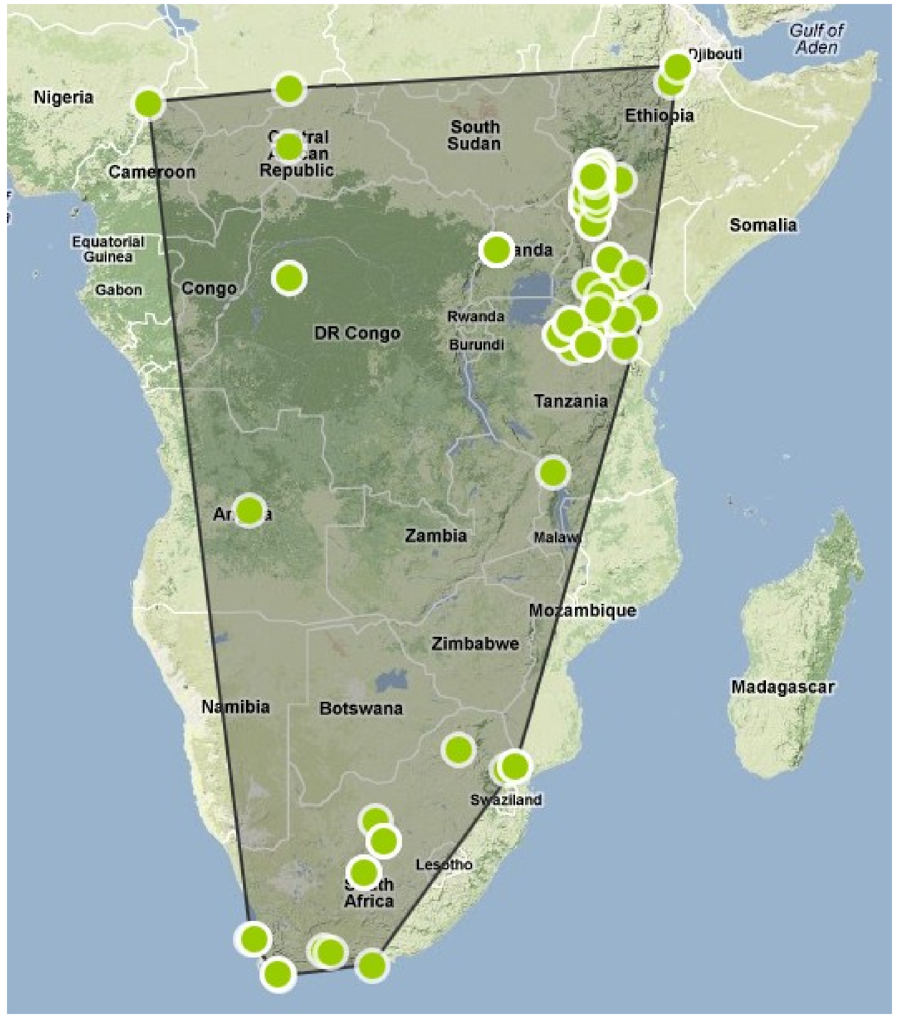
Convex Hulls

Geometric Algorithms

Wildlife analysis: extent of occurrence



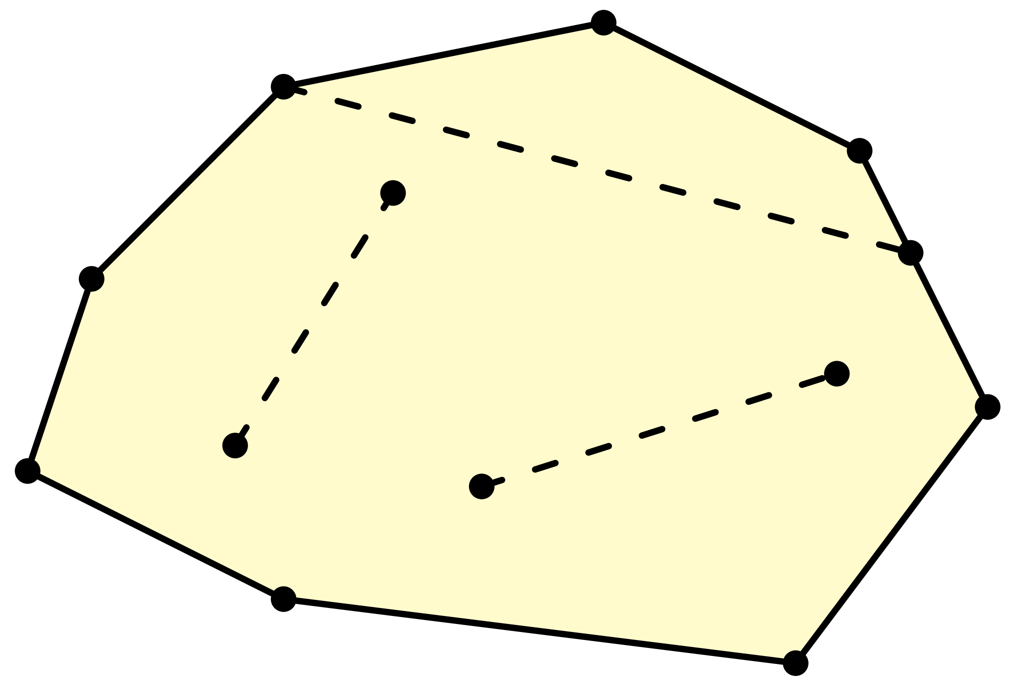
Black Rhino (Diceros bicornis)

http://geocat.kew.org/

http://www.iucnredlist.org/

Convexity

Definition: A shape or set is convex if for any two points that are part of the shape, the whole connecting line segment is also part of the shape



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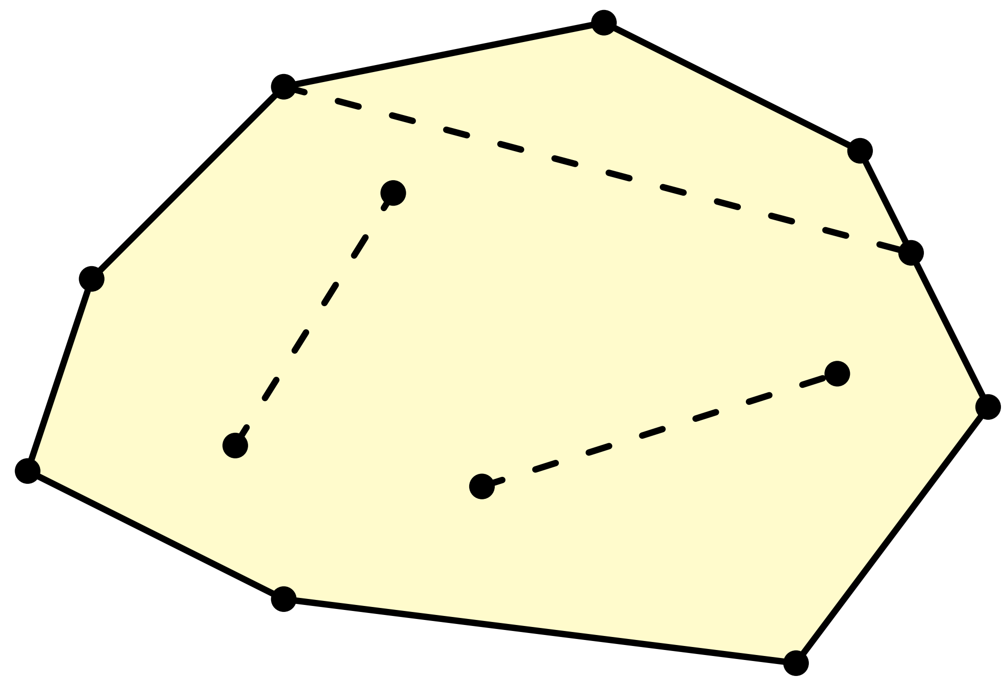
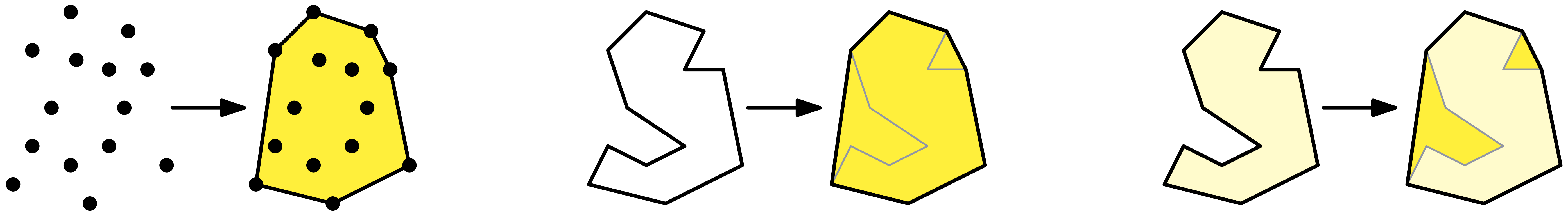


Question: Which of the following shapes are convex?

- point
- line segment
- line
- circle
- disk
- quadrant

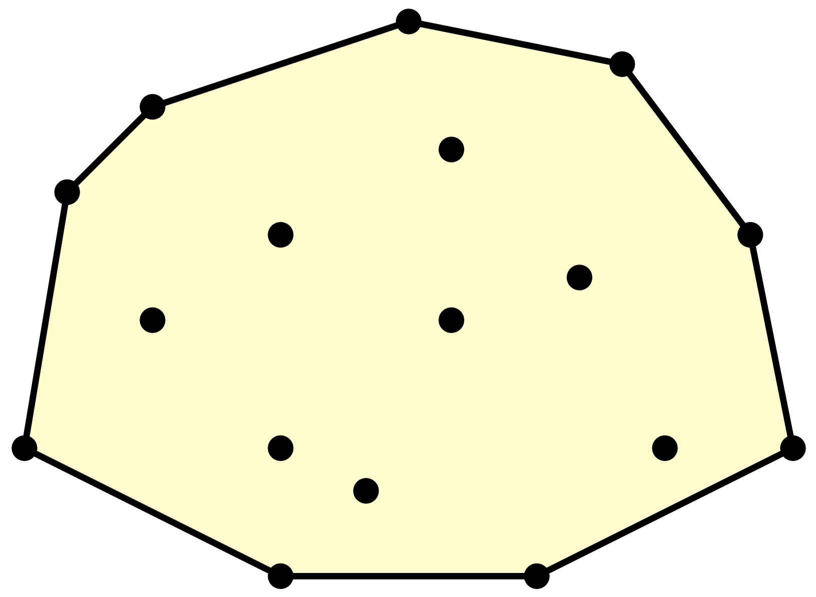
Convex hull

Definition: For any subset of the plane (set of points, polygonal chain, simple polygon), its convex hull is the smallest convex set that contains that subset



Convex hull problem

Problem: Give an algorithm that computes the convex hull of any given set of n points in the plane efficiently



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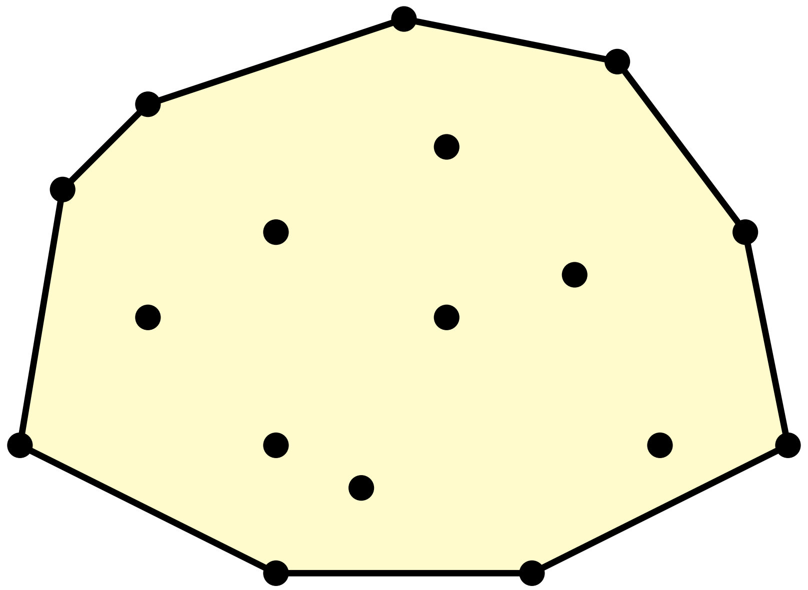
- input has $2n$ coordinates, so $O(n)$ size



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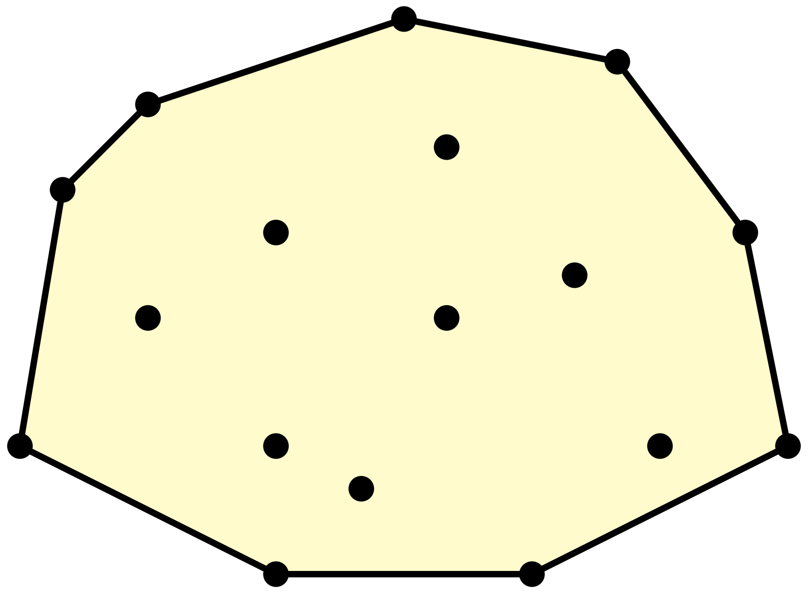

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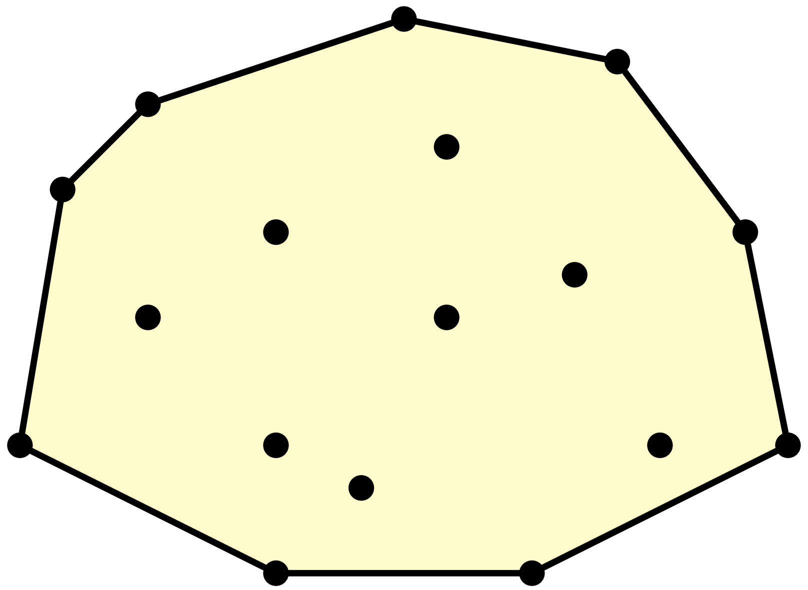

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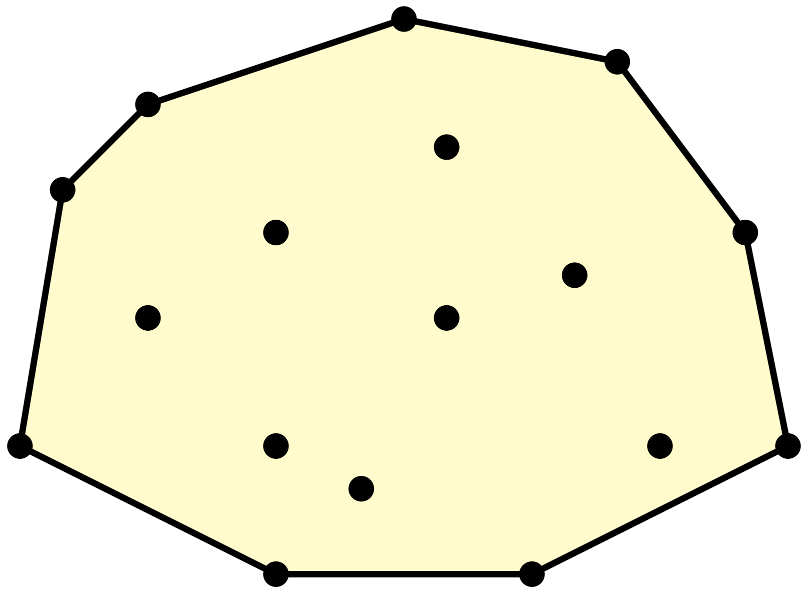

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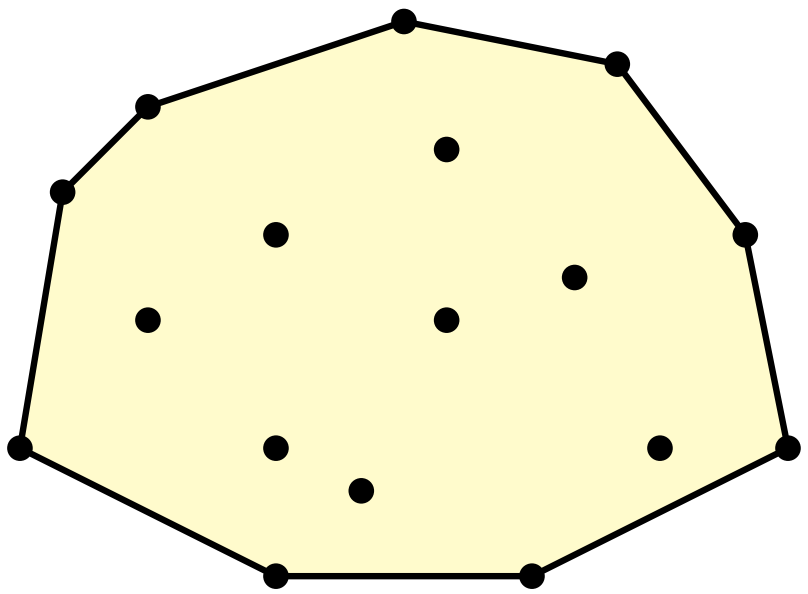
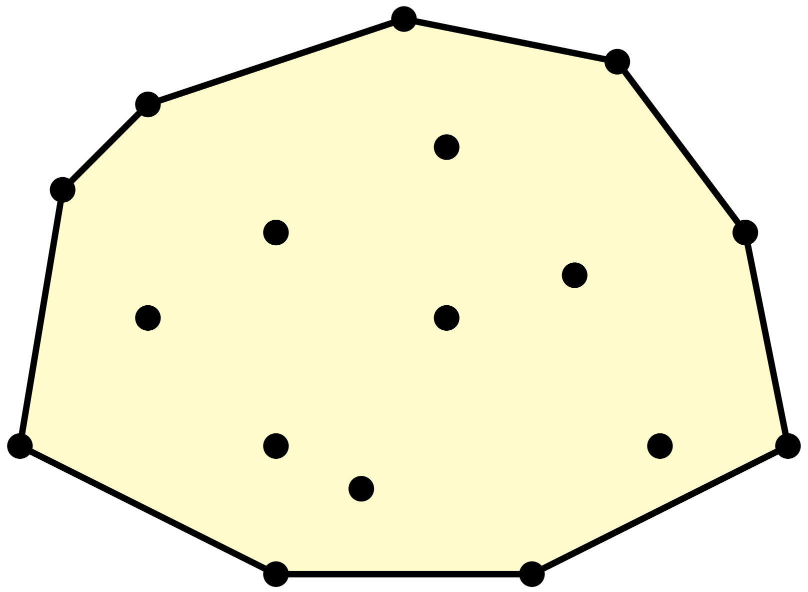


- input has $2n$ coordinates, so $O(n)$ size
- output?
  - assume the n points are distinct
  - output has at least 2 and at most n points
  - output size is between $O(1)$ and $O(n)$

Convex hull problem: questions

Problem: Give an algorithm that computes the convex hull of any given set of n points in the plane efficiently

Question: Is there an algorithm computing a convex hull faster than $O(n)$ time in the worst case?



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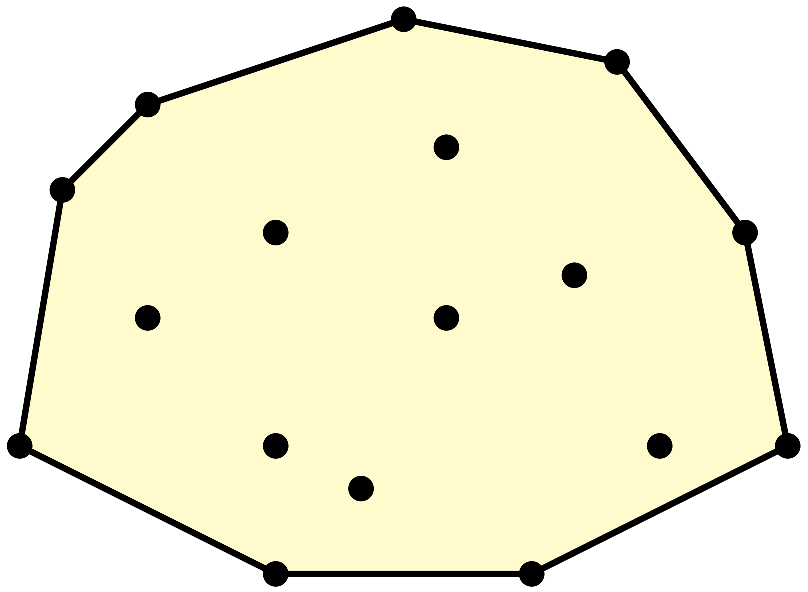

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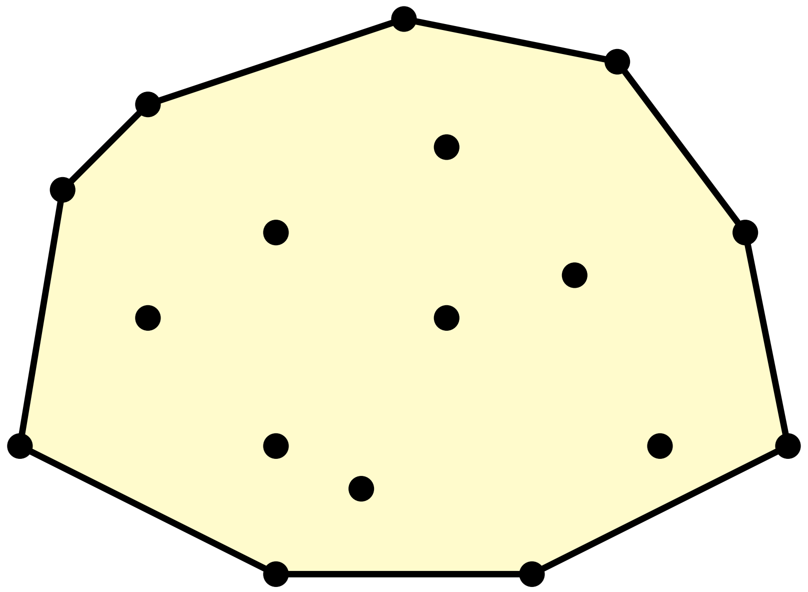

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Question: Is there any hope of finding an $O(n)$ time algorithm?

Developing an algorithm

To develop an algorithm, draw many sketches to gain insight, make various observations, find useful properties

Convex hull algorithm

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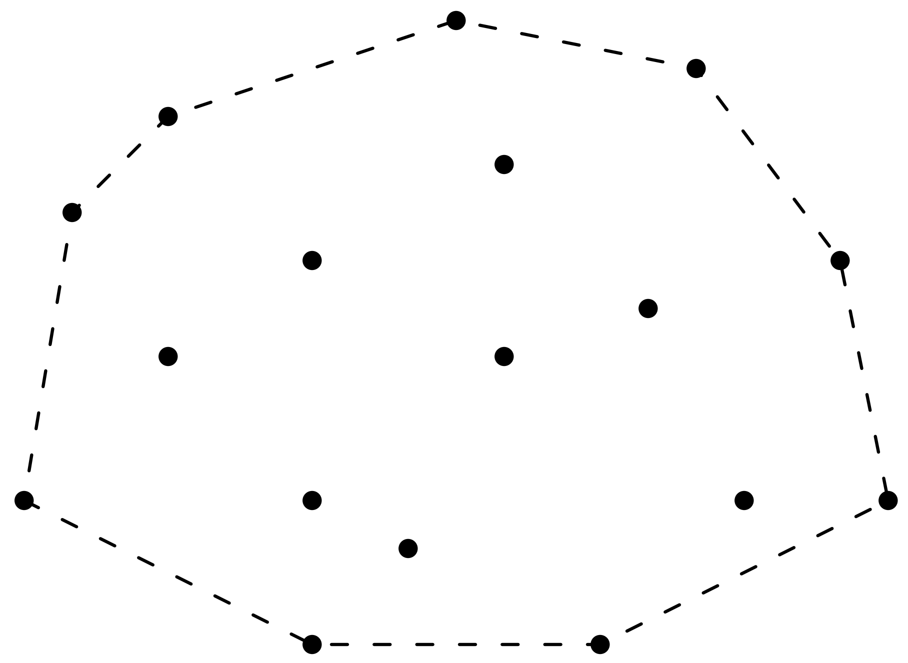

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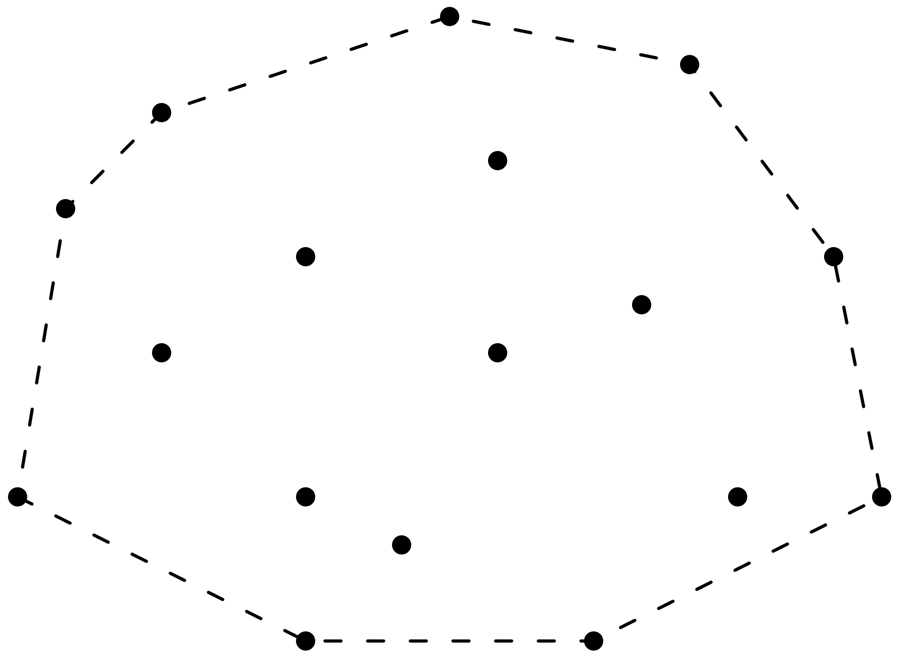

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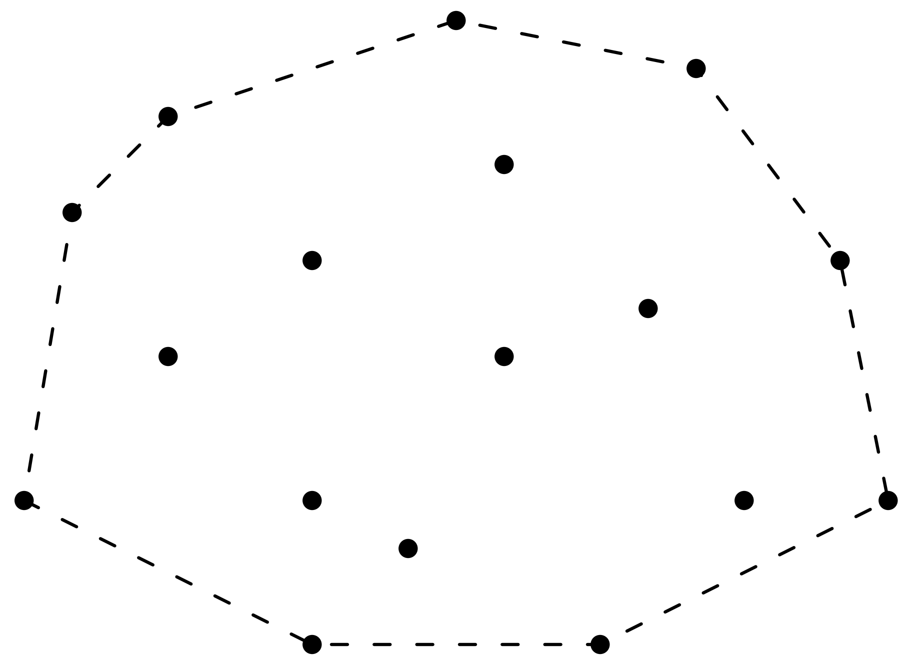

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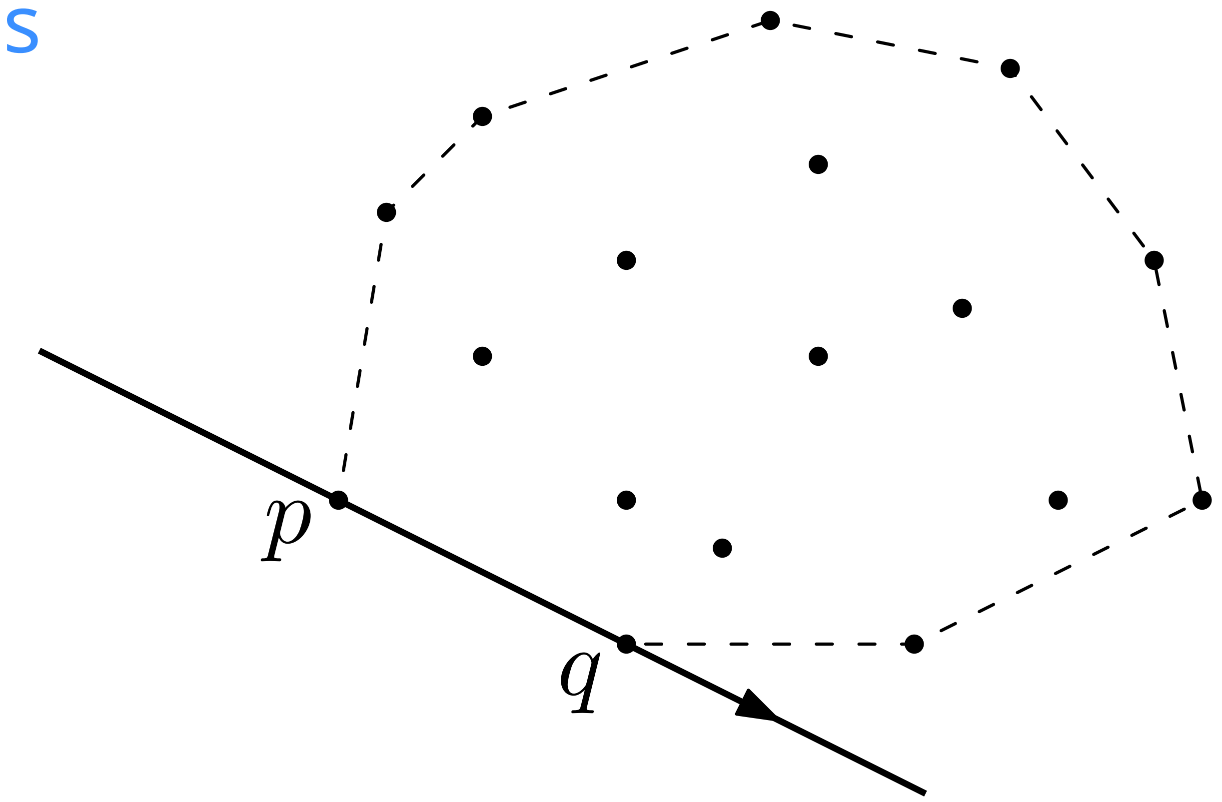

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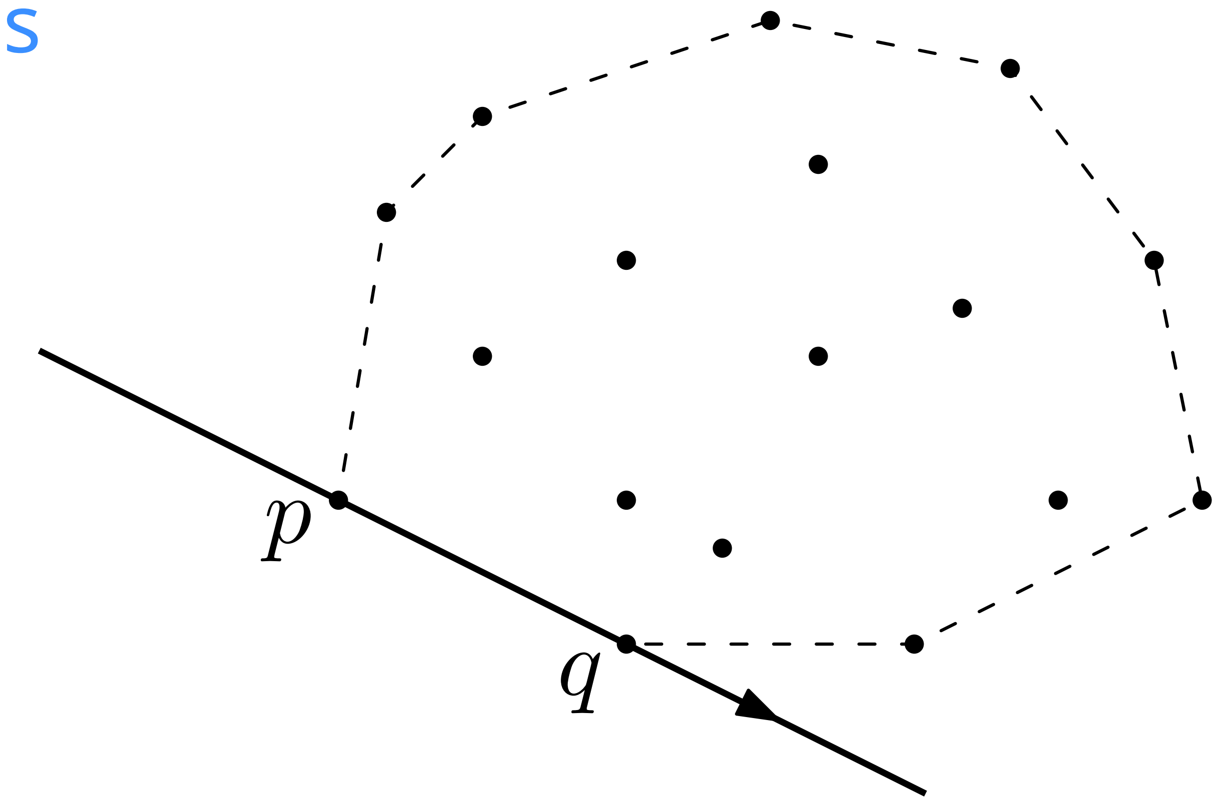

all points lie left of the directed line from p to q (if the edge from p to q is a CCW convex hull edge)

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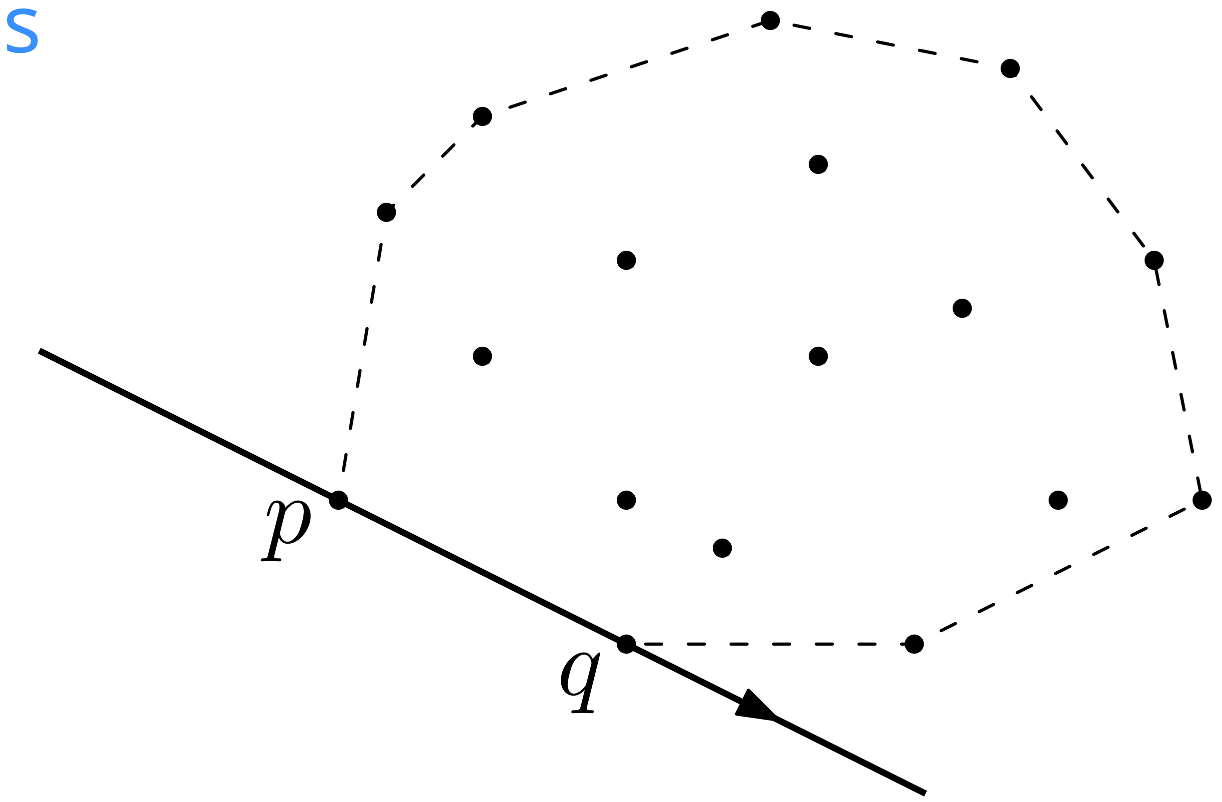

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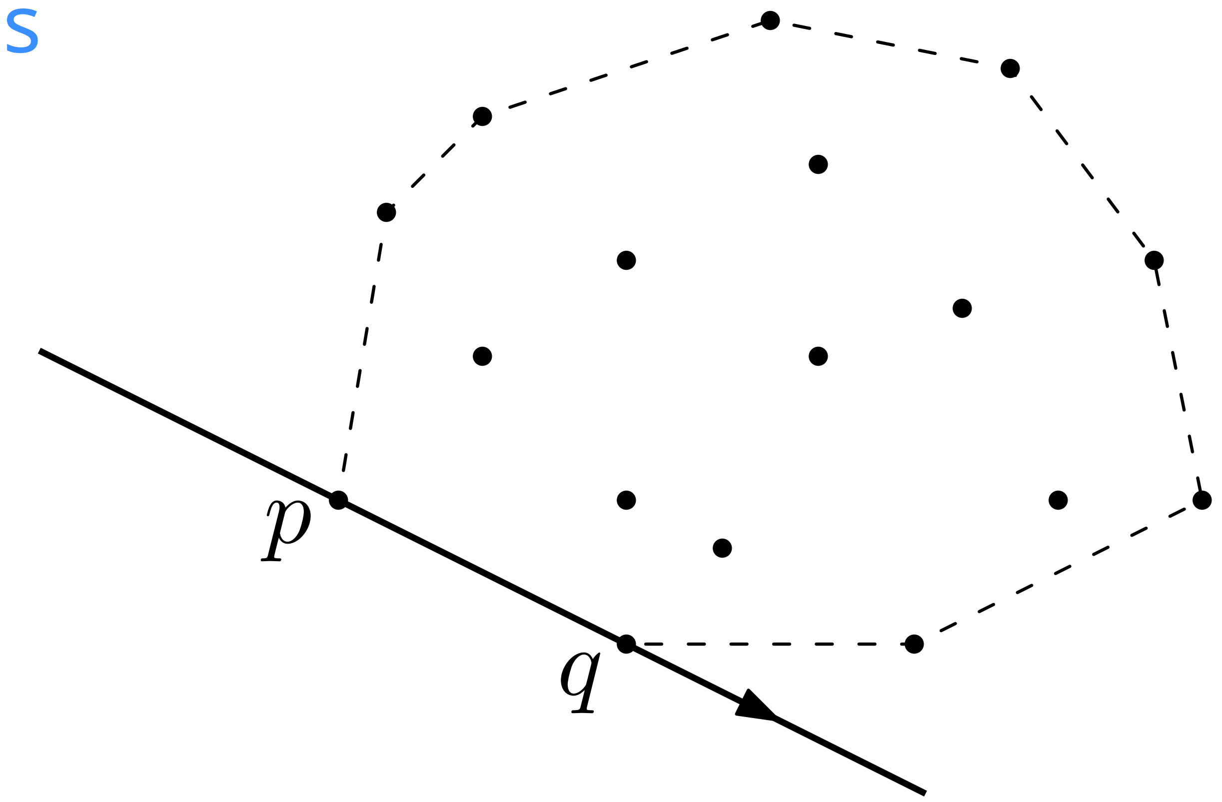

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Simple algorithm?

Convex hull algorithm

Algorithm SlowConvexHull(P)

Input: set P of distinct points in the plane

Output: list L with vertices of $CH(P)$ in counter-clockwise order

$E \leftarrow \varnothing$

for all ordered pairs $(p, q) \in P \times P$ with $p \neq q$ **do**

    $valid \leftarrow true$

    for all points $r \in P$ not equal to p or q **do**

        if r lies right of the directed line from p to q **then**

            $valid \leftarrow false$

    if $valid$ **then**

        add directed edge $\overrightarrow{pq}$ to E

from the set E of edges construct a list L of vertices of $CH(P)$, sorted in counter-clockwise order

Questions to keep in mind

Question: How must line 5 (if r lies right of $\overline{pq} \ldots$) be interpreted to make the algorithm correct, i.e., how do we handle degeneracies?

Question: How efficient is the algorithm?

Question: Is the algorithm robust against rounding errors?

Note: Robustness is a huge issue when implementing geometric algorithms

Convex hull algorithm 2

Another approach: incremental, from left to right

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Let's first compute the upper boundary of the convex hull this way (property: on the upper hull, points appear in x-order)

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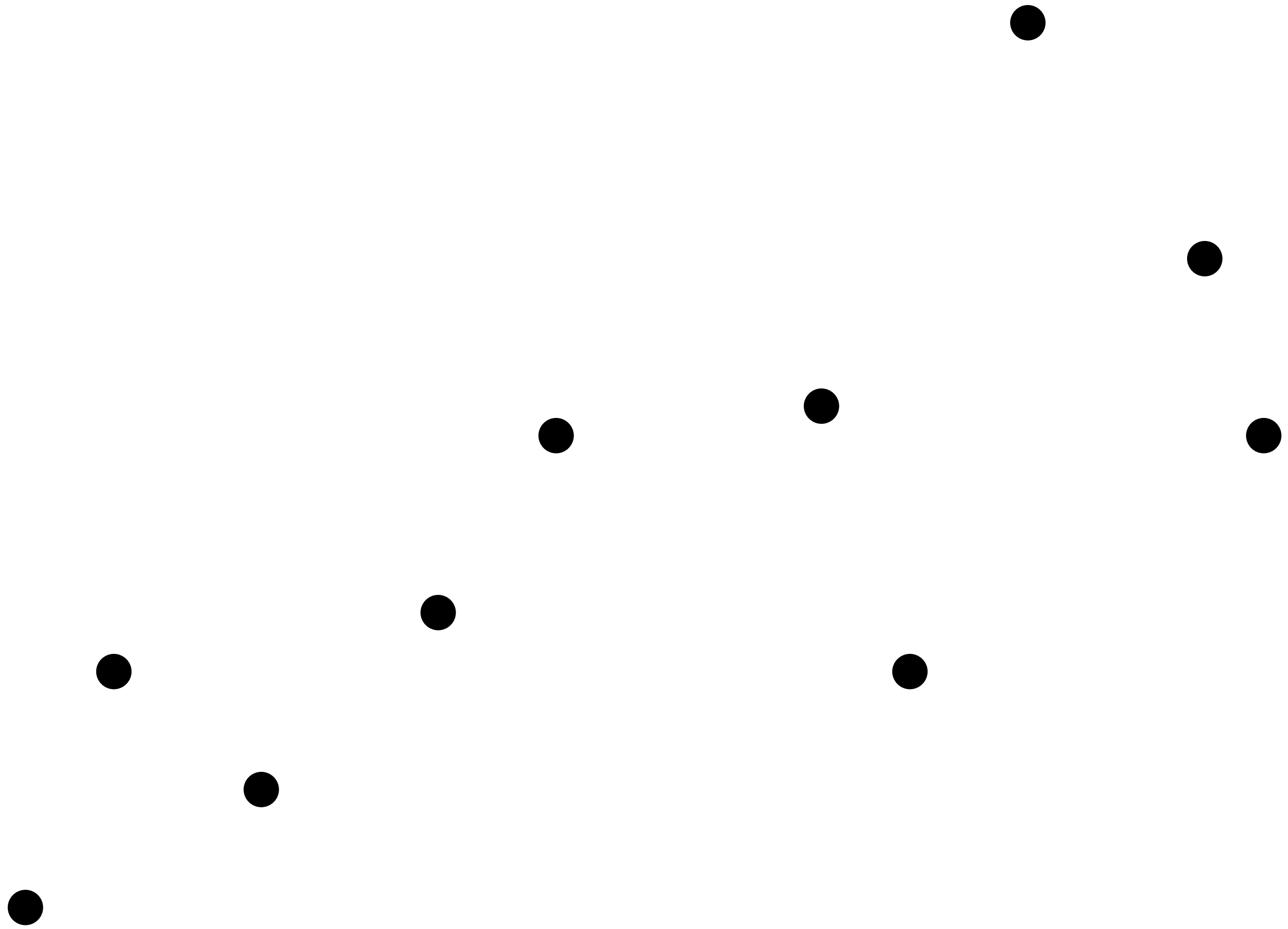
Let's first compute the upper boundary of the convex hull this way (property: on the upper hull, points appear in x-order)

Main idea: Sort the points from left to right (= by x-coordinate). Then insert the points in this order, and maintain the upper hull so far

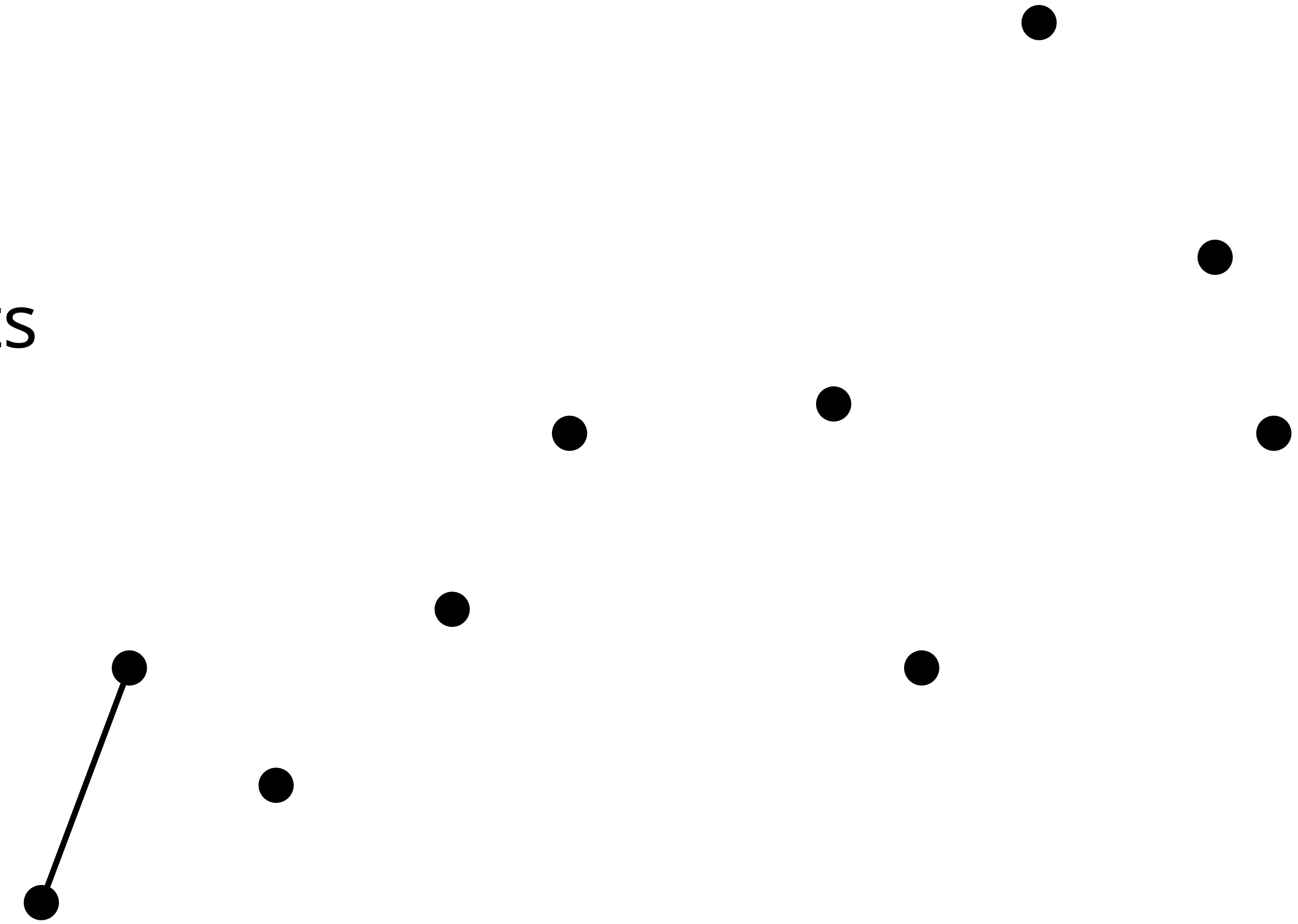
Convex hull algorithm 2

Observation: from left to right, there are only right turns on the upper hull



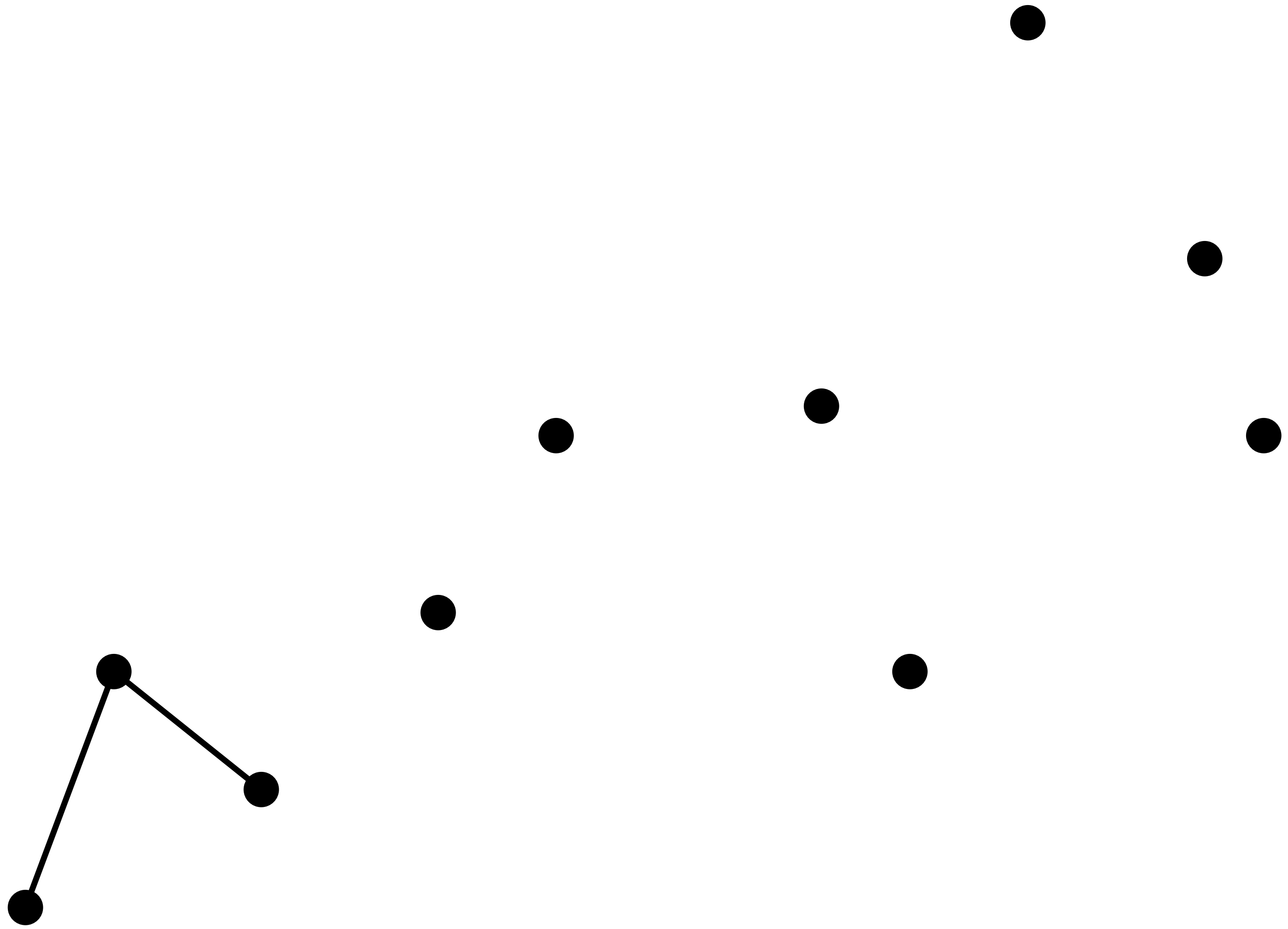
Convex hull algorithm 2

Initialize by inserting the leftmost two points



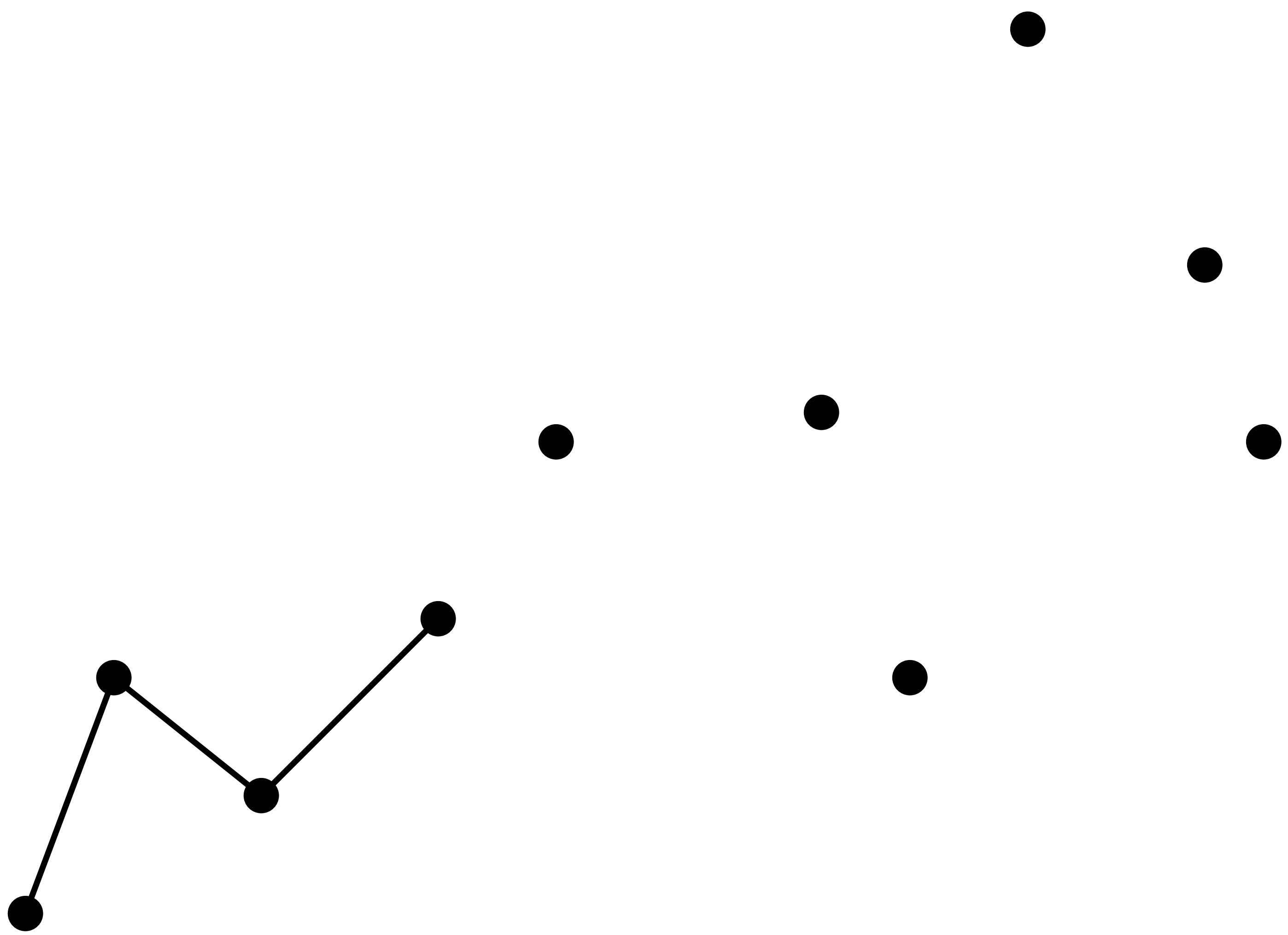
Convex hull algorithm 2

When we consider the third point there will be a right turn at the previous point, so we add it



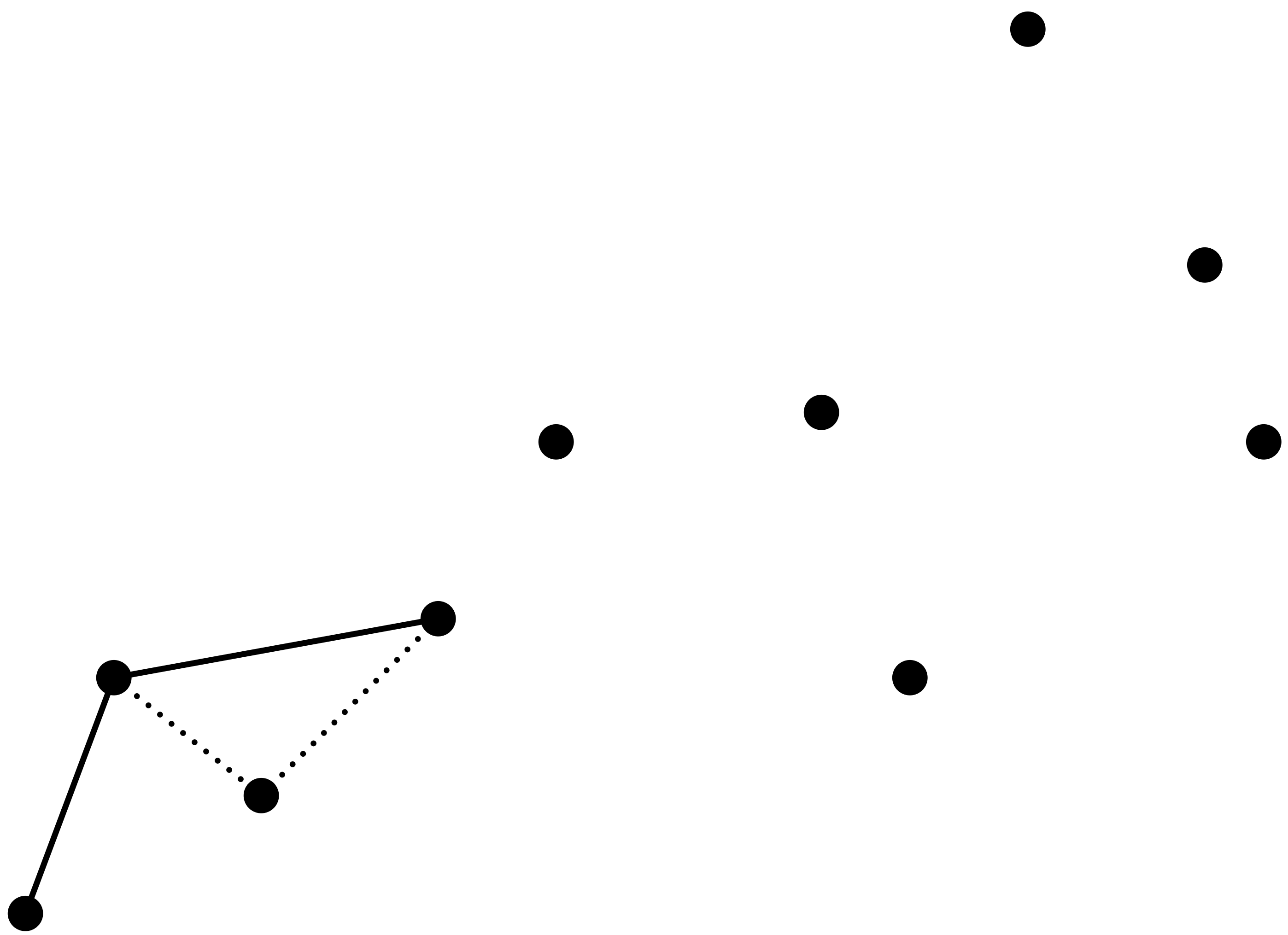
Convex hull algorithm 2

When we consider the fourth point we get a left turn at the third point



Convex hull algorithm 2

. . . so we remove the third point from the upper hull when we add the fourth



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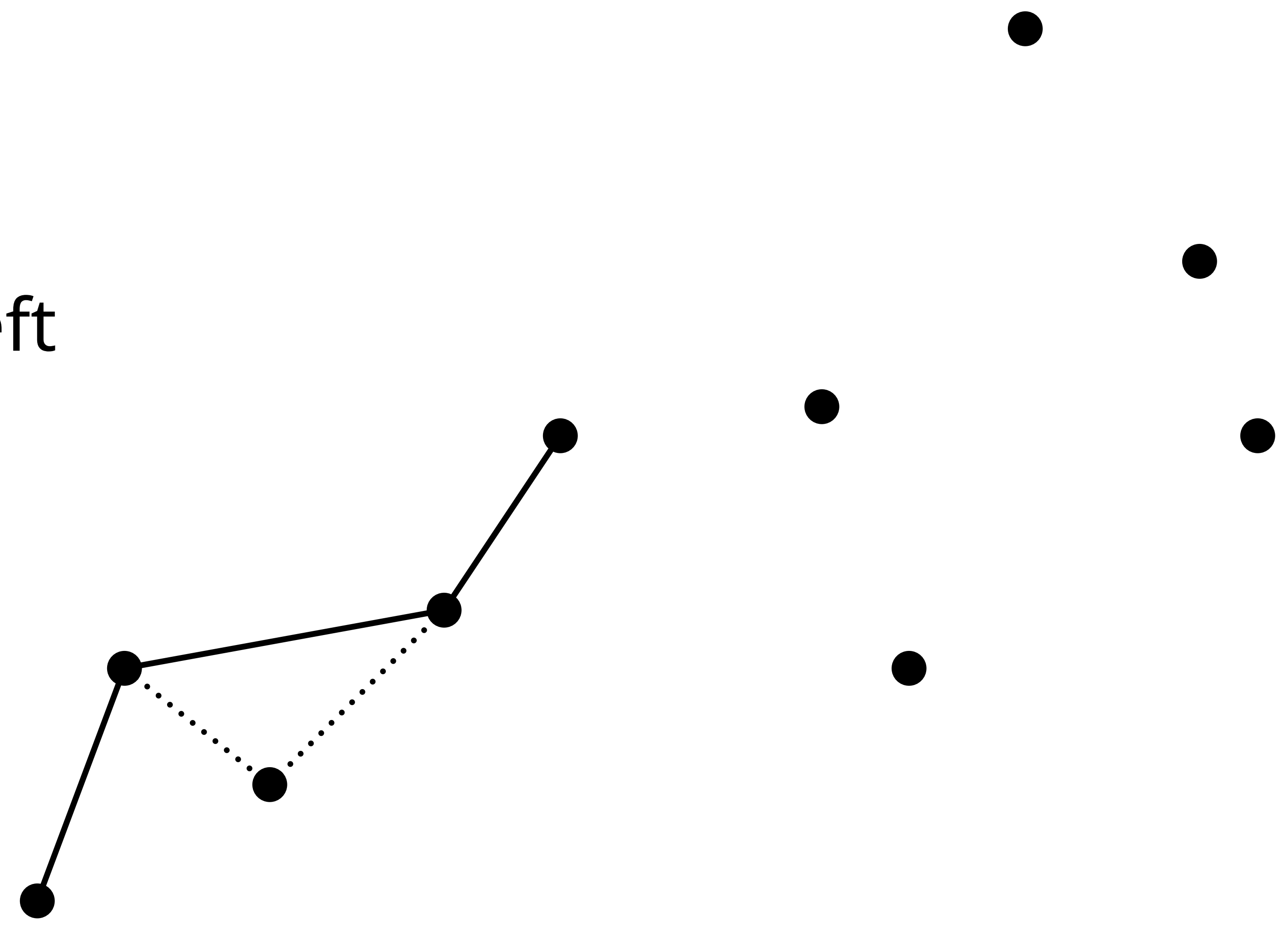
When we consider the fifth point we get a left turn at the fourth point



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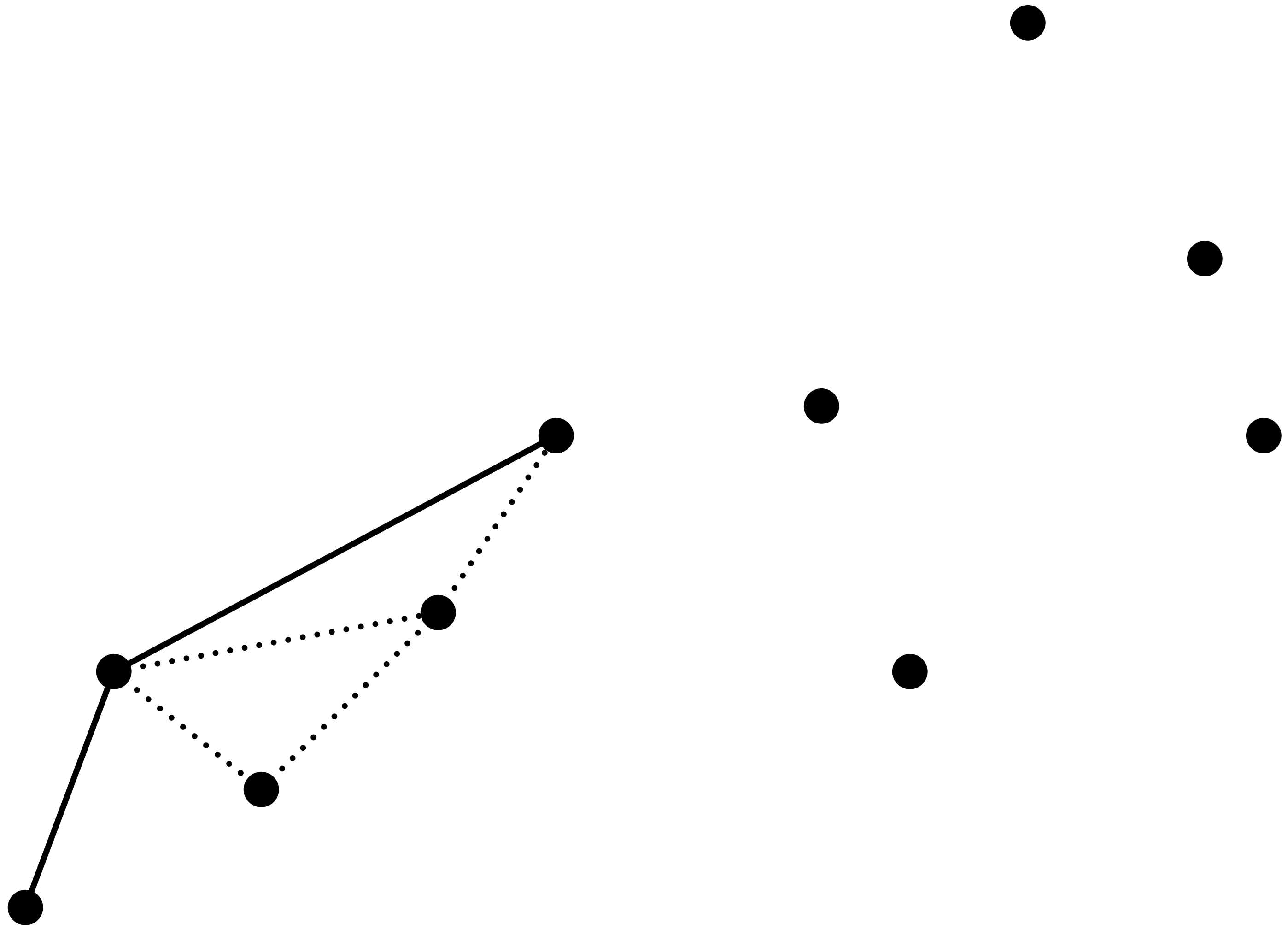
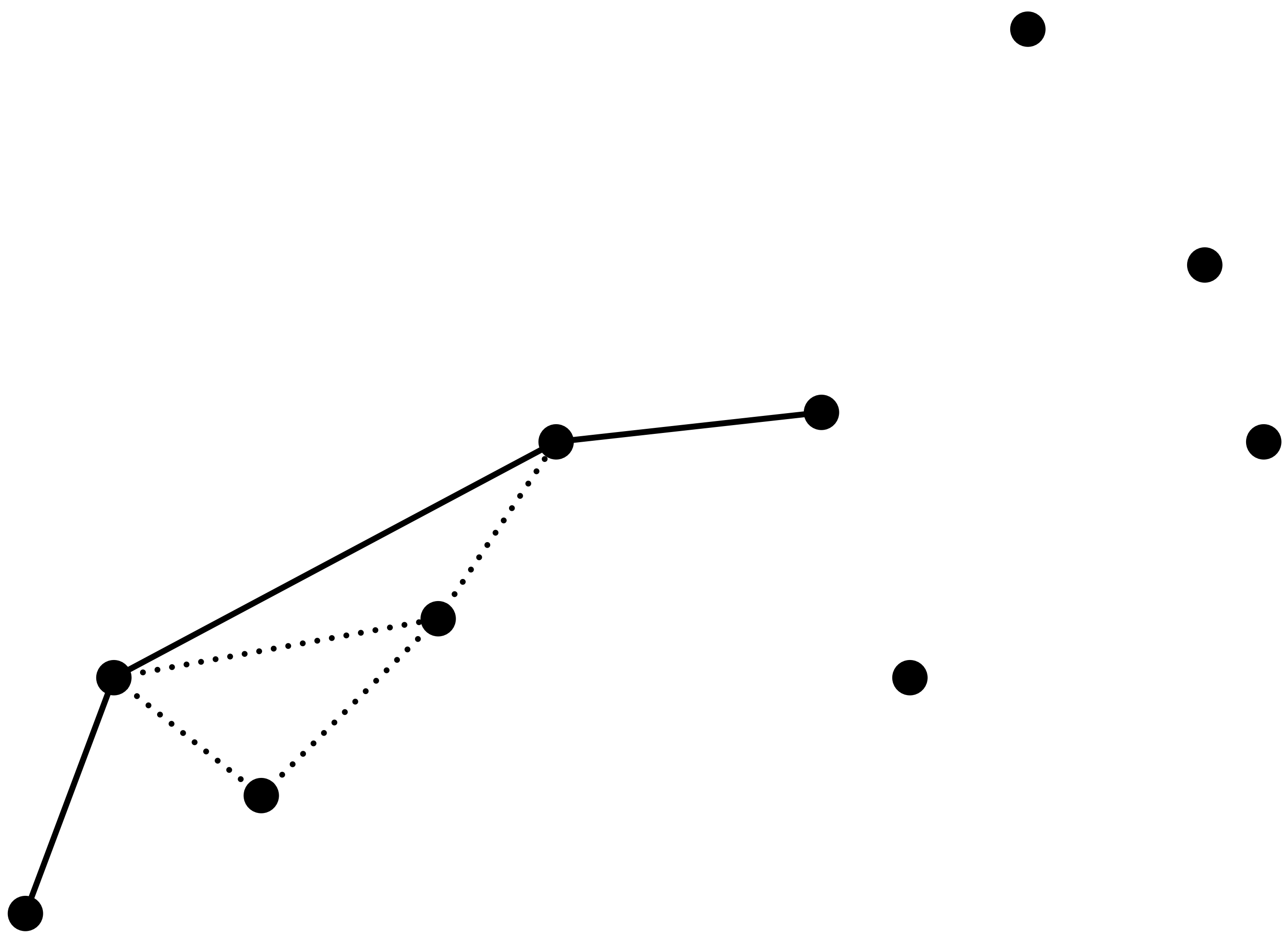


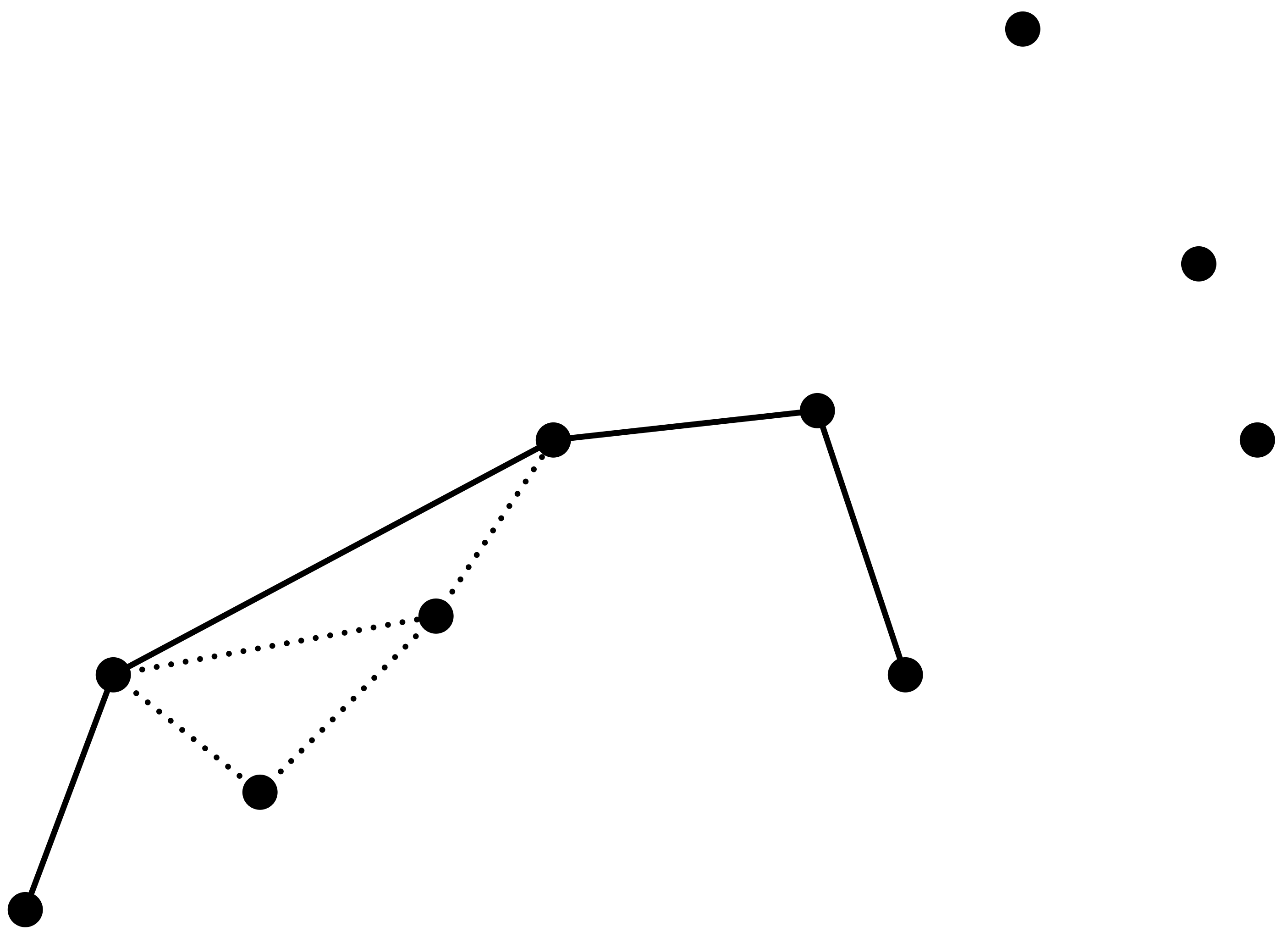
Convex hull algorithm 2

When we consider the sixth point we get a right turn at the fifth point, so we just add it



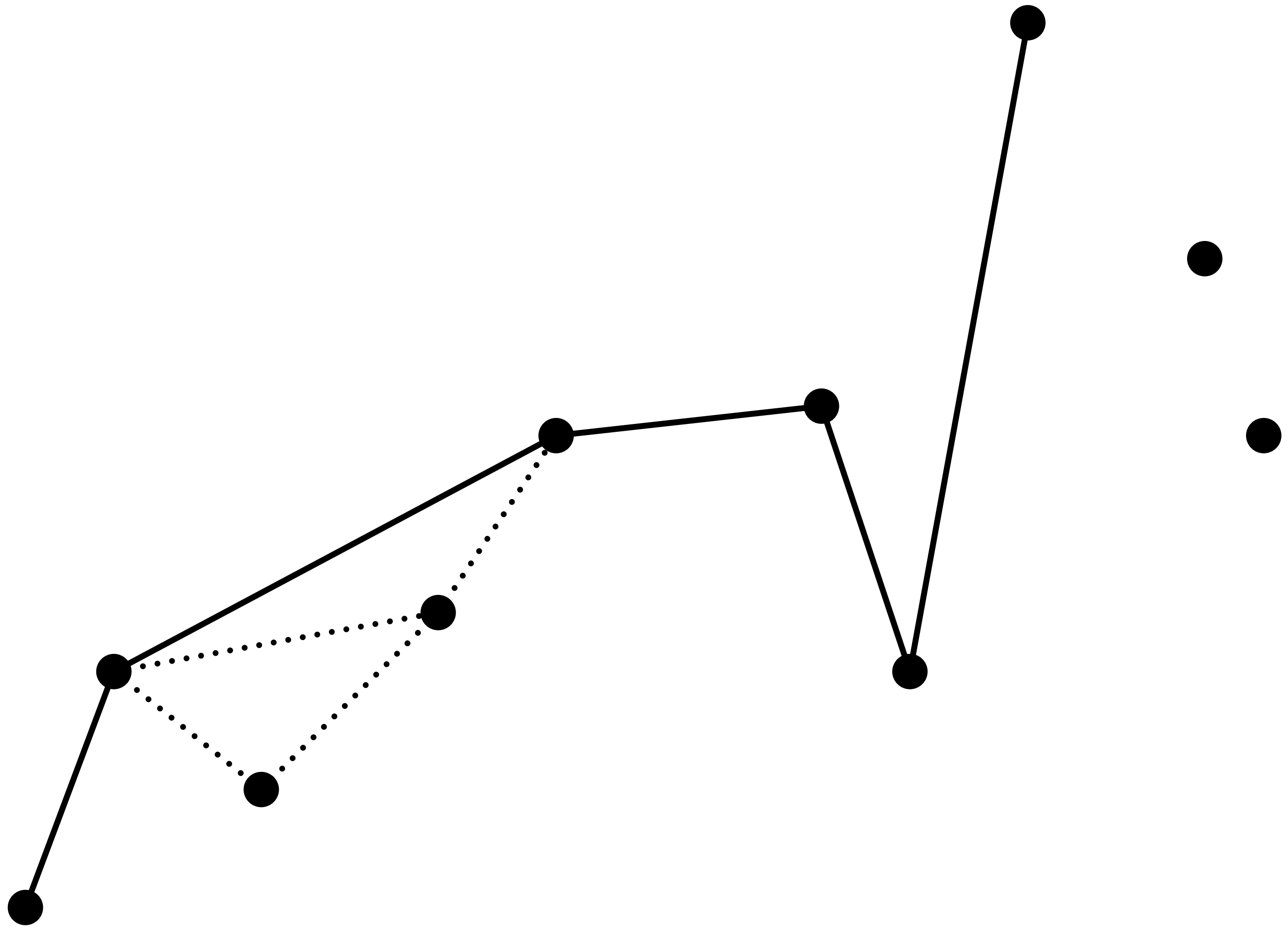
Convex hull algorithm 2

We also just add the seventh point



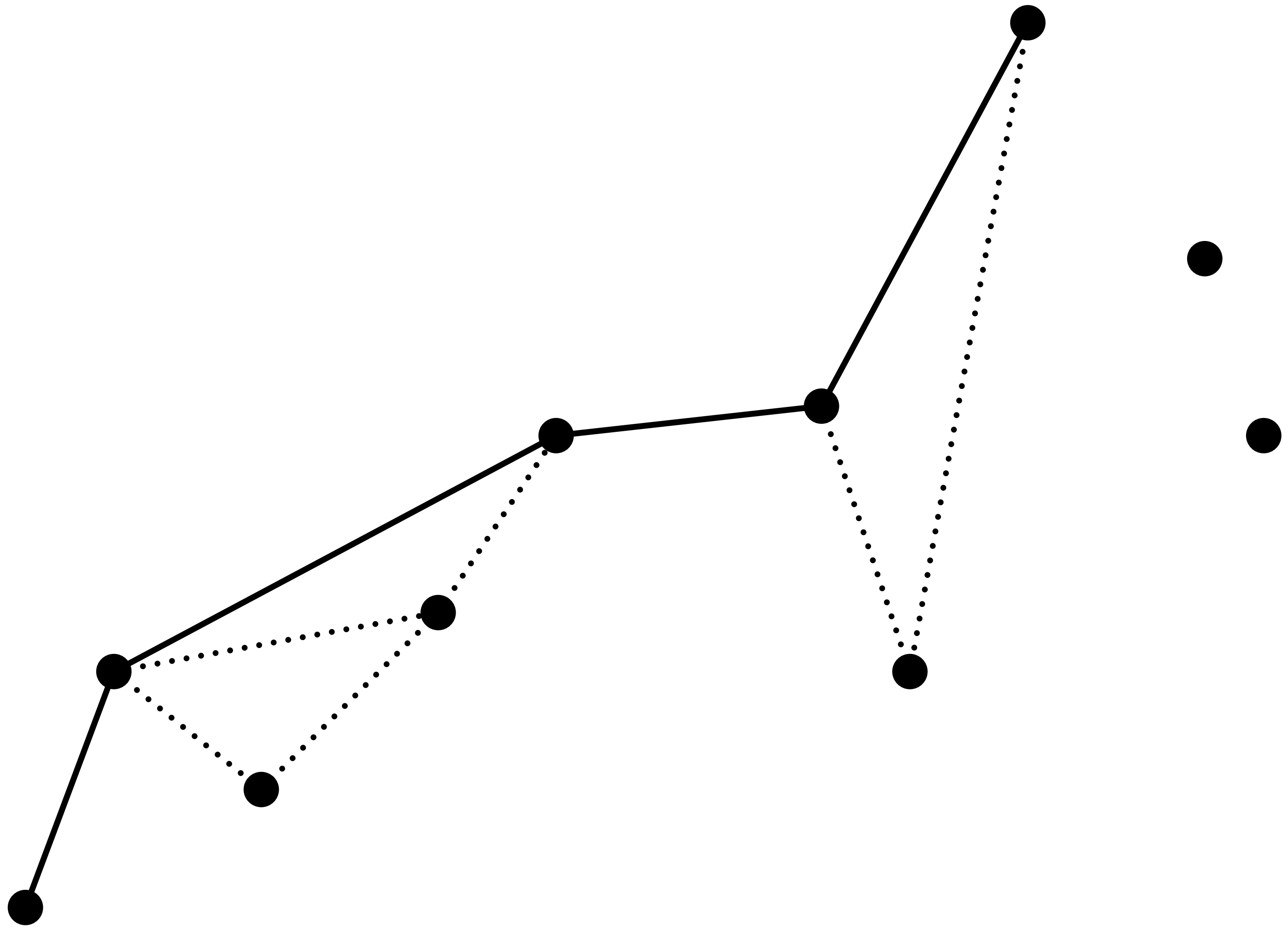
Convex hull algorithm 2

When considering the eighth point. . .



Convex hull algorithm 2

. . . we remove the seventh point



Convex hull algorithm 2

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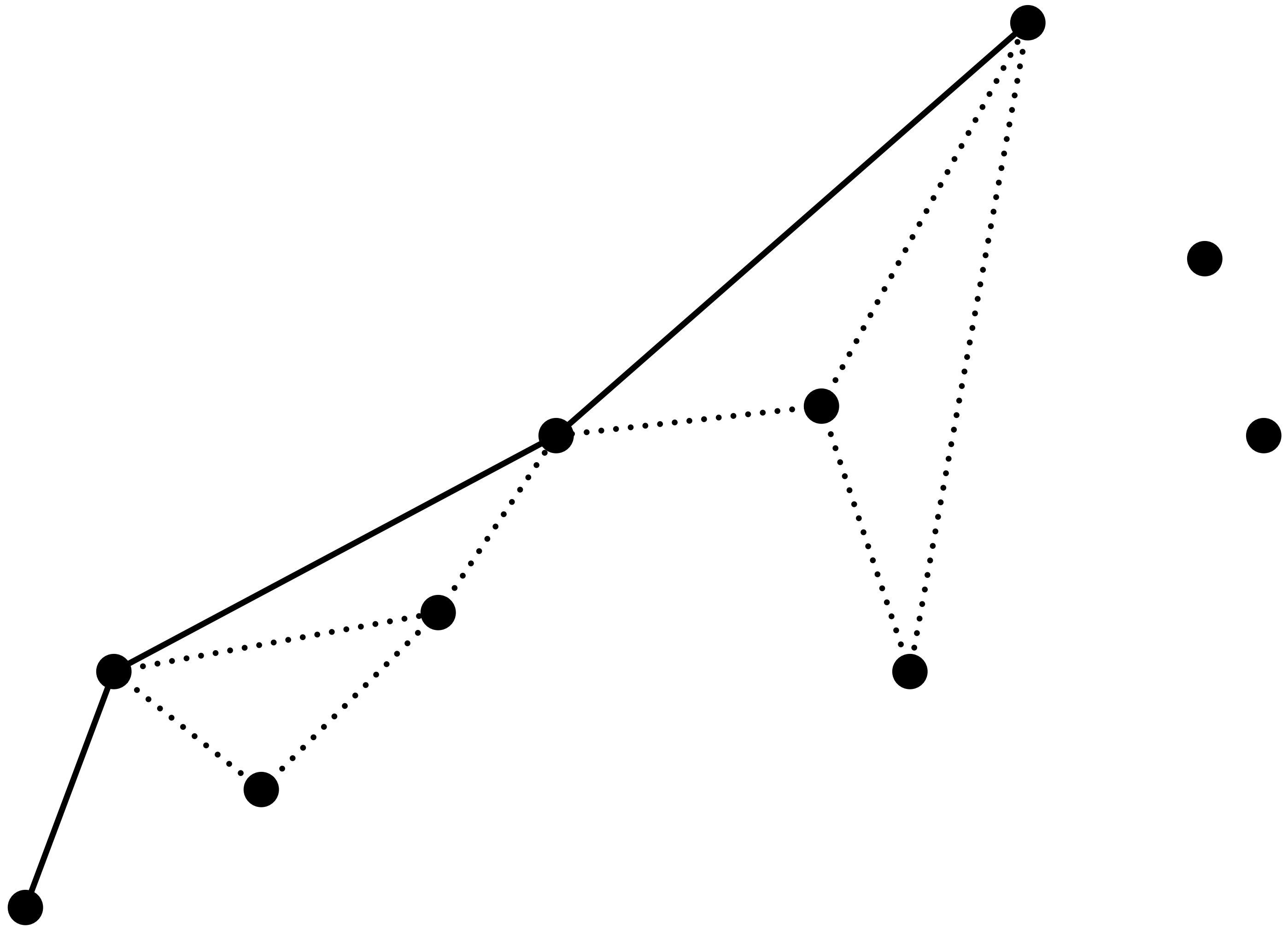

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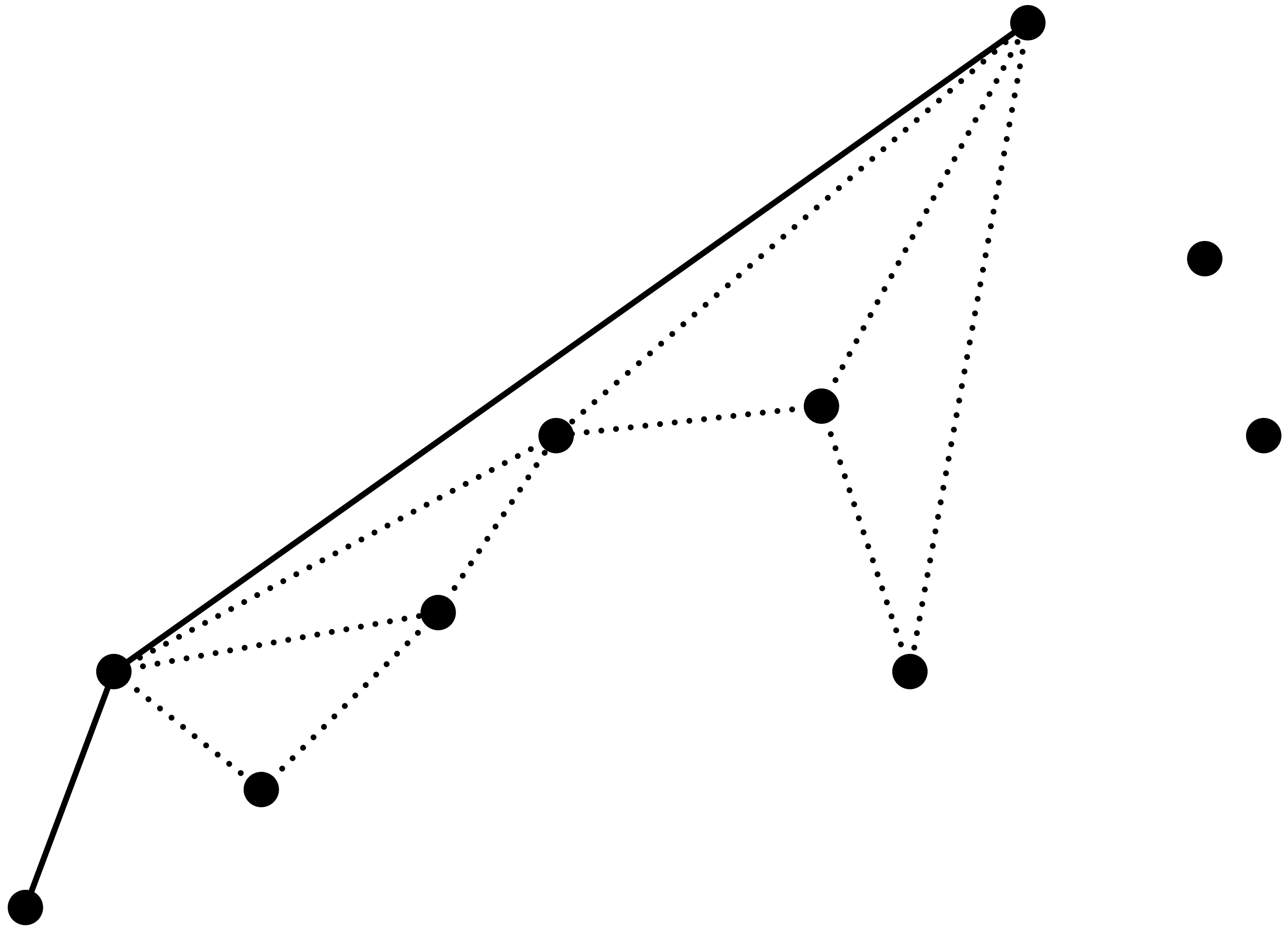
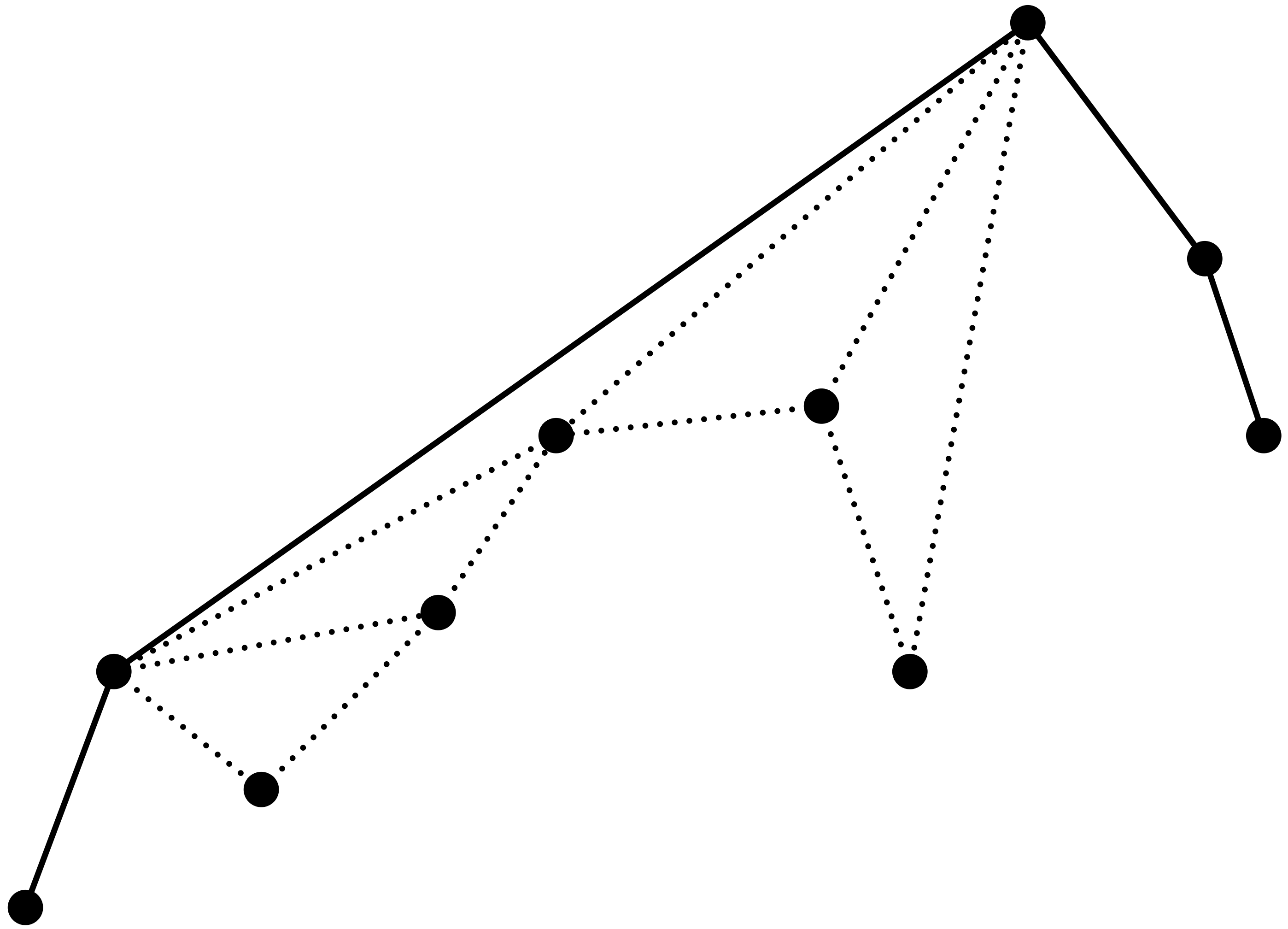


Convex hull algorithm 2

after two more steps we get the upper hull



Convex hull algorithm 2: Graham Scan

Algorithm GRAHAMSCAN(P)

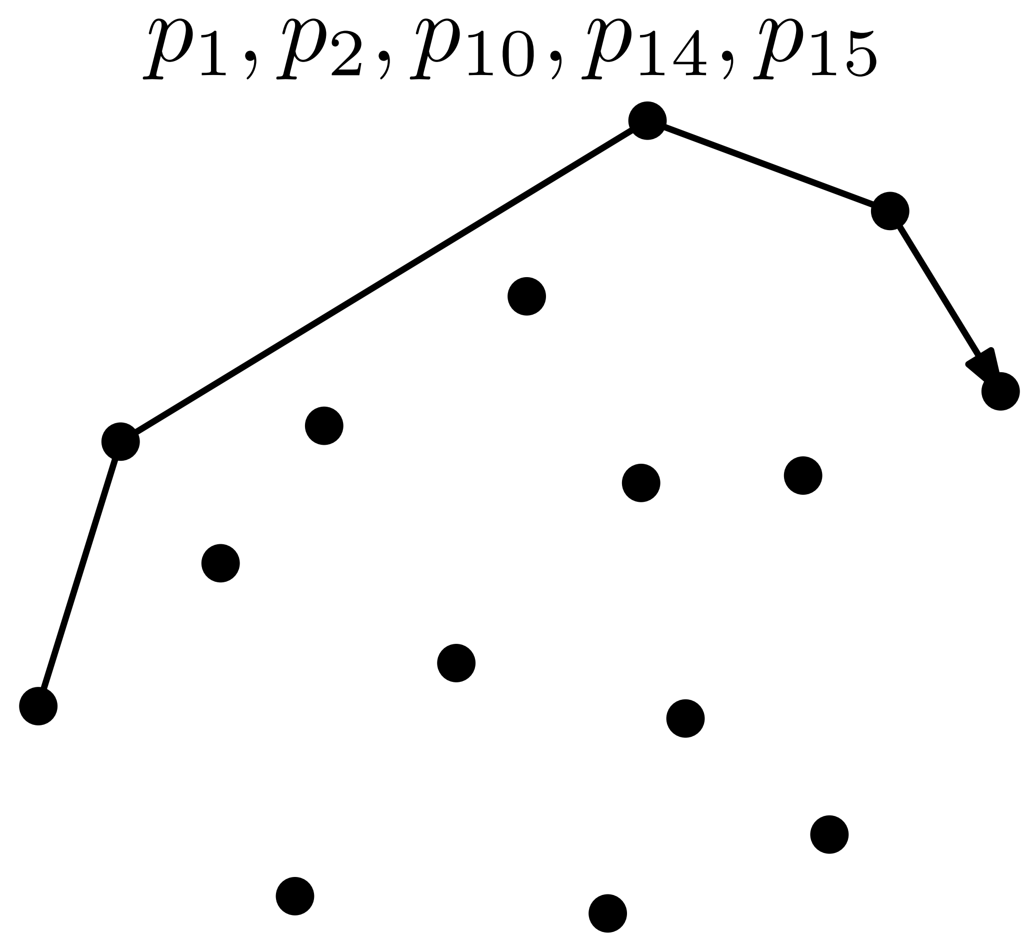
Input: set P of points in the plane

Output: list L containing vertices of $CH(P)$ in clockwise order

```
sort the points by x-coordinate, resulting in a sequence p_1, ..., p_n
put the points p_1 and p_2 in a list L_upper, with p_1 as the first point
for i ← 3 to n do
    append p_i to L_upper
    while L_upper contains more than two points and the last three points do
    not make a right turn do
        delete the middle of the last three points from L_upper
```

Convex hull algorithm 2: Graham Scan

We have computed the upper convex hull



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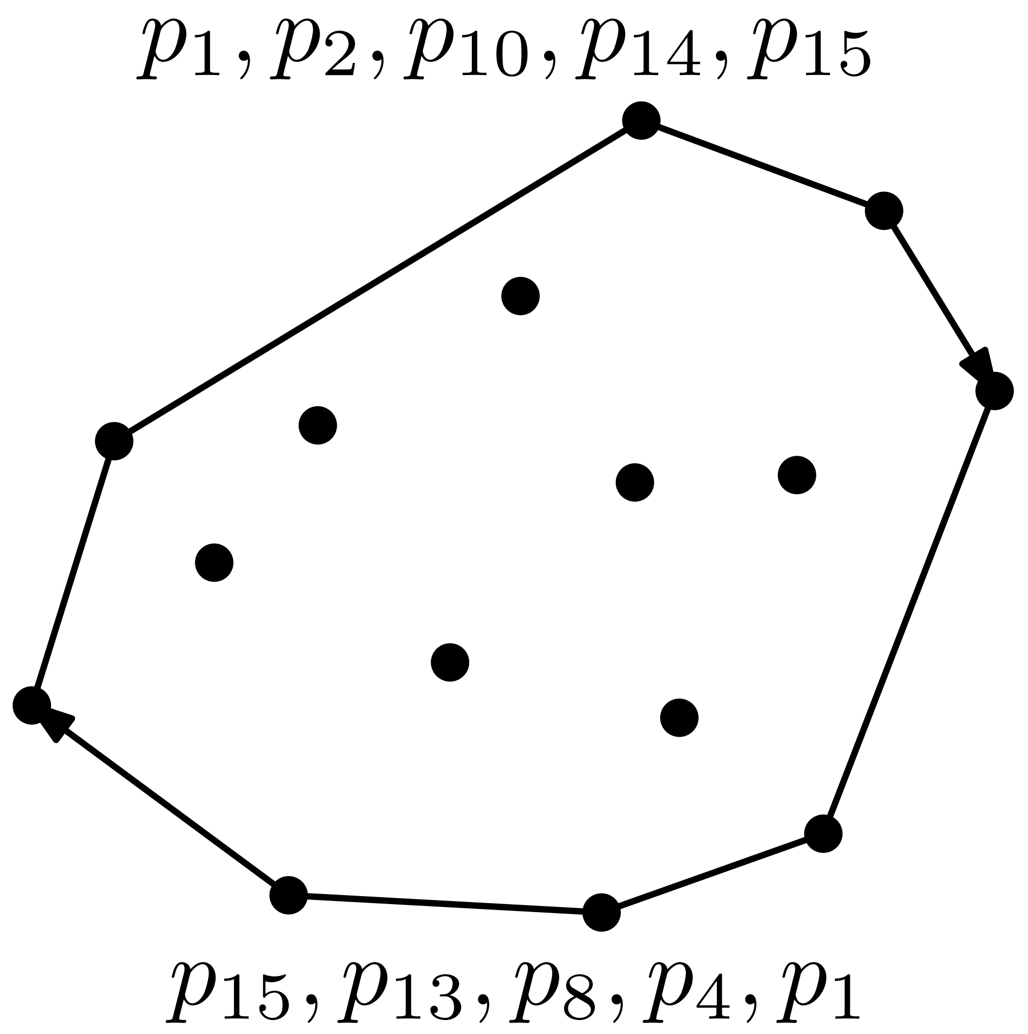
We have computed the upper convex hull

Then we do the same for the lower convex hull, from right to left



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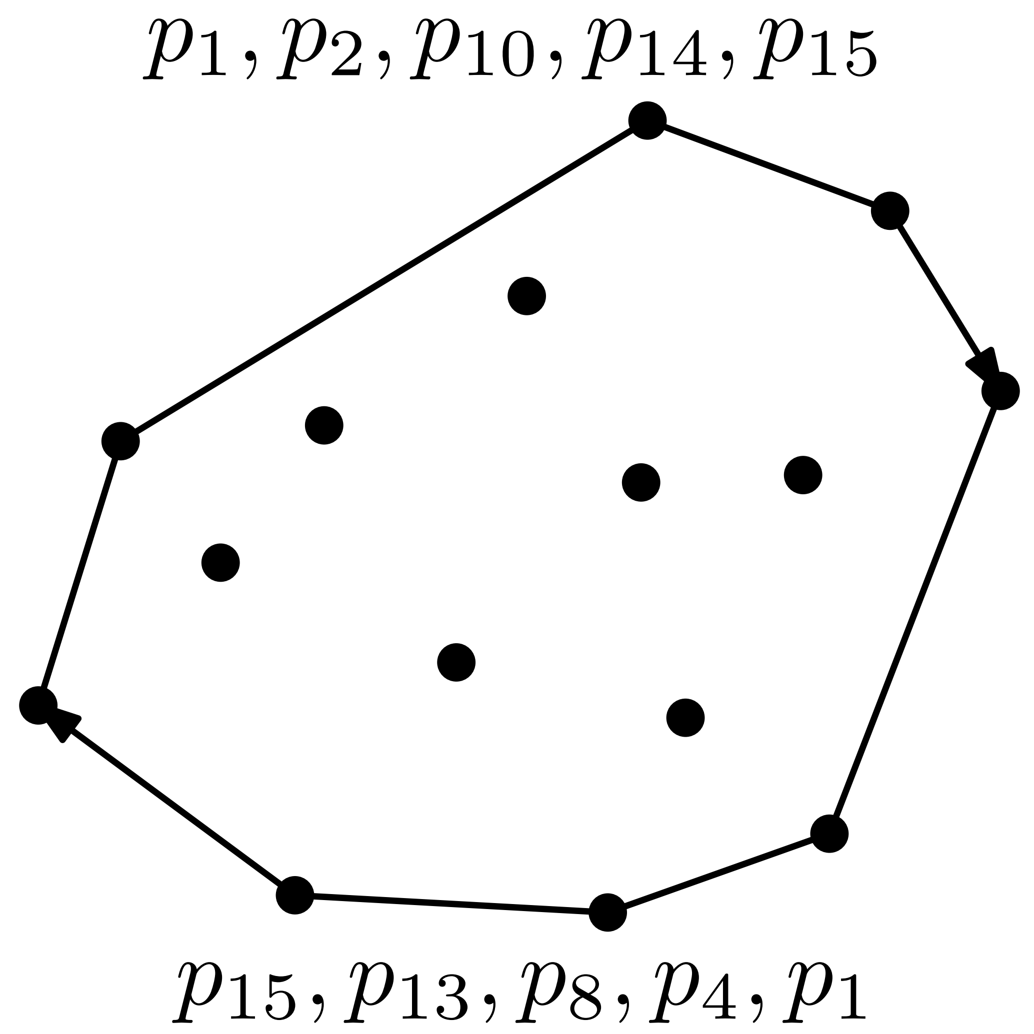
We remove the first and last points of the lower convex hull



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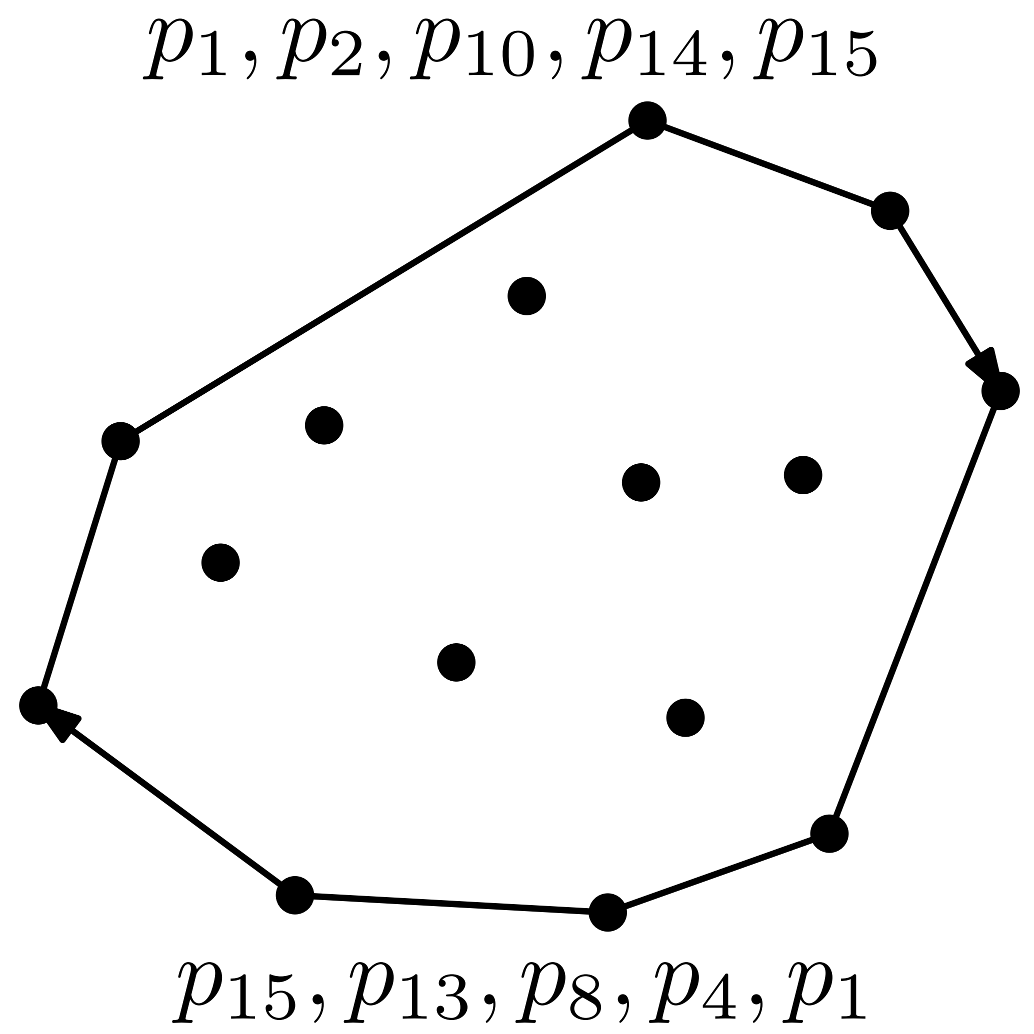
We remove the first and last points of the lower convex hull . . . and concatenate the two lists in one



Algorithm analysis

Algorithm analysis generally has two components

- proof of correctness
- efficiency analysis, proof of running time

Correctness

Are the general observations on which the algorithm is based correct?

Does the algorithm handle degenerate cases correctly?

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Does the algorithm handle degenerate cases correctly?

Here:

- Does the sorted order matter if two or more points have the same x-coordinate?
- What happens if there are three or more collinear points, in particular on the convex hull?

Efficiency

For each line of pseudocode identify

- how much time it takes
- how many times it is executed once

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Sometimes there are global arguments why an algorithm is more efficient than it seems at first

Efficiency

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1: sort the points by x-coordinate, resulting in a sequence $p_1, \ldots, p_n$
2: put the points p_1 and p_2 in a list L_{upper}, with p_1 as the first point
3: **for** $i \leftarrow 3$ to n **do**
4: append p_i to L_{upper}
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    delete the middle of the last three points from L_{upper} $\quad O(1)$ $\}\times k_i$

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1: sort the points by x-coordinate, resulting in a sequence $p_1, \ldots, p_n$ — $O(n \log n)$
2: put the points p_1 and p_2 in a list L_{upper}, with p_1 as the first point — $O(1)$
3: **for** $i \leftarrow 3$ to n **do** — $\times n-2$
4: append p_i to L_{upper} — $O(1)$
5: **while** L_{upper} contains more than two points **and** the last three points do not make a right turn **do**
6: delete the middle of the last three points from L_{upper} — $O(1)$ $\}\times k_i$

Total time:

$$O(n \log n) + O(1) + \sum_{i=3}^{n} (O(1) + k_i \cdot O(1)) =$$

Efficiency

Algorithm GRAHAMSCAN(P)

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$\}\times n-2$ (lines 4–6)

Total time:

$$O(n \log n) + O(1) + \sum_{i=3}^{n} (O(1) + k_i \cdot O(1)) = O(n \log n) + \sum_{i=3}^{n} O(1 + k_i)$$

Efficiency: attempt 1

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Question: Is this analysis tight?

Efficiency: attempt 1

Total time: $$O(n \log n) + \sum_{i=3}^{n} O(1 + k_i)$$

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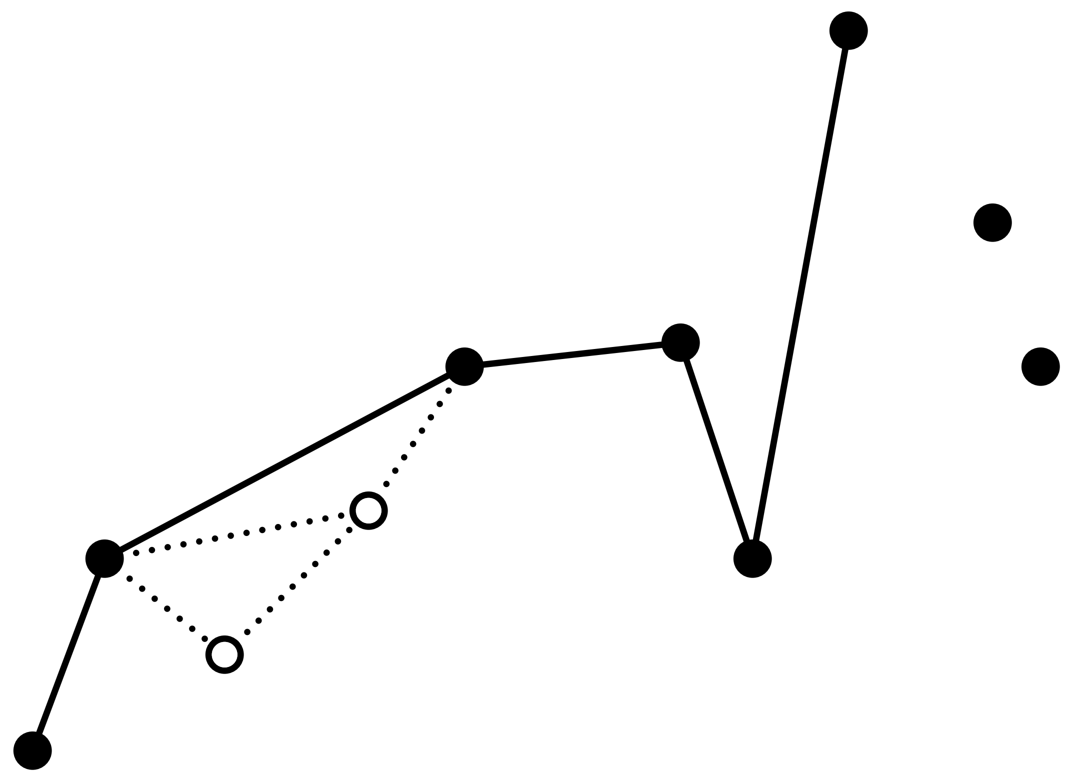
Question: Is this analysis tight?

Sometimes there are global arguments why an algorithm is more efficient than it seems at first

Efficiency: attempt 2

Total time: $$O(n \log n) + \sum_{i=3}^{n} O(1 + k_i)$$



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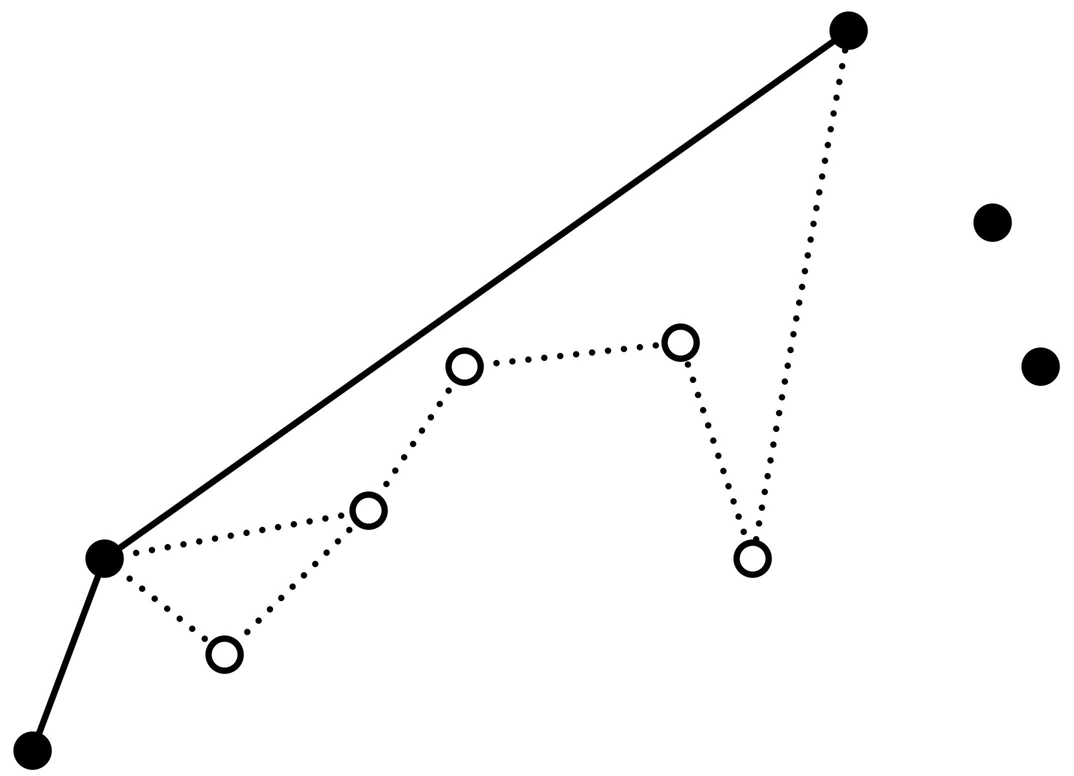

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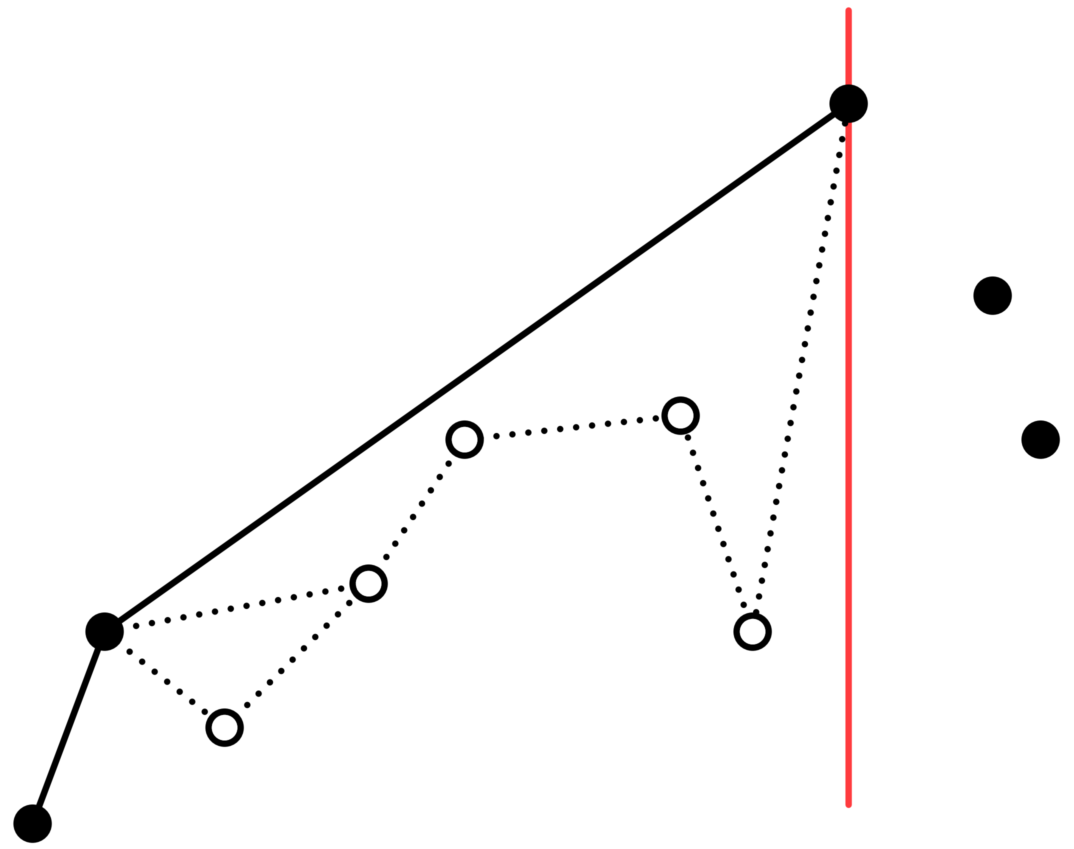

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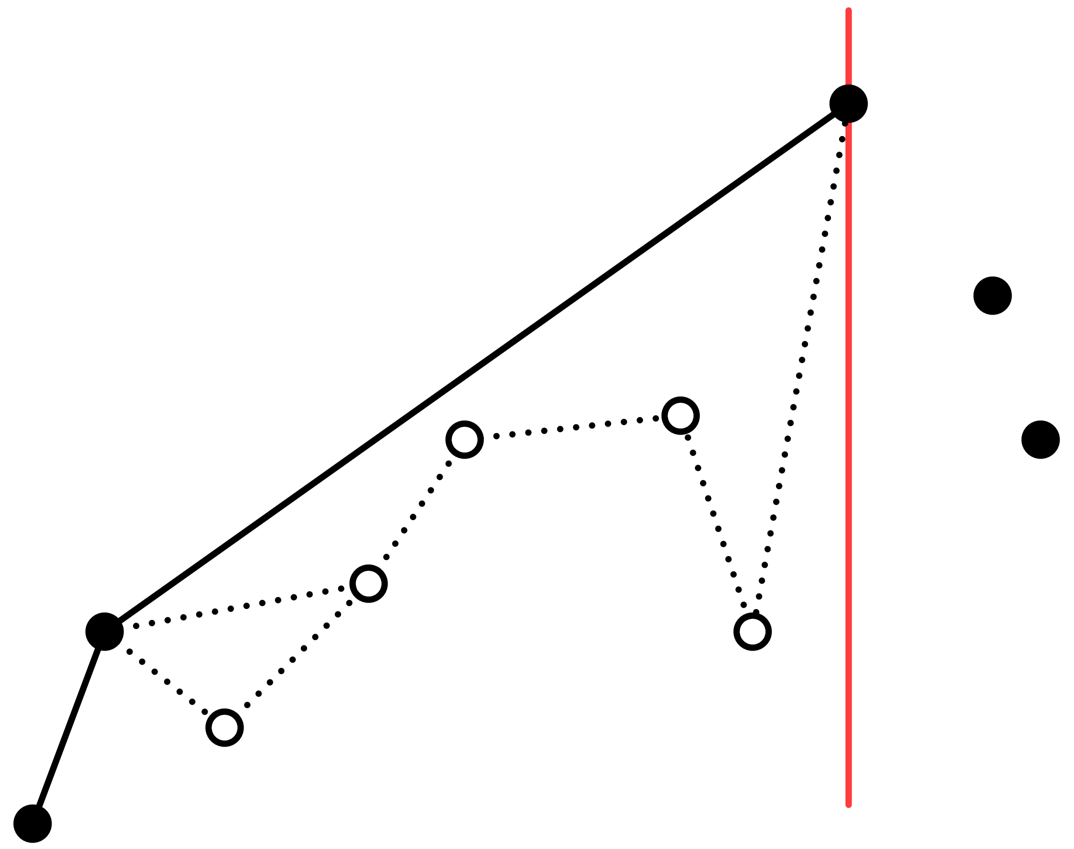
Global argument: each point can be removed only once from the upper hull



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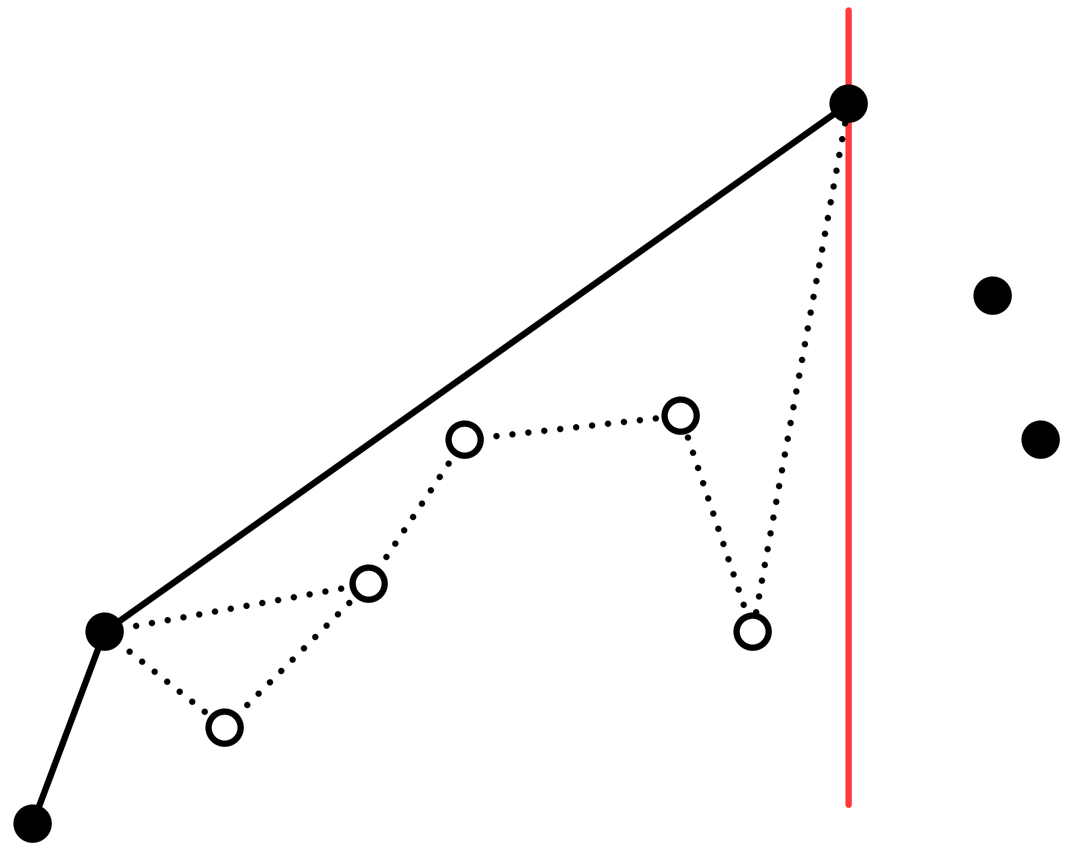

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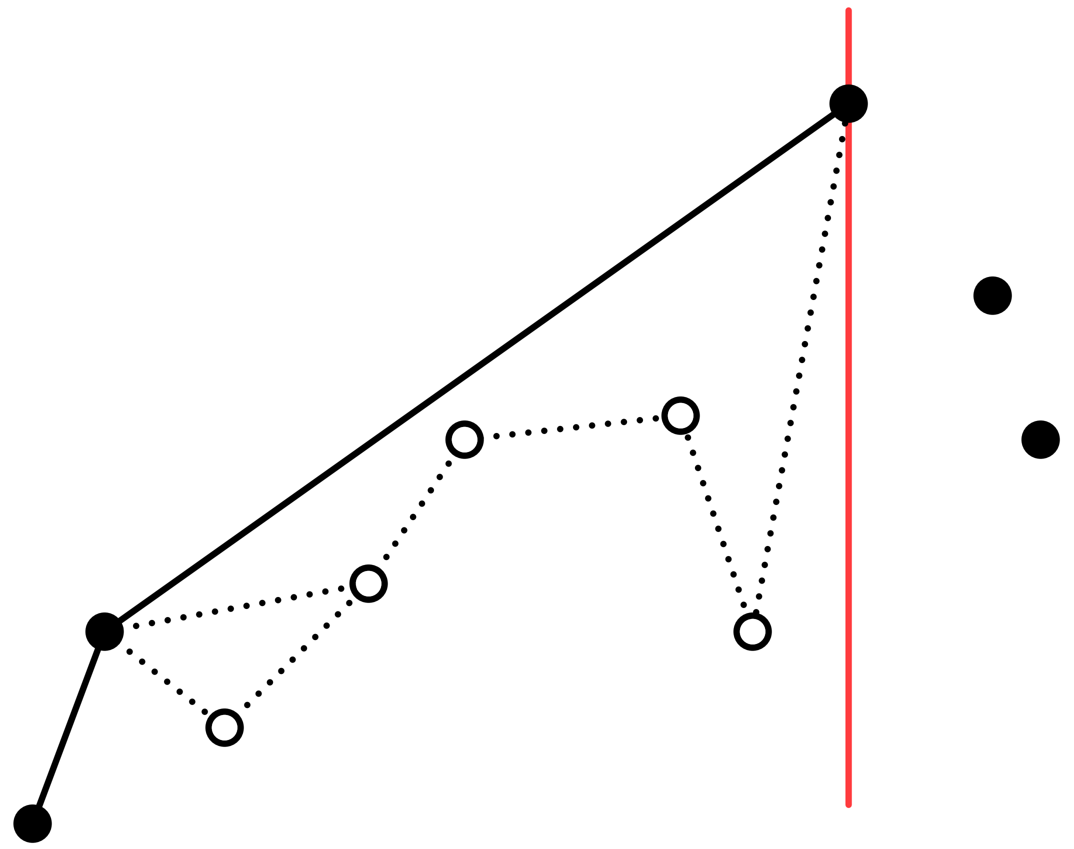

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Hence,

$$O(n \log n) + \sum_{i=3}^{n} O(1 + k_i) = O(n \log n) + O(n) = O(n \log n)$$

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The convex hull of a set of n points in the plane can be computed in $O(n \log n)$ time, and this is worst-case optimal.

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Question: Can we do better?

$O(nh)$ if the convex hull has h vertices?
$O(n \log h)$? Yes we can!

An output-sensitive algorithm

Idea: start on the convex hull and wrap around the convex hull

An output-sensitive algorithm

Algorithm GIFTWRAPPING(P)

Input: set P of points in the plane

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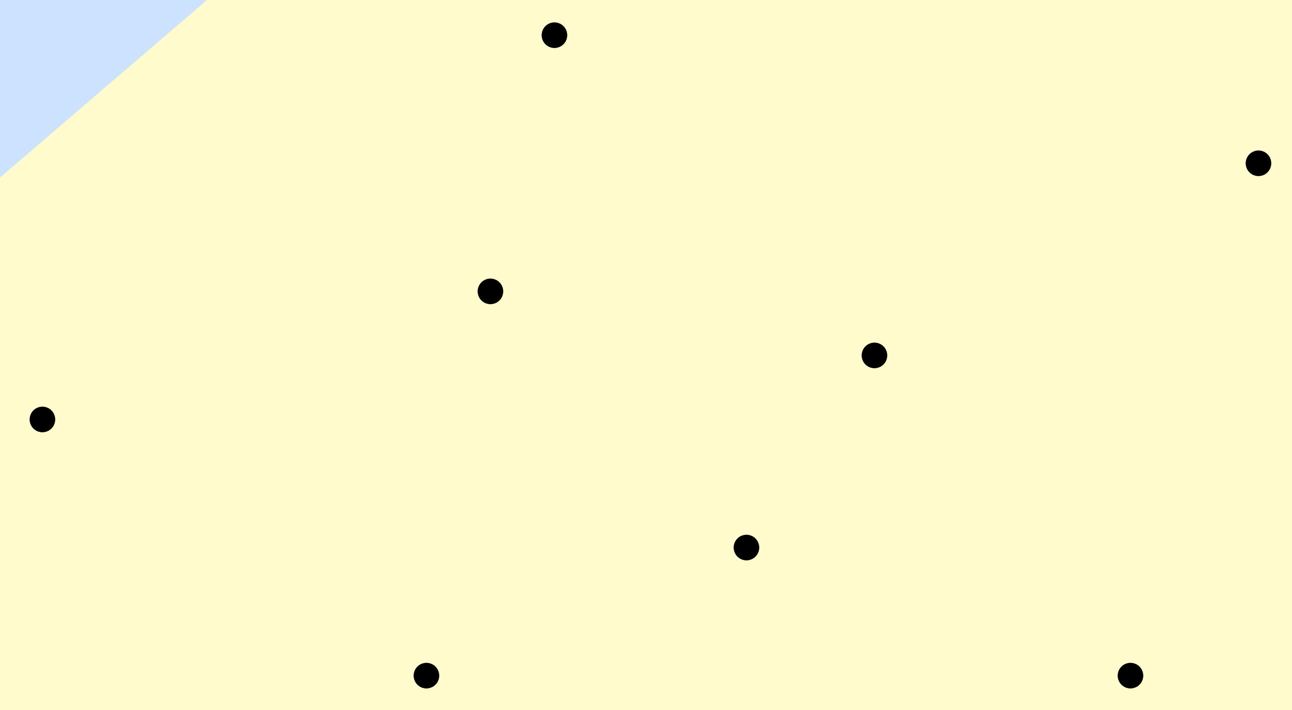

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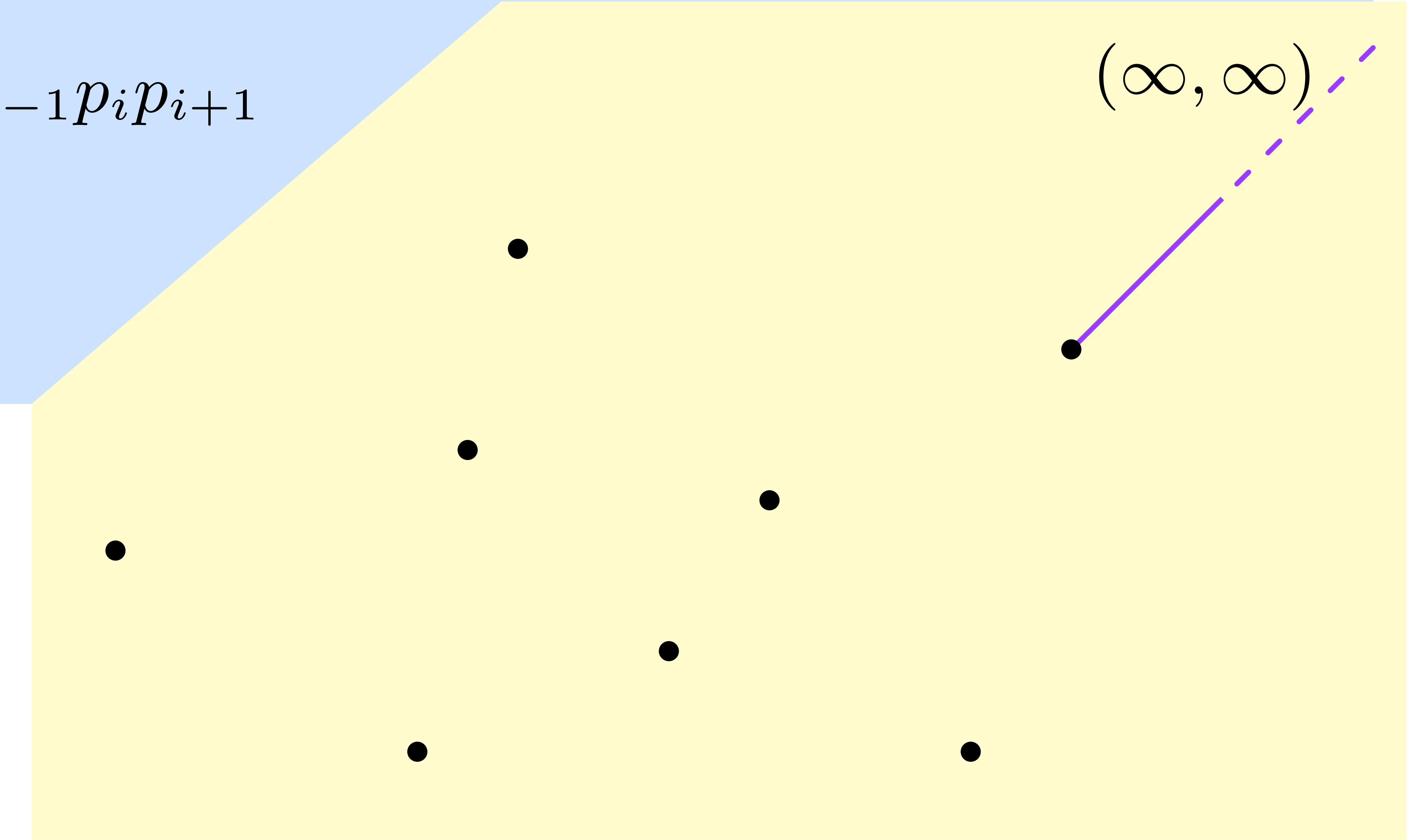

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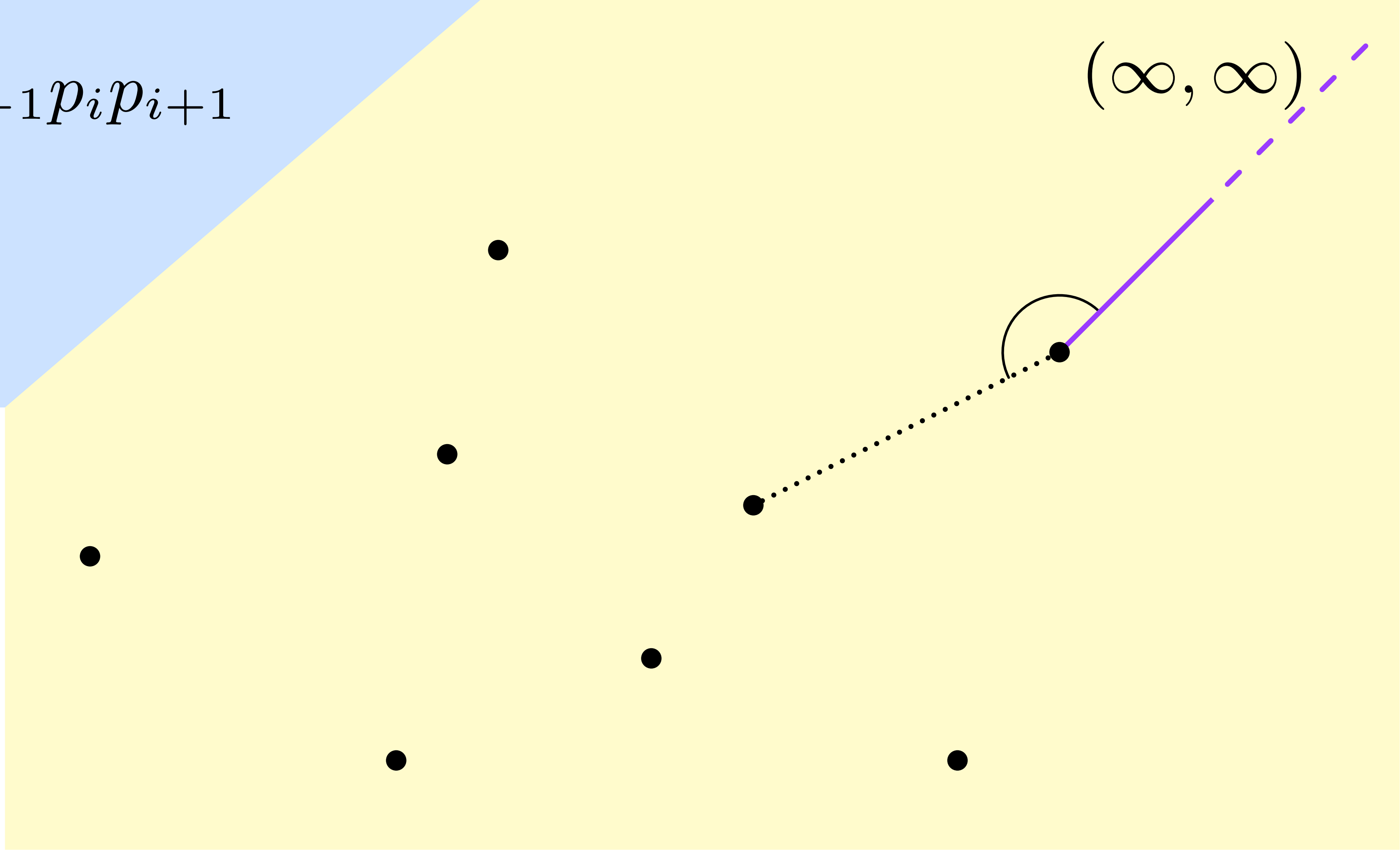

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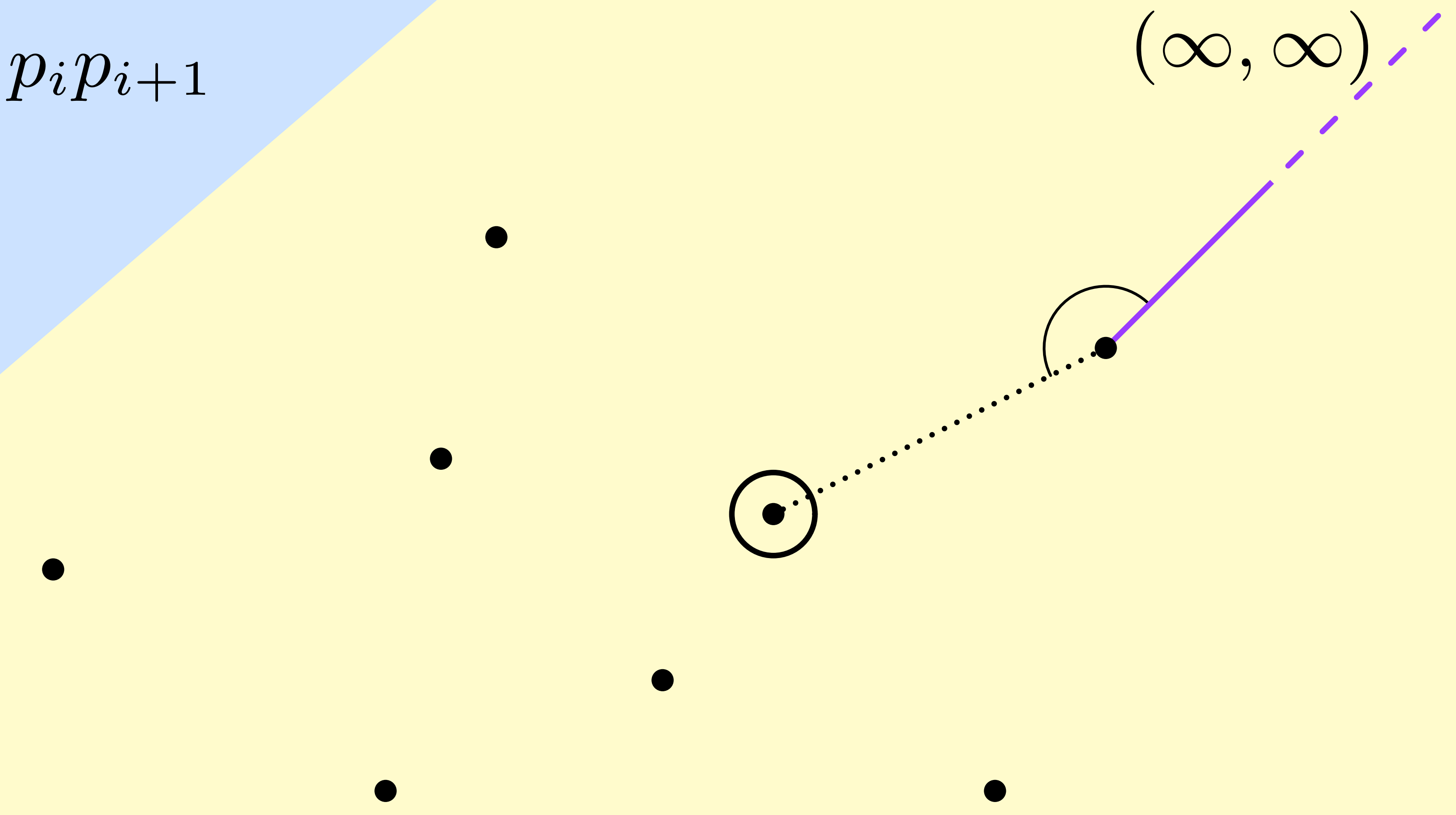

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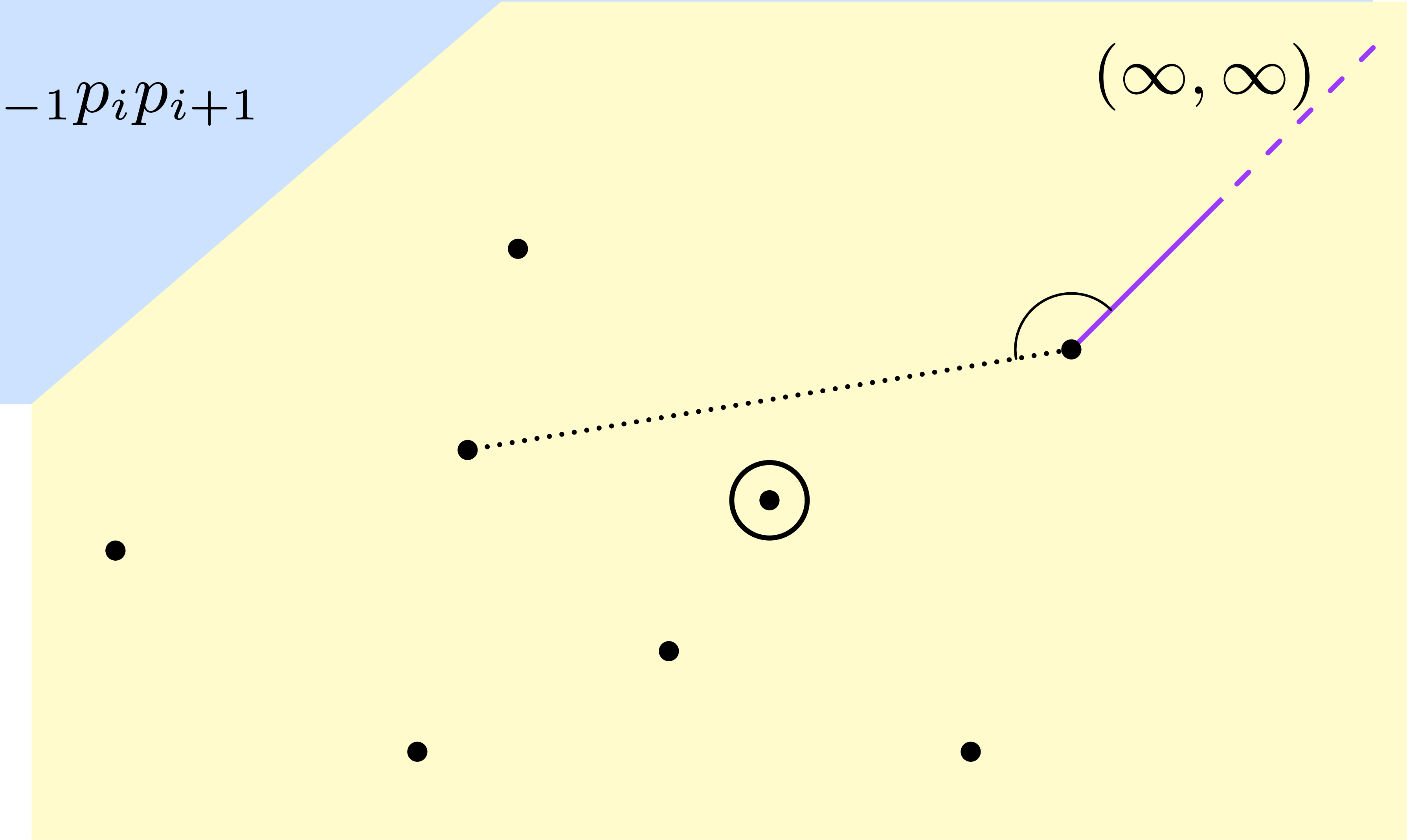

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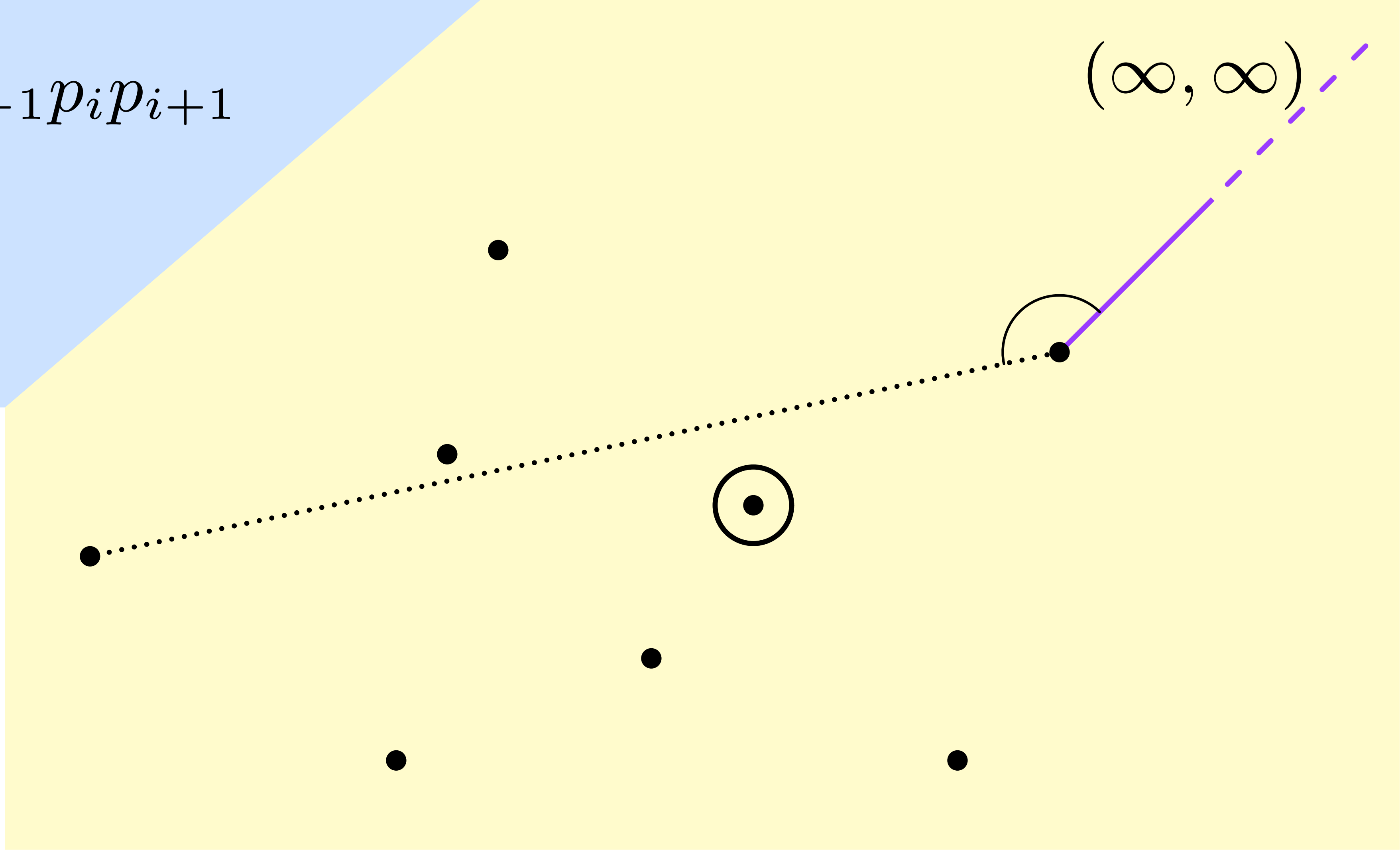

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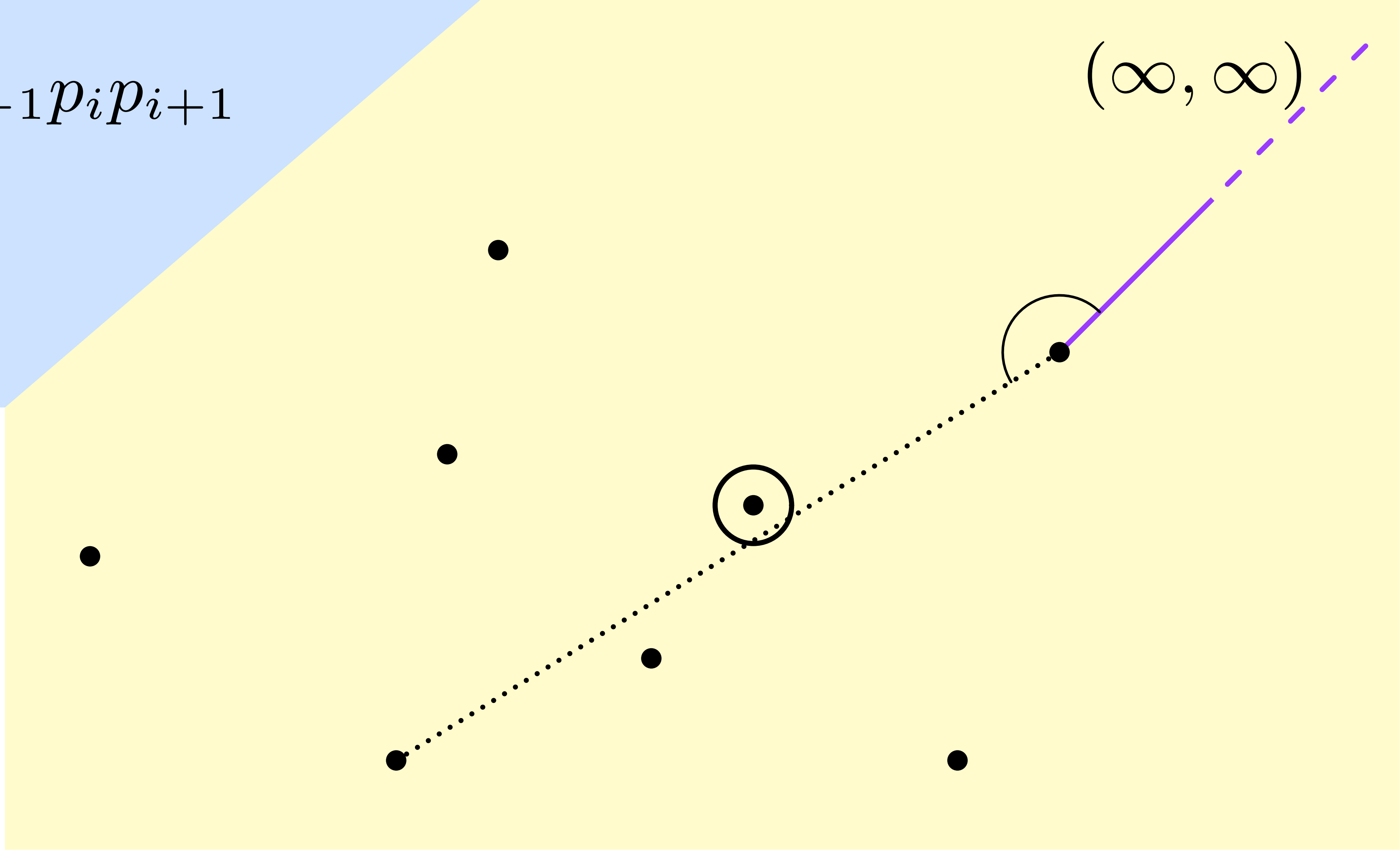

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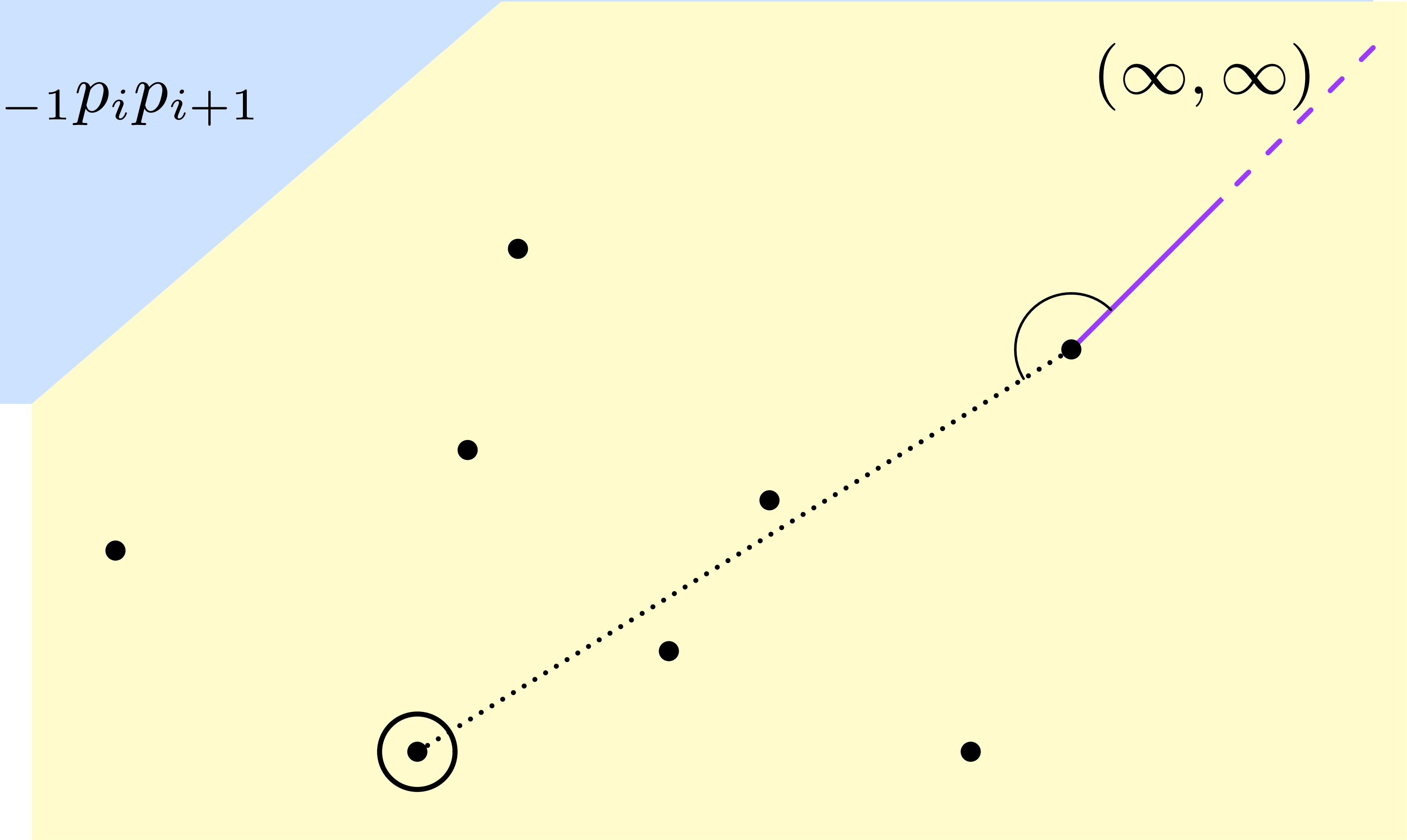

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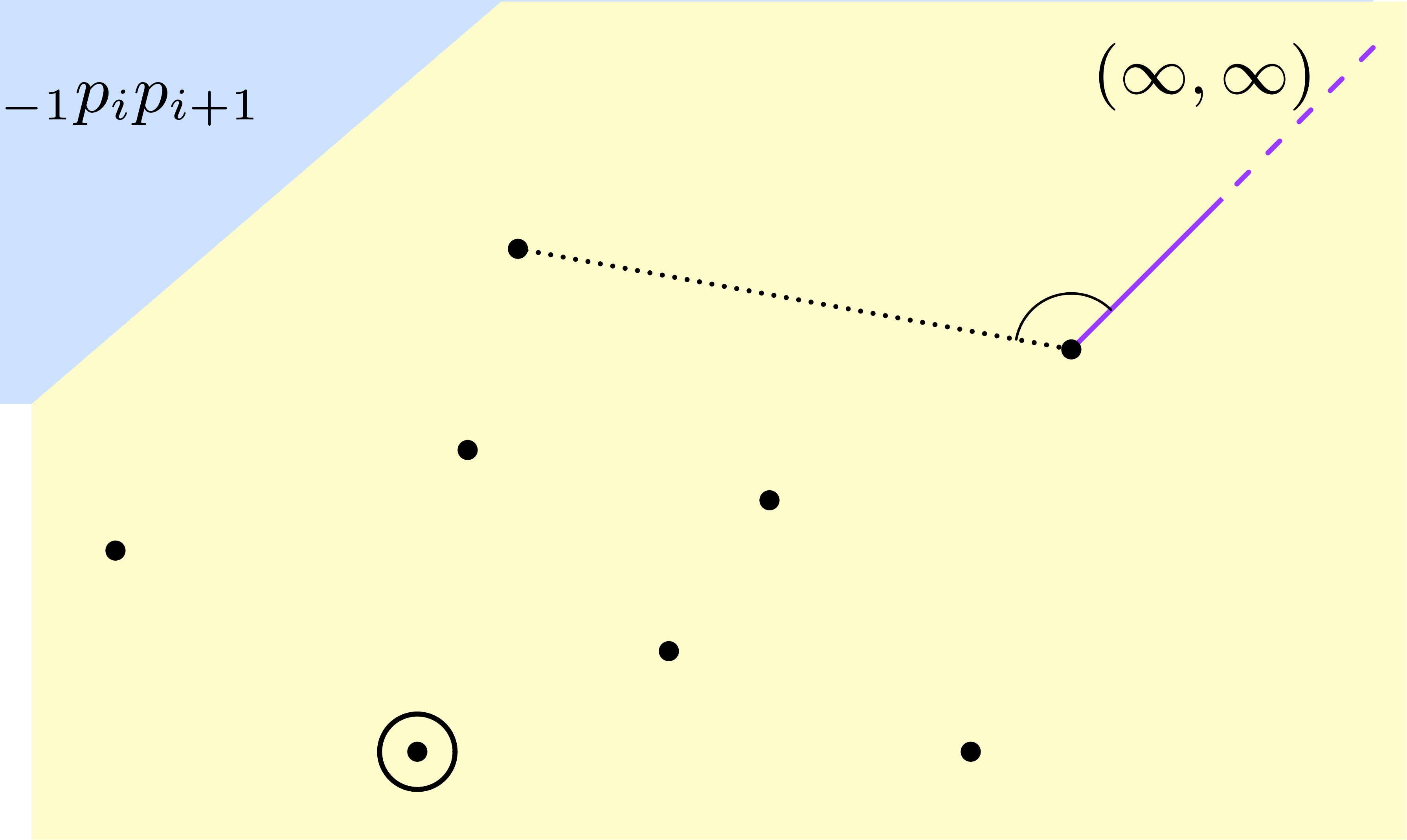

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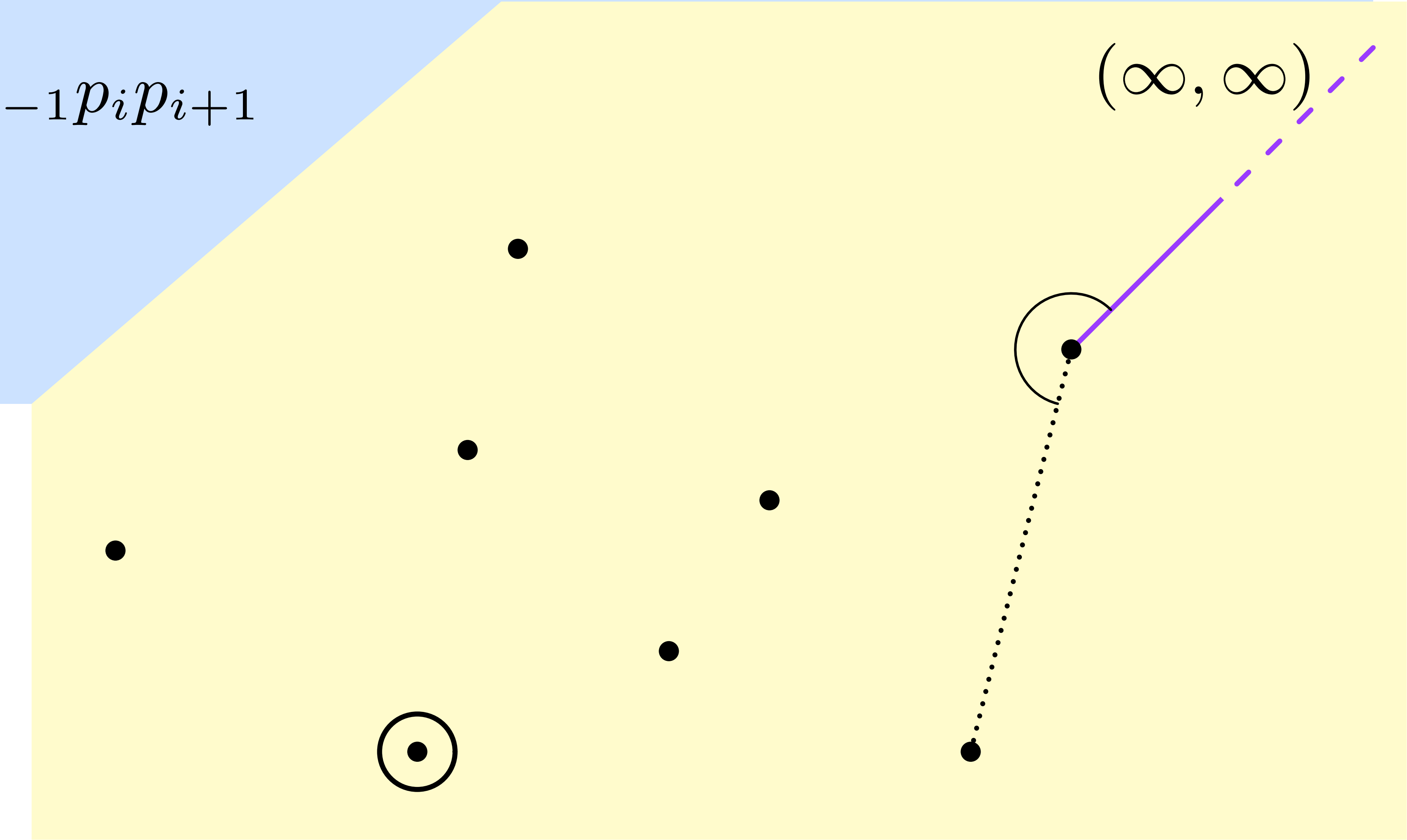

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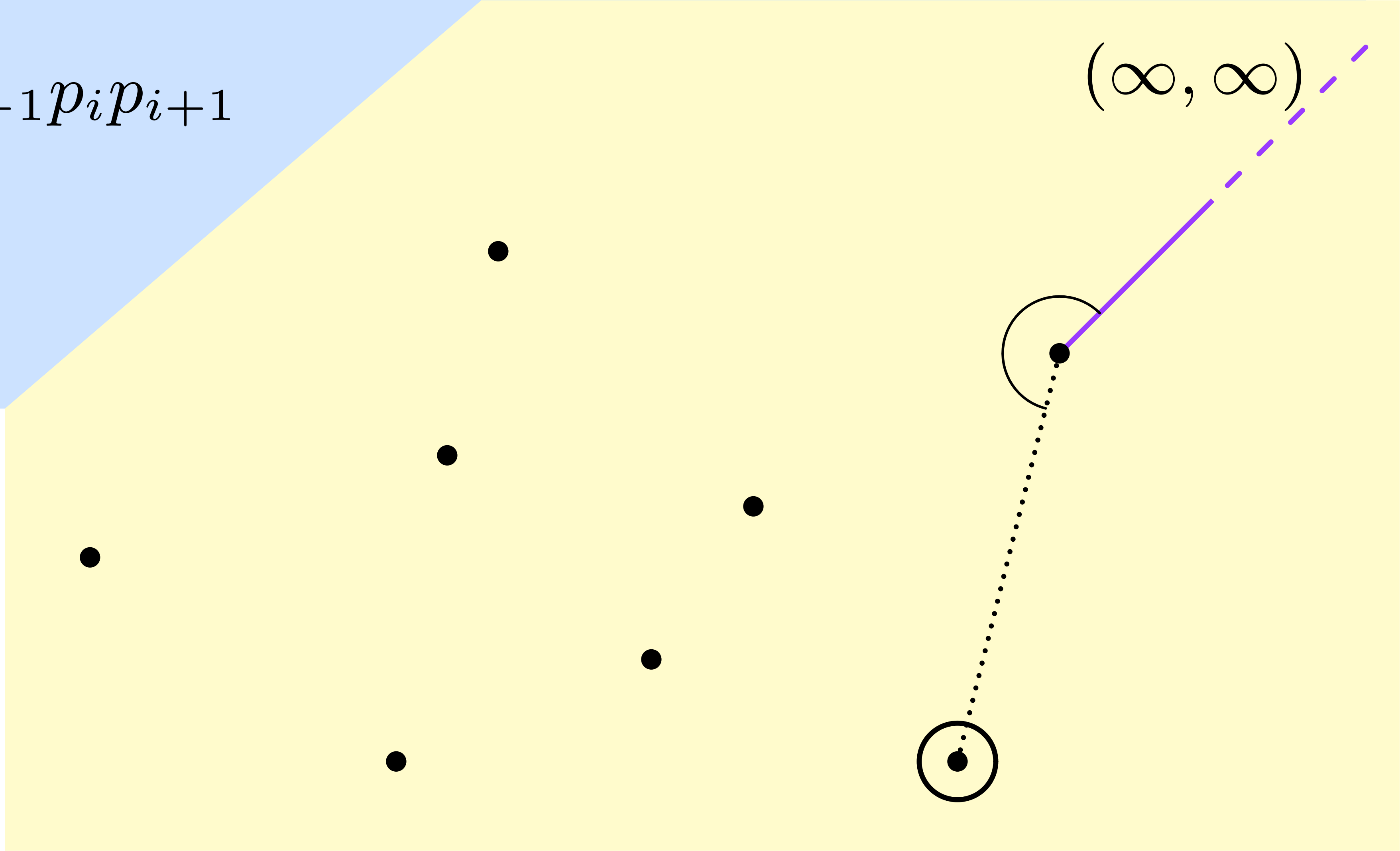

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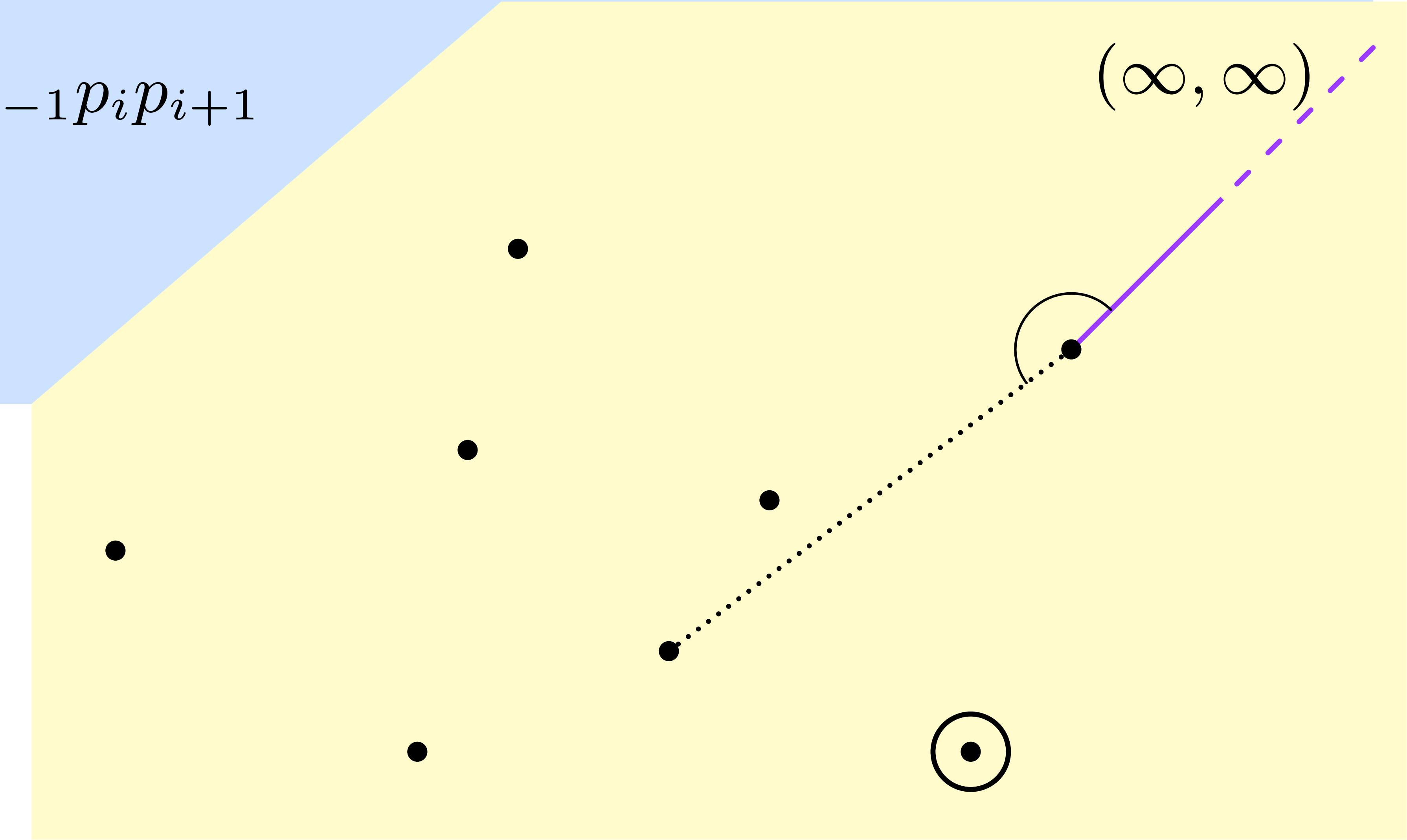

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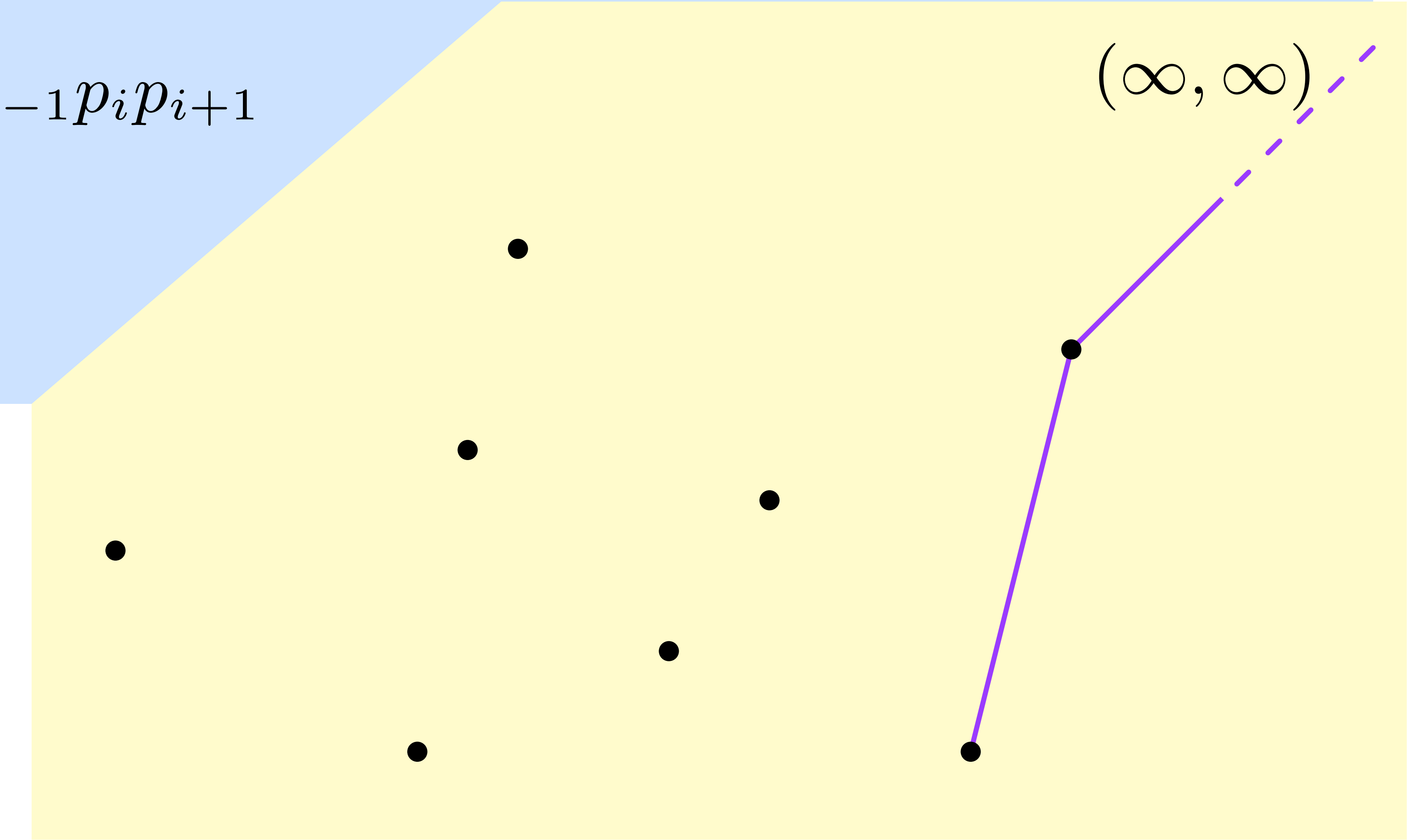

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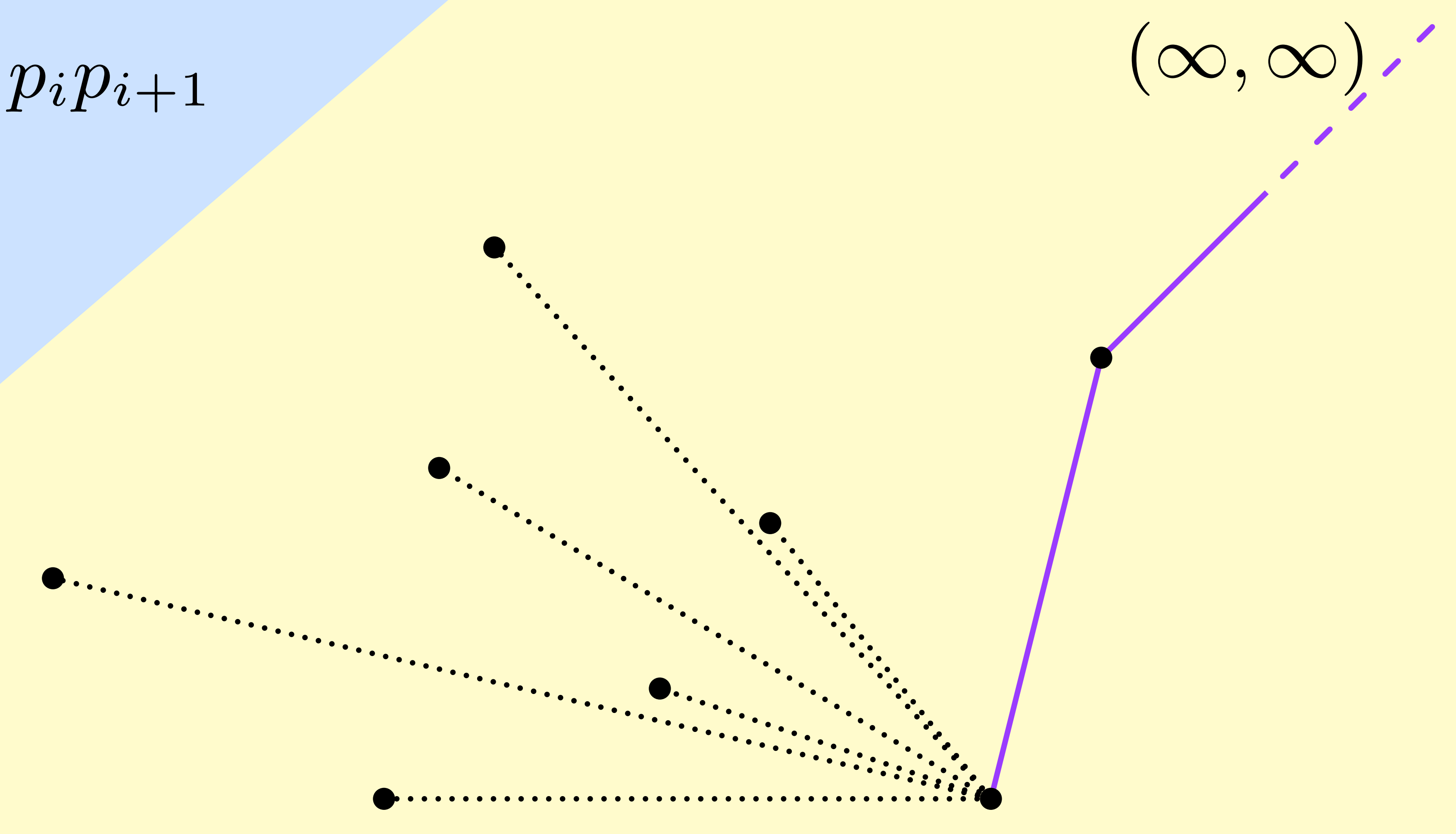

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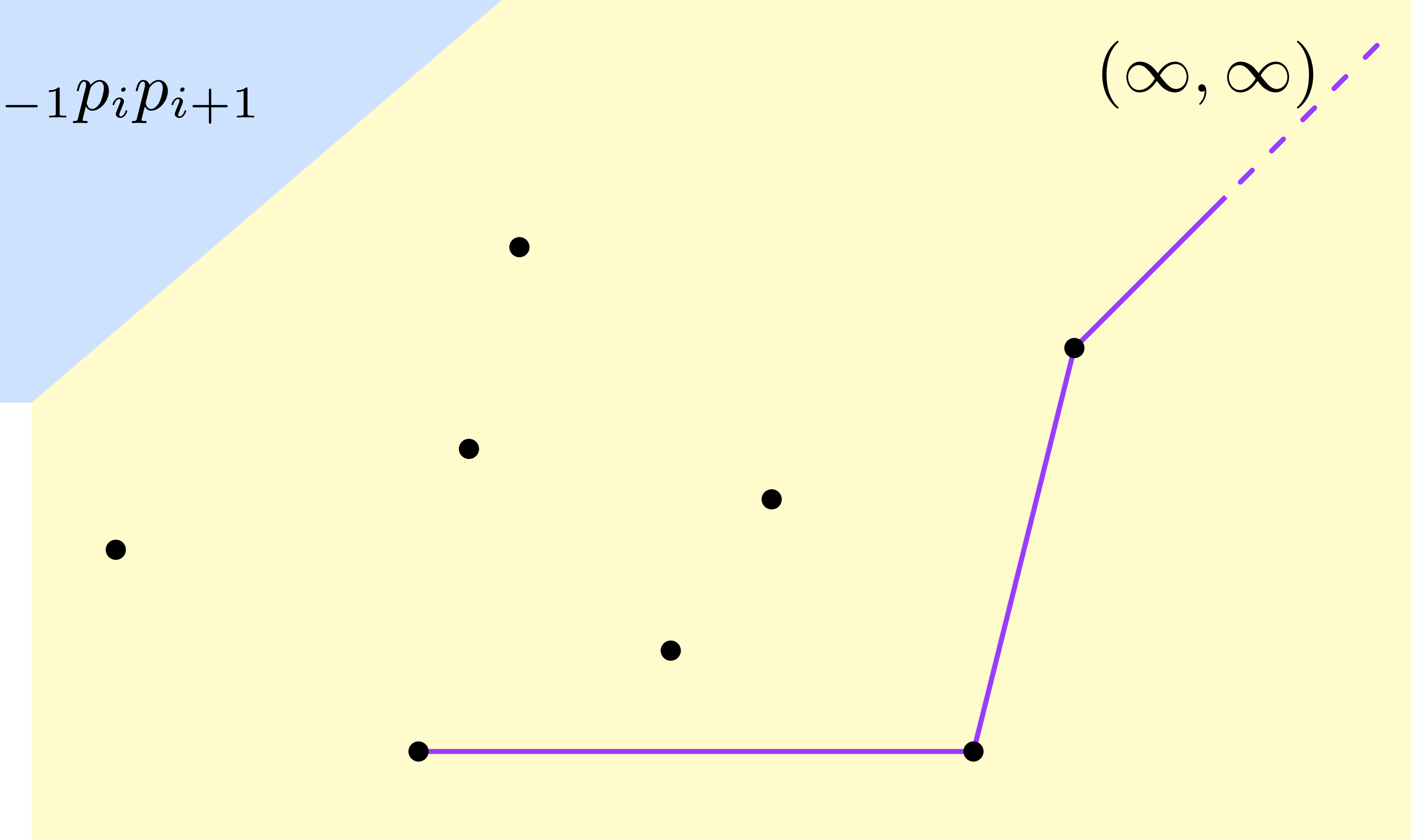

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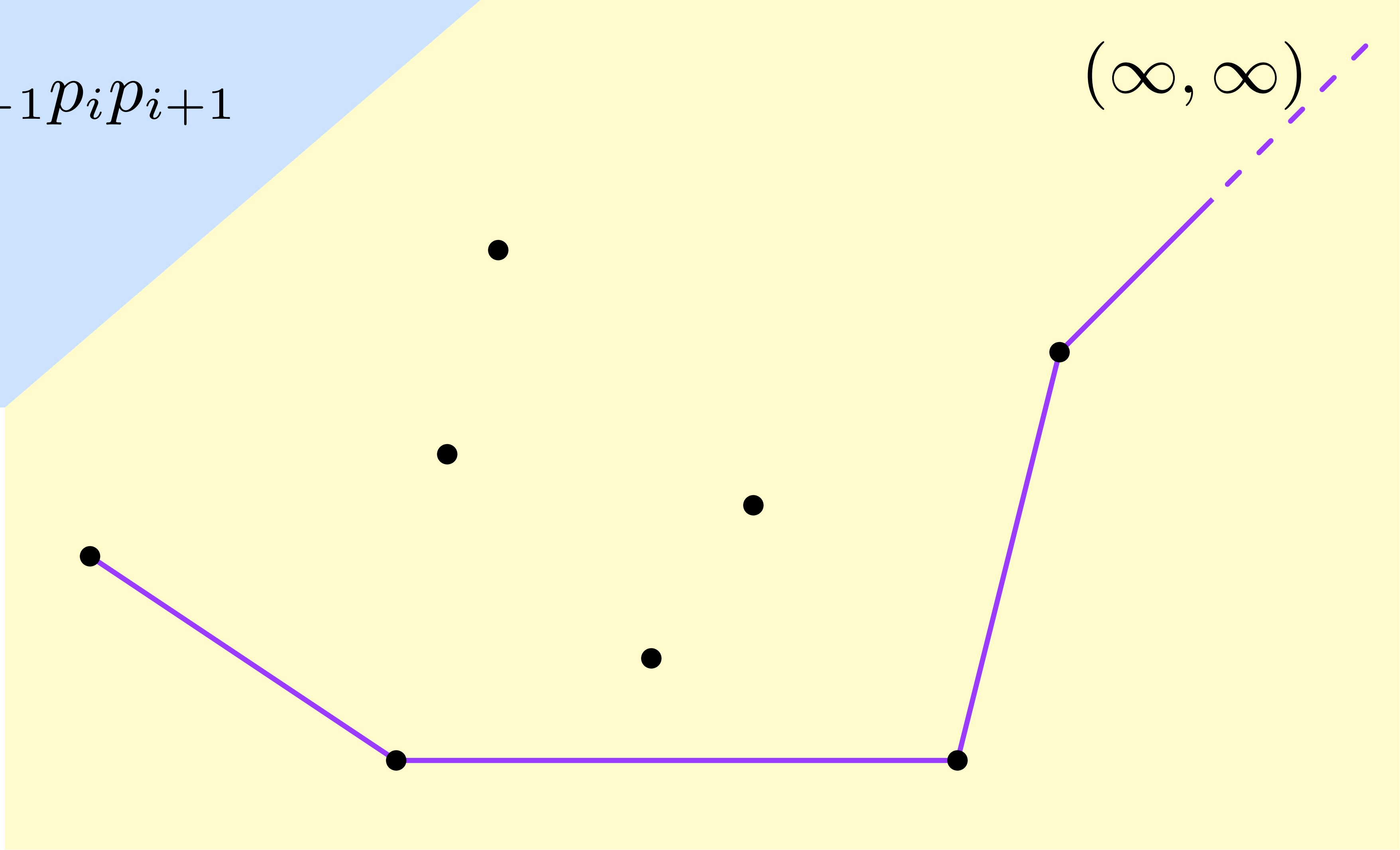

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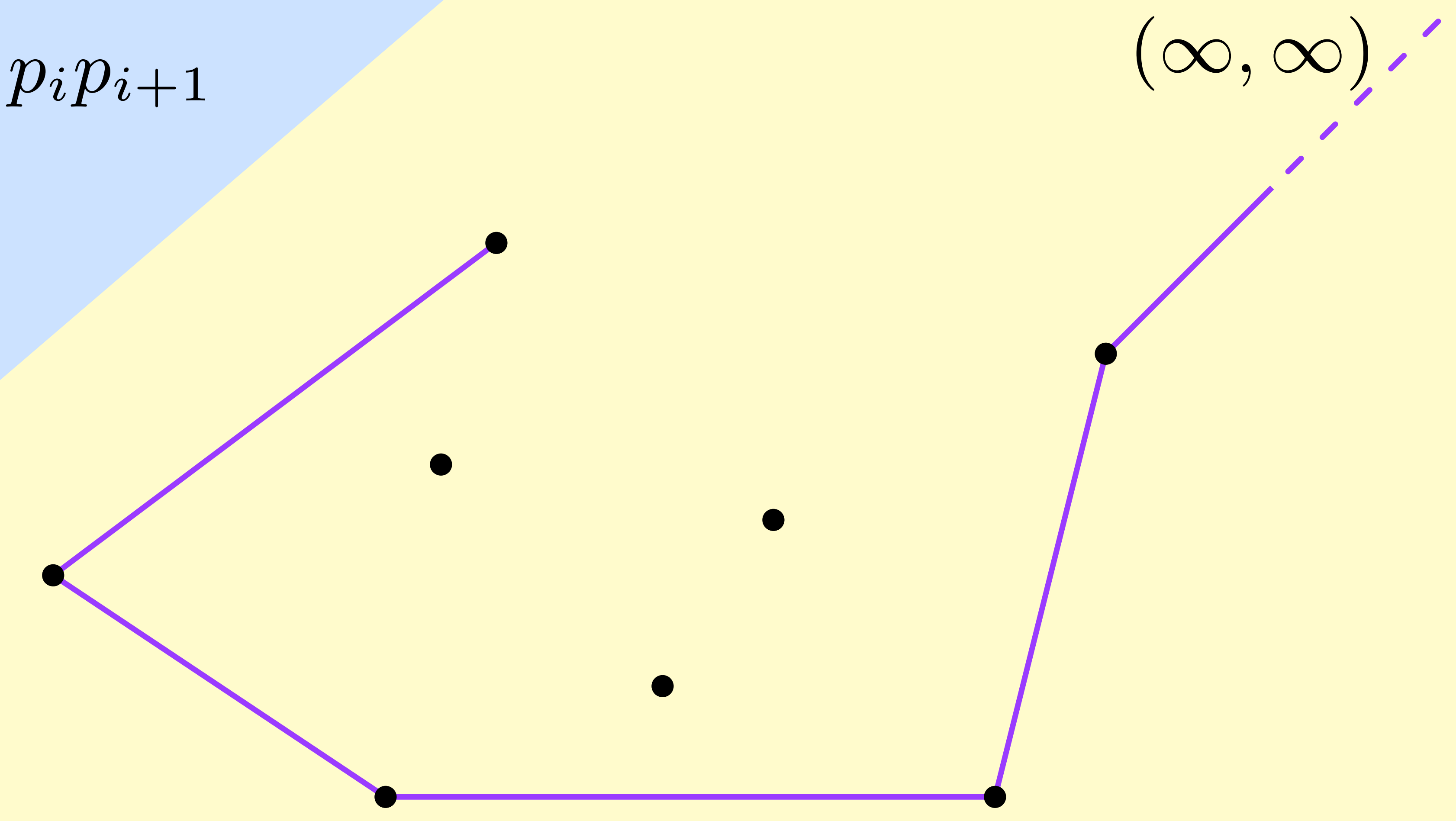

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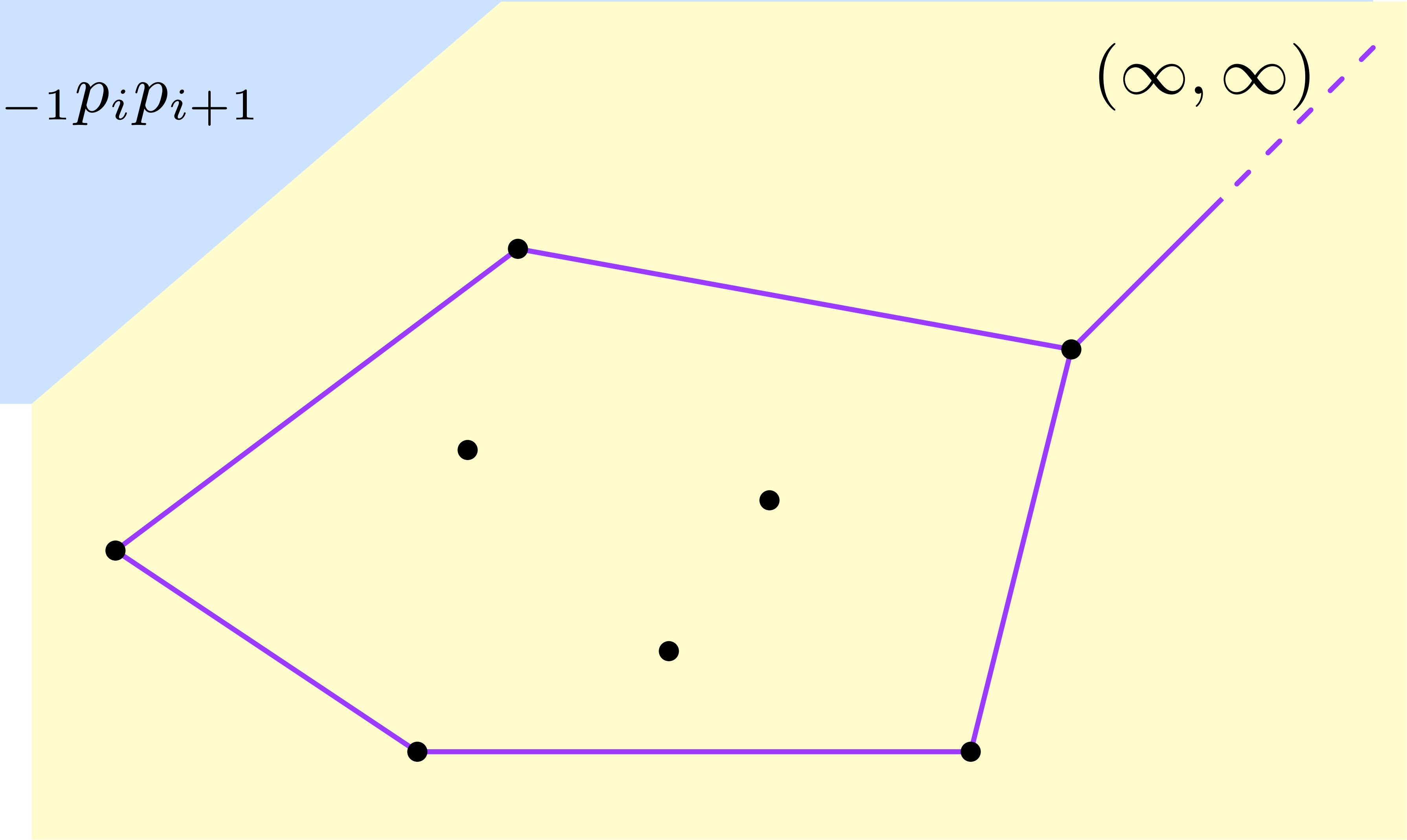

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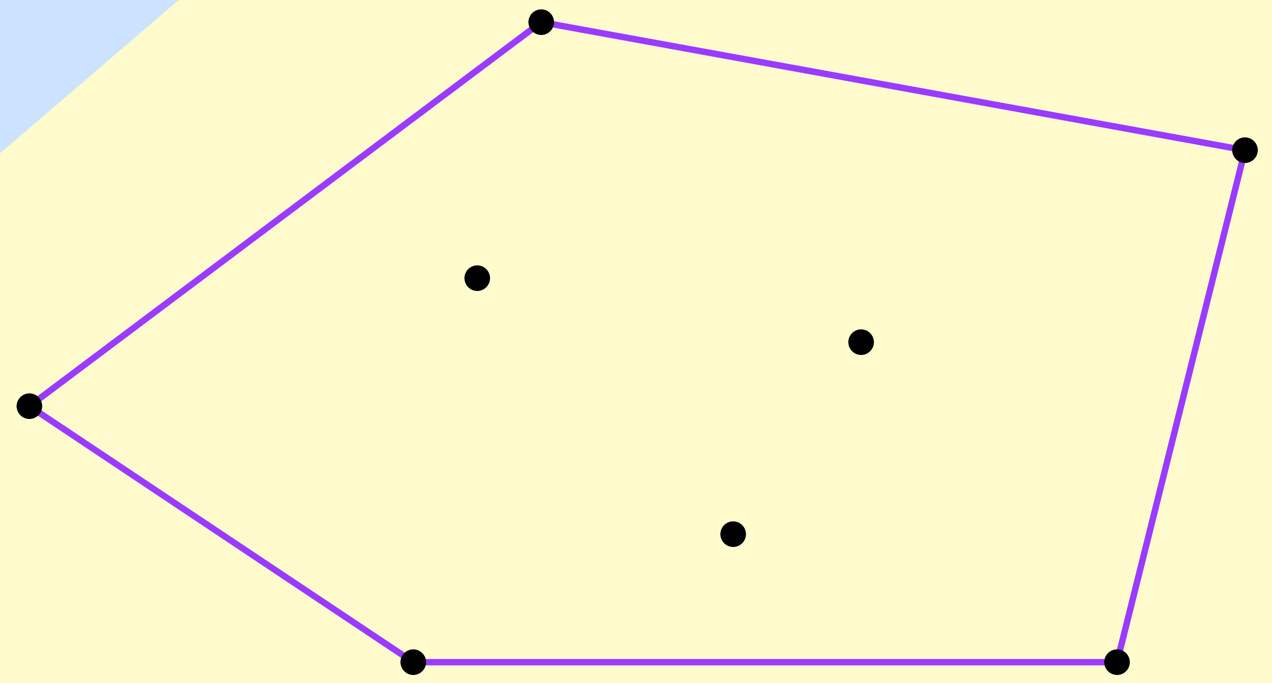

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- initialization: p_1 is on the convex hull
- maintenance: by construction, all points lie to the right of the line p_i and p_{i+1}

Gift wrapping algorithm: efficiency

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Input: set P of points in the plane

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1: $p_0 \leftarrow (\infty, \infty)$, $p_1 \leftarrow$ right-most vertex in P, insert p_1 into L
2: **while** $true$ **do**
3:     choose $p_{i+1} \in P$ maximizing $\angle p_{i-1} p_i p_{i+1}$
4:     **if** $p_{i+1} = p_1$ **then**
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Gift wrapping algorithm: efficiency

Algorithm GIFTWRAPPING(P)

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    if p_{i+1} = p_1 then
        break
    insert p_{i+1} into L
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$\times h$

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  insert p_{i+1} into L — $O(1)$

$\times h$

Total time:

$$O(n) + h \cdot (O(n) + O(1) + O(1)) = O(hn)$$

Optimality?

When should we choose which algorithm?

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- many points on $CH(P)$: $O(n \log n)$ Graham Scan
- few points on $CH(P)$: $O(nh)$ Gift Wrapping

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Can we do better?

Optimality?

When should we choose which algorithm?

- many points on $CH(P)$: $O(n \log n)$ Graham Scan
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Can we do better? yes, by combining both algorithms

Chan’s Algorithms

Chan's Algorithms

Chan's Algorithms

Suppose we know h:

Chan's Algorithms

Suppose we know h:

Algorithm ChanHull(P,h)

```
partition P into sets P_i with h points each
for i ← 1 to n/h do
    L_i ← GrahamScan(P_i)
p_0 ← (∞, ∞), p_1 ← right-most point in P, insert p_1 into L
for j ← 1 to h − 1 do
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    p_{j+1} ← max_∠{q_1, ..., q_{n/h}}
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$O(h \log h)$ $\}\times n/h$ $\}\ O(n \log h)$

$O(\log h)$

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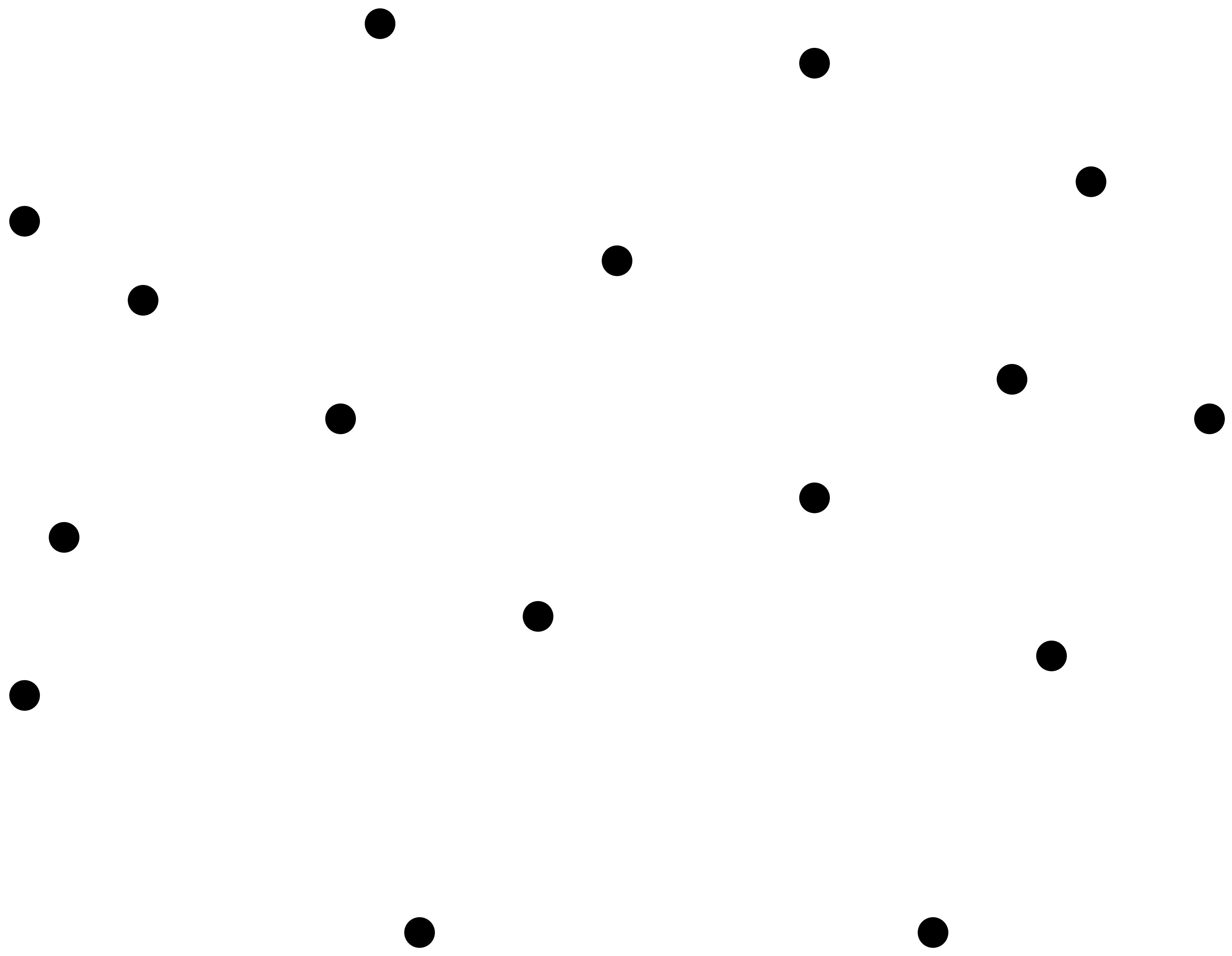
return L

Total running time: $O(n \log h)$

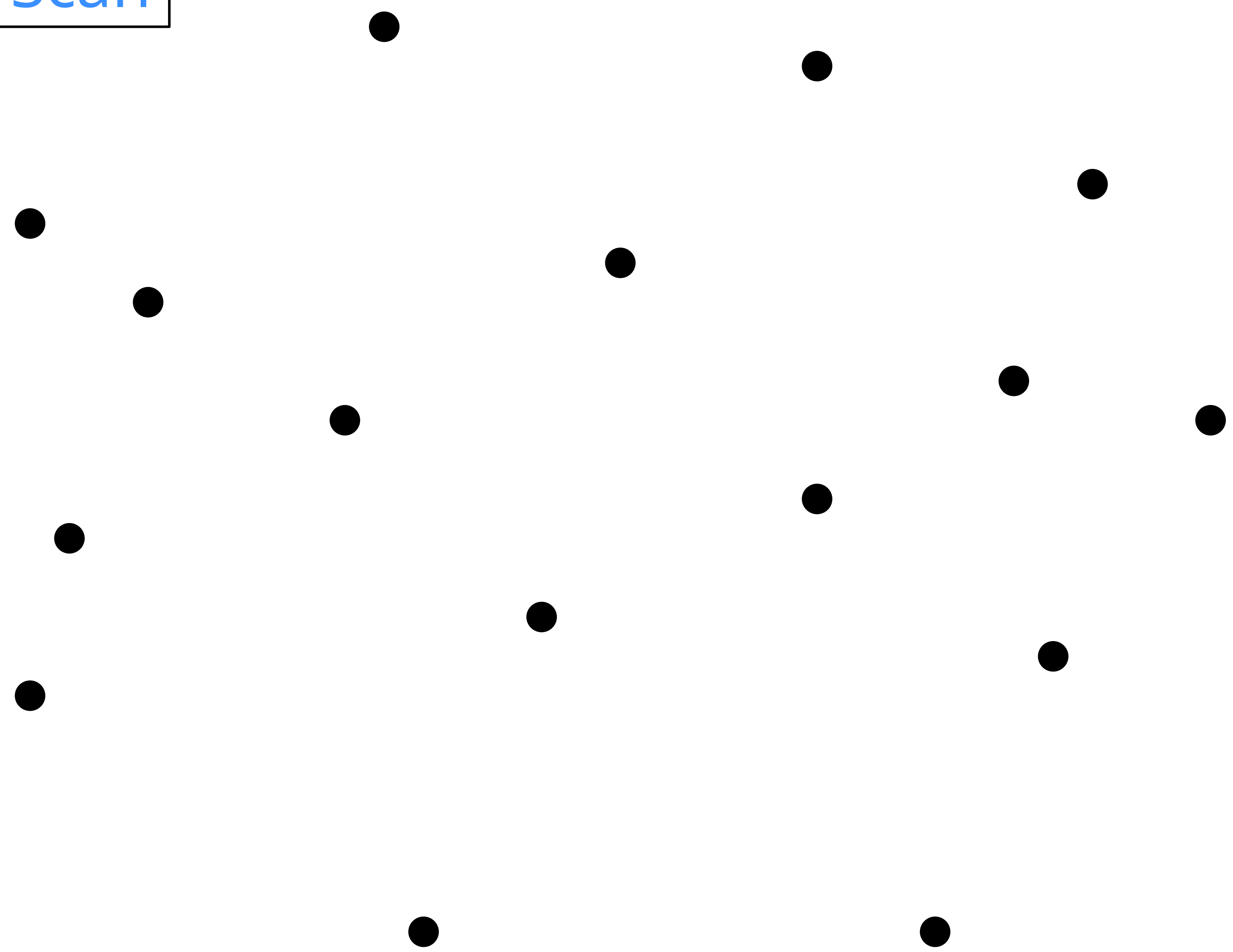
Example



$n = 16$

Example

Graham Scan



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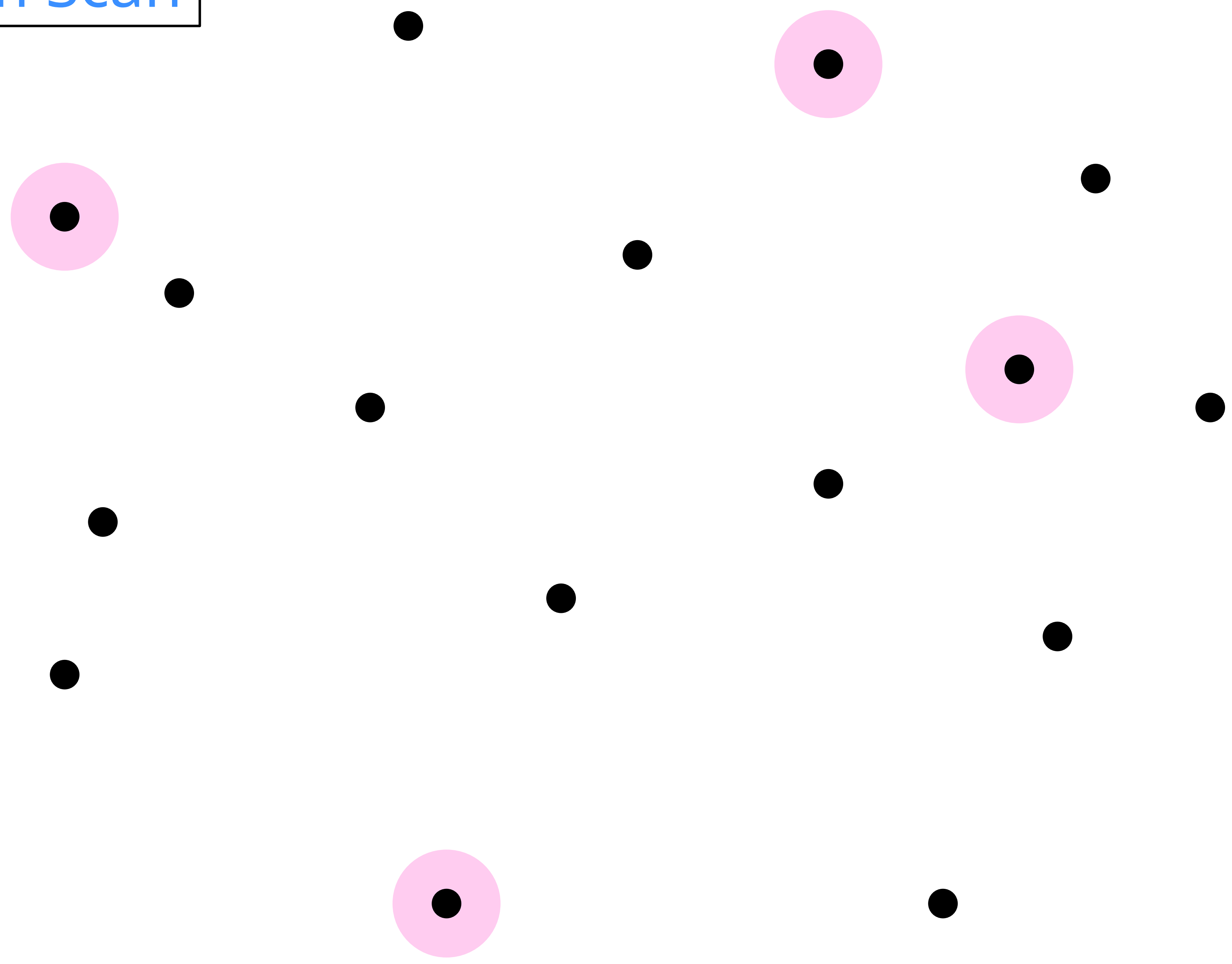

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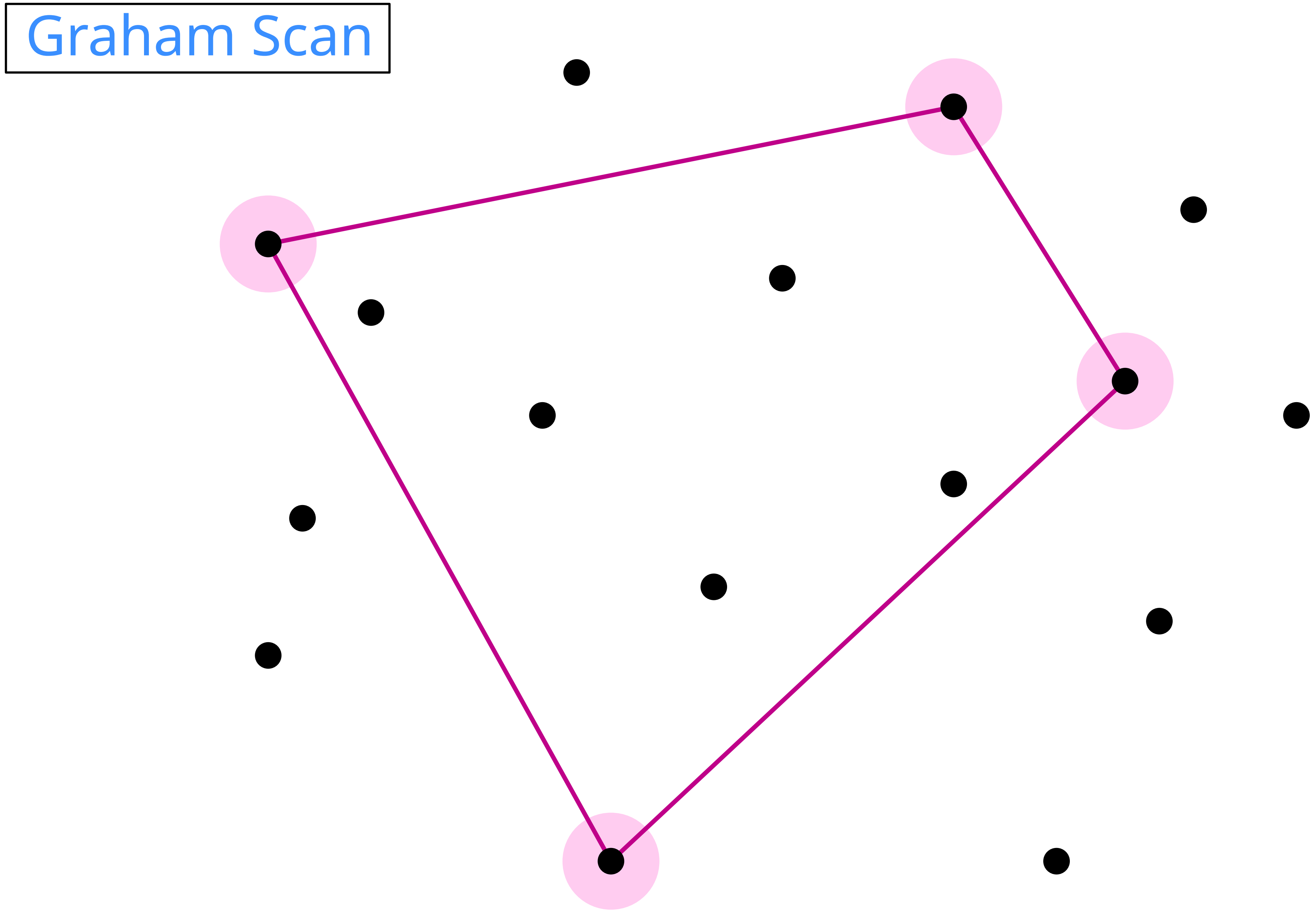

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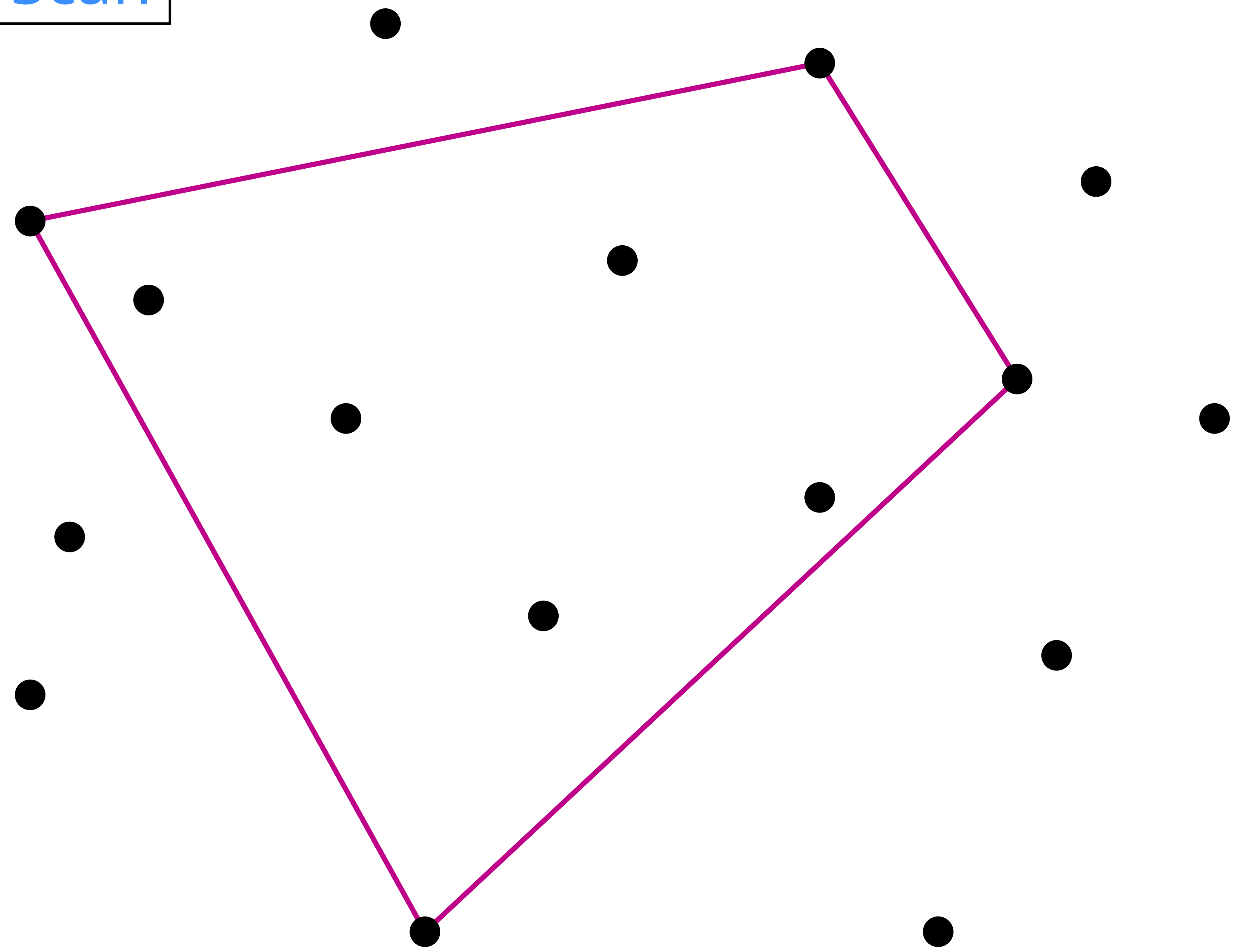

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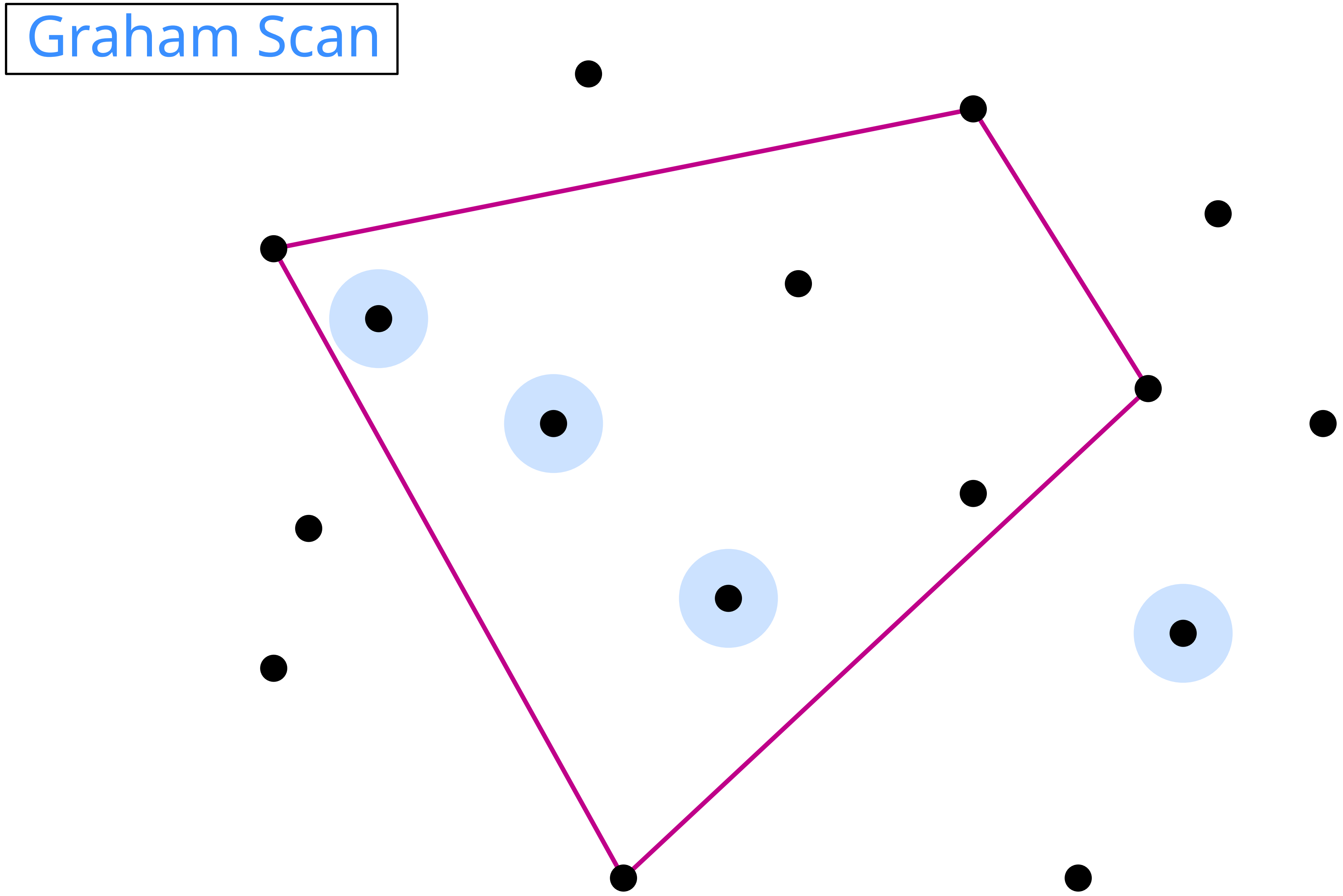

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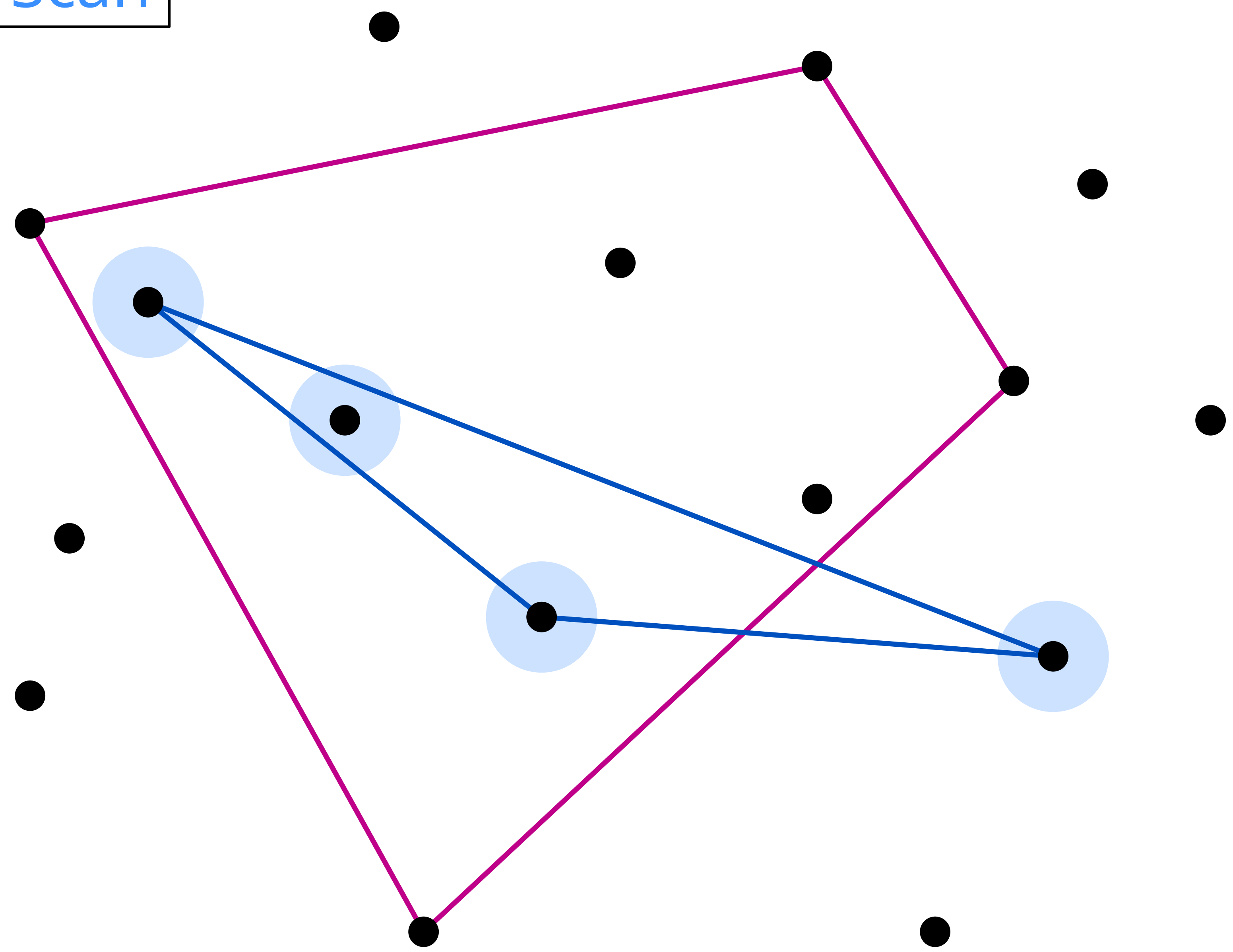

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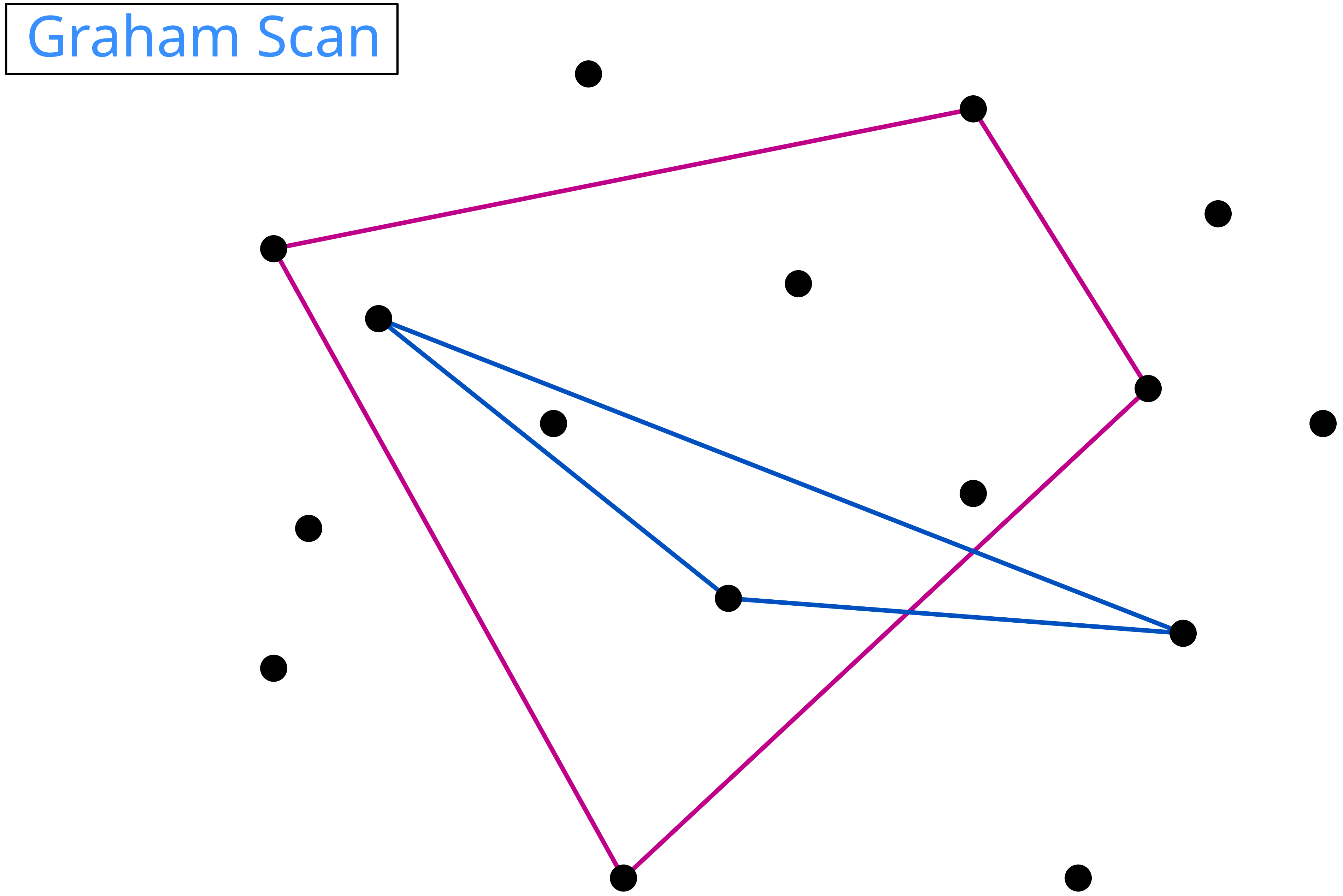

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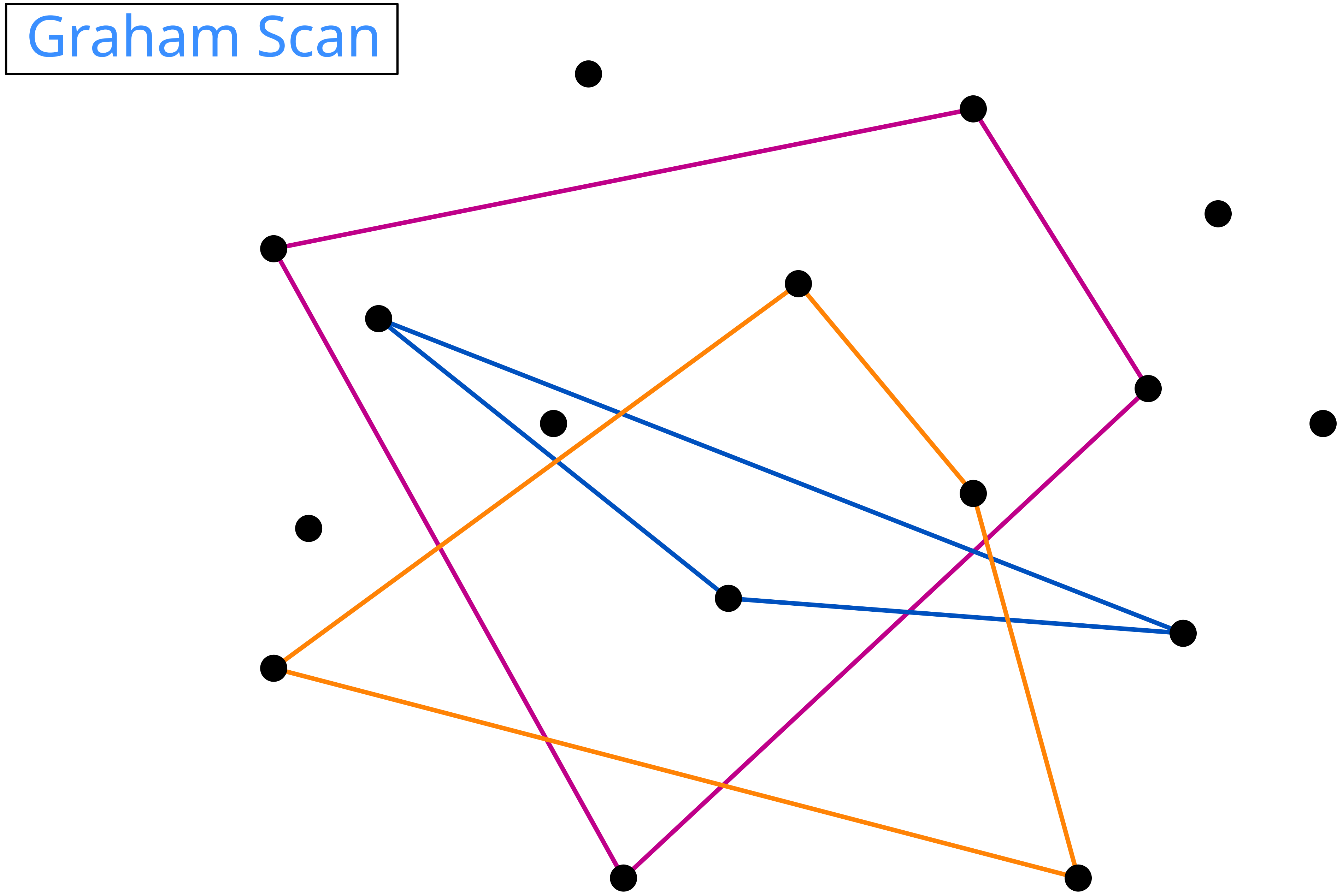

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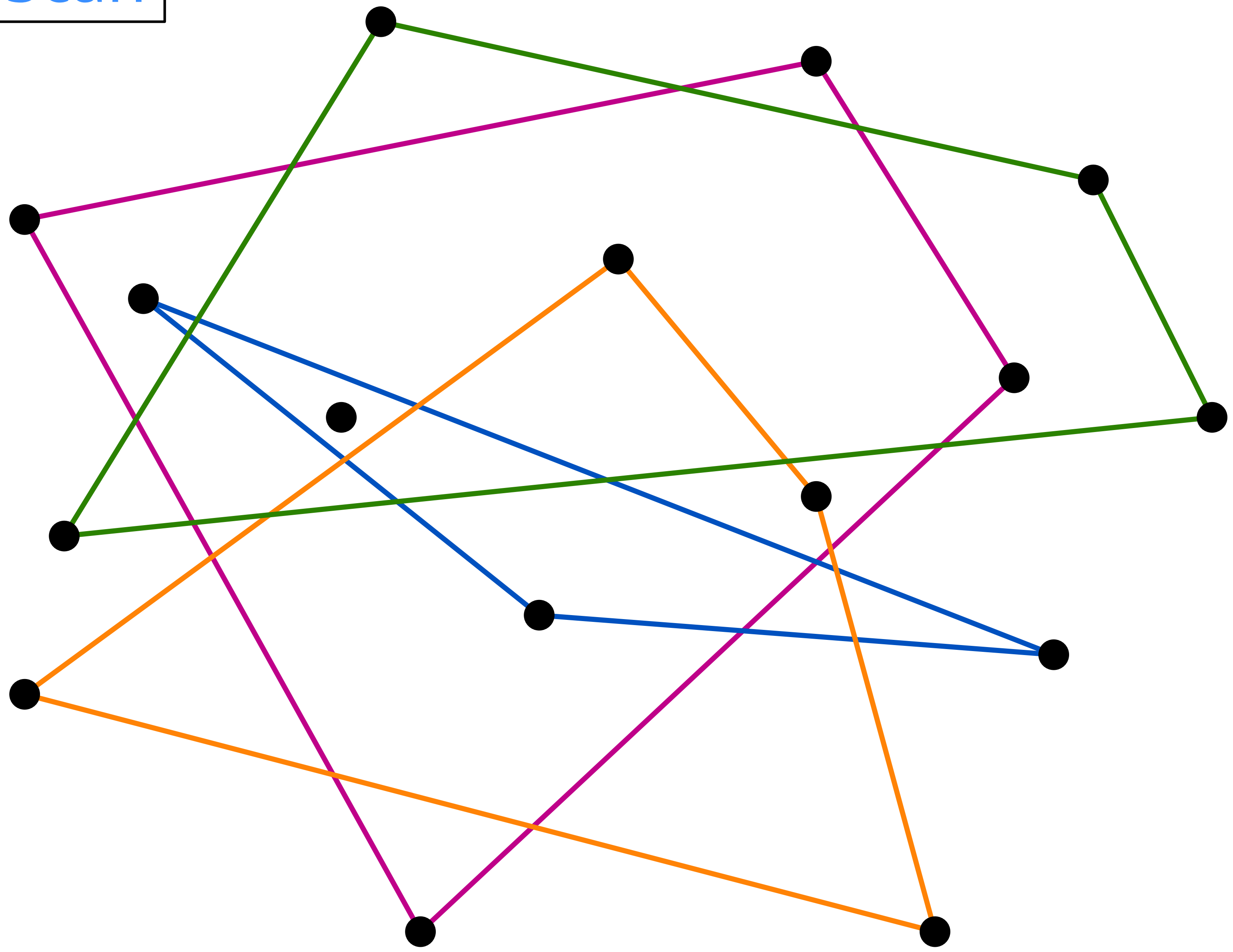

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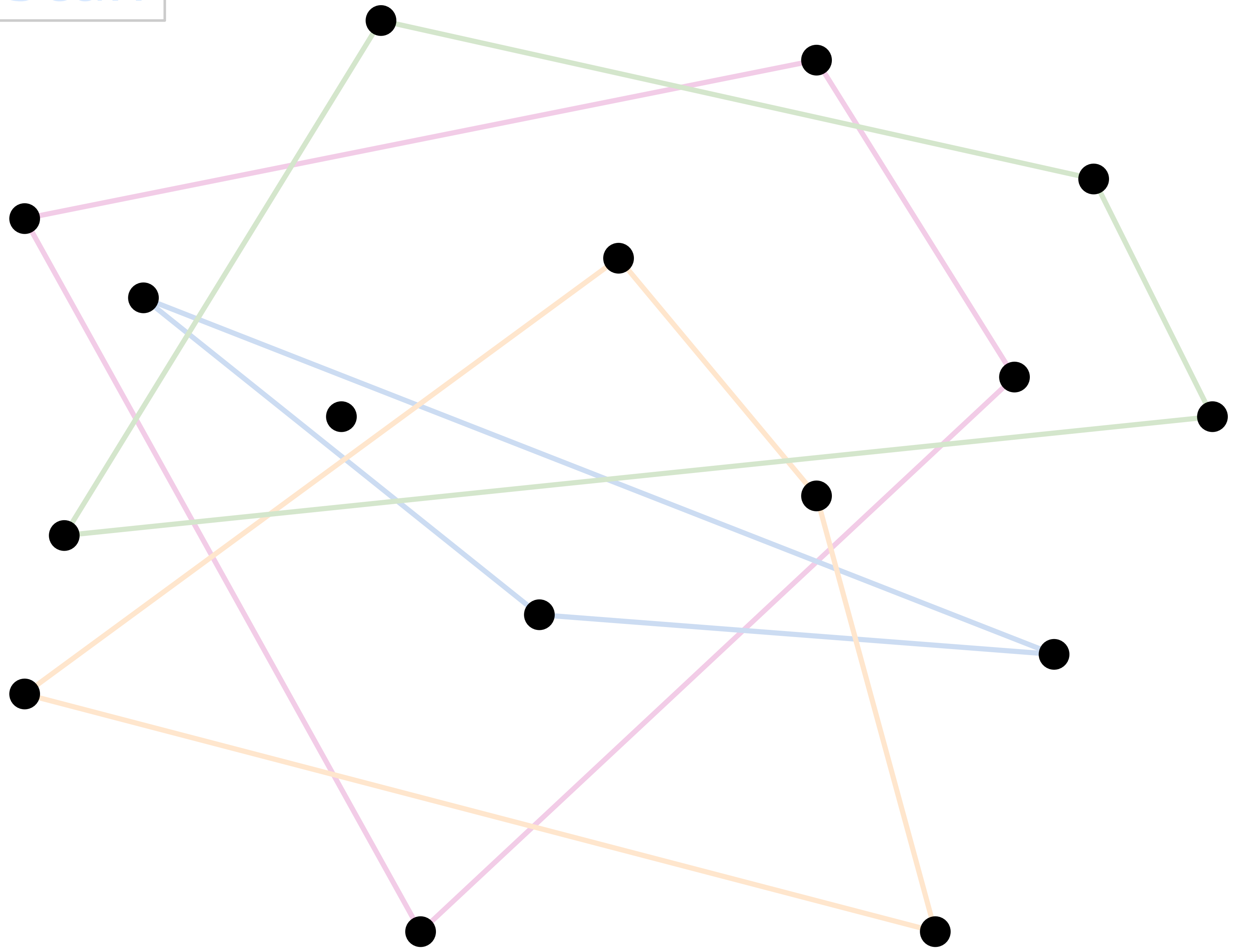

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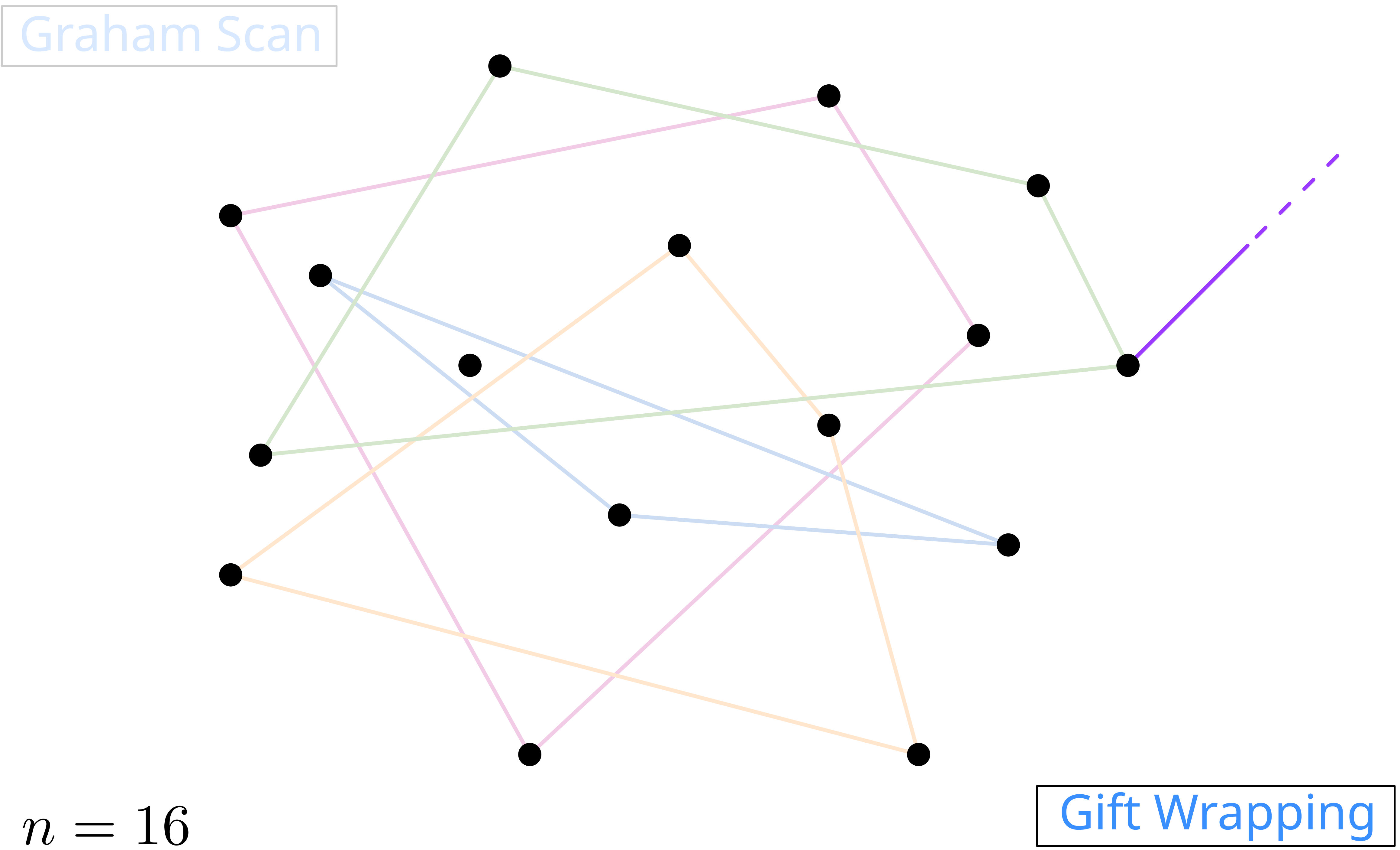
Gift Wrapping

Example



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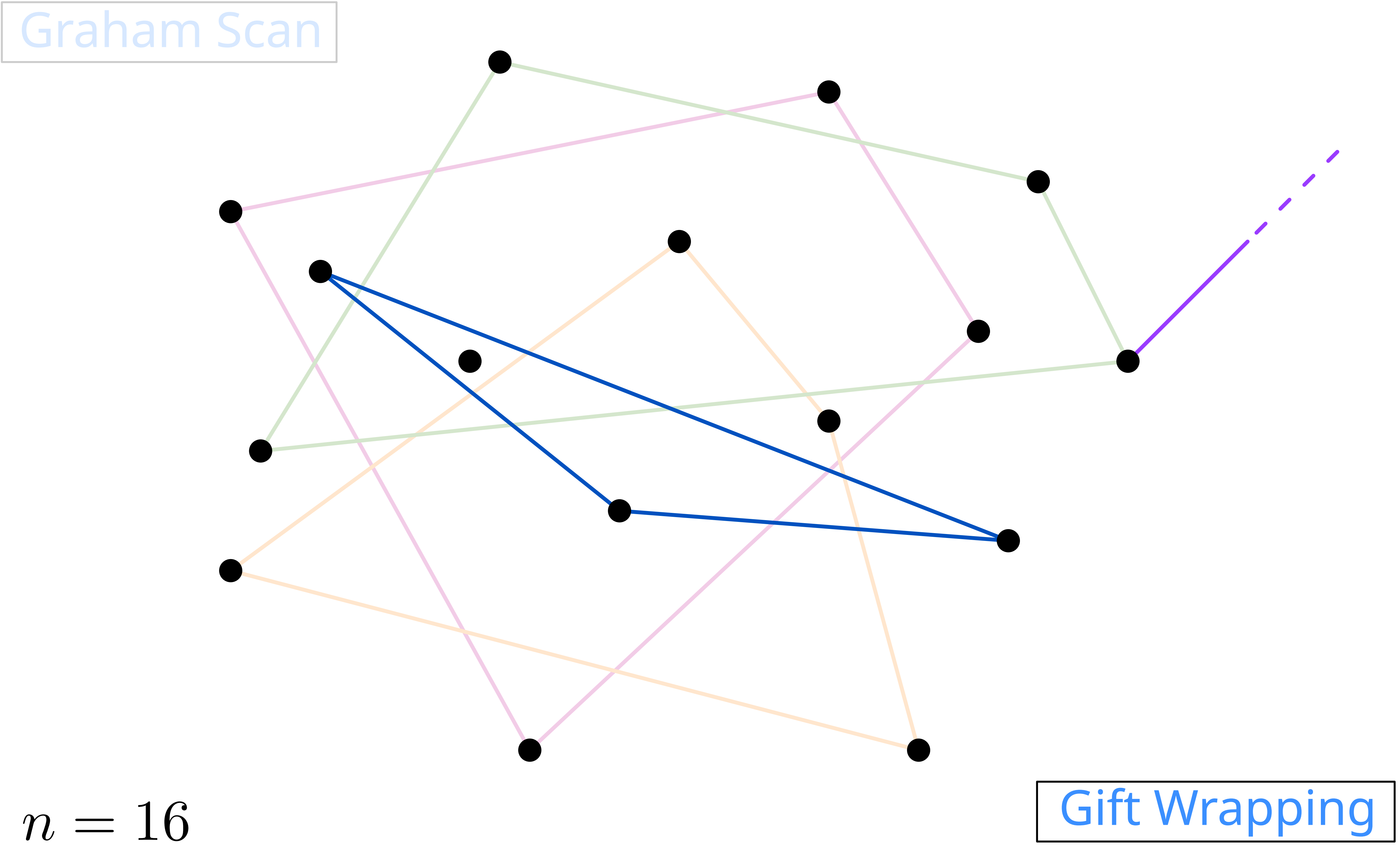

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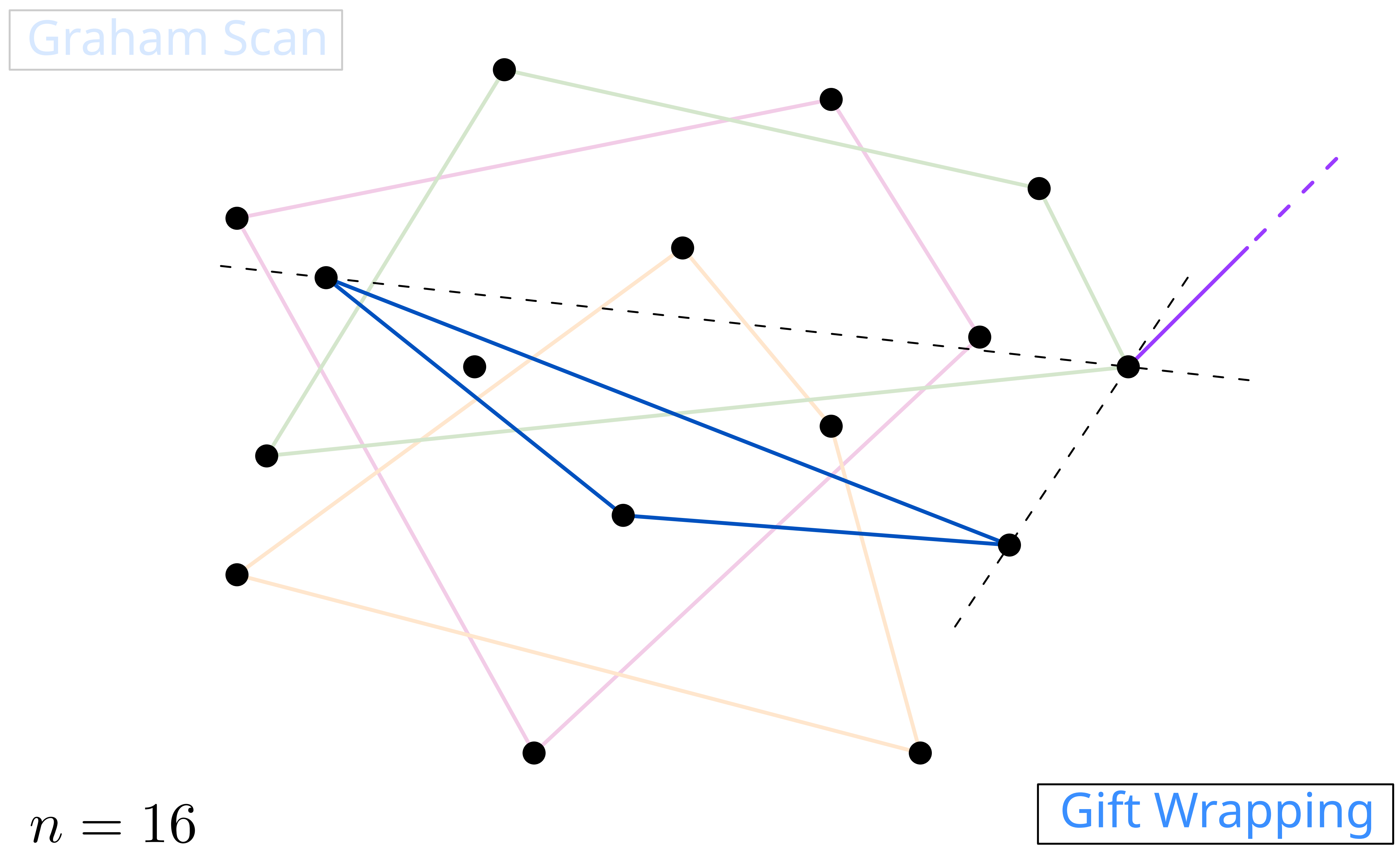

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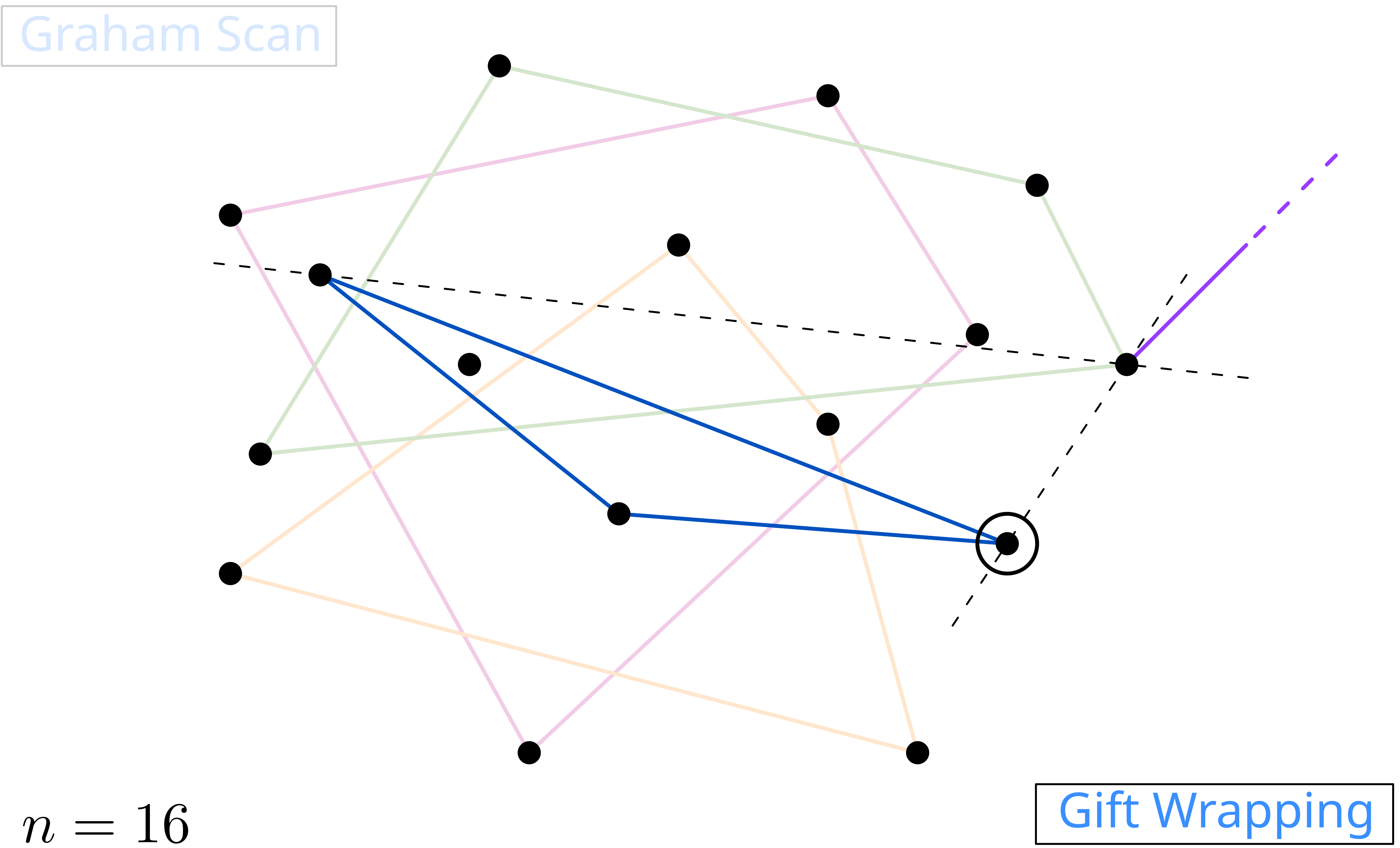

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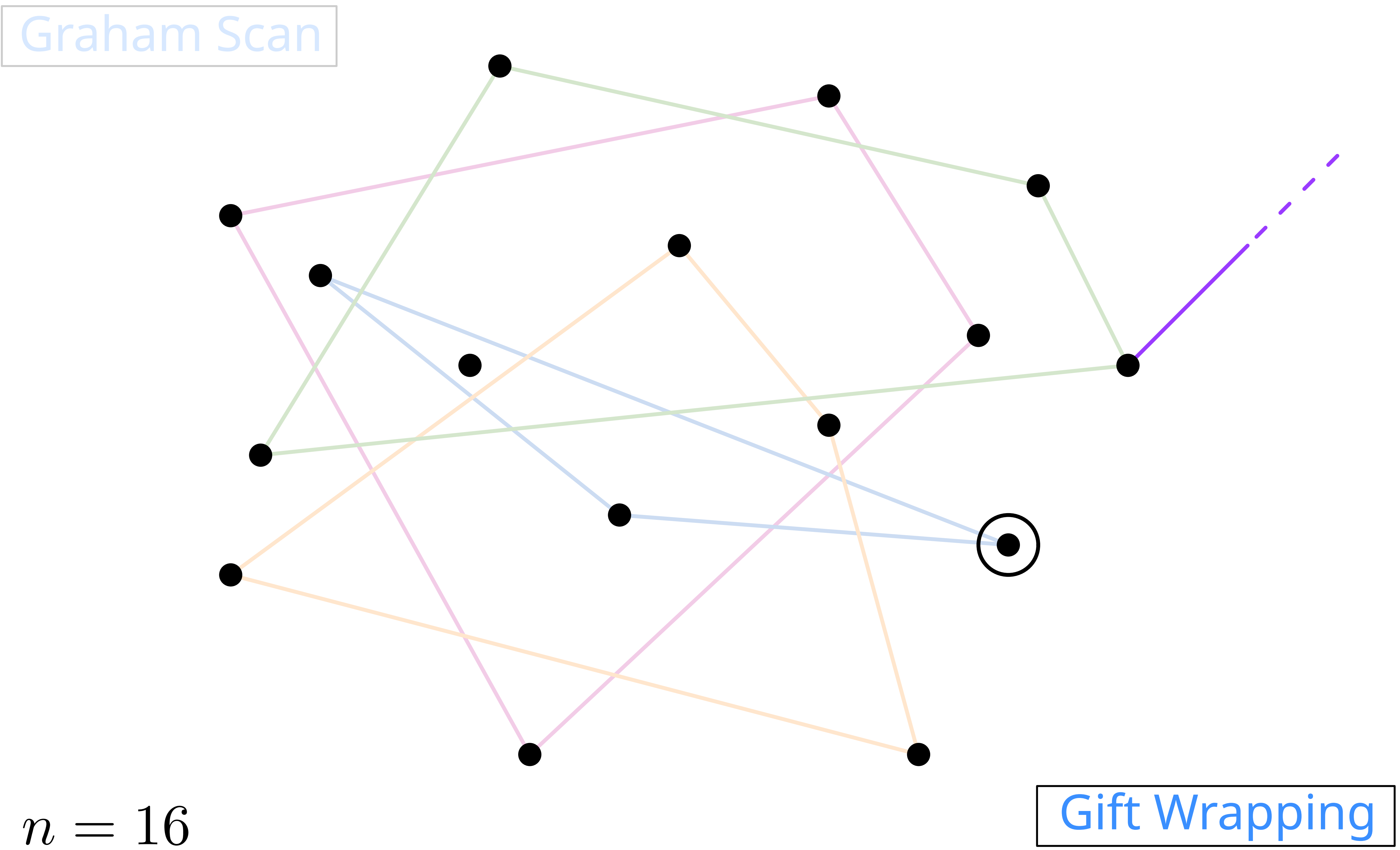

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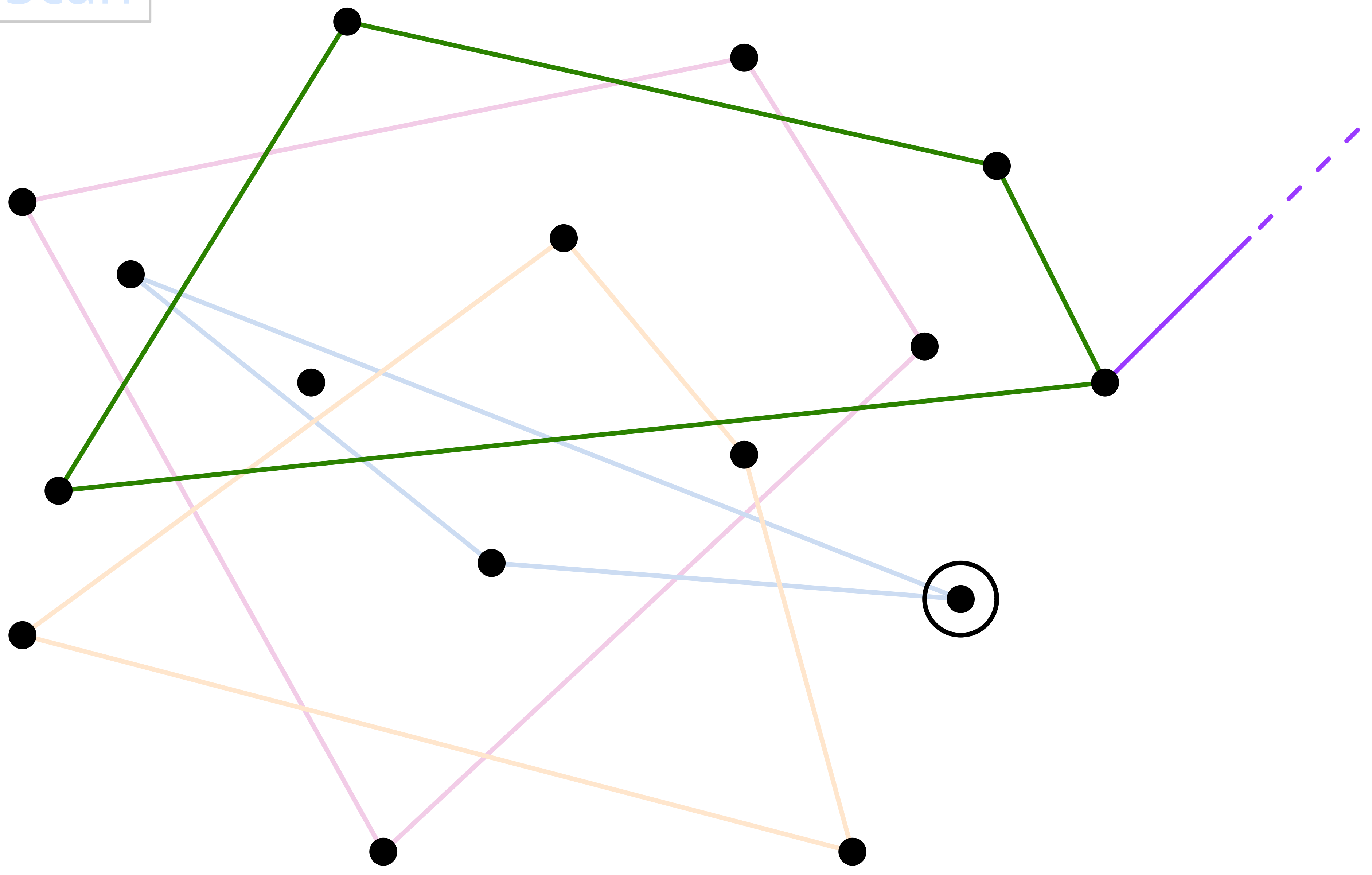

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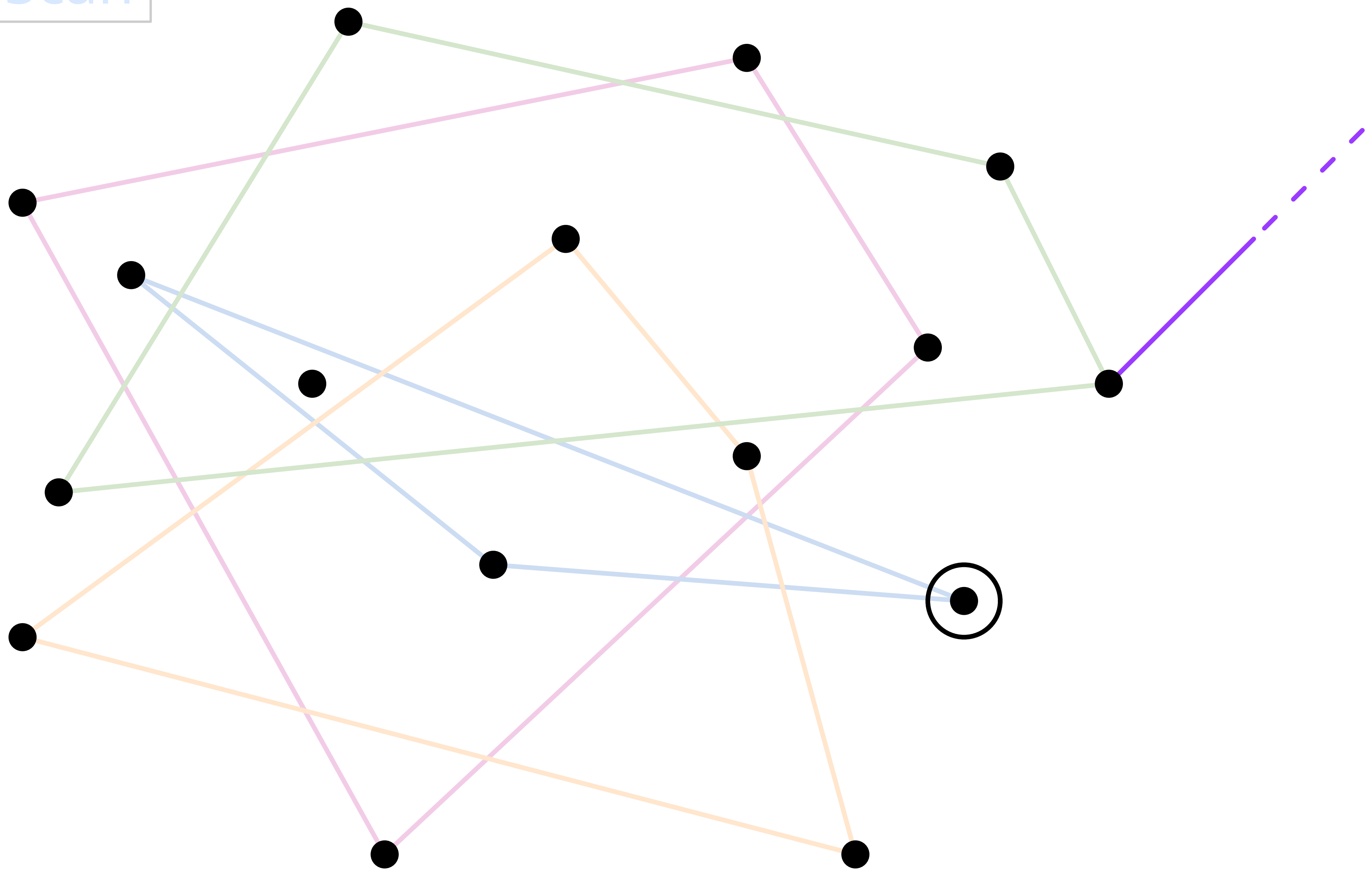

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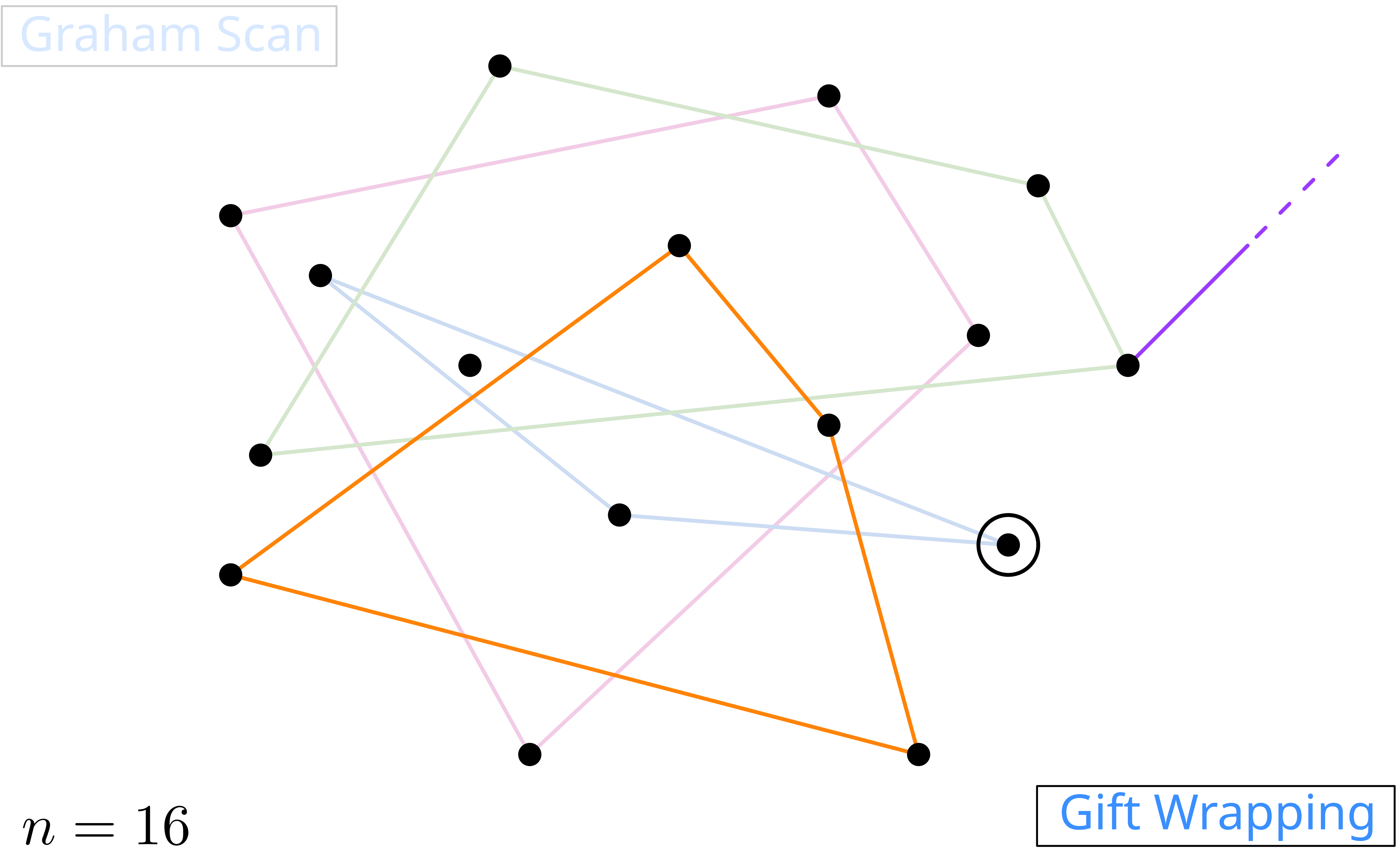

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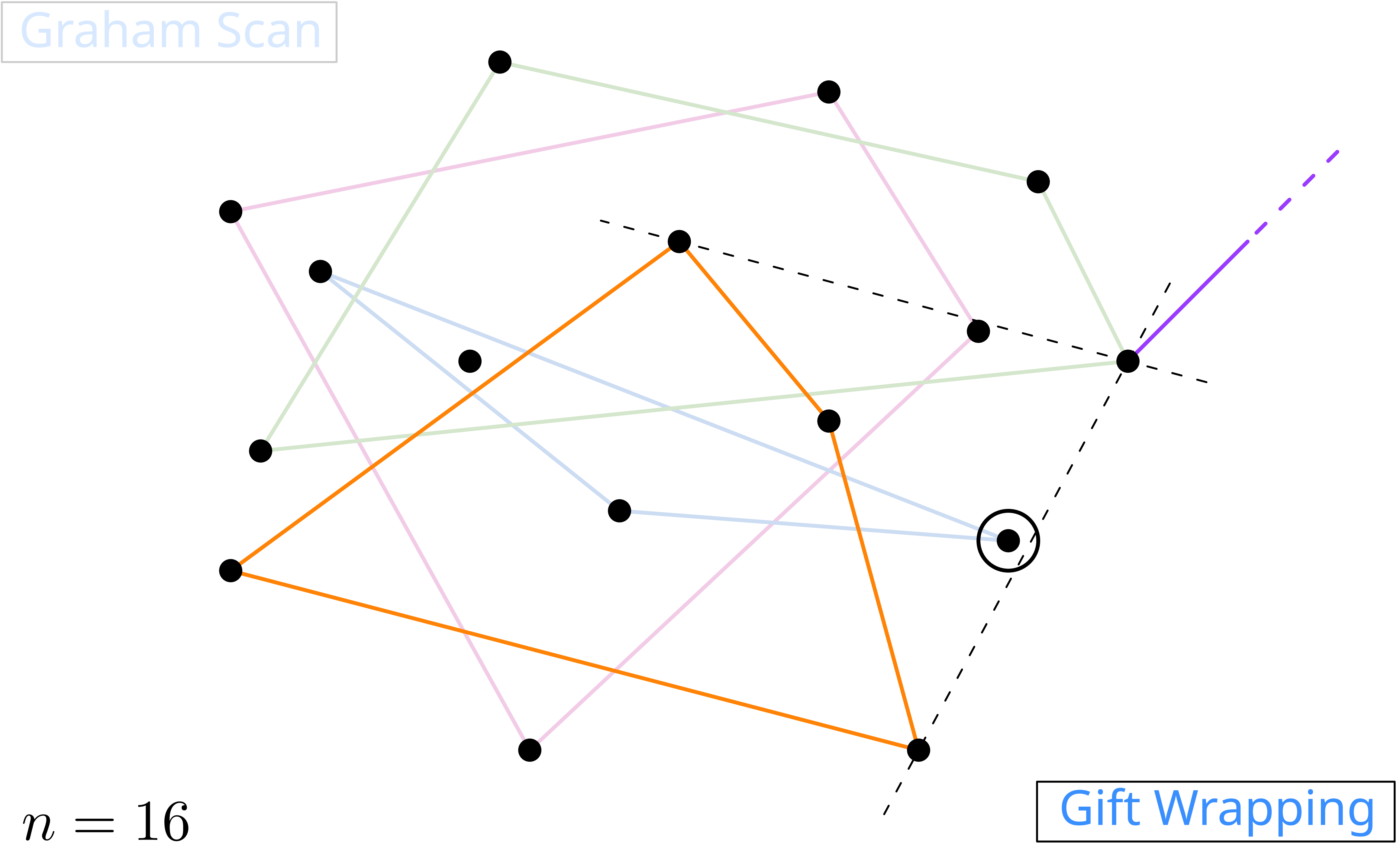

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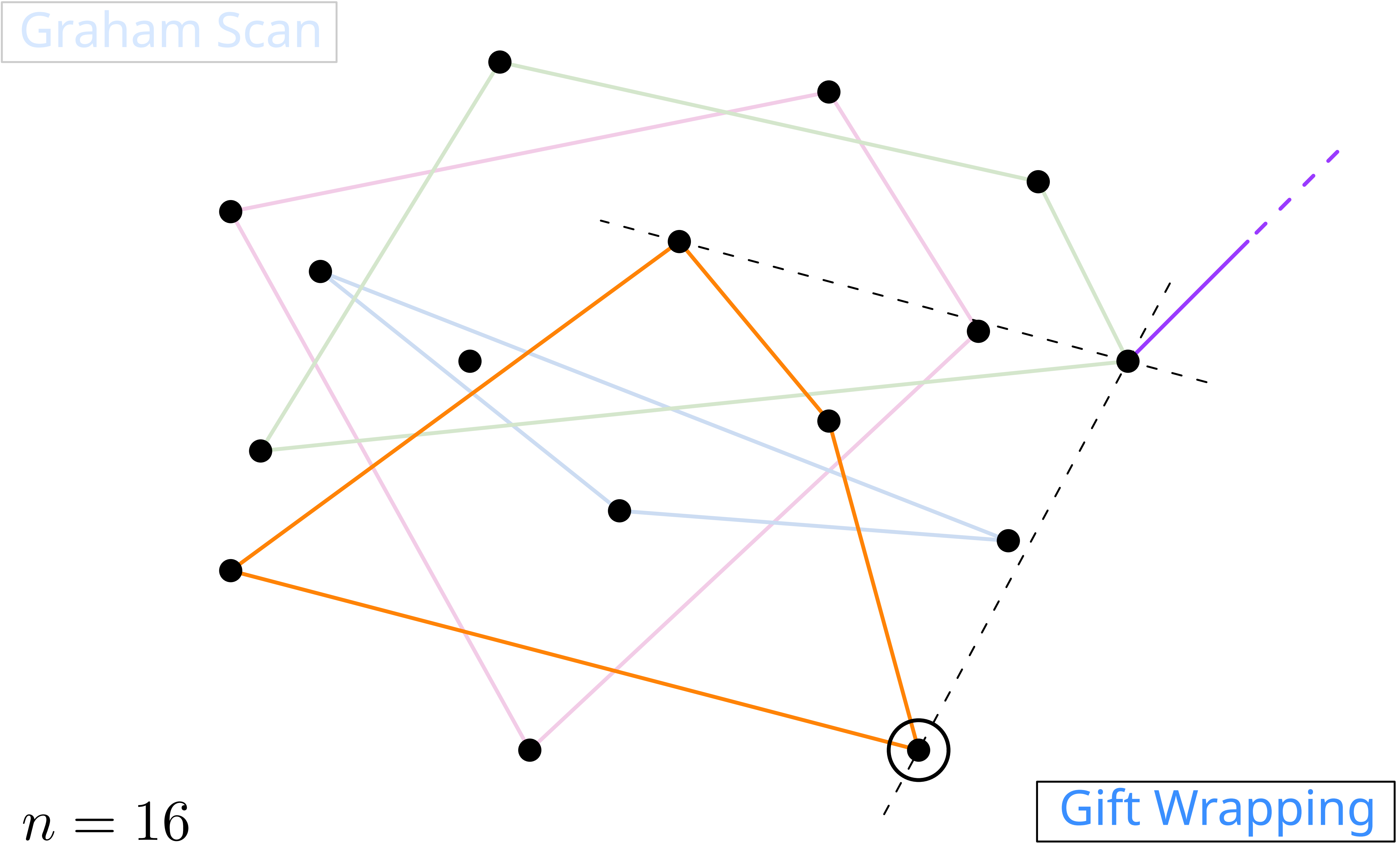

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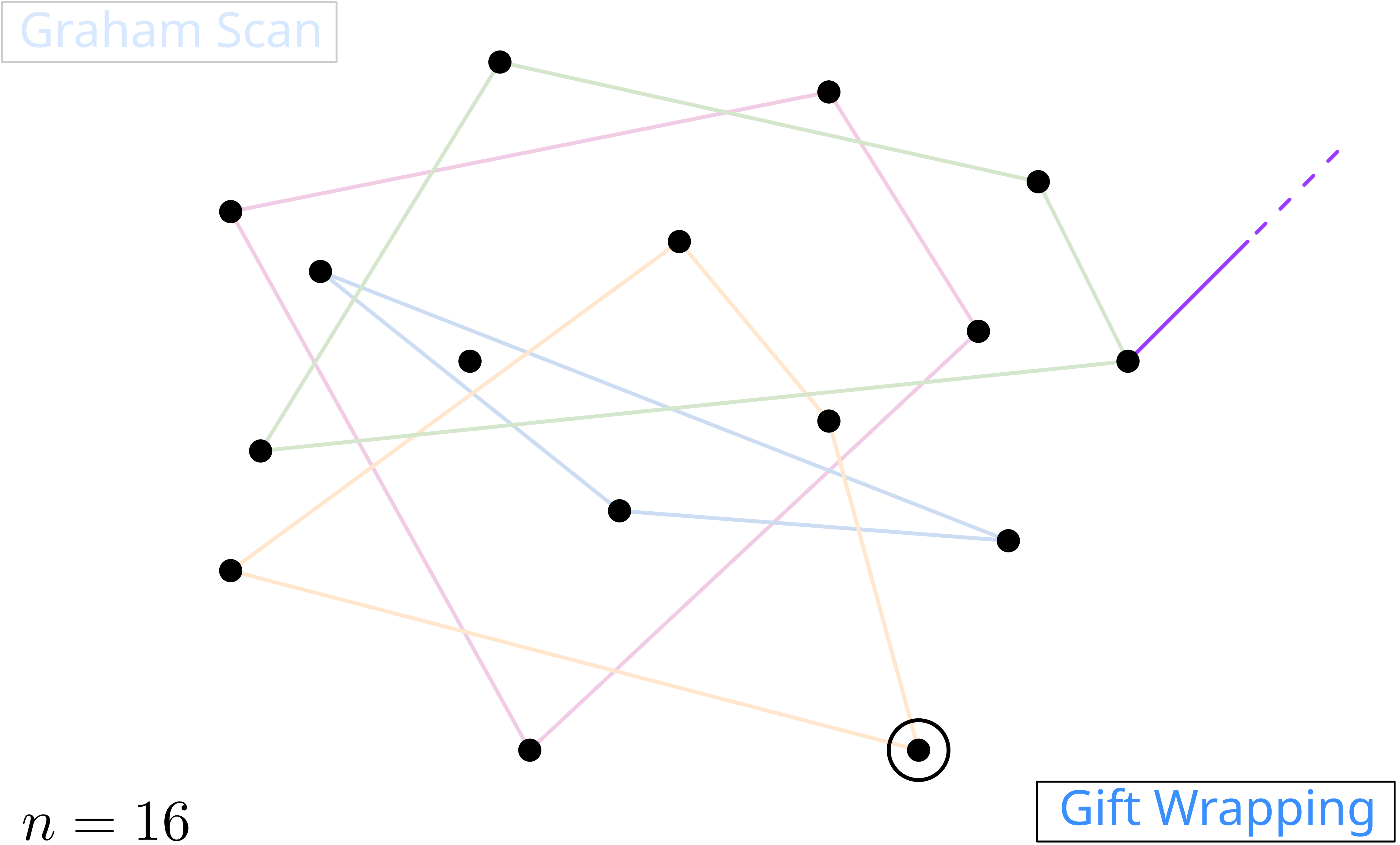

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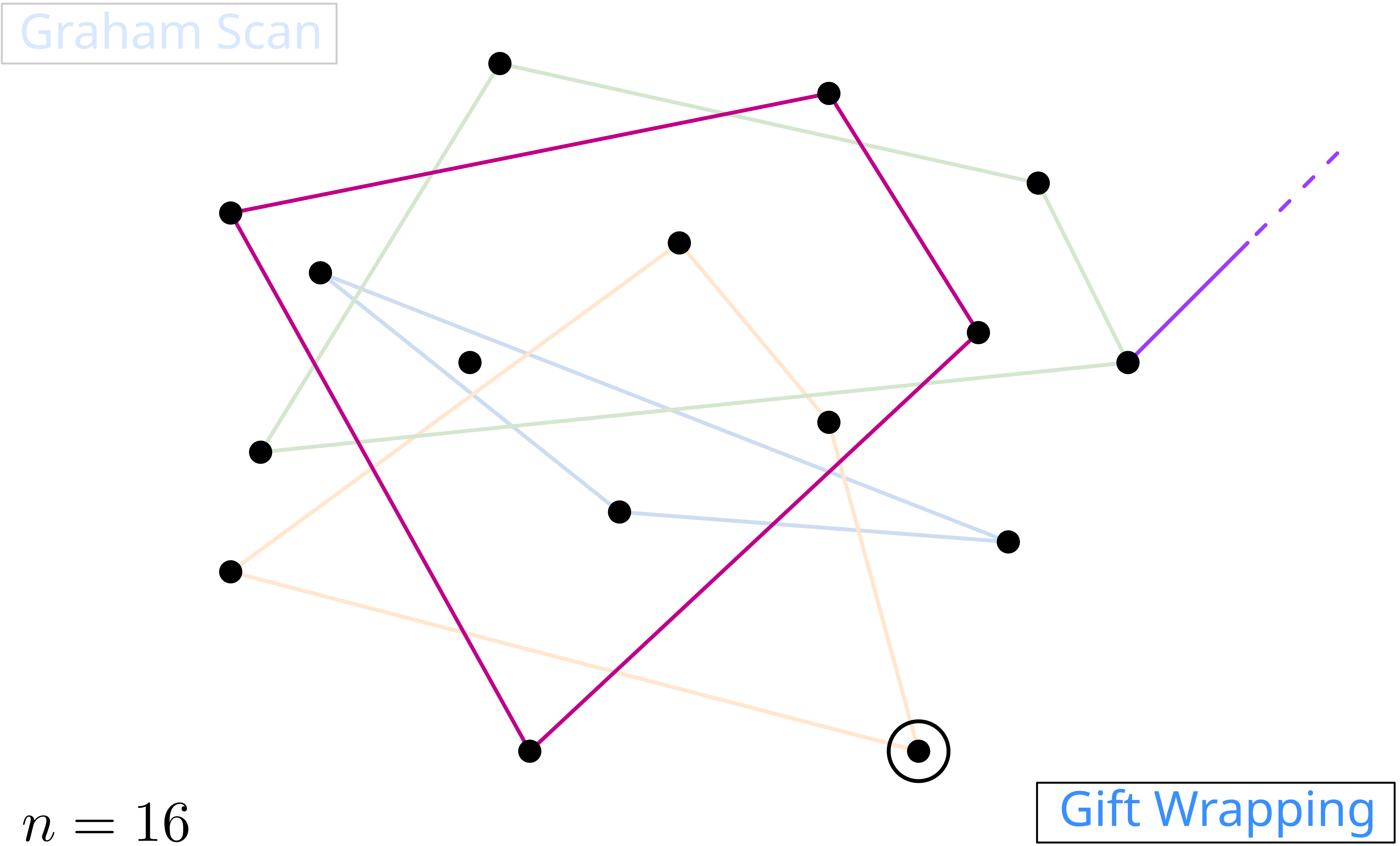

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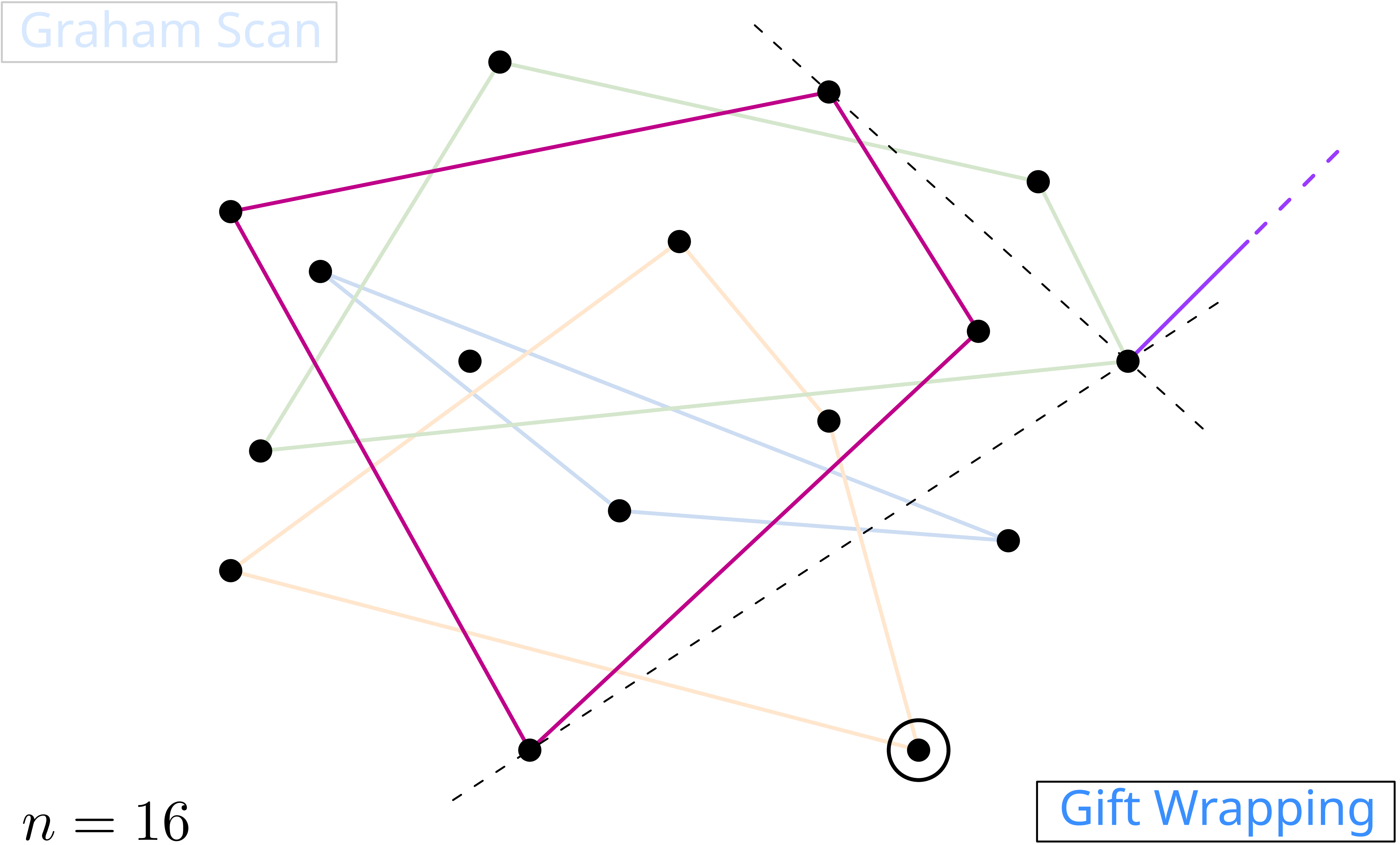

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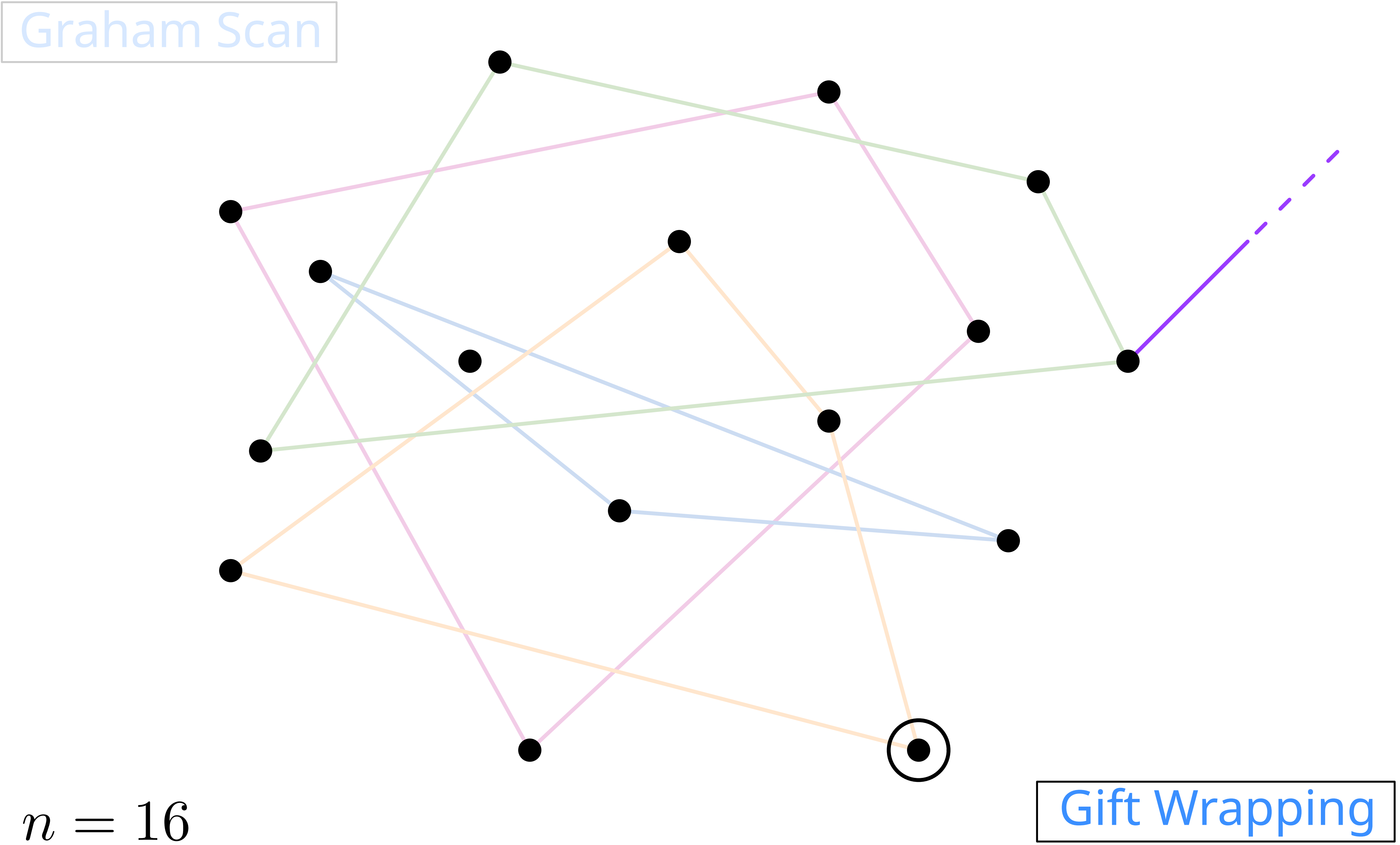

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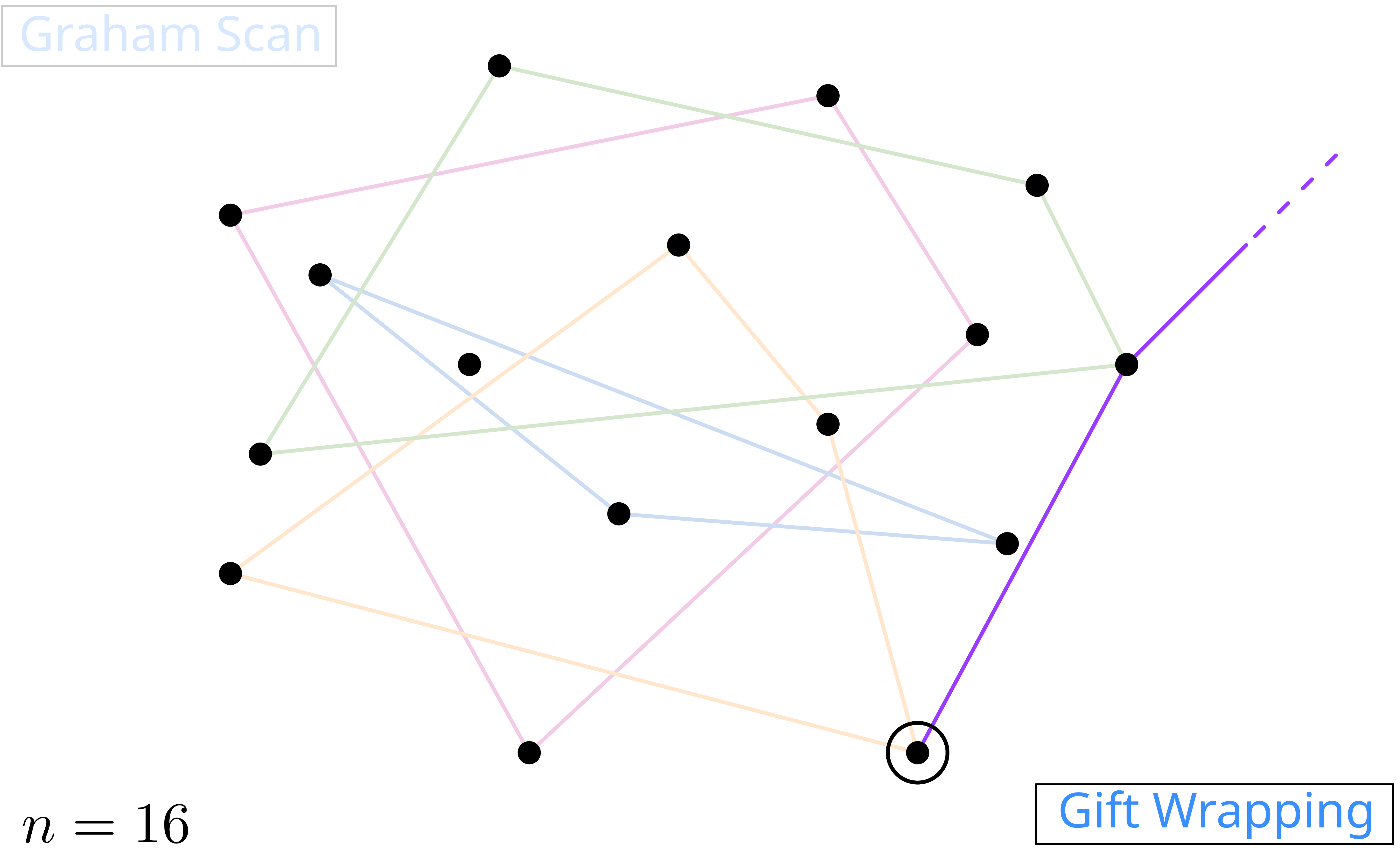

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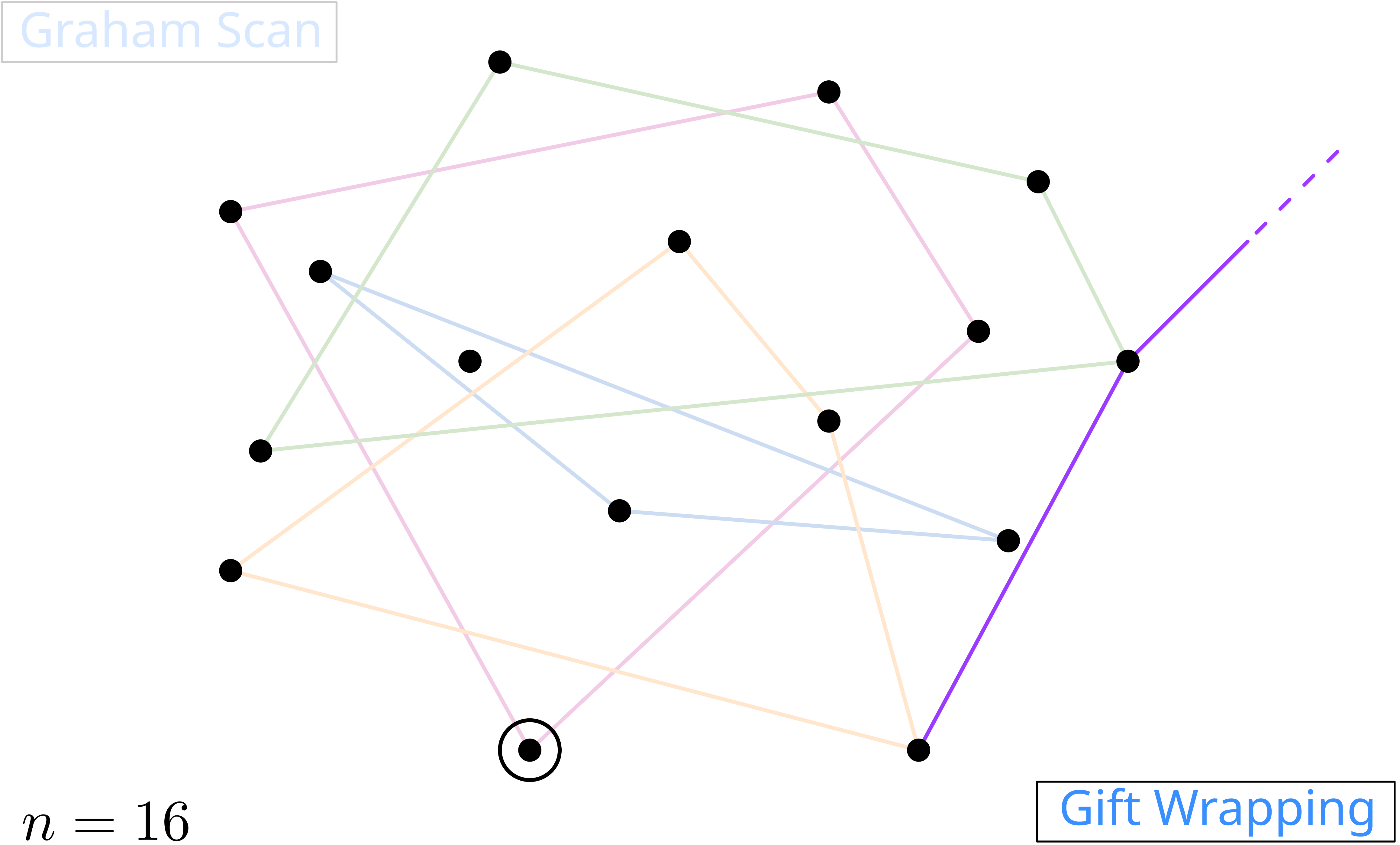

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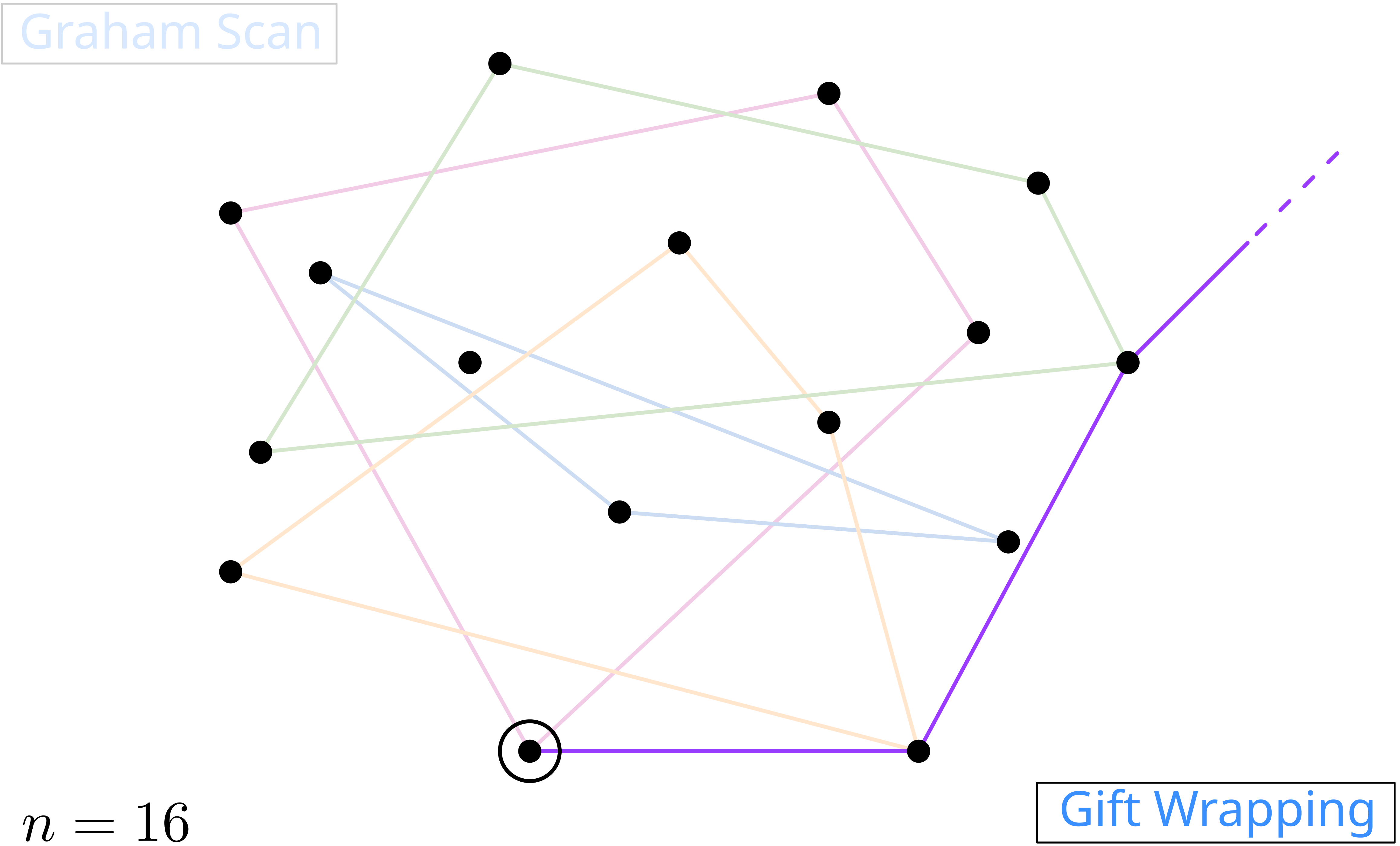

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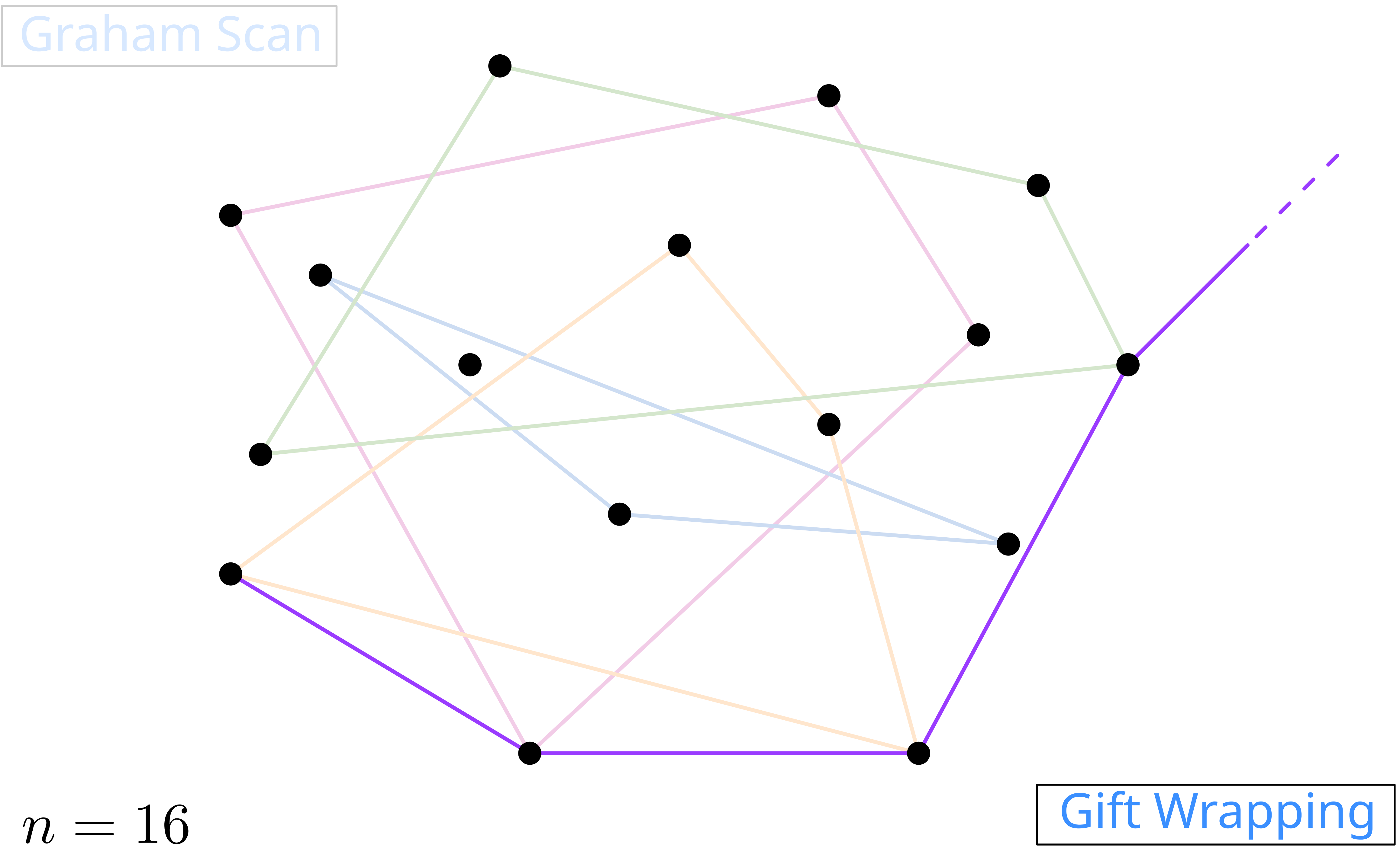

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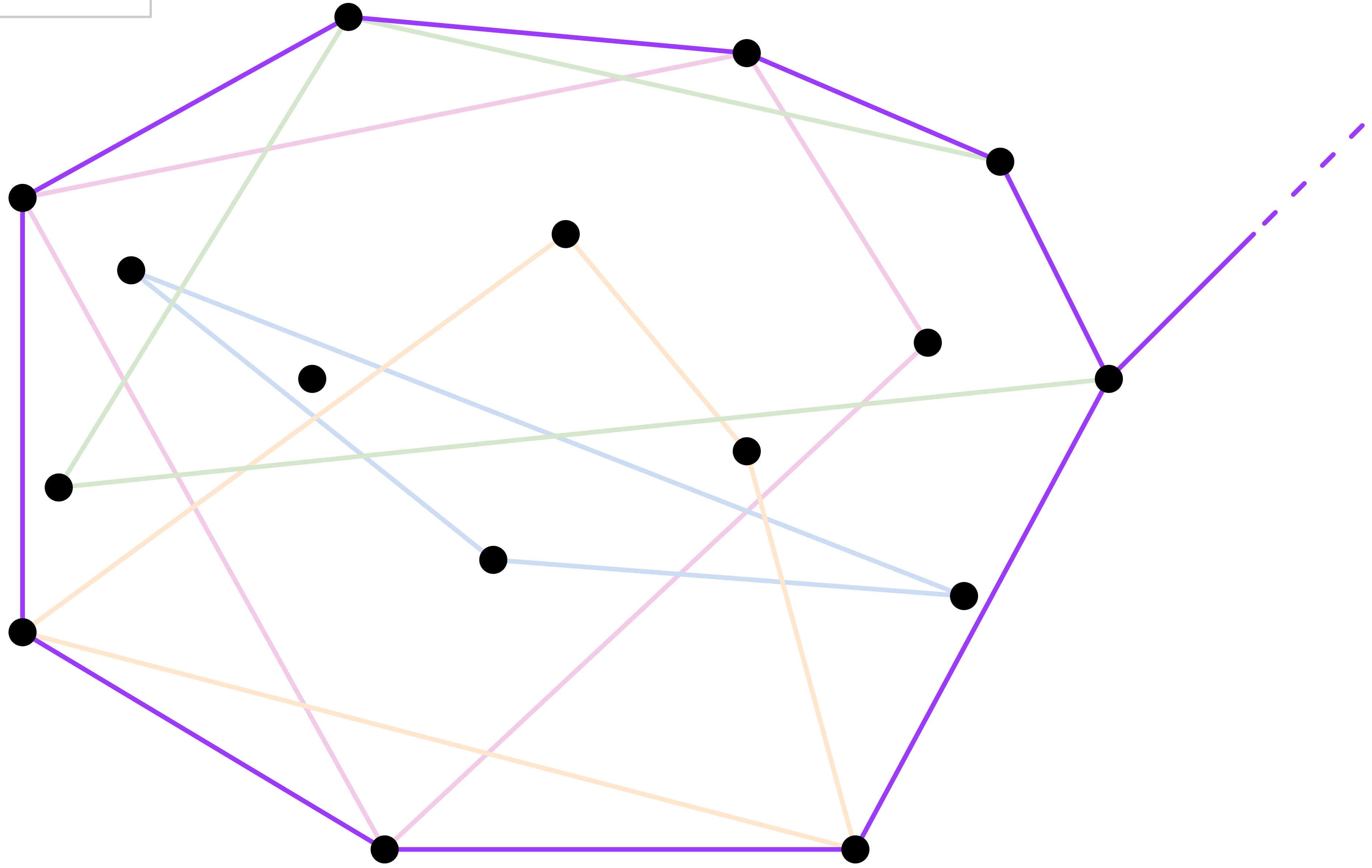

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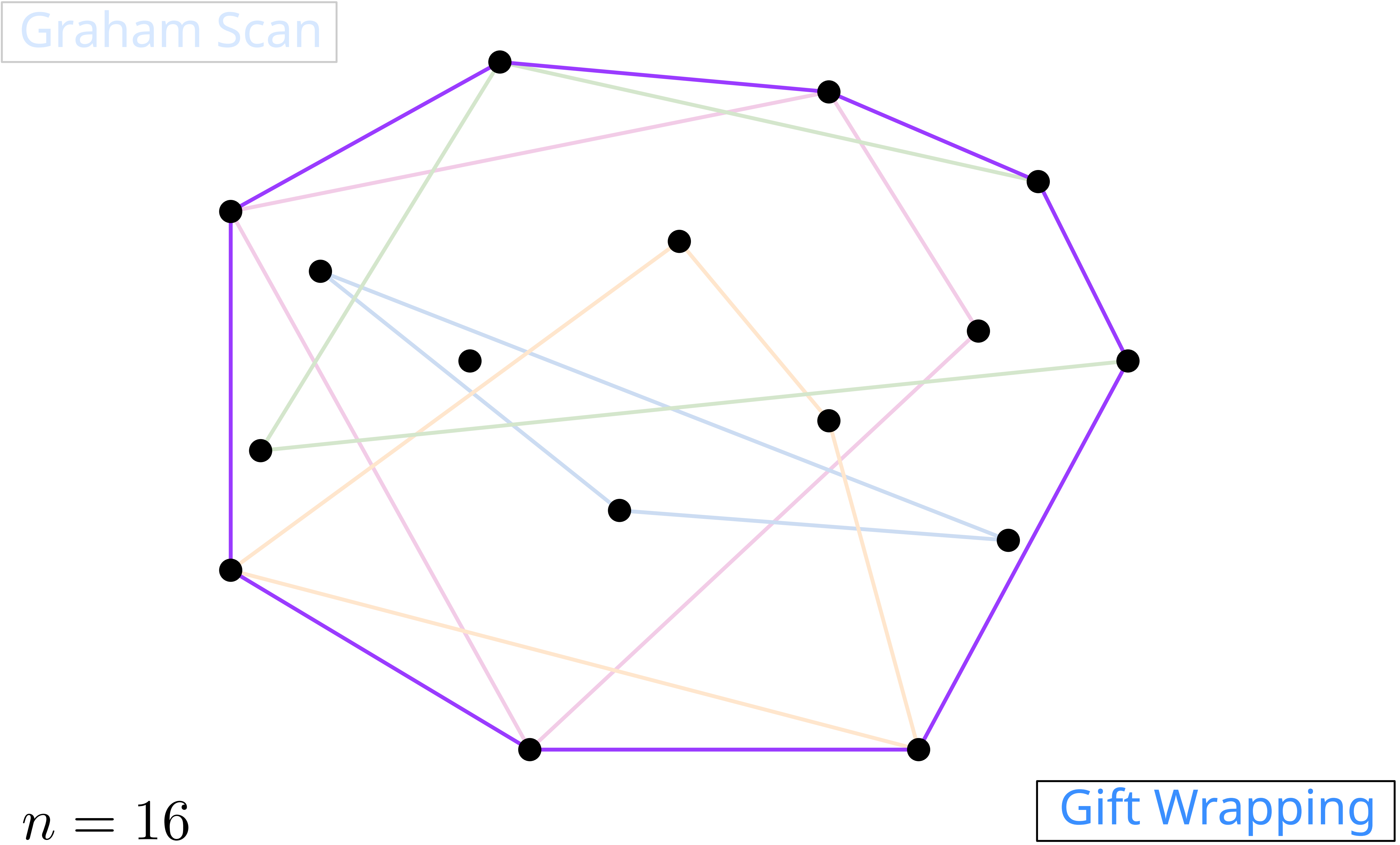
Graham Scan

Gift Wrapping

Example



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Chan’s Algorithms

But in general we do not know h

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Algorithm CHANHULL(P, m)

partition P into sets P_i with m points each

for $i \leftarrow 1$ to n/m **do**

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    $p_{j+1} \leftarrow \max_{\angle}\{q_1, \ldots, q_{n/m}\}$
    insert p_{j+1} into L
    if $p_{j+1} = p_1$ **then**
        return L
return failure

What's up with m?

Suggestions?

What's up with m?

Algorithm FULLCHANHULL(P)

for $t \leftarrow 0, 1, 2, \ldots$ **do**
    $m = \ldots$
    $result \leftarrow$ CHANHULL(P, m)
    if $result \neq$ failure **then**
        break
return $result$

What's up with m?

Algorithm FULLCHANHULL(P)

for $t \leftarrow 0, 1, 2, \ldots$ **do**
    $m = \min\{n, 2^{2^t}\}$
    $result \leftarrow$ CHANHULL(P, m)
    if $result \neq$ failure **then**
        break
return $result$

What's up with m?

Algorithm FULLCHANHULL(P)

for $t \leftarrow 0, 1, 2, \ldots$ **do**
  $m = \min\{n, 2^{2^t}\}$
  $result \leftarrow$ CHANHULL(P, m) $O(n \log m) = O(n \log 2^{2^t})$
  if $result \neq$ failure **then**
    break
return $result$

What's up with m?

Algorithm FULLCHANHULL(P)

1: **for** $t \leftarrow 0, 1, 2, \ldots$ **do**
2: $\quad m = \min\{n, 2^{2^t}\}$
3: $\quad result \leftarrow$ CHANHULL(P, m) $\quad O(n \log m) = O(n \log 2^{2^t})$
4: $\quad$ **if** $result \neq$ failure **then**
5: $\quad\quad$ **break**
6: **return** $result$

$\times O(\log \log h)$

What's up with m?

Algorithm FullChanHull(P)

for $t \leftarrow 0, 1, 2, \ldots$ **do**
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$\times O(\log \log h)$

return $result$

Running time:

$$\sum_{t=0}^{\log \log h} O(n \log 2^{2^t})$$

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  - **break**

$\times O(\log \log h)$

return $result$

Running time:

$$\sum_{t=0}^{\log \log h} O(n \log 2^{2^t}) = O(n) \sum_{t=0}^{\log \log h} O(2^t)$$

What's up with m?

Algorithm FULLCHANHULL(P)

for $t \leftarrow 0, 1, 2, \ldots$ **do**
  $m = \min\{n, 2^{2^t}\}$
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return result

$\times O(\log \log h)$

Running time:

$$\sum_{t=0}^{\log\log h} O(n \log 2^{2^t}) = O(n) \sum_{t=0}^{\log\log h} O(2^t) \leq O(n) \cdot O(2^{\log\log h + 1})$$

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$\times O(\log \log h)$

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Running time:

$$\sum_{t=0}^{\log \log h} O(n \log 2^{2^t}) = O(n) \sum_{t=0}^{\log \log h} O(2^t) \leq O(n) \cdot O(2^{\log \log h + 1})$$

$$= O(n) \cdot O(\log h) = O(n \log h)$$

Summary

Slow Convex Hull $O(n^3)$

Graham Scan $O(n \log n)$

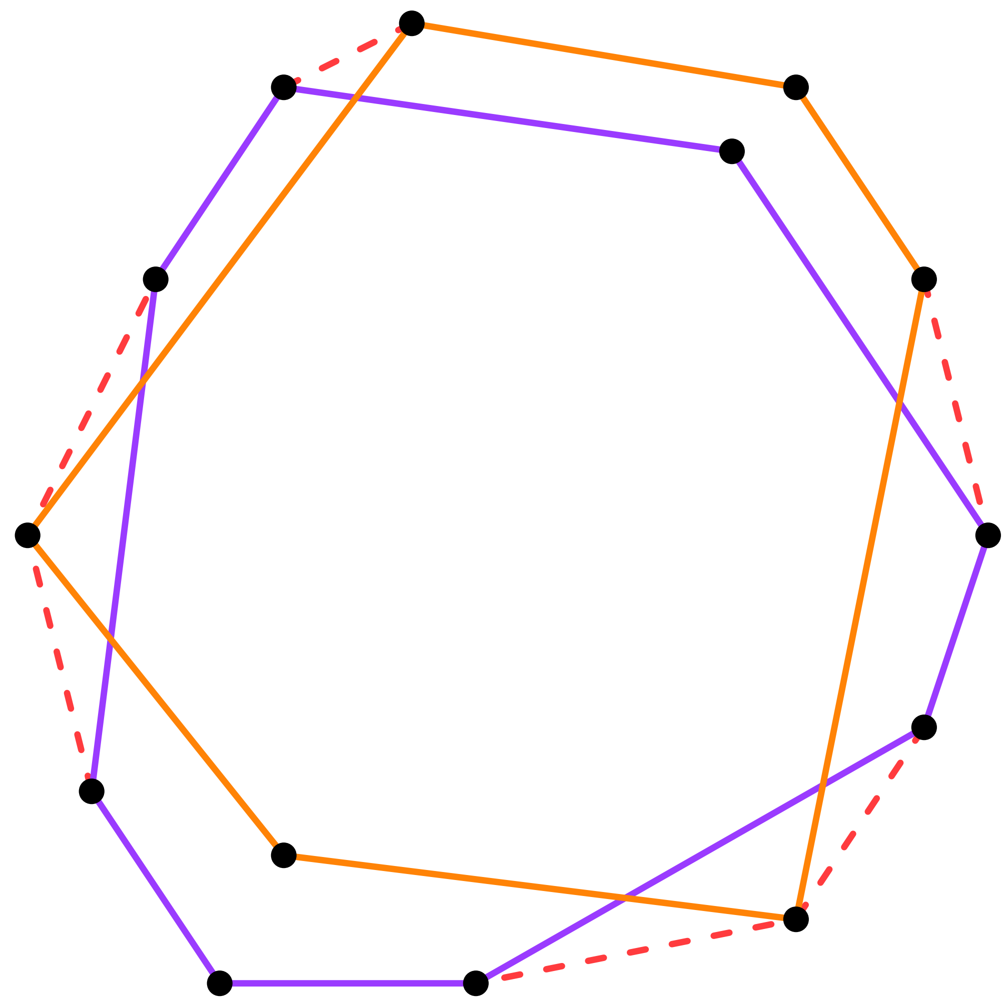
Gift Wrapping $O(nh)$

Chan's Hull $O(n \log h)$

Other approaches: divide and conquer

Split the point set in two halves, compute the convex hulls recursively, and merge

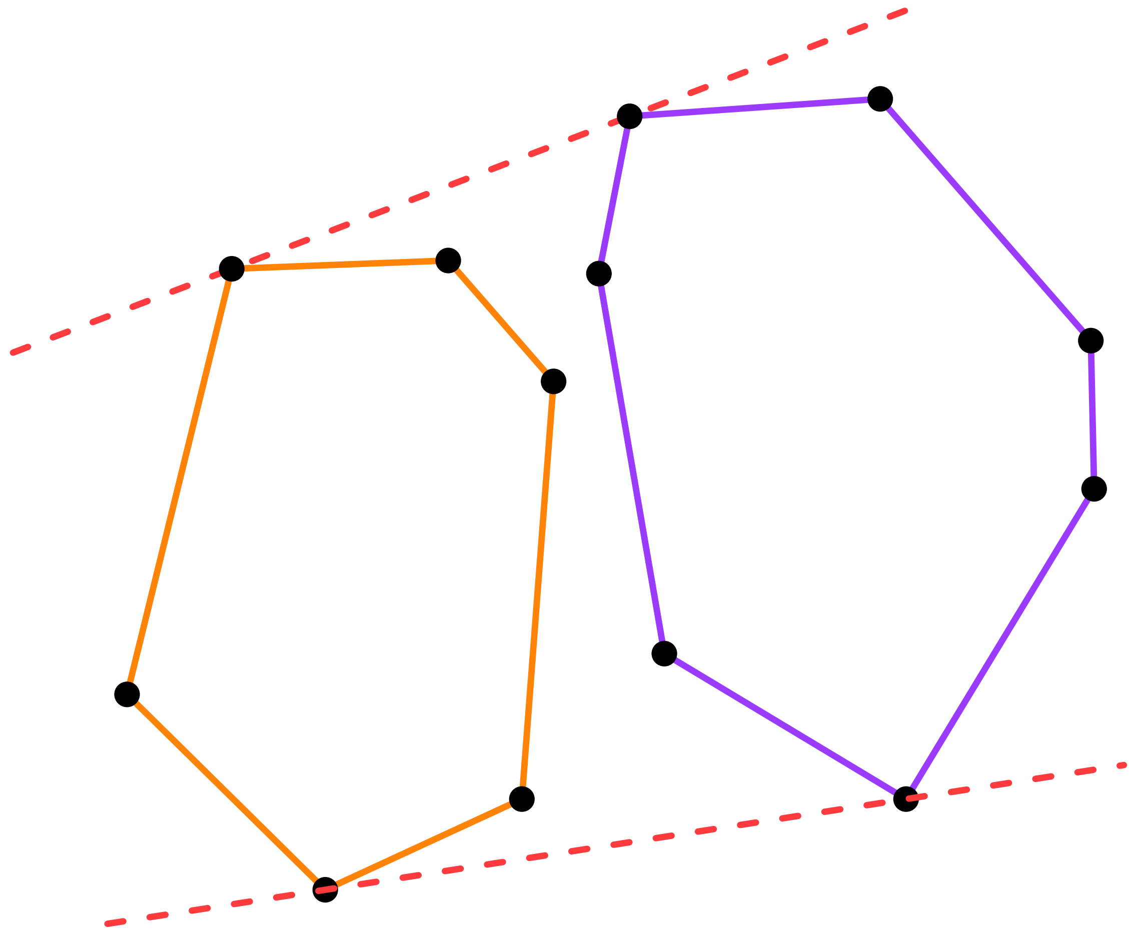
The merge step involves finding “extreme vertices” in every direction



Other approaches: divide and conquer

Alternatively: split the point set in two halves on x-coordinate, compute the convex hulls recursively, and merge

The merge step now comes down to finding two common tangent lines



Convex hulls in 3D

For a 3-dimensional point set, the convex hull is a convex polyhedron

It has vertices (0-dim.), edges (1-dim.), and facets (2-dim.) on its boundary, and a 3-dimensional interior

The boundary is a planar graph, so it has $O(n)$ vertices, edges, and facets

Convex hulls in 4D

For a 4-dimensional point set, the convex hull is a convex polyhedron

It has vertices (0-dim.), edges (1-dim.), 2-facets (2-dim.), and 3-facets (3-dim.) on its boundary, and a 4-dimensional interior

Its boundary can have $\Theta(n^2)$ facets in the worst case!