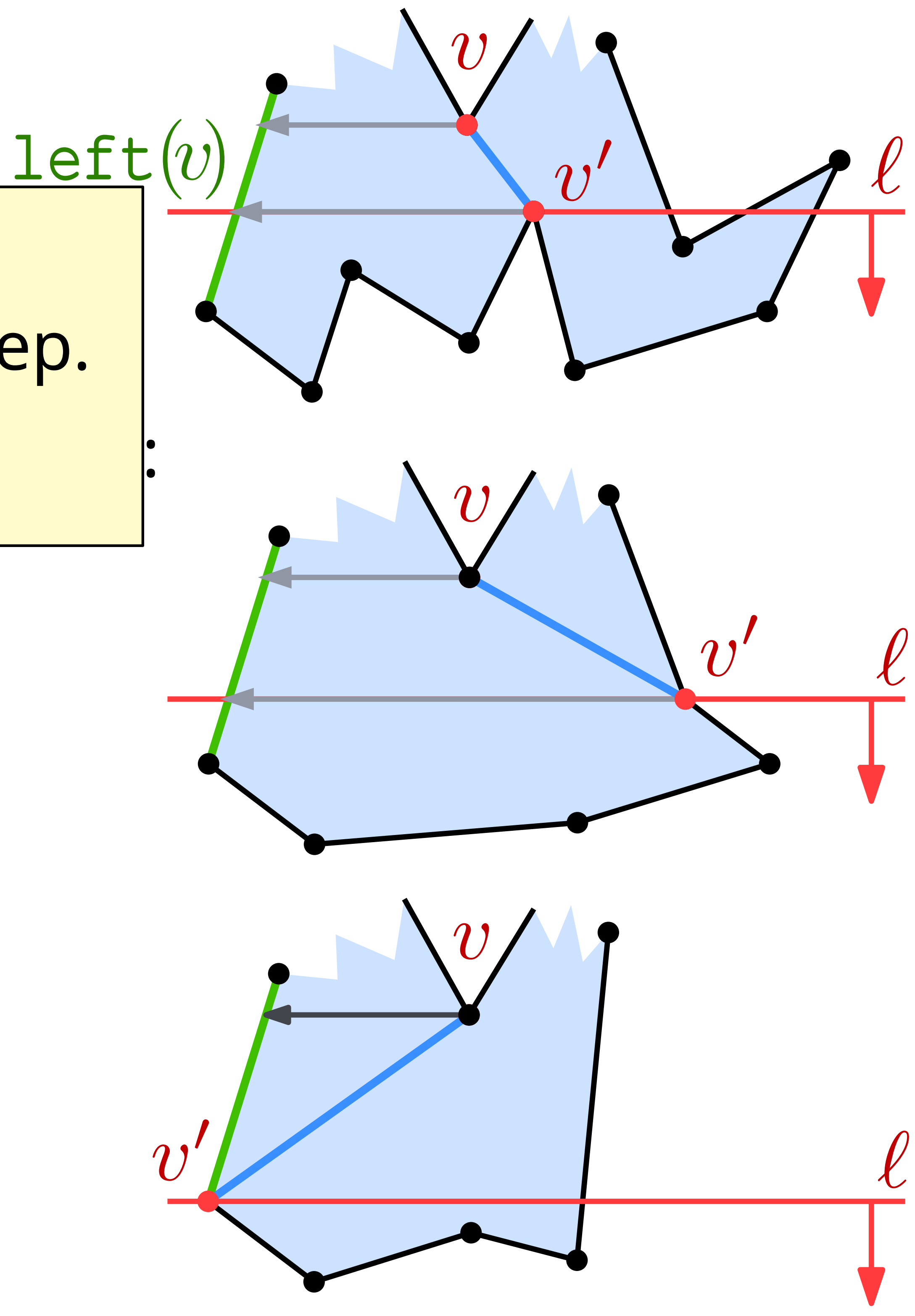

Towards a Sweepline Algorithm

Events:

Status:

Towards a Sweepline Algorithm

Events: Vertices of P in lexicographical order

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Towards a Sweepline Algorithm

Events: Vertices of P in lexicographical order

Status: Components of P intersected by ℓ: for each component store left edge e and vertex $v = \texttt{helper}(e)$

Algorithm MakeMonotone(P)

Algorithm MAKEMONOTONE(polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $E(P)$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically
$\mathcal{T} \leftarrow \varnothing$ (binary search tree for status of sweepline)
while $\mathcal{Q} \neq \varnothing$ **do**
    $v \leftarrow \mathcal{Q}$.popVertex()
    HANDLEVERTEX(v)
return $\mathcal{D}$

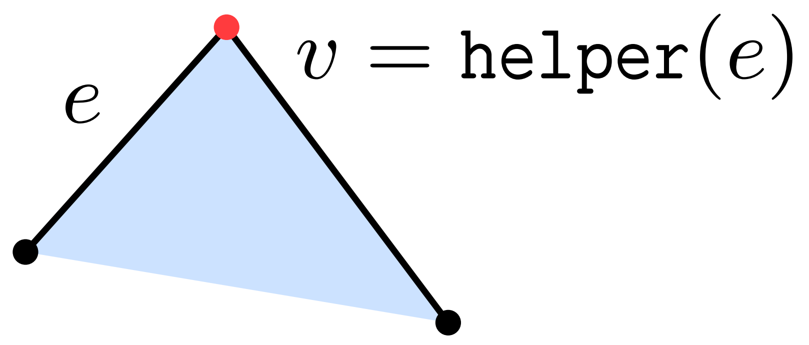
Algorithm MakeMonotone(P)

Algorithm MAKEMONOTONE(polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $E(P)$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically
$\mathcal{T} \leftarrow \emptyset$ (binary search tree for status of sweepline)
while $\mathcal{Q} \neq \emptyset$ **do**
    $v \leftarrow \mathcal{Q}$.popVertex()
    HANDLEVERTEX(v)
return $\mathcal{D}$

HANDLESTARTVERTEX(vertex v)

$\mathcal{T} \leftarrow$ insert left edge e
$\texttt{helper}(e) \leftarrow v$



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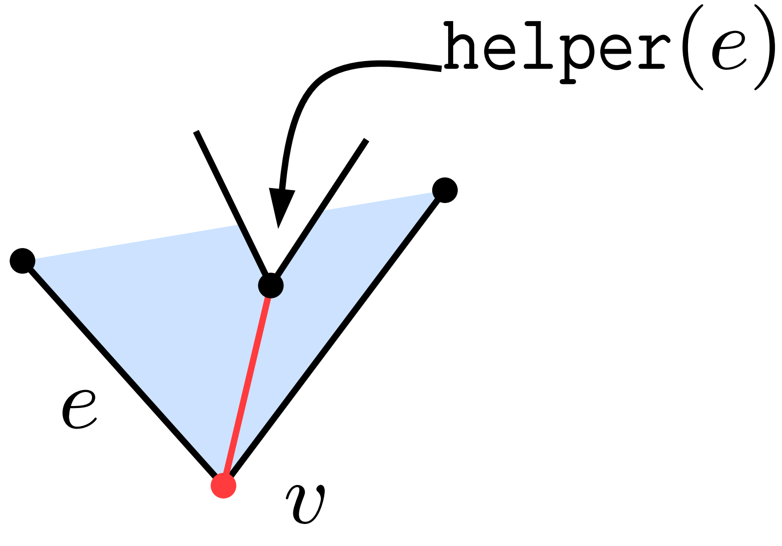
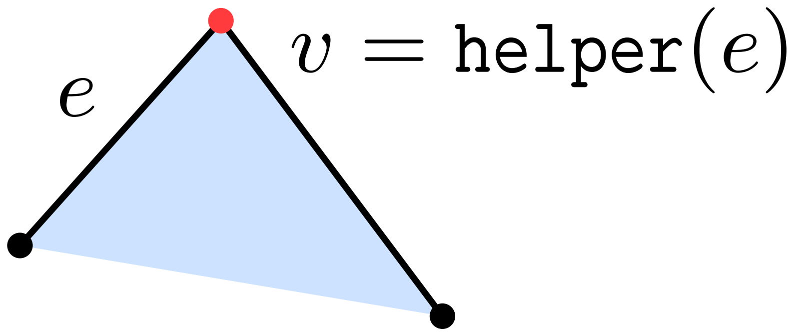
Algorithm MAKEMONOTONE(polygon P)

1: $\mathcal{D} \leftarrow$ doubly-connected edge list for $E(P)$
2: $\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically
3: $\mathcal{T} \leftarrow \varnothing$ (binary search tree for status of sweepline)
4: **while** $\mathcal{Q} \neq \varnothing$ **do**
5: $\quad v \leftarrow \mathcal{Q}$.popVertex()
6: $\quad$ HANDLEVERTEX(v)
7: **return** $\mathcal{D}$



HANDLESTARTVERTEX(vertex v)

1: $\mathcal{T} \leftarrow$ insert left edge e
2: $\texttt{helper}(e) \leftarrow v$



HANDLEENDVERTEX(vertex v)

1: $e \leftarrow$ left edge
2: **if** isMergeVertex($\texttt{helper}(e)$) **then**
3: $\quad \mathcal{D} \leftarrow$ insert $(\texttt{helper}(e), v)$
4: delete e from $\mathcal{T}$

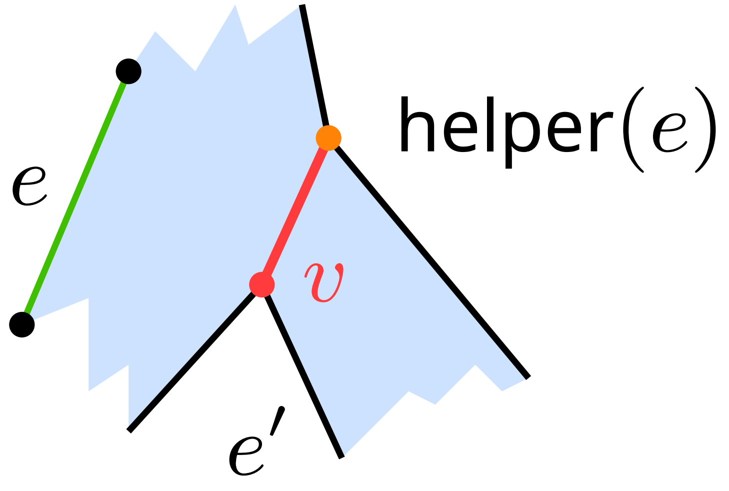
Algorithm MakeMonotone(P)

Algorithm MAKEMONOTONE(polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $E(P)$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically
$\mathcal{T} \leftarrow \varnothing$ (binary search tree for status of sweepline)
while $\mathcal{Q} \neq \varnothing$ **do**
    $v \leftarrow \mathcal{Q}$.popVertex()
    HANDLEVERTEX(v)
return $\mathcal{D}$

HANDLESPLITVERTEX(vertex v)

$e \leftarrow$ edge left of v in $\mathcal{T}$
$\mathcal{D} \leftarrow$ insert $(\texttt{helper}(e), v)$
$\texttt{helper}(e) \leftarrow v$
$\mathcal{T} \leftarrow$ insert right edge e' of v
$\texttt{helper}(e') \leftarrow v$



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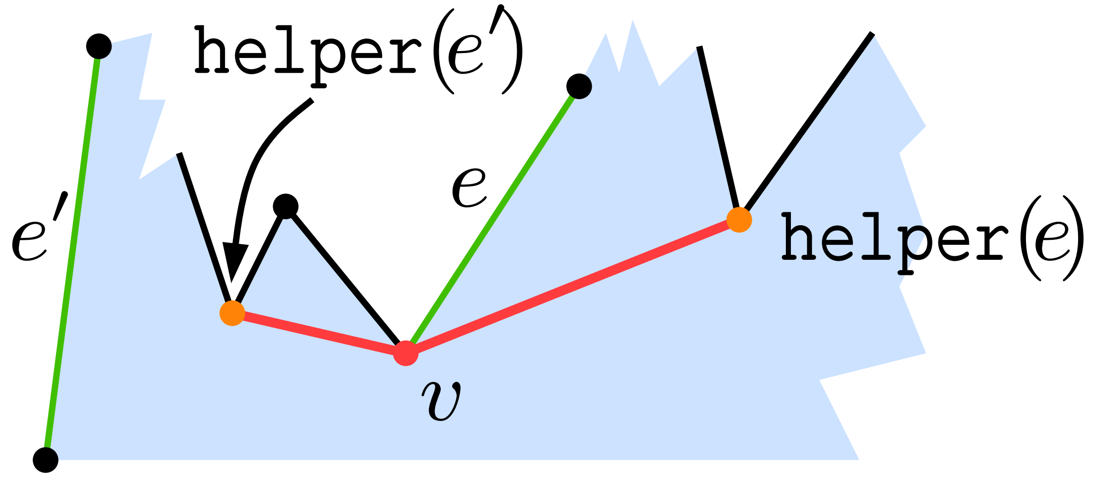
while $\mathcal{Q} \neq \varnothing$ **do**
    $v \leftarrow \mathcal{Q}$.popVertex()
    HANDLEVERTEX(v)
return $\mathcal{D}$

HANDLEMERGEVERTEX(vertex v)

$e \leftarrow$ right edge
if isMergeVertex($\texttt{helper}(e)$) **then**
    $\mathcal{D} \leftarrow$ insert $(\texttt{helper}(e), v)$
delete e from $\mathcal{T}$
$e' \leftarrow$ edge left of v in $\mathcal{T}$
if isMergeVertex($\texttt{helper}(e')$) **then**
    $\mathcal{D} \leftarrow$ insert $(\texttt{helper}(e'), v)$
$\texttt{helper}(e') \leftarrow v$



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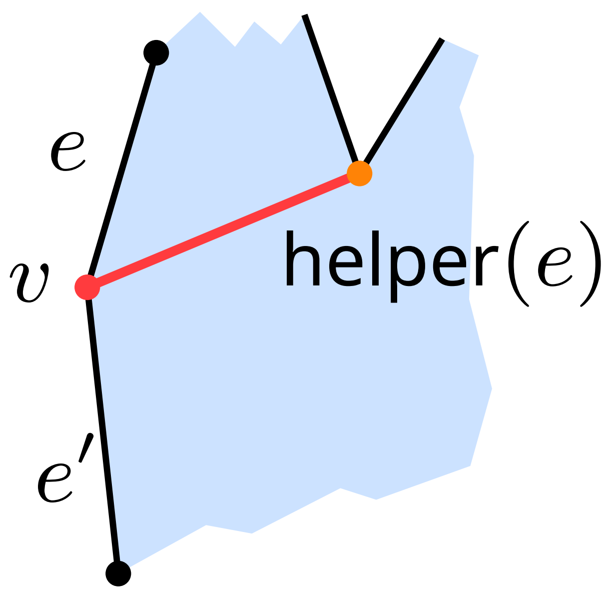
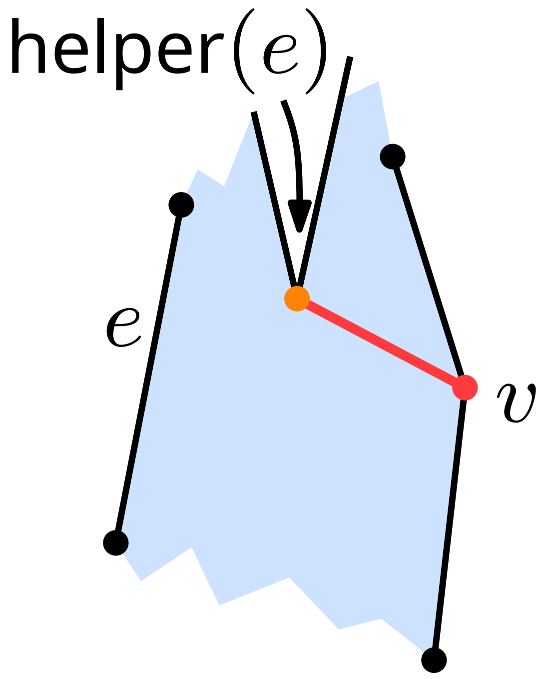
$\mathcal{T} \leftarrow \varnothing$ (binary search tree for status of sweepline)
while $\mathcal{Q} \neq \varnothing$ **do**
    $v \leftarrow \mathcal{Q}$.popVertex()
    HANDLEVERTEX(v)
return $\mathcal{D}$

HANDLEREGULARVERTEX(vertex v)

if P lies locally right of v **then**
    $e, e' \leftarrow$ upper, lower edge
    if isMergeVertex(helper(e)) **then**
        $\mathcal{D} \leftarrow$ insert (helper$(e), v$)
    delete e from $\mathcal{T}$
    $\mathcal{T} \leftarrow$ insert e'; helper$(e') \leftarrow v$
else
    $e \leftarrow$ edge left of v in $\mathcal{T}$
    if isMergeVertex(helper(e)) **then**
        $\mathcal{D} \leftarrow$ insert (helper$(e), v$)
    helper$(e) \leftarrow v$





Analysis

Lemma 2: Algorithm MAKEMONOTONE inserts a set of crossing-free diagonals in P that partition P into y-monotone polygons.

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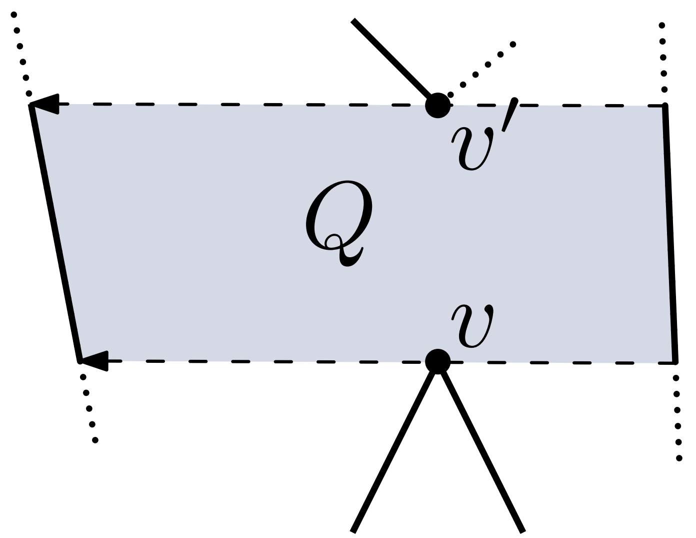

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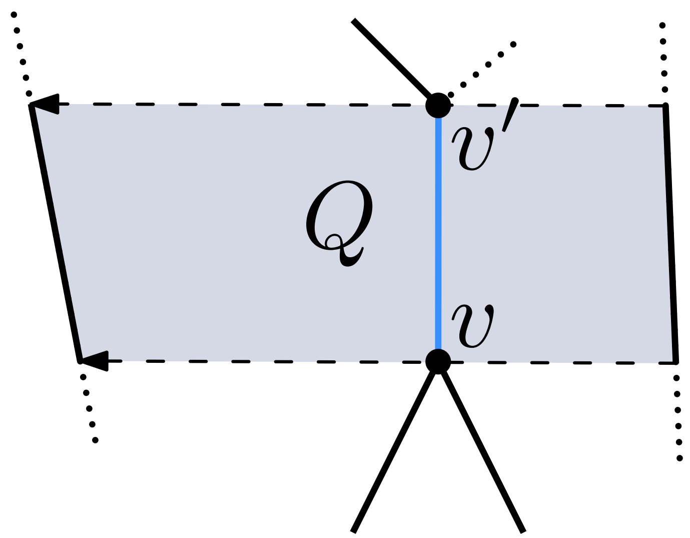

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- create priority queue $\mathcal{Q}$
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- space: *obviously?* $O(n)$

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Lemma 2: Algorithm MAKEMONOTONE inserts a set of crossing-free diagonals in P that partition P into y-monotone polygons.

Theorem 3: A simple polygon with n vertices can be partitioned in $O(n \log n)$ time using $O(n)$ space into y-monotone polygons.

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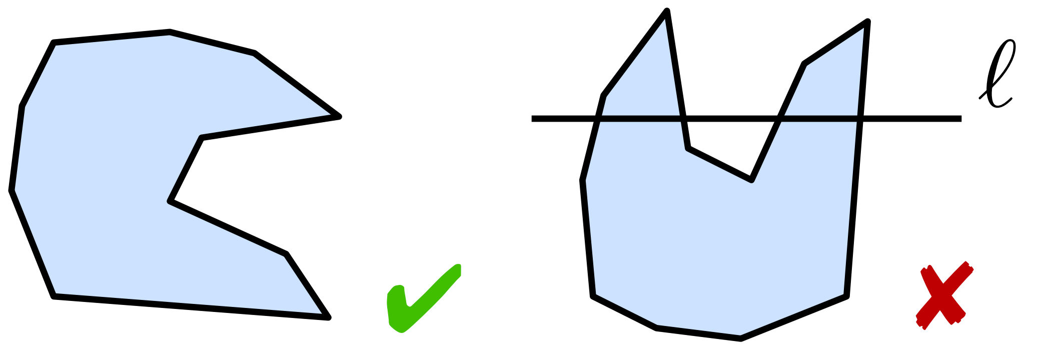
  - search, delete, insert elements of $\mathcal{T}$ — $O(\log n)$ time
  - add ≤ 2 diagonals to $\mathcal{D}$ — $O(1)$ time
- space: — $O(n)$

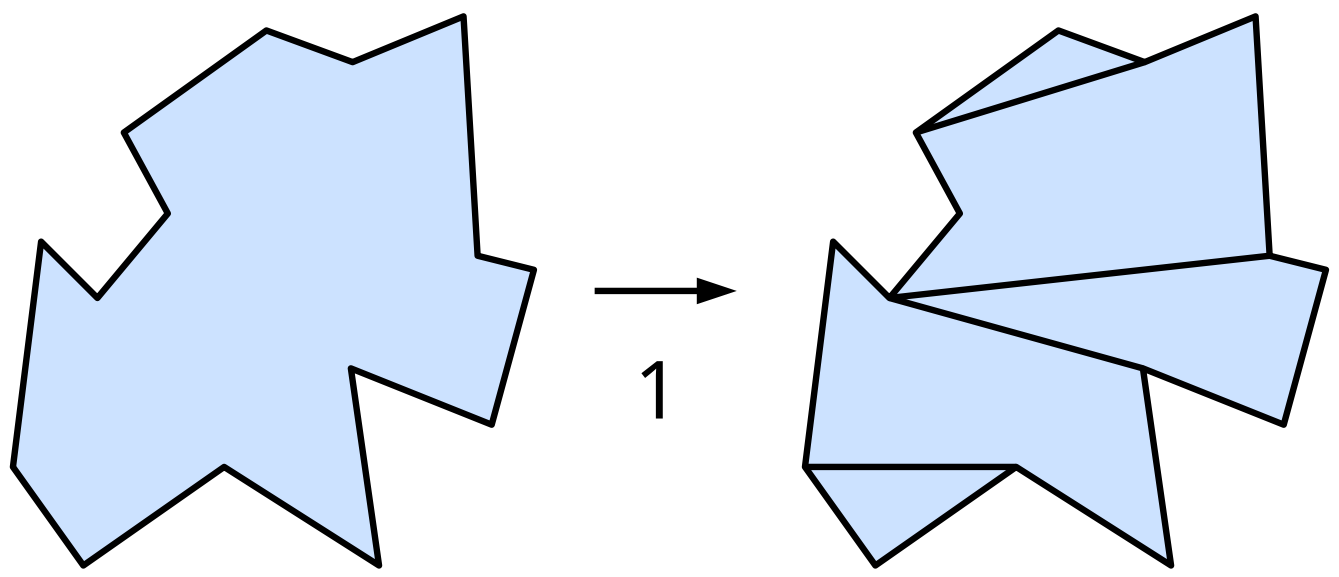
Triangulation: Overview

2-step procedure:

- step 1: partition P into y-monotone subpolygons

Definition: A polygon P is y-monotone if, for every horizontal line ℓ, the intersection $\ell \cap P$ is connected.





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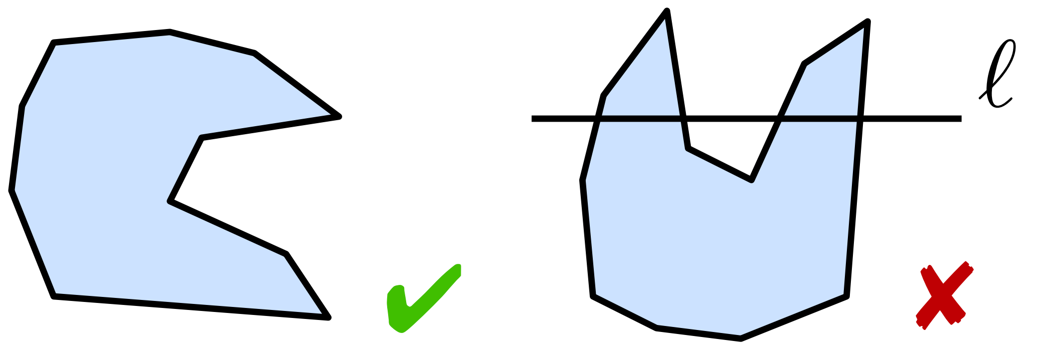
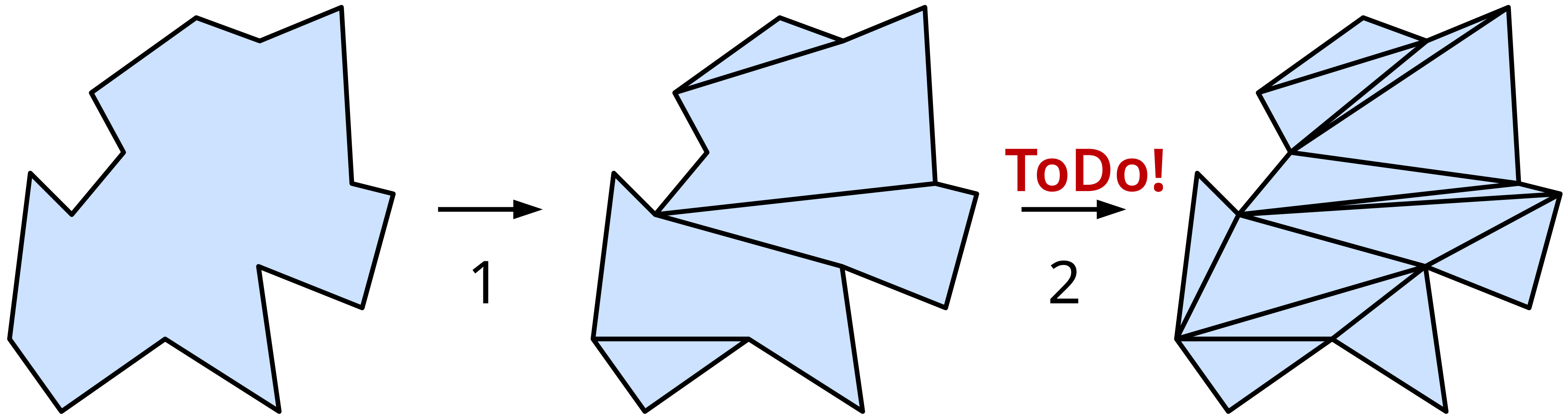


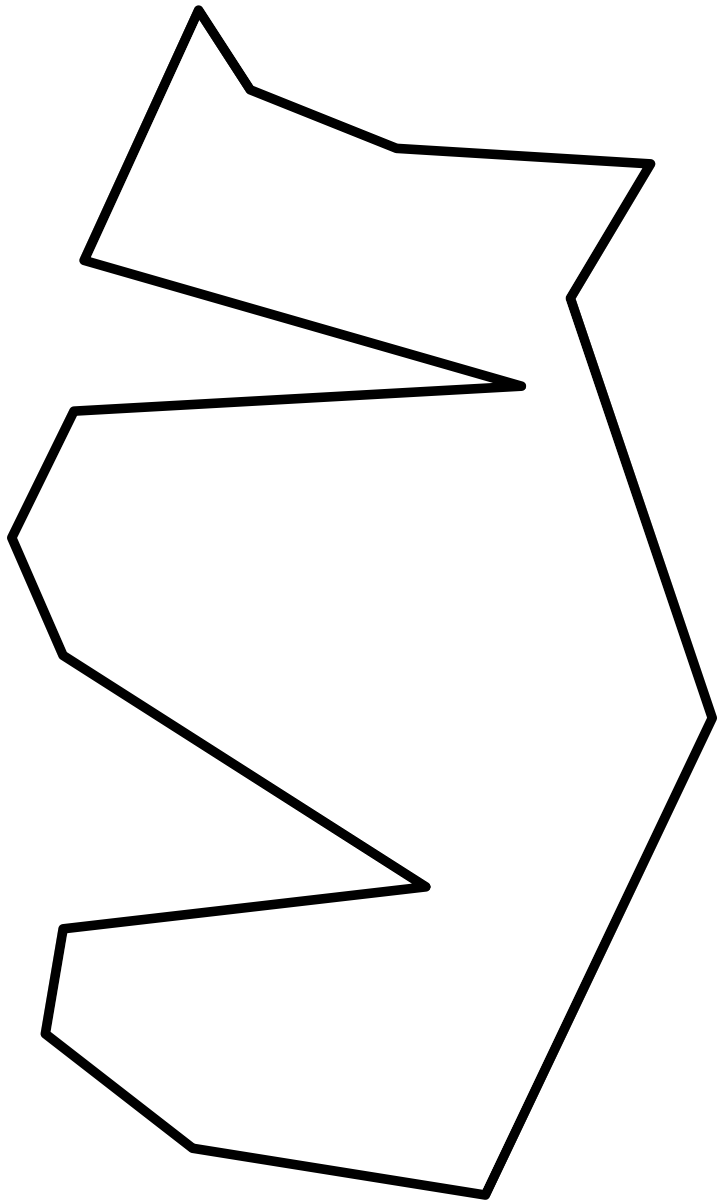
- step 2: triangulate y-monotone subpolyons



Triangulating a y-monotone Polygon

reminder: boundary chains from top to bottom only go down

approach: greedy, on both sides top-down



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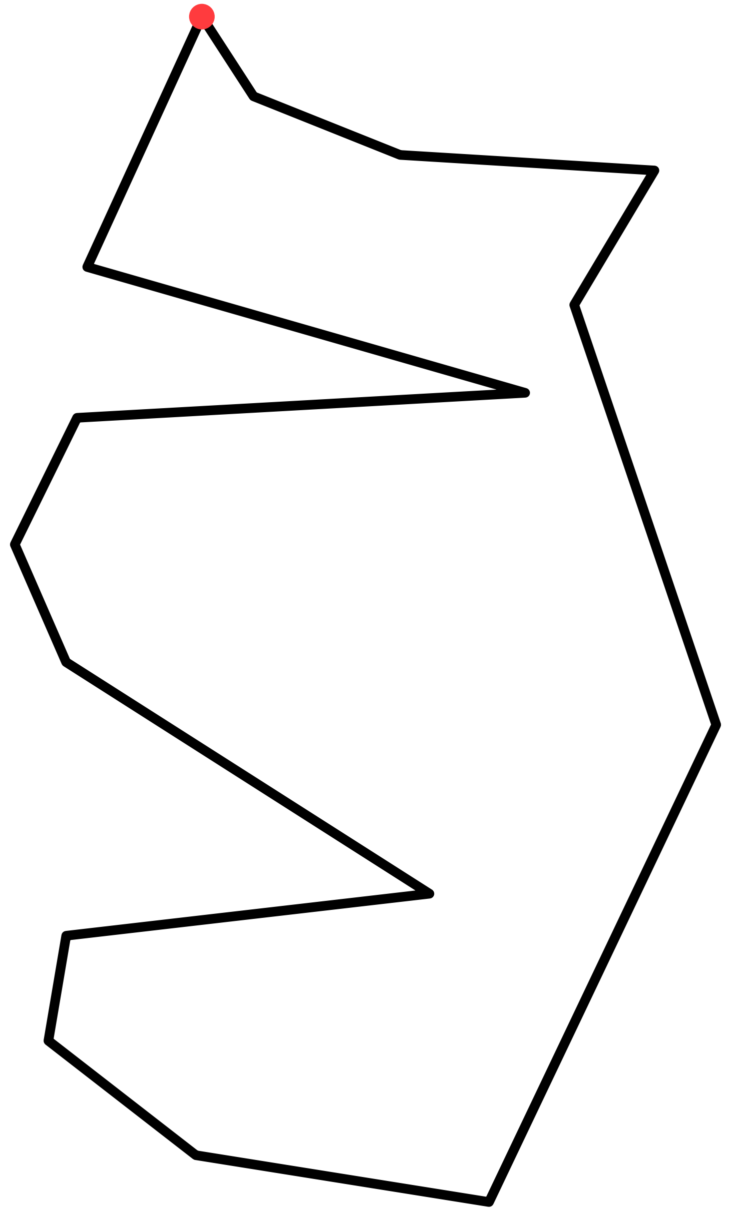

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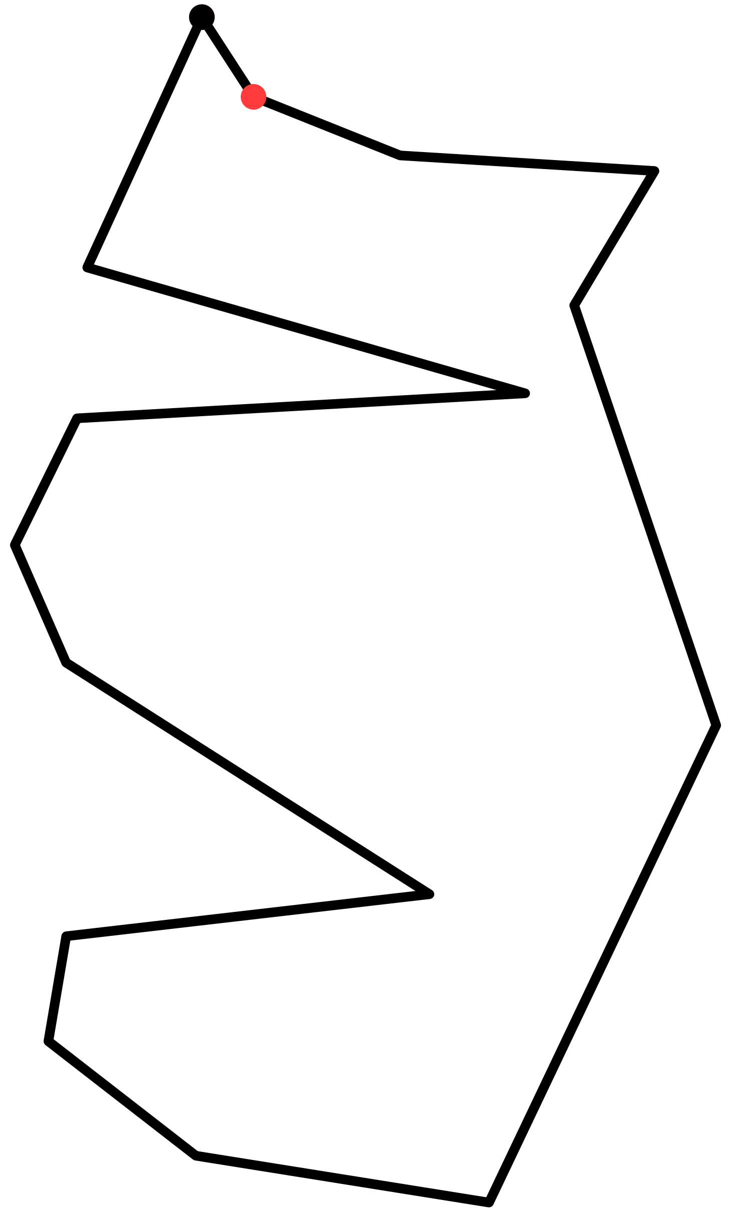

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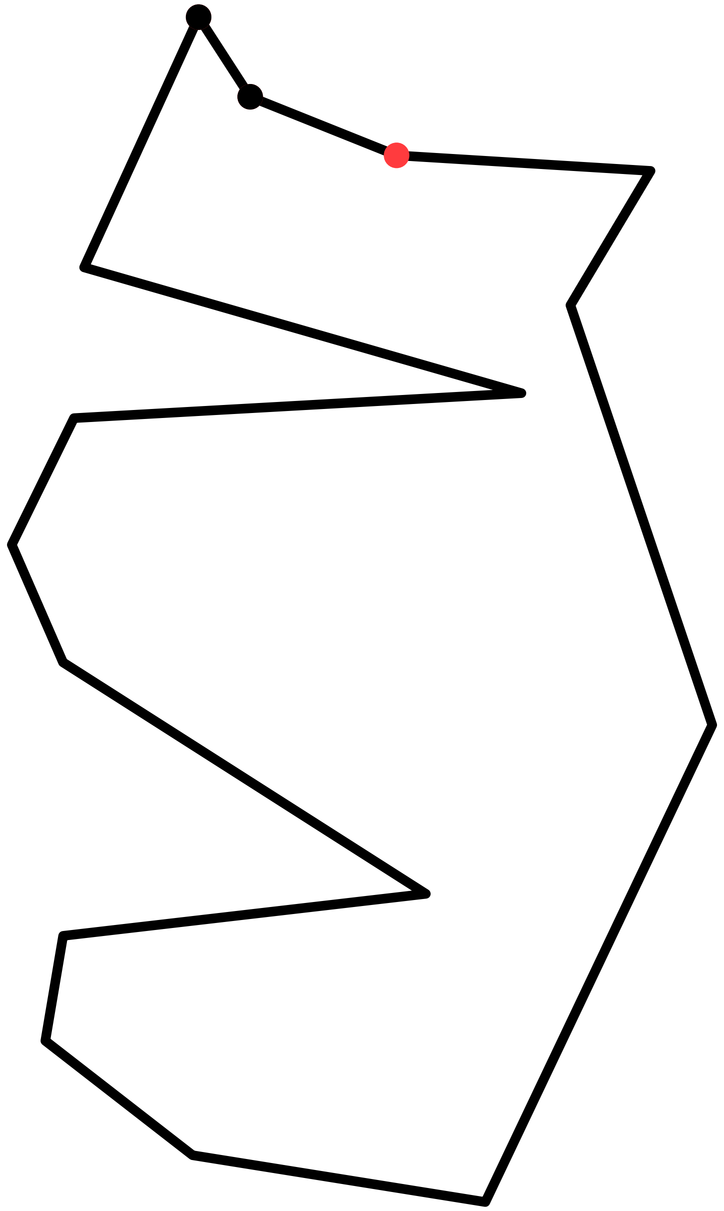

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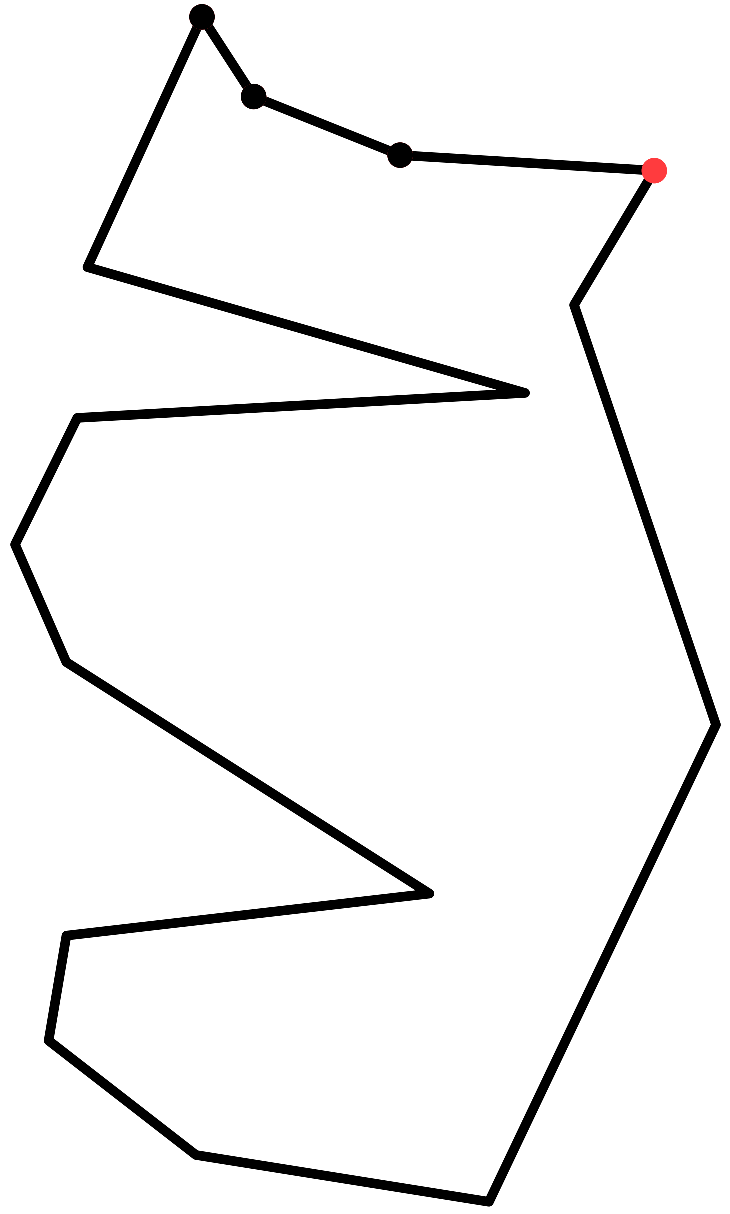

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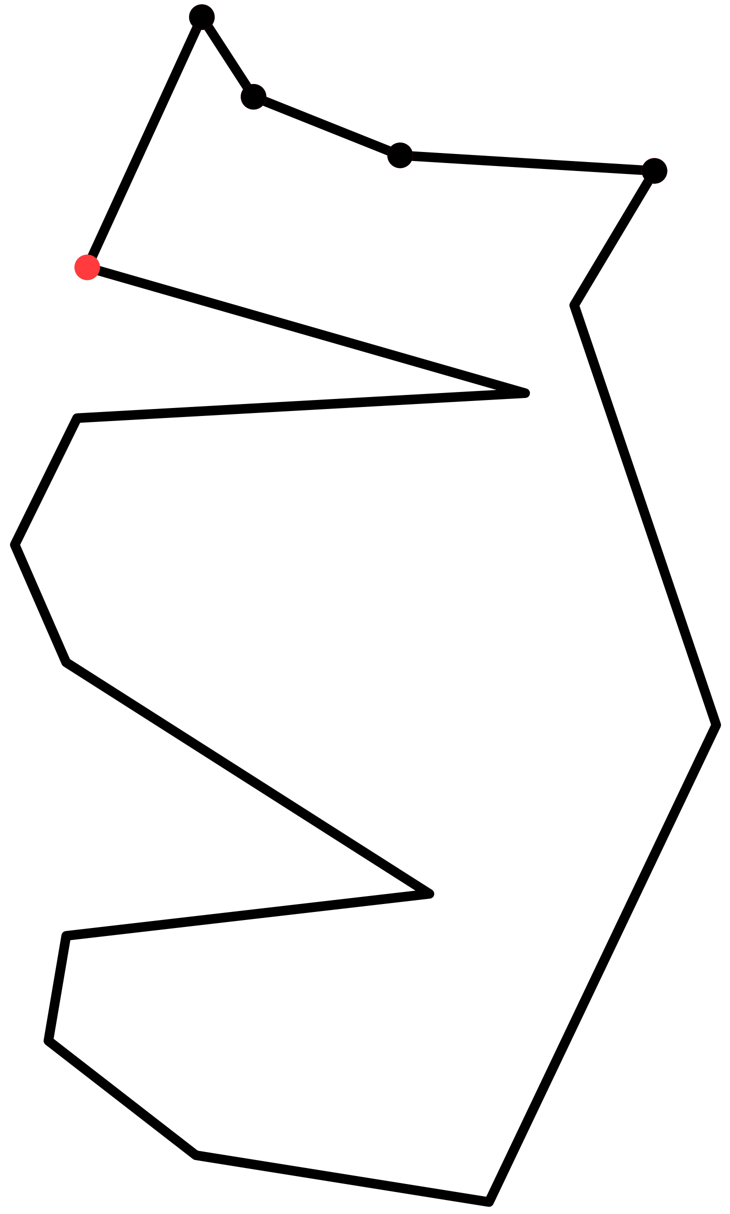

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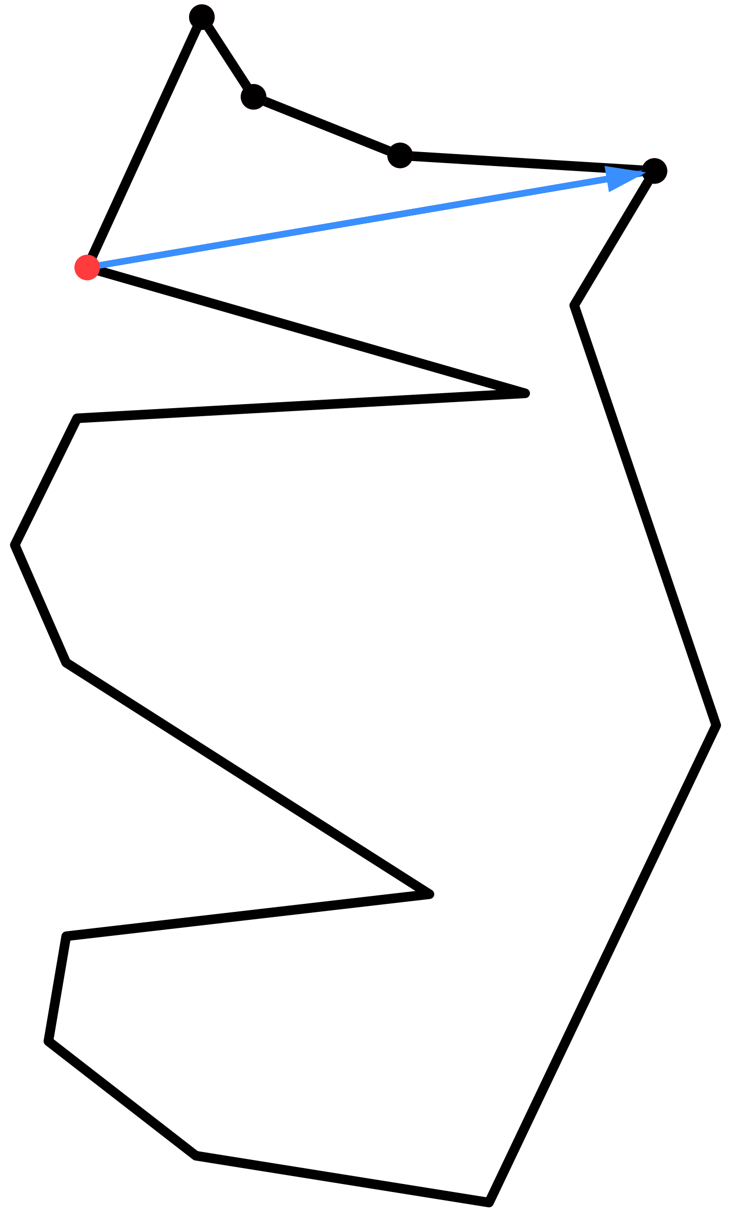

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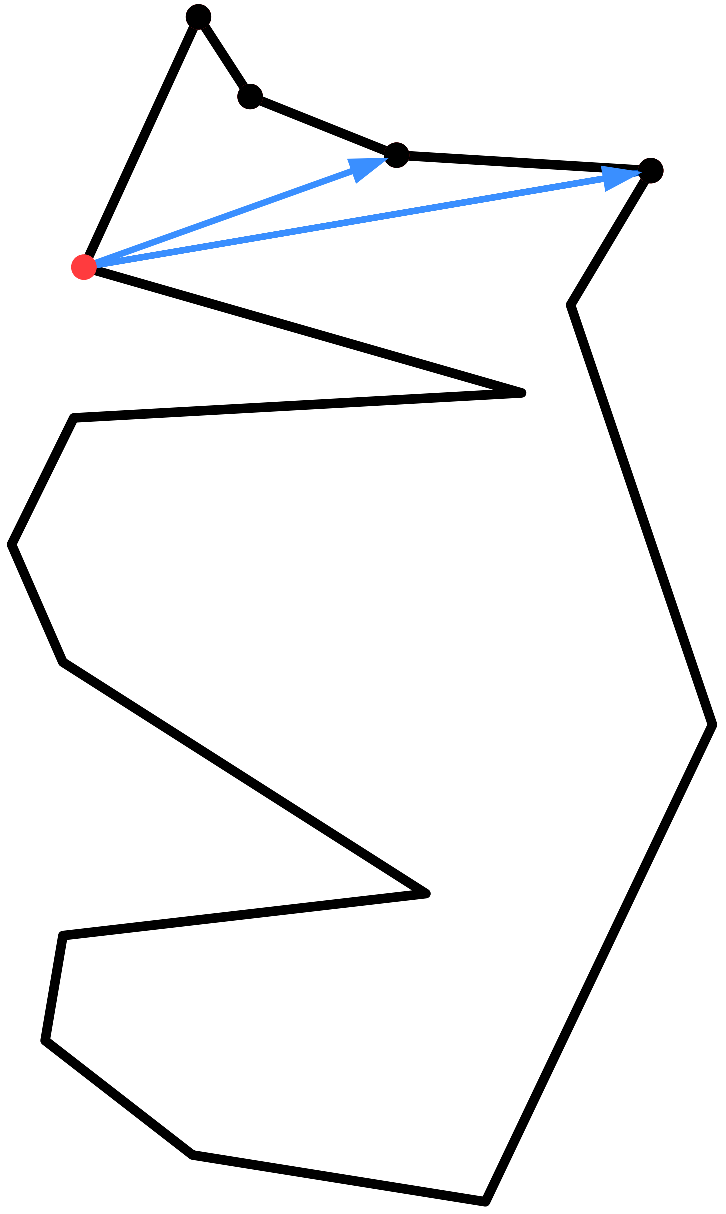

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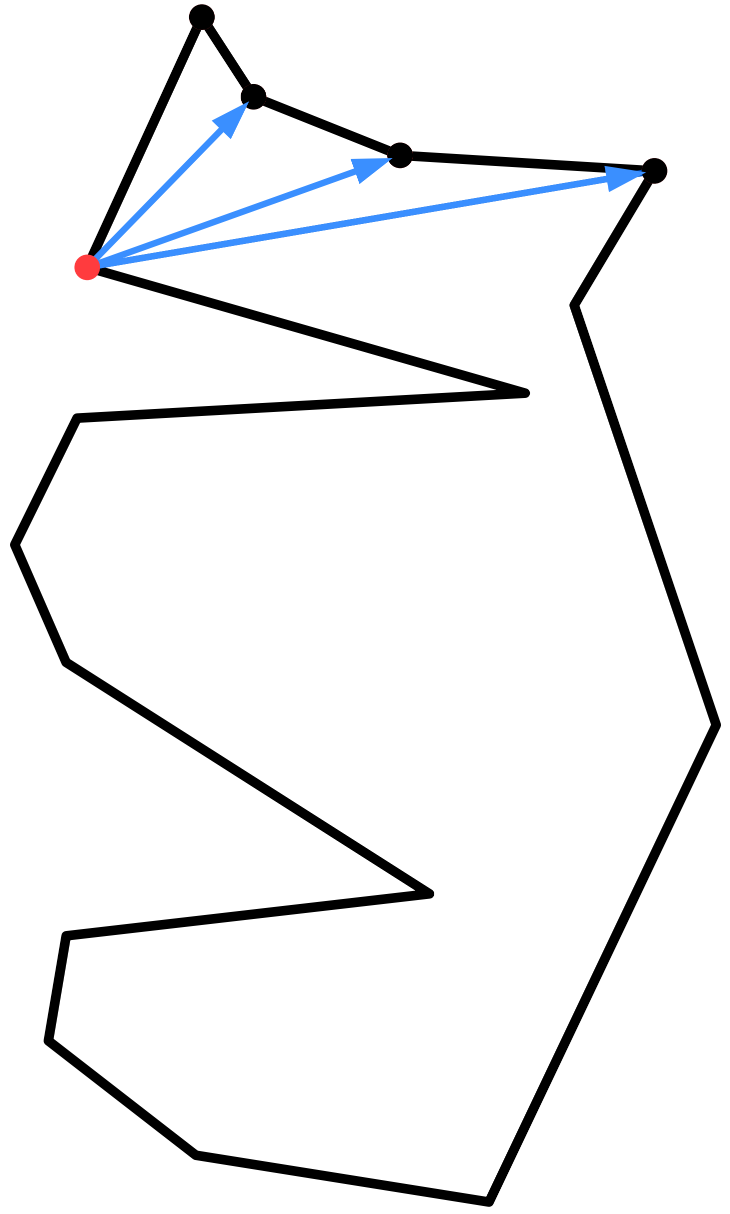

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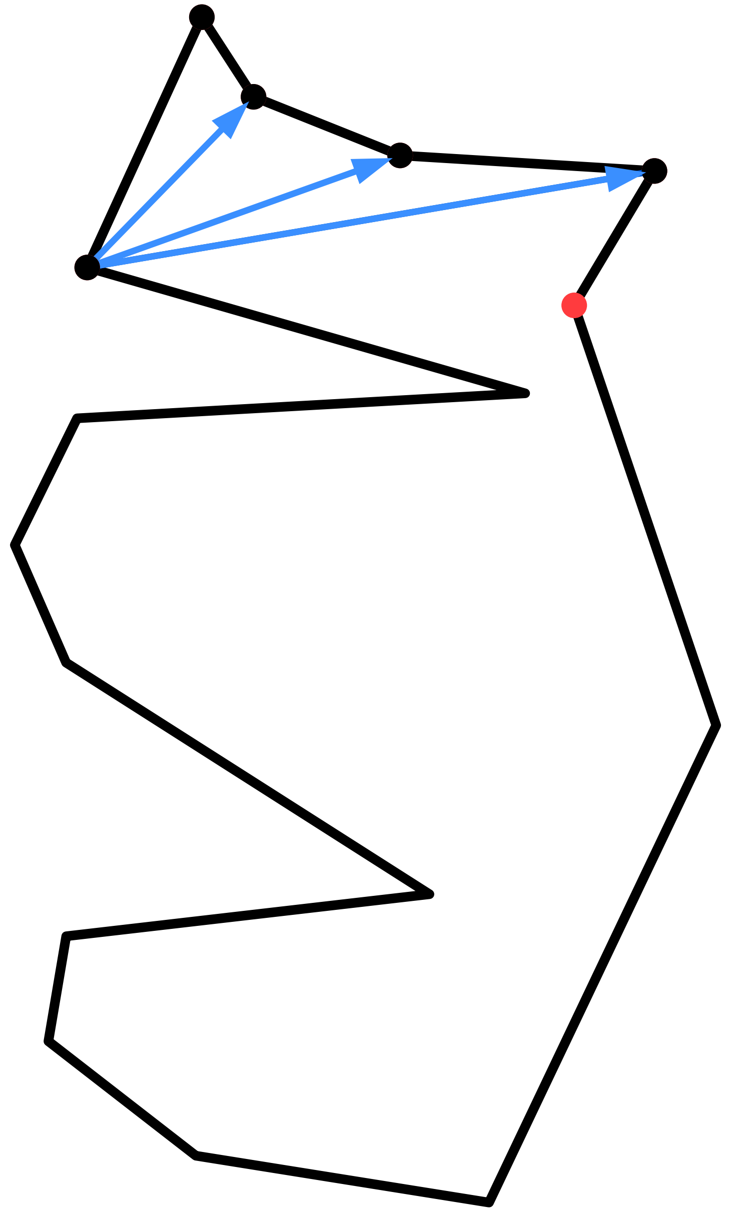

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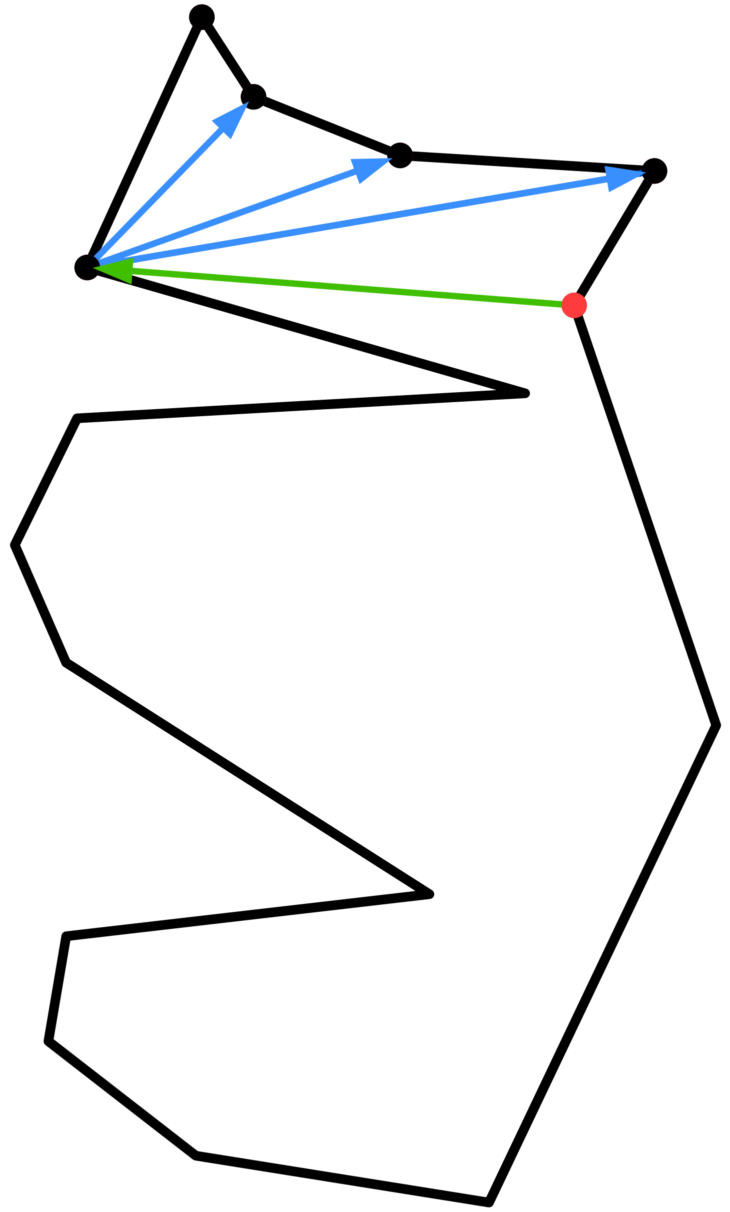

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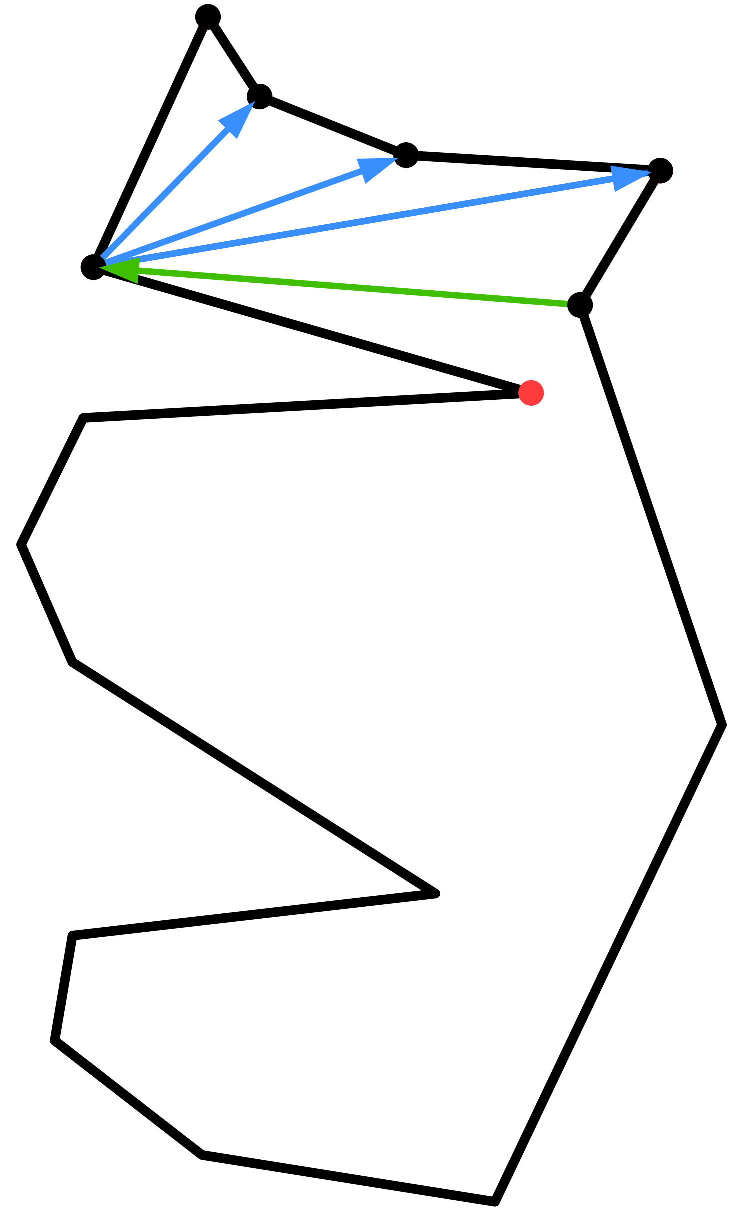

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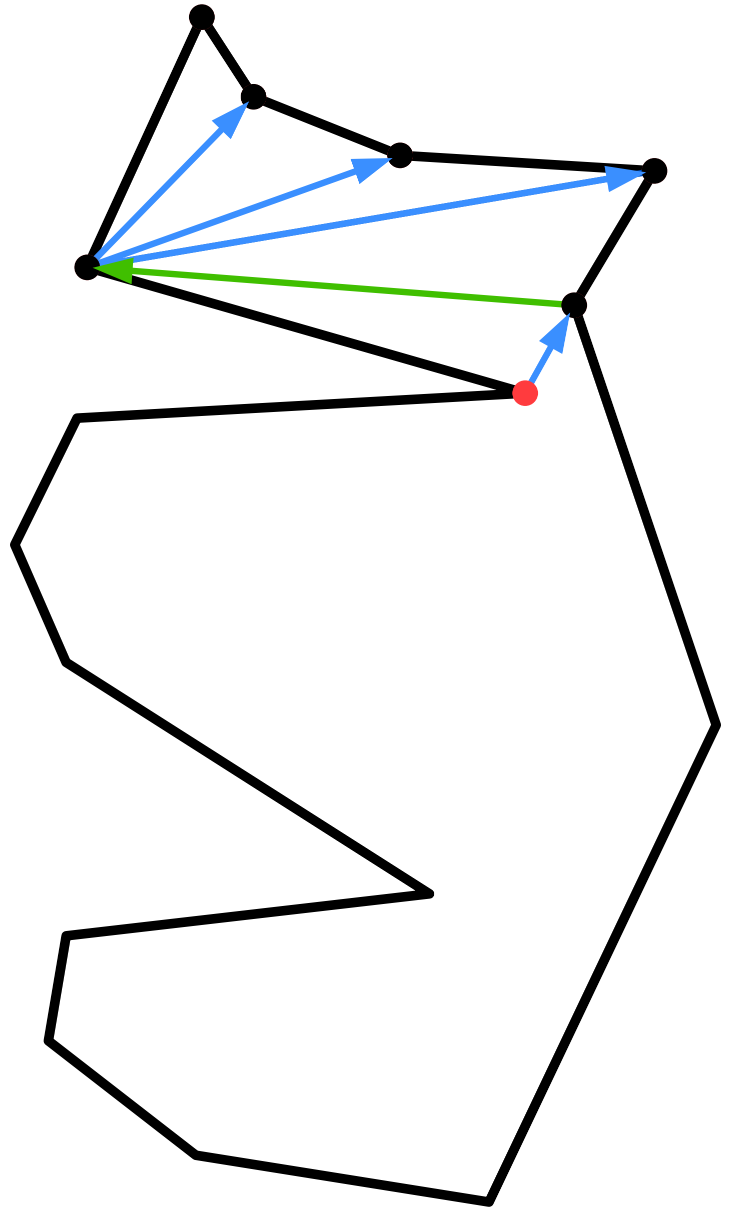

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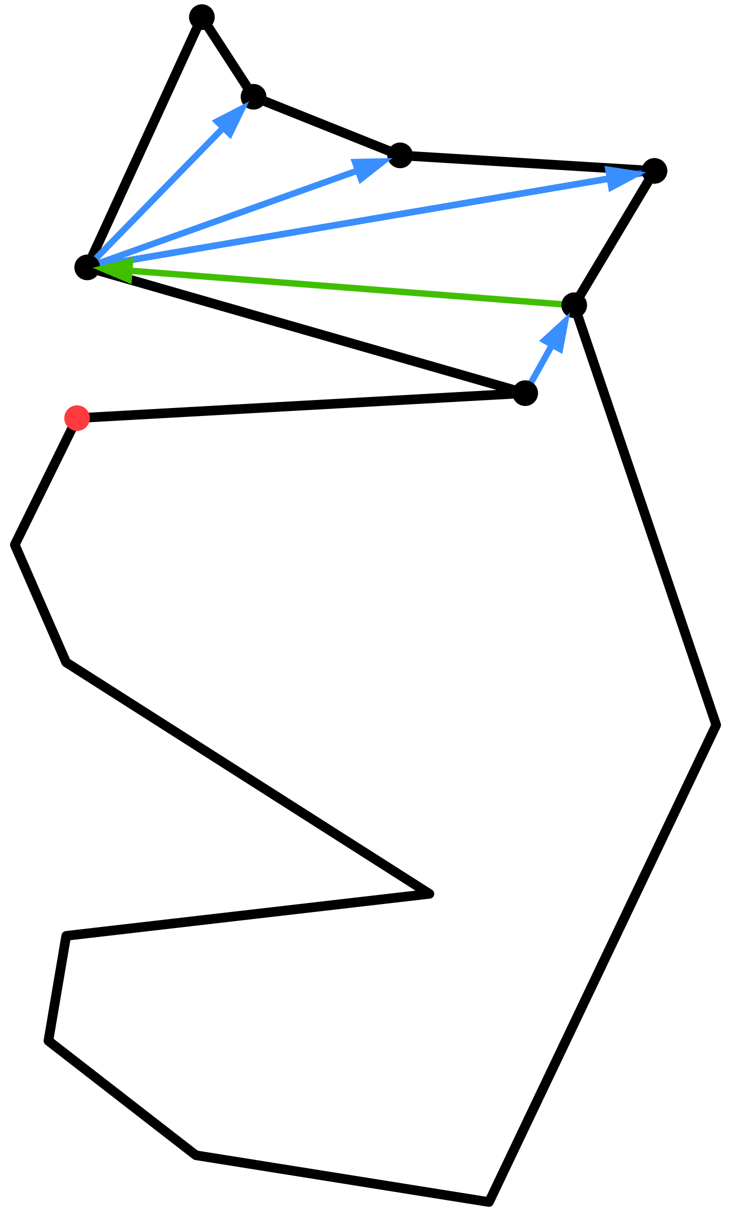

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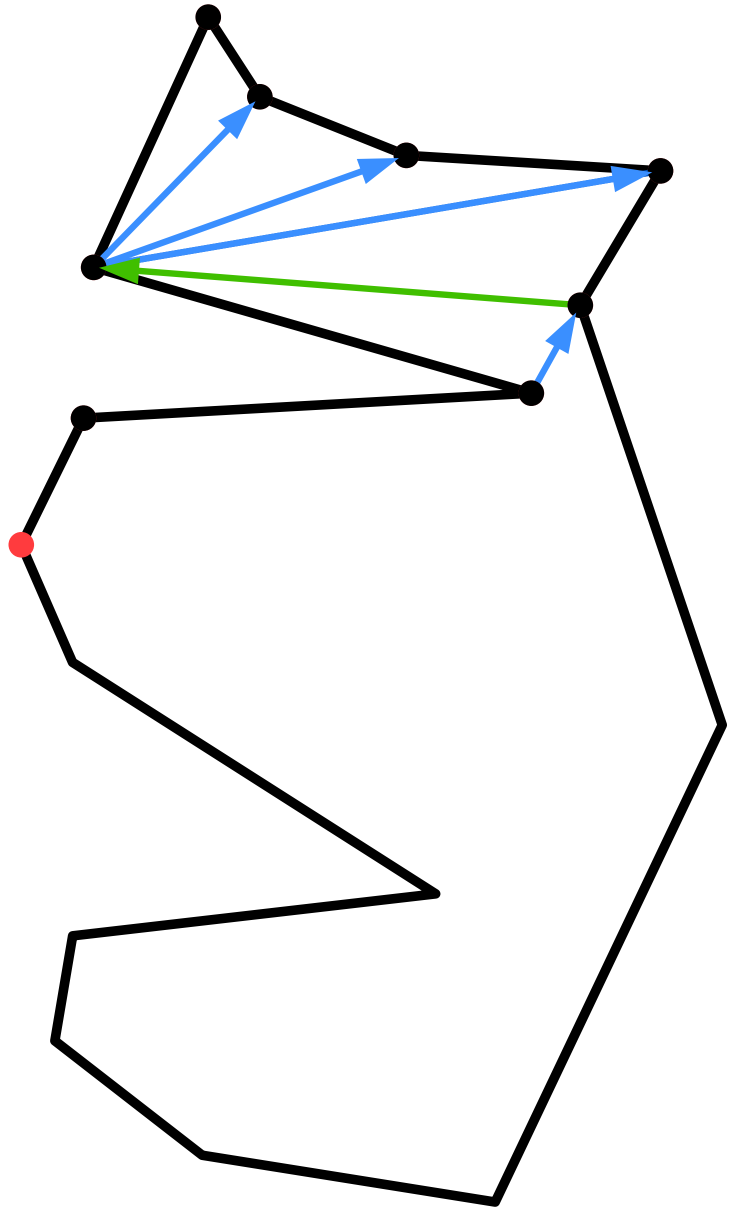

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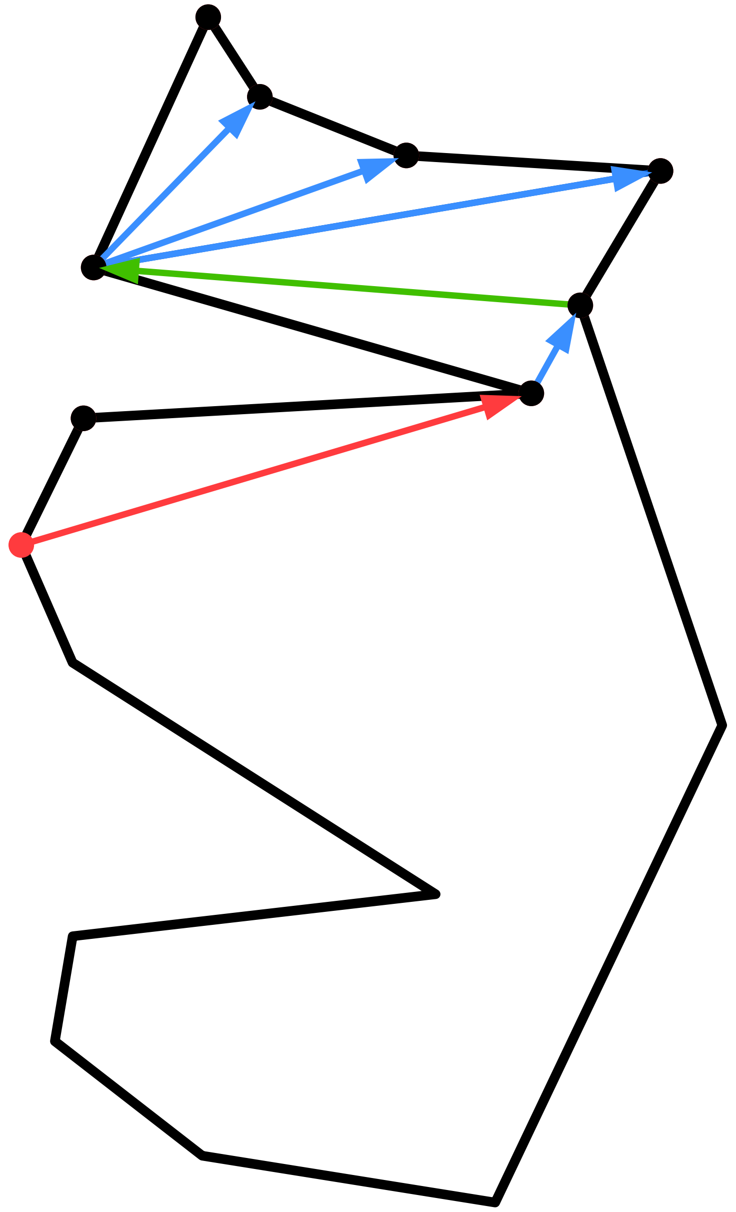

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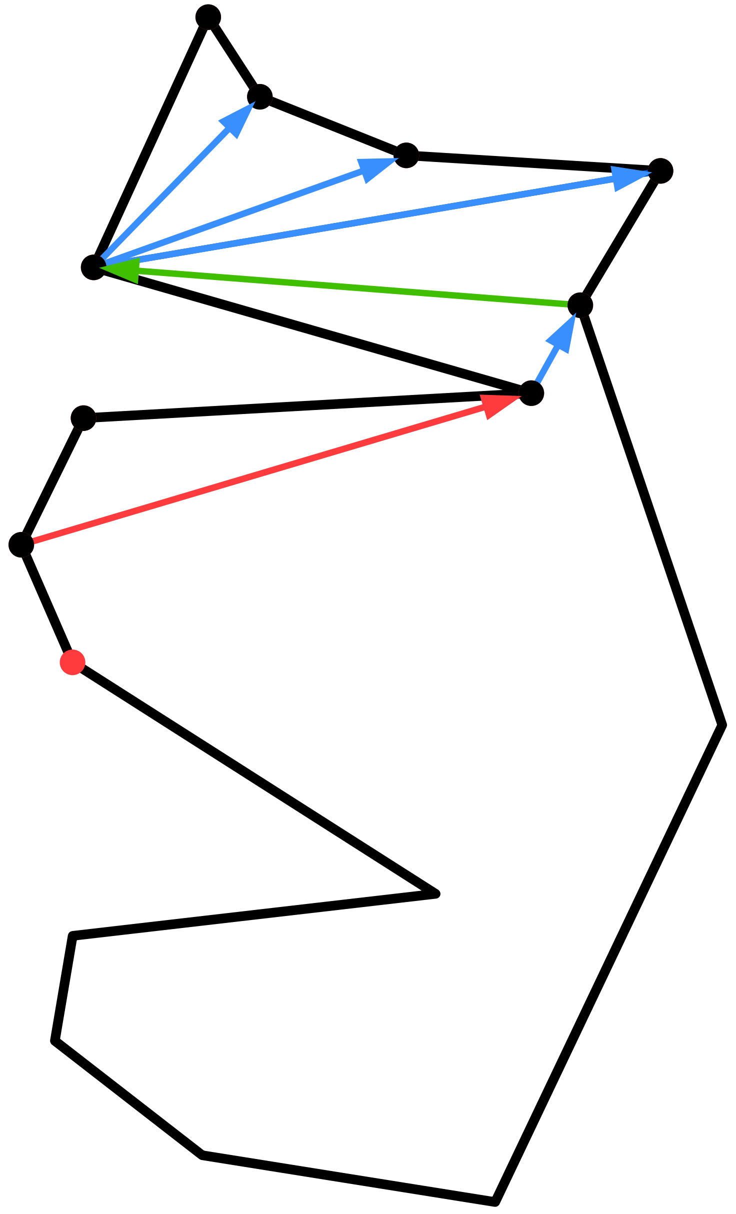

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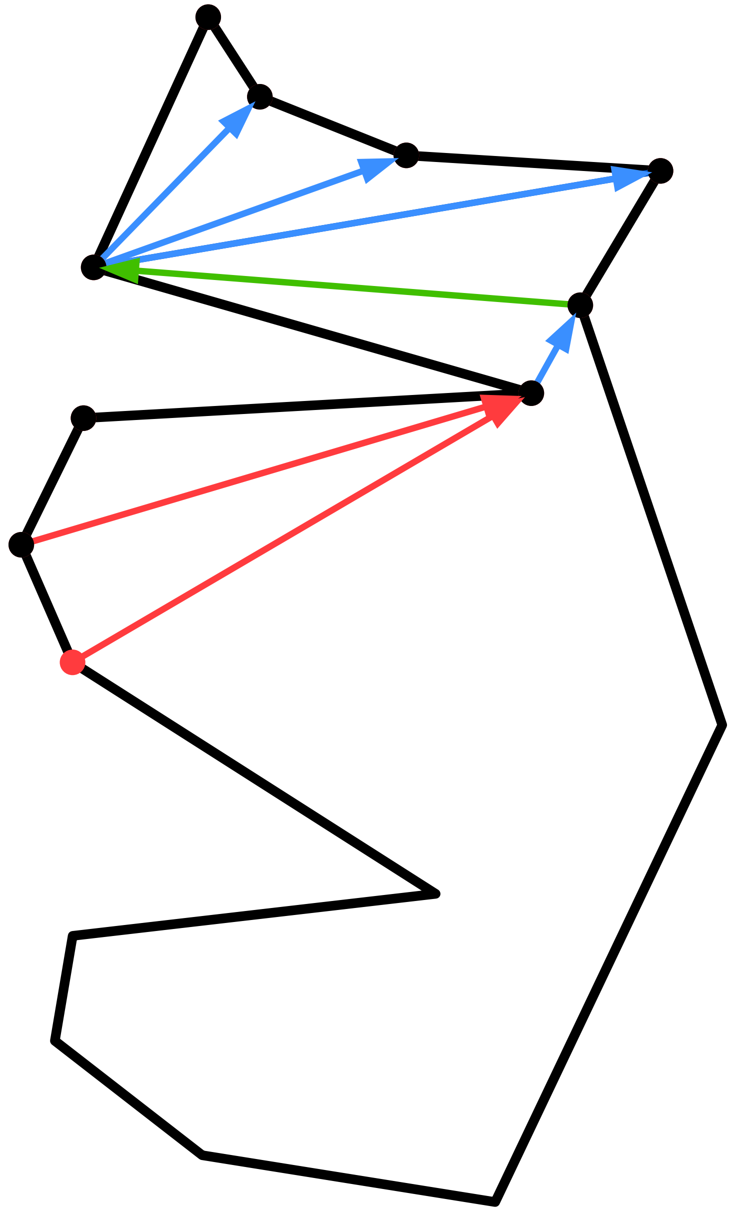

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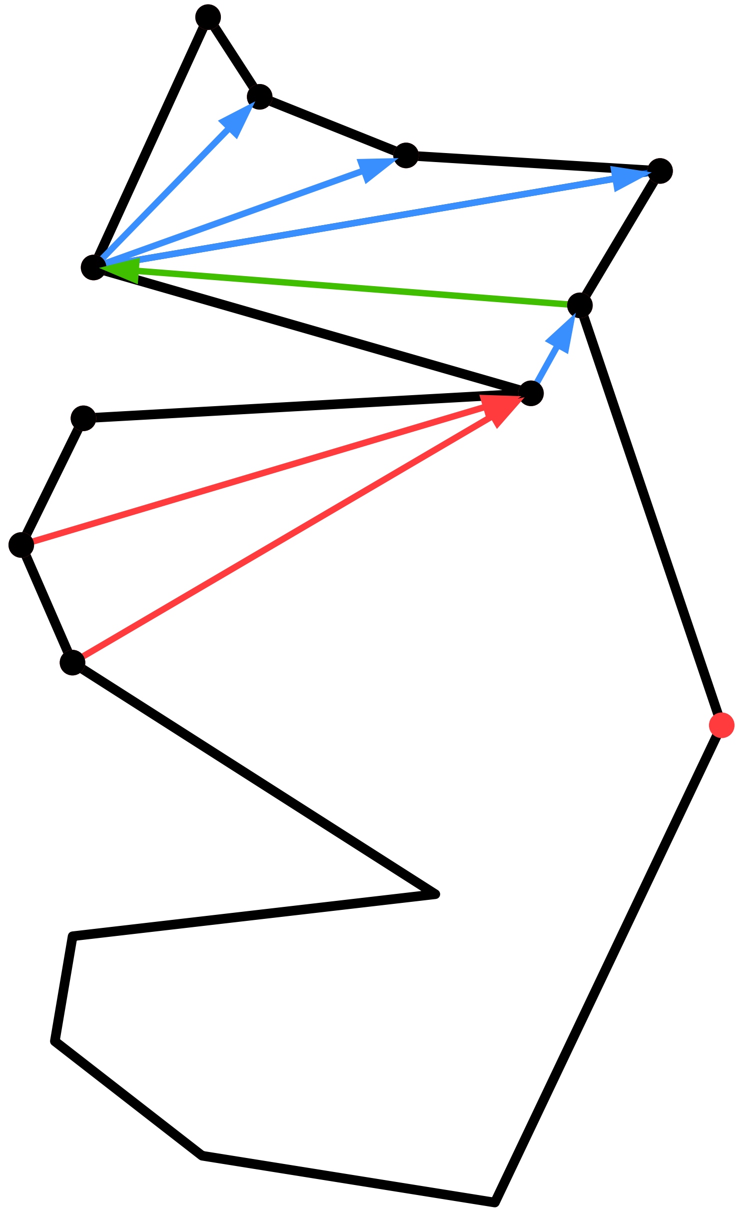

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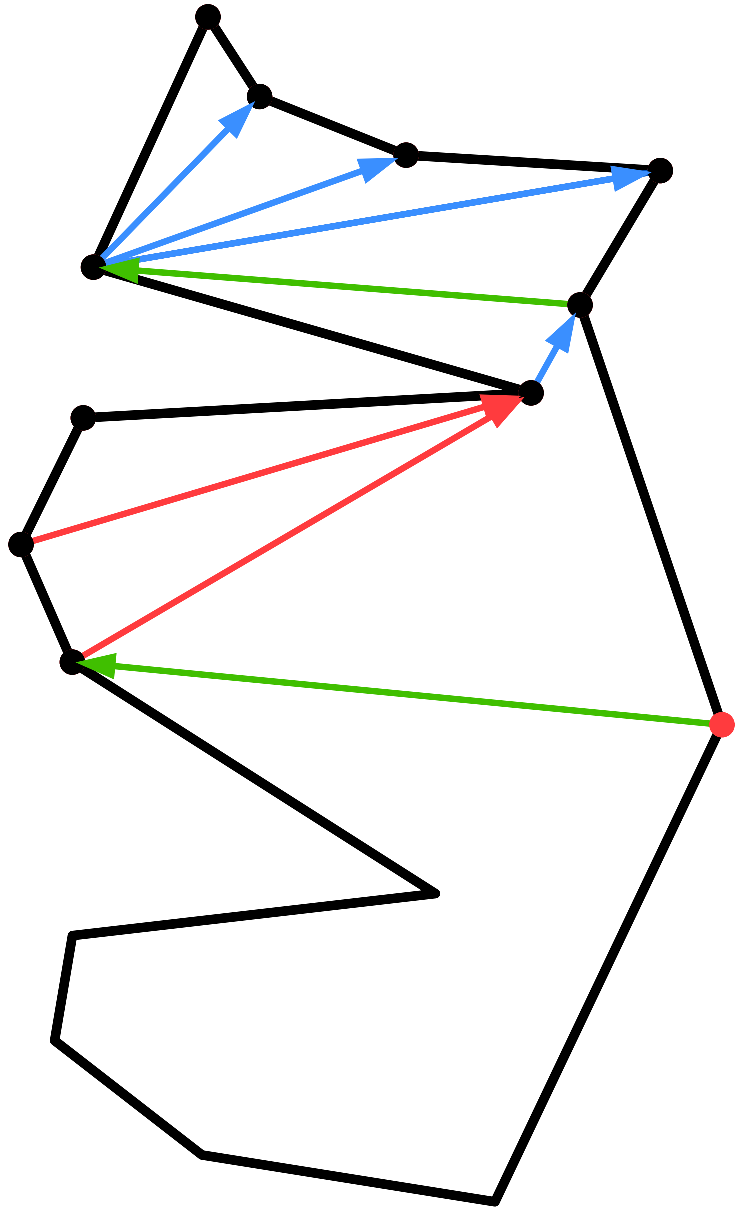

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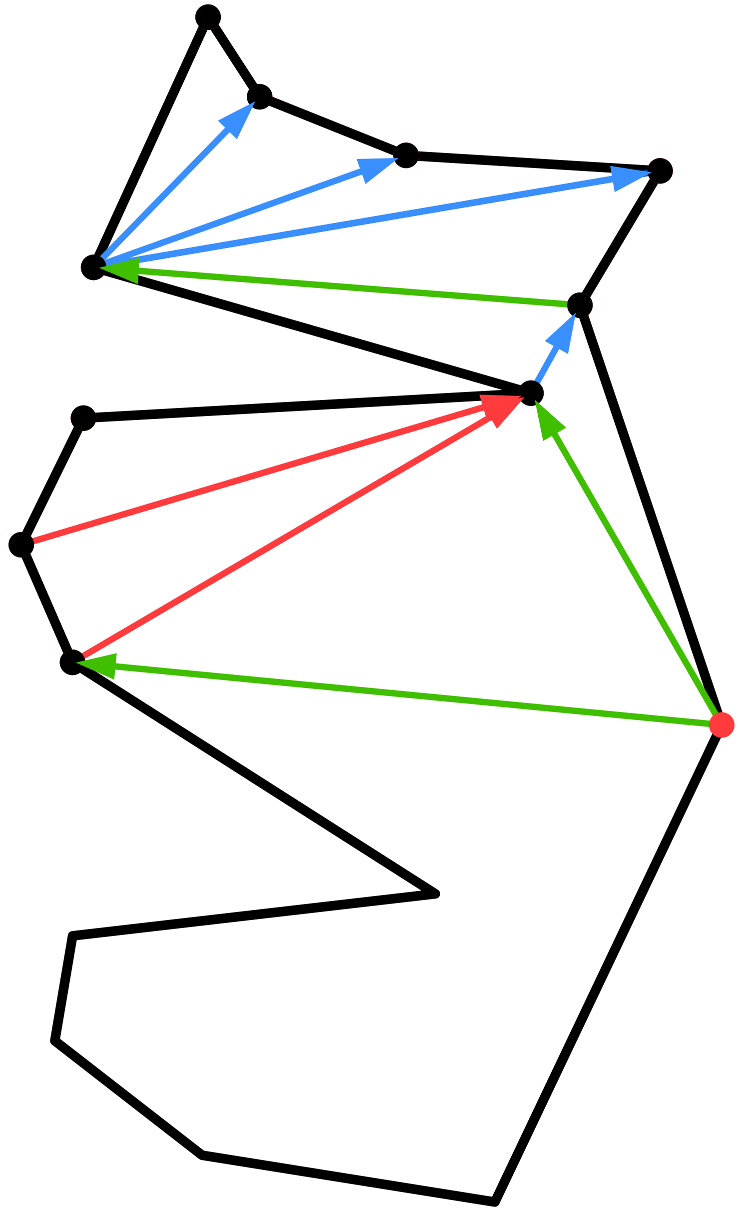
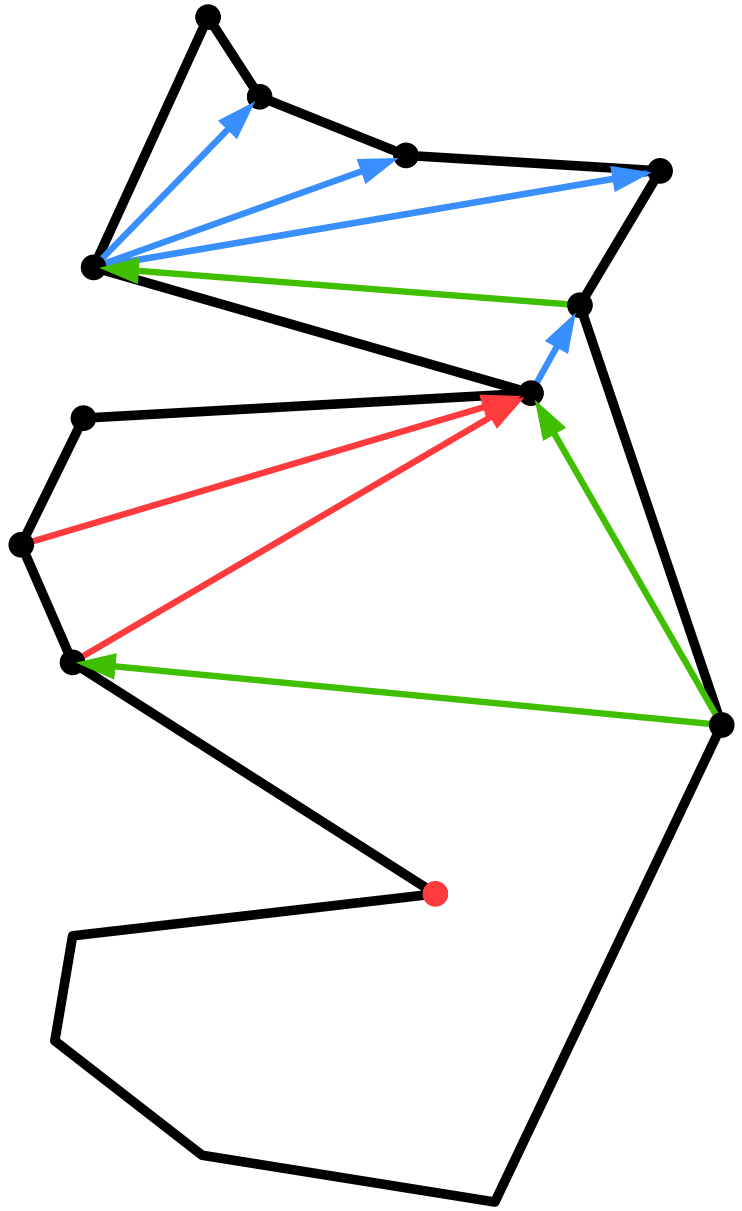


Triangulating a y-monotone Polygon

reminder: boundary chains from top to bottom only go down

approach: greedy, on both sides top-down



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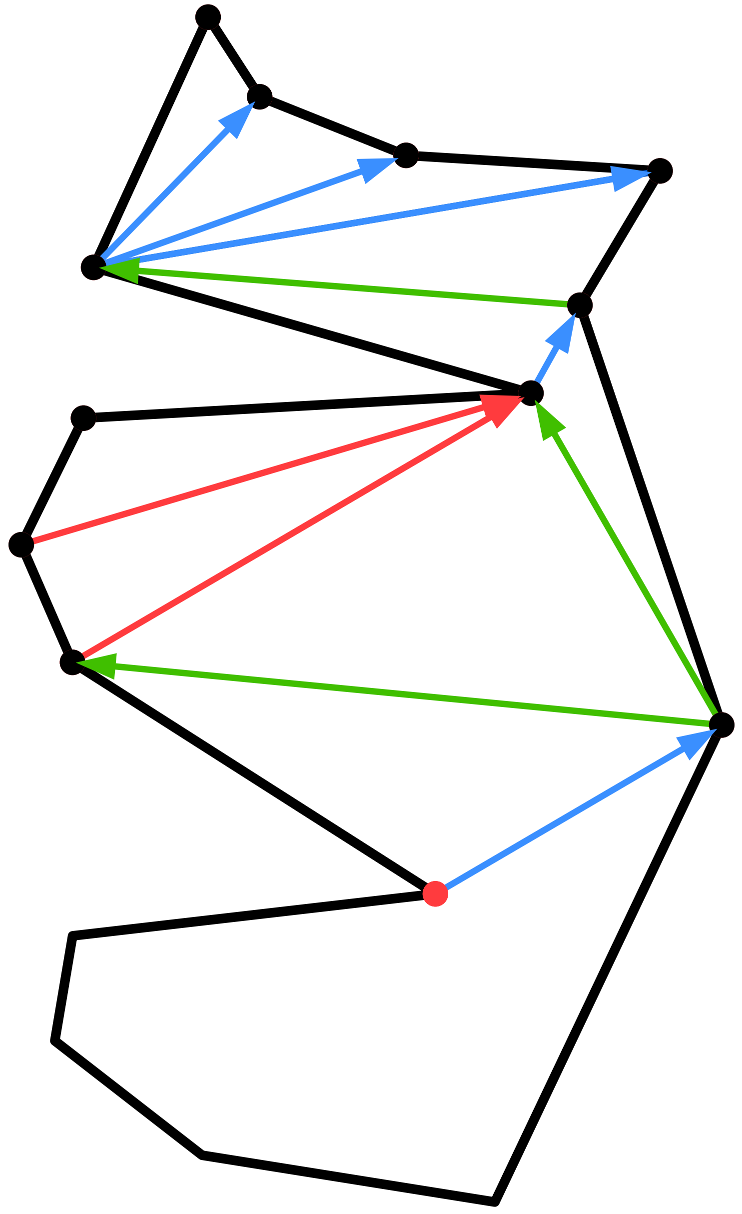

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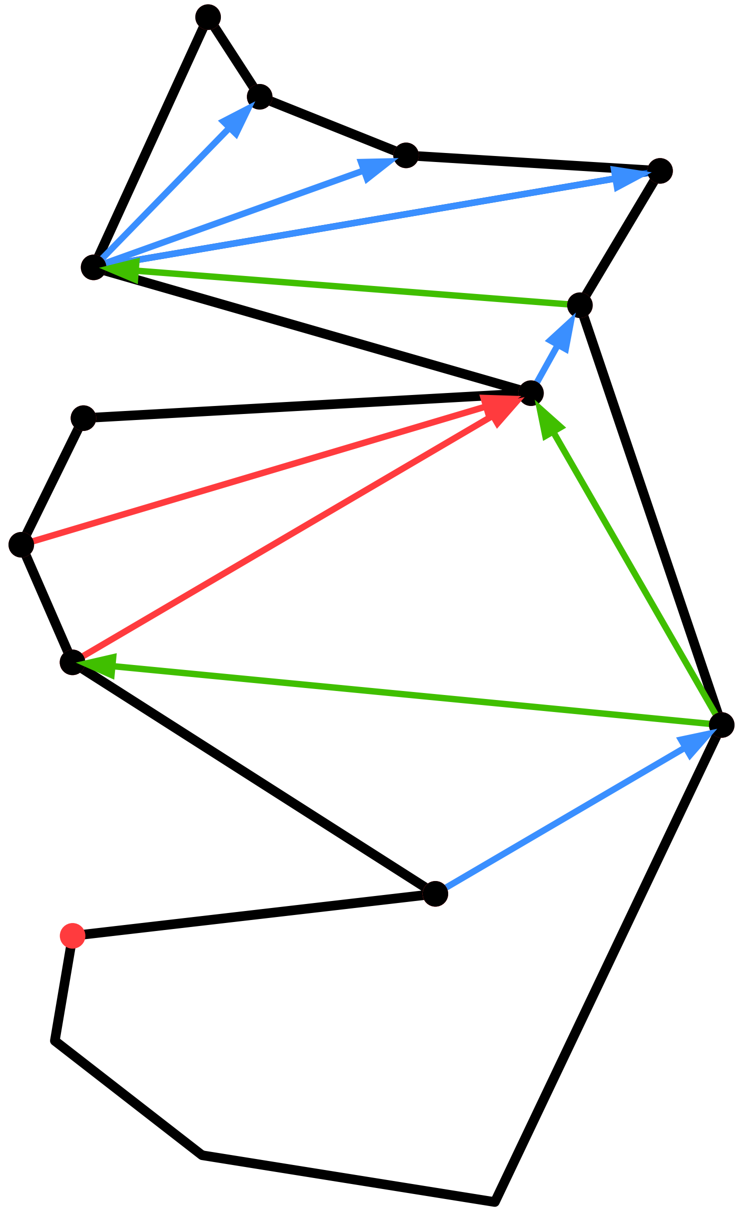

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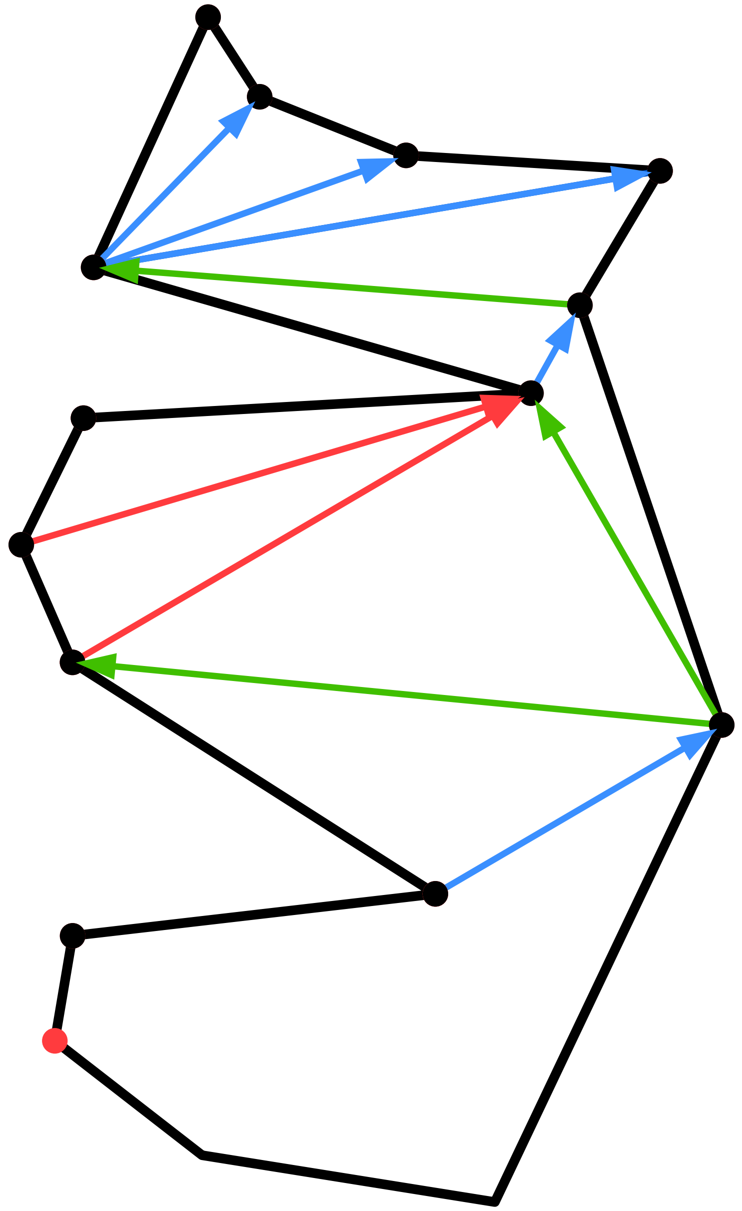

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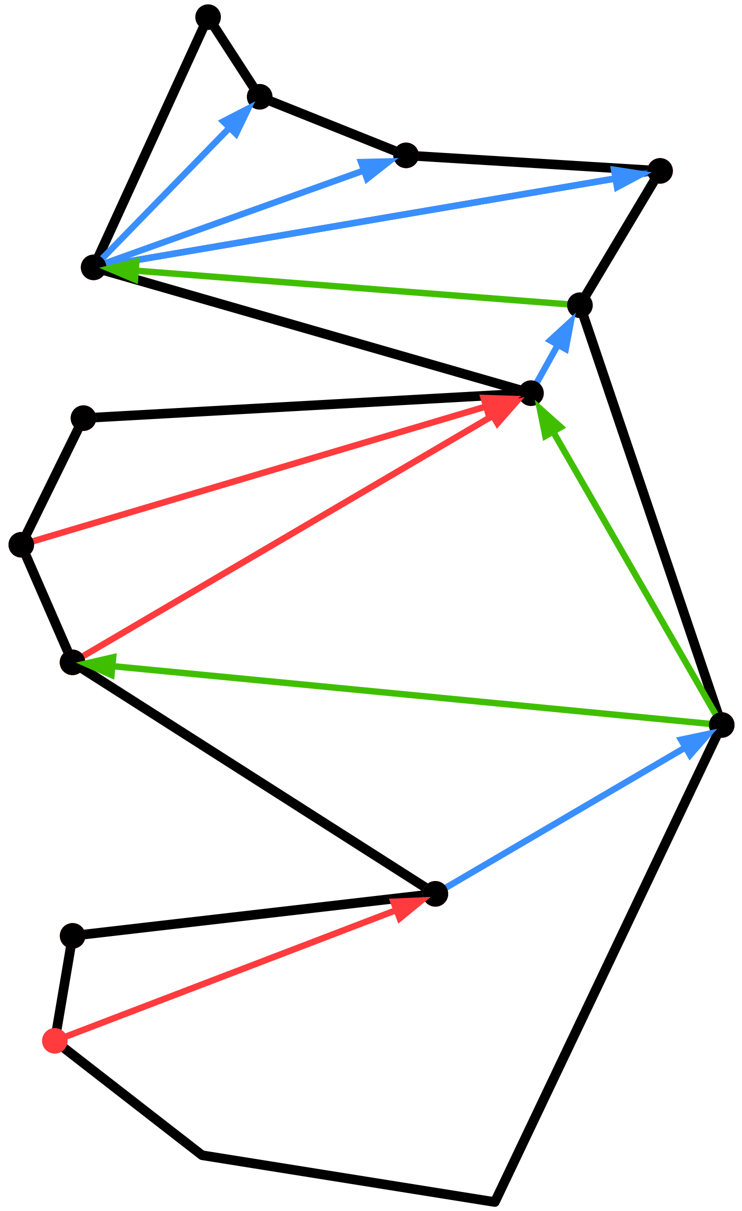

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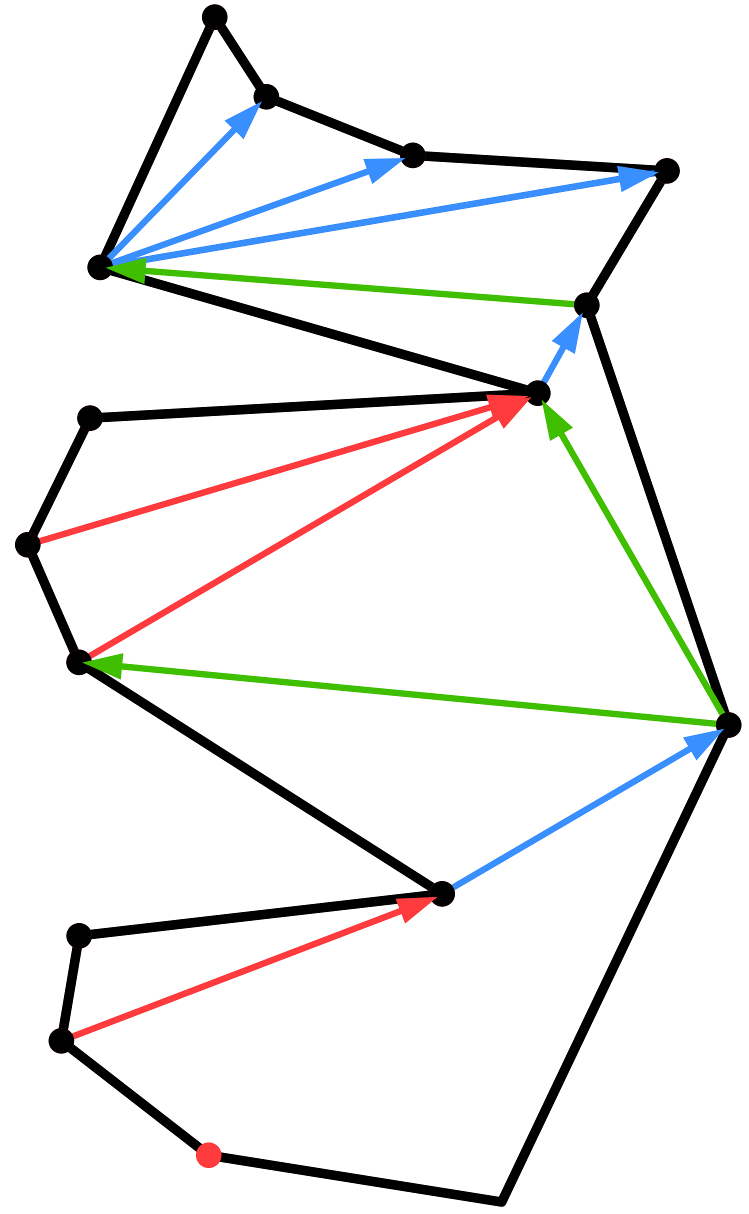

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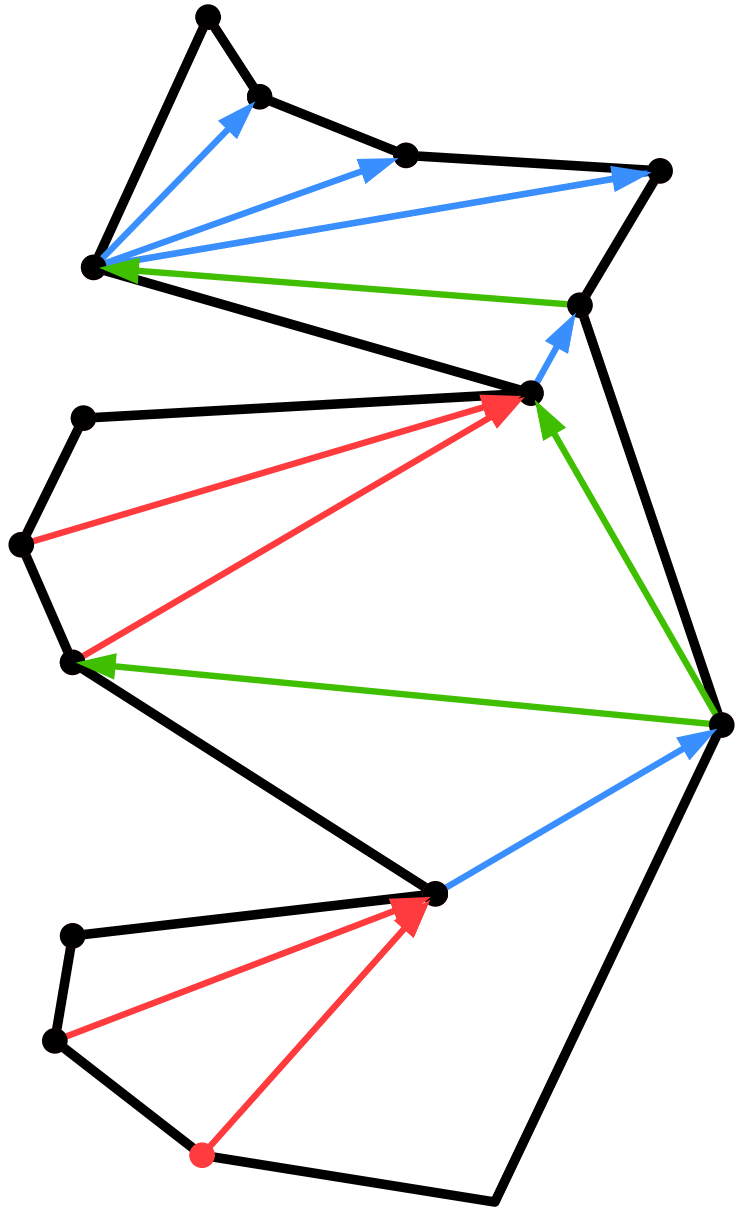

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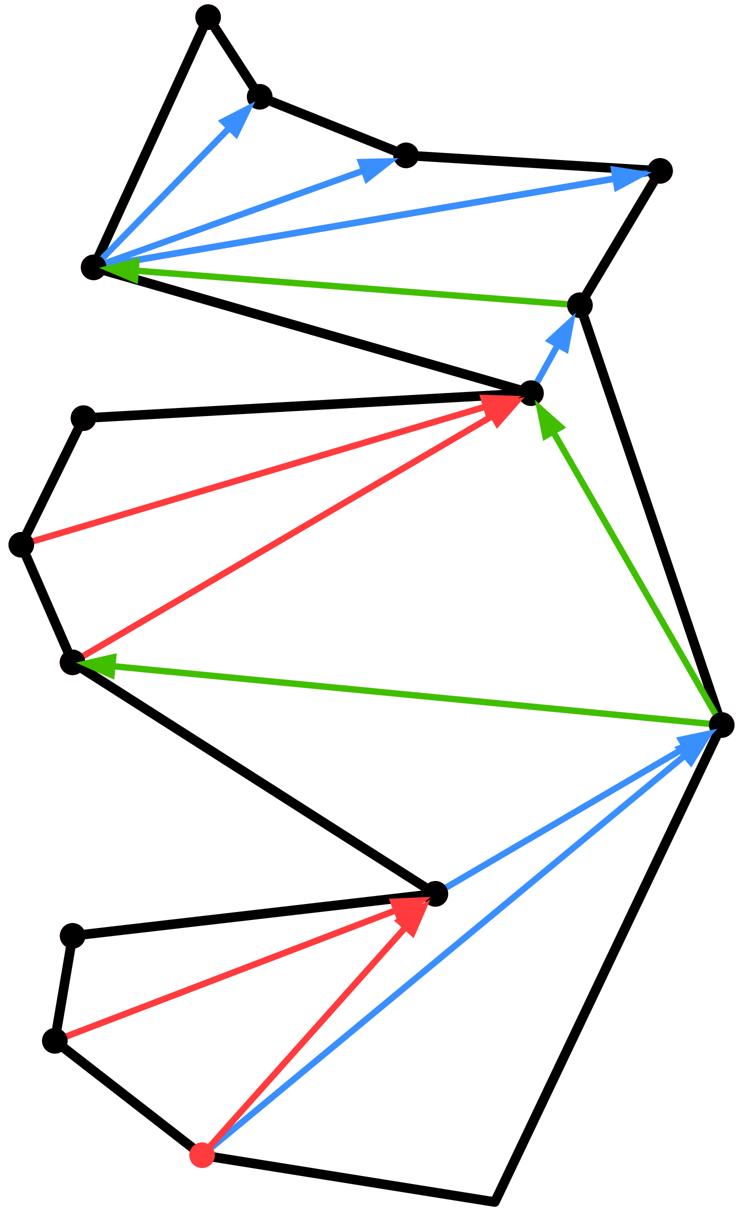

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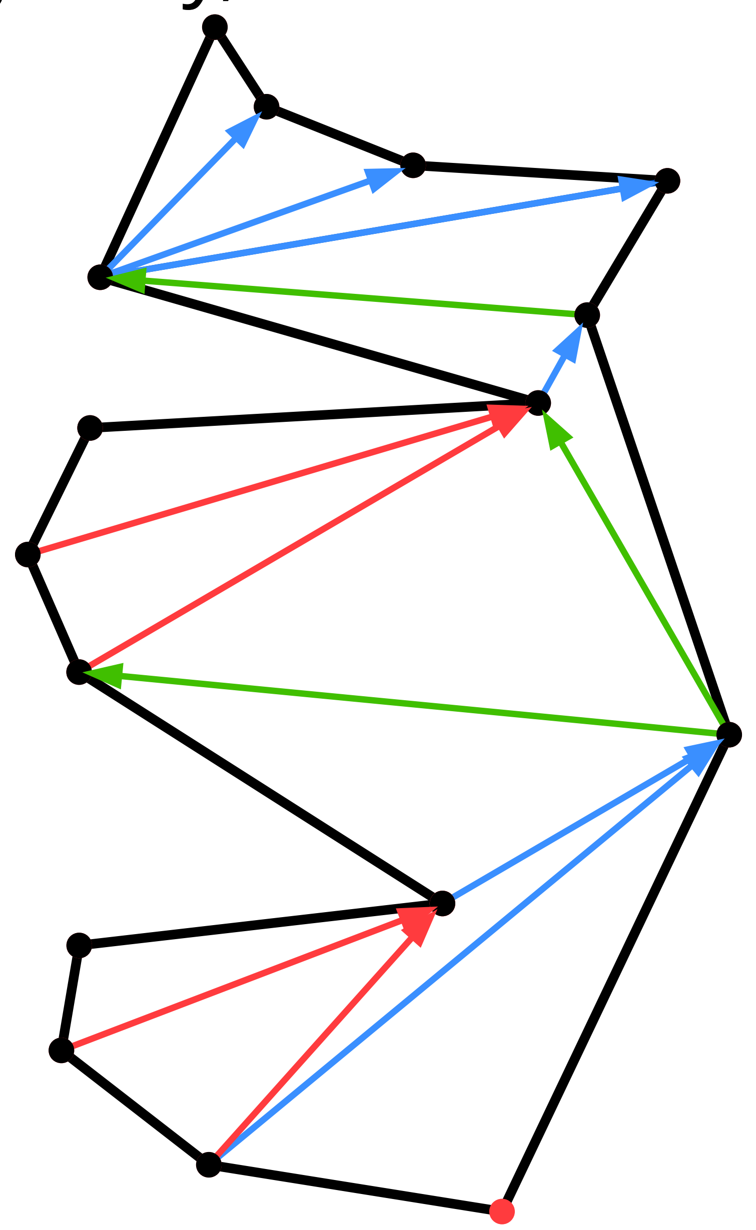

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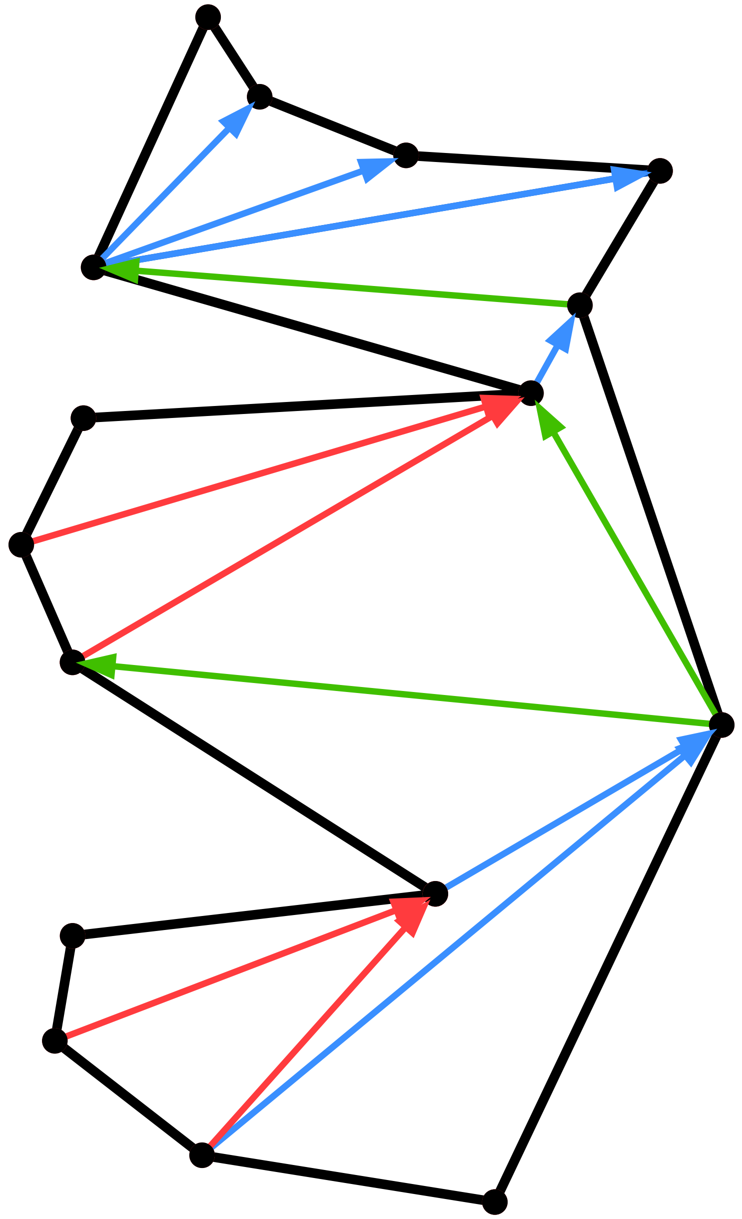

invariant?

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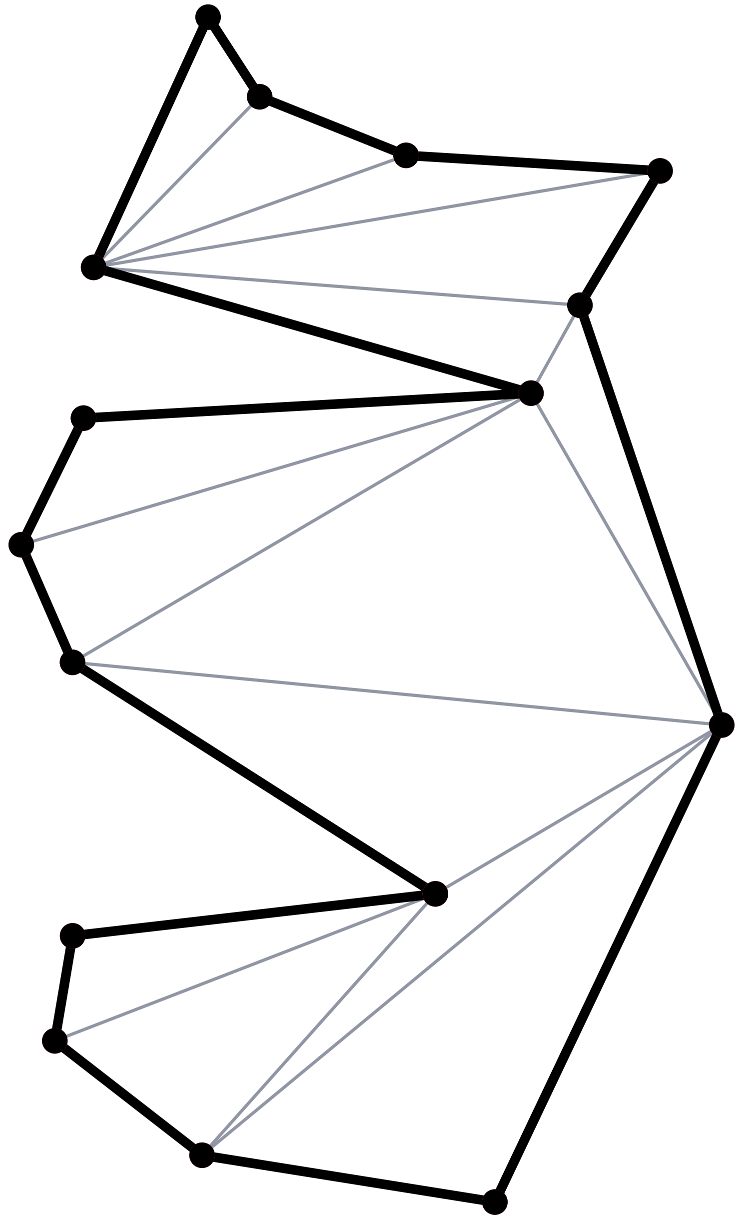

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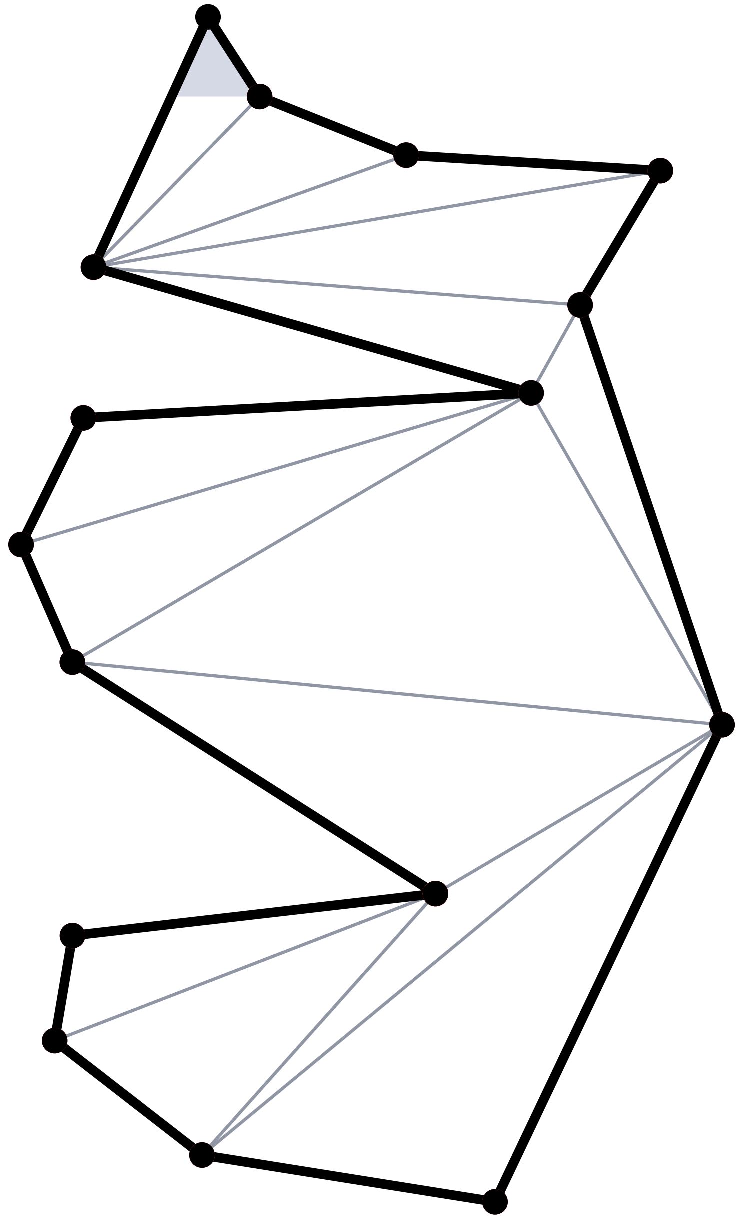

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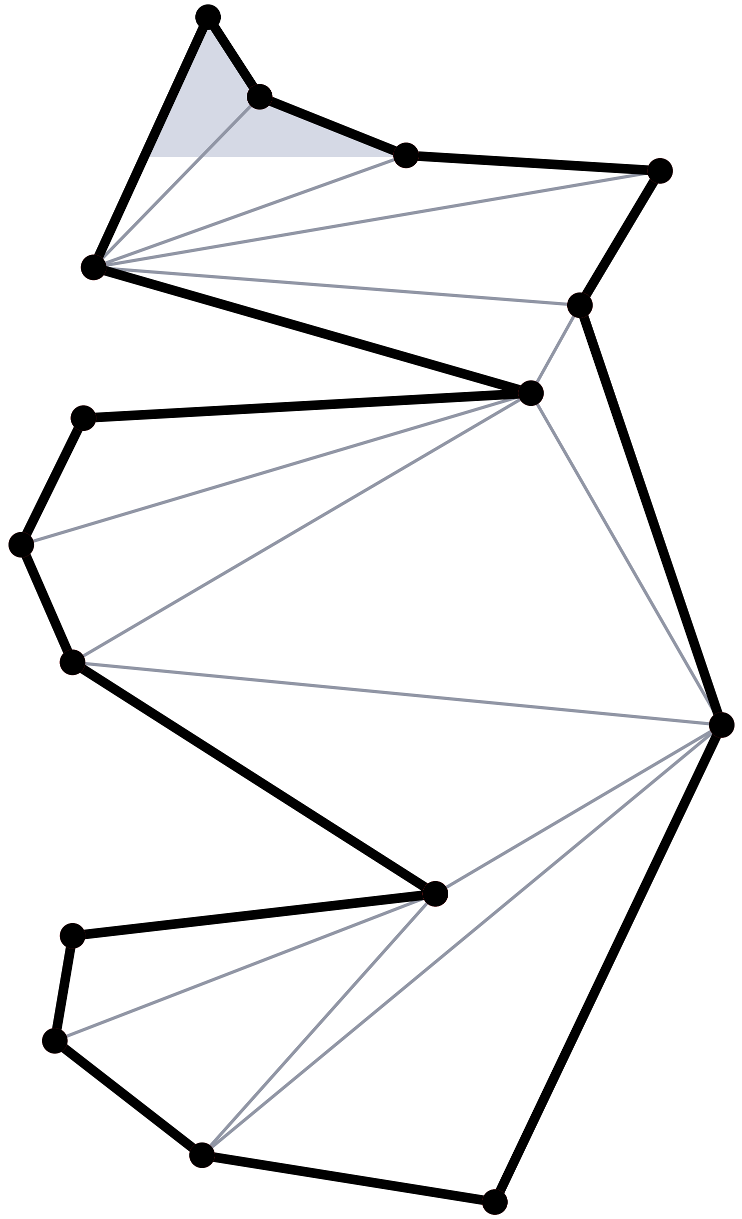

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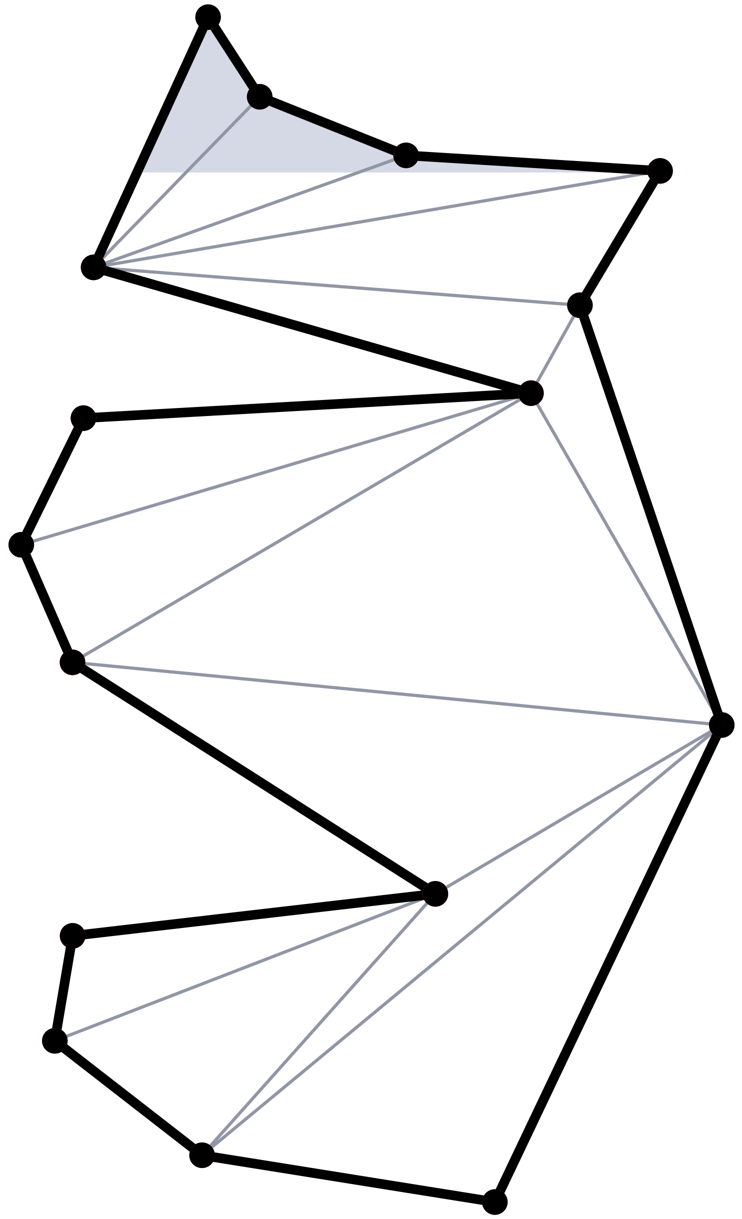

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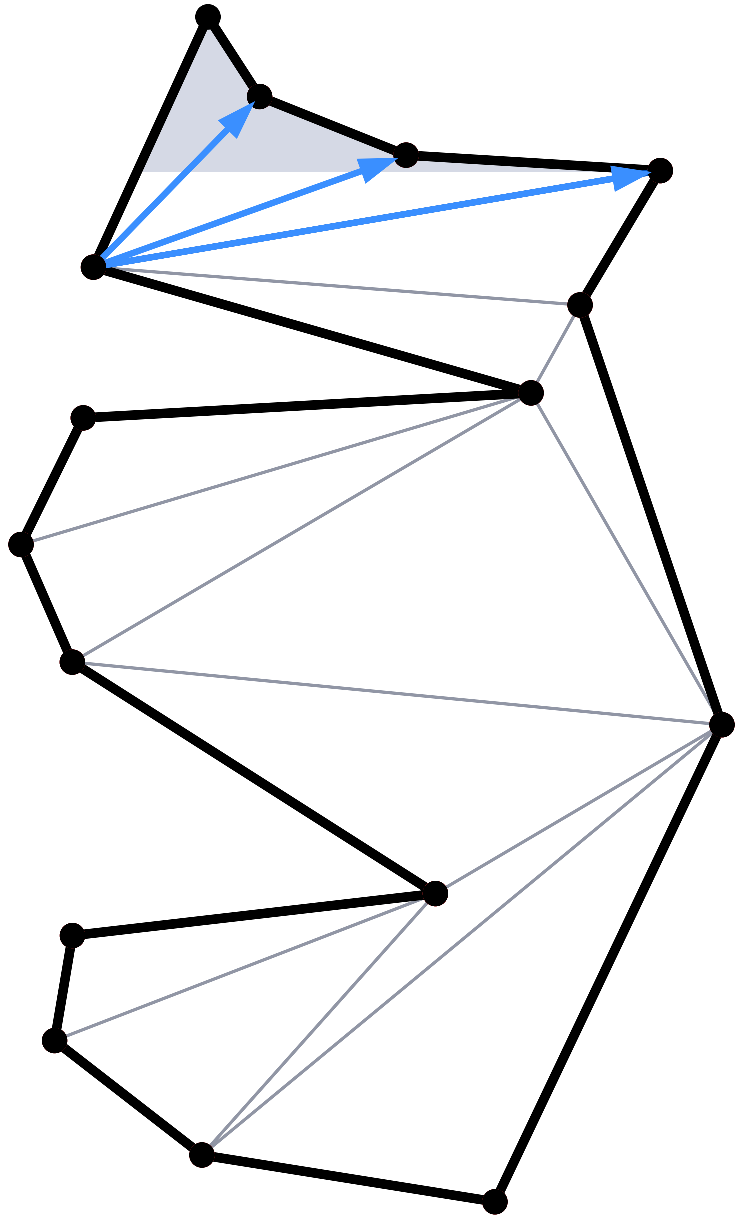

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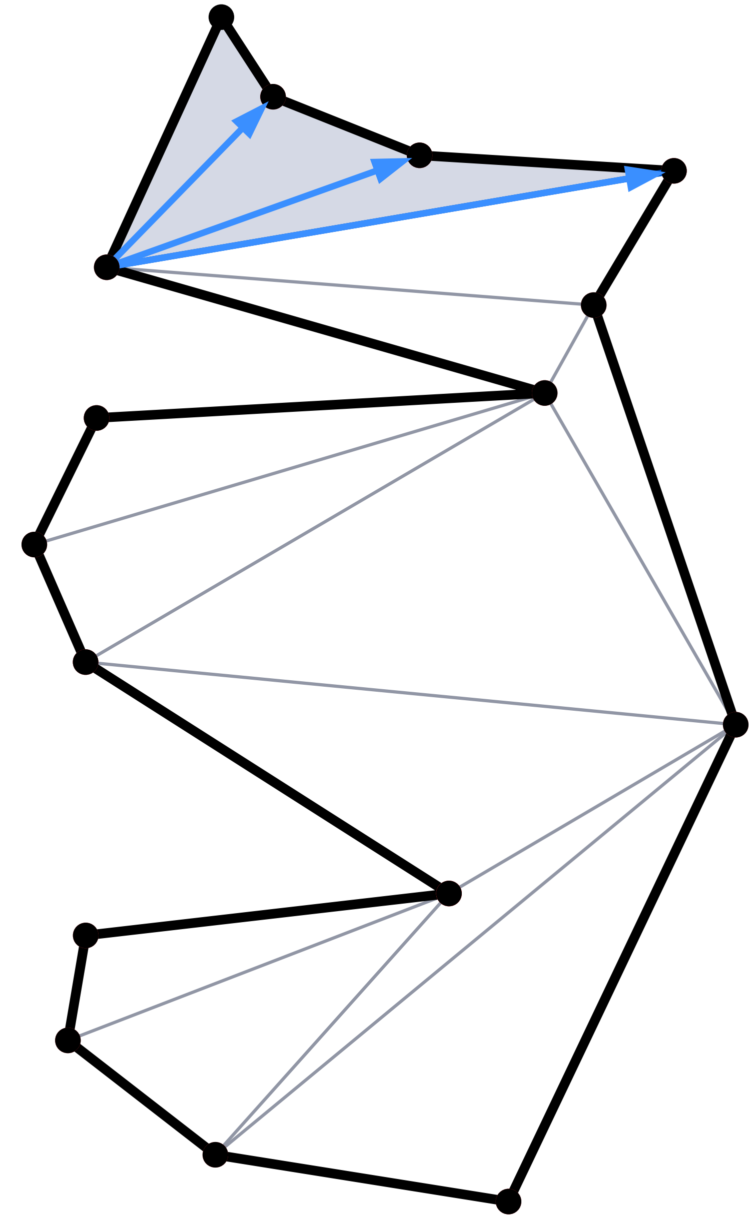
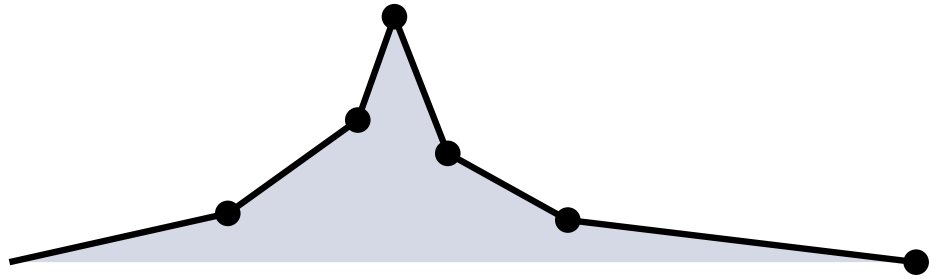


invariant?

untriangulated part above current vertex is an upside-down funnel



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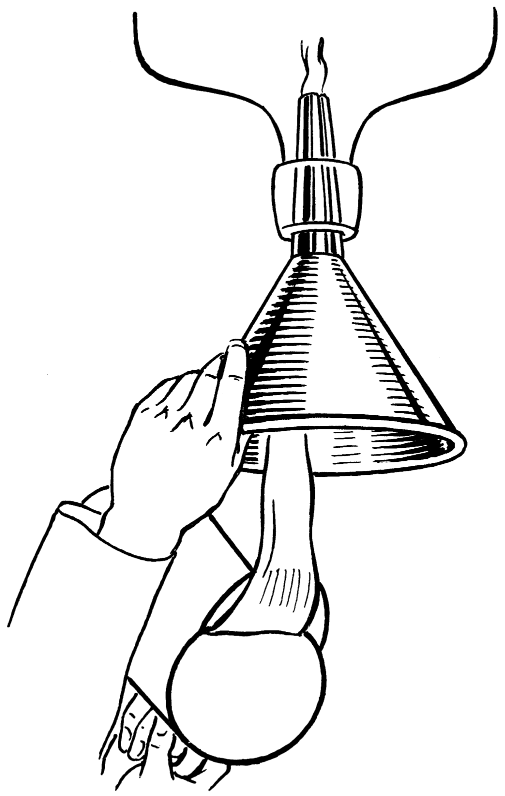

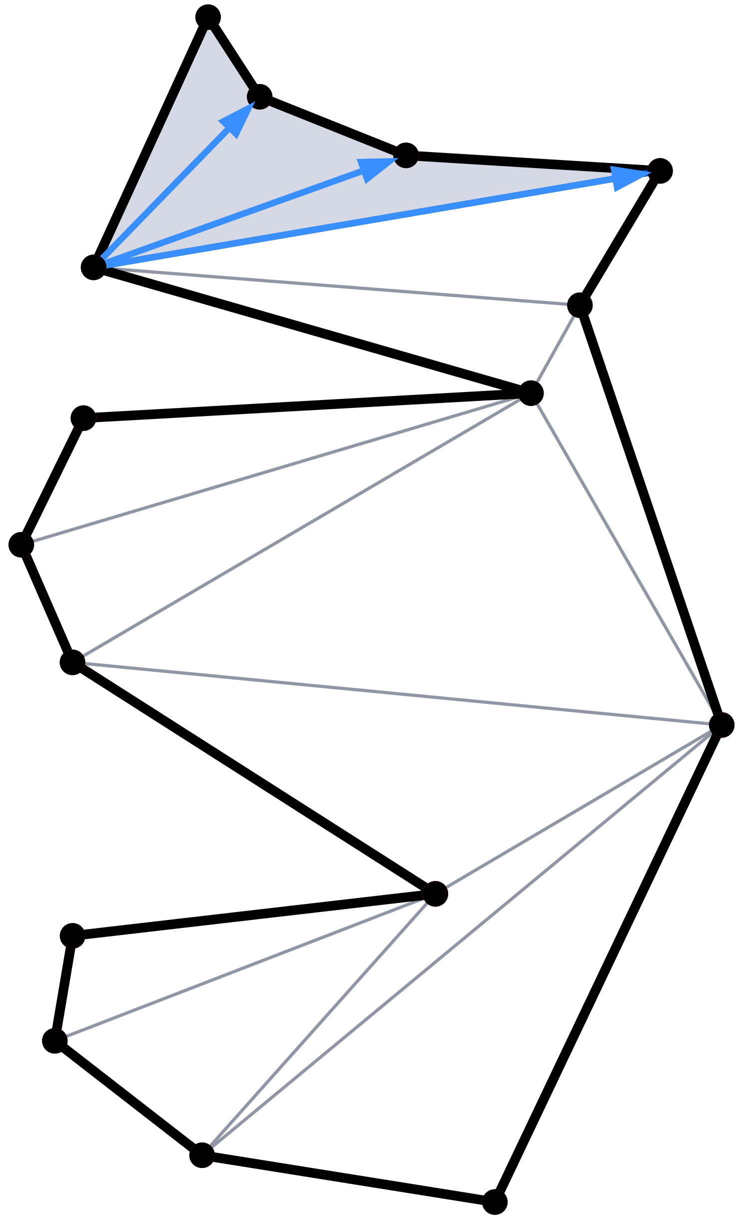

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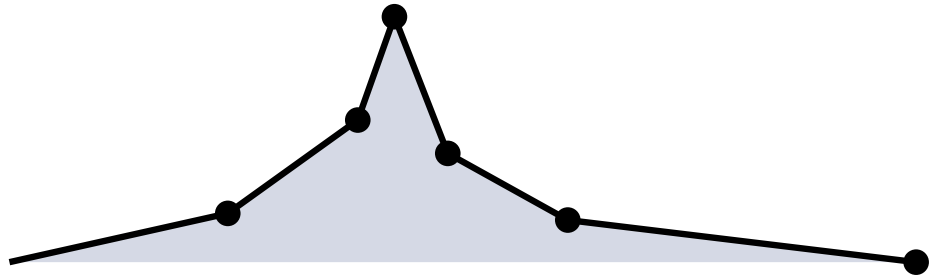

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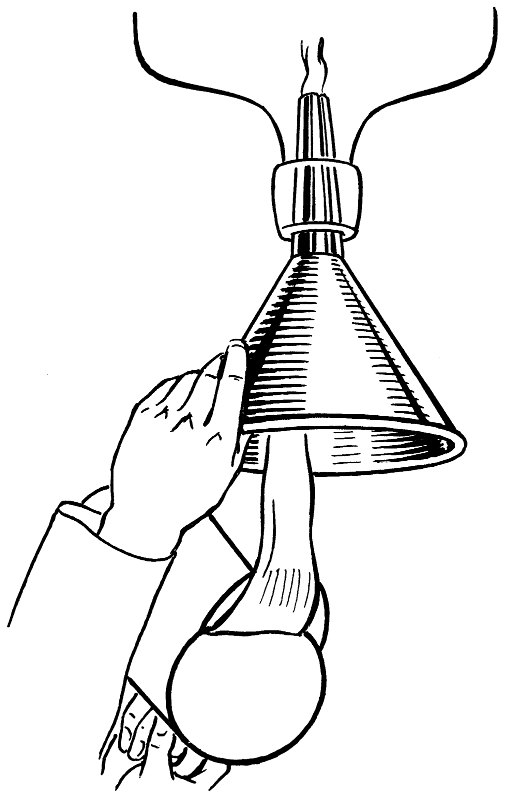

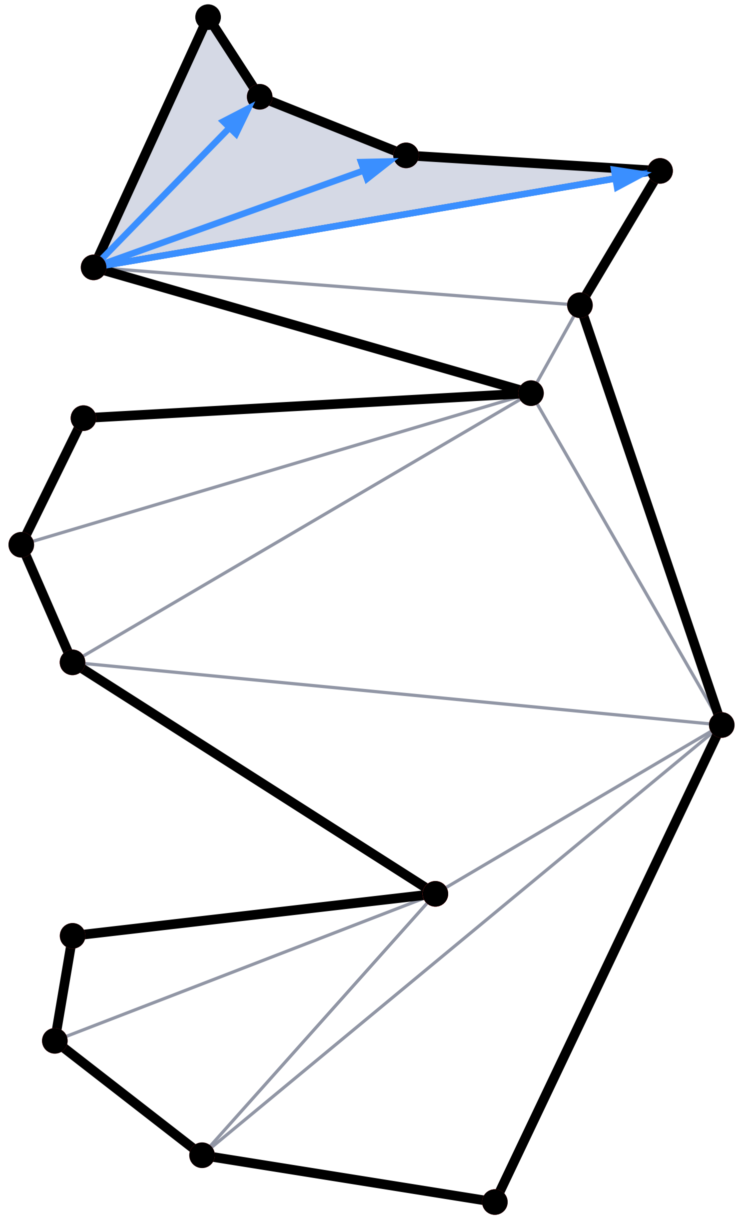

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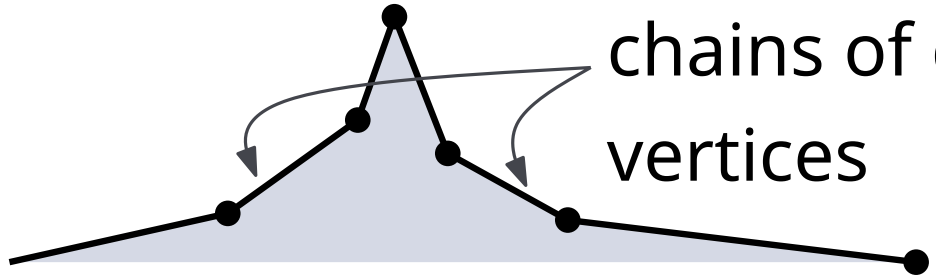

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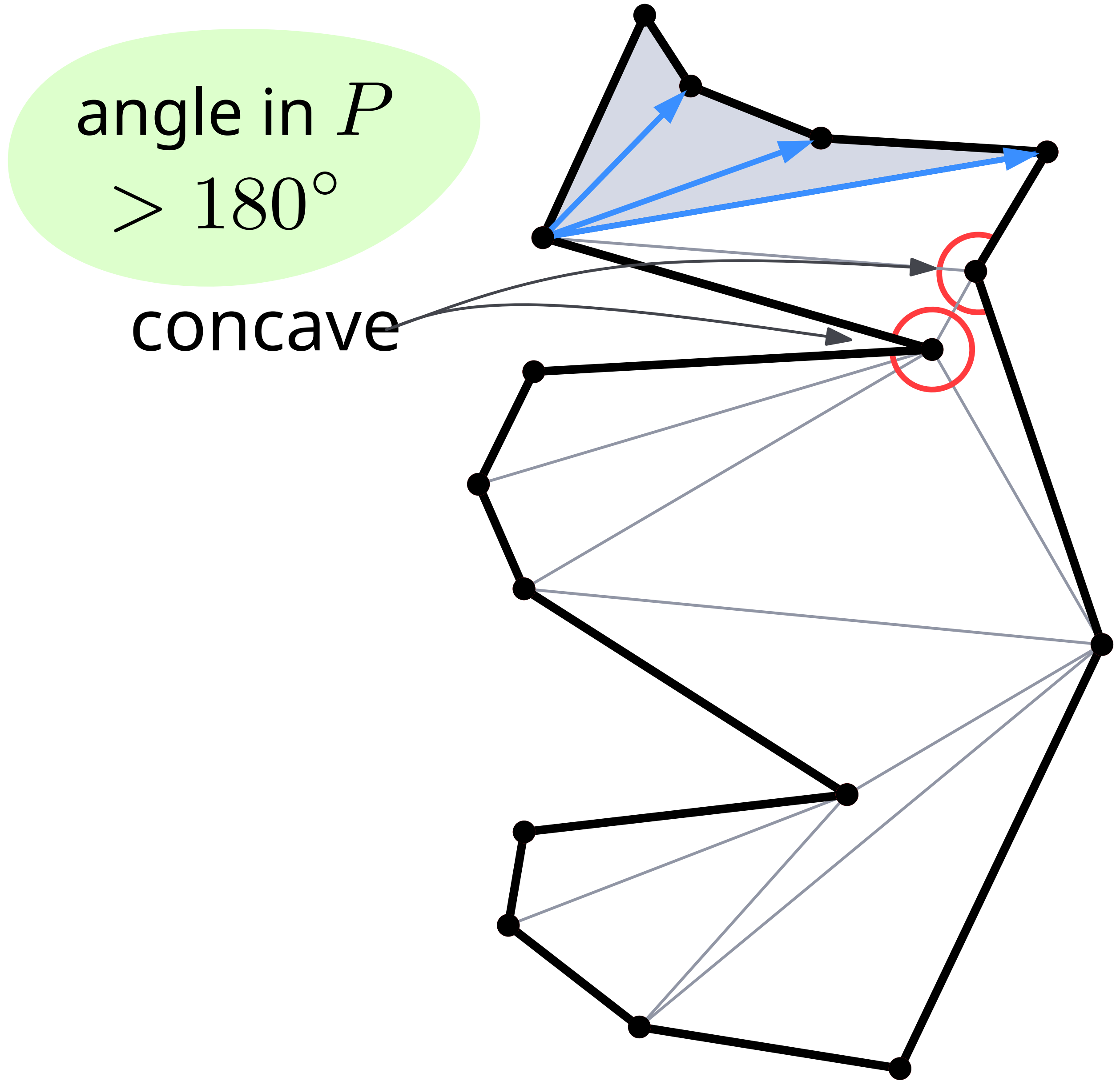

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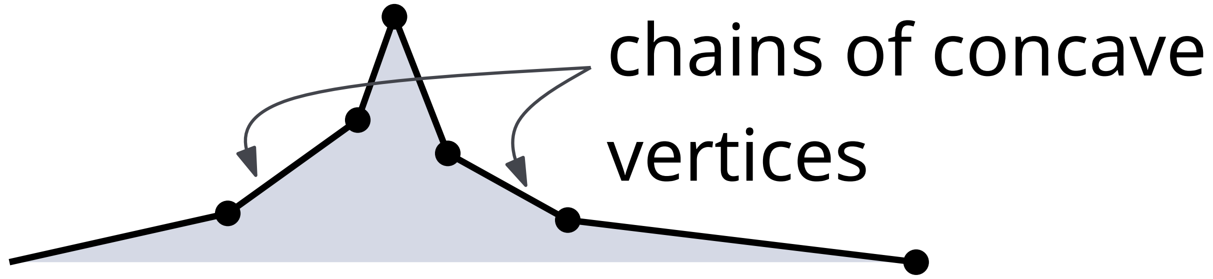

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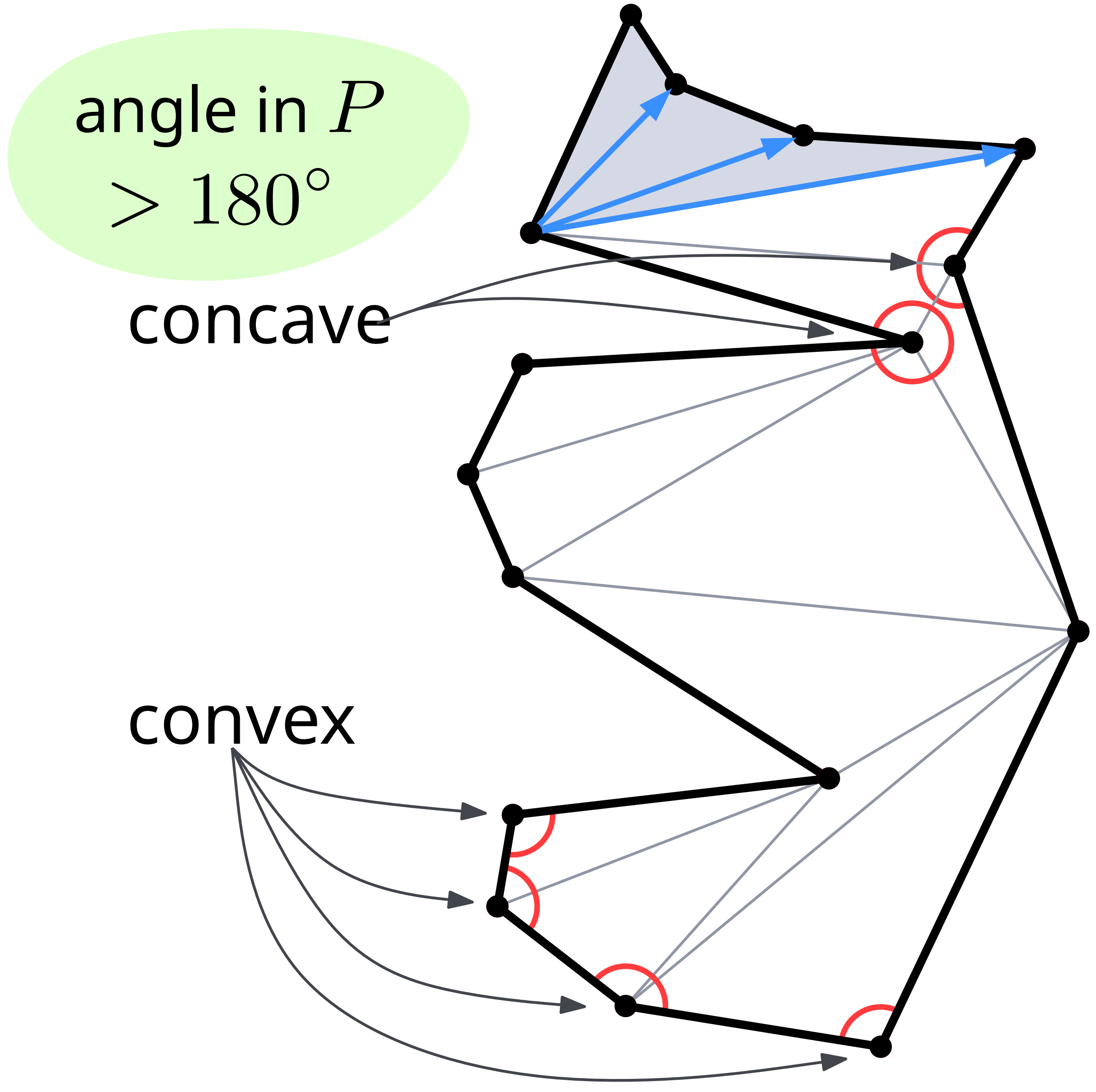

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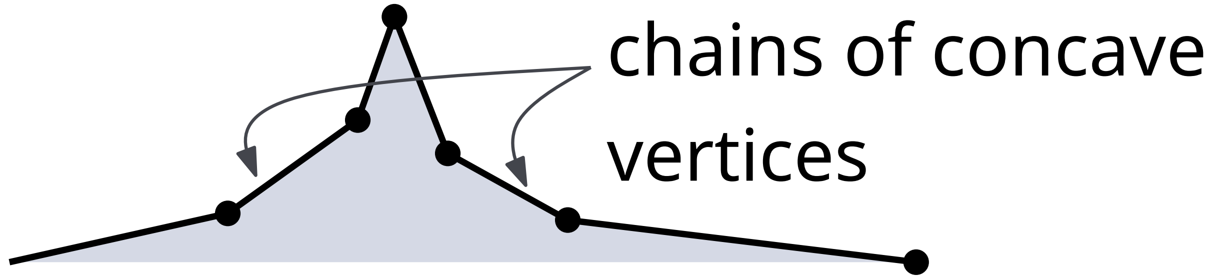

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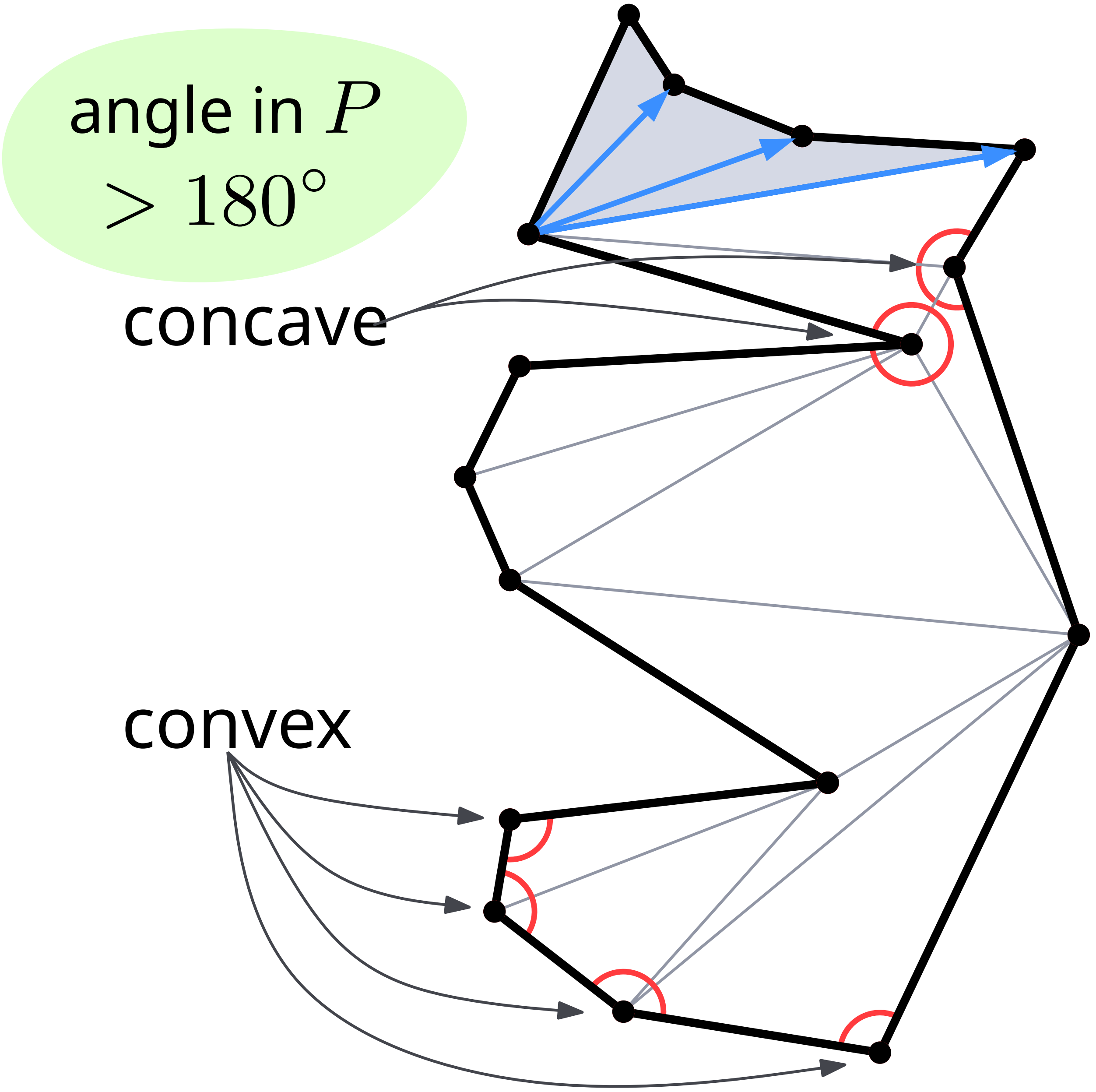

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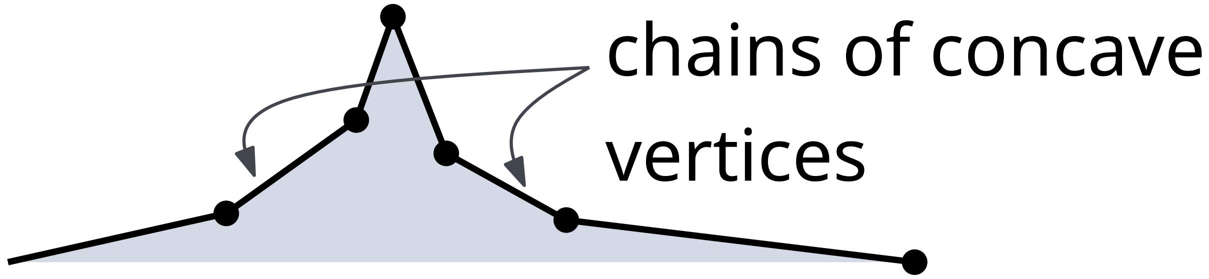
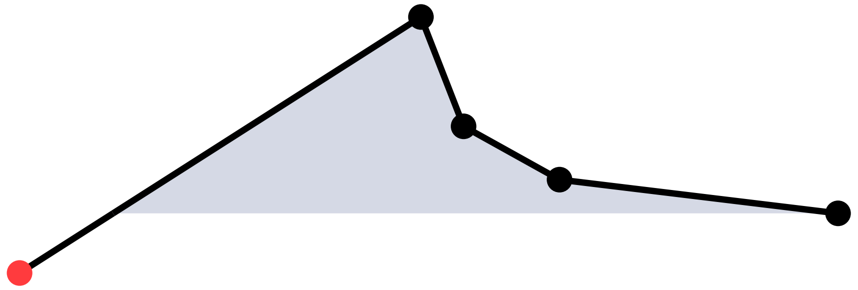


more precisely:



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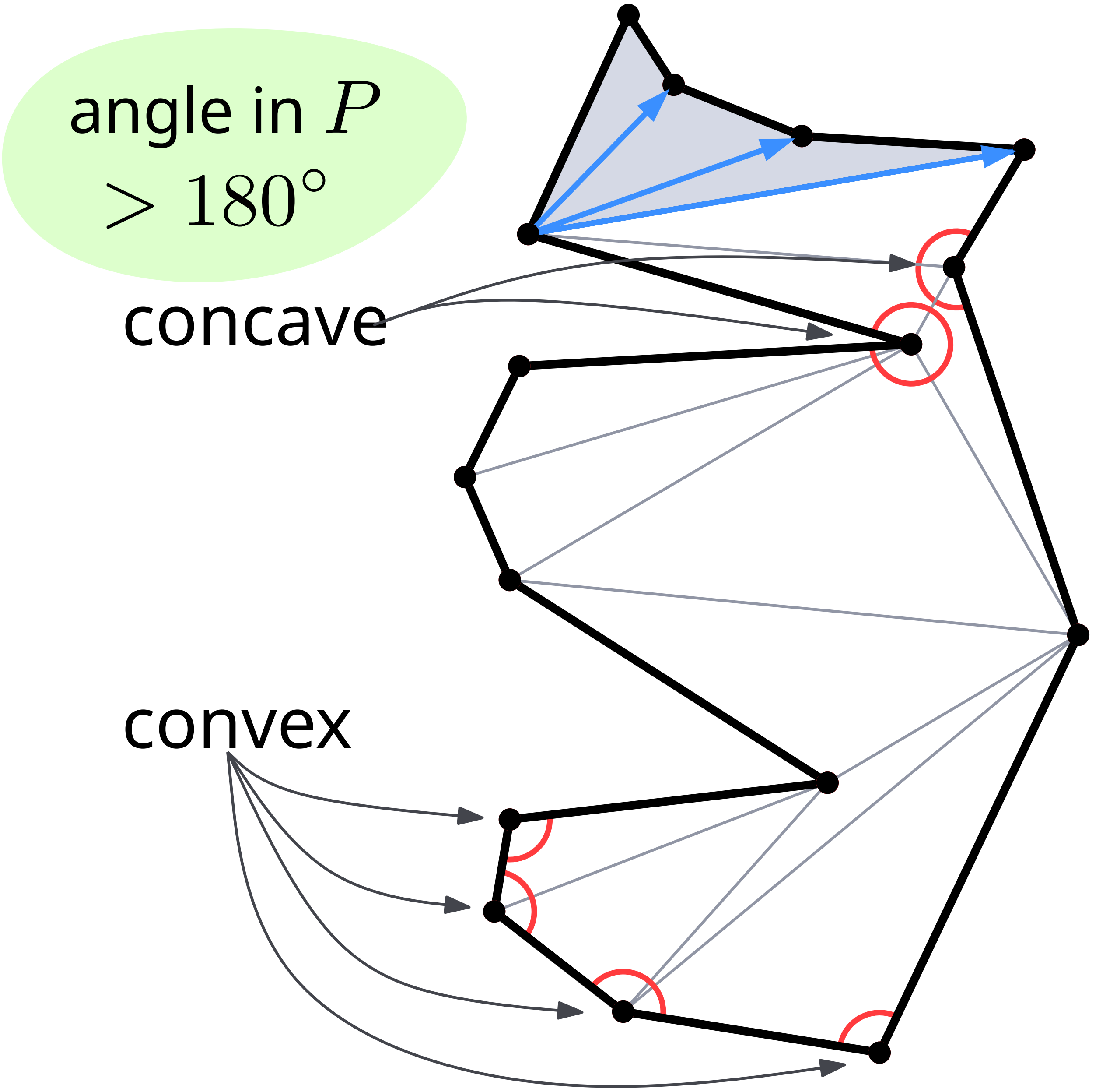

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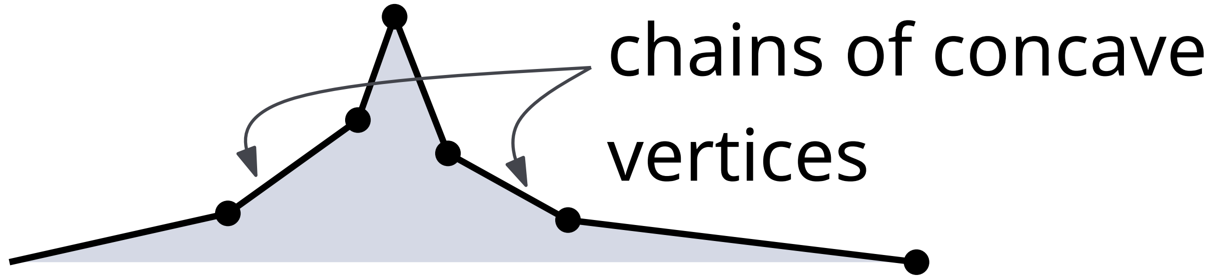

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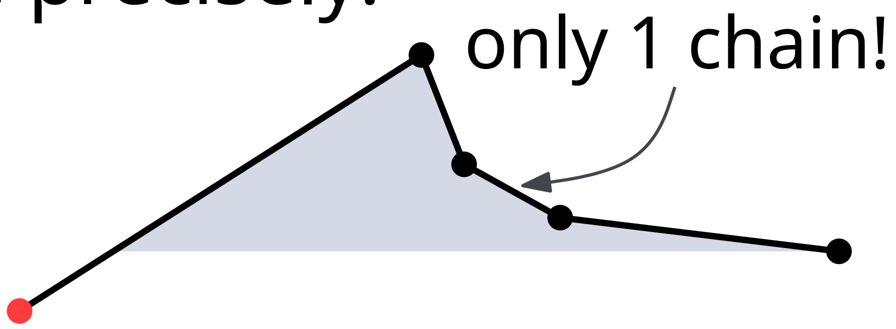

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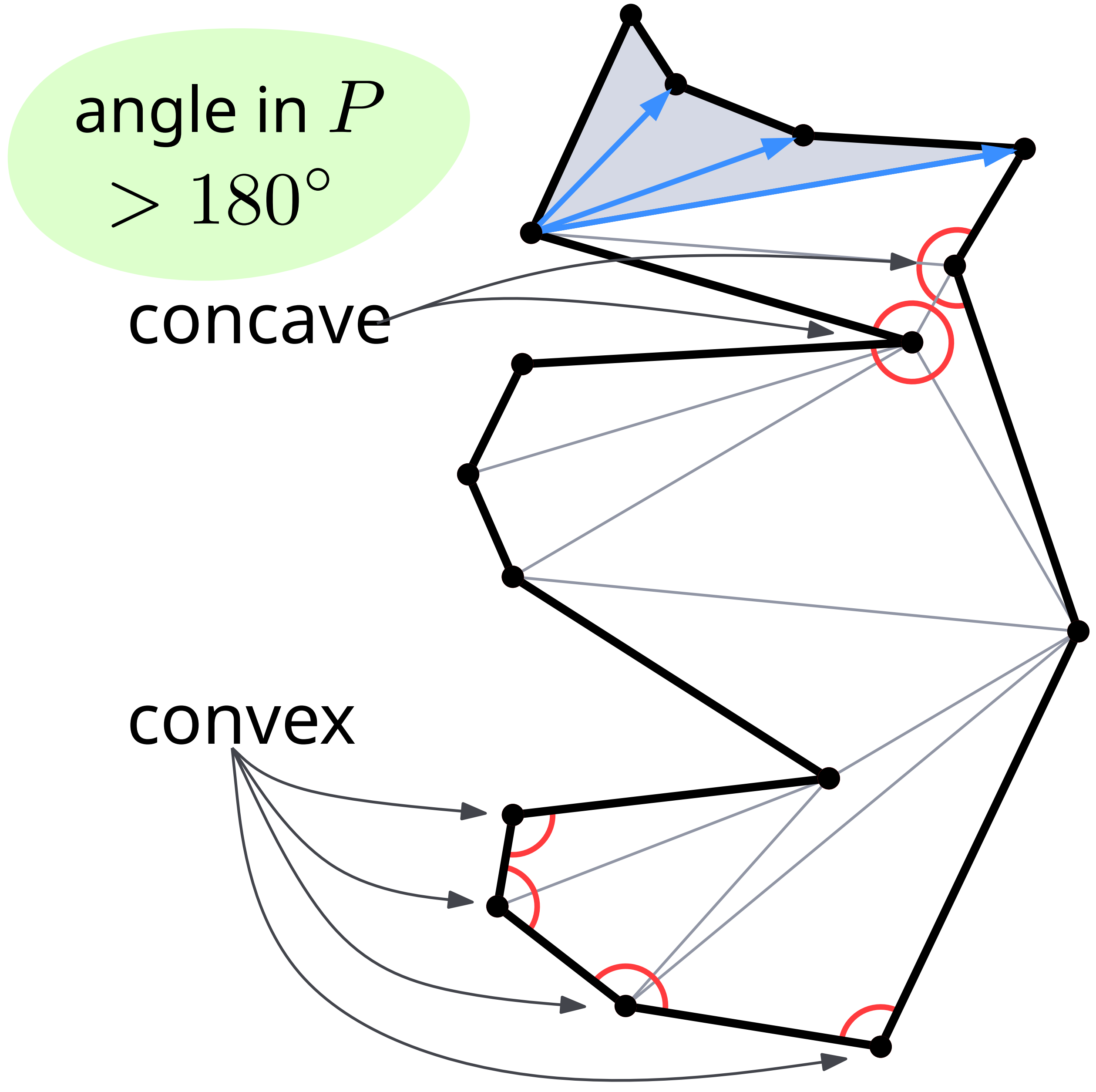

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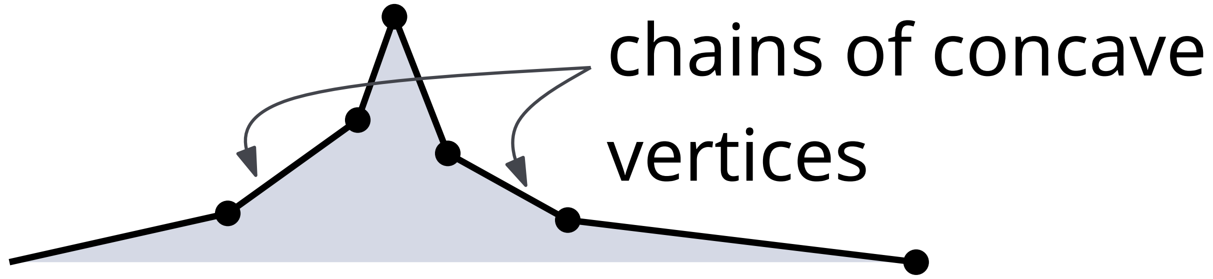

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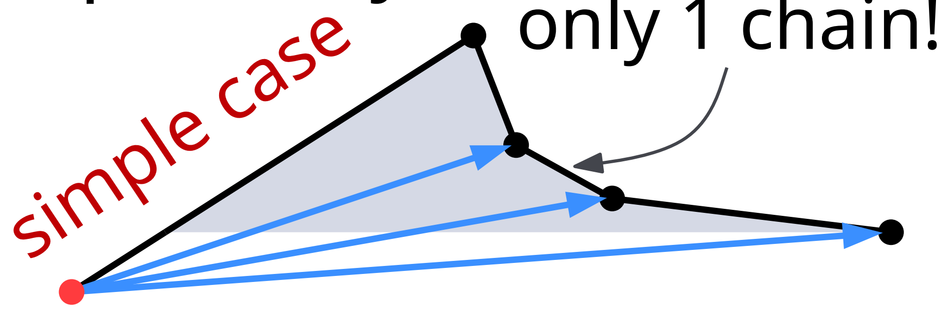

Algorithm TriangulateMonotonePolygon

TRIANGULATEMONOTONEPOLYGON(polygon P as DCEL)

1: merge vertices of left/right boundary $\rightarrow$ decreasing seq. $u_1, \ldots, u_n$
2: stack $S \leftarrow \emptyset$; S.push(u_1); S.push(u_2)

Algorithm TriangulateMonotonePolygon

TriangulateMonotonePolygon(polygon P as DCEL)

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merge vertices of left/right boundary → decreasing seq. u_1, ..., u_n
stack S ← ∅; S.push(u_1); S.push(u_2)
for j ← 3 to n − 1 do
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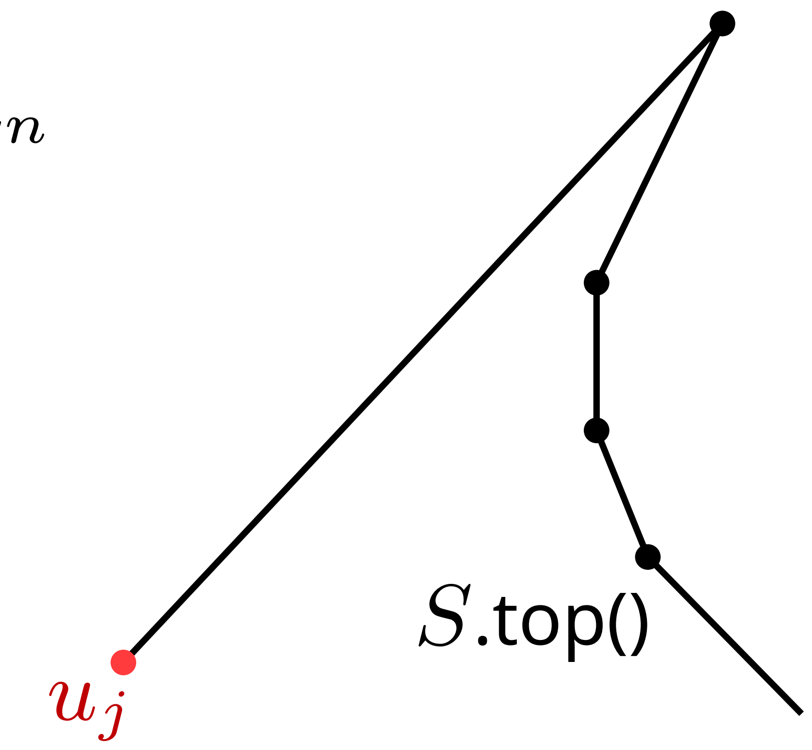
            if S is not empty then
                add (u_j, v)
        S.push(u_{j−1}); S.push(u_j)
    else
```



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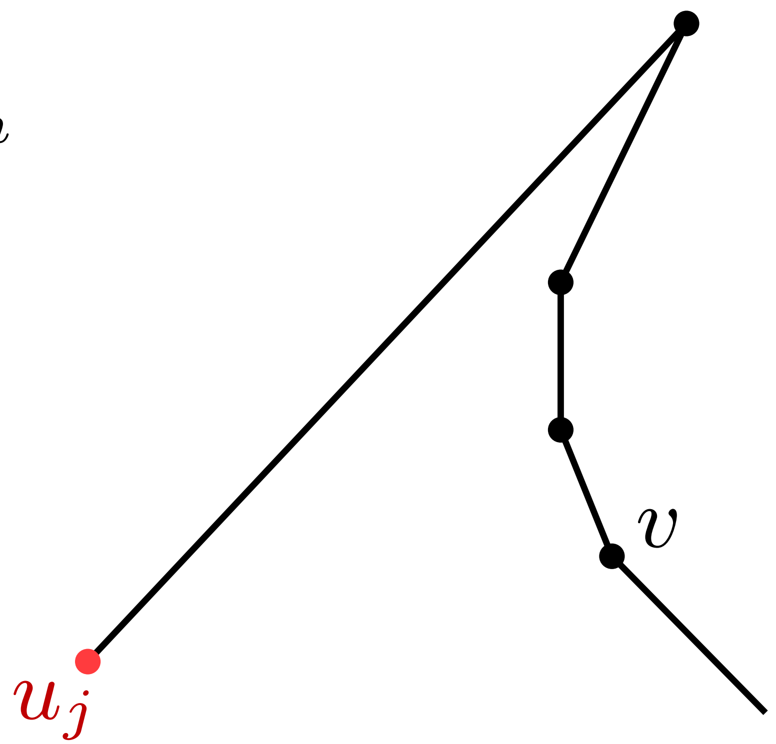

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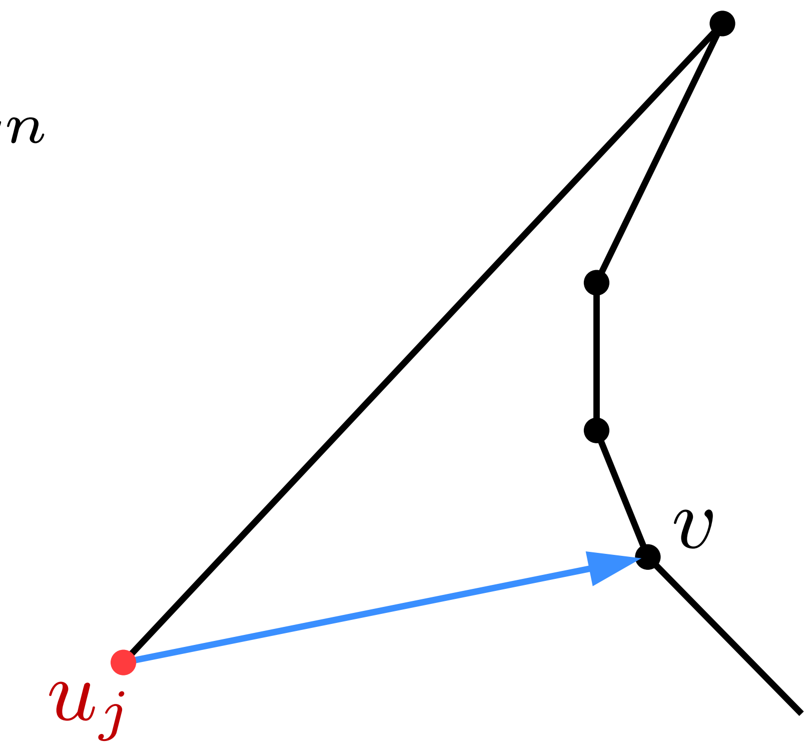

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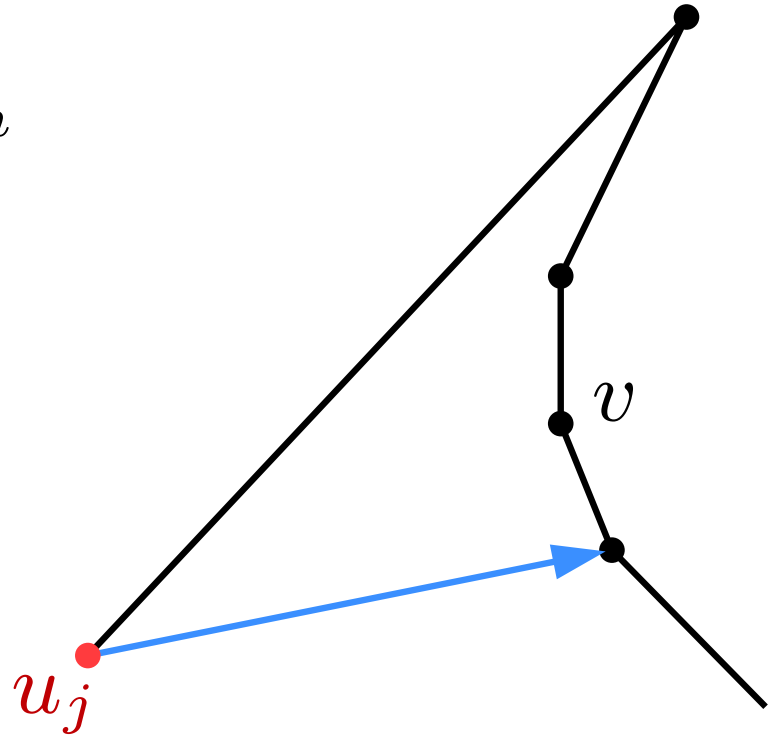

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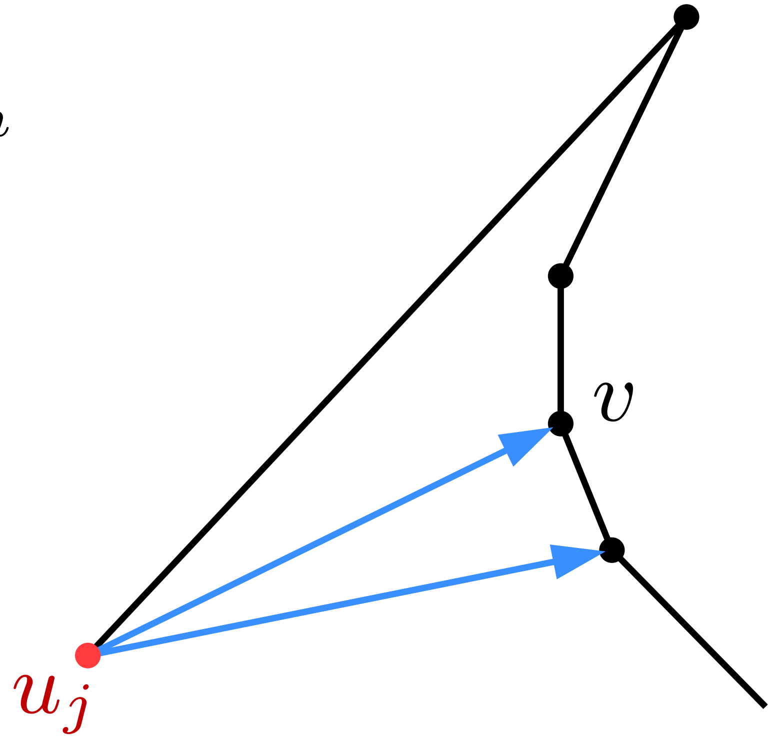

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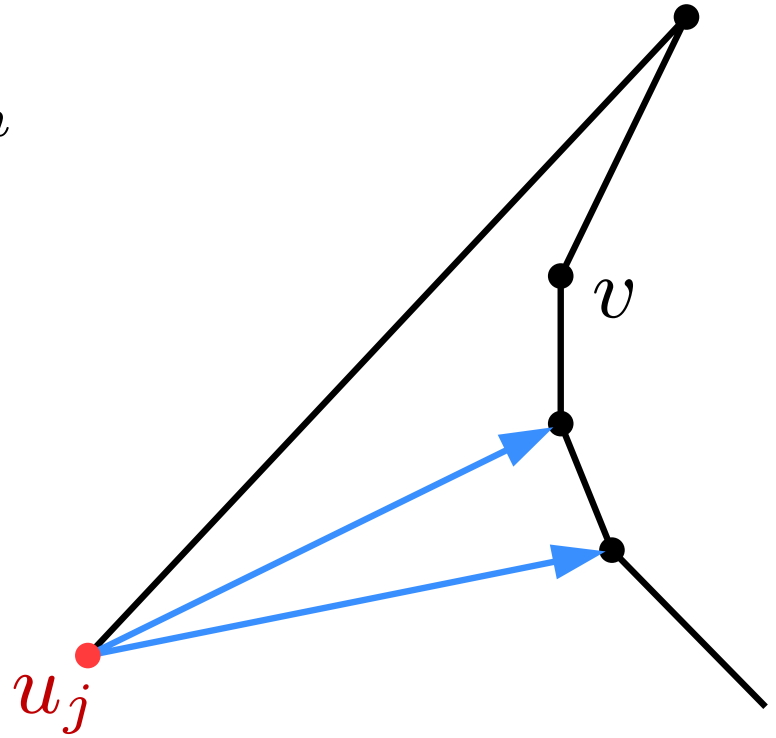

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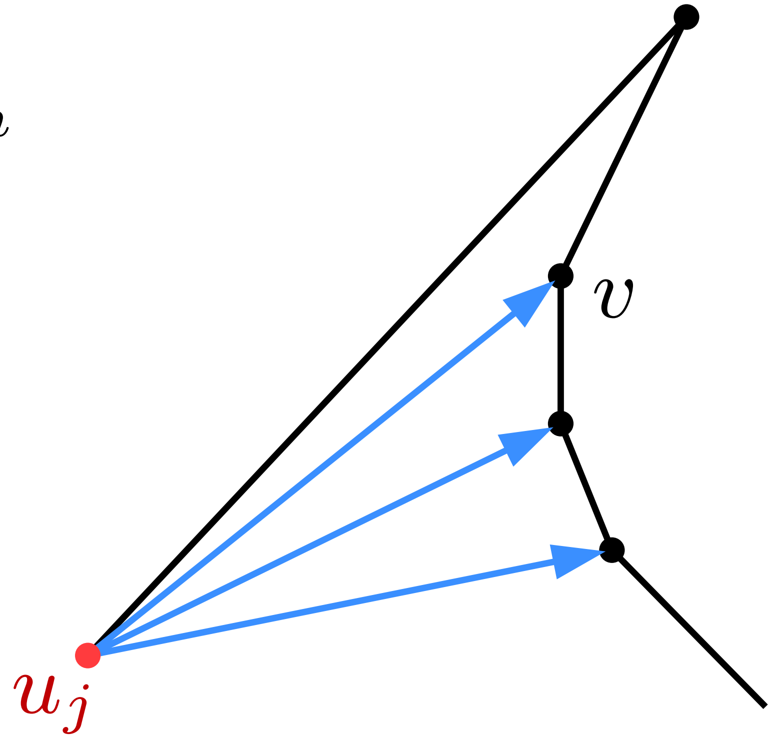

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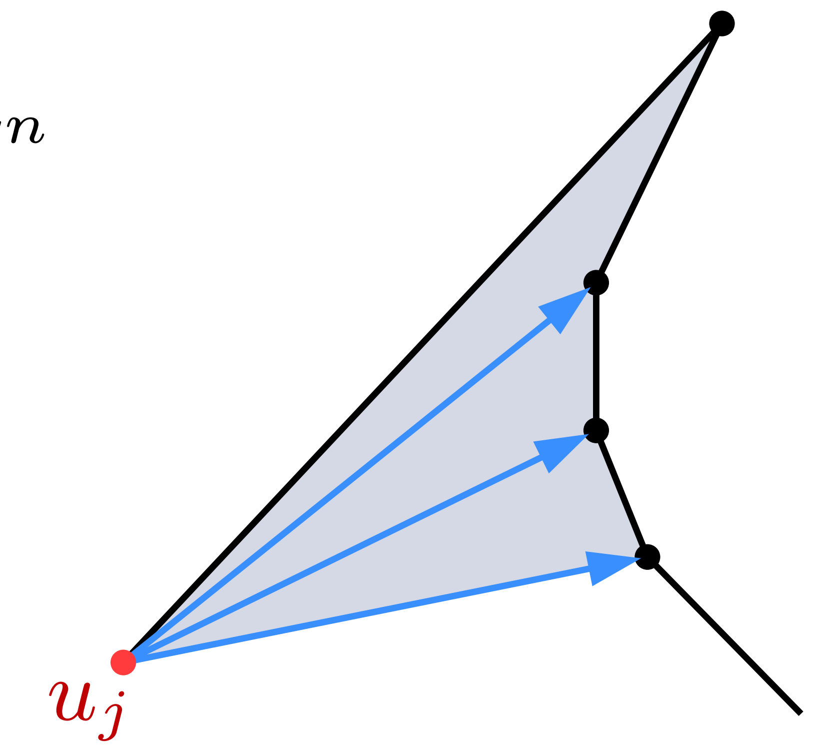

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stack $S \leftarrow \varnothing$; S.push(u_1); S.push(u_2)
for $j \leftarrow 3$ to $n-1$ **do**
    if u_j and S.top() on different boundaries **then**
        while S is not empty **do**
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            if S is not empty **then**
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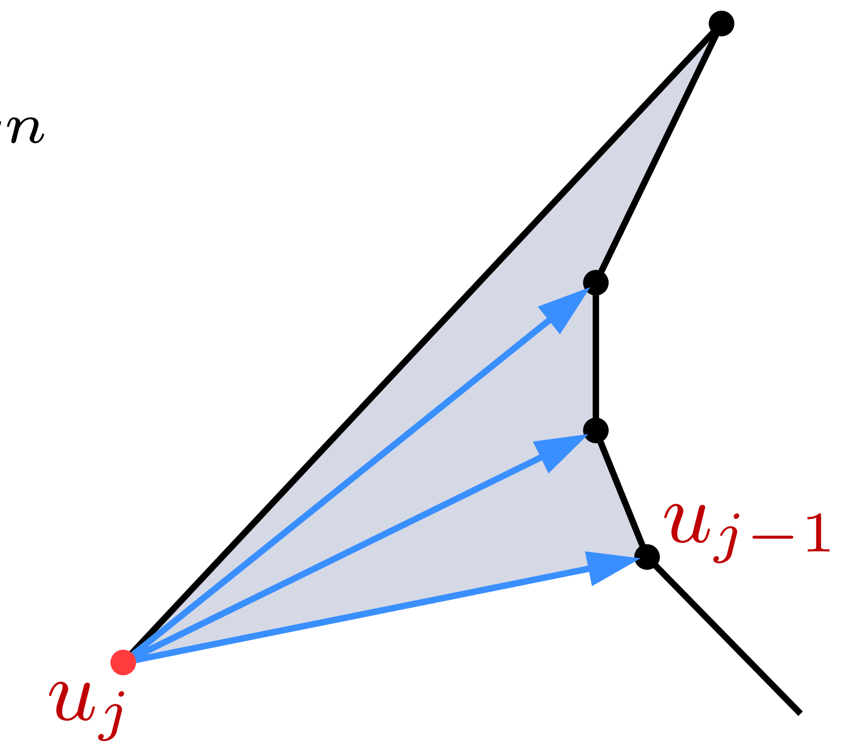
    else



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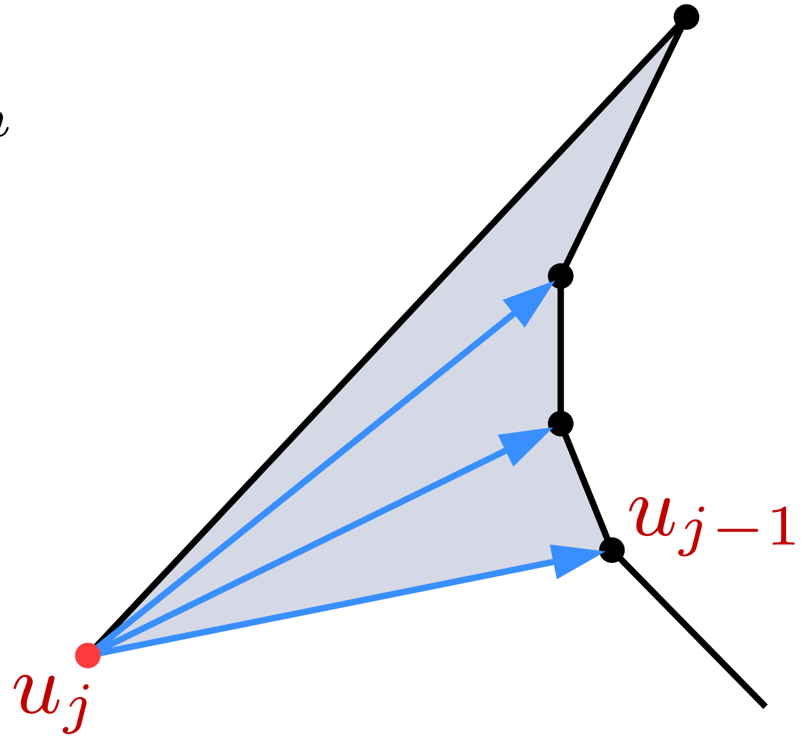
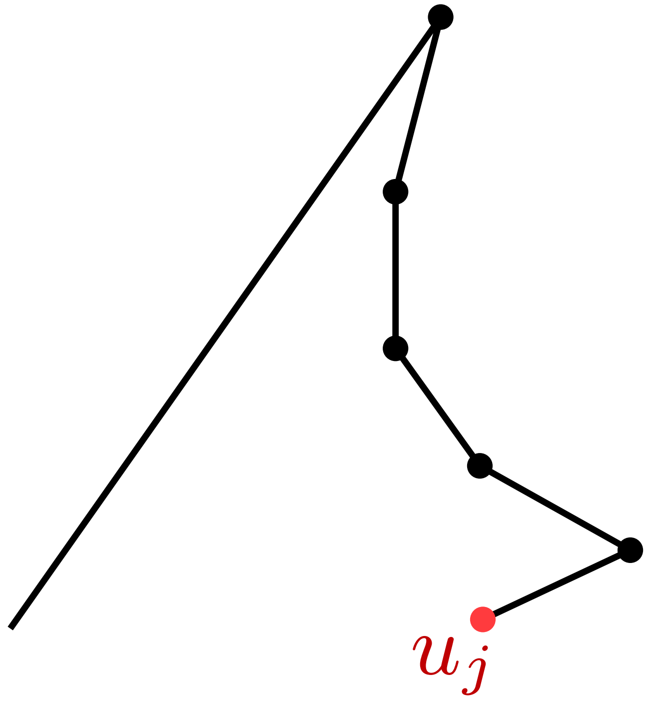
    if u_j and S.top() on different boundaries **then**
        while S is not empty **do**
            $v \leftarrow S$.pop()
            if S is not empty **then**
                add (u_j, v)
        S.push(u_{j-1}); S.push(u_j)
    else
        $v \leftarrow S$.pop()
        while S is not empty **and** u_j sees S.top() **do**
            $v \leftarrow S$.pop()
            add diagonal (u_j, v)
        S.push(v); S.push(u_j)





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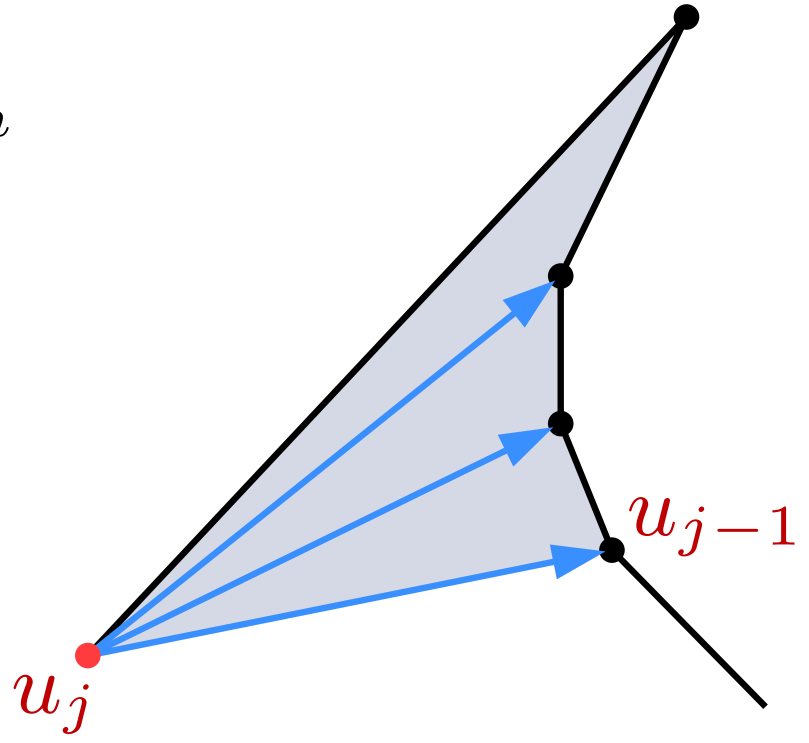

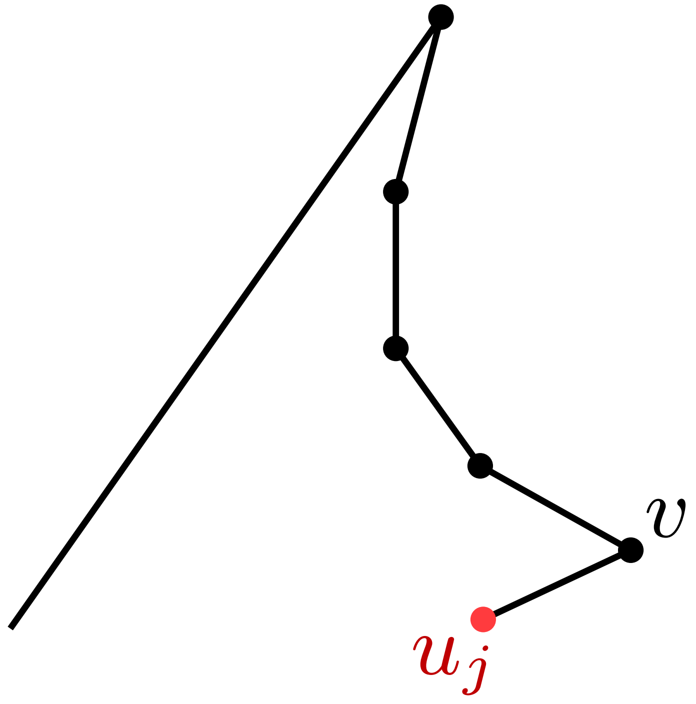

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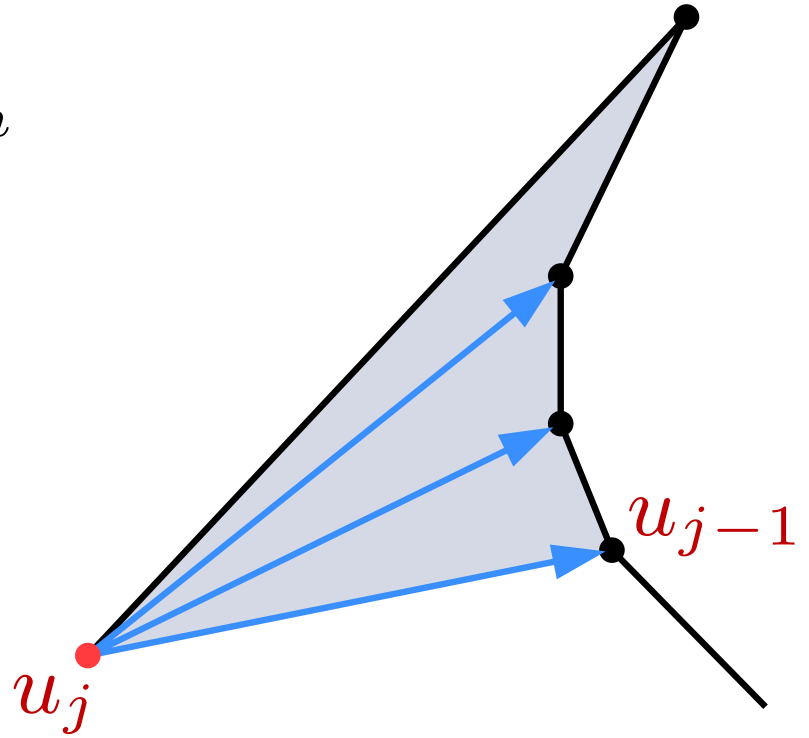

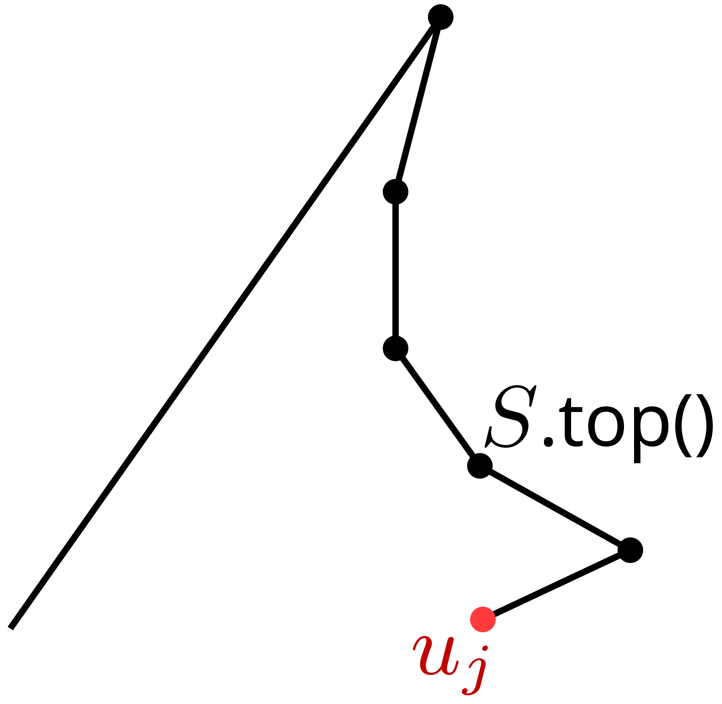

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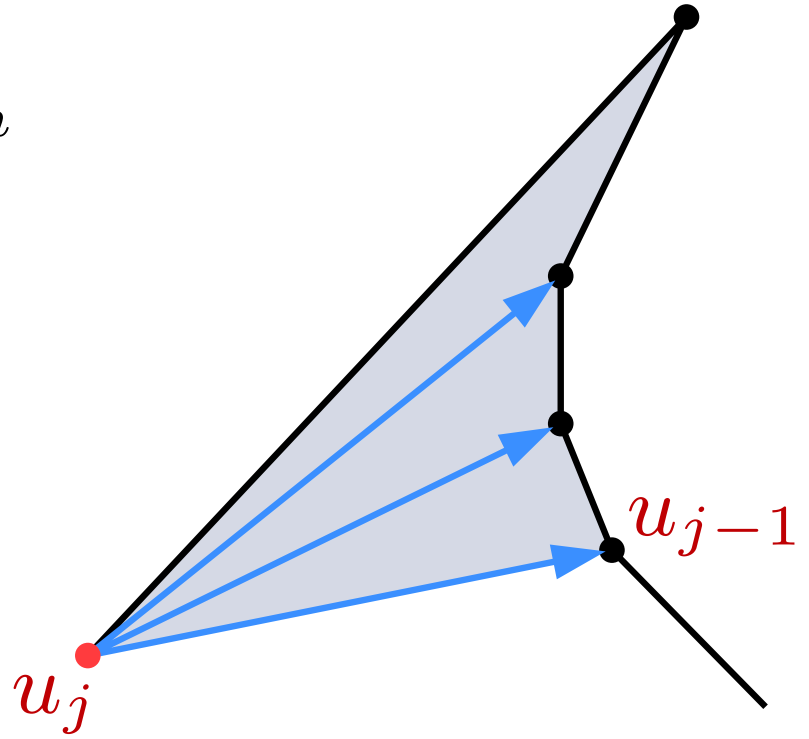

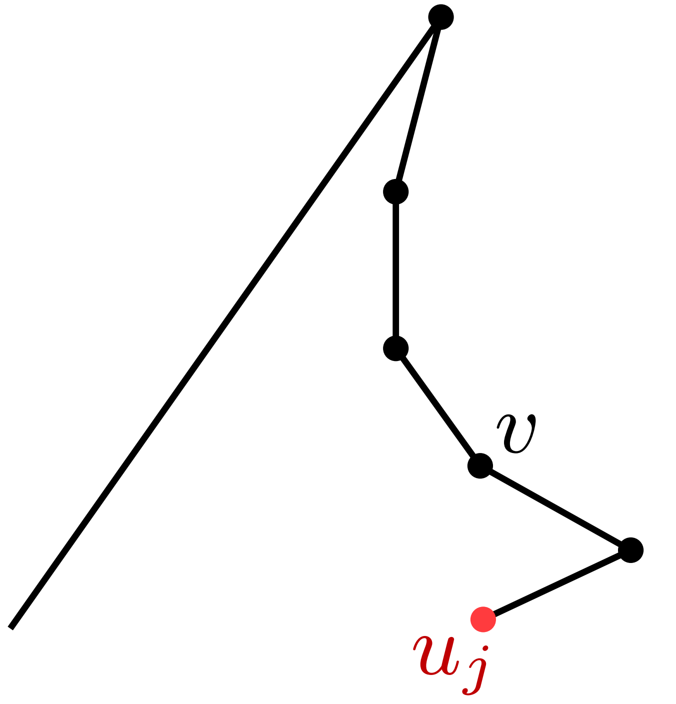

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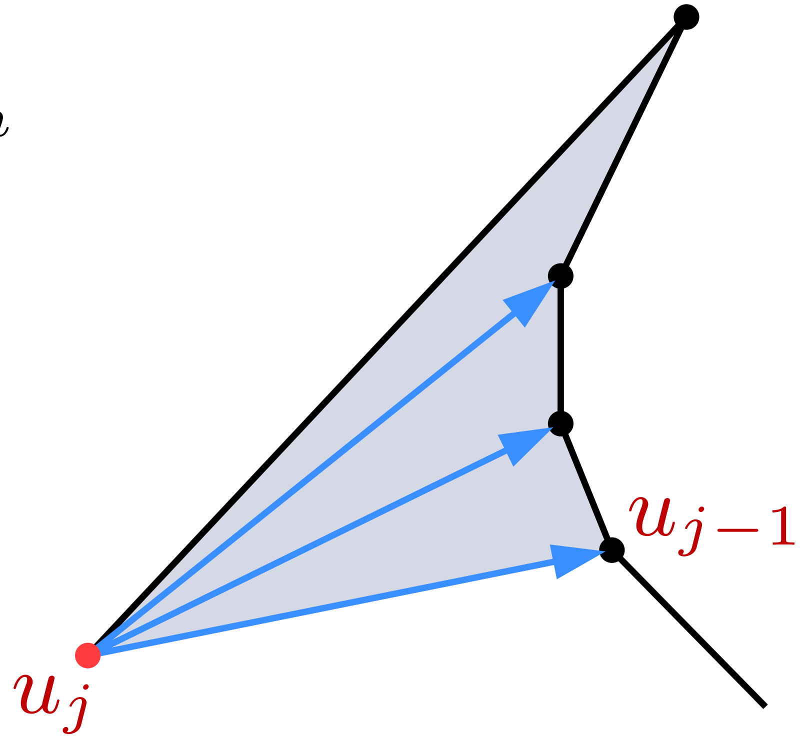

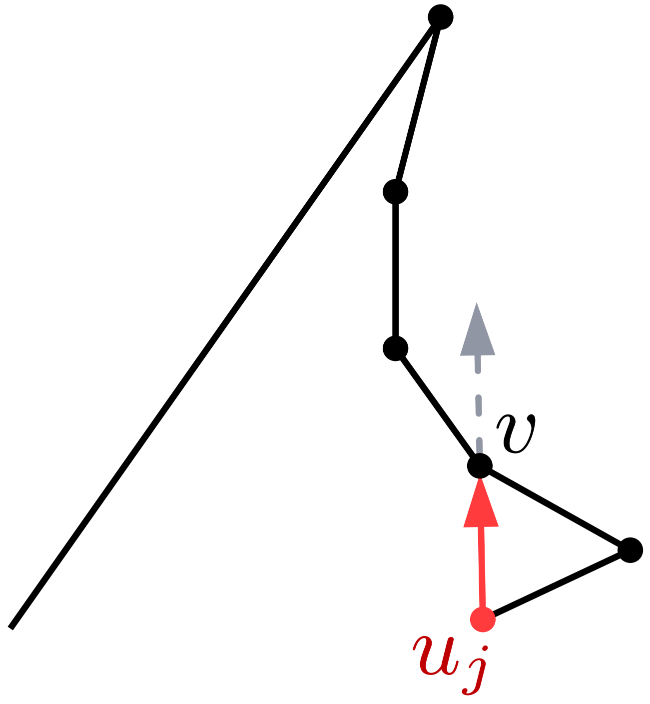

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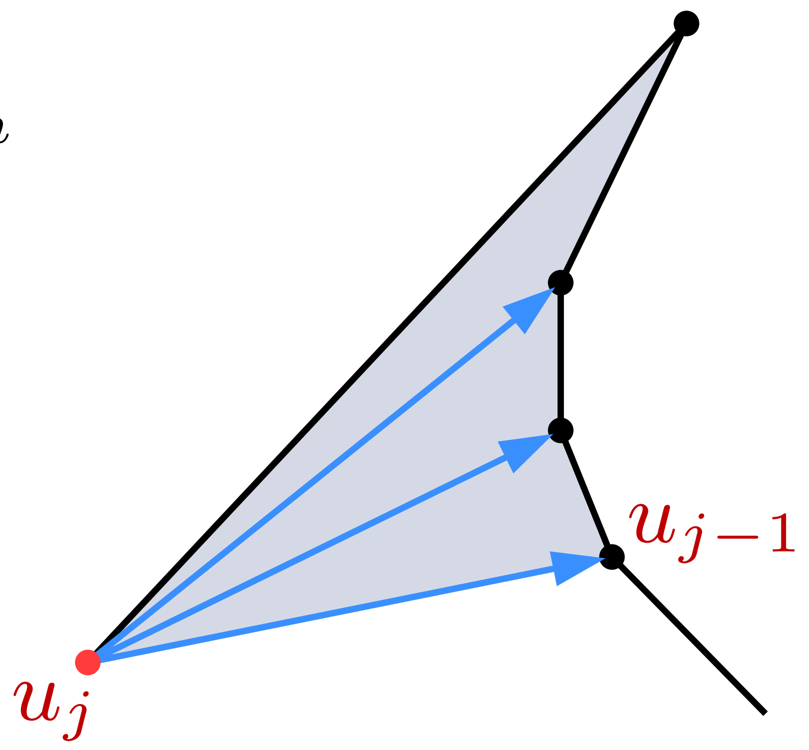

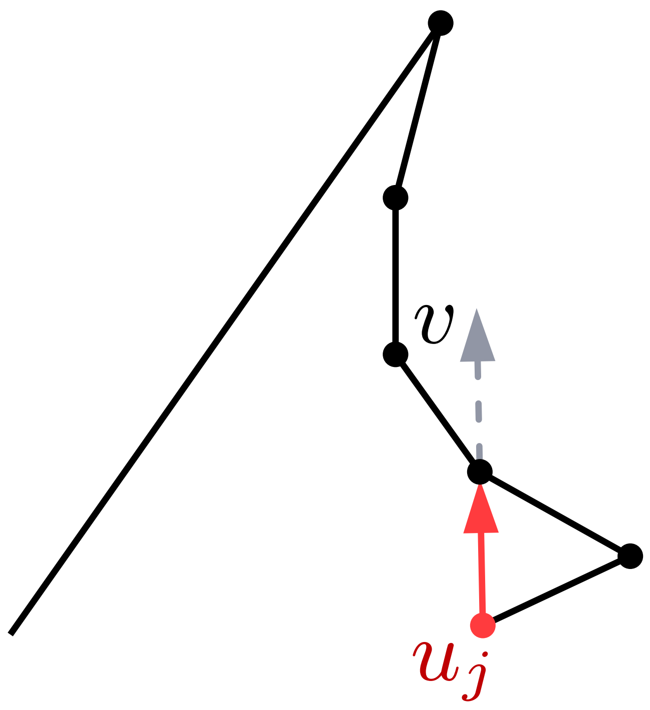

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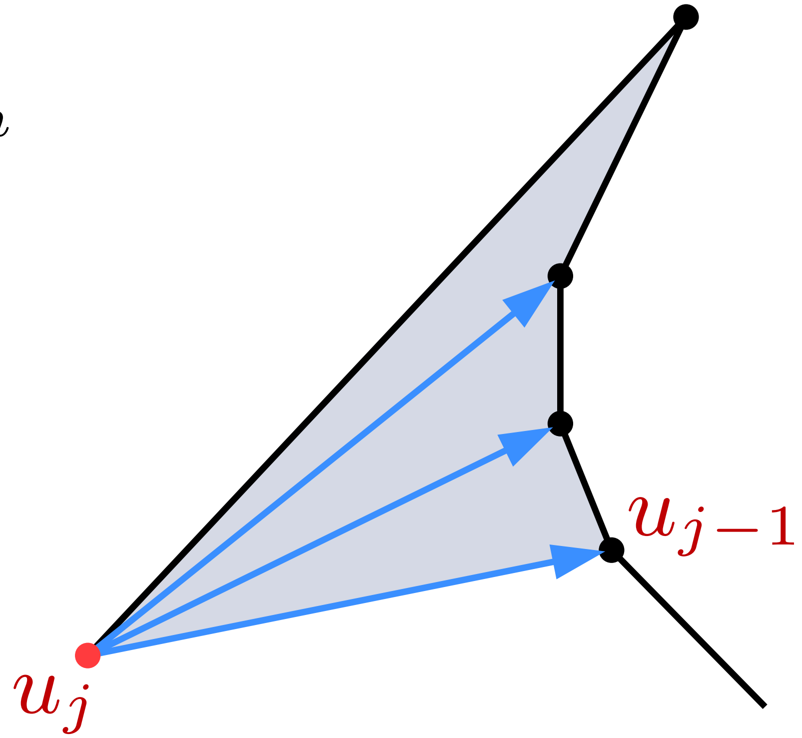

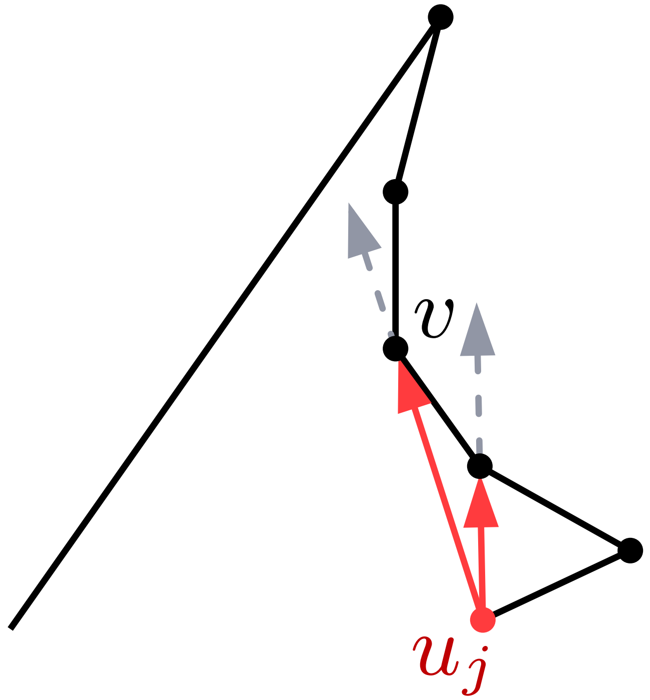

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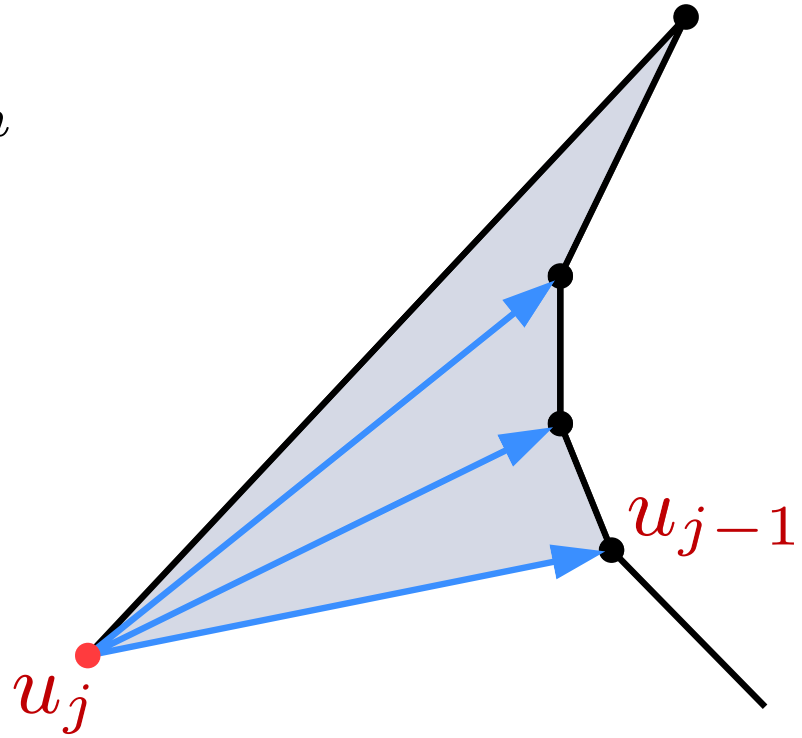

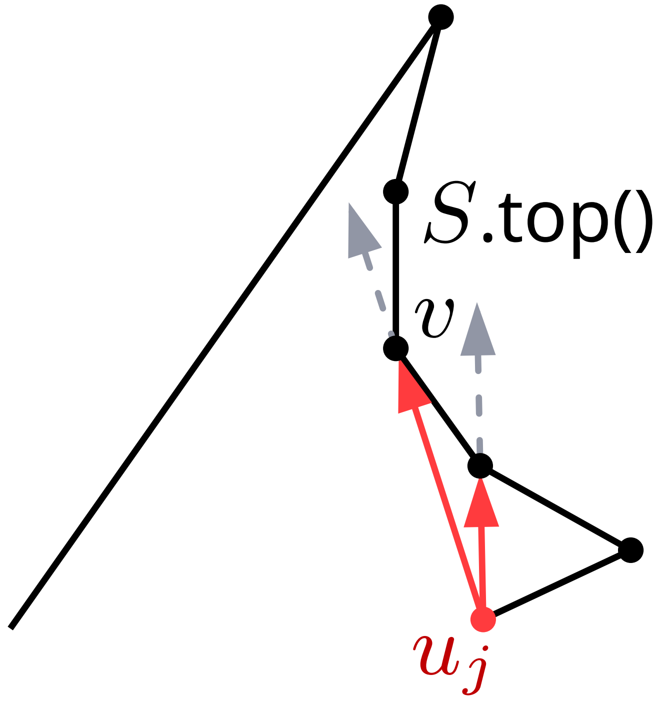

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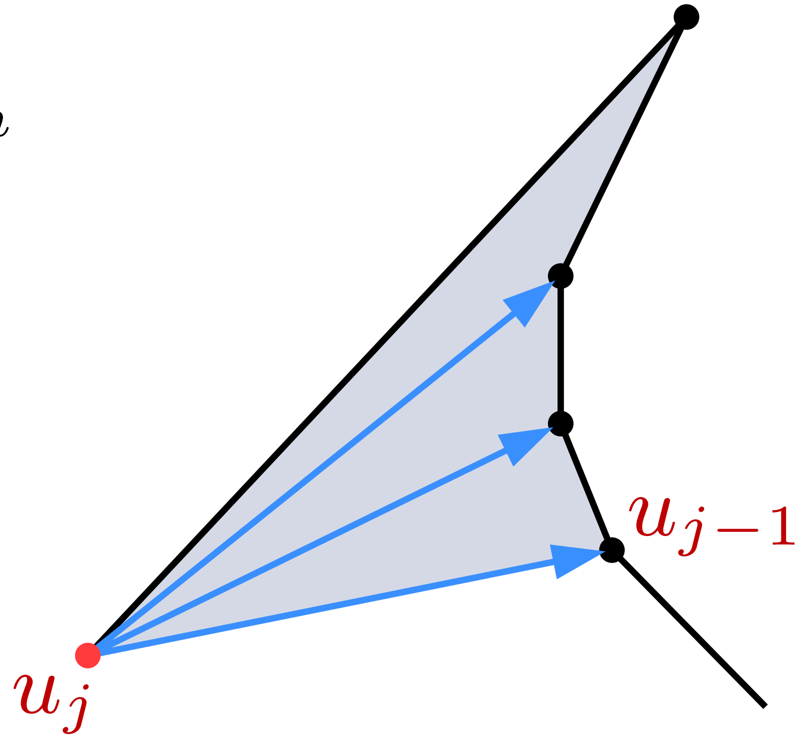

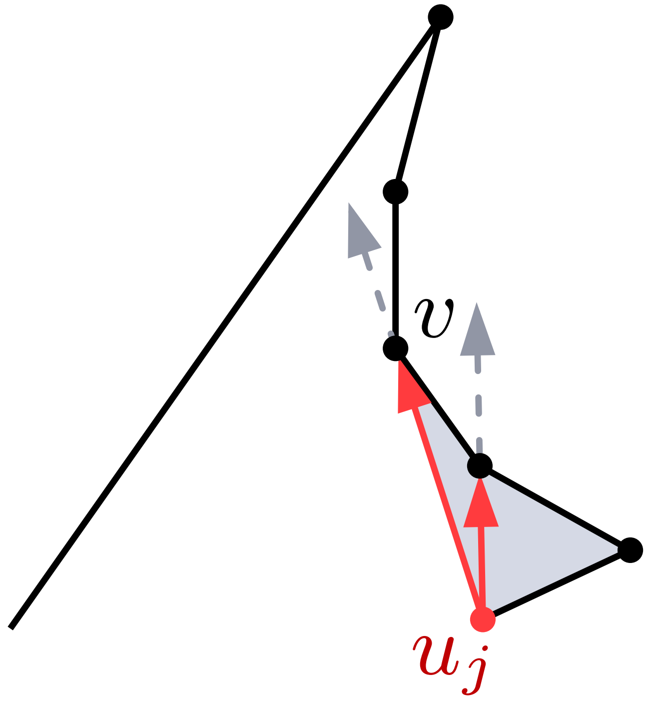

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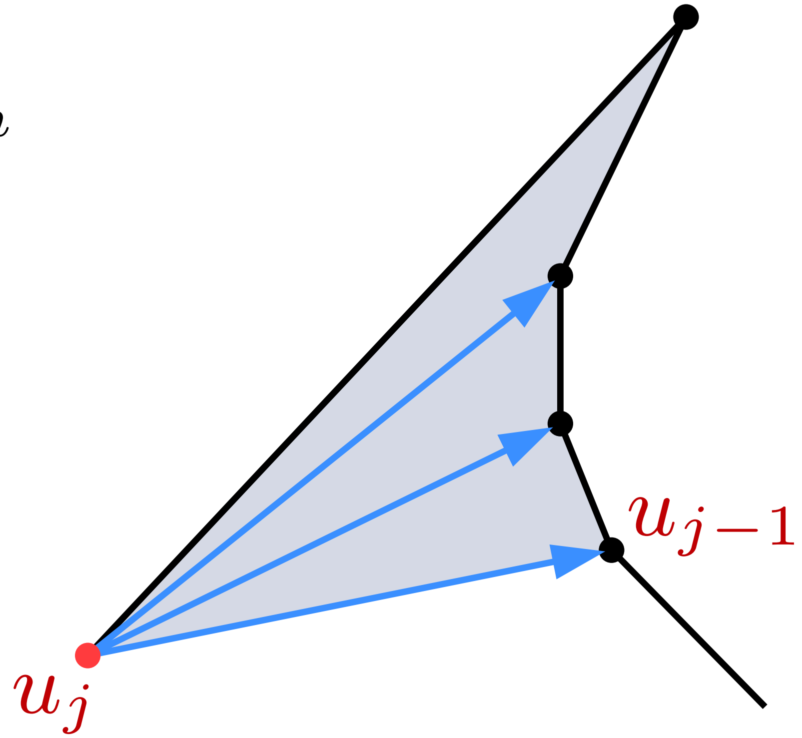

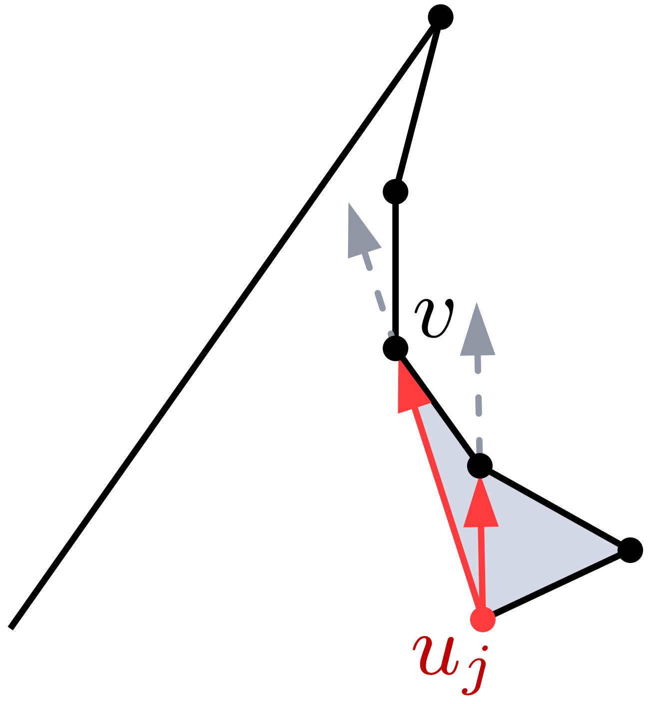

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Question:
running time?

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$\Downarrow$ Does this follow immediately?

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at most $n - 3$ diagonals added,
$\Downarrow$ each is part of 2 y-monotone polygons
$\Rightarrow$ summed complexity of y-monotone polygons is $O(n)$

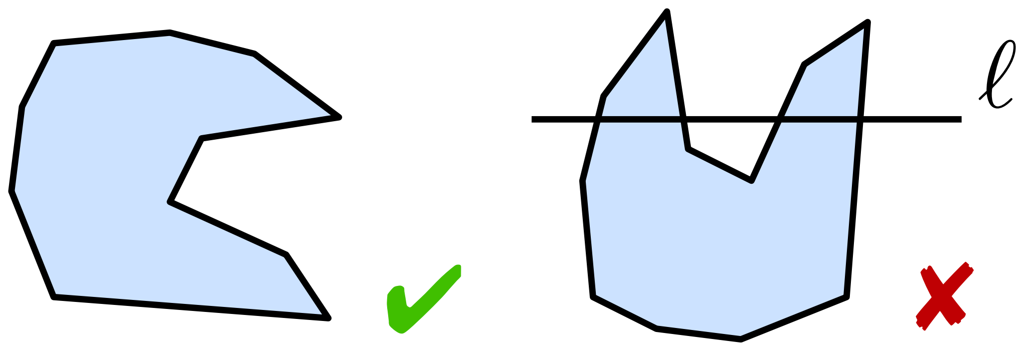
Theorem 5: A simple polygon with n vertices can be triangulated in $O(n \log n)$ time using $O(n)$ space.

Summary (Art Gallery Problem)

2-step procedure:

- step 1: partition P into y-monotone subpolygons

Definition: A polygon P is y-monotone if, for every horizontal line ℓ, the intersection $\ell \cap P$ is connected.



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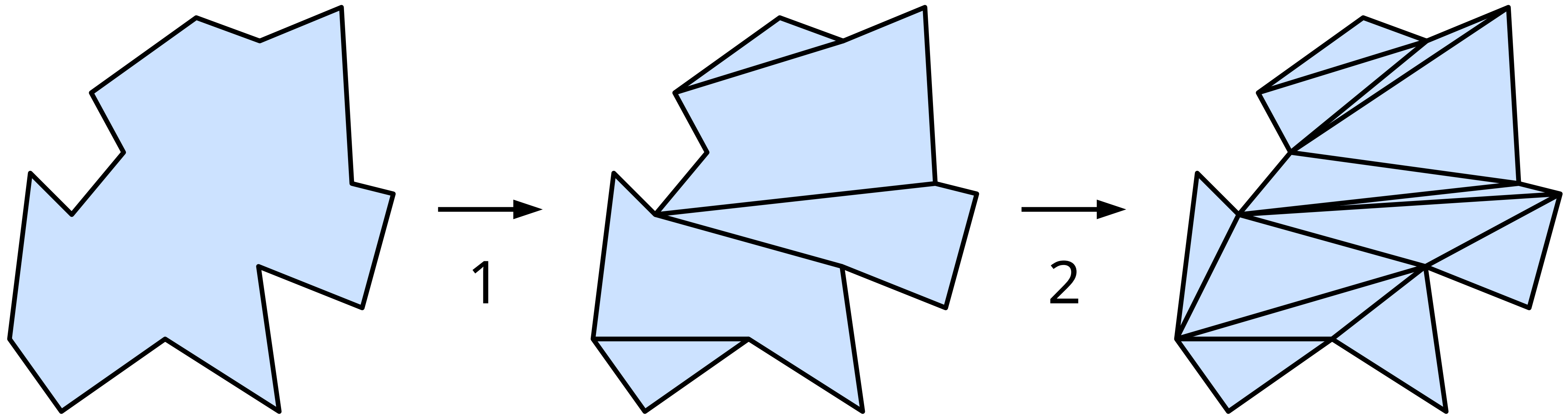

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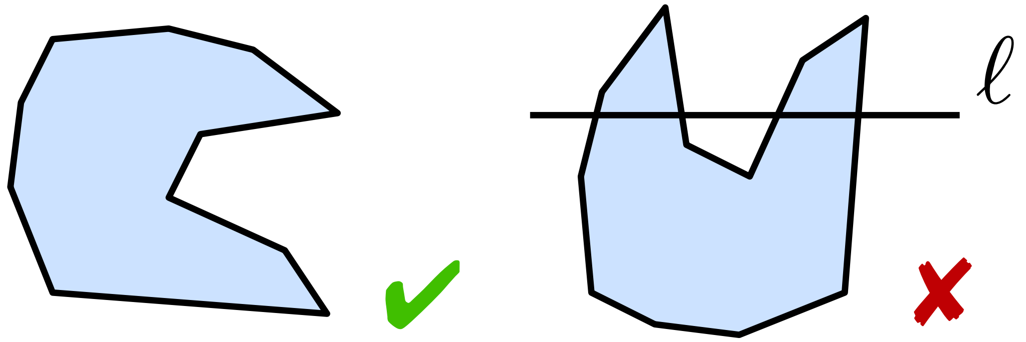

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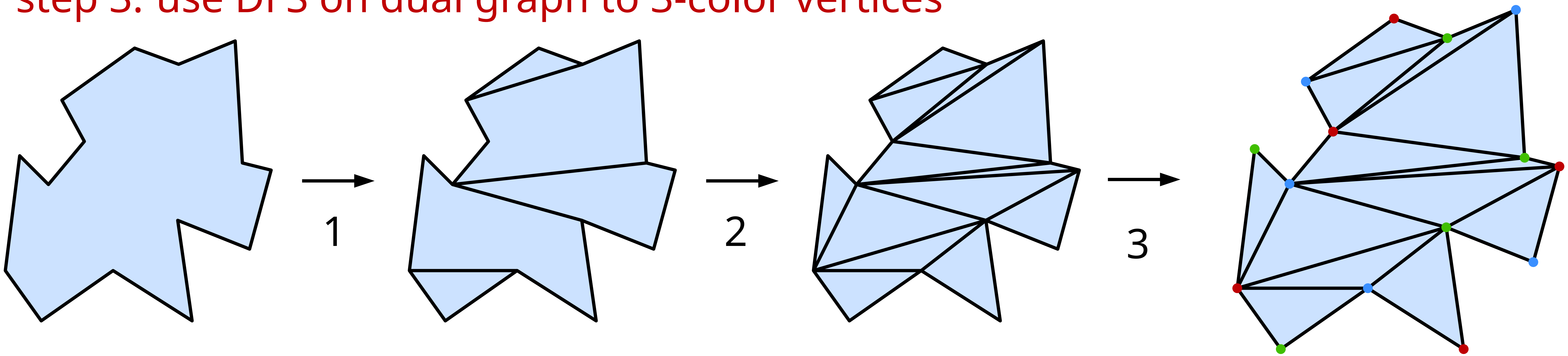

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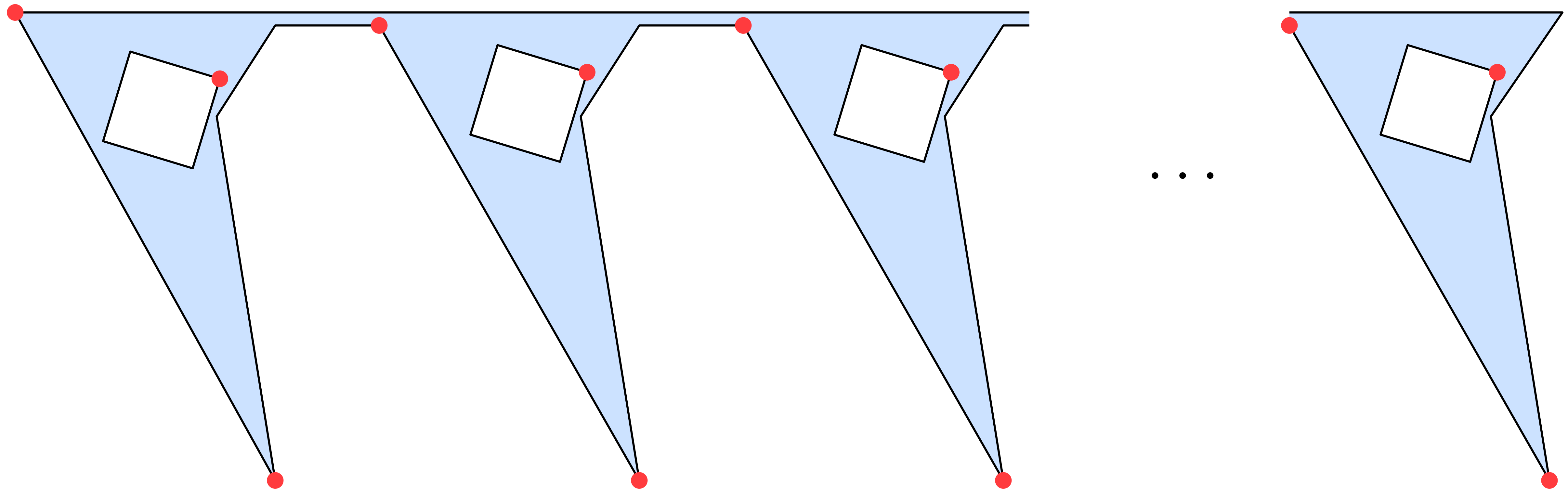

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Can we triangulate general simple polygons faster?

- Yes. This was an open problem for a long time, until increasingly faster (randomized) algorithms were developed by the end of 1980s
- $O(n)$-time algorithm by Chazelle [1990] (complicated)
- There is an elegant $O(n \log^* n)$ expected-time algorithm [Seidel 1991] (simple)