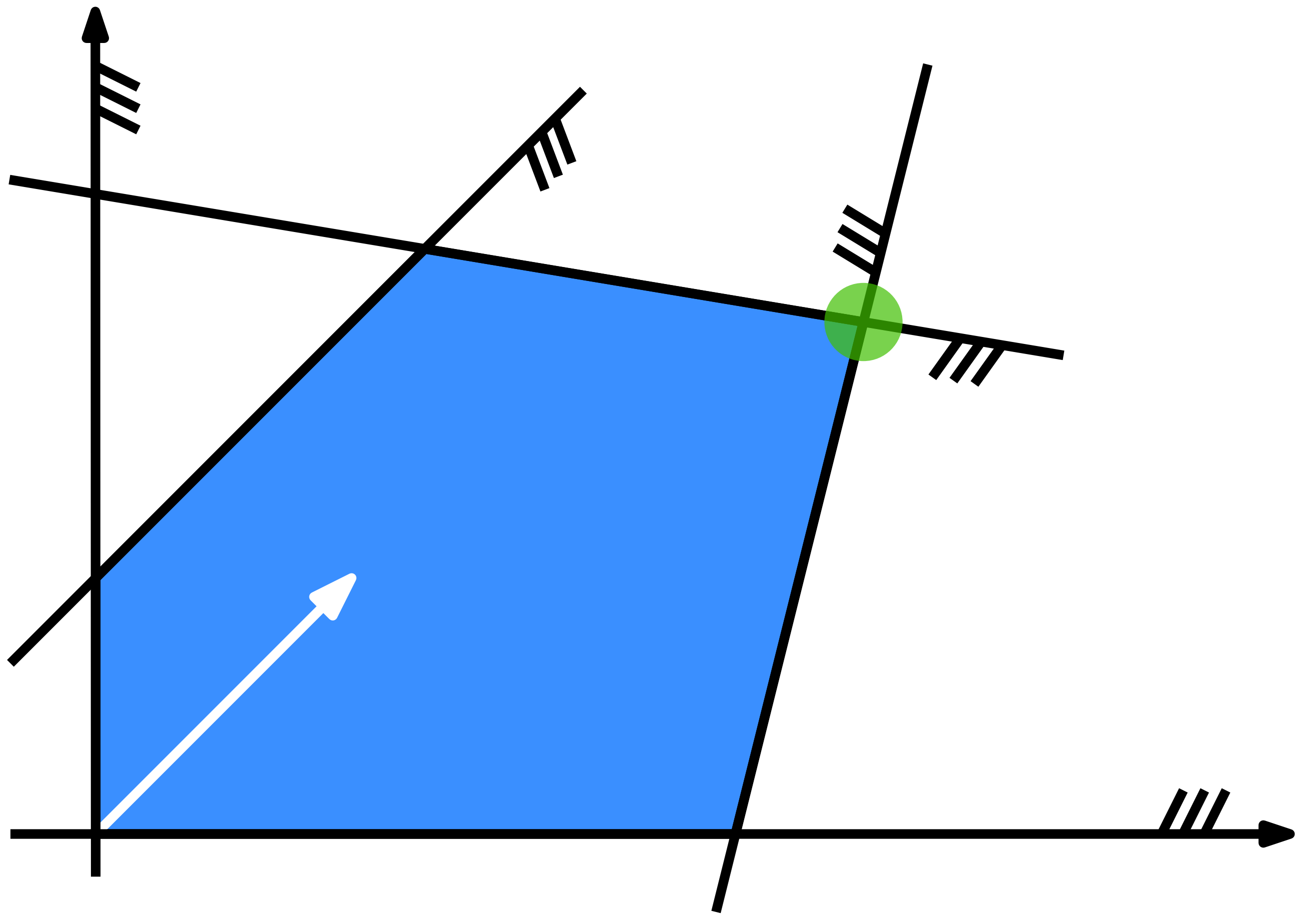
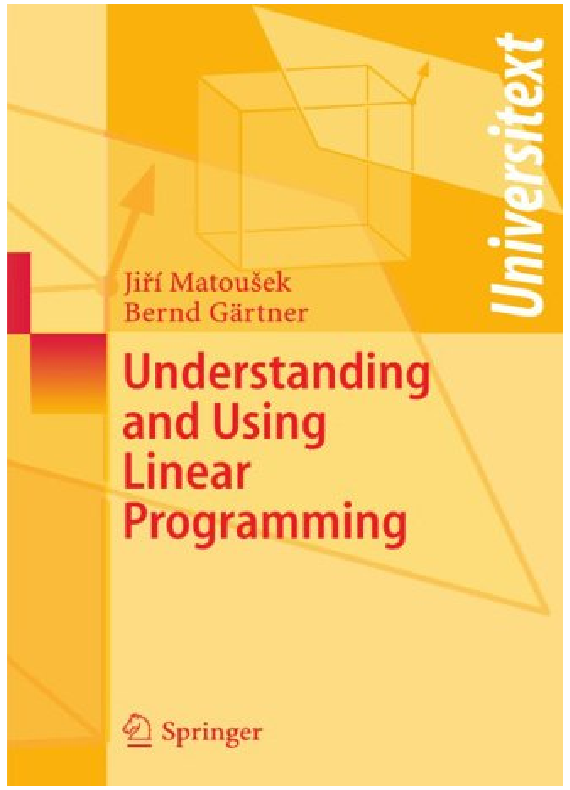
Linear Programming

Introduction and Examples



Background

Following "Understanding and Using Linear Programming" by Matoušek and Gärtner



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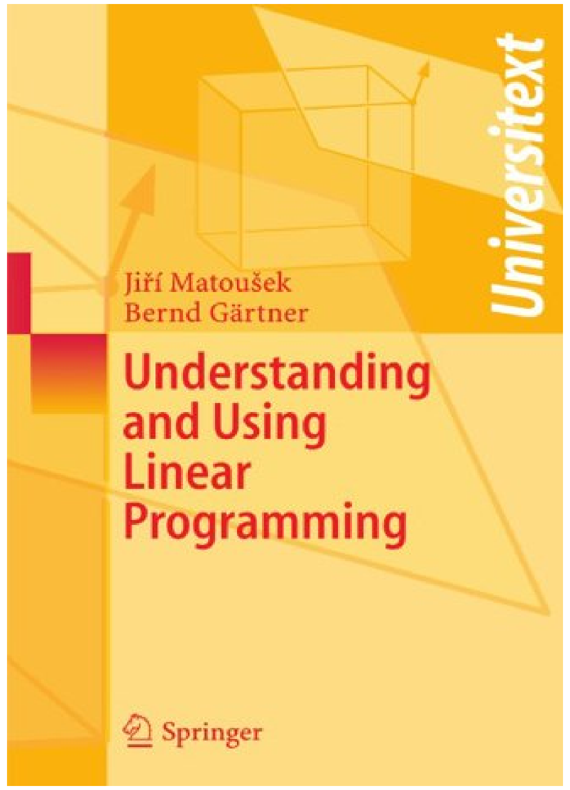

Etymology of Linear Programming:

- 1950's, programming is military term
  $\approx$ planning schedules, supply, deployment
- other name: linear optimization

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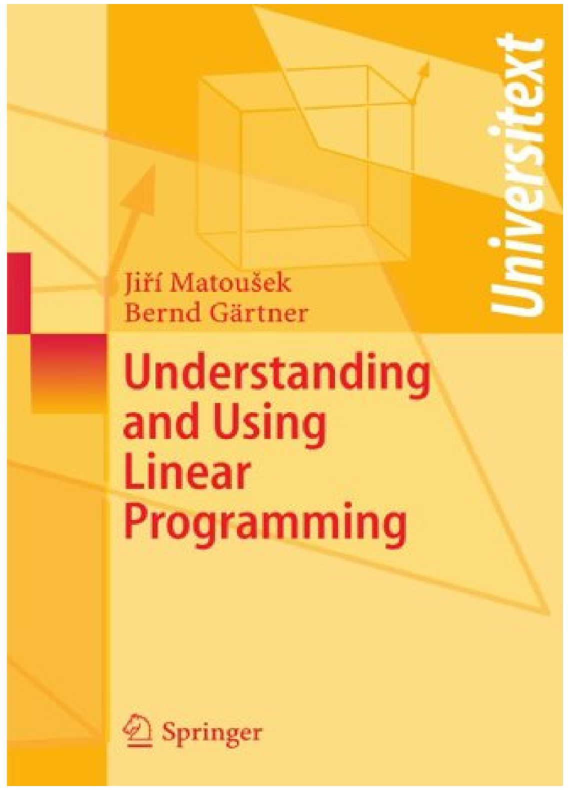

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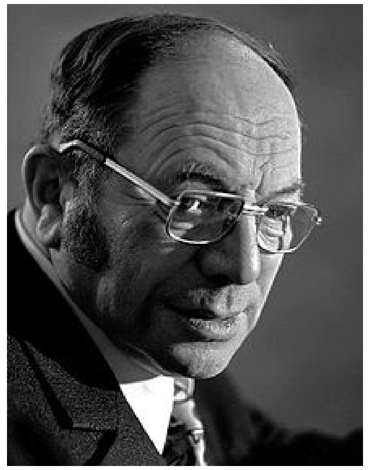
Motivation: many problems can be modelled and solved using linear programming

- in economics and industrial applications
- in computer science, in particular integer linear programs for combinatorial optimization problems
  $\rightarrow$ approximation algorithms (and more)

History

- Leonid Witaljewitsch Kantorowitsch (1939): linear programming for production planning (1975: nobel prize)



Leonid Kantorowitsch
(1912 – 1986)

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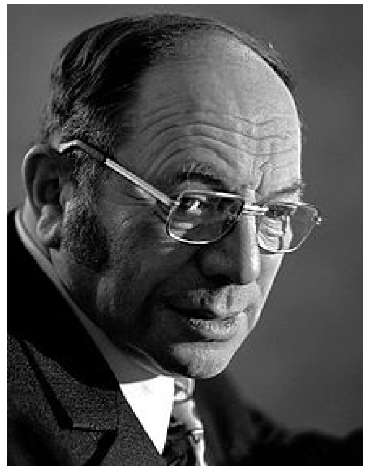

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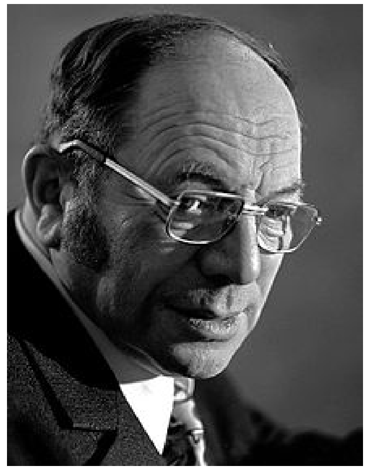
- 1950s: important for oil refineries
- 1970s: more and more industries use linear programming (e. g. airlines)



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Introduction to Linear Programming by Example

maximize $x_1 + x_2$

for $x_1, x_2 \in \mathbb{R}$ satisfying

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ -x_1 + x_2 &\leq 1 \\ x_1 + 6x_2 &\leq 15 \\ 4x_1 - x_2 &\leq 10 \end{aligned}$$

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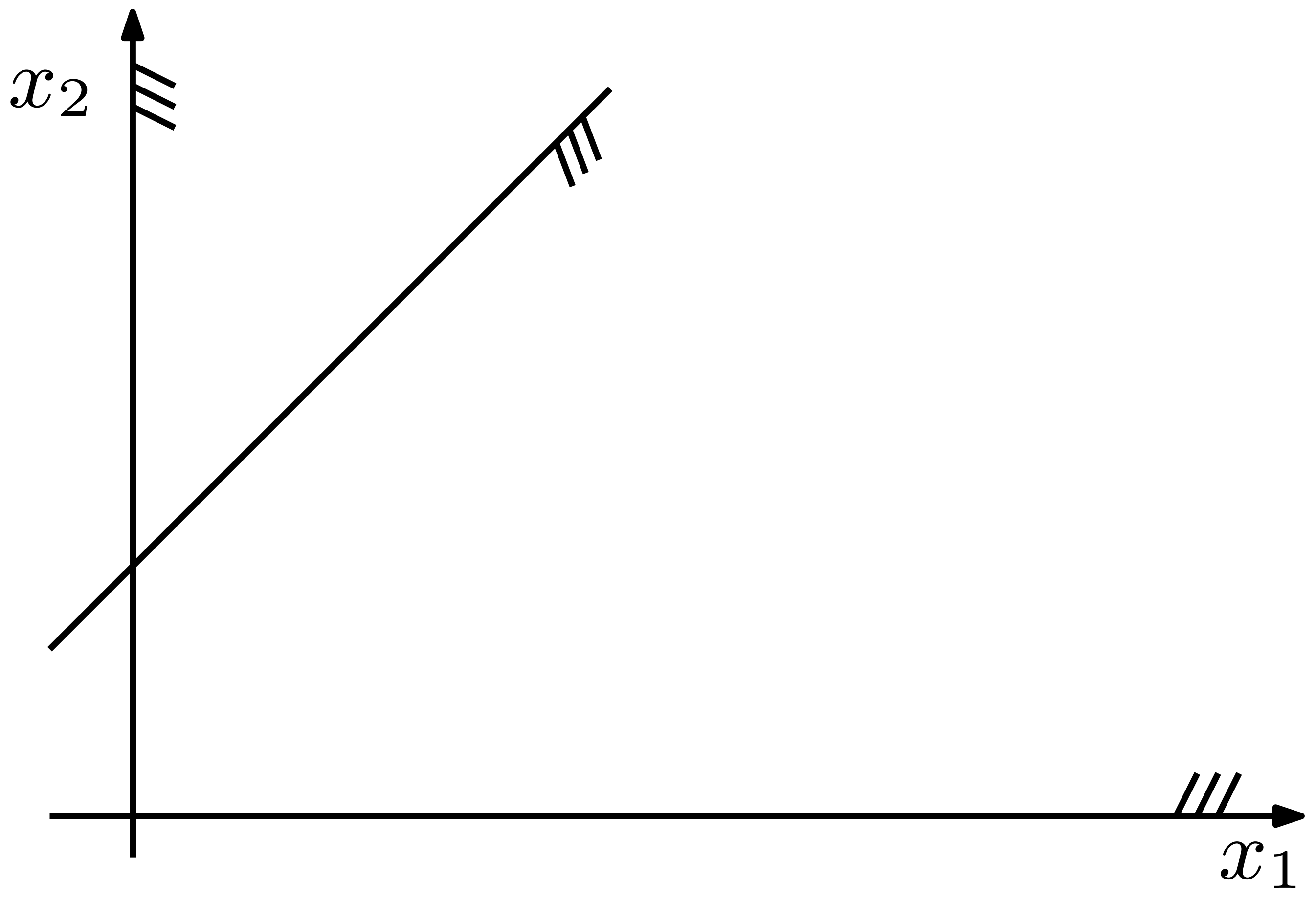

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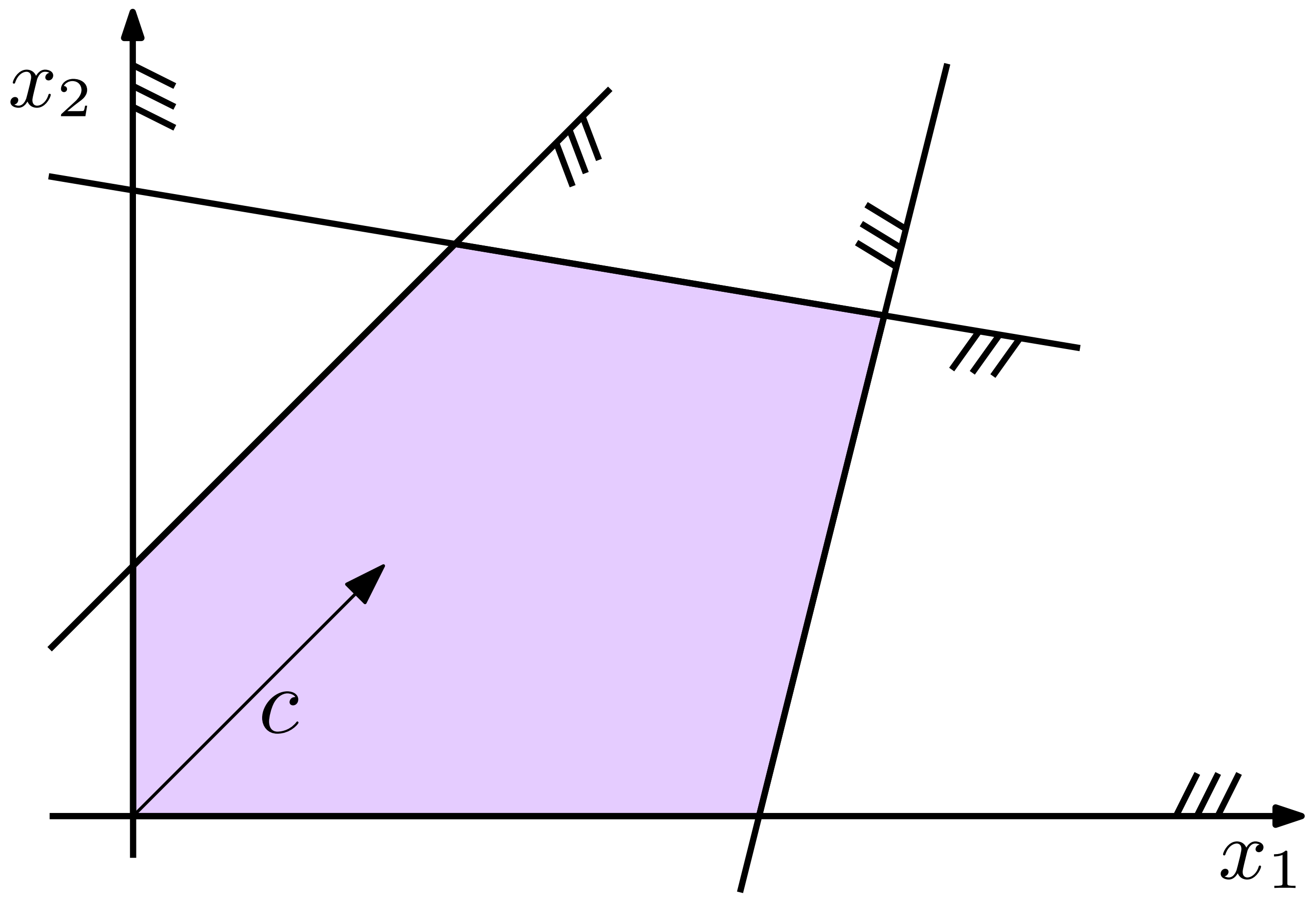

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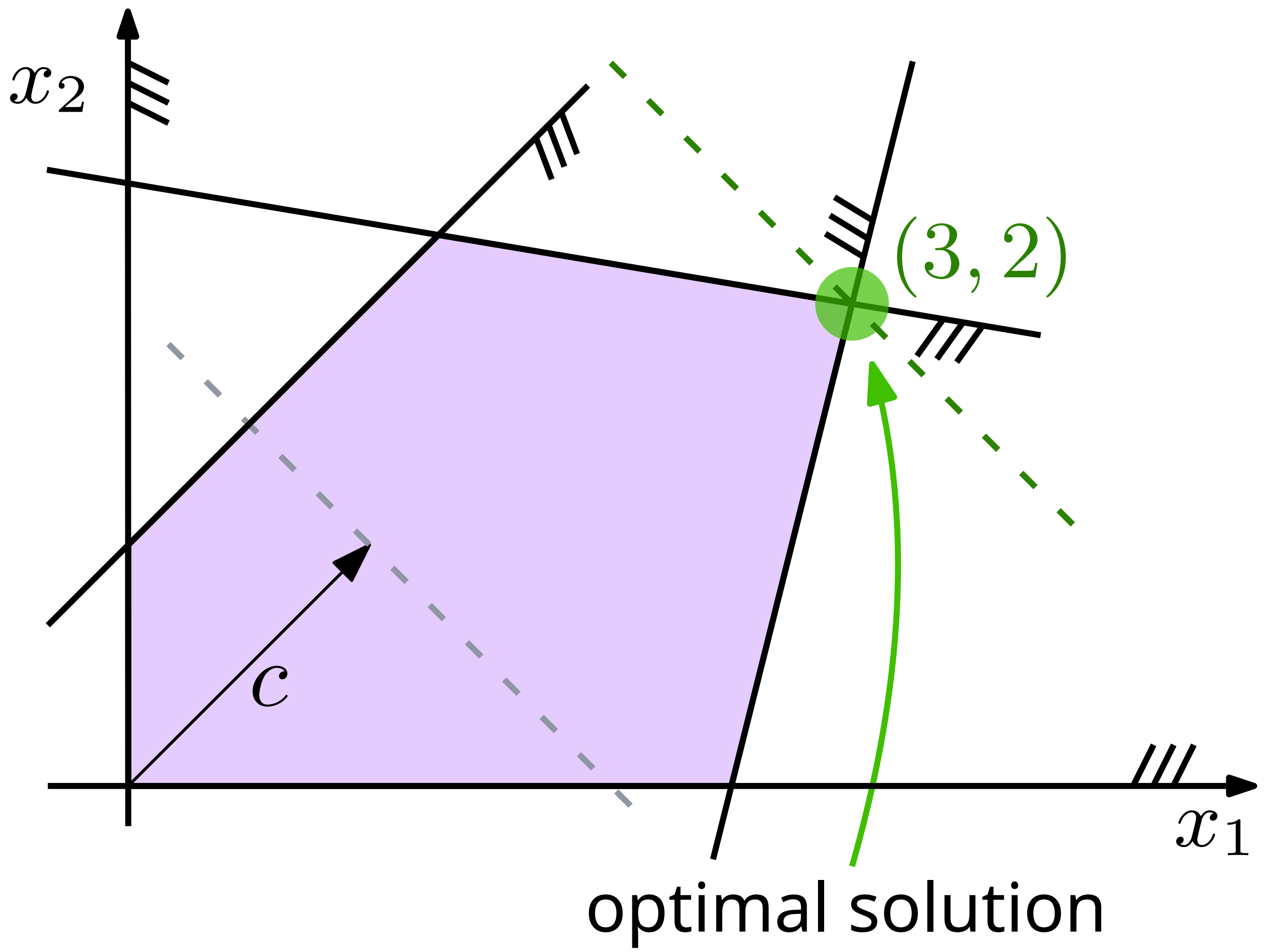

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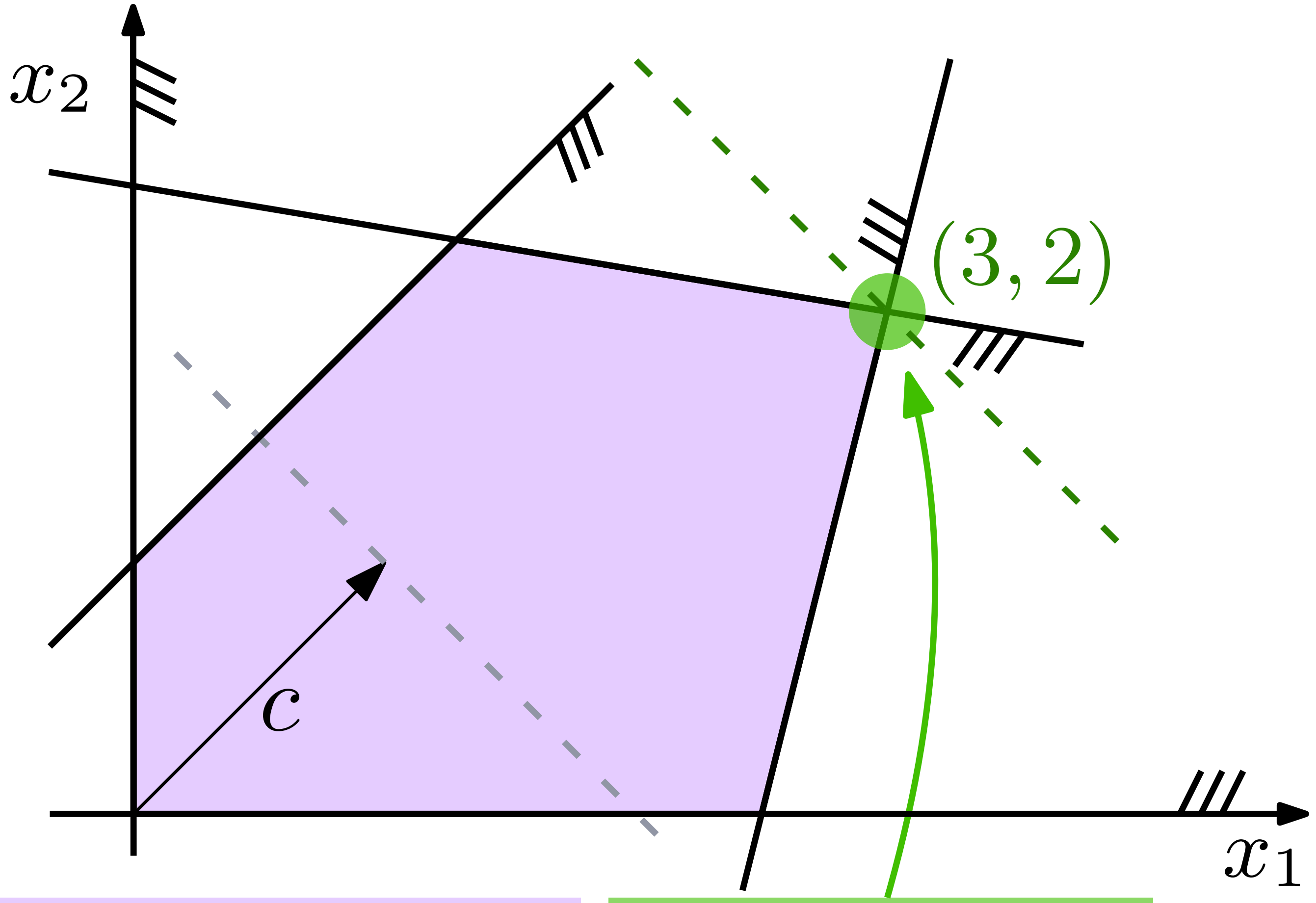

Vocabulary: objective function, constraints, feasible solutions, optimal solution

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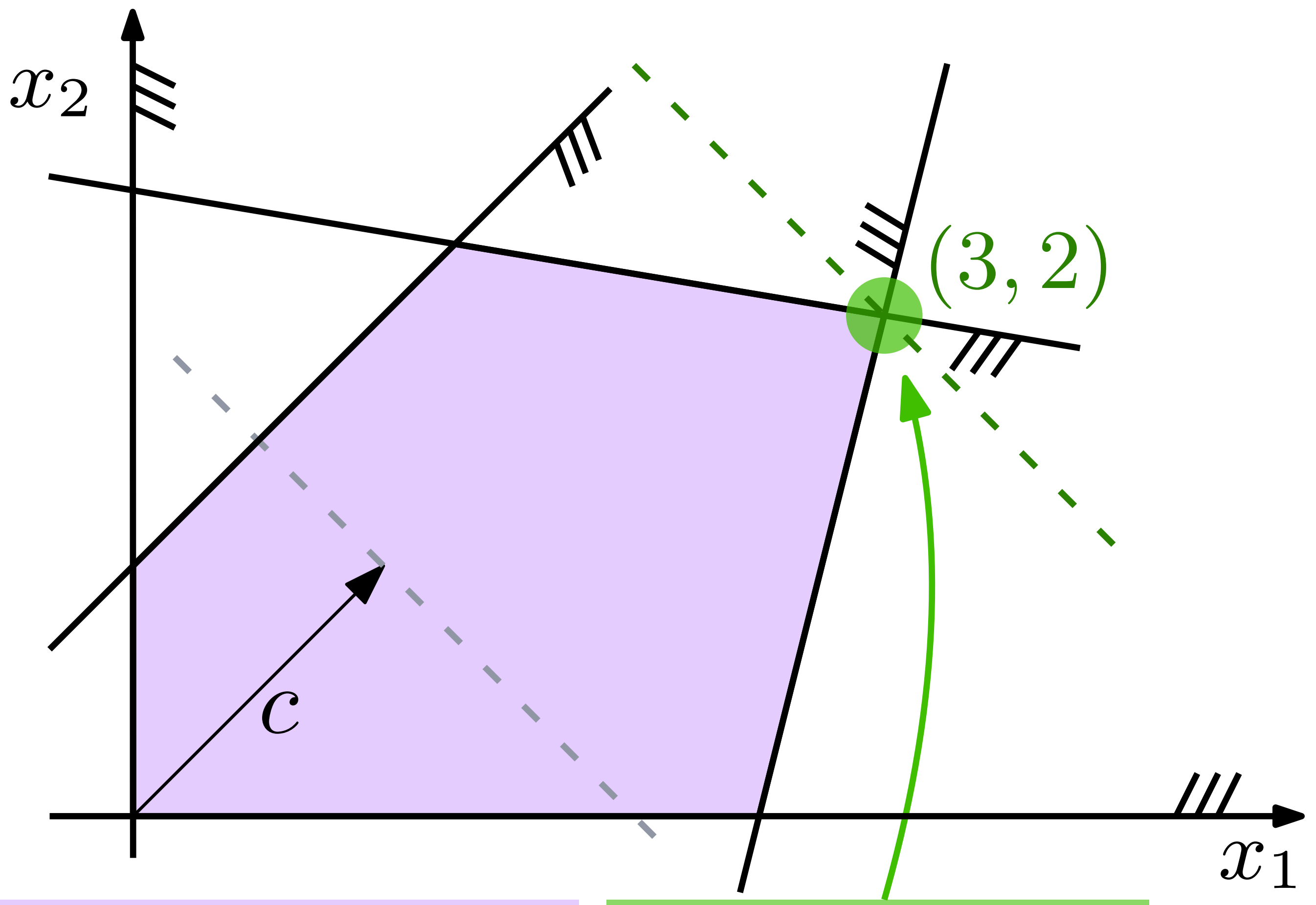

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More generally:

$$\begin{aligned} &\text{maximize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Here $x \in \mathbb{R}^n$ encodes the variables
and $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$ are given.

Introduction to Linear Programming

maximize $\cancel{x_1 + x_2}$ $x_1/6 + x_2$

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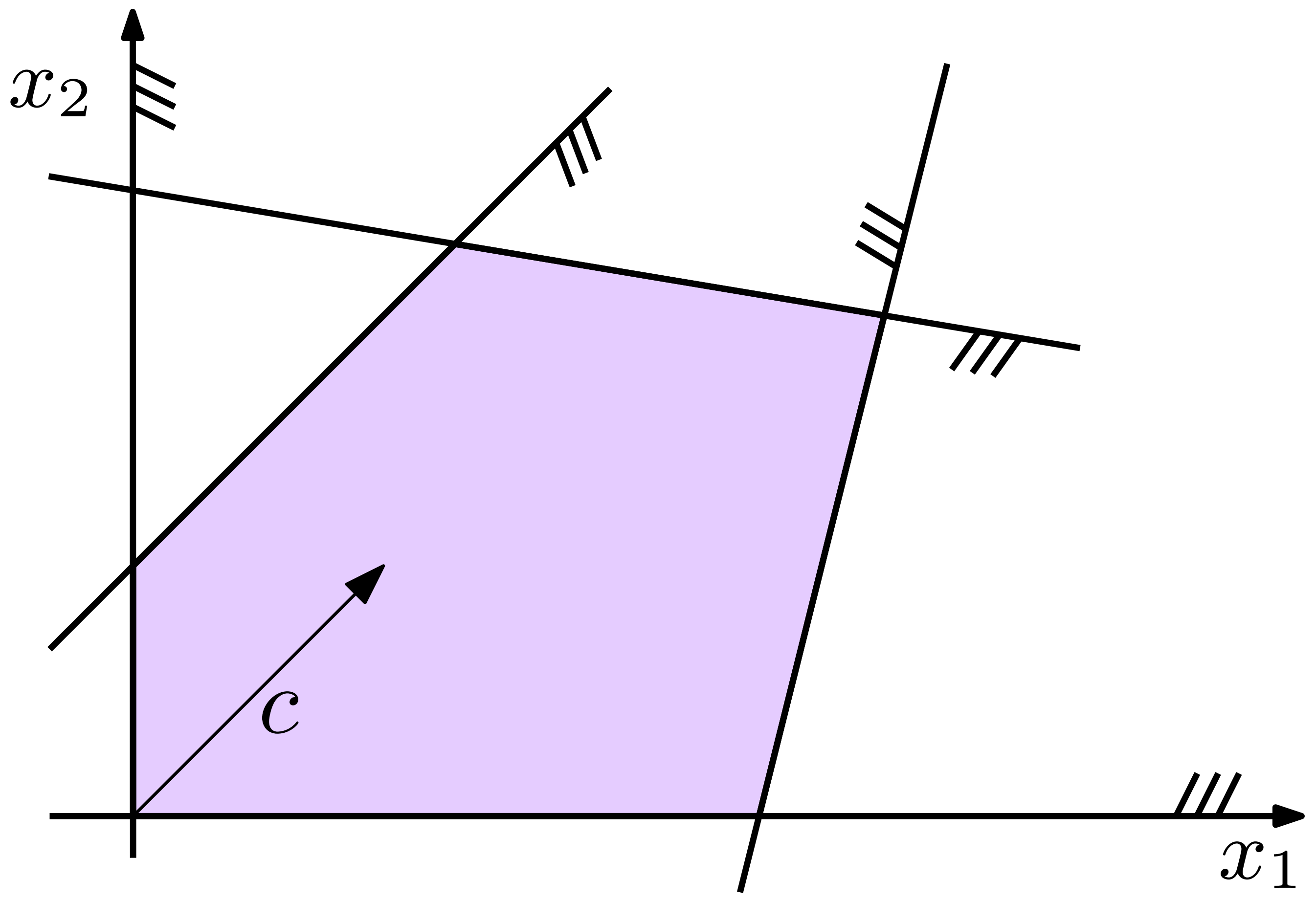

What if we change c to $(\frac{1}{6}, 1)$?

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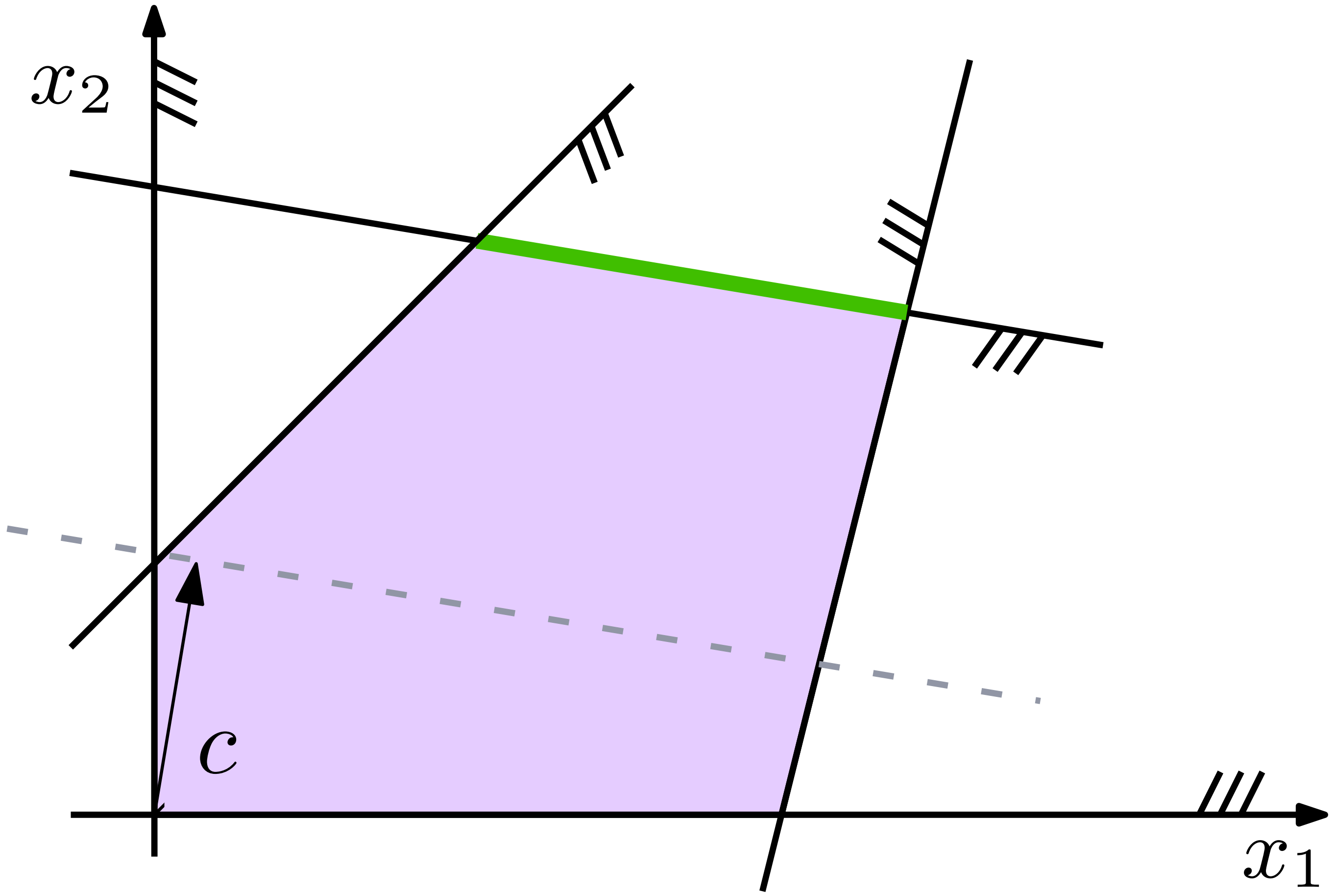

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Line segment as optimal solution.

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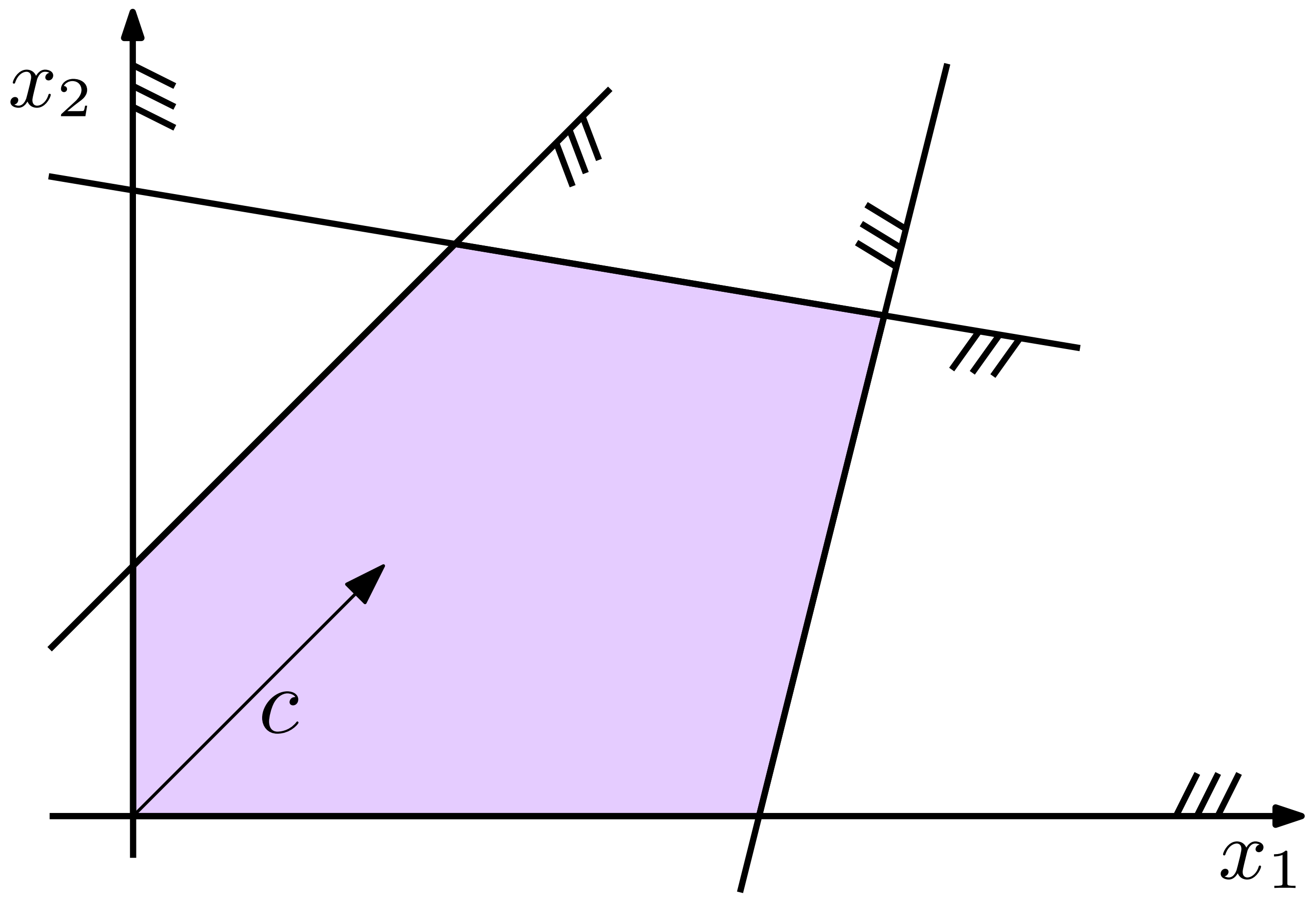

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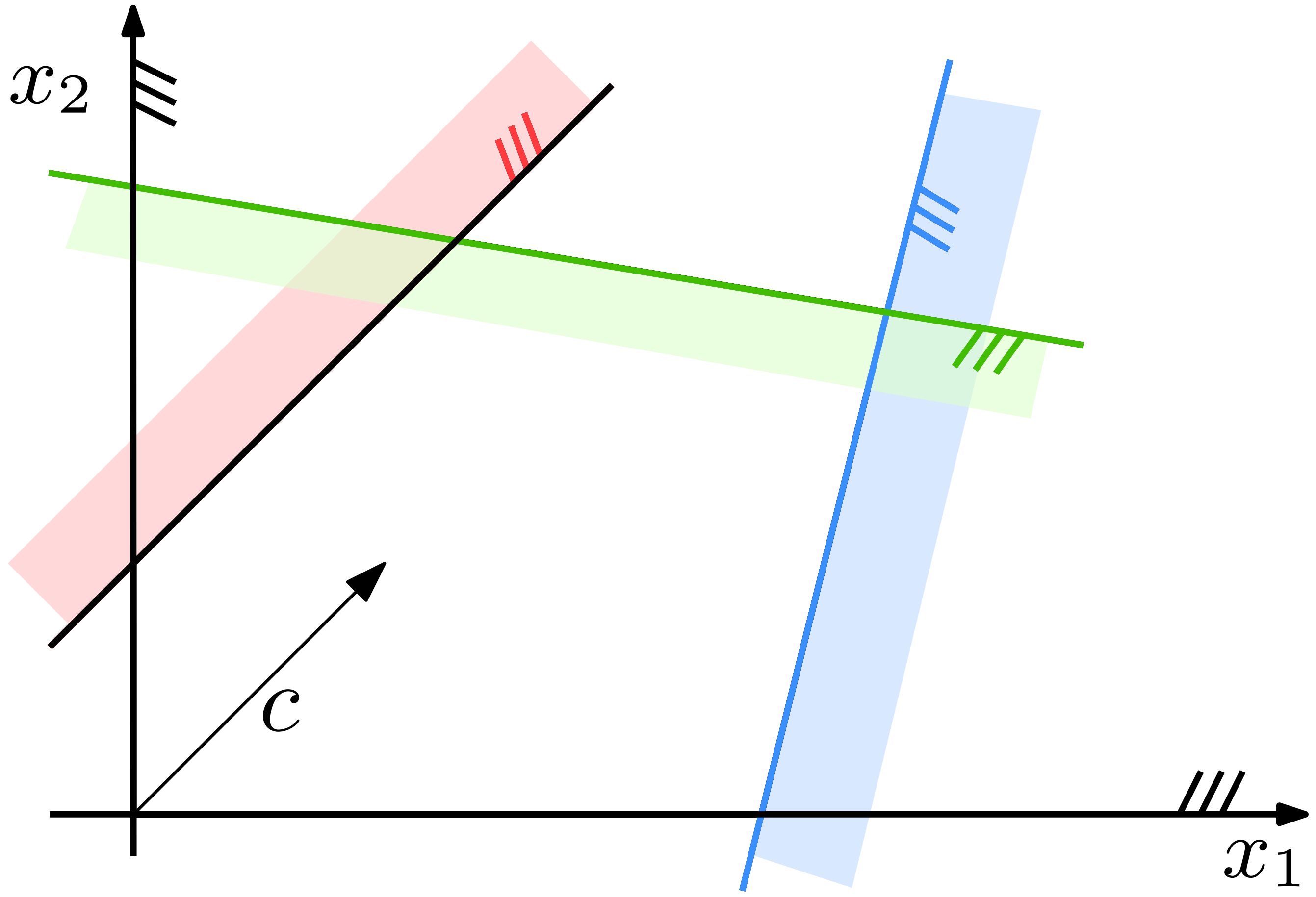

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Infeasible.

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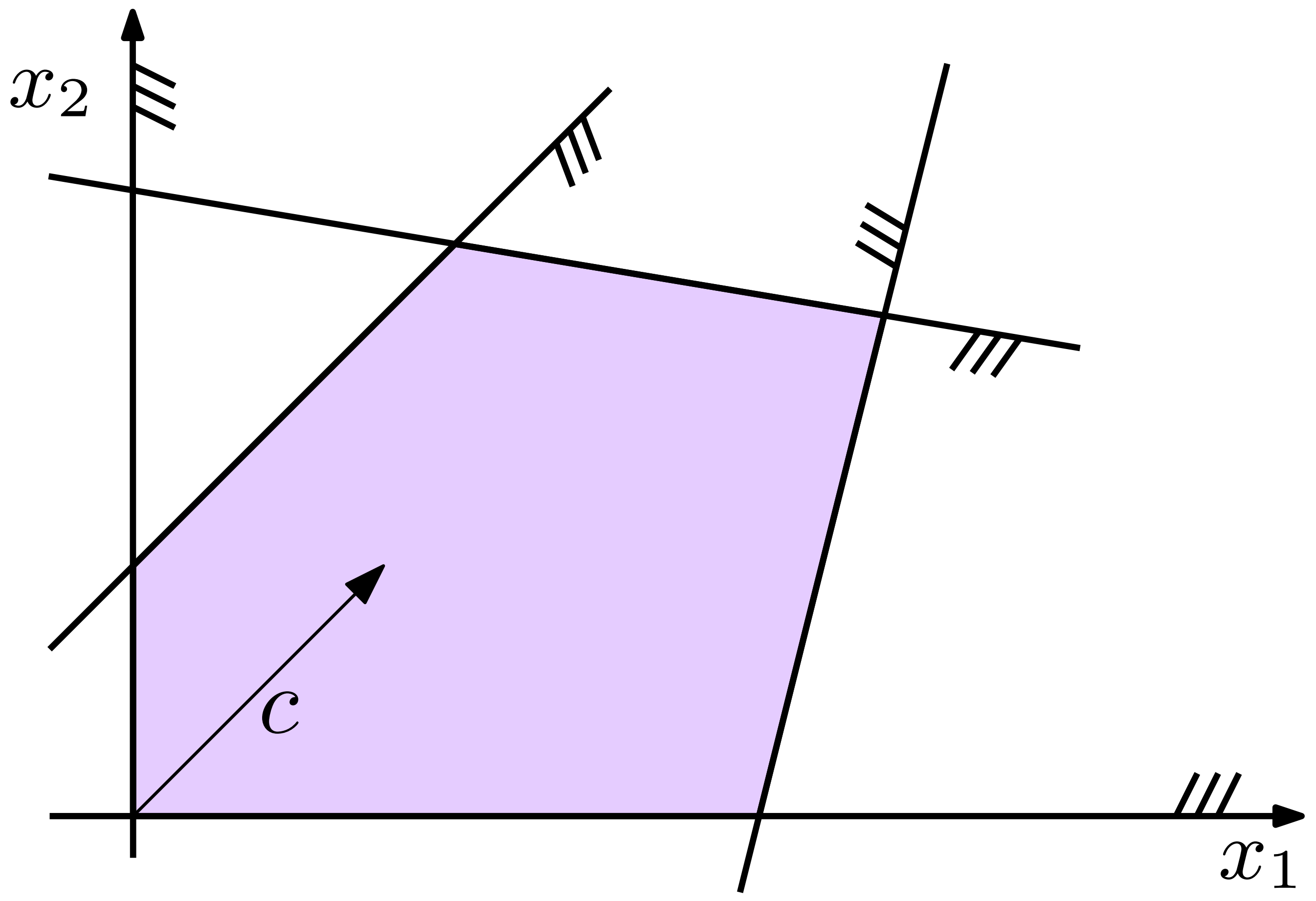

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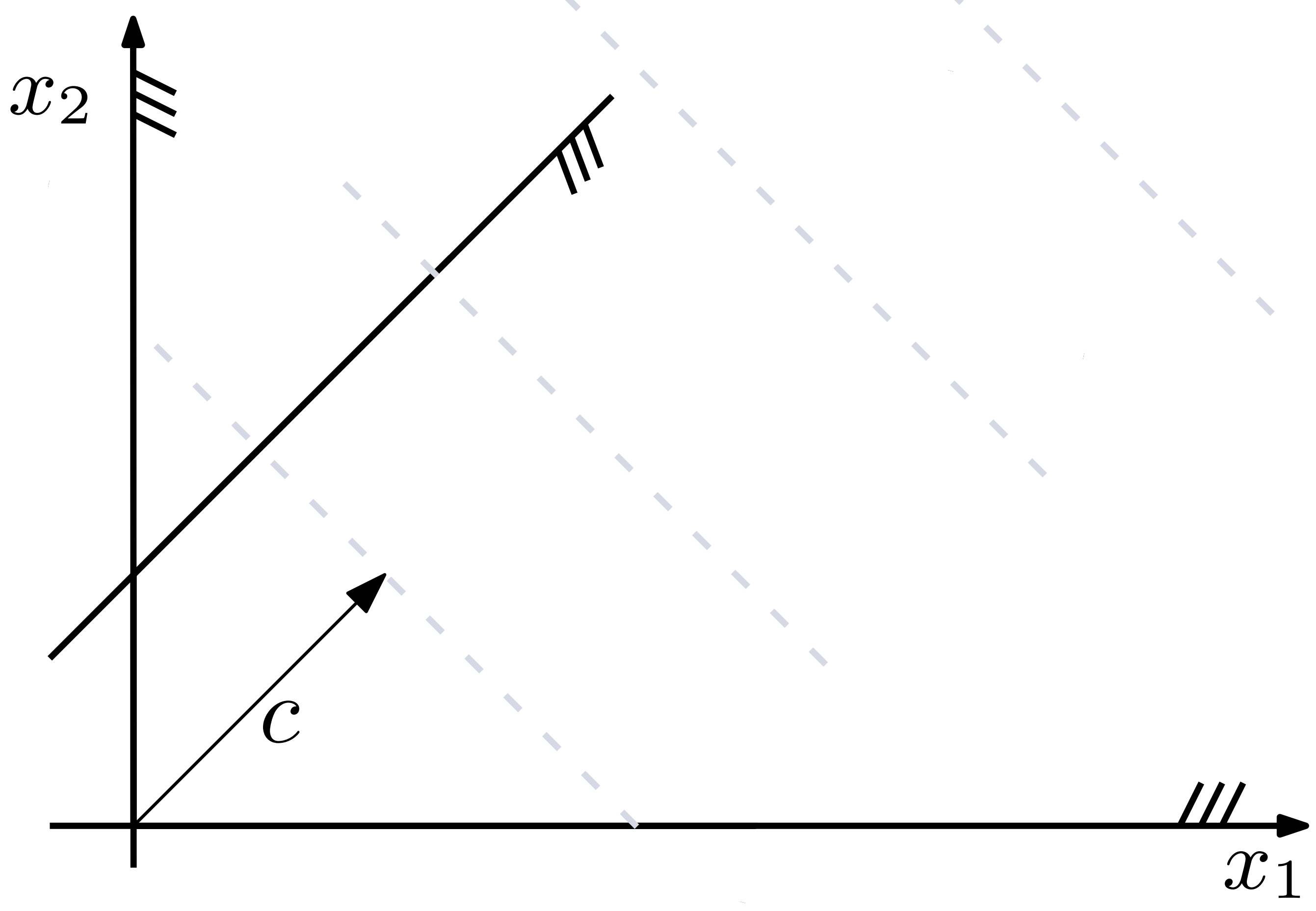

What if we remove the last two constraints?

Unbounded.

Efficiency of Linear Programming

Linear programs are efficiently solvable both

- in practice
  (good software, thousands of variables and constraints)
  simplex method: worst-case exponential time, but typically fast
- in theory
  (algorithms bounded in time by polynomial functions of inputs)
  ellipsoid method: polynomial time

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Note: Algorithms good for (i) are not the same as those for (ii)!

History of Algorithms for Linear Programming

simplex algorithm (Dantzig 1947):

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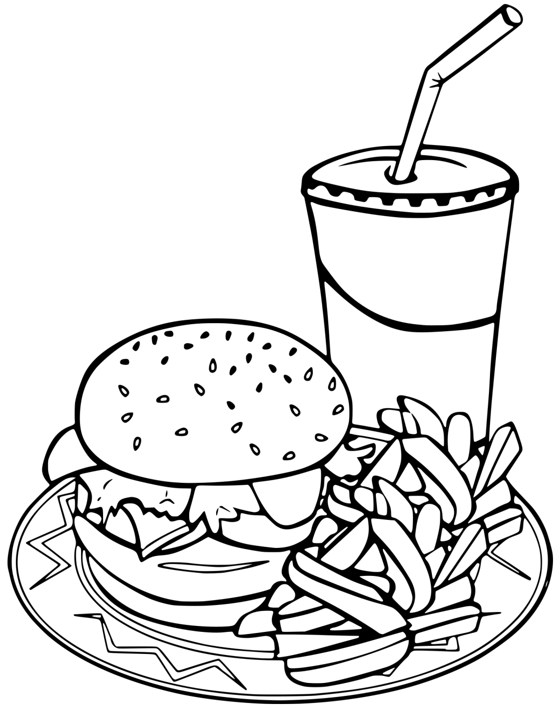
practically efficient software:

- starting 1990s: Robert E. Bixby: fast, stable simplex code (CPLEX), can solve very large LPs. Today: CPLEX, Gurobi and other solvers

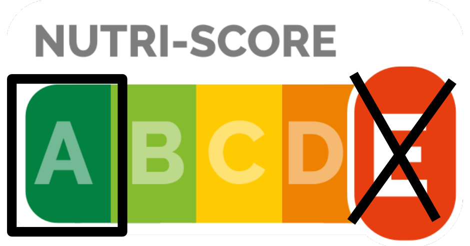
Examples

How to model problems as LPs

Example: Optimized Diet



A restaurant needs side dishes to fulfill minimimum nutritional value of a meal.
They want to do so using carrots, cabbage, and pickles.
They aim to be as cheap as possible!



Example: Optimized Diet

Nutrition \ Food	Carrot, Raw	White Cabbage, Raw	Cucumber, Pickled	Required per dish
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
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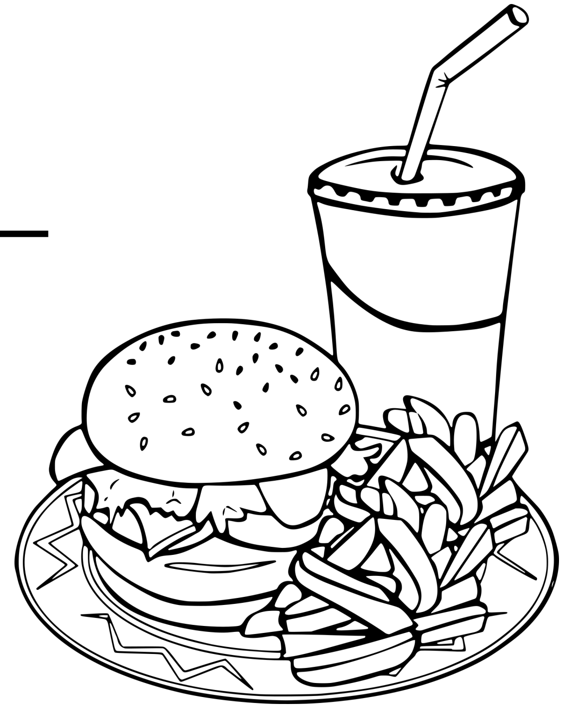
price [€/kg]	0.75	0.5	0.15	–



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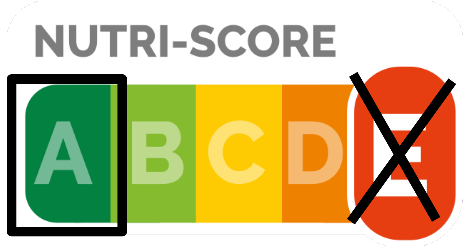

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Optimal solution:

$x_1 = 9.5g, x_2 = 38g, x_3 = 295g$

Cost: 0.07€

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Moral: Modeling is hard!

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Around 5:00 PM, Anne called, "Nu, it's five and you haven't called. What should I be cooking?" I replied that she didn't really want to know. I then read off the amounts of foods in the optimal diet. Her reaction: "The diet is a bit weird but conceivable. Is that it?"

"Not exactly," I replied, "AND 500 gallons of vinegar." She thought it funny and laughed.

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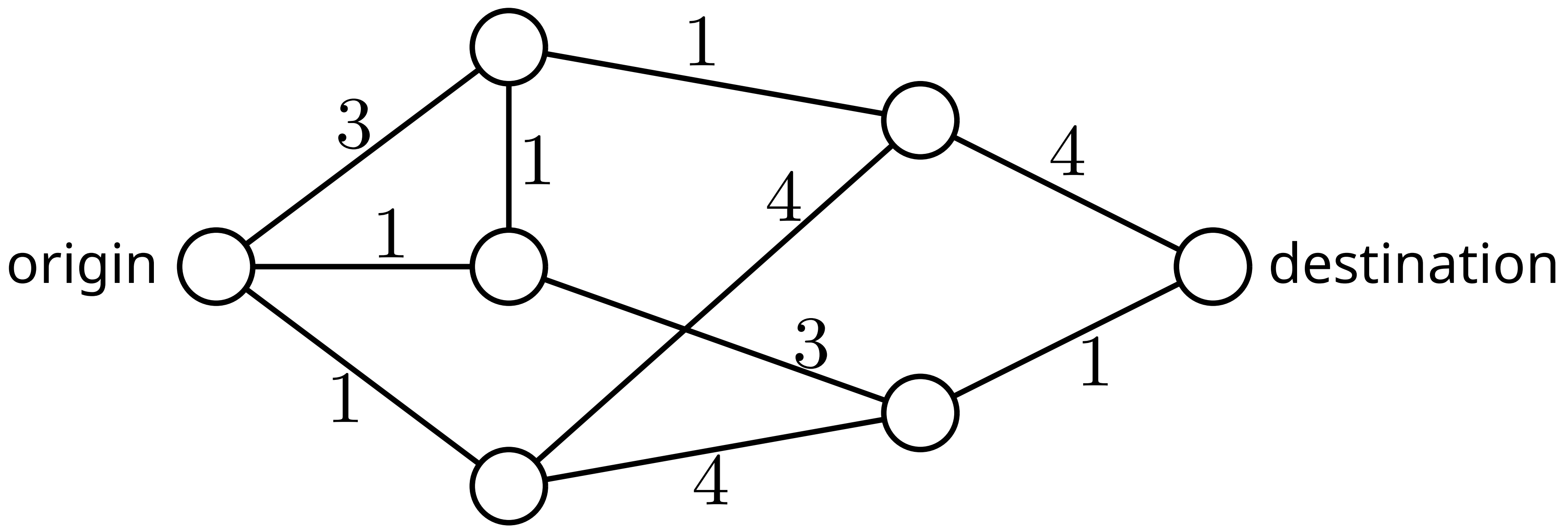
from: Dantzig, George B. "The diet problem." Interfaces 20.4 (1990): 43-47.

Example: Flow in a Network

How to send as much data as possible over a local network?



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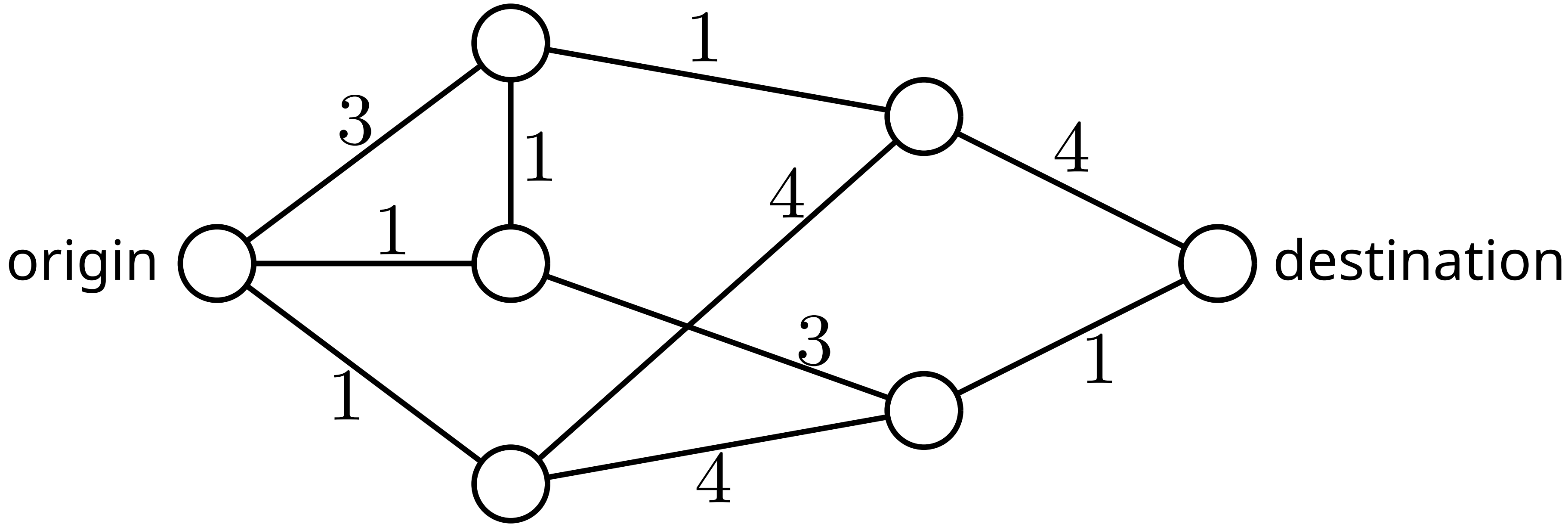

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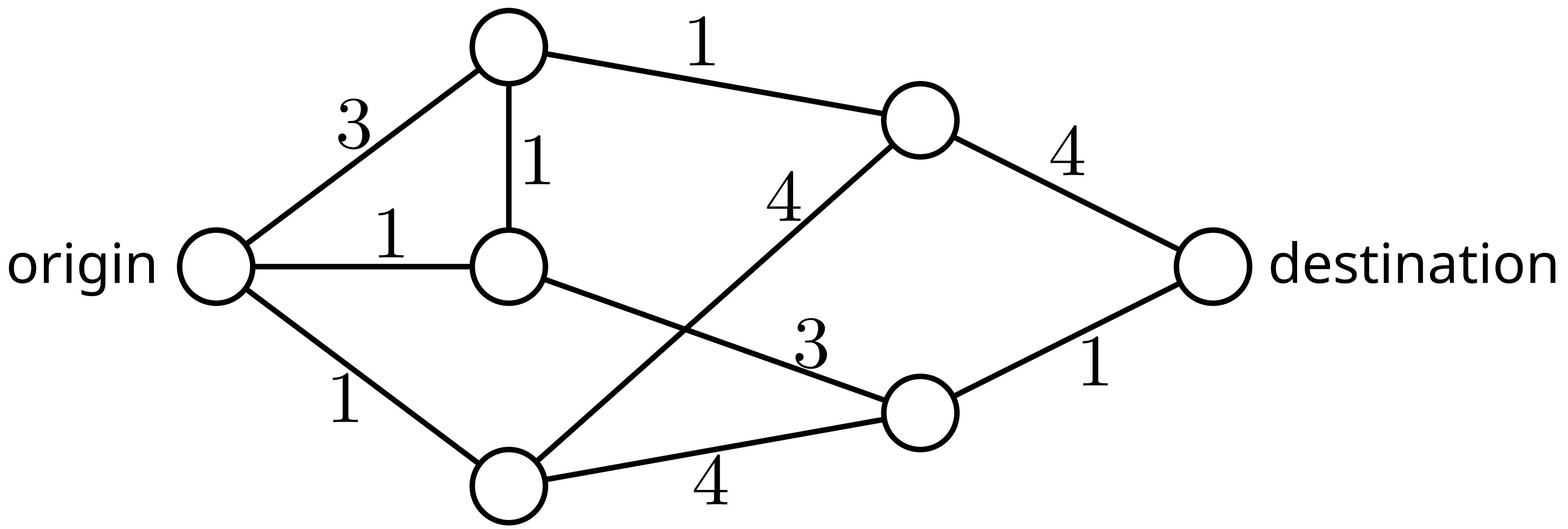

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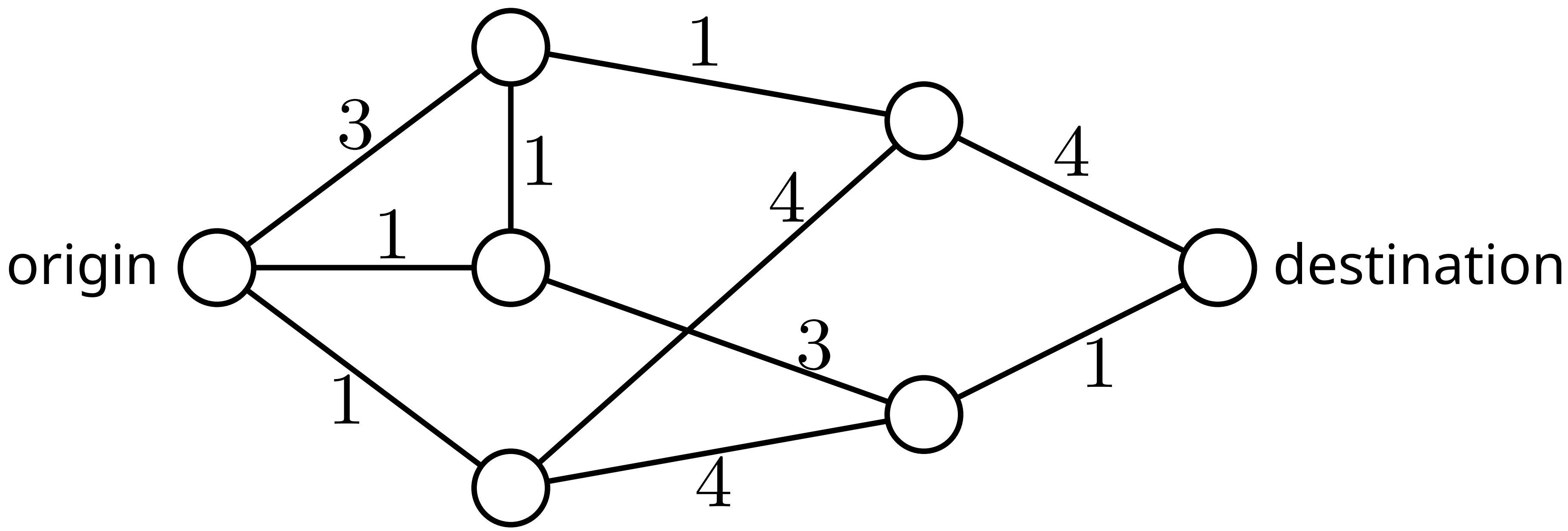

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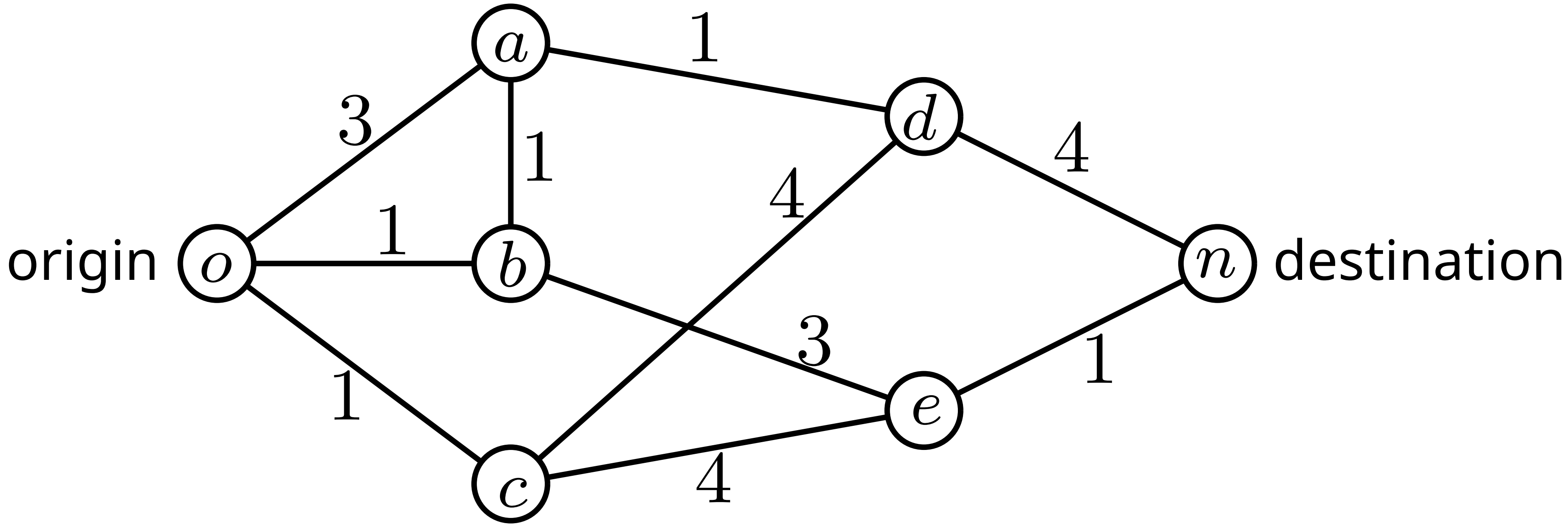

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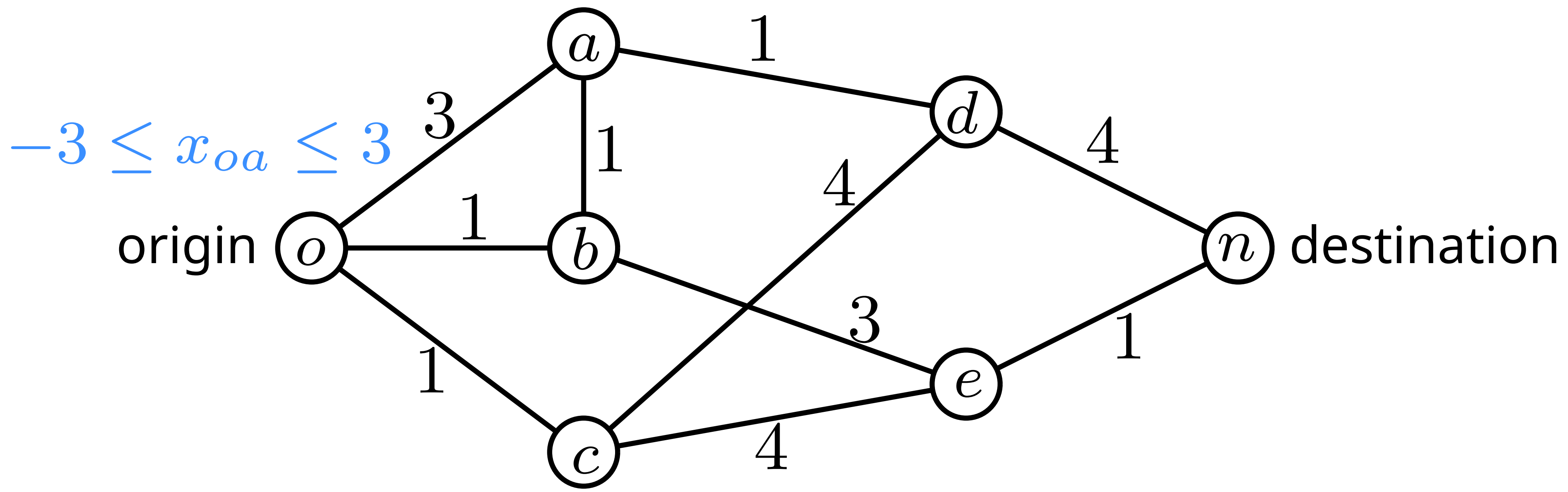

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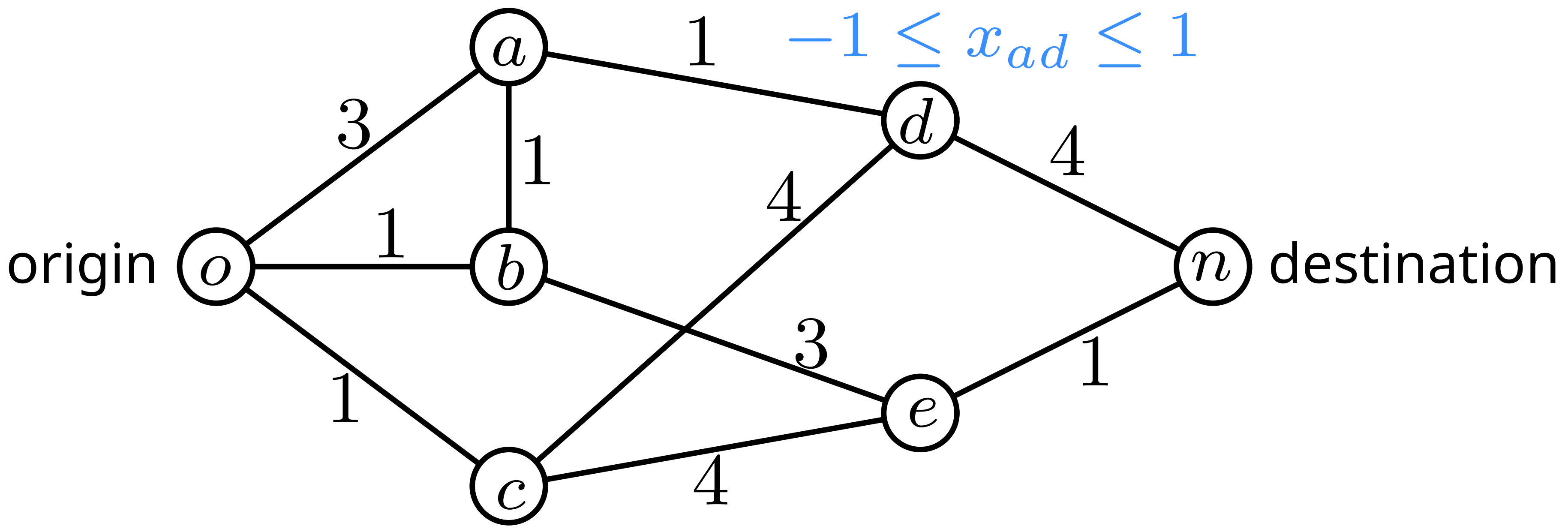

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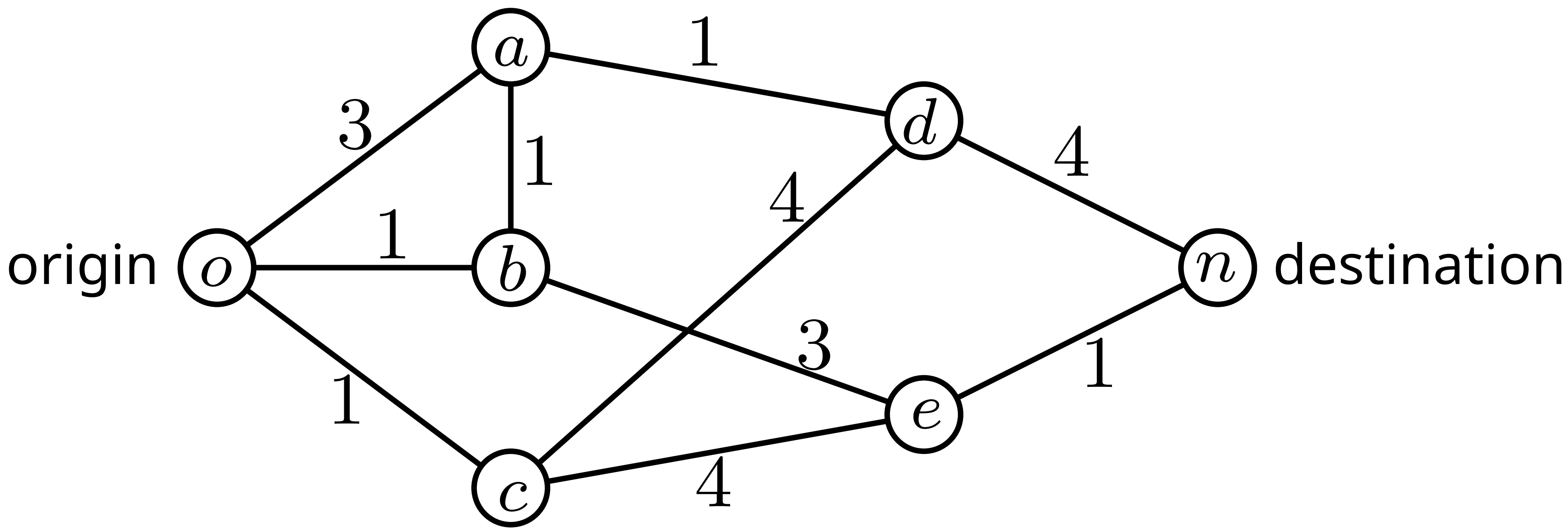

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1. flow $\leq$ capacities on edges how?
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Example: Flow in a Network

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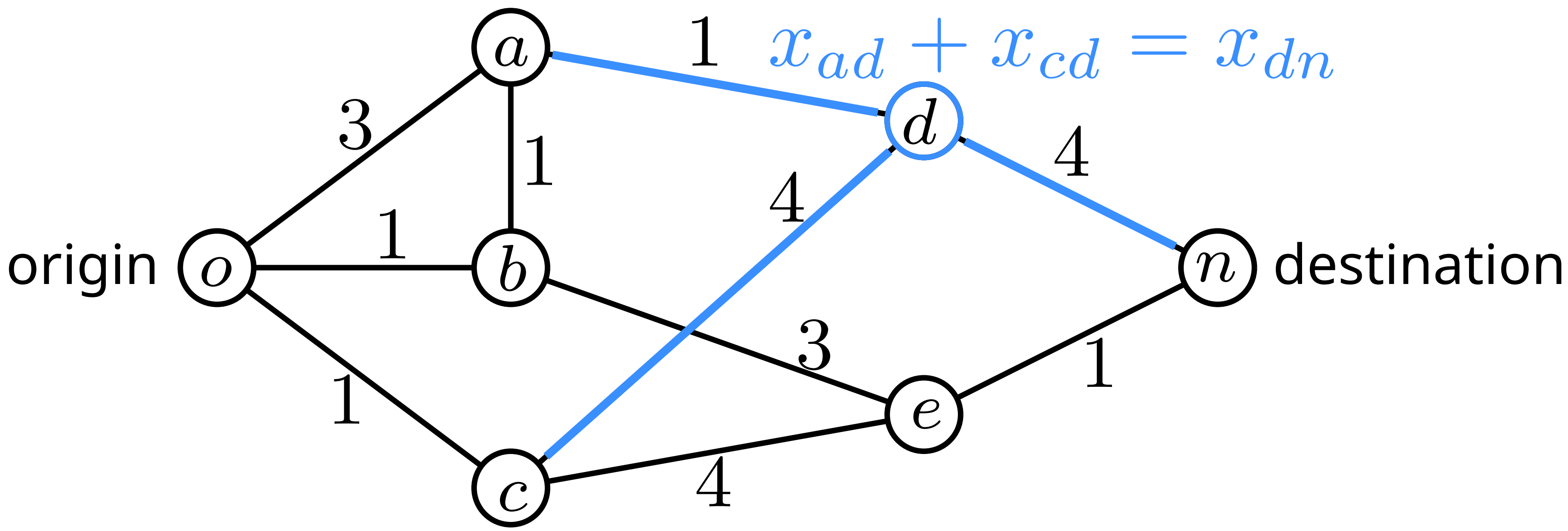

nodes cannot store data and links can transport in only one direction

$\rightarrow$ need to determine orientation and amount per edge (with direction)

Can we formulate this as LP?

$\rightarrow$ introduce variable x_{uv} for each edge (u, v) and require

1. flow $\leq$ capacities on edges how?
2. inflow = outflow on all nodes (except origin, destination)

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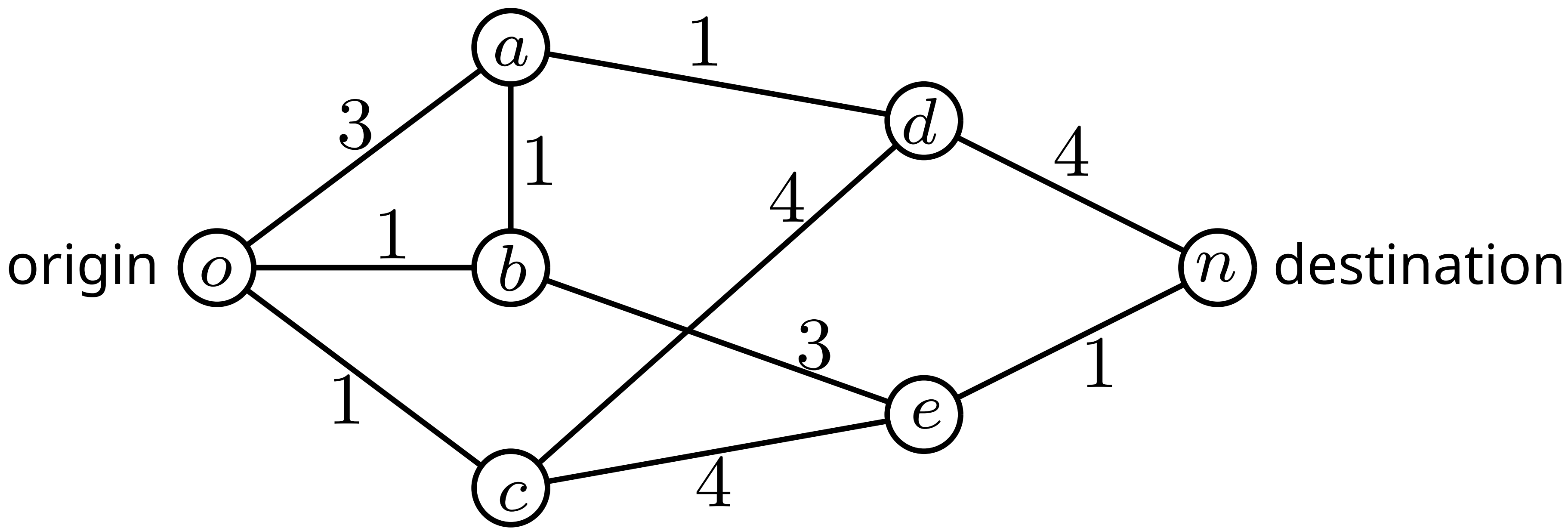

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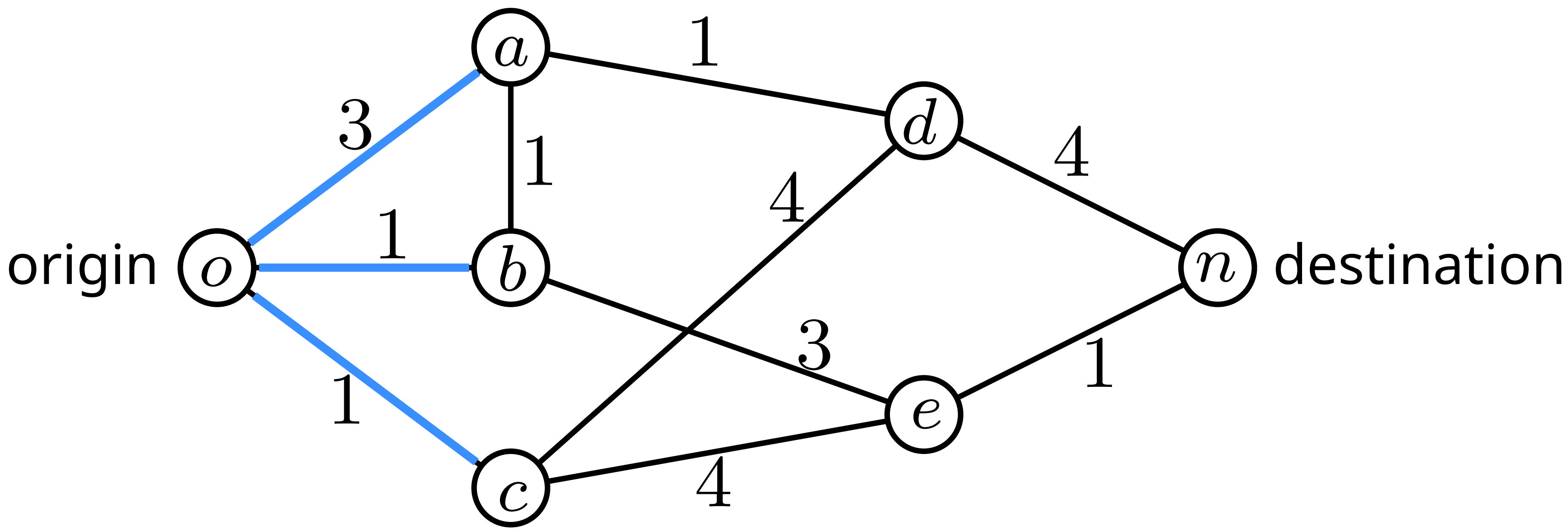

Maximize . . .?

$x_{oa} + x_{ob} + x_{oc}$

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Example: Flow in a Network

Linear Program Formulation

maximize $x_{oa} + x_{ob} + x_{oc}$

subject to
$$-3 \leq x_{oa} \leq 3,\ -1 \leq x_{ob} \leq 1,\ -1 \leq x_{oc} \leq 1$$
$$-1 \leq x_{ab} \leq 1,\ -1 \leq x_{ad} \leq 1,\ -3 \leq x_{be} \leq 3$$
$$-4 \leq x_{cd} \leq 4,\ -4 \leq x_{ce} \leq 4,\ -4 \leq x_{dn} \leq 4$$
$$-1 \leq x_{en} \leq 1$$
$$x_{oa} = x_{ab} + x_{ad}$$
$$x_{ob} + x_{ab} = x_{be}$$
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$$x_{ad} + x_{cd} = x_{dn}$$
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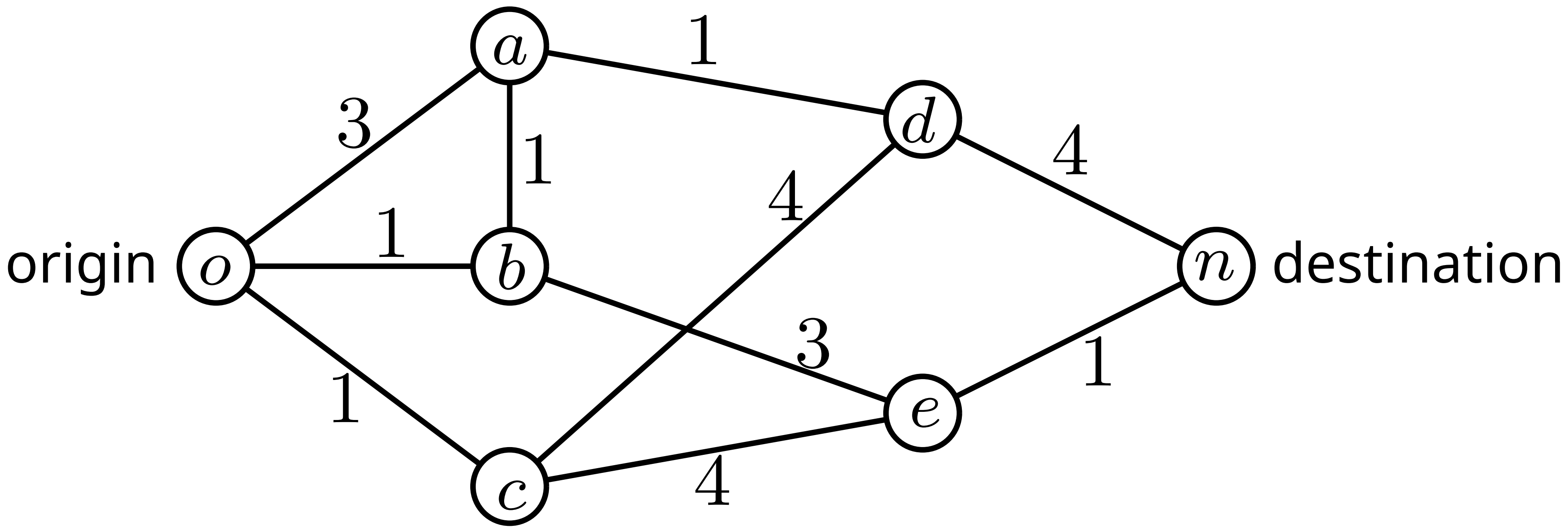

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&x_{oa} = x_{ab} + x_{ad}\\
&x_{ob} + x_{ab} = x_{be}\\
&x_{oc} = x_{cd} + x_{ce}\\
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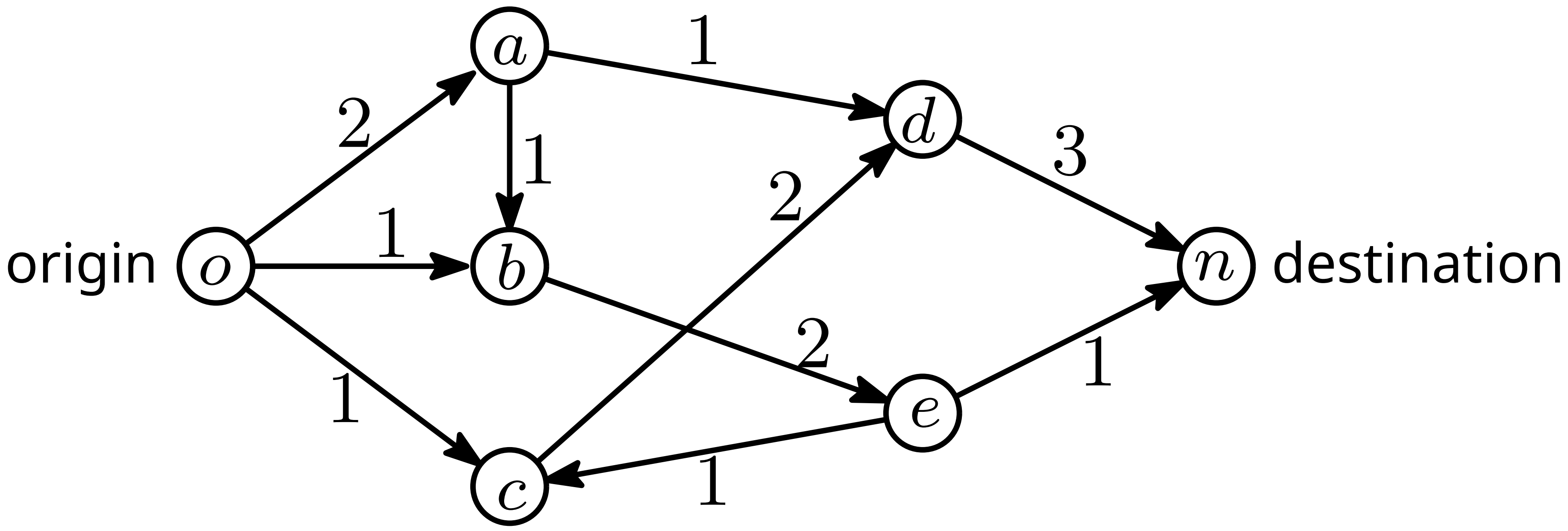
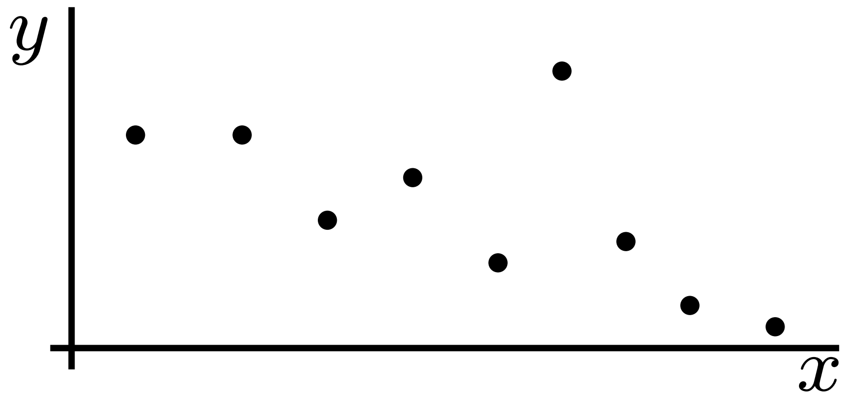
&x_{be} + x_{ce} = x_{en}
\end{aligned}
$$

Optimal solution: 4



Example: Fitting a Line

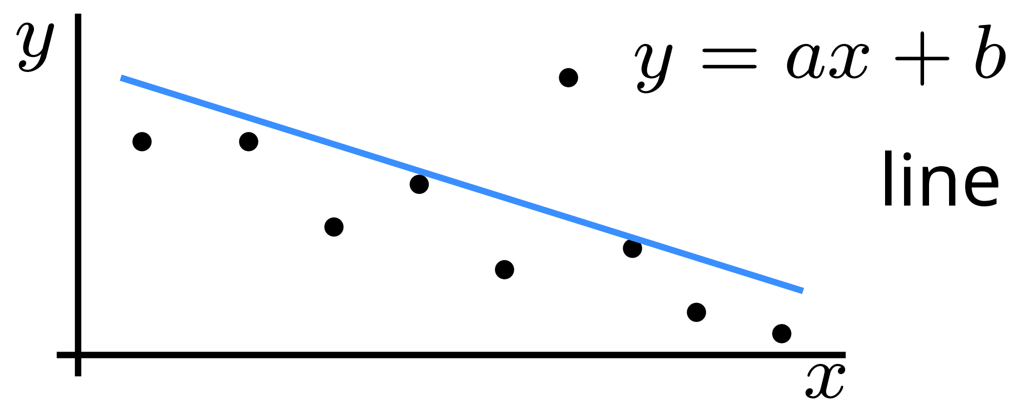
Given 2d data points, whose coordinates have a linear dependency (e.g., physical measurements),



Example: Fitting a Line

Given 2d data points, whose coordinates have a linear dependency (e.g., physical measurements),

we want to quantify the linear dependency by fitting a line through the points.



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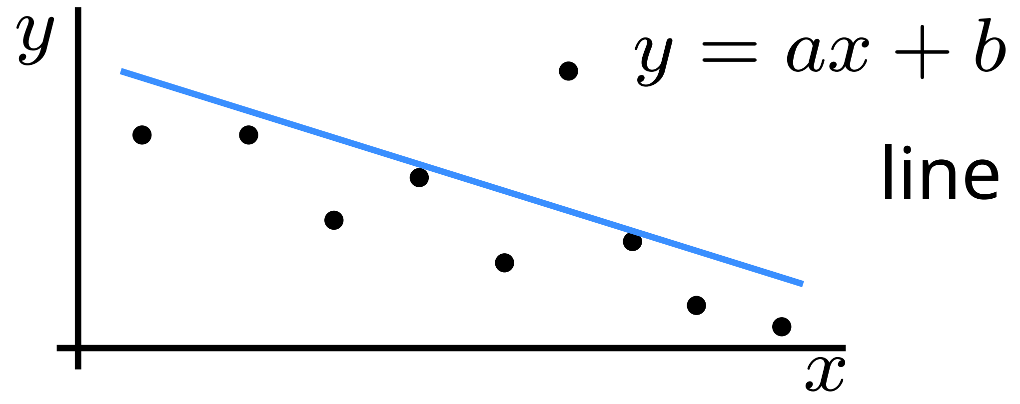


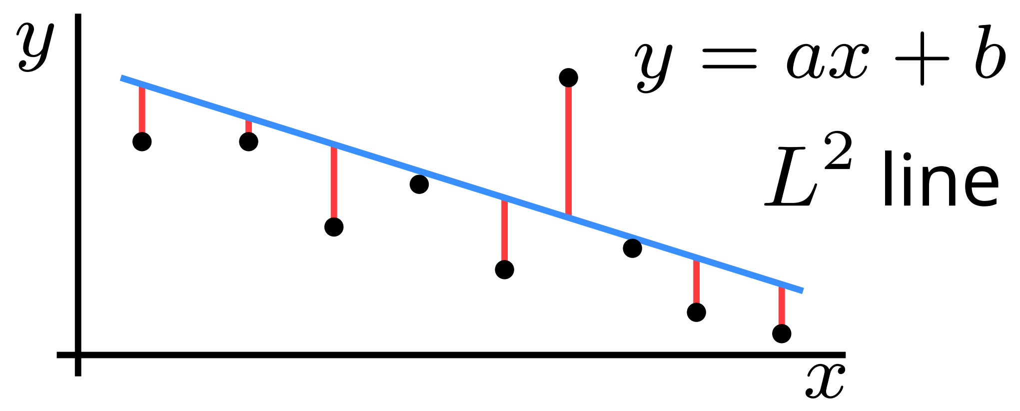
How to best fit a line, i.e., measure the error to a line?

Example: Fitting a Line

L^2 error:

minimize $\sum_{i=1}^{n}(ax_i + b - y_i)^2$

(sensitive to extreme outliers)



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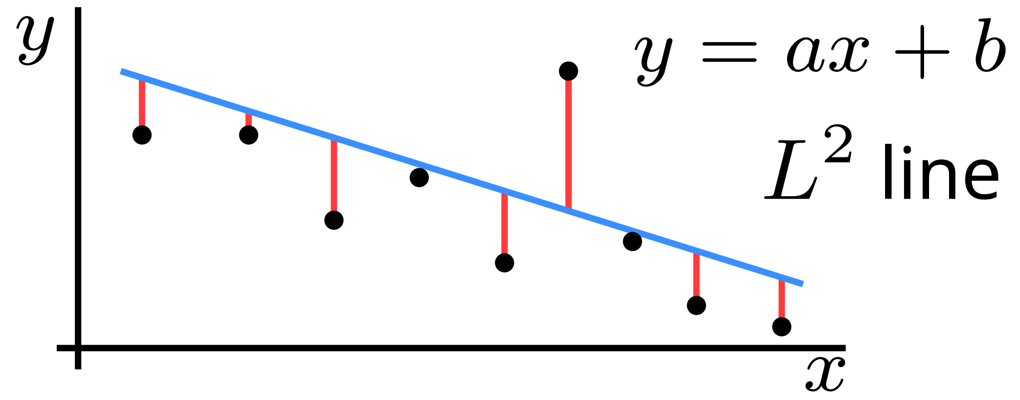

L' error:

minimize $\sum_{i=1}^{n}|ax_i + b - y_i|$

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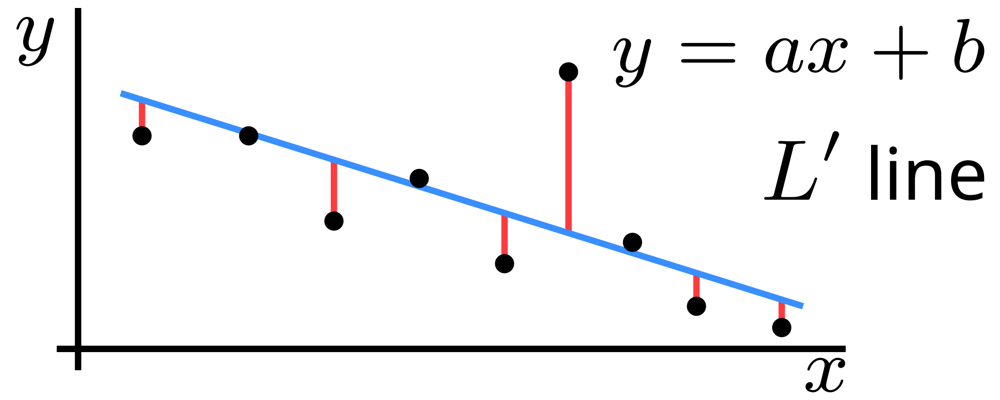

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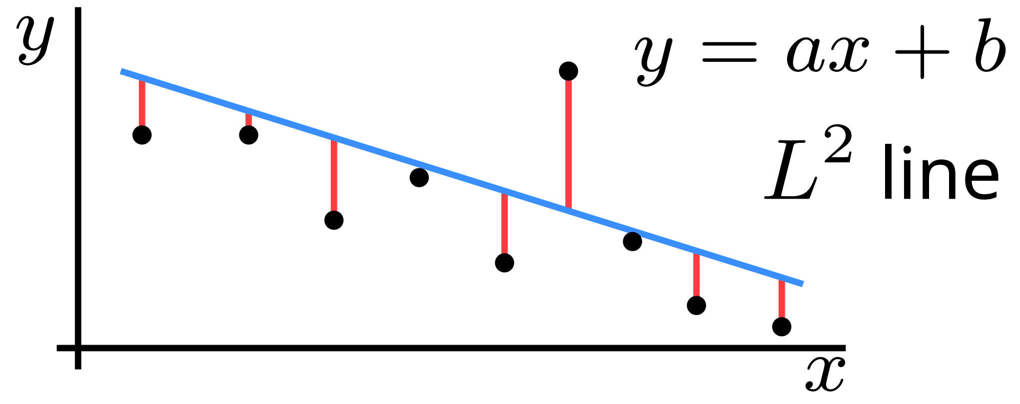

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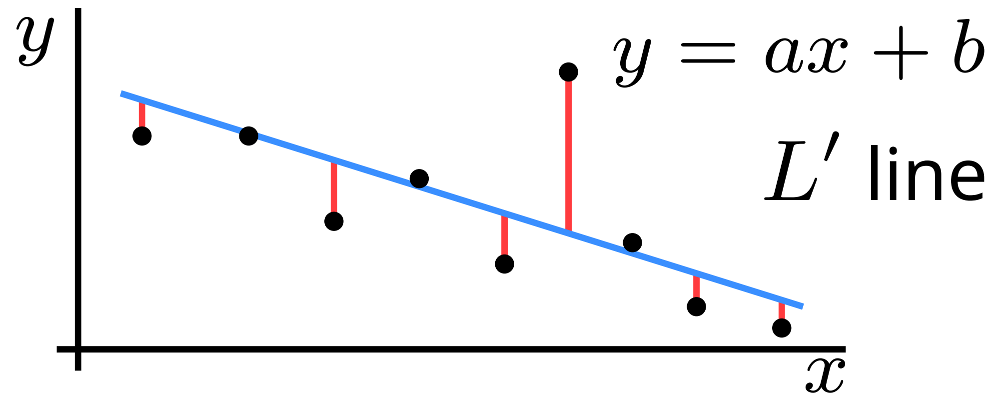

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minimize $e_1 + e_2 + \ldots + e_n$

subject to $e_i = |ax_i + b - y_i|$ for all i

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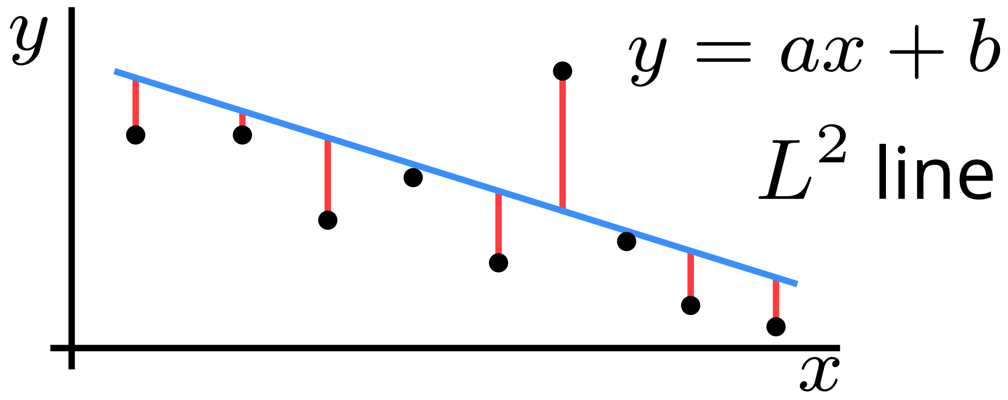

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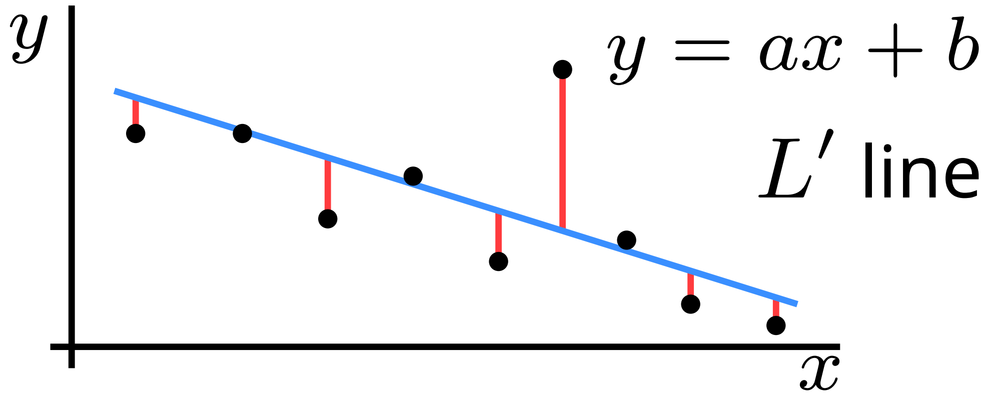
not (yet) an LP!



Example: Fitting a Line

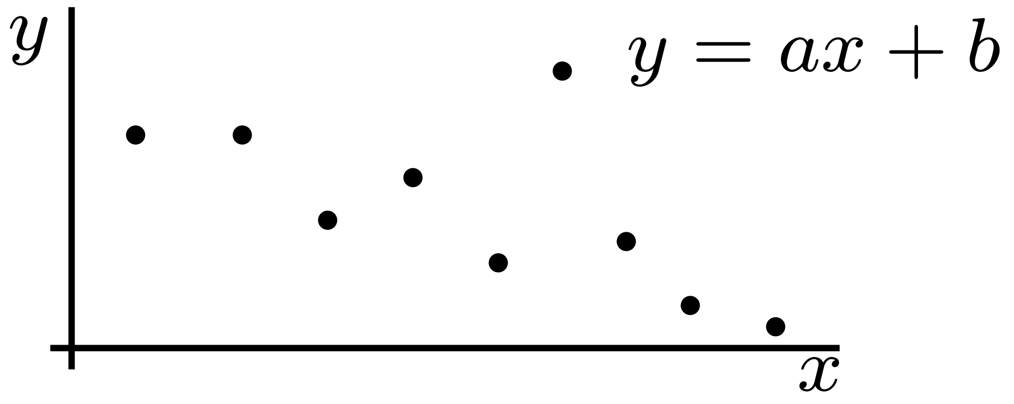
Linear program:

minimize $e_1 + e_2 + ... + e_n$

subject to $e_i \geq ax_i + b - y_i$

$e_i \geq -(ax_i + b - y_i)$

for all i



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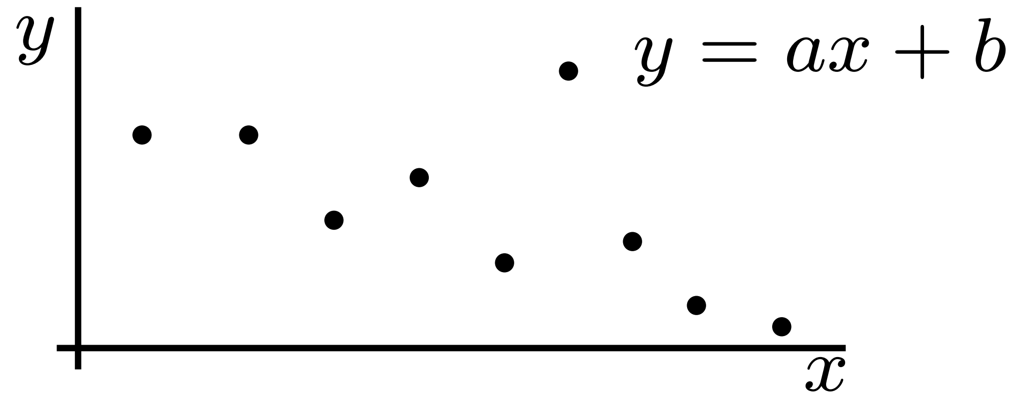

The constraints guarantee
$e_i \geq \max\{ax_i+b-y_i, -(ax_i+b-y_i)\} = |ax_i+b-y_i|$.

In an optimal solution, is satisfied with equality.

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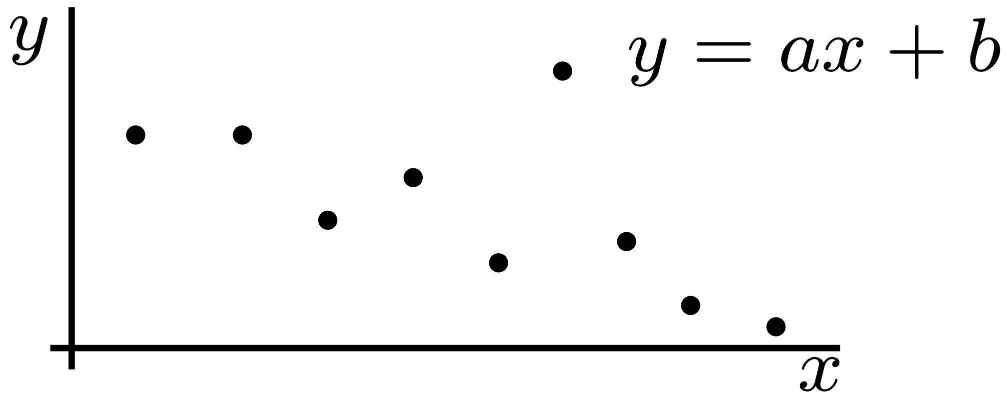

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How many variables do we have for n data points?

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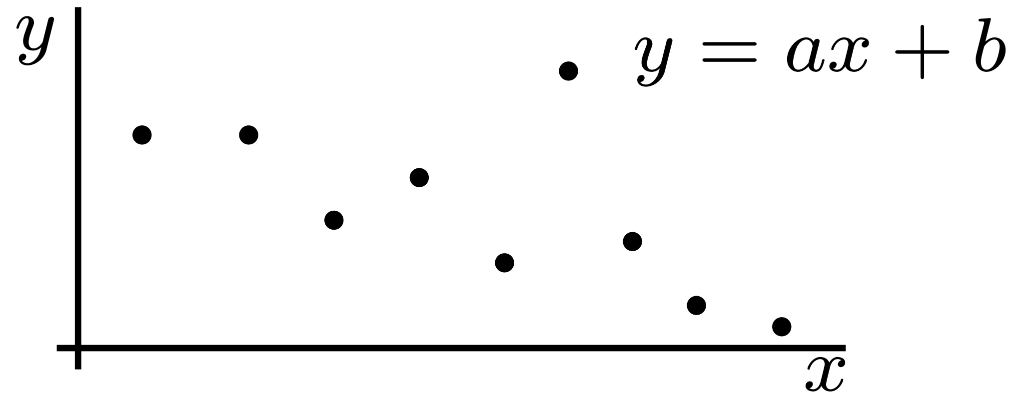

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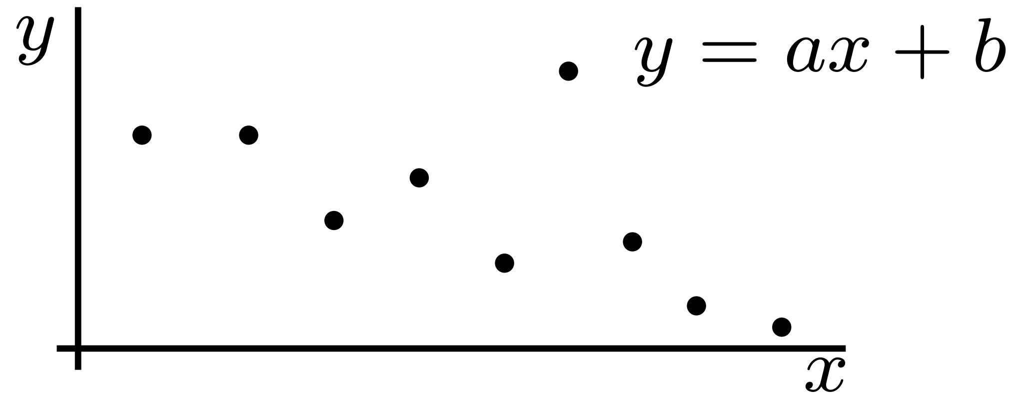

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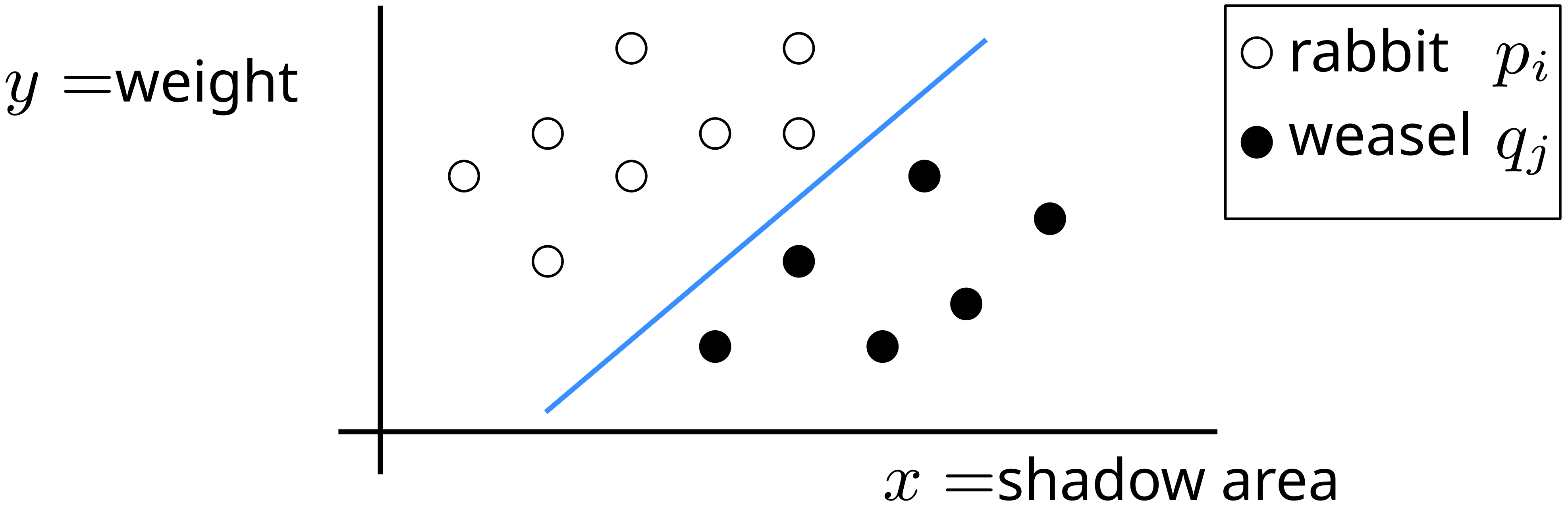
Moral: Objective functions or constraints with absolute values can often be handled by introducing extra variables or constraints.

Example: Separation of points



Does there exist a separating line $y = ax + b$?

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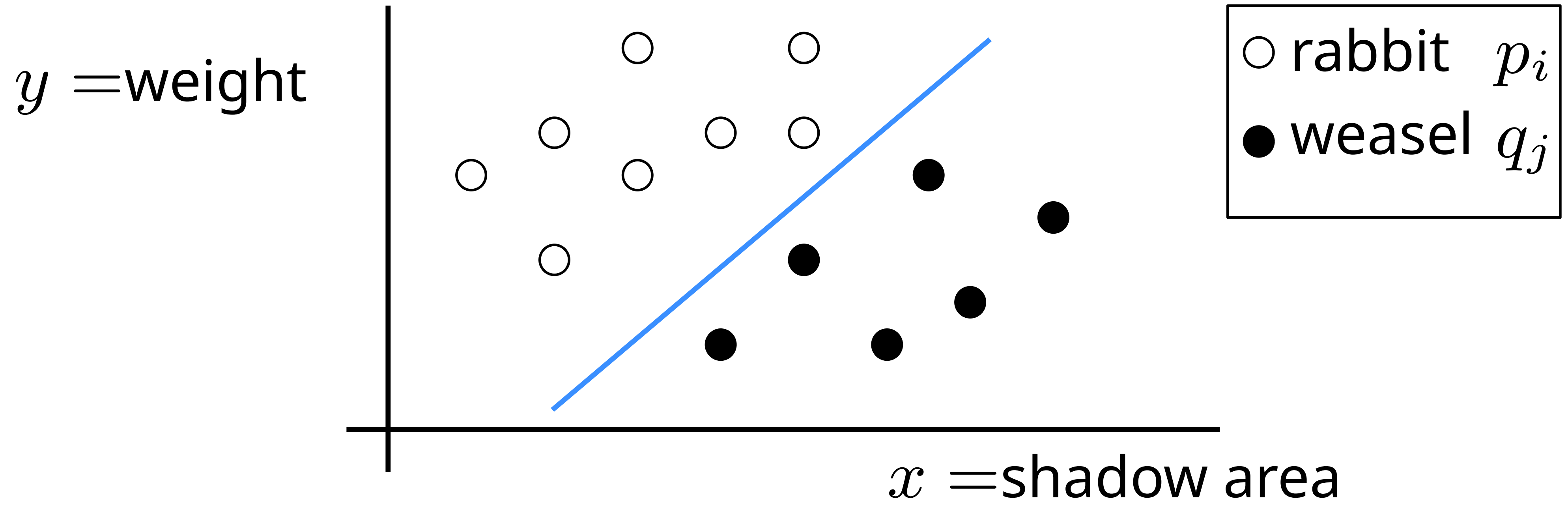

Does there exist a separating line $y = ax + b$?

Case p_i points "above" q_j points:

$$y(p_i) > ax(p_i) + b \qquad \text{for all } i$$
$$y(q_i) < ax(q_j) + b \qquad \text{for all } j$$

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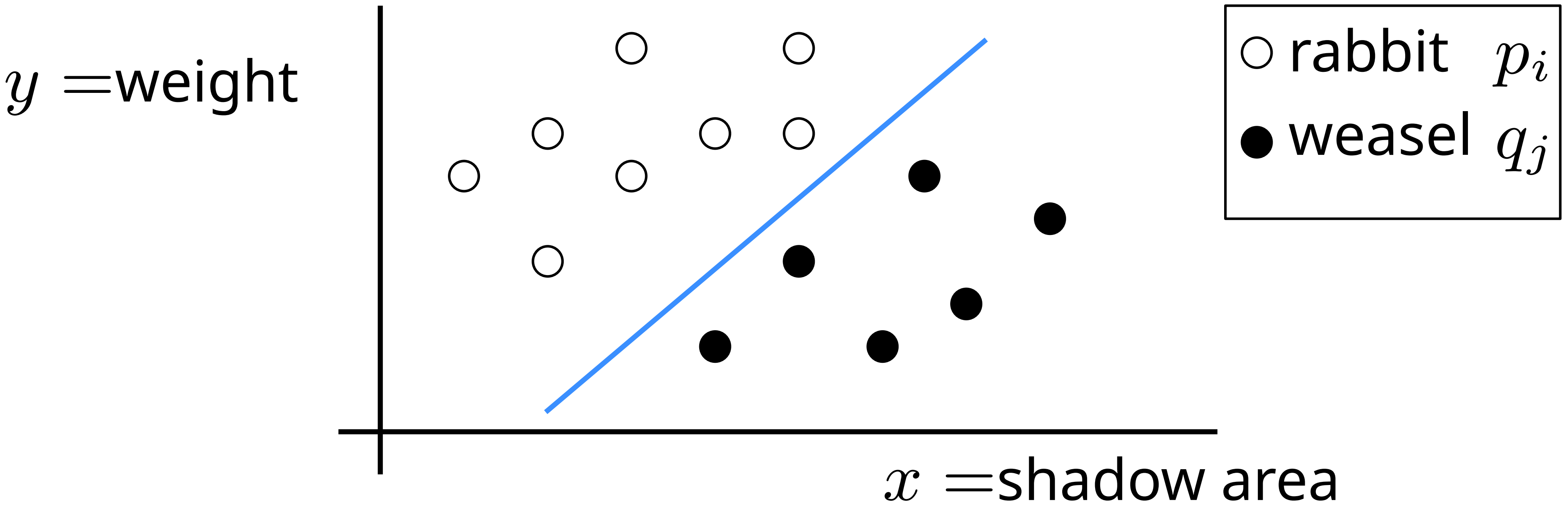

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strict inequalities
not allowed

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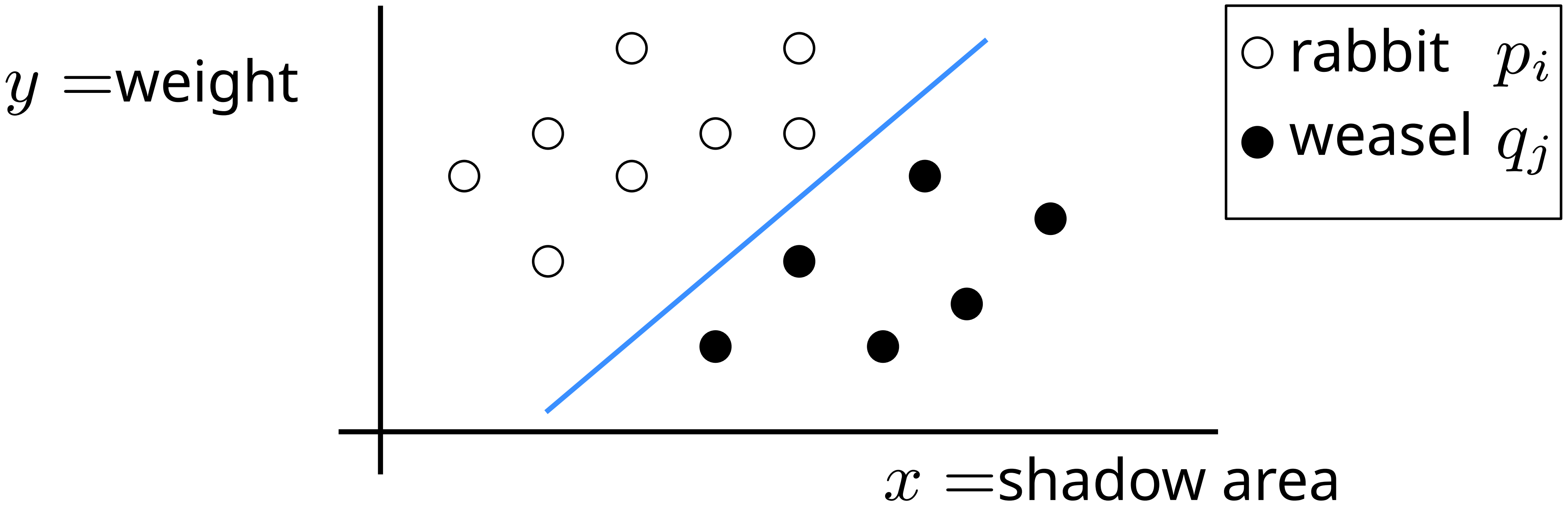

Does there exist a separating line $y = ax + b$?

Case p_i points "above" q_j points:

introduce variable δ for separation

maximize δ

subject to $y(p_i) \geq ax(p_i) + b + \delta$ for all i

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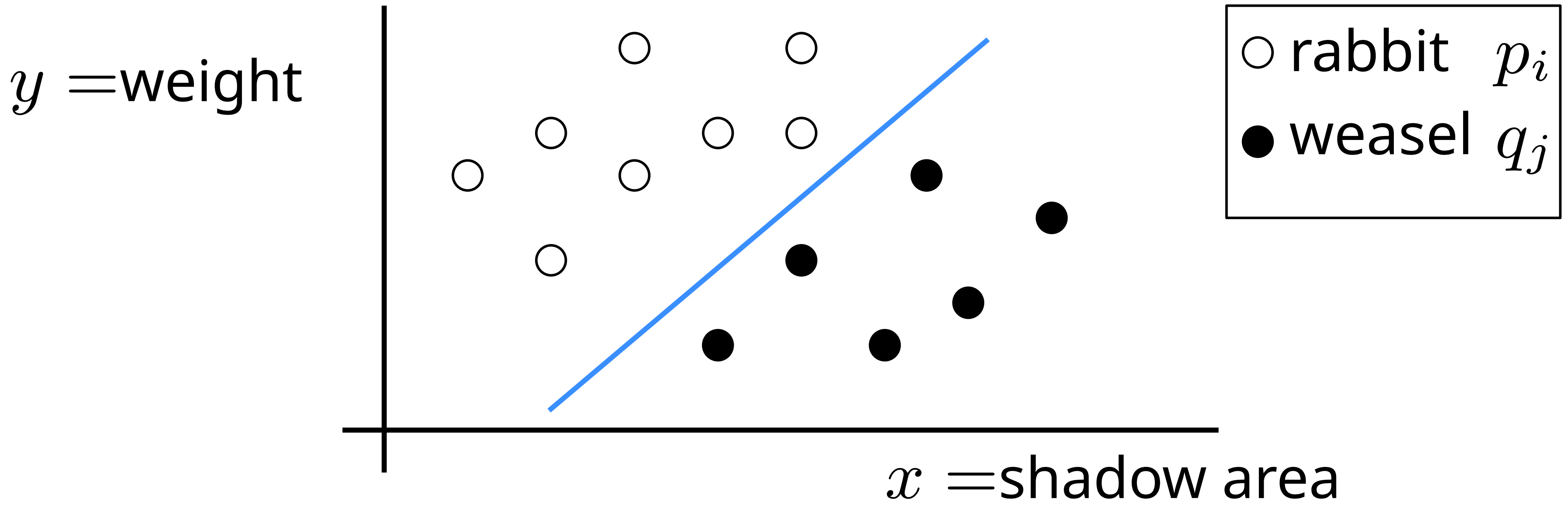

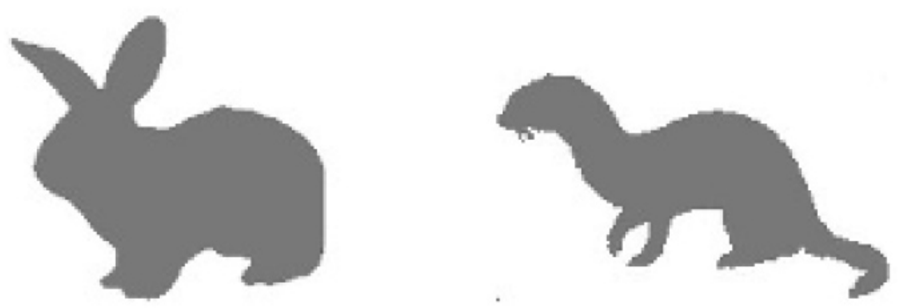

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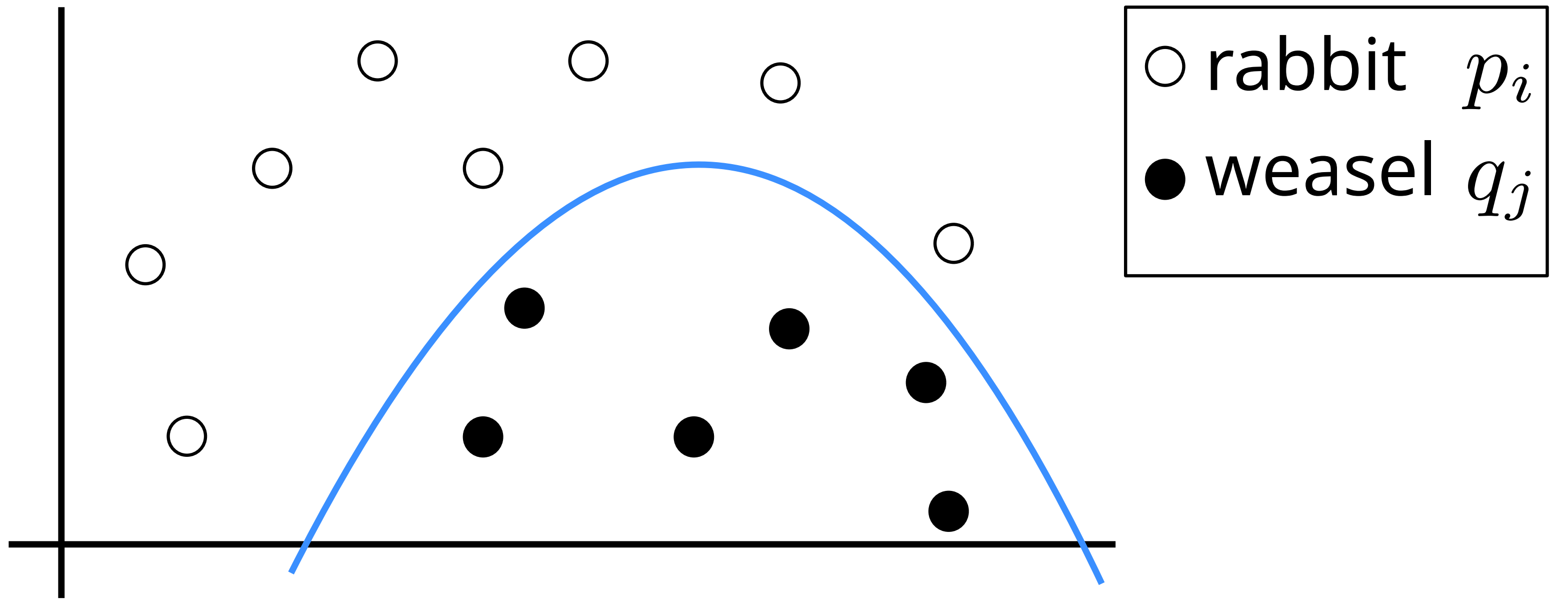
If solution gives $\delta = 0$, then no solution for strict inequalities

Example: Separation of points



Does there exist a separating parabola $y = ax^2 + bx + c$?

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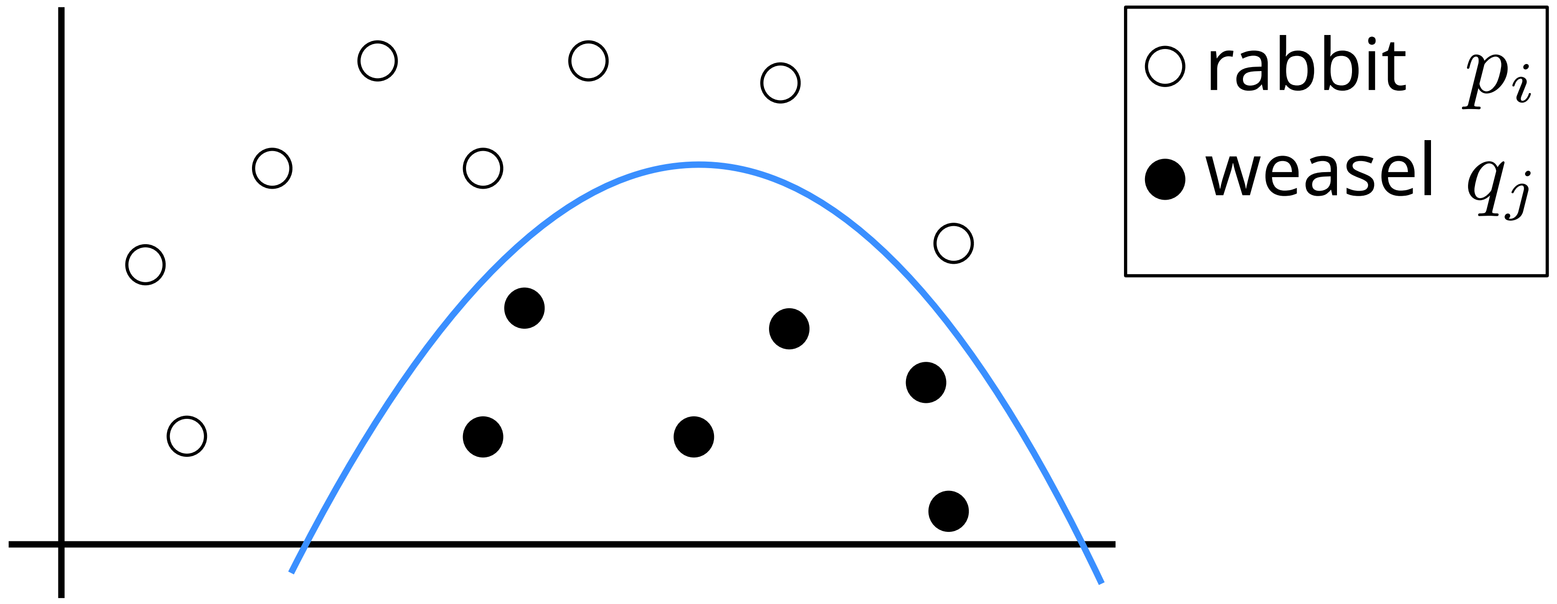

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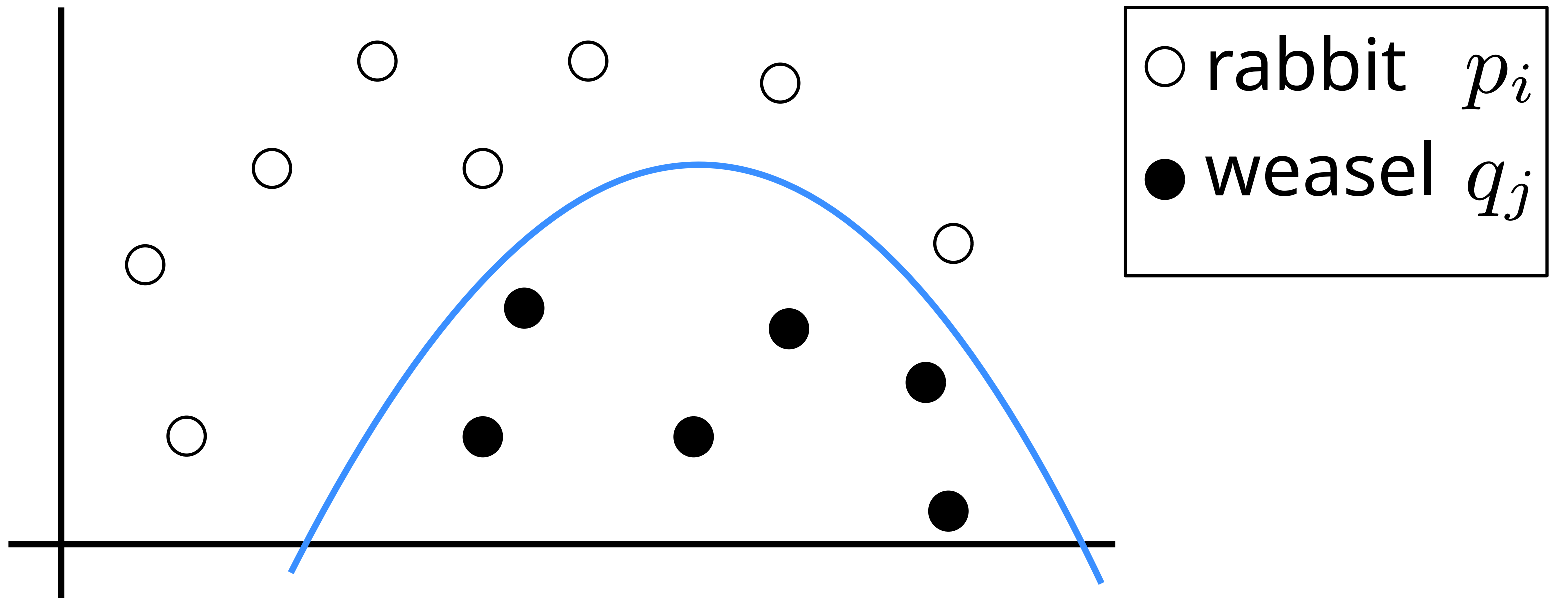

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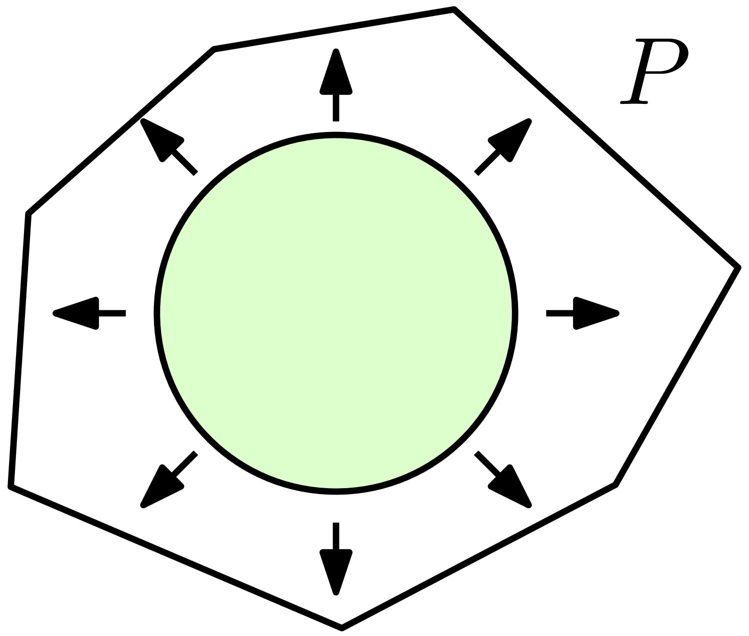
Morals: Strict inequalities can be modeled by extra variable.
Also: Non-linear problems can sometimes be incorporated into coefficients of LP.

Example: Largest Disk in a Polygon



Given a polygon P, find a disk of maximum radius inside P.

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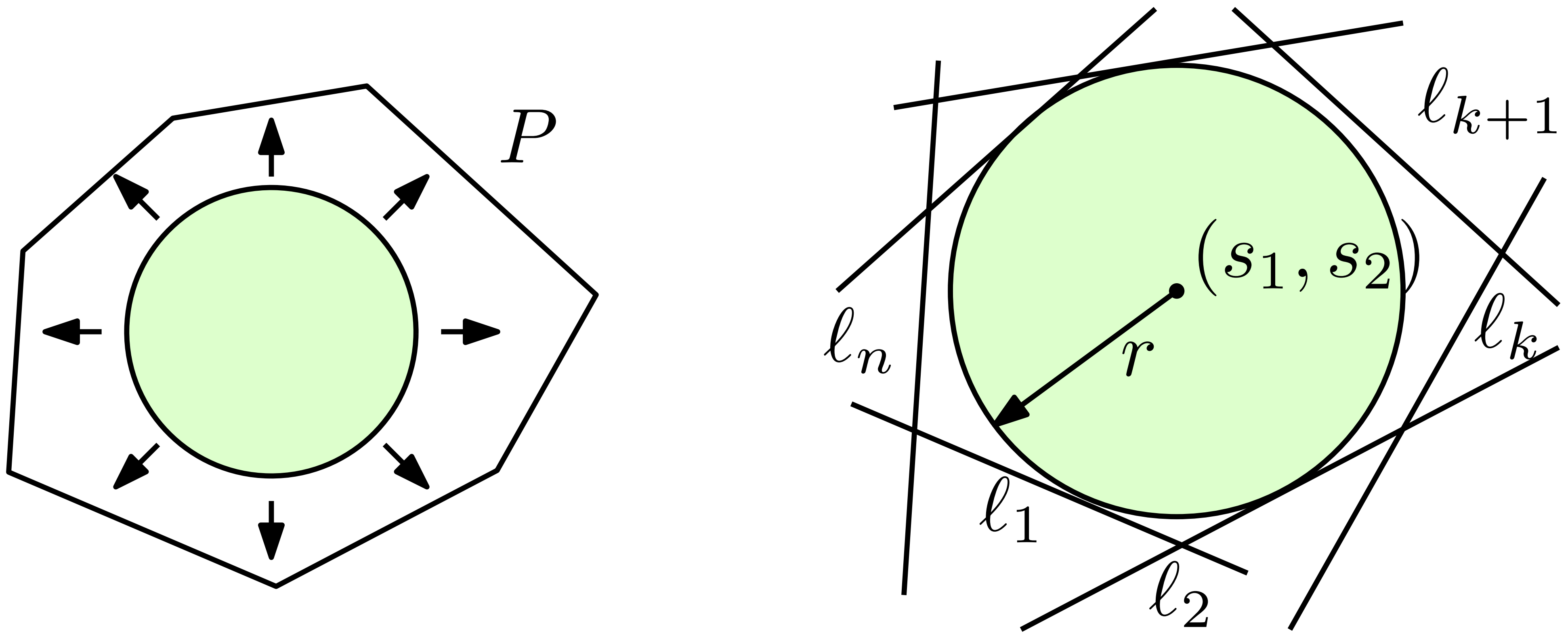

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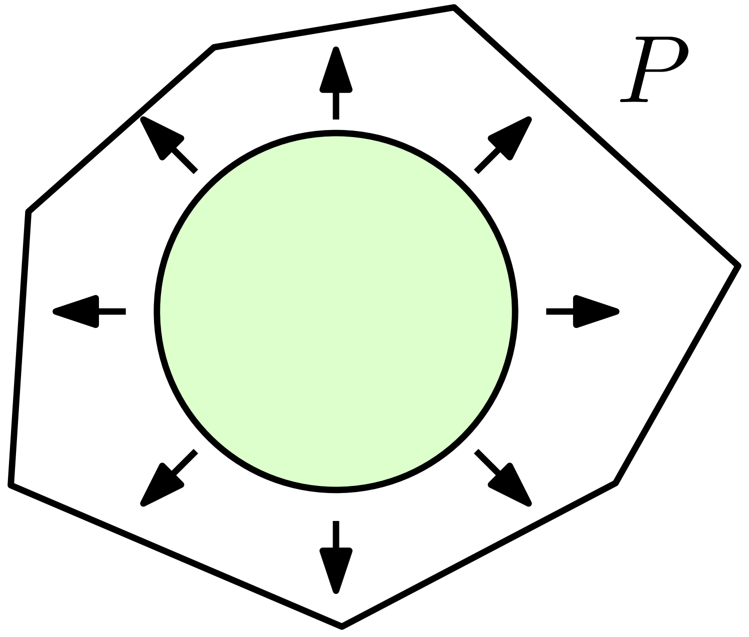
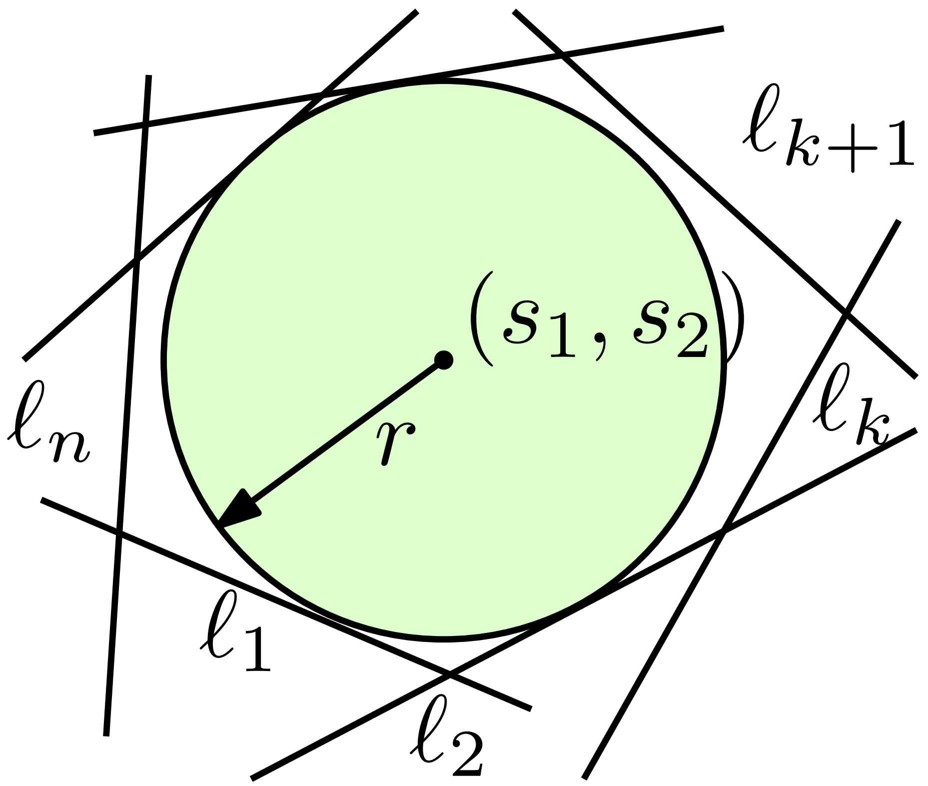
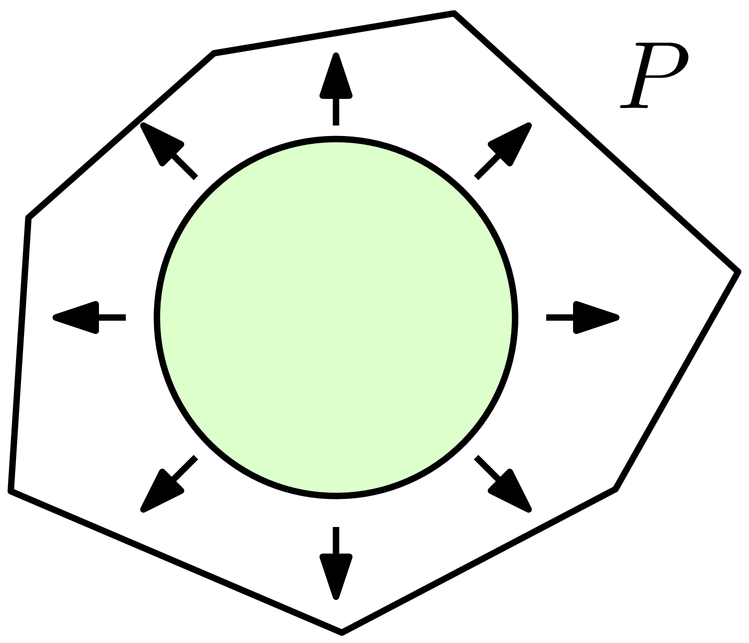


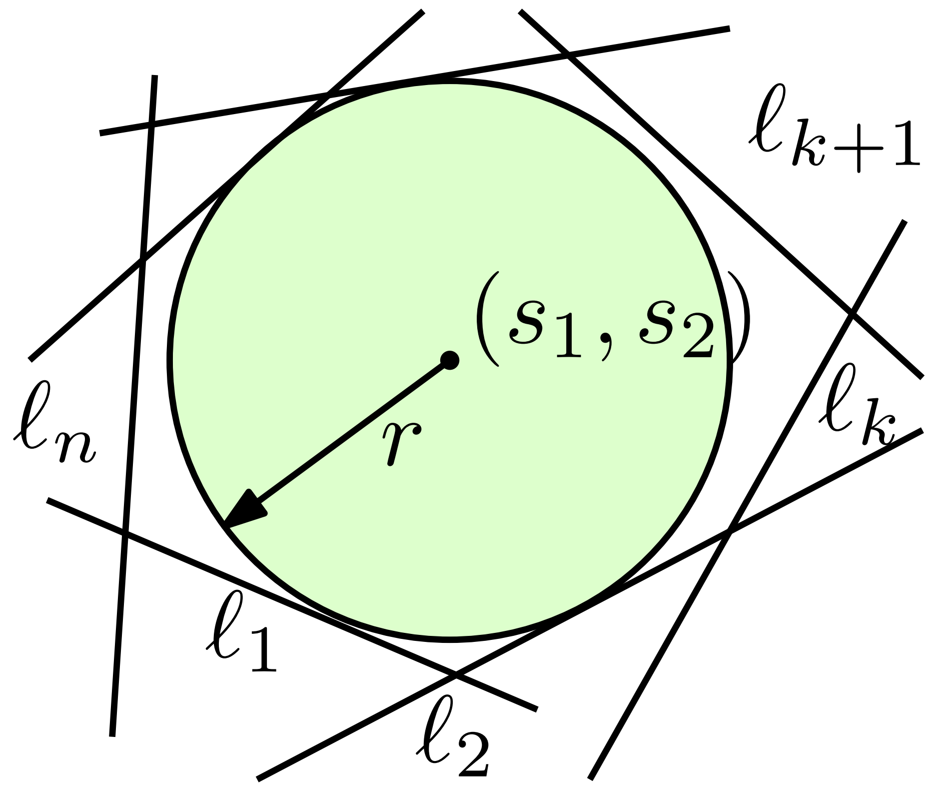


Given a polygon P, find a disk of maximum radius inside P.

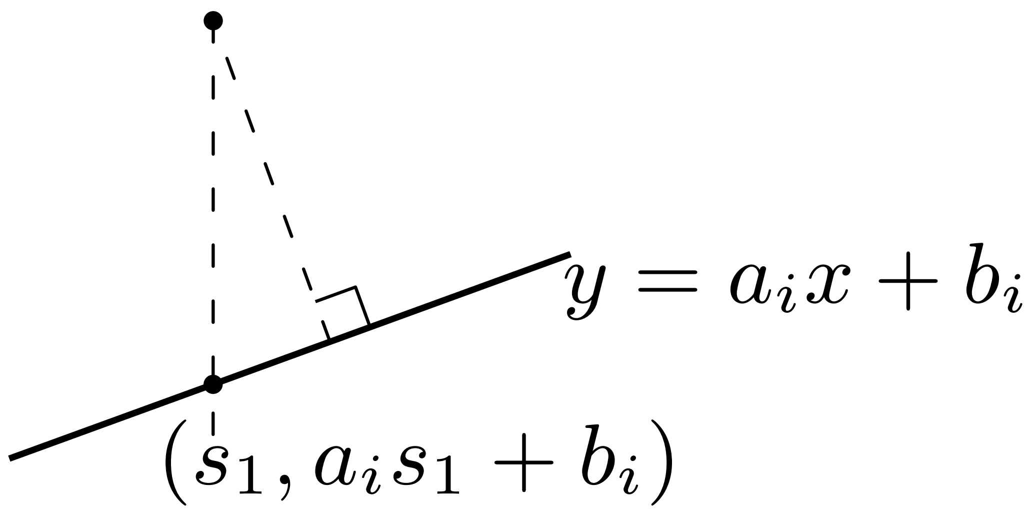
What is the distance of the center point (s_1, s_2) to a line ℓ_i?

Example: Largest Disk in a Polygon





The distance from the center point (s_1, s_2) to the line ℓ_i ($y = a_i x + b_i$) is the absolute value of $\frac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}}$.



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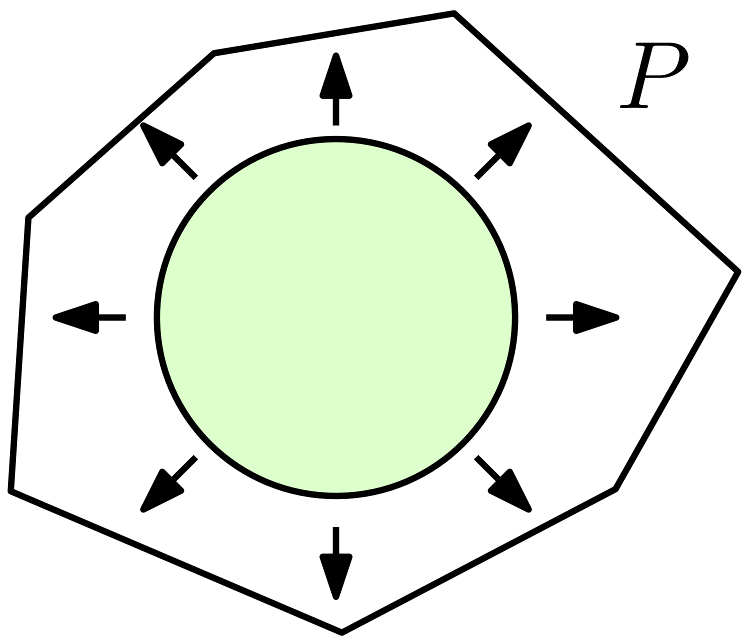

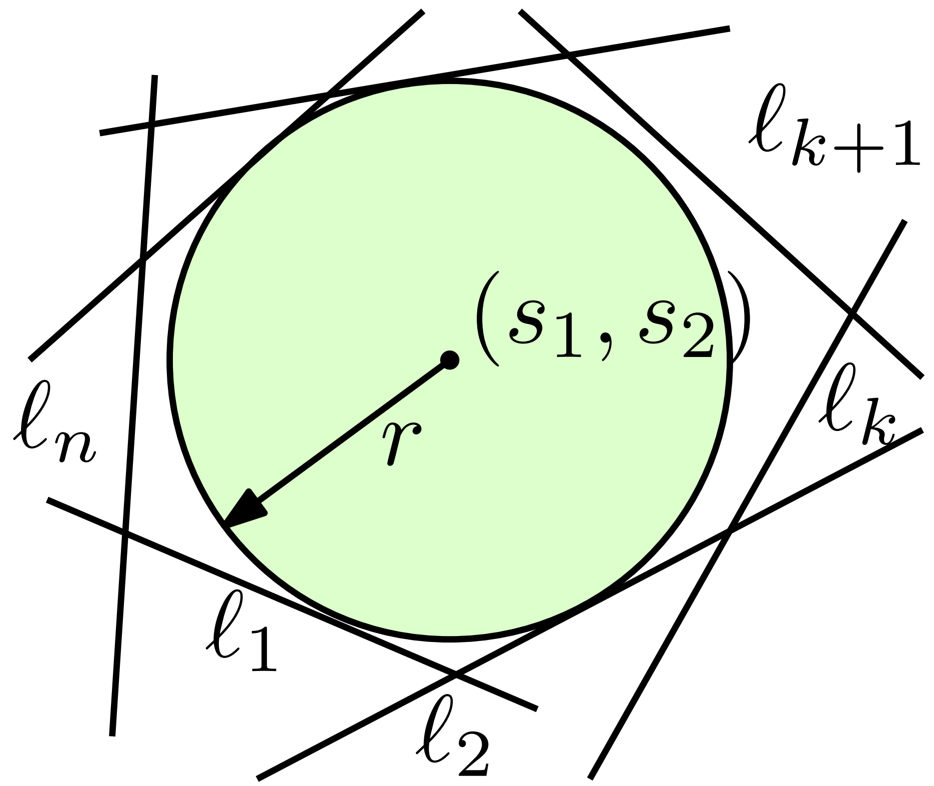

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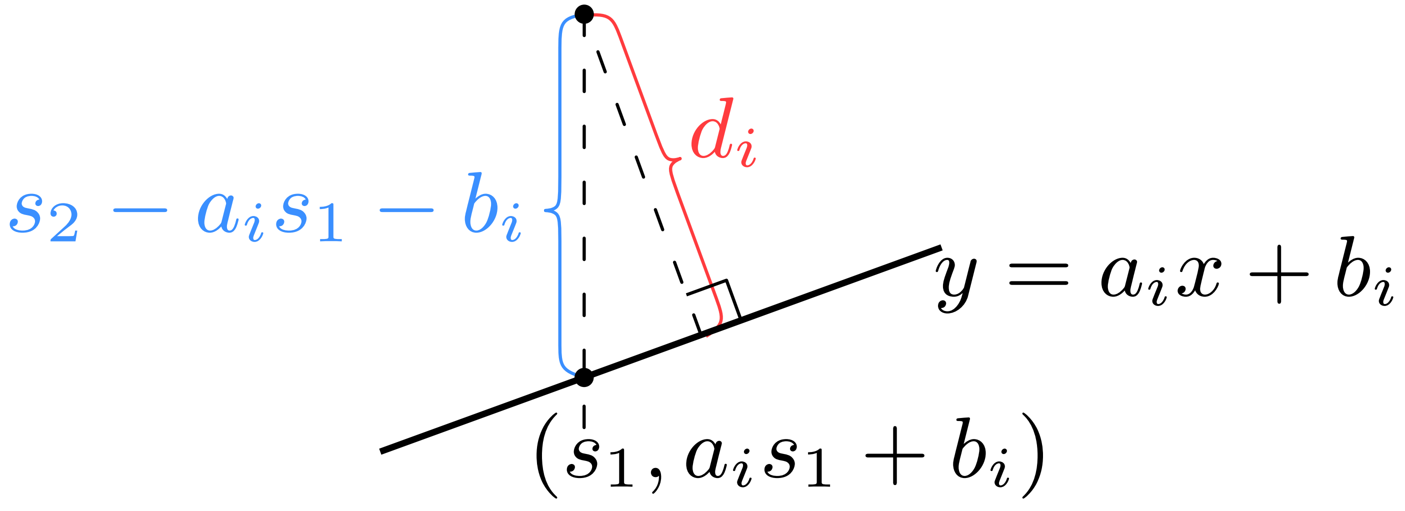

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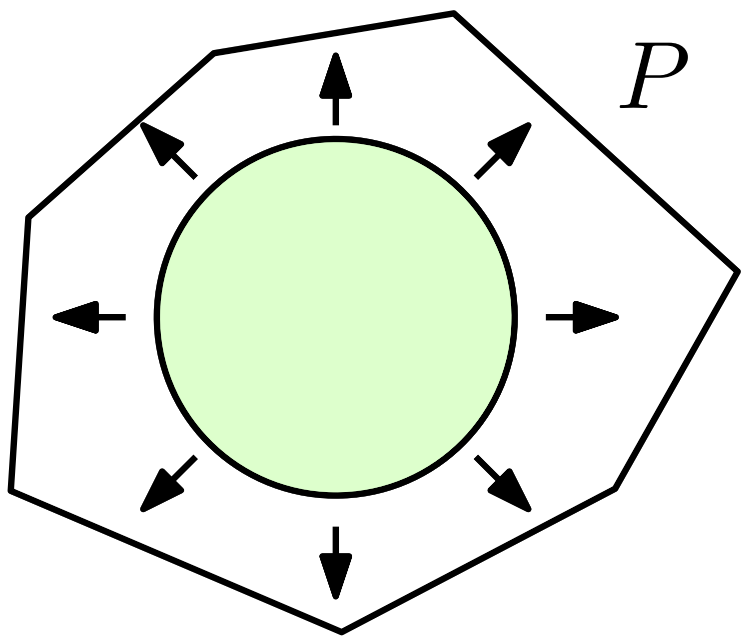

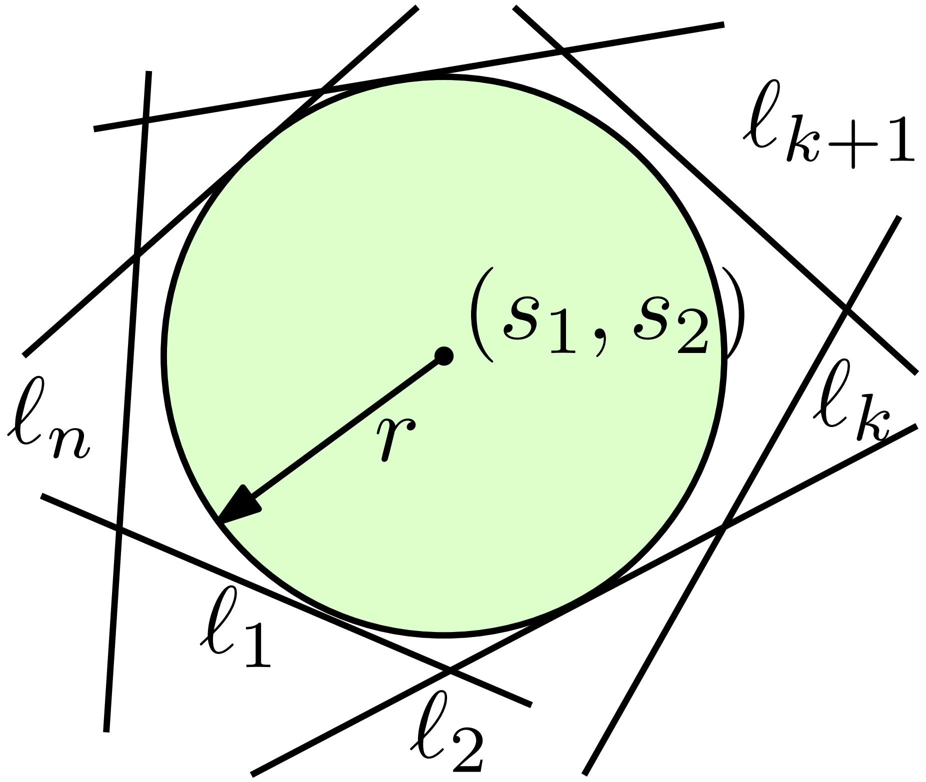

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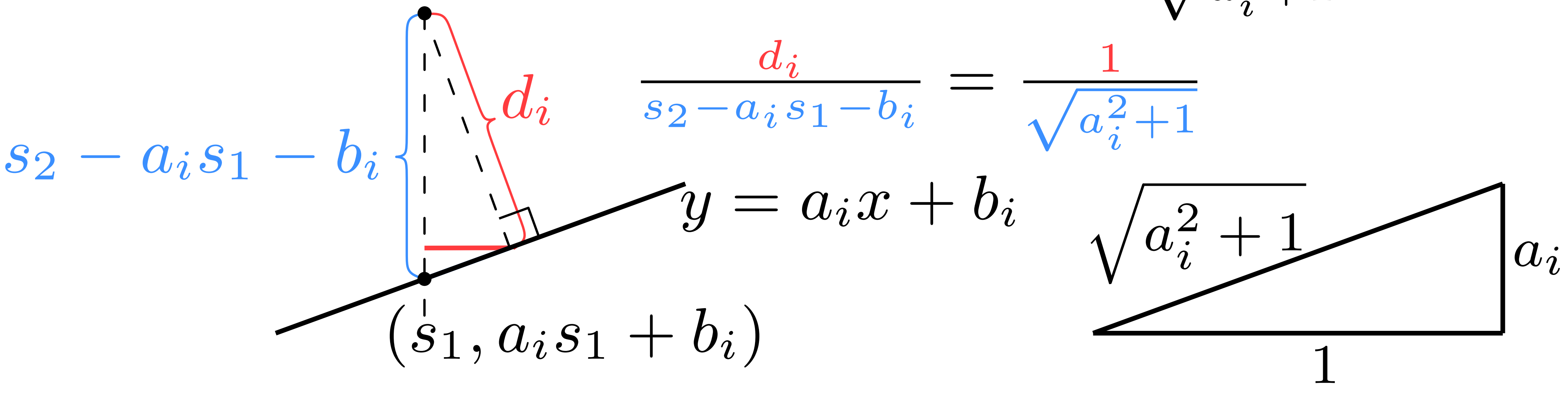

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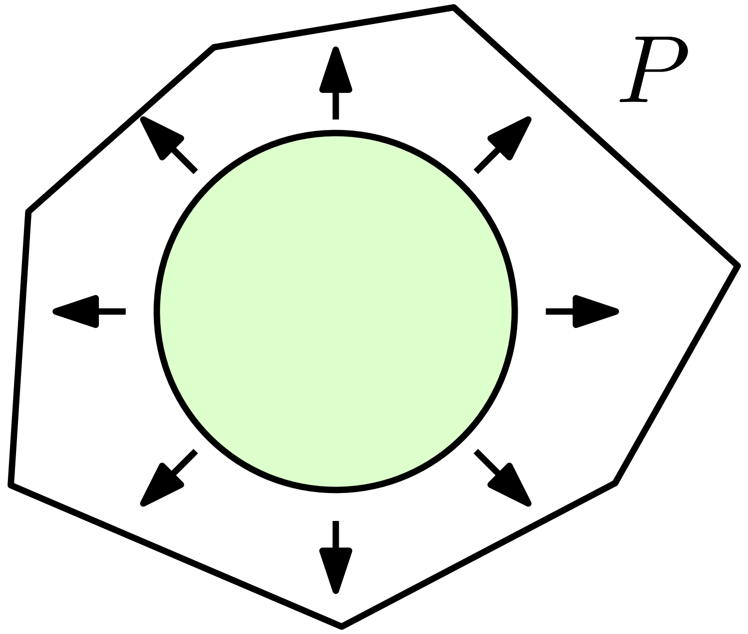

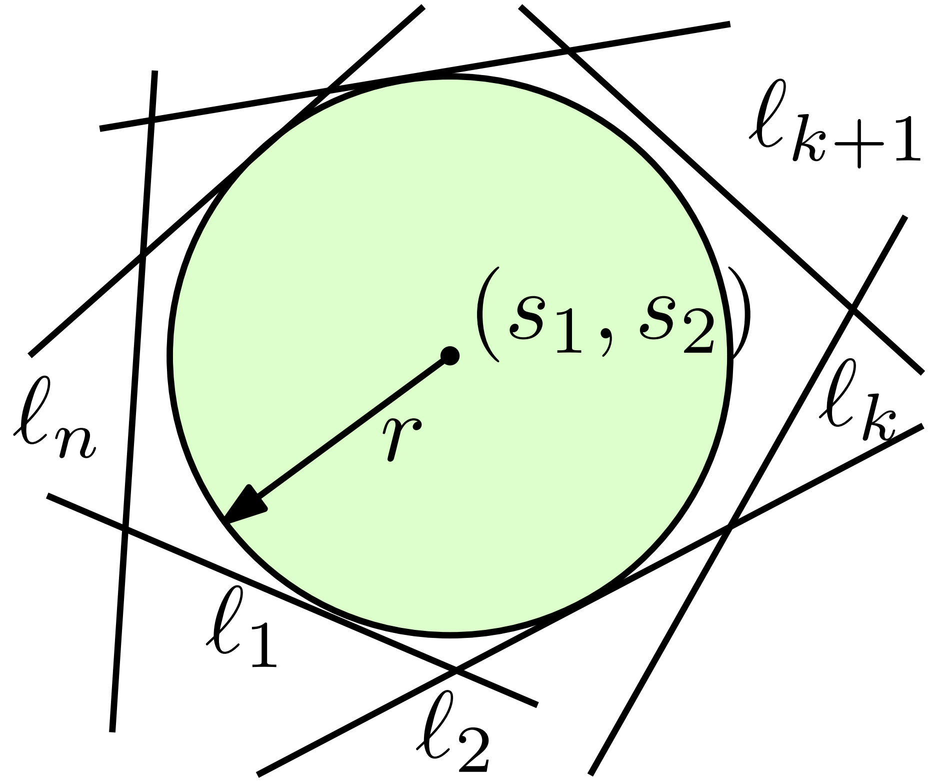

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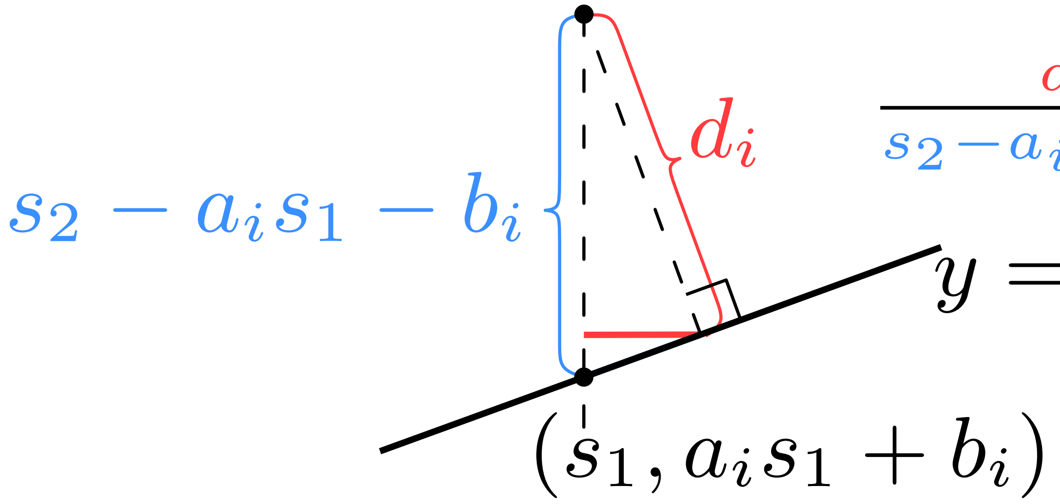
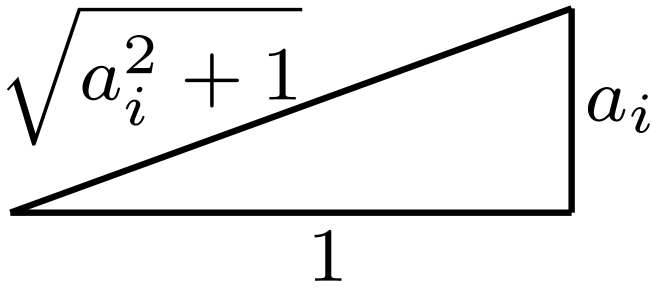


$\frac{d_i}{s_2 - a_i s_1 - b_i} = \frac{1}{\sqrt{a_i^2+1}}$ so $d_i = \frac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2+1}}$



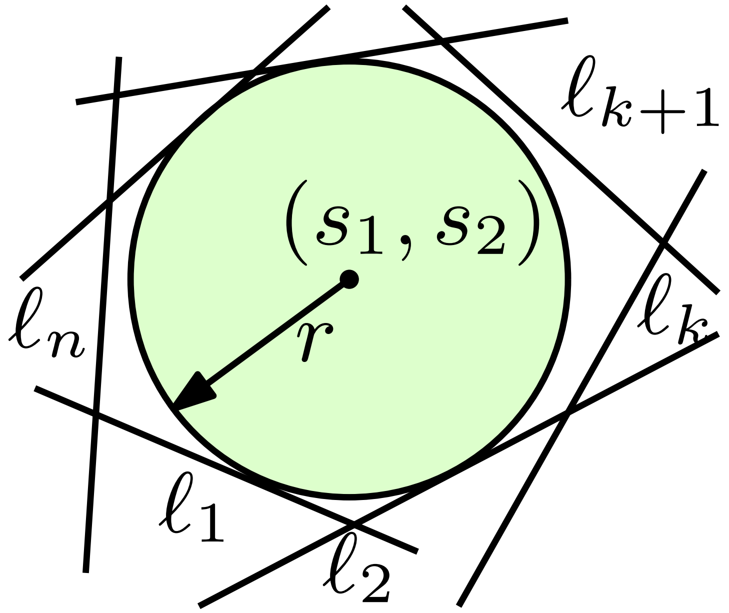
Example: Largest Disk in a Polygon

We get the following linear programm:

maximize r

subject to $\dfrac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}} \geq r$ for $i = 1, 2, ..., k$

$\dfrac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}} \leq -r$ for $i = k+1, k+2, ..., n$



There are three variables: s_1, s_2 and r.

An optimal solution yields the desired largest disk contained in P.

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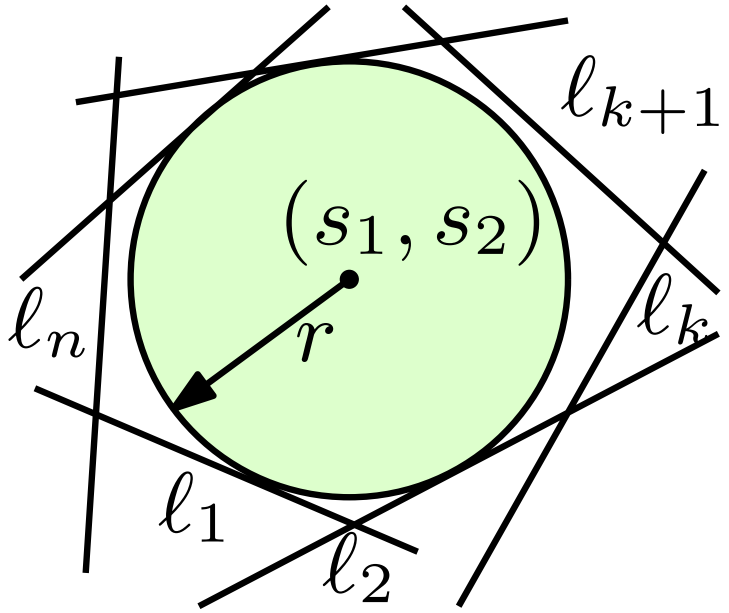

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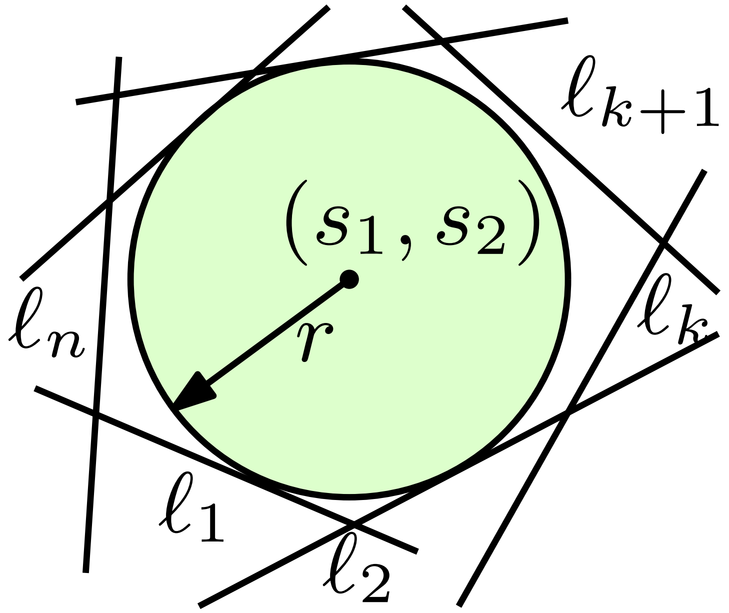

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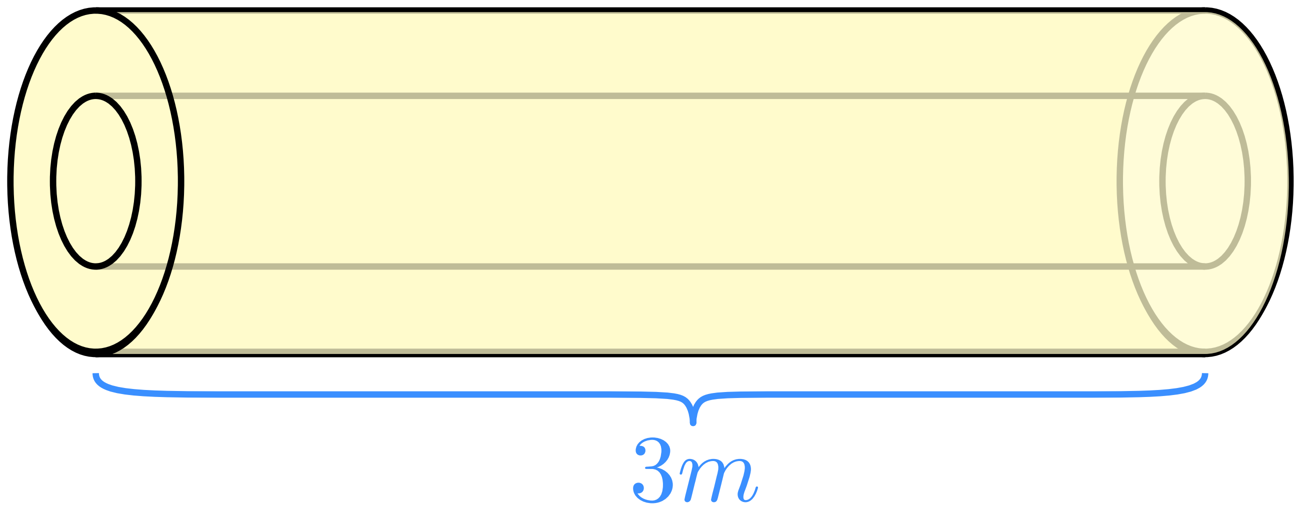
Note: Finding the smallest disk containing a polygon is not linear but is convex optimization problem.

Example: Cutting Paper Rolls

Paper mill makes 3 meter paper rolls.



What's the fewest number of rolls need to satisfy an order of:

- 97 rolls width 135cm
- 610 rolls width 108cm
- 395 rolls width 93cm
- 211 rolls width 42cm

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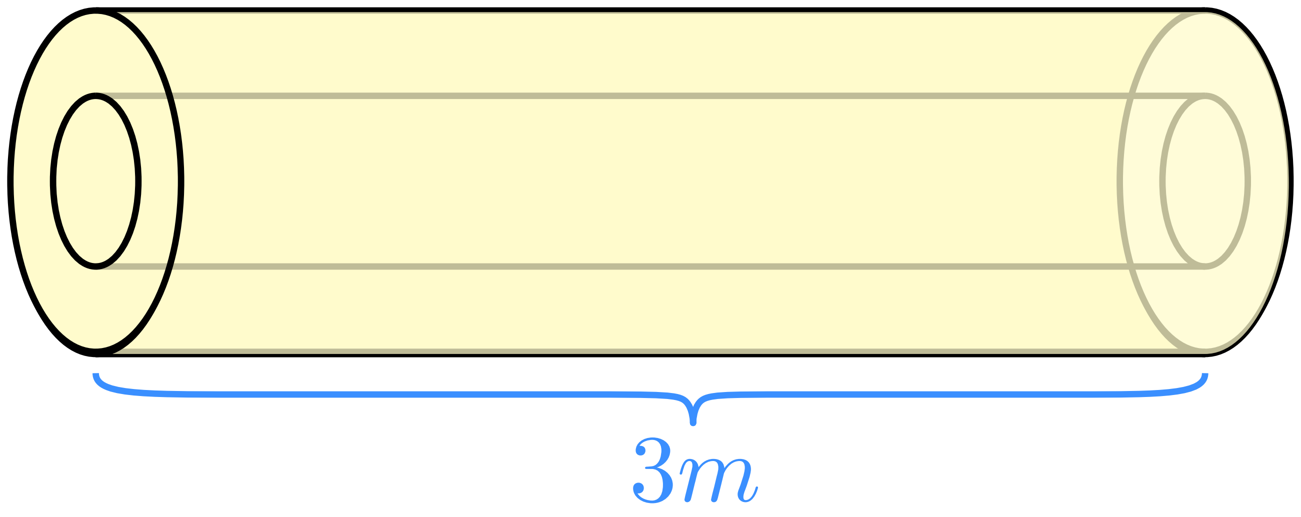

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How to formulate this as an LP?

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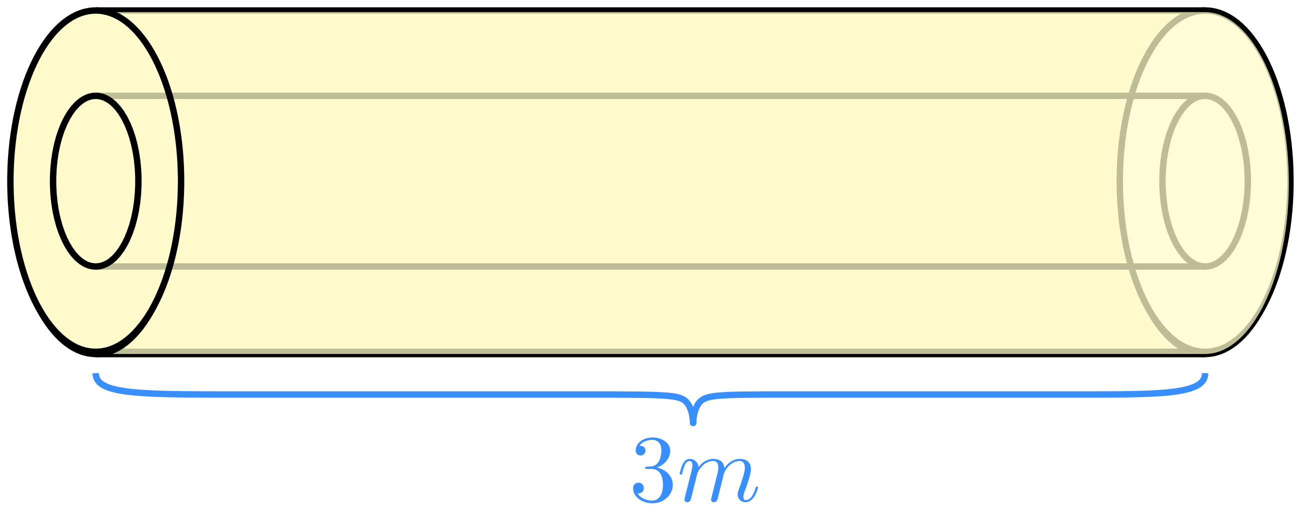

Possible ways to cut roll with <42cm wasted:

P1: $2 \cdot 135$

P2: $135 + 108 + 42$

P3: $135 + 93 + 42$

P4: $135 + 3 \cdot 42$

P5: $2 \cdot 108 + 2 \cdot 42$

P6: $108 + 2 \cdot 93$

P7: $108 + 93 + 2 \cdot 42$

P8: $108 + 4 \cdot 42$

P9: $2 \cdot 93$

P10: $2 \cdot 93 + 2 \cdot 42$

P11: $93 + 4 \cdot 42$

P12: $7 \cdot 42$

can be generated by computer

Example: Cutting Paper Rolls

For each possibility P_j, add a variable $x_j \geq 0$ representing # rolls cut that way.

minimize $\sum_{j=1}^{12} x_j$ (total # of rolls cut)

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next lecture: integer linear programming

Summary

A linear program (LP) is the problem of maximizing a given linear function over the set of all vectors that satisfy a given system of linear equations and inequalities.

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Modelling problems as LPs:

- many problems have natural LP formulation (e.g. diet problem)
- important algorithmic problems, e.g., network flow
- some non-linear concepts can be handled by additional variables, e.g., absolute value
- geometric problems: e.g., separation problems, fitting disk in polygon
- encoding many possibilities by variables (paper cutting);
  integer variables ?!

Attributions

- book cover: "Understanding and Using Linear Programming" by Matoušek and Gärtner
- image sources (history): de.wikipedia.org, news.standford.de
- image source (burger): https://www.publicdomainpictures.net/en/view-image.php?image=408431\&picture=fast-food-food-snack
- image source (shadows): "Understanding and Using Linear Programming"