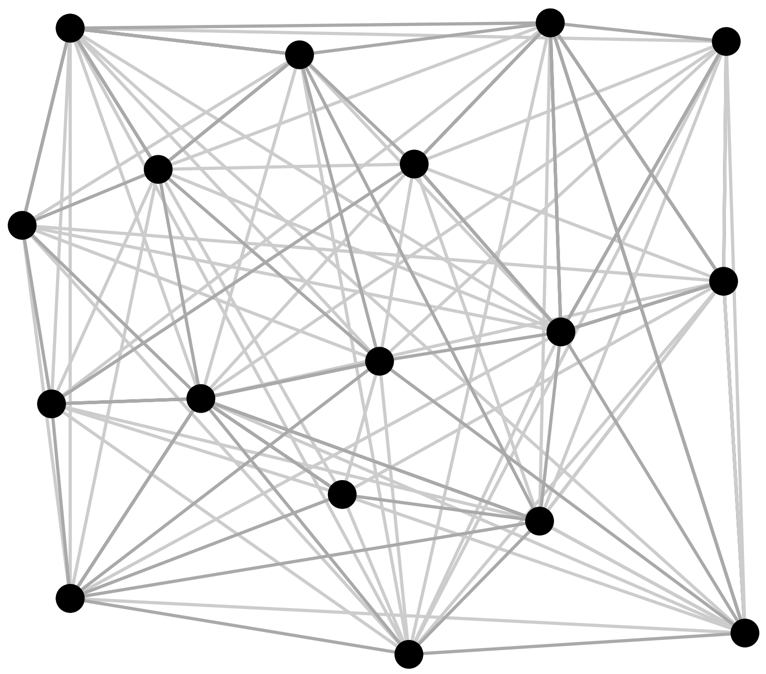
k-CENTER via Parametric Pruning

Metric k-CENTER

Given: A complete graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$
satisfying the triangle inequality



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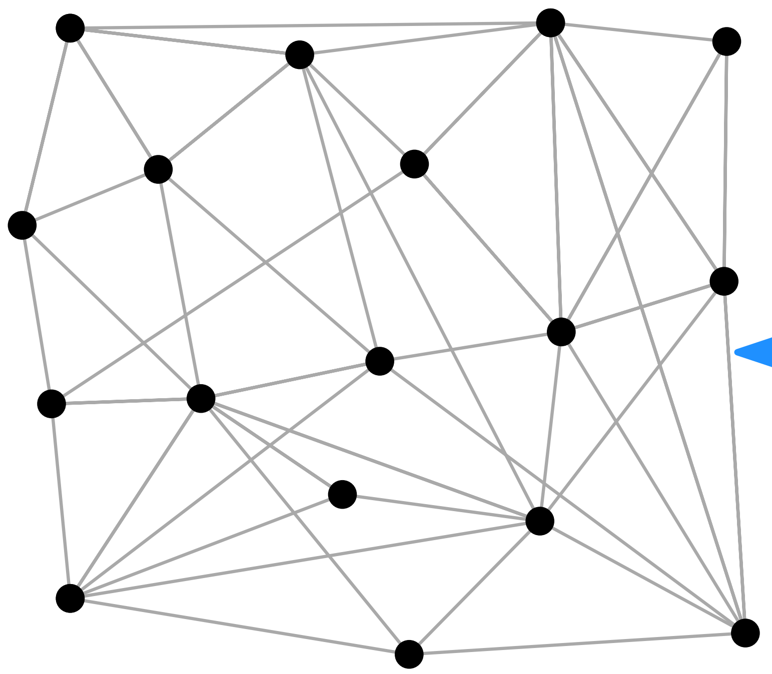

for readability:
not all edges shown

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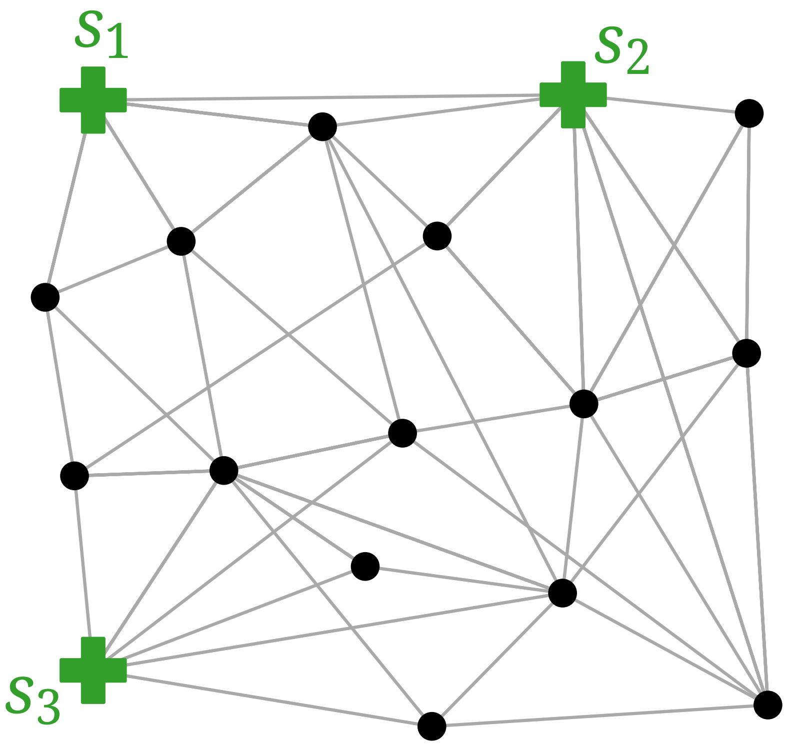
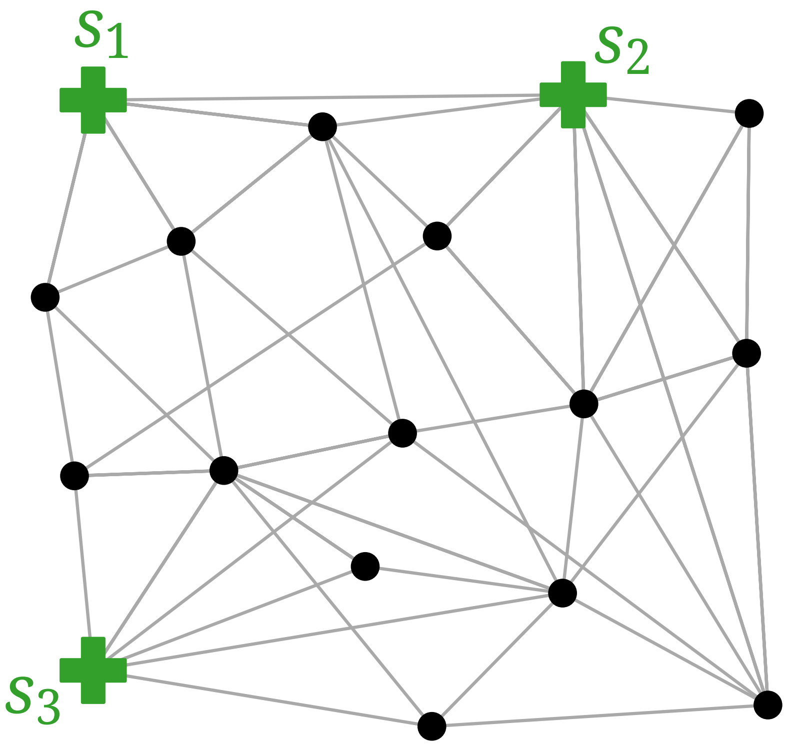
For each vertex set $S \subseteq V$



Metric k-CENTER

Given: A complete graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality

For each vertex set $S \subseteq V$,
$c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.



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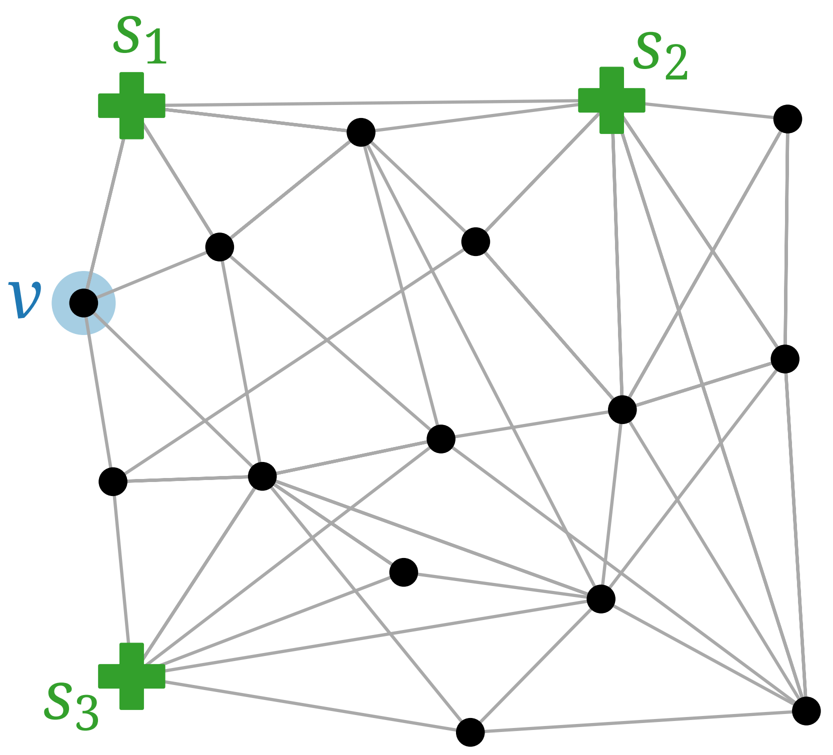

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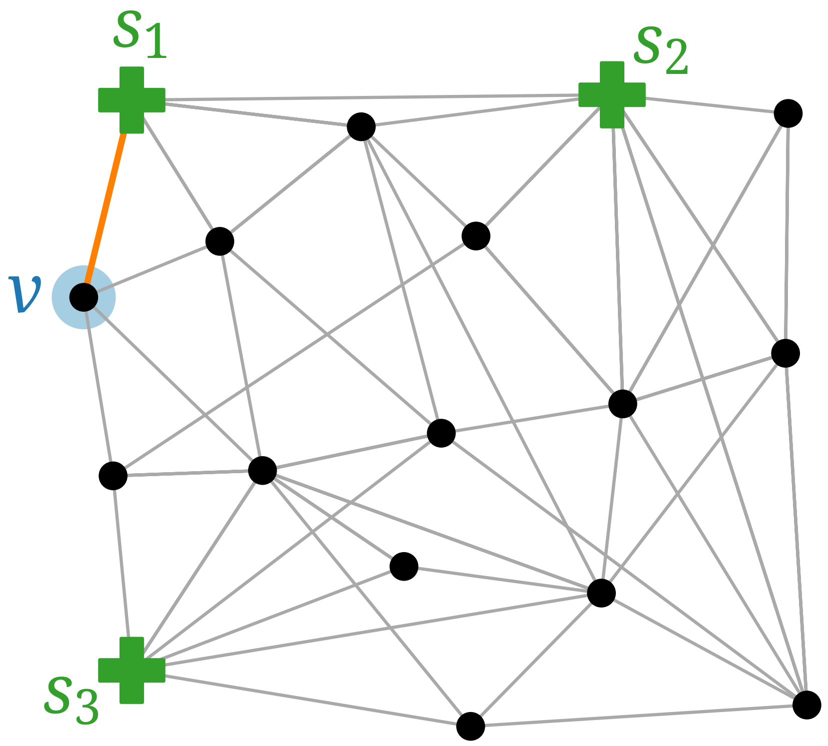

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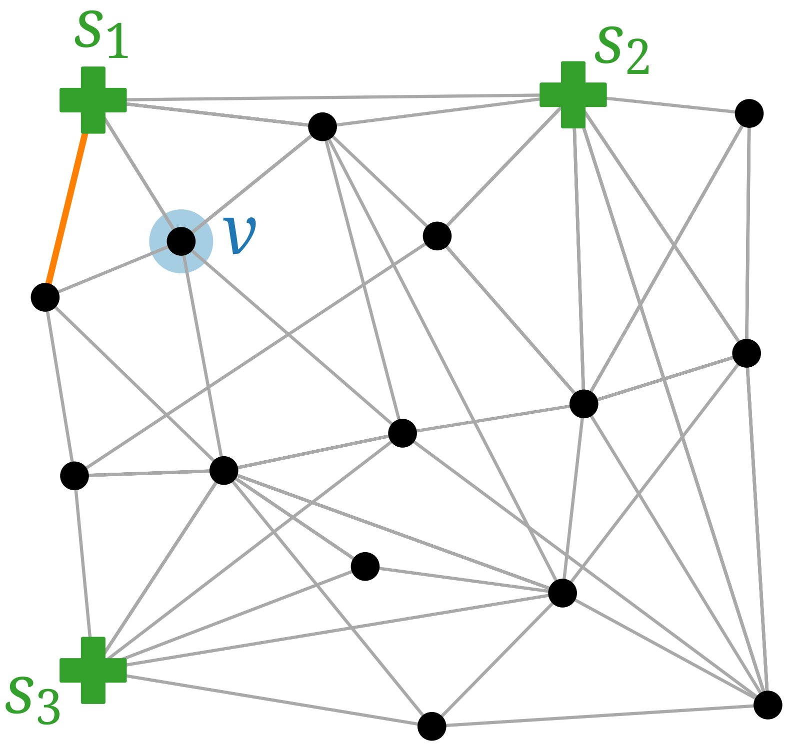

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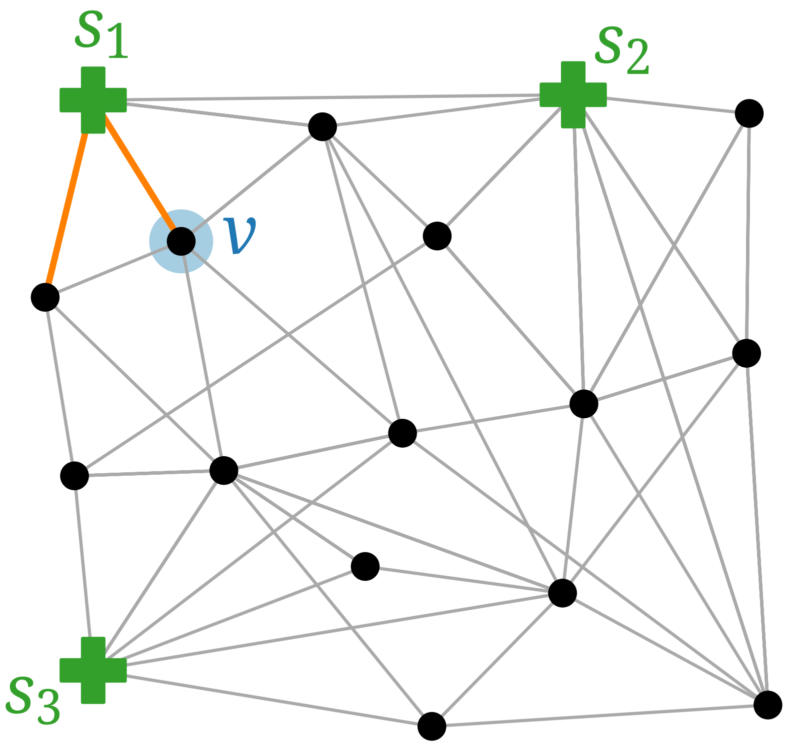

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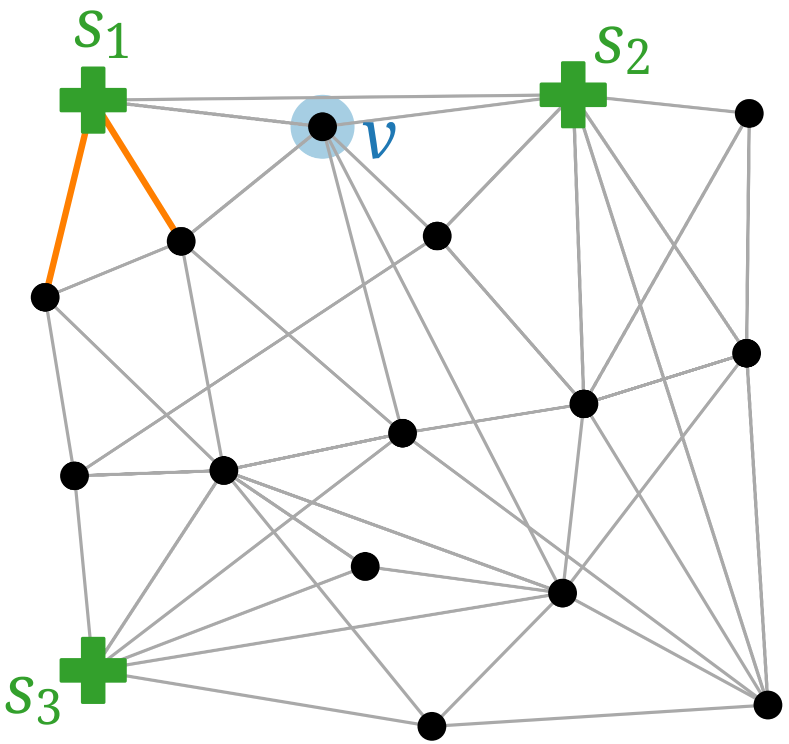

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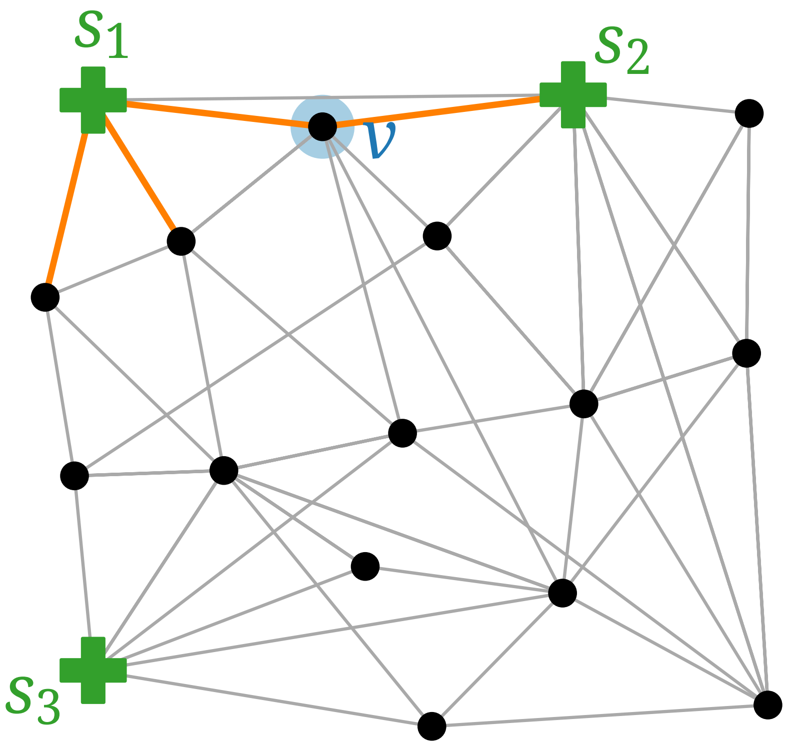

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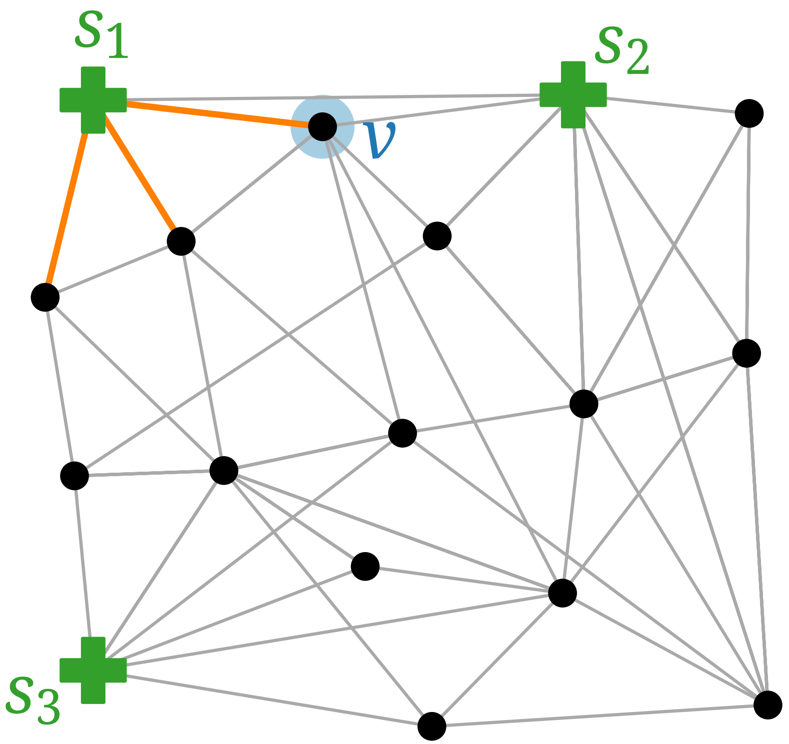

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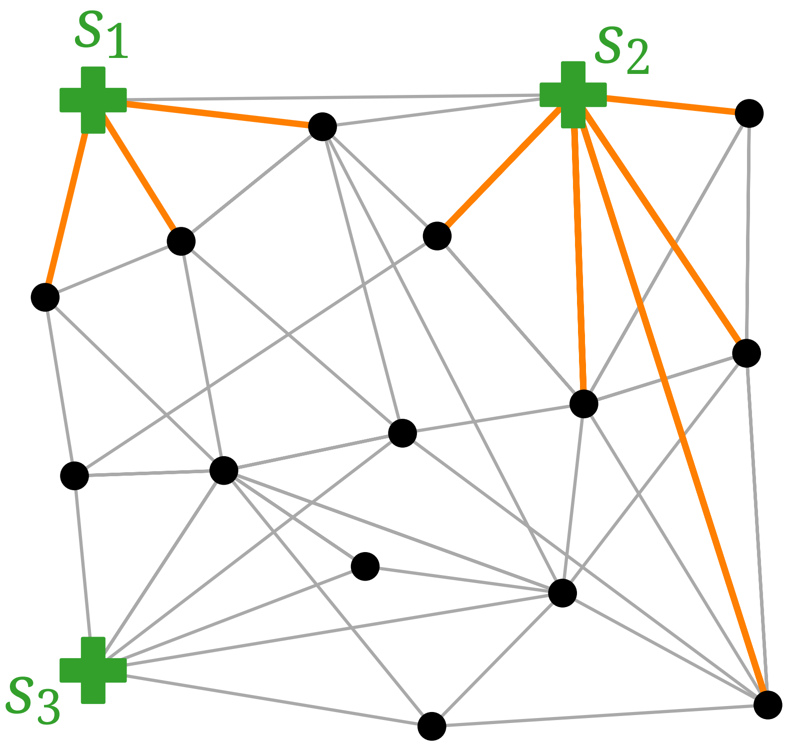

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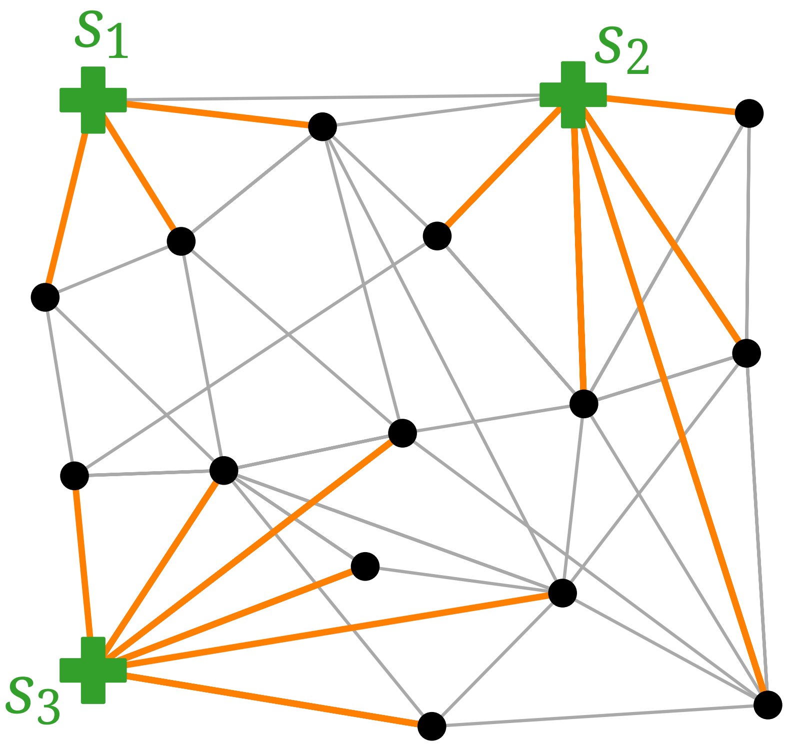

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For each vertex set $S \subseteq V$,
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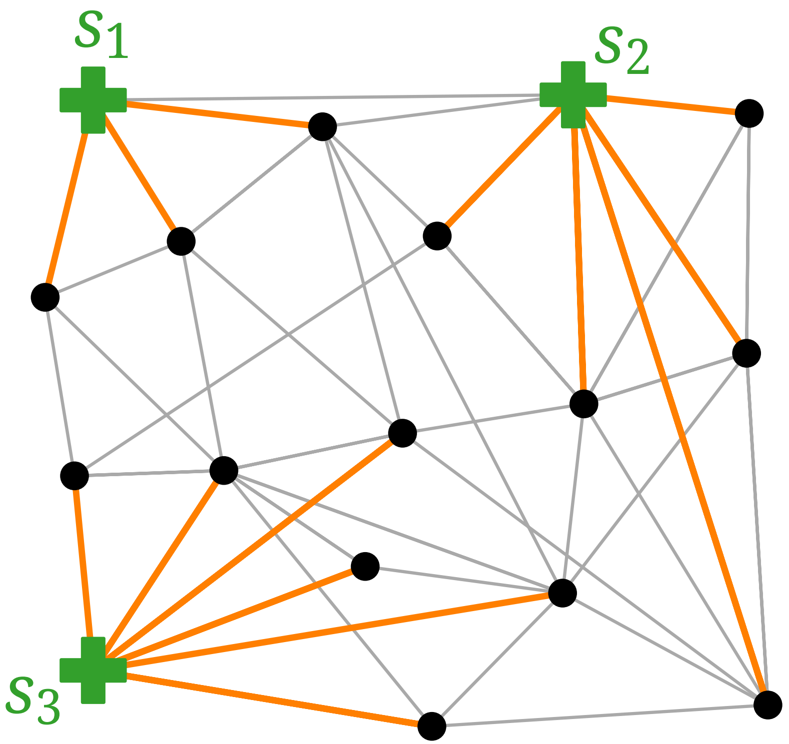
$\mathrm{cost}(S) := \max_{v \in V} c(v, S)$



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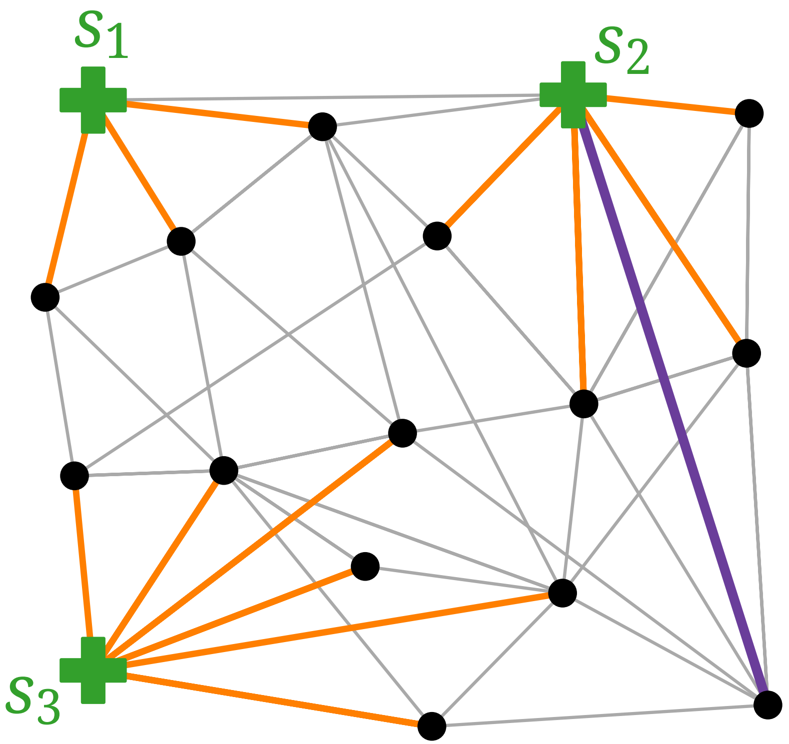

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Find: A vertex set S such that
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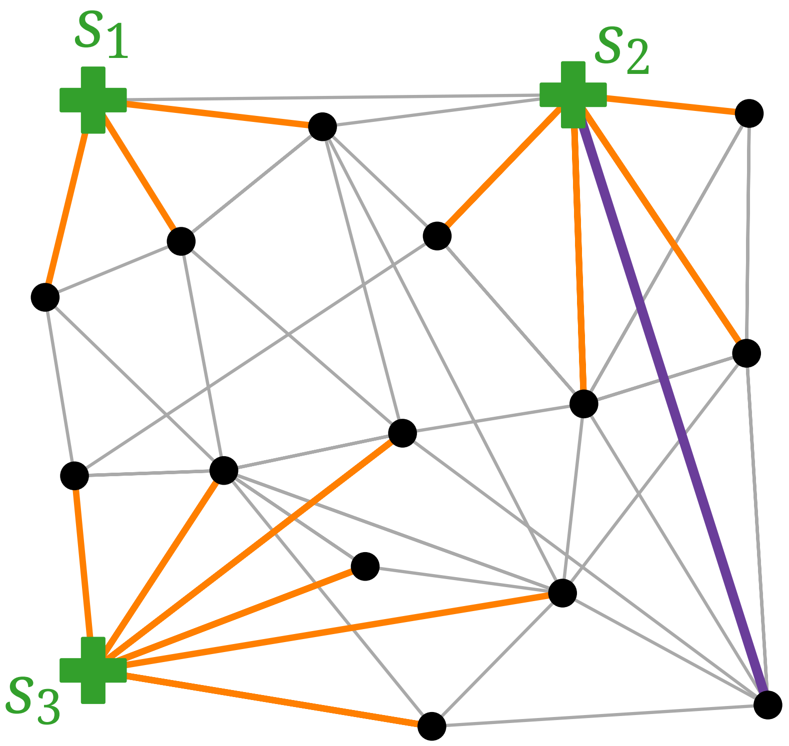

Metric k-Center

Given: A complete graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality and a natural number $k \leq |V|$.

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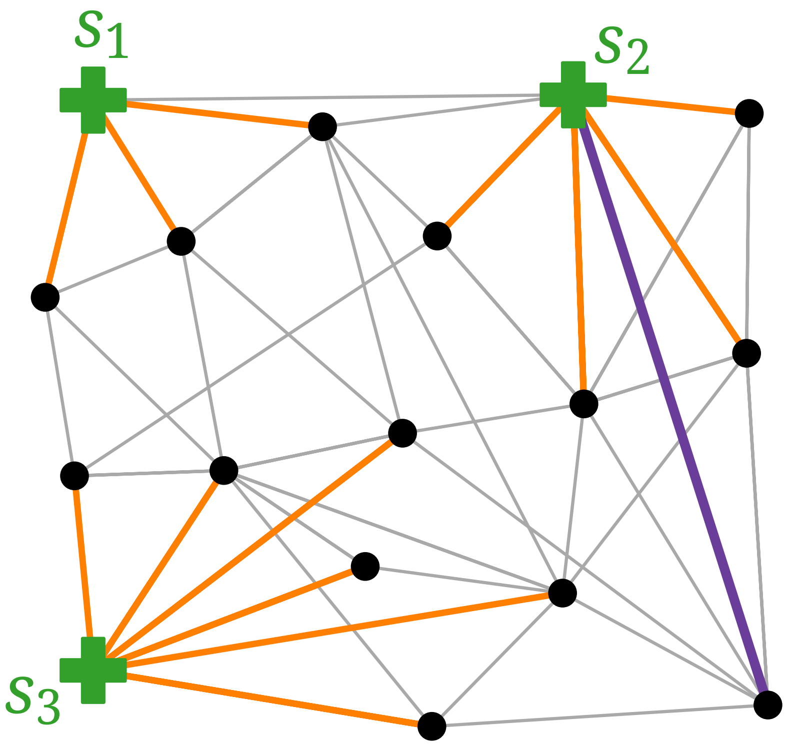

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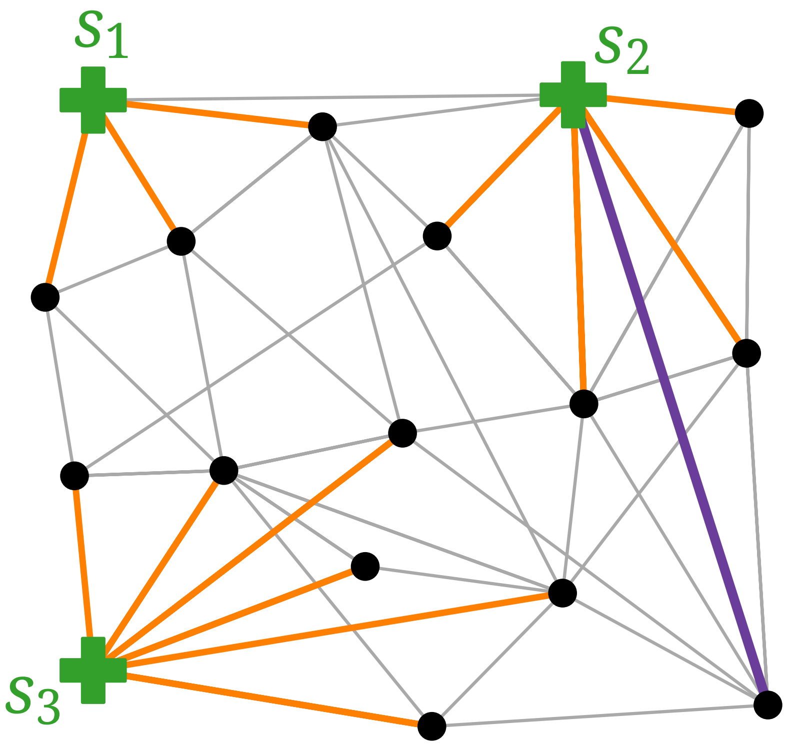

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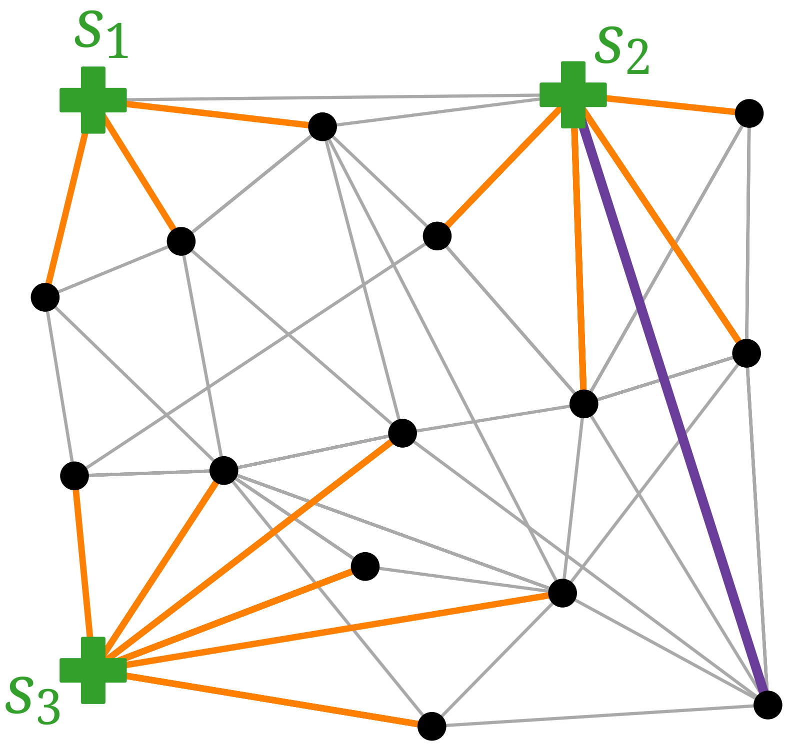

optimal for $k = 3$?   better solution?

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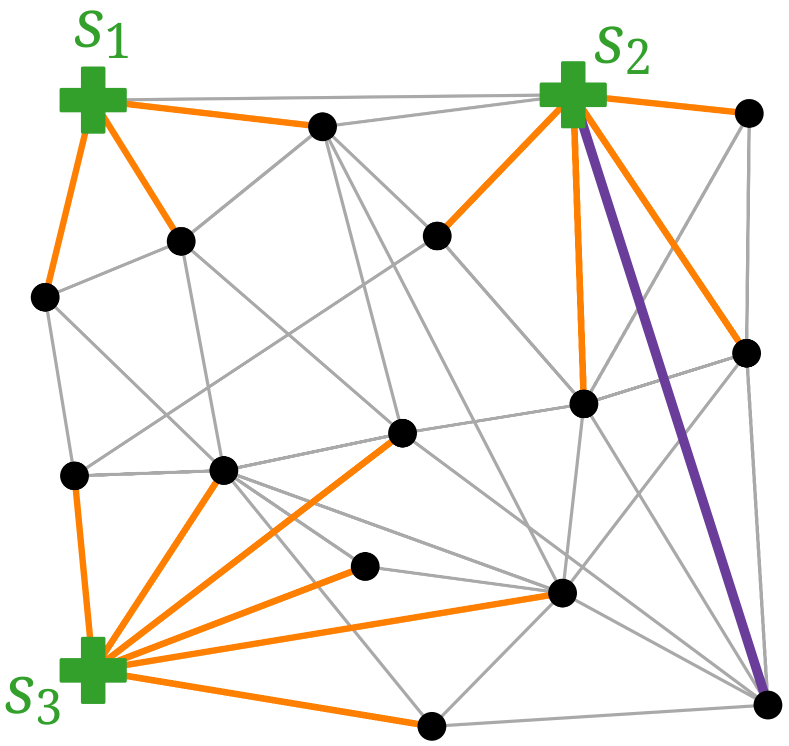

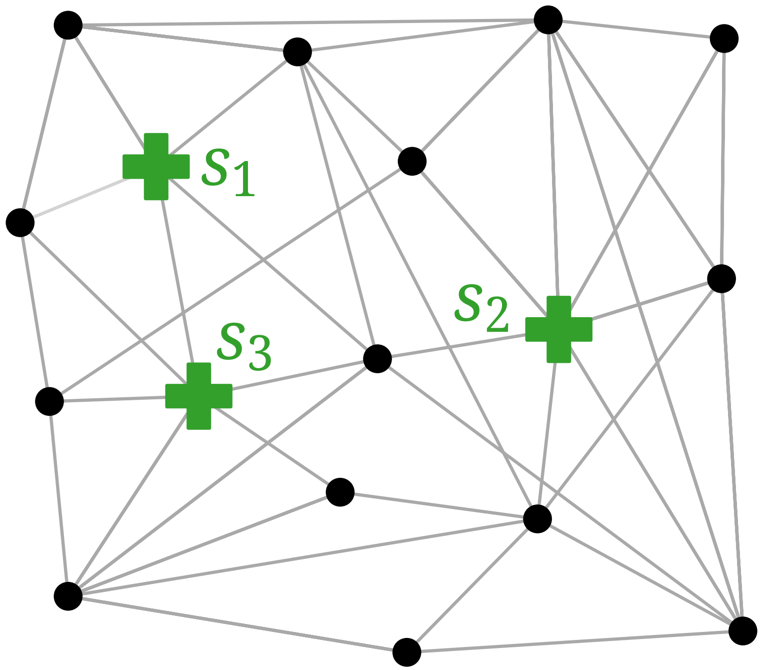

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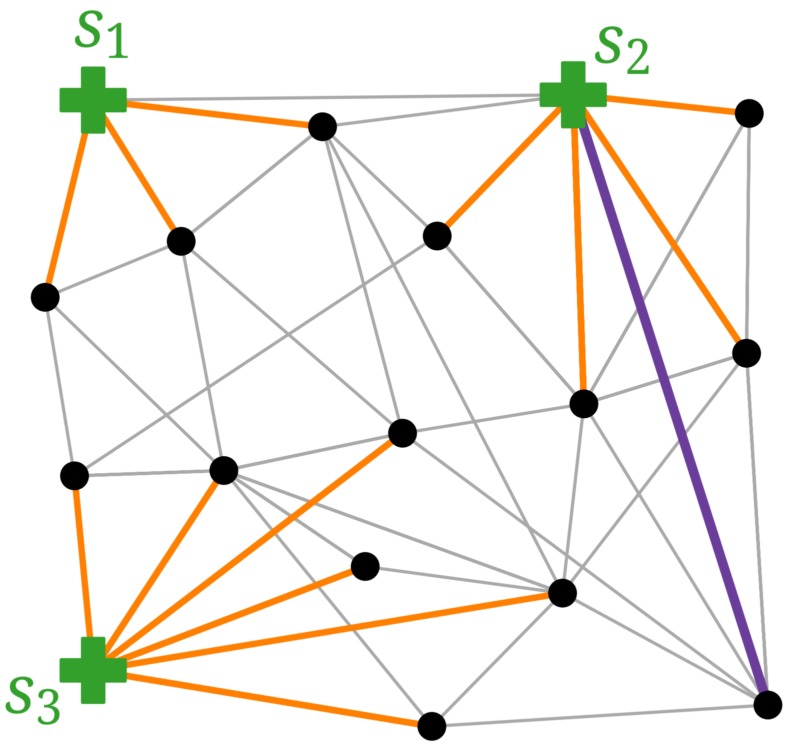

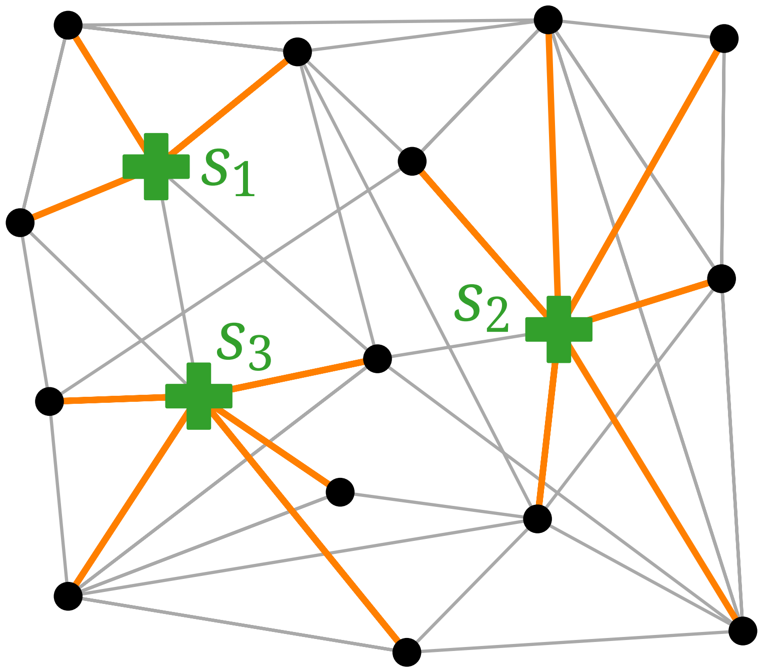

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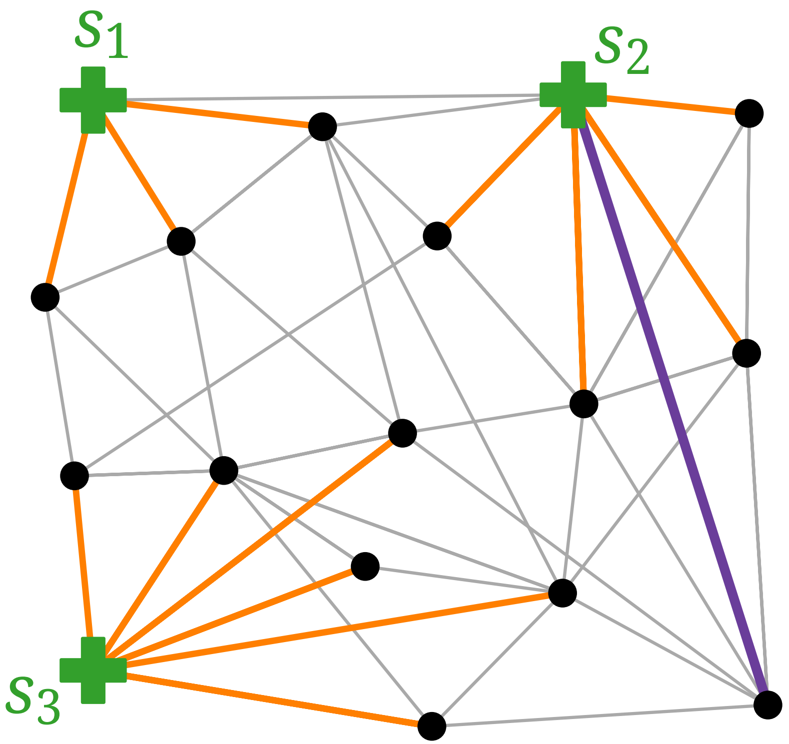

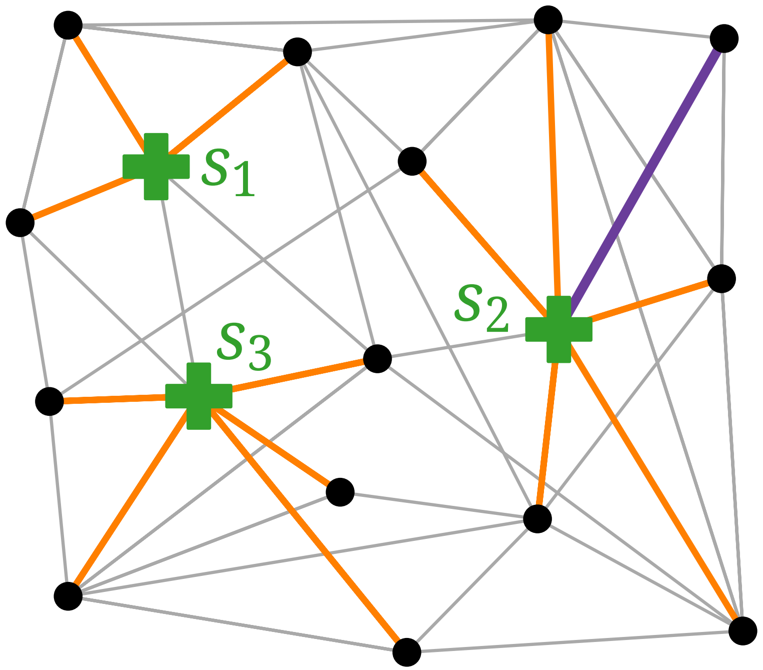

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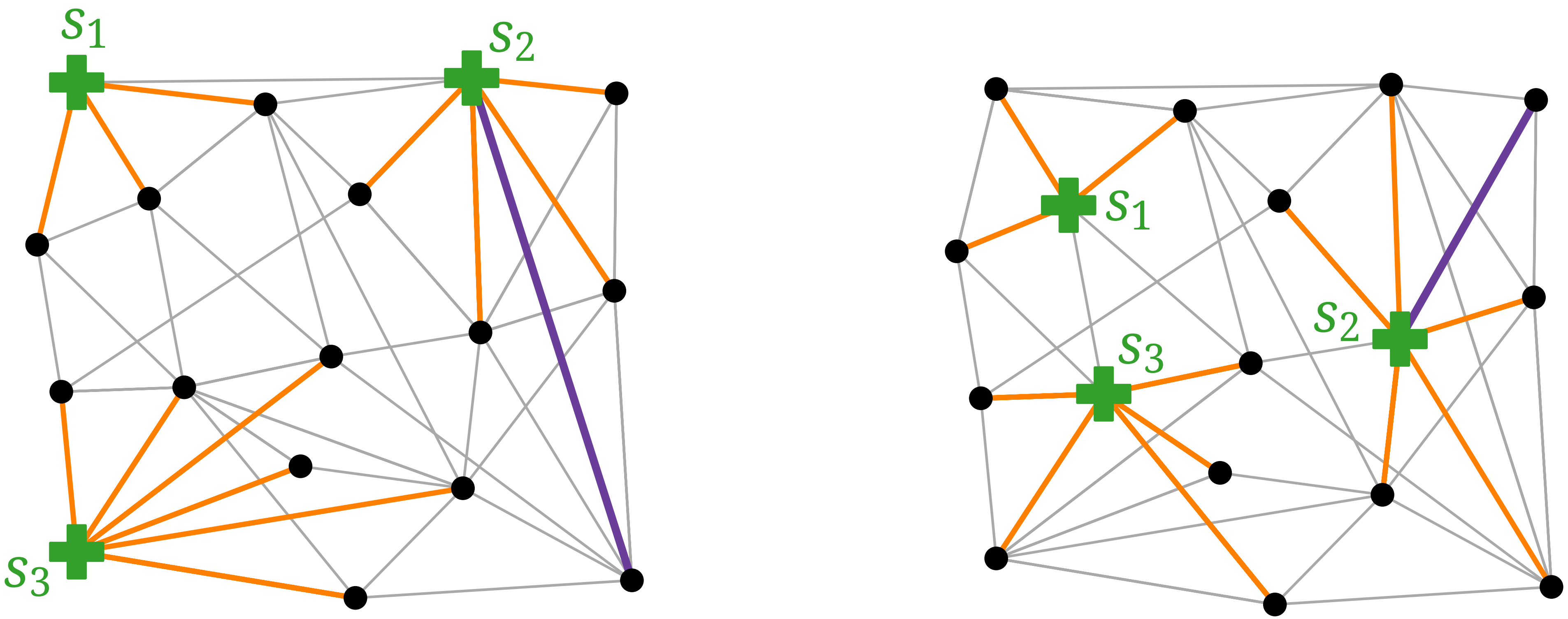
today: 2-approximation of metric k-center using *parametric pruning.*



Technique: Parametric Pruning

Idea 1: Reduce optimization problem to decision problem

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optimization: minimize cost of solution (or maximize)

decision (t): find answer to: Is there a feasible solution with cost $\leq t$?

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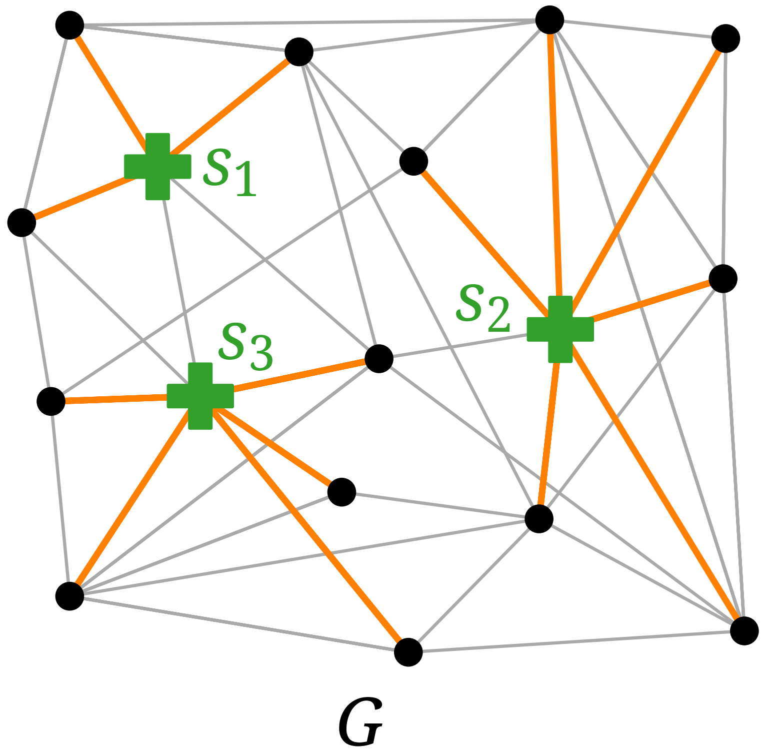
Idea 3 (lower bound): factor-α approximation algorithm consists of two steps

1.) use family of instances $I(t)$ to compute lower bound t^* for OPT

2.) find solution in instance $I(\alpha t^*)$

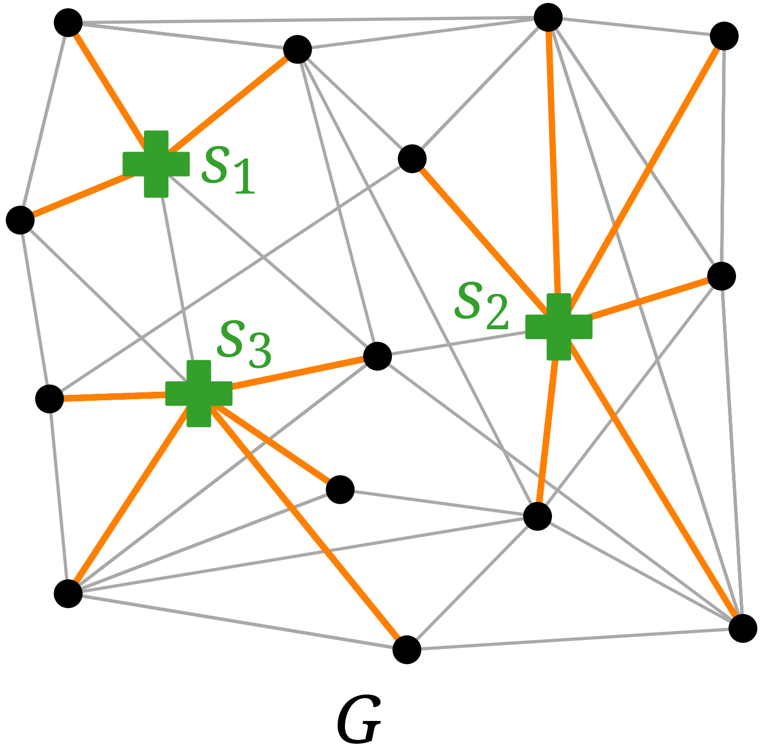
Parametric Pruning: Metric k-center

Metric k-center: What is the decision problem? How do the pruned instances $I(t)$ look like?



Parametric Pruning: Metric k-center

Let $E = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \ldots \leq c(e_m)$.



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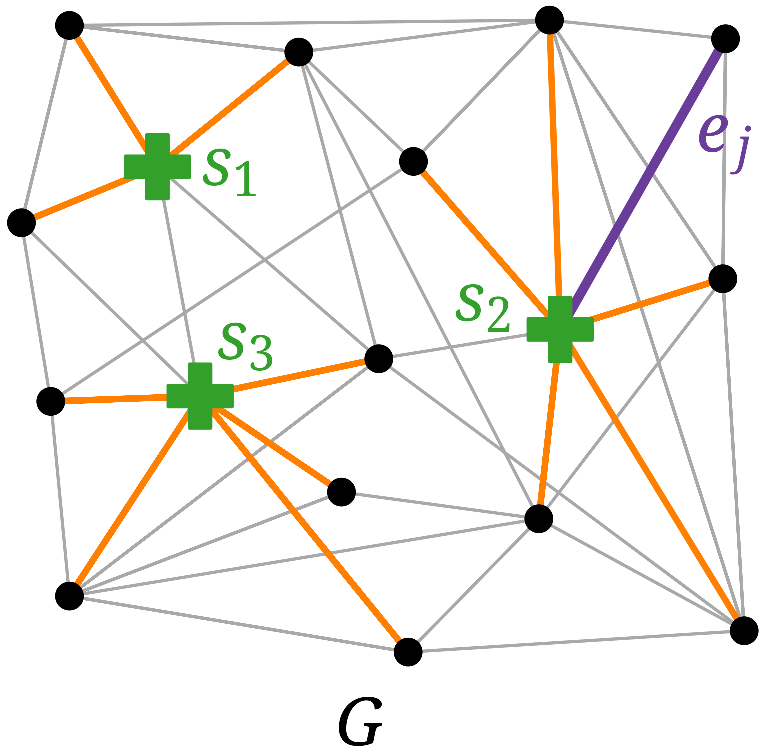

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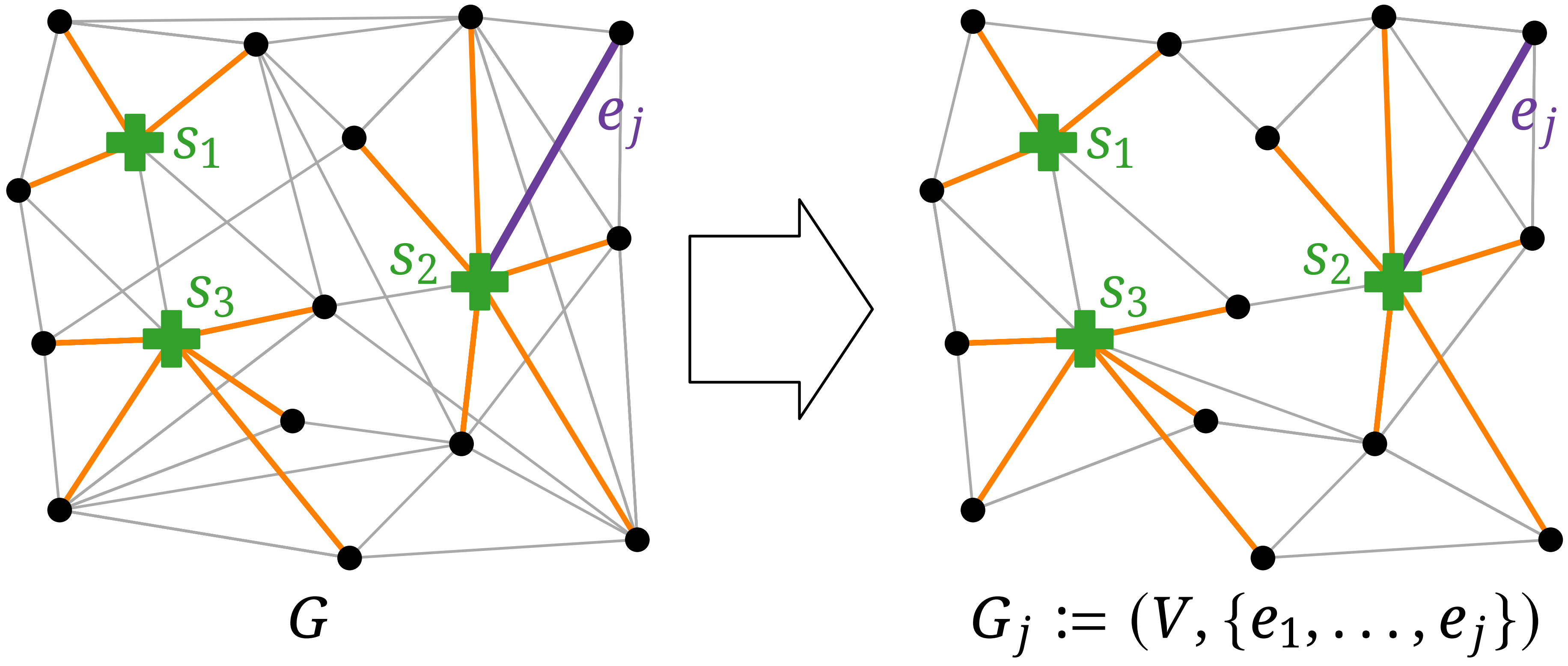

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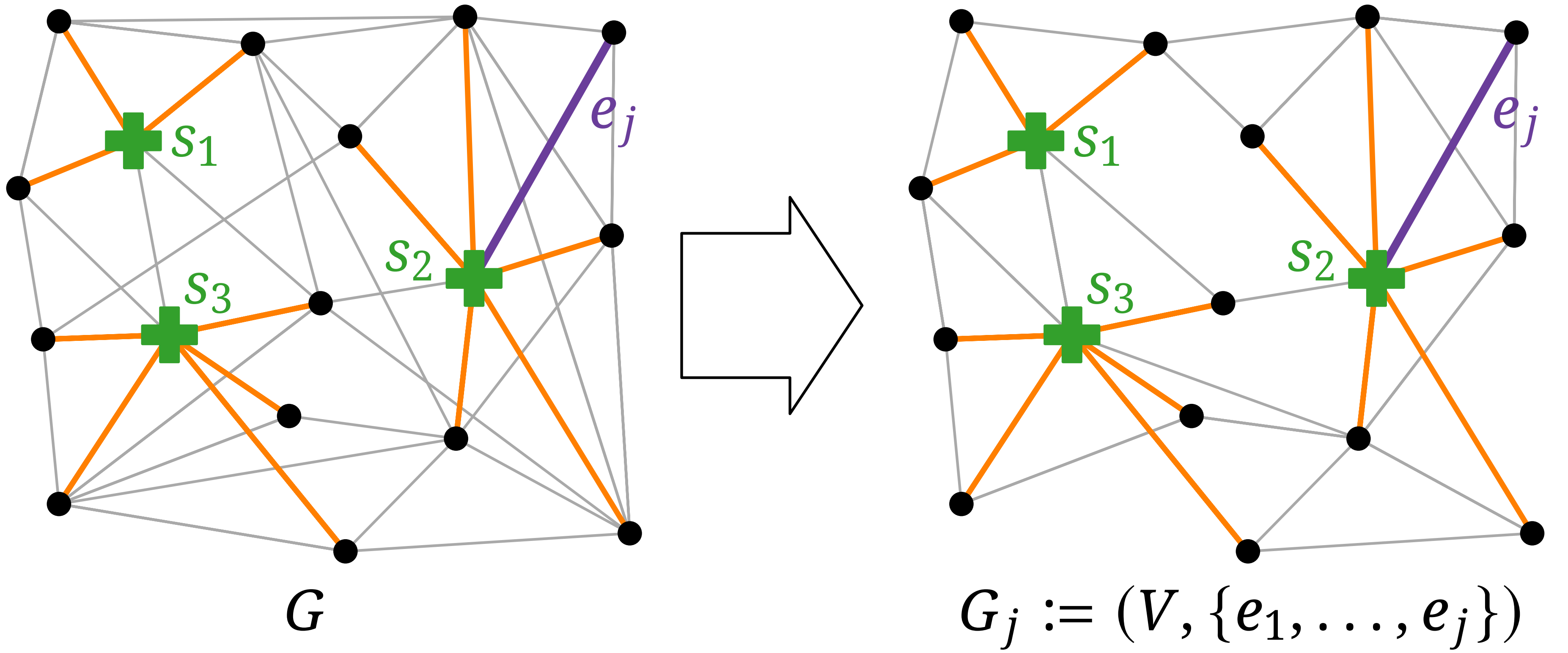

…try each pruned instance G_j.

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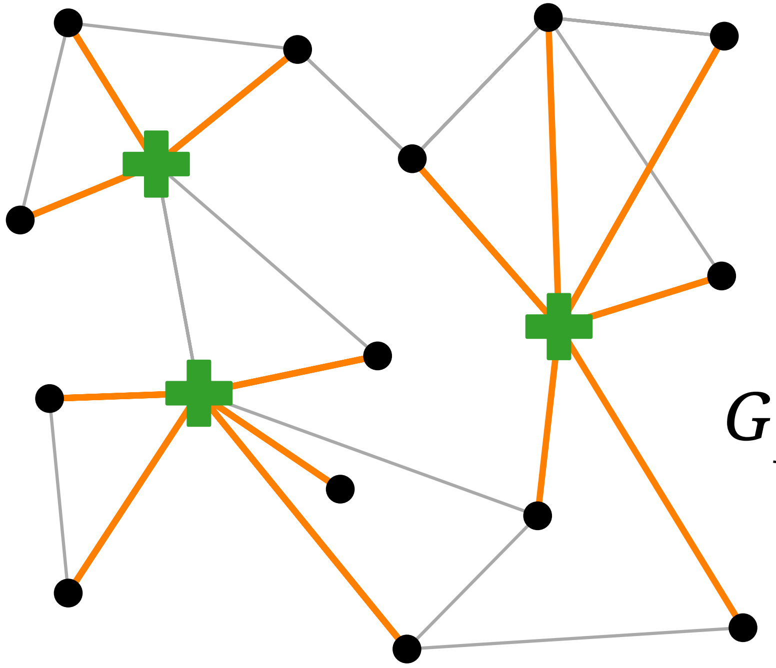
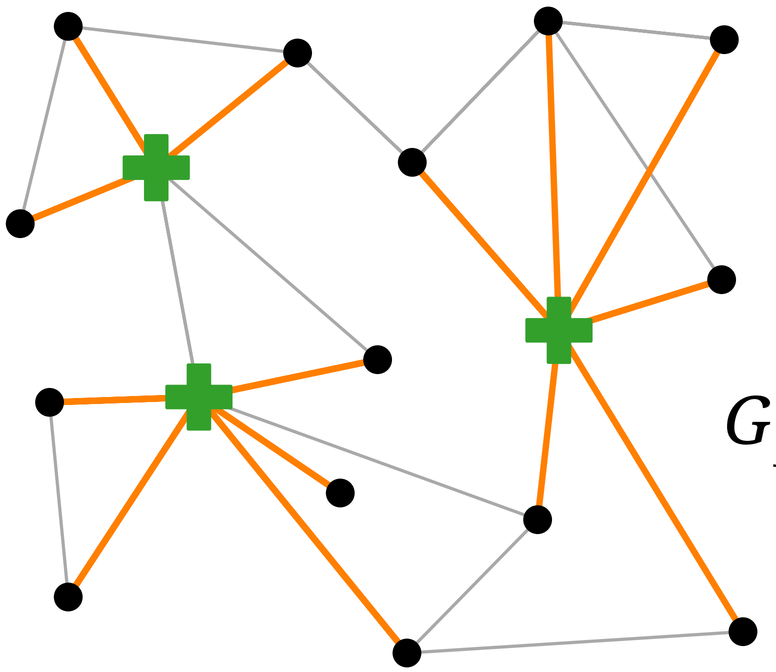
Def.



$G_j := (V, \{e_1, \ldots, e_j\})$

…try each G_j.

Def. A vertex set D of a graph H is dominating if each vertex is either in D or adjacent to a vertex in D.



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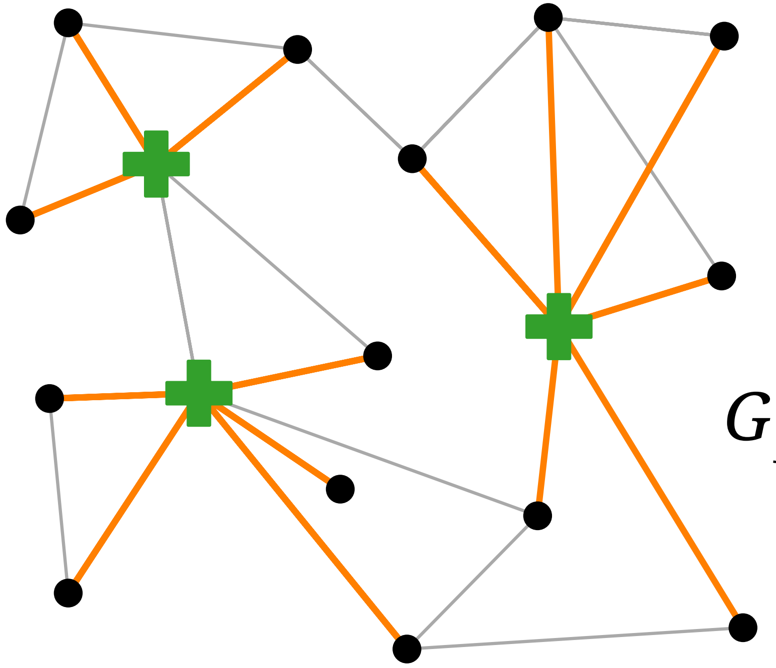

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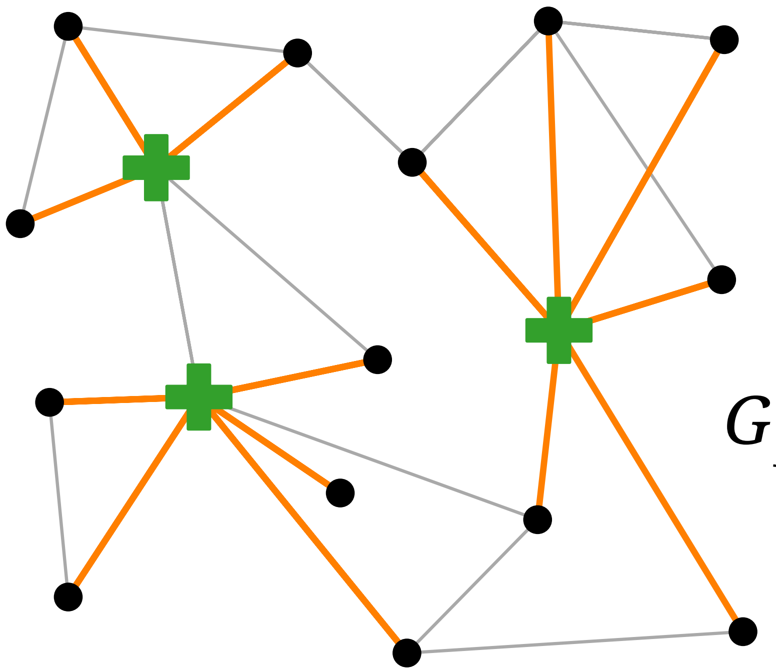

decision problem for k-Center:
$\mathrm{dom}(G_j) \leq k$?

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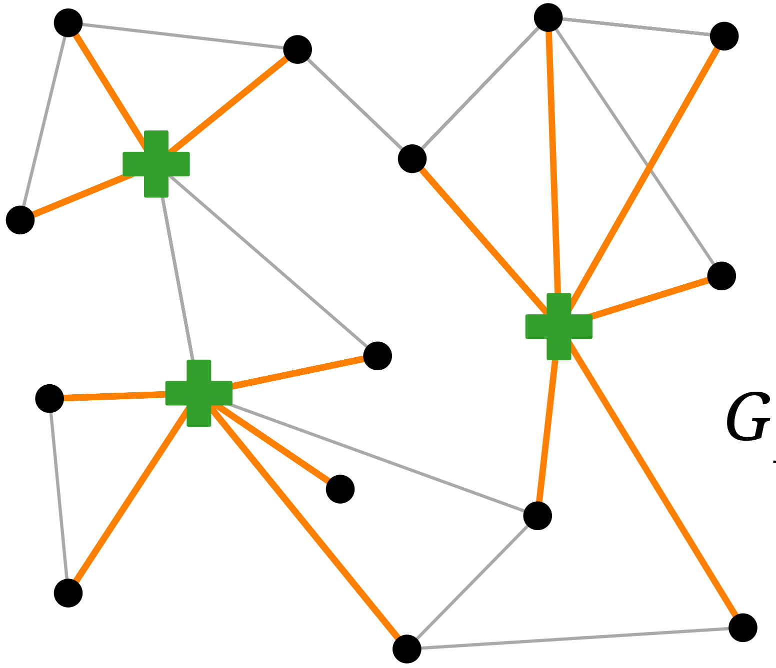

decision problem for k-Center:
$\mathrm{dom}(G_j) \leq k$?

$$G_j := (V, \{e_1, \dots, e_j\})$$

…but computing $\mathrm{dom}(H)$ is NP-hard.

Square of a Graph – Lower bounding k-Center

Square of a Graph

Idea: step 1.) Find a maximal set of vertices that have to connect to different centers

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Def. The square H^2 of a graph H has the same vertex set as H.

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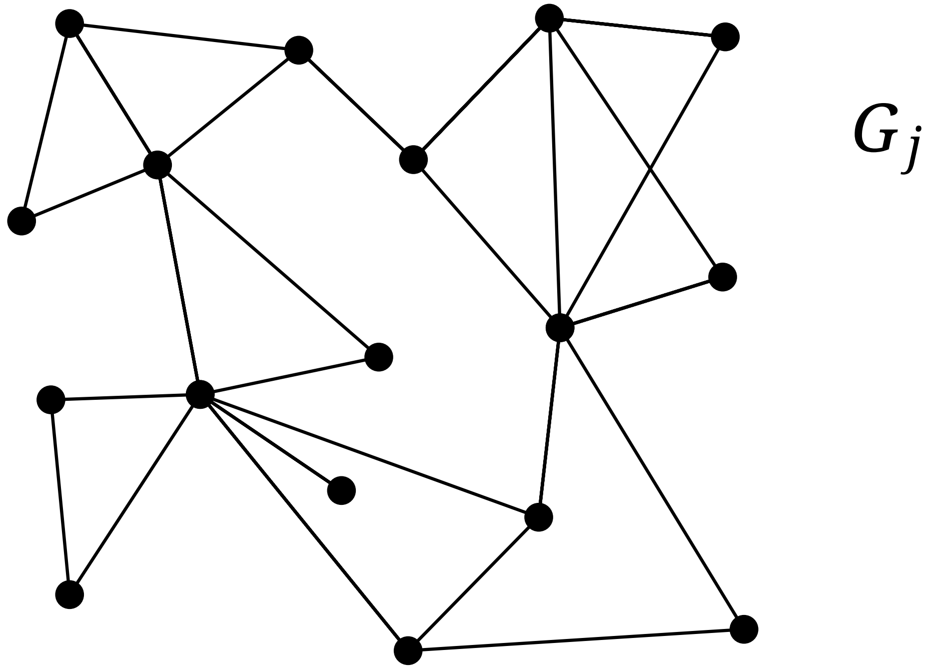

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Idea: step 1.) Find a maximal set of vertices that have to connect to different centers
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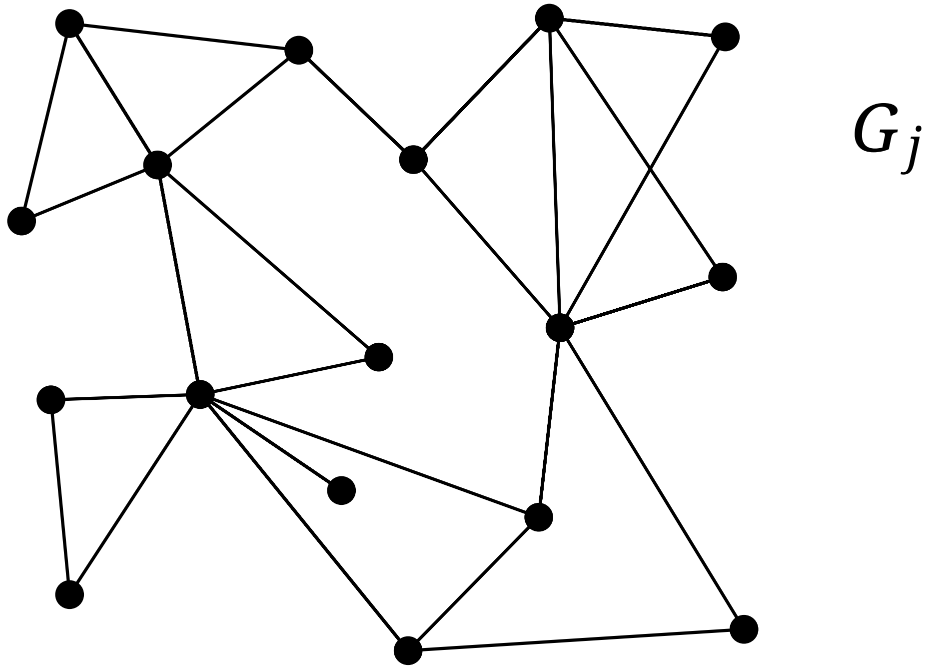
Ingredient 1: square of $G_j \approx$ "coarsened" G_j.

Def. The square H^2 of a graph H has the same vertex set as H. Two vertices $u \neq v$ are adjacent in H^2 iff they are within (graph-)distance at most two in H.



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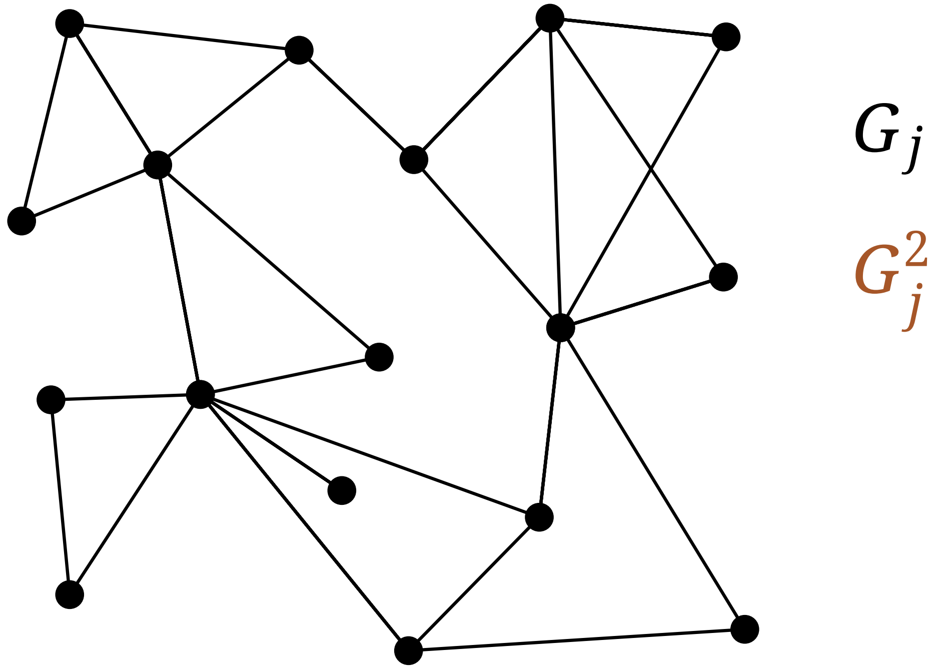

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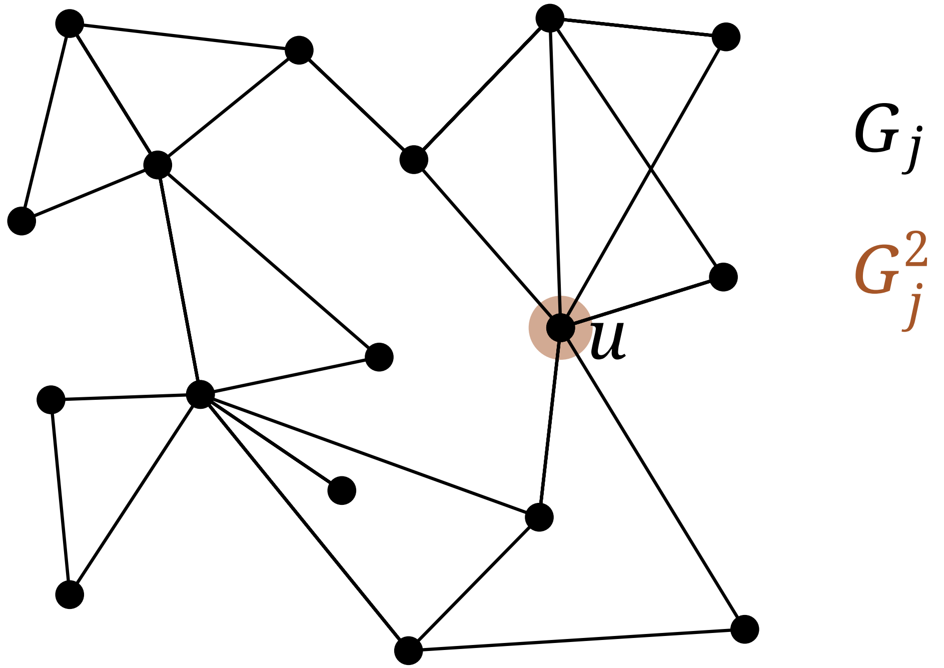

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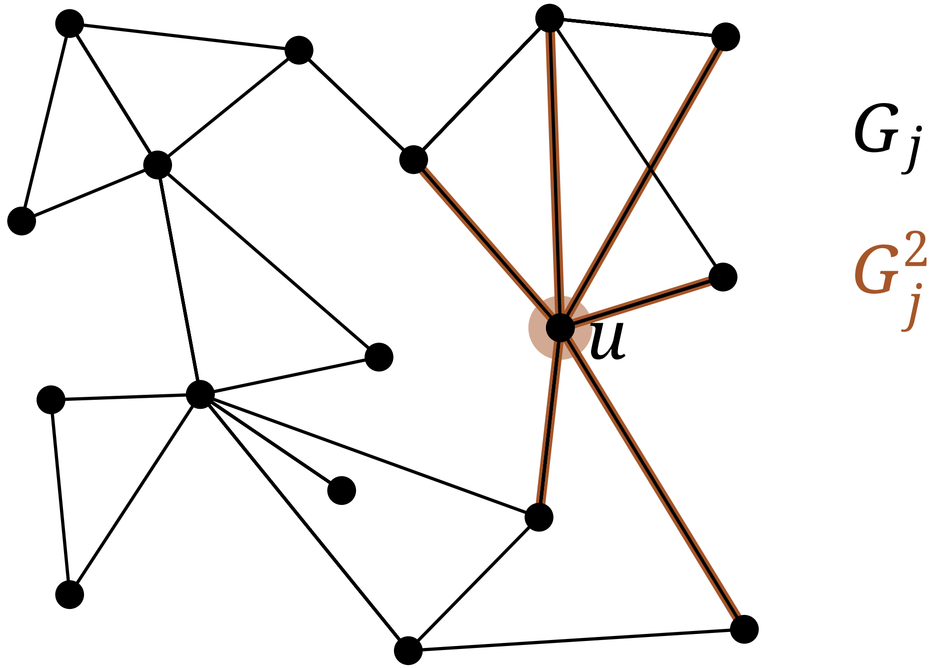

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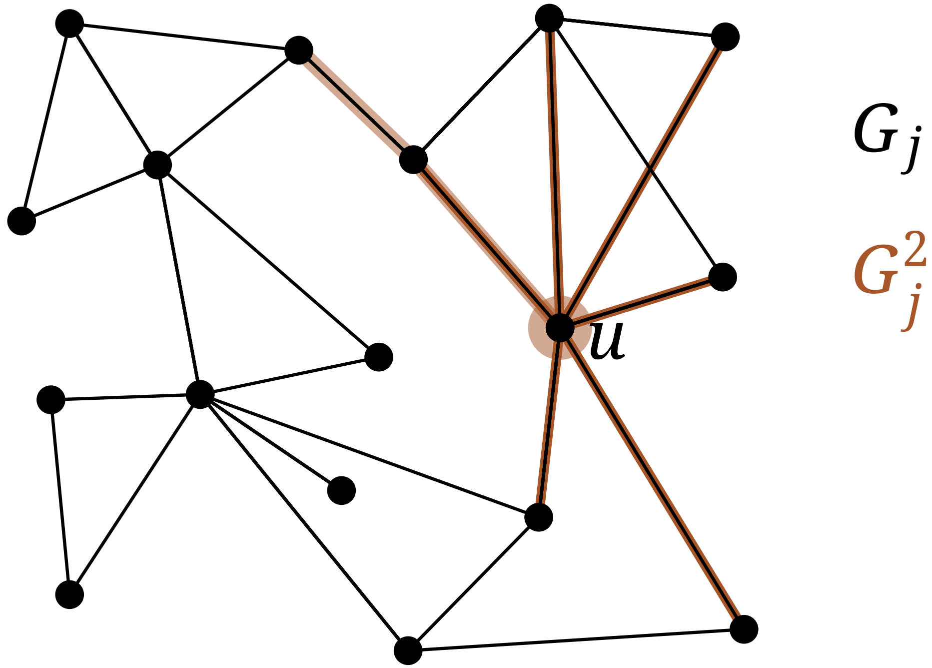

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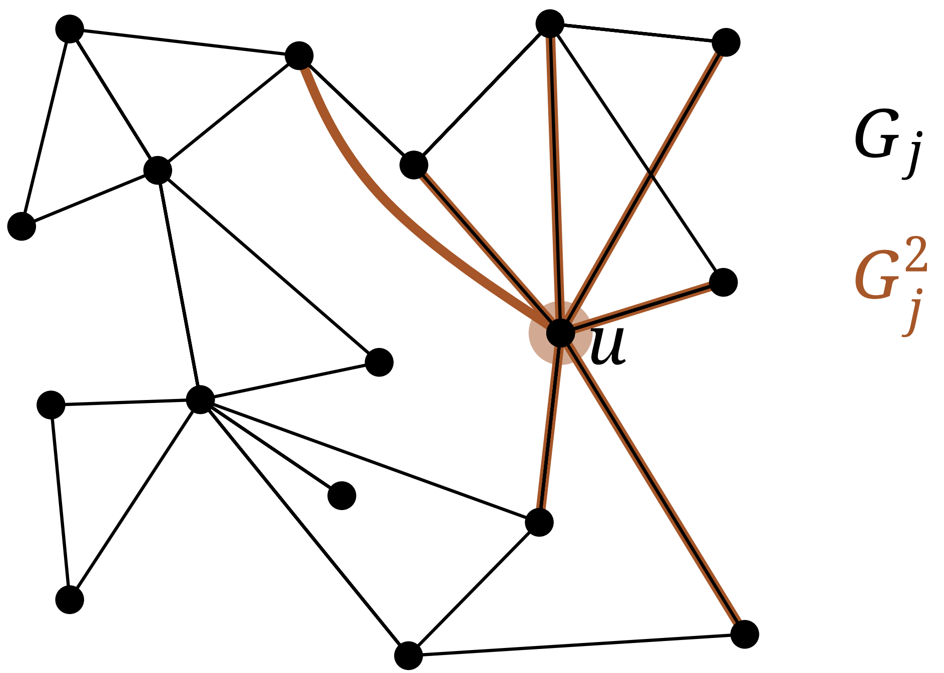

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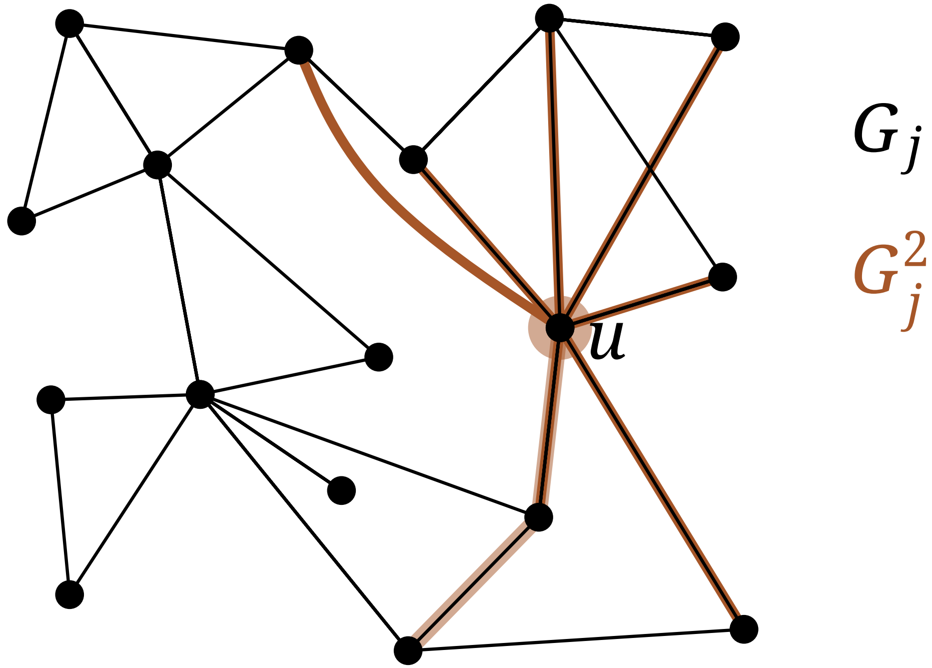

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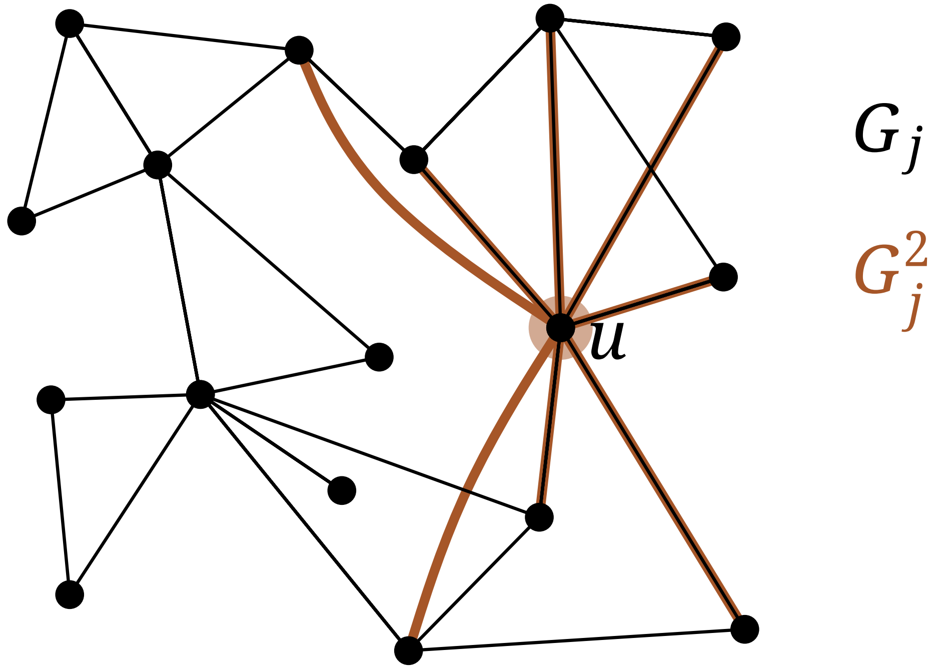

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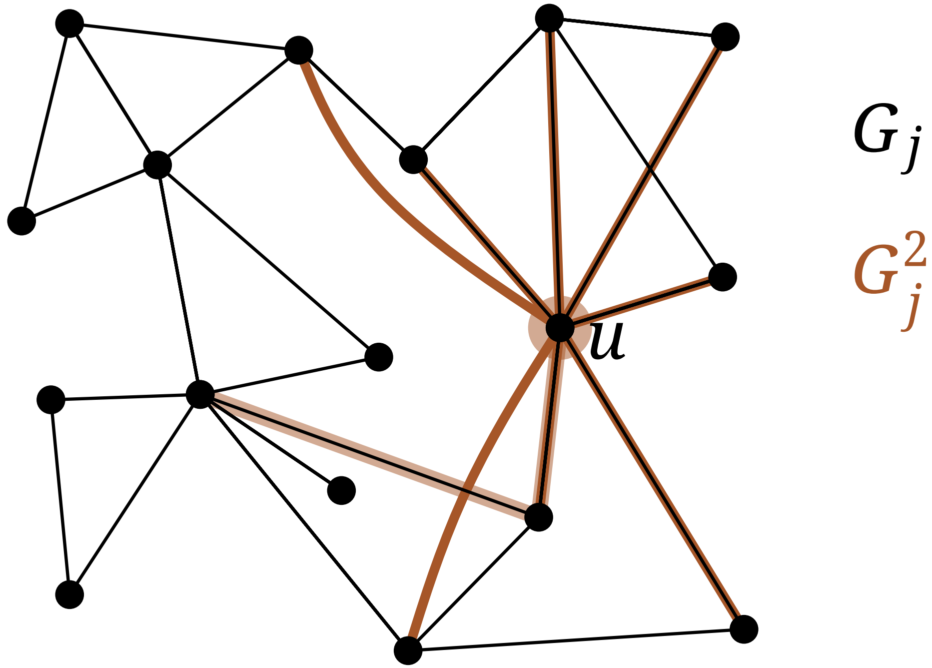

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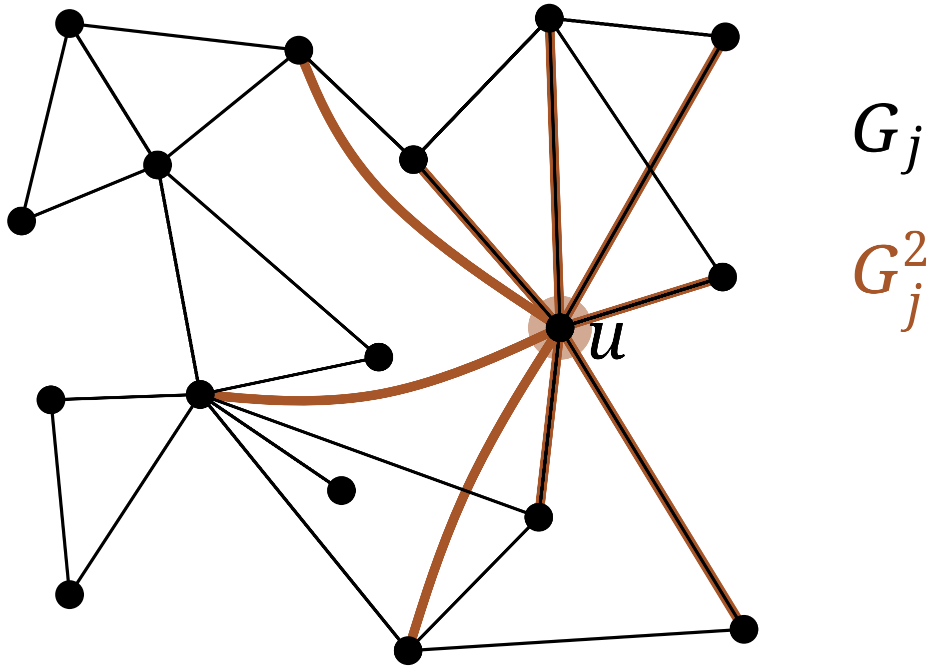

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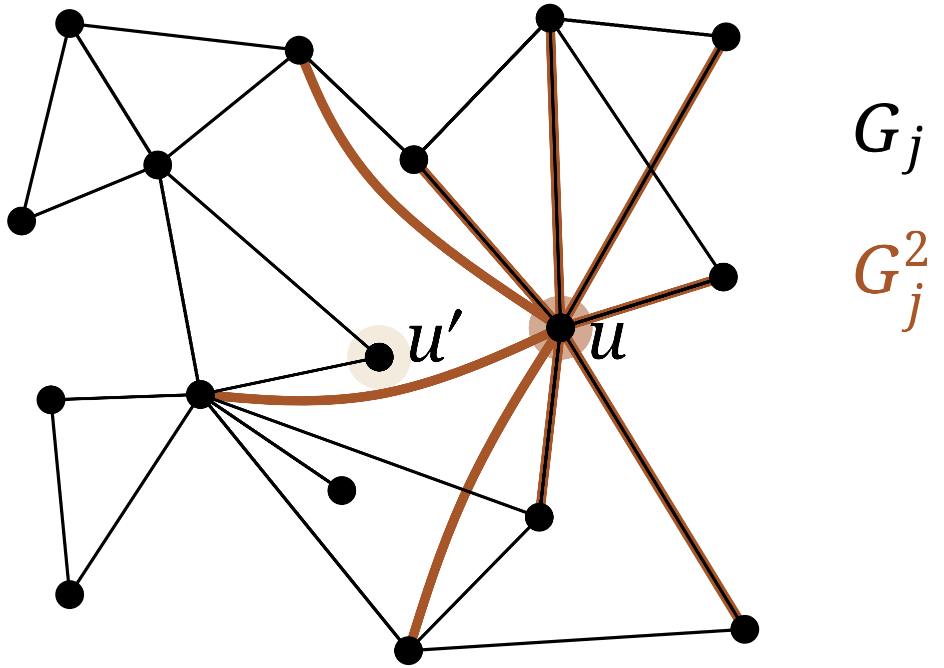

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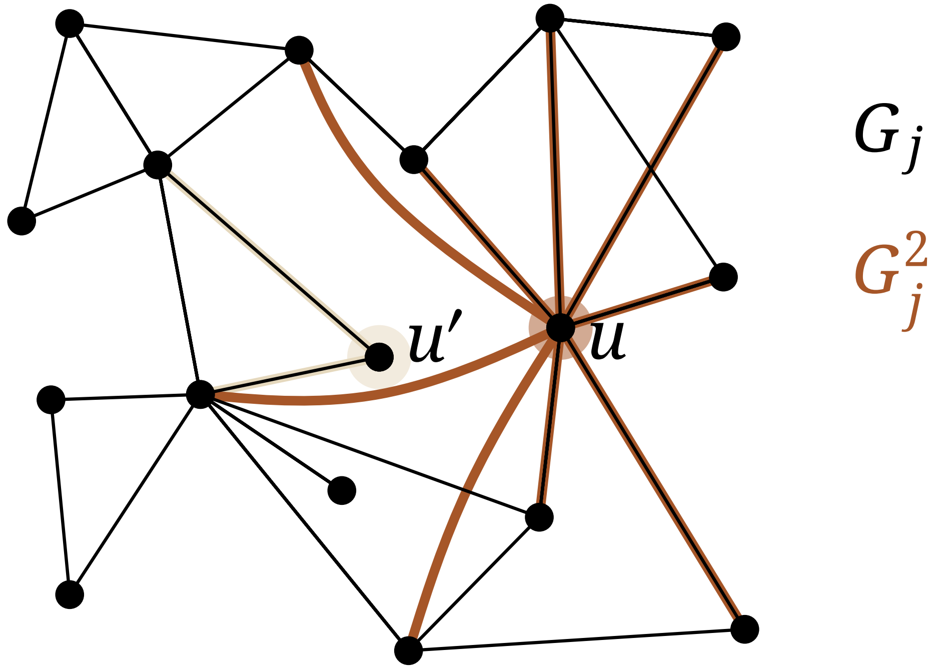

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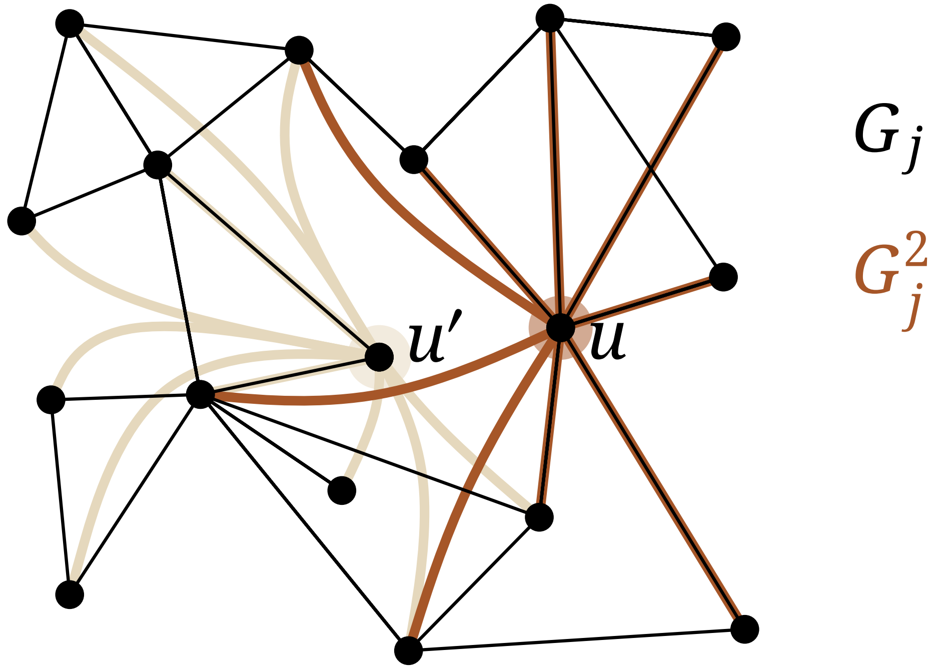

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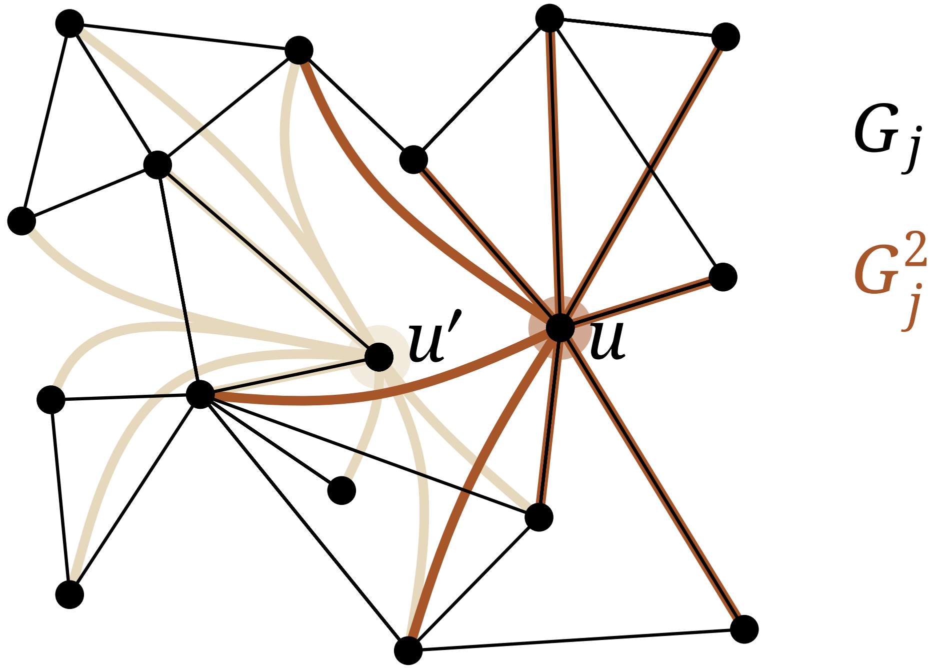

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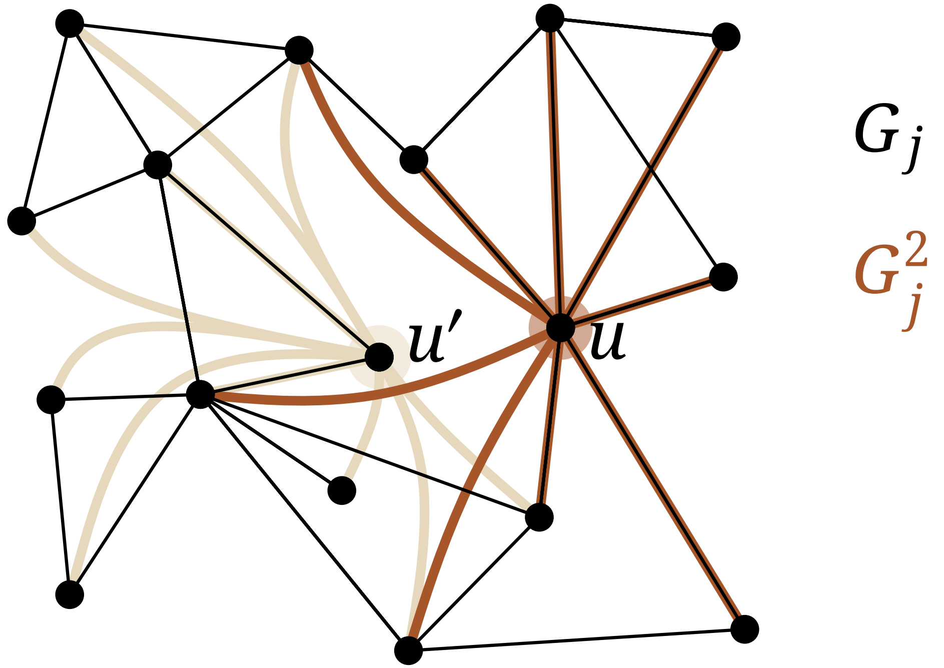

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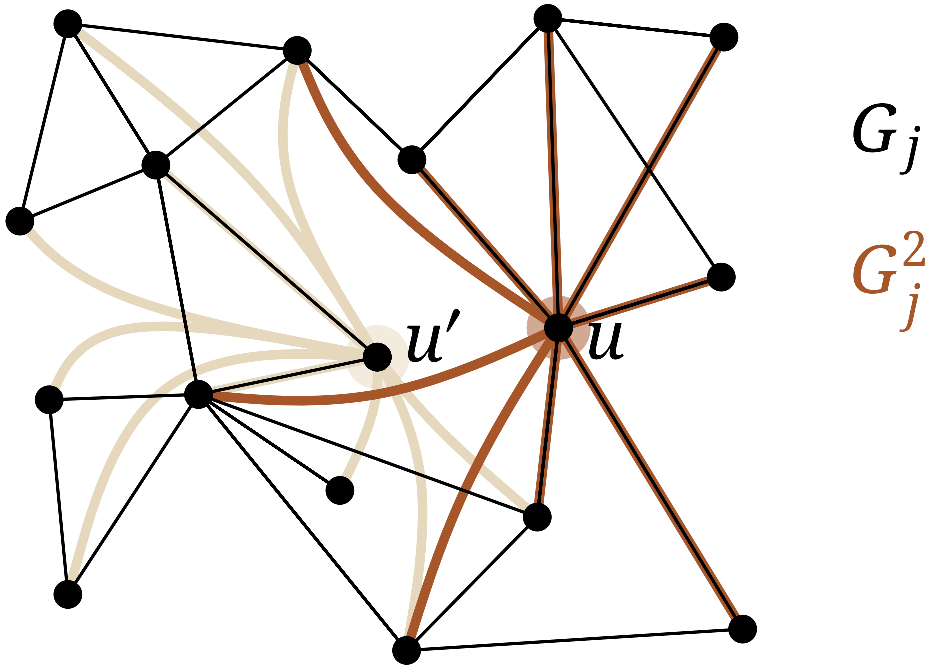

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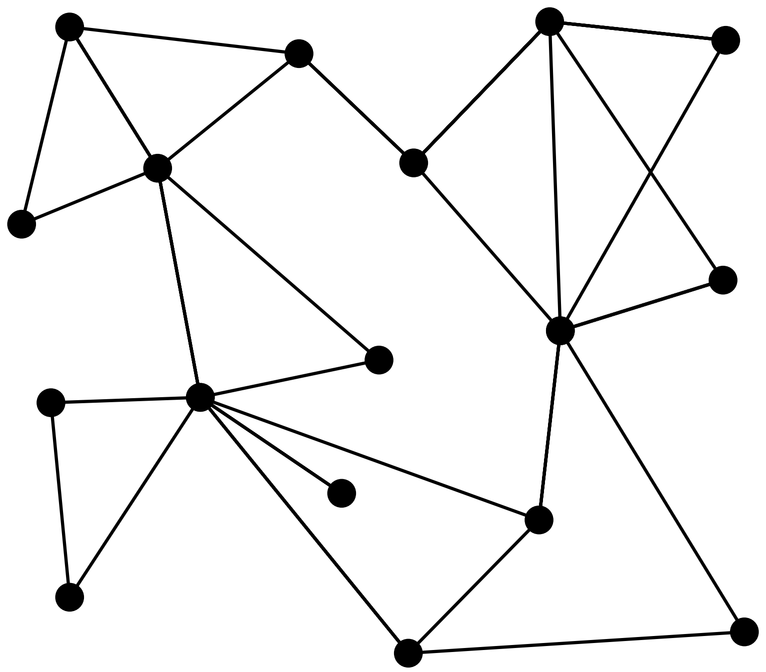

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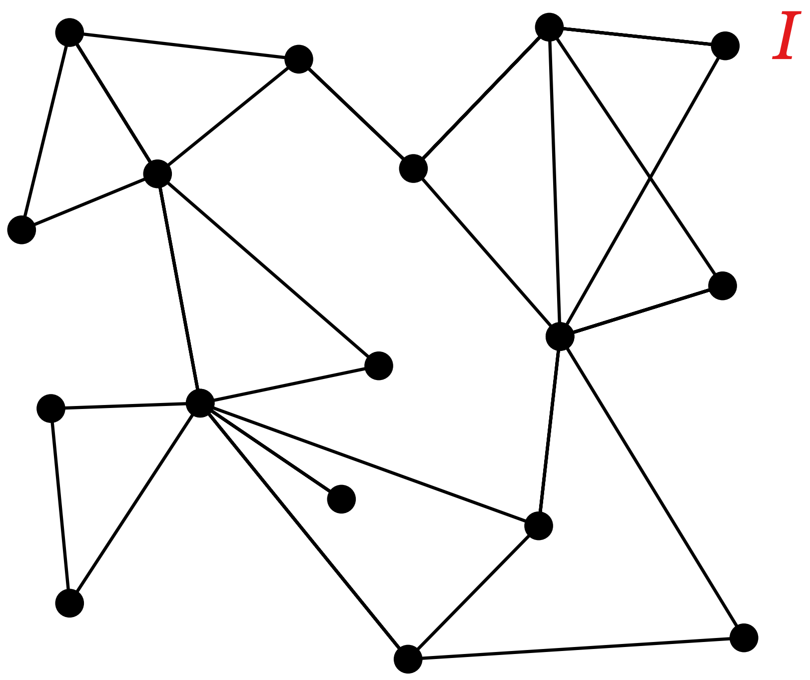

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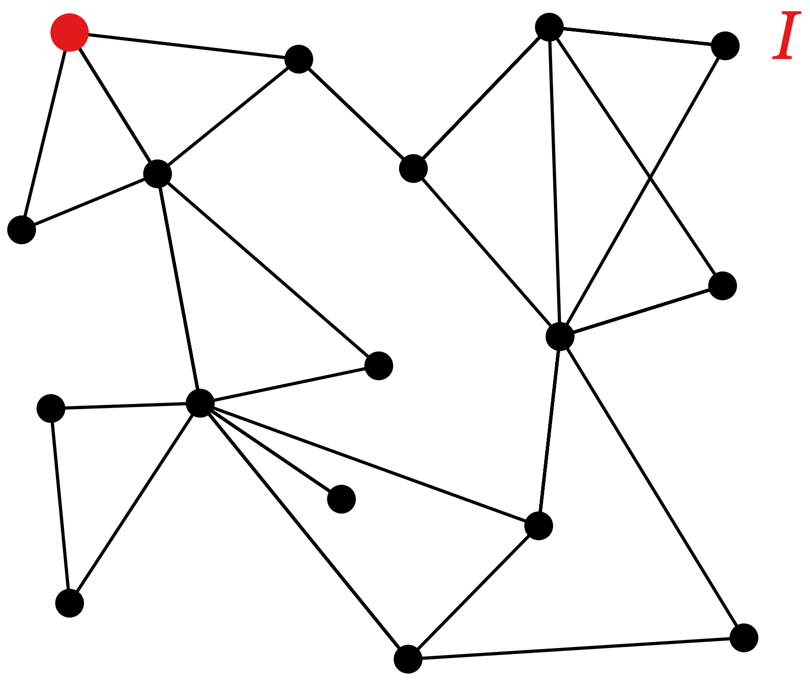

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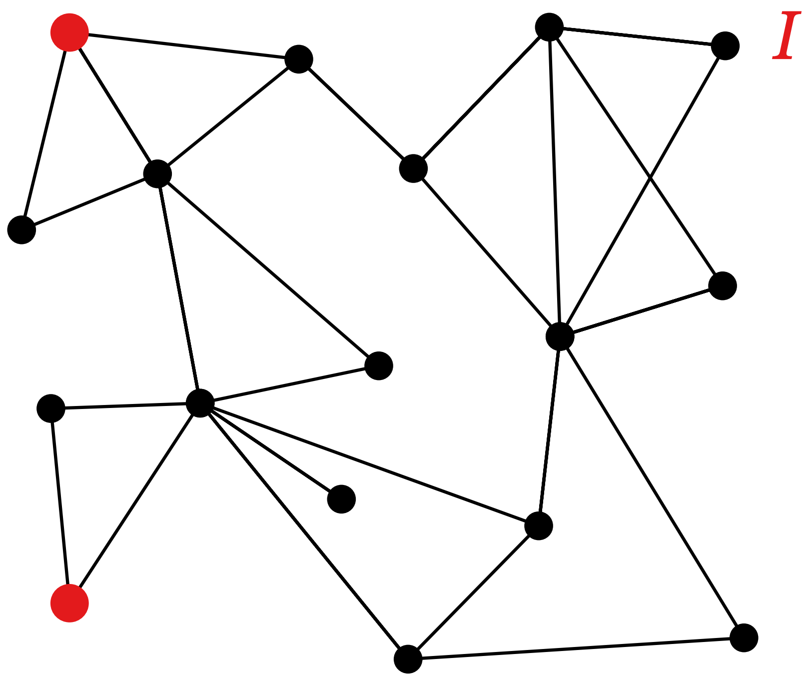

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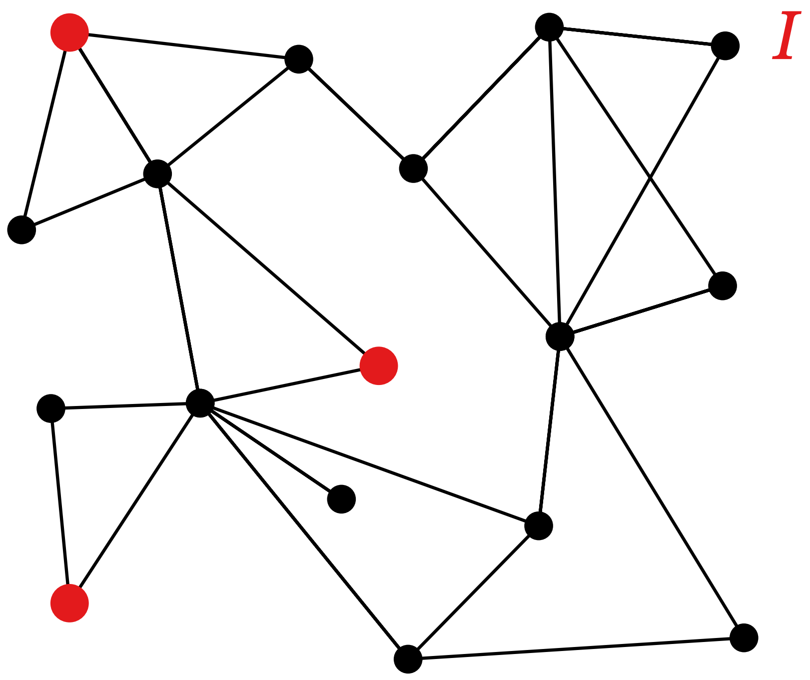

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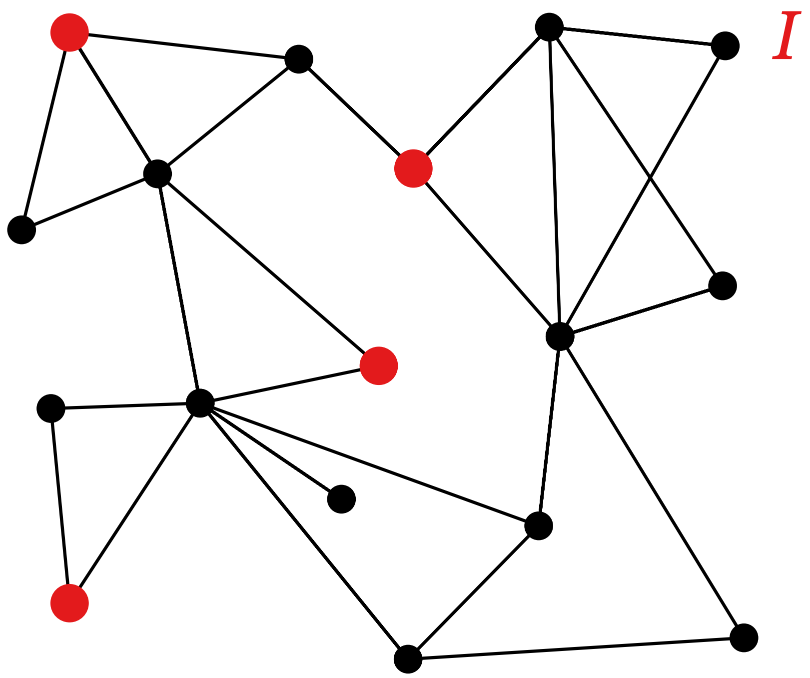

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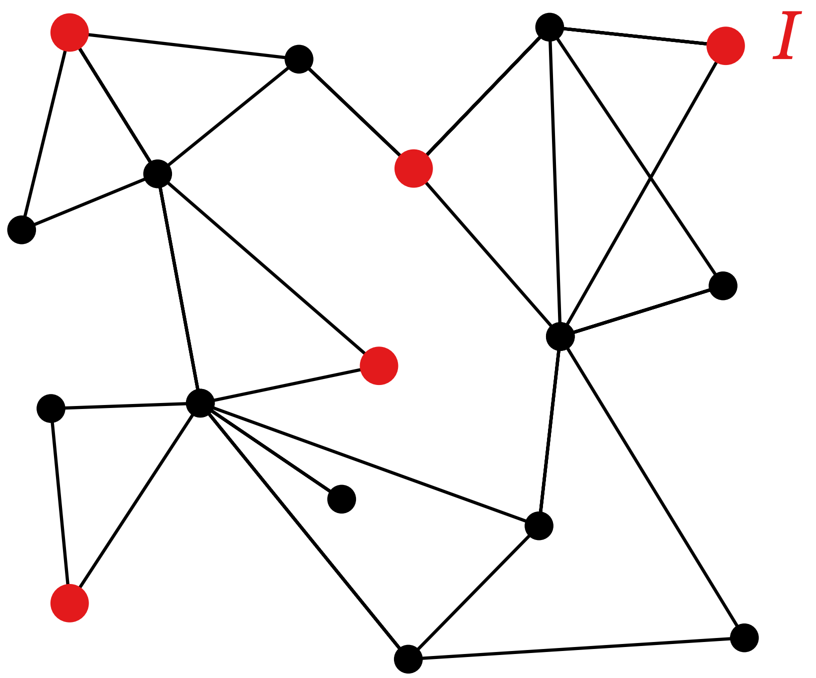

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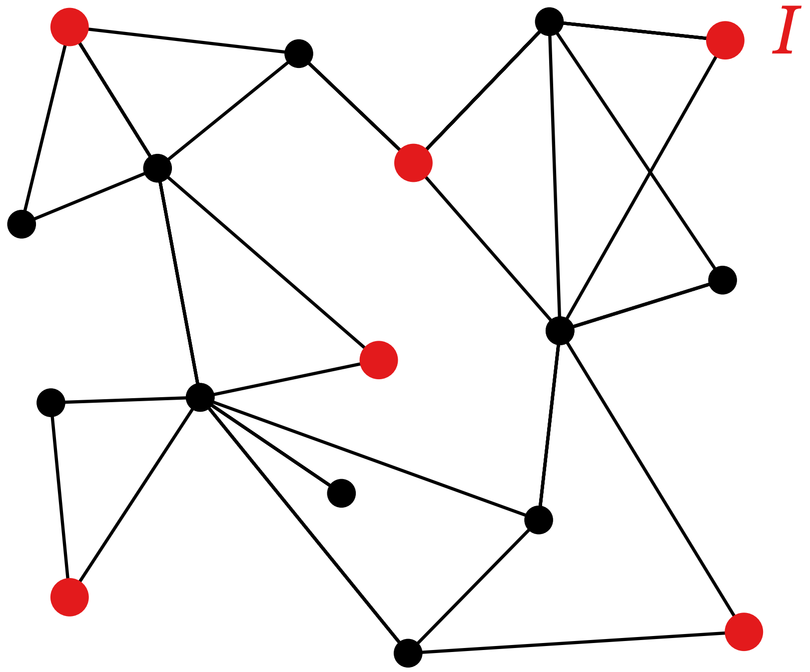

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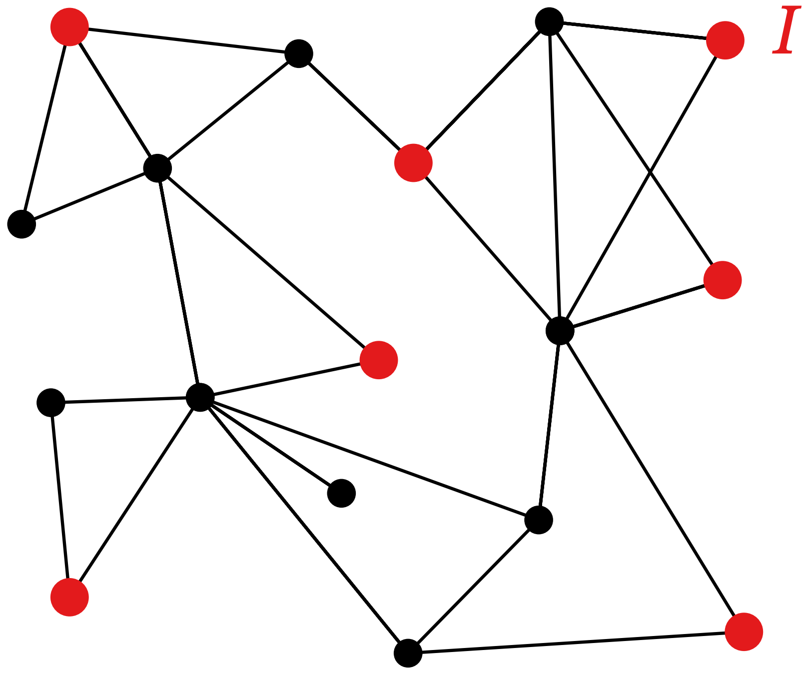

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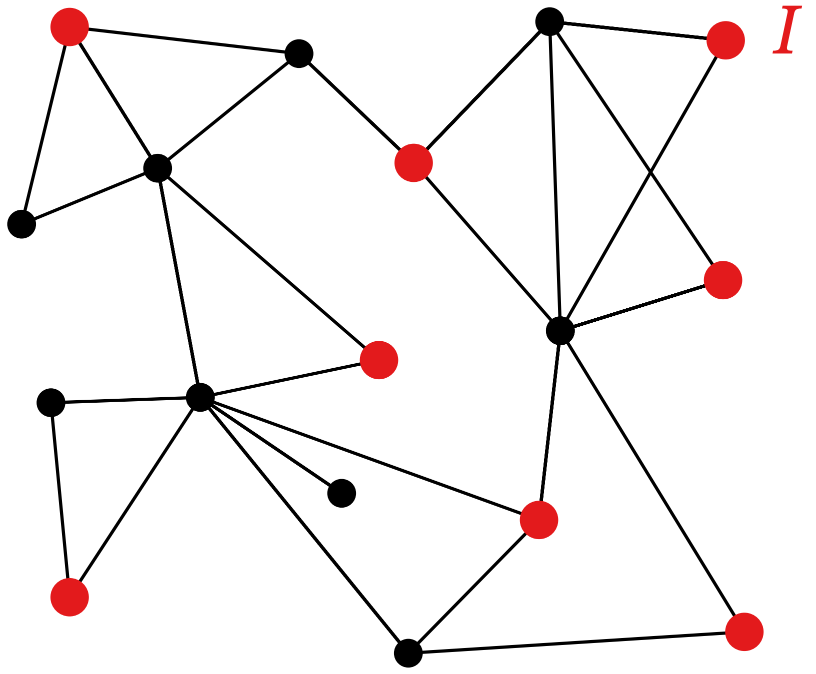

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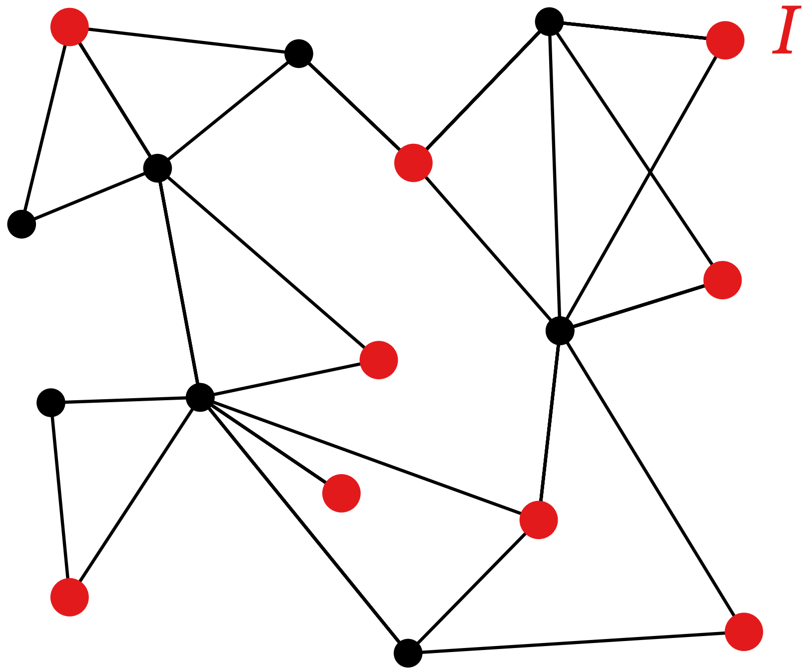

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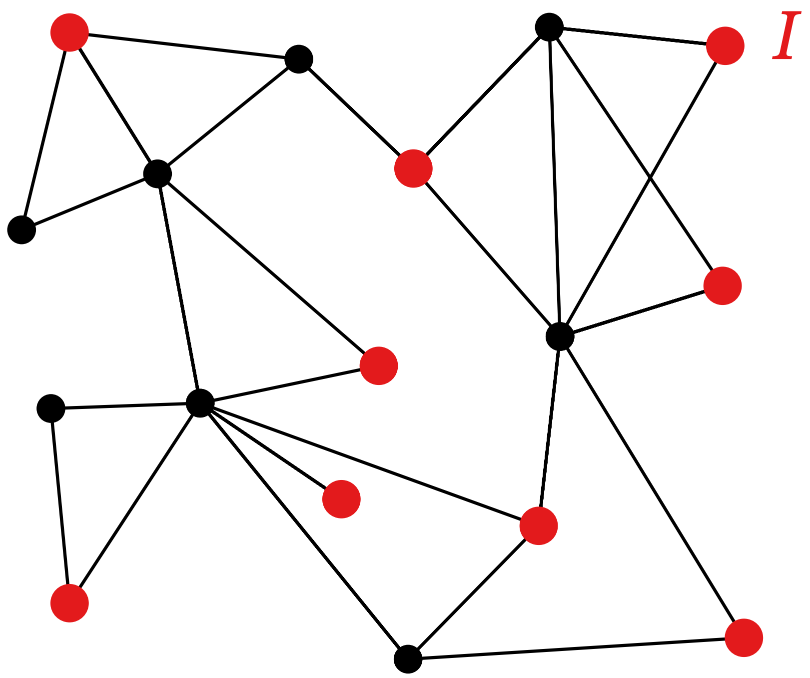

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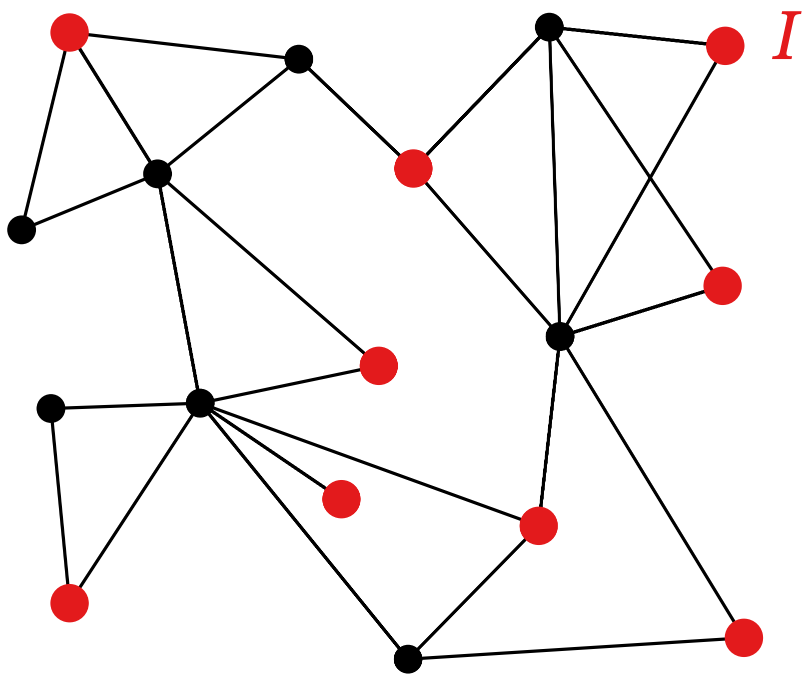

Obs. Maximal independent sets are dominating sets!

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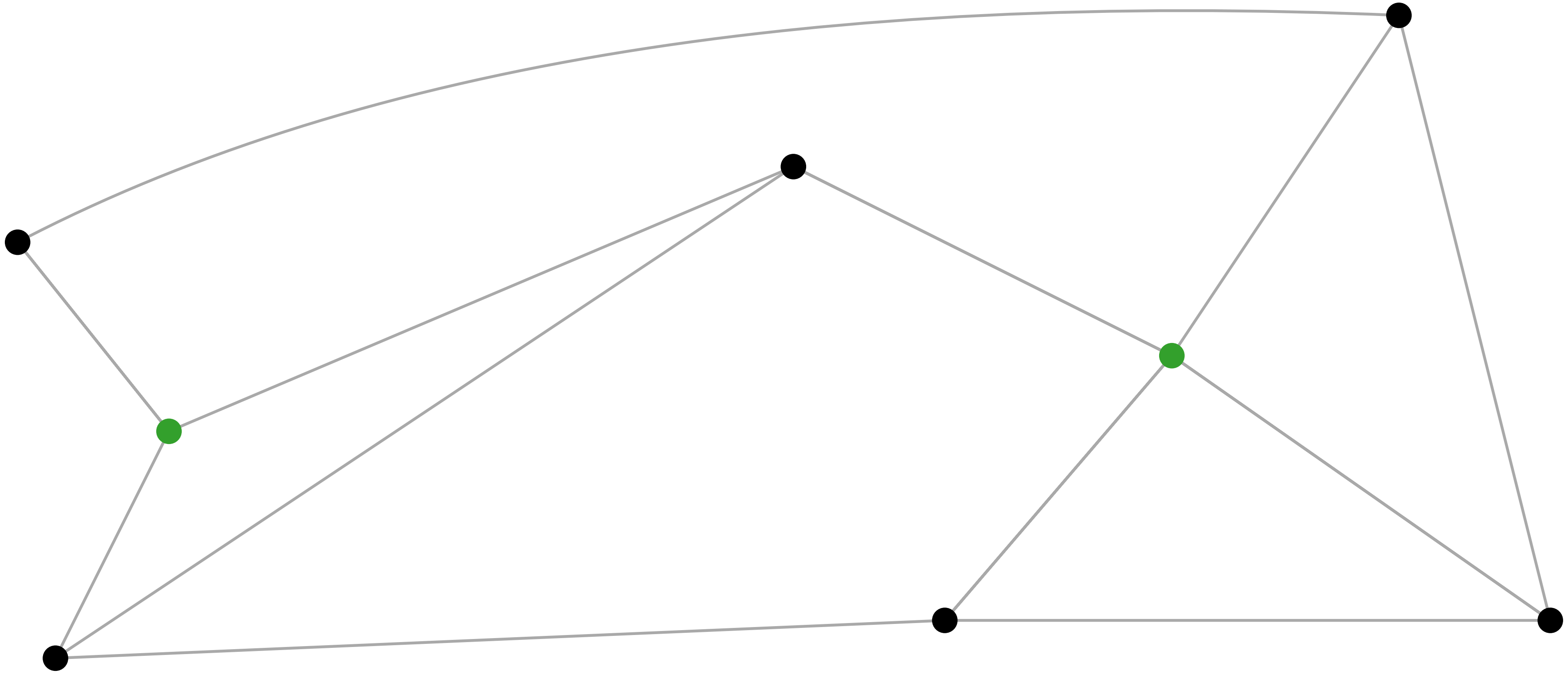

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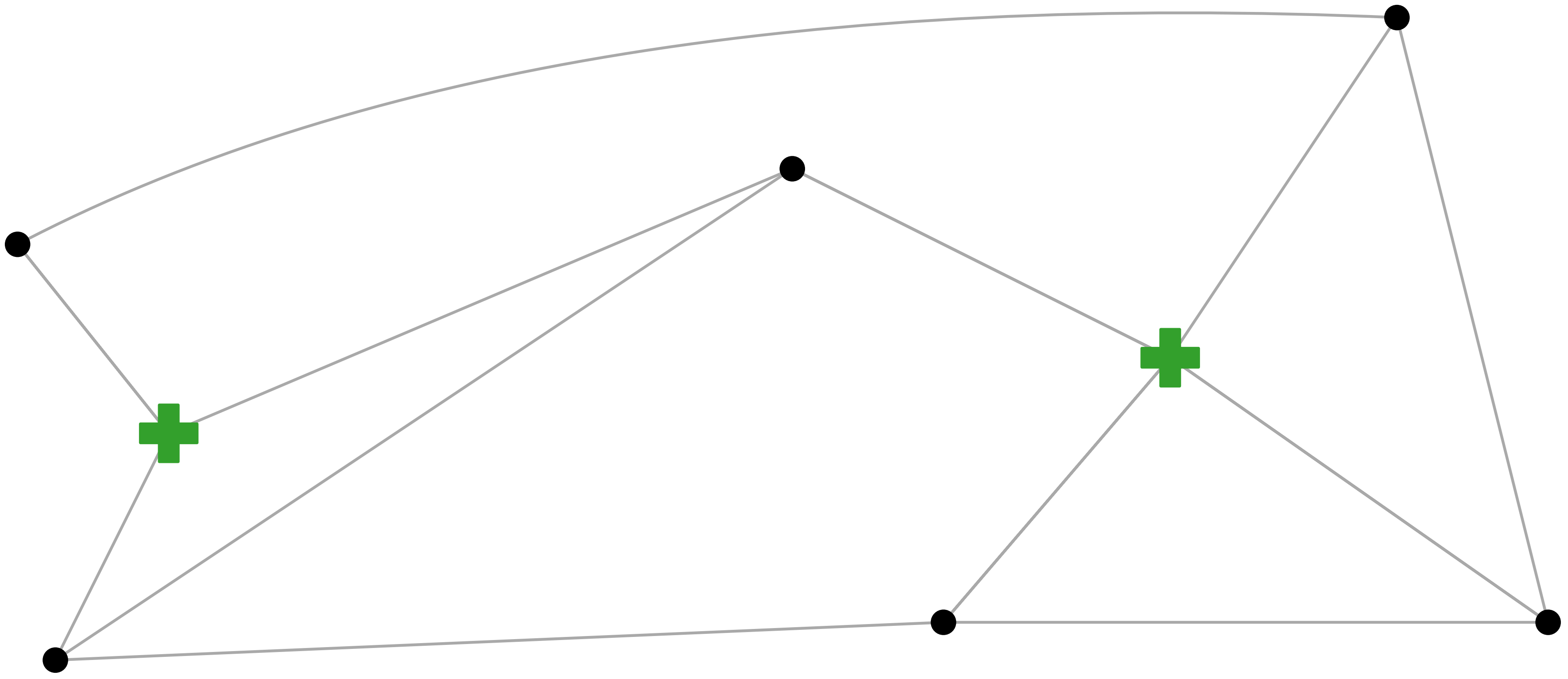

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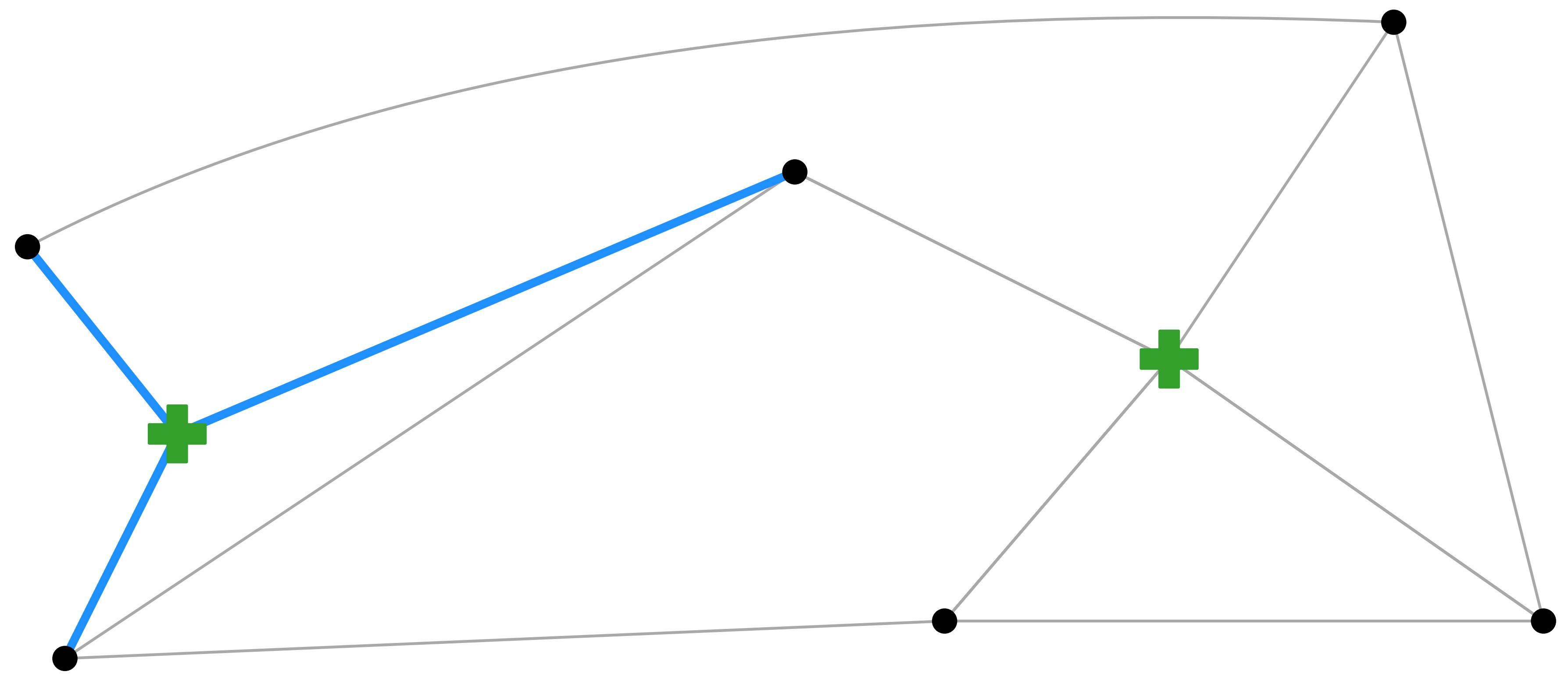

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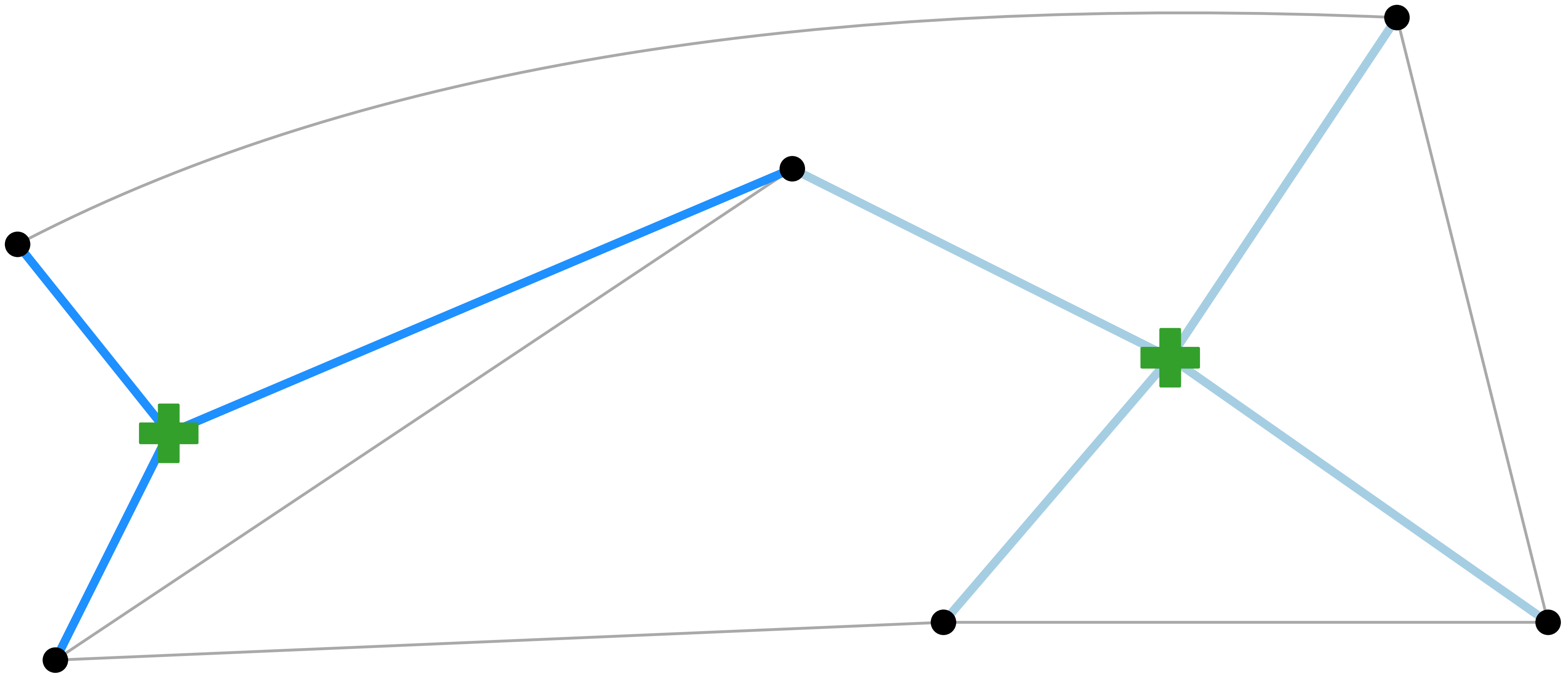

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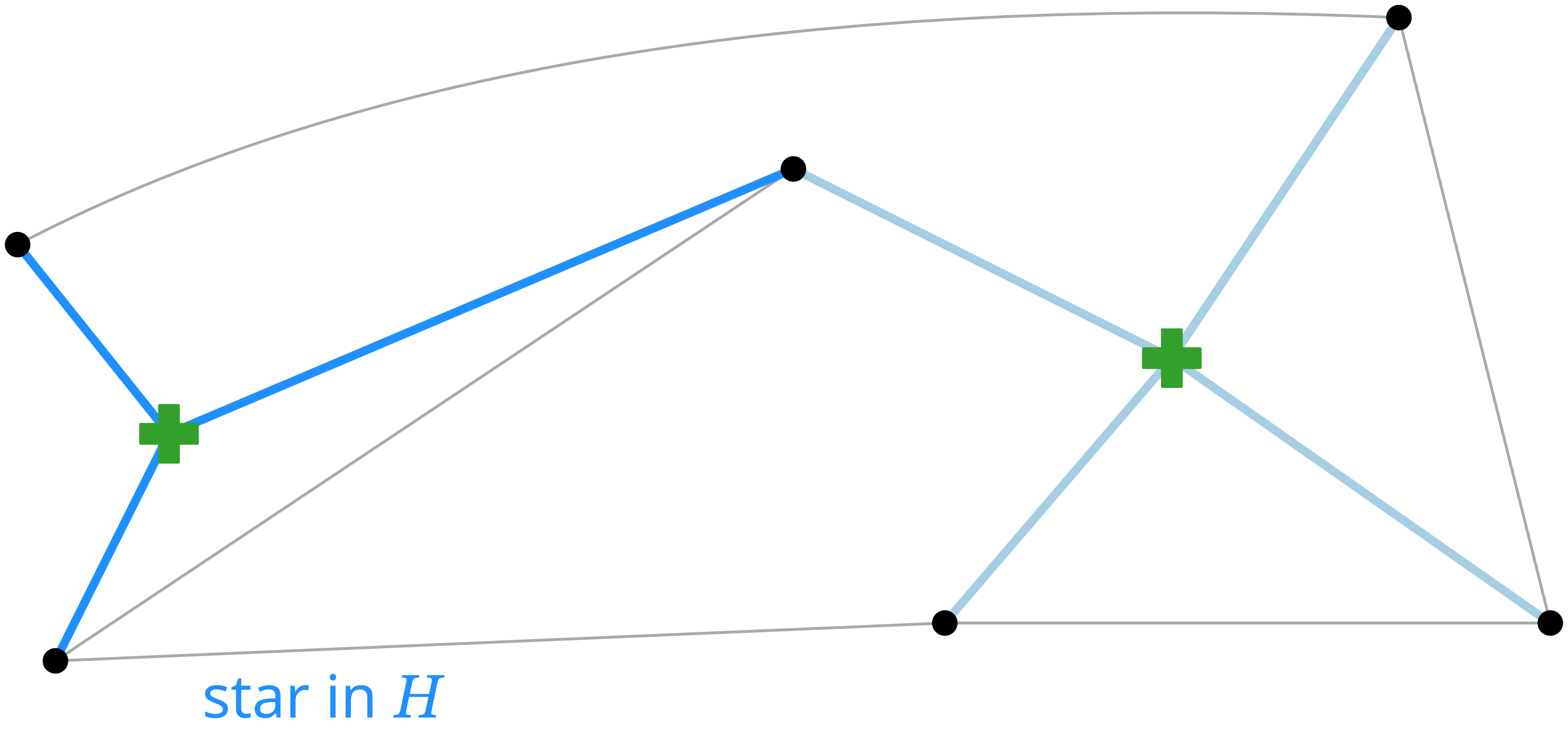

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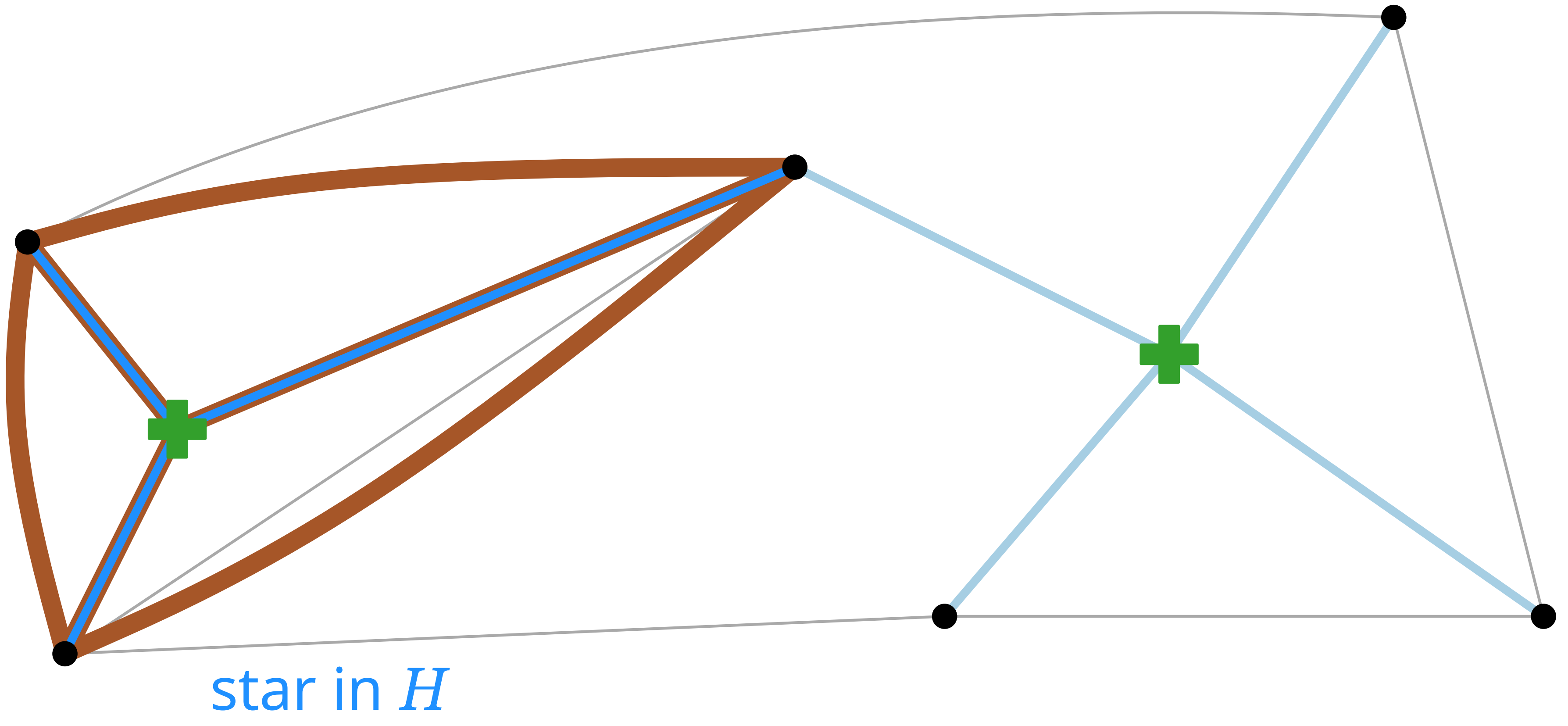

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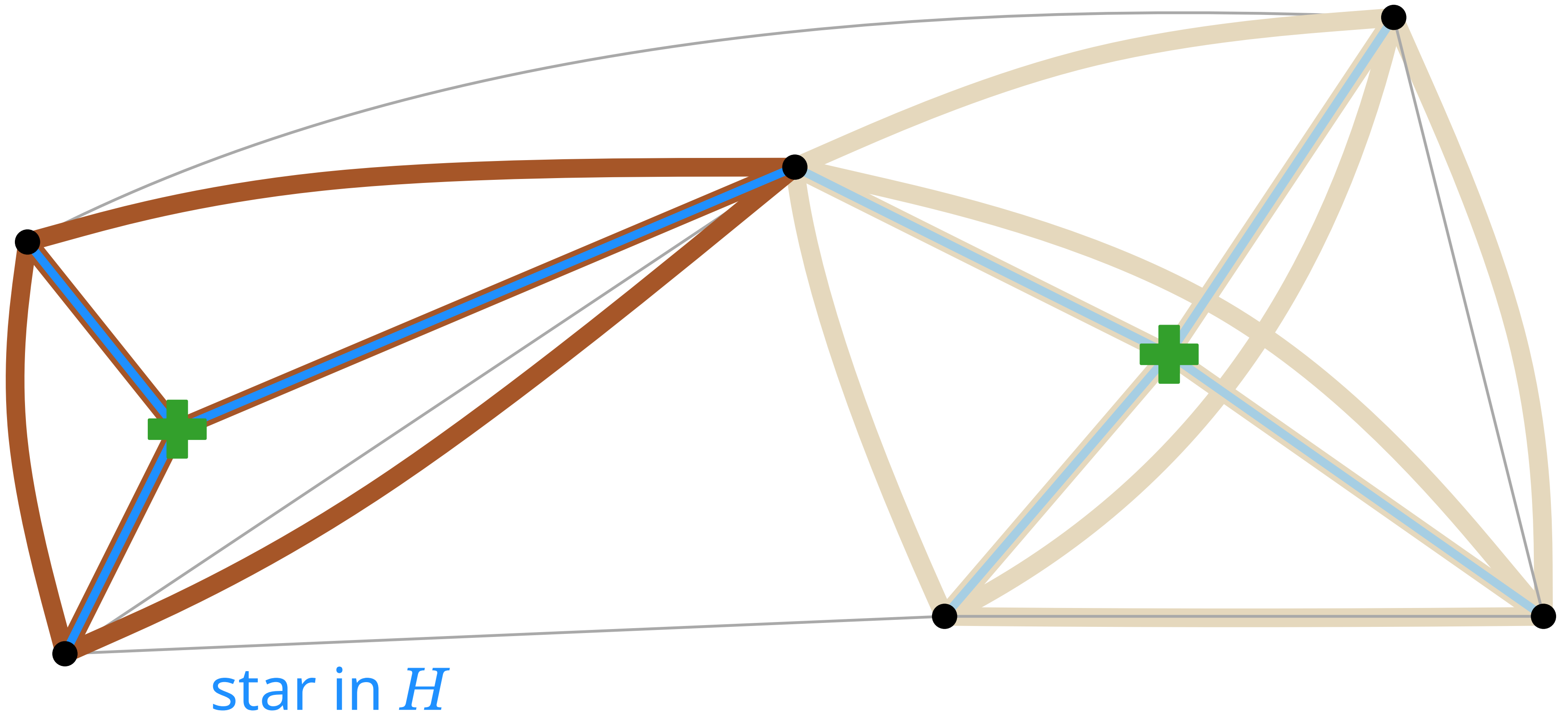

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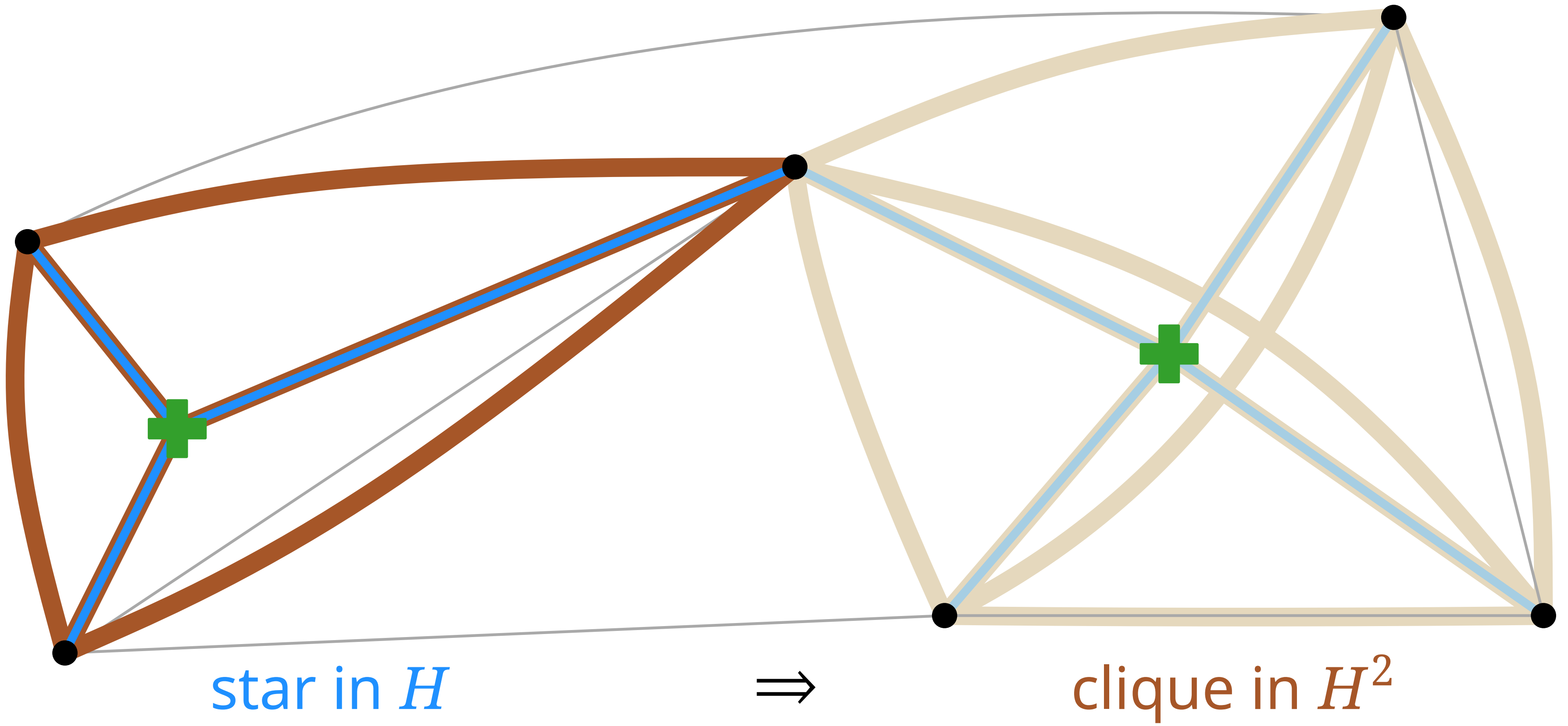

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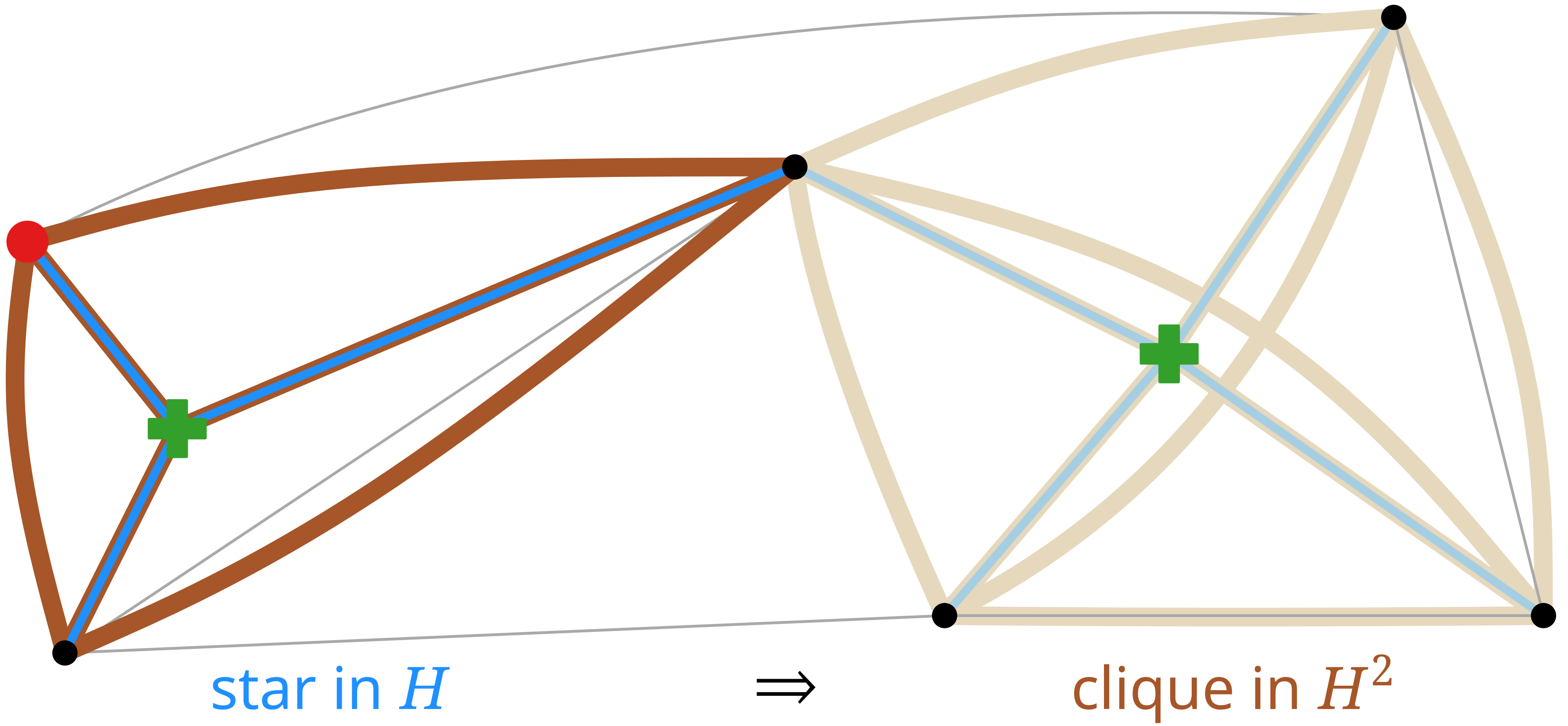

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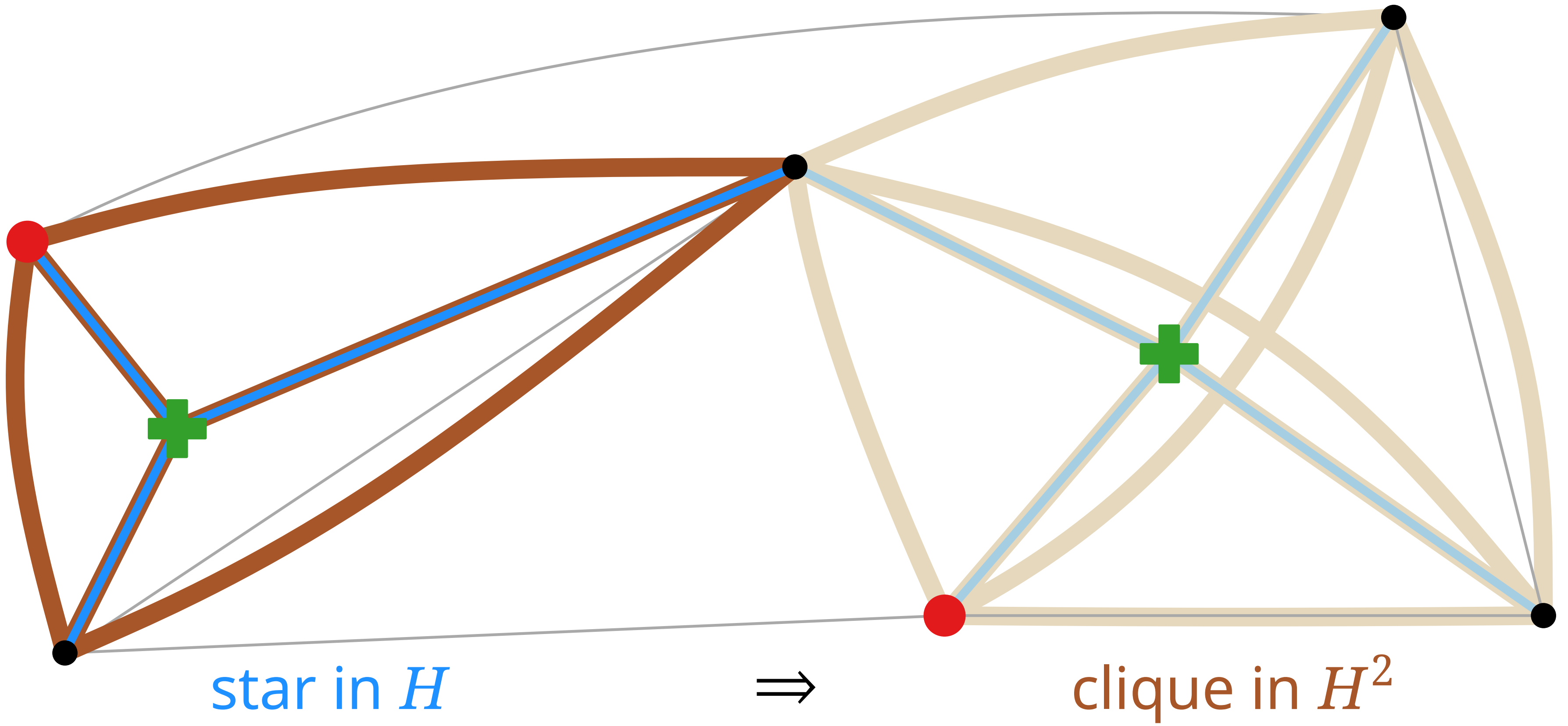

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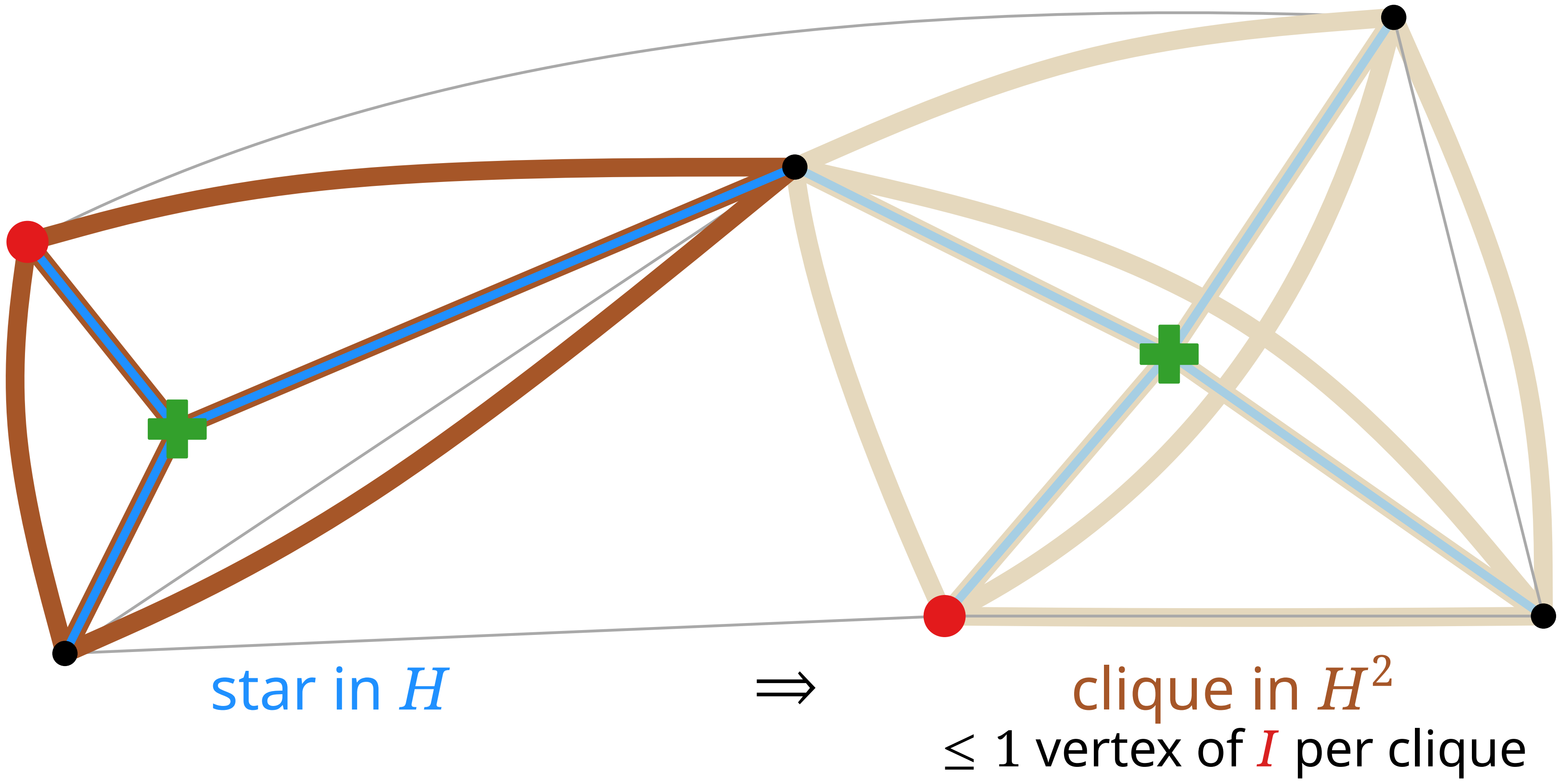

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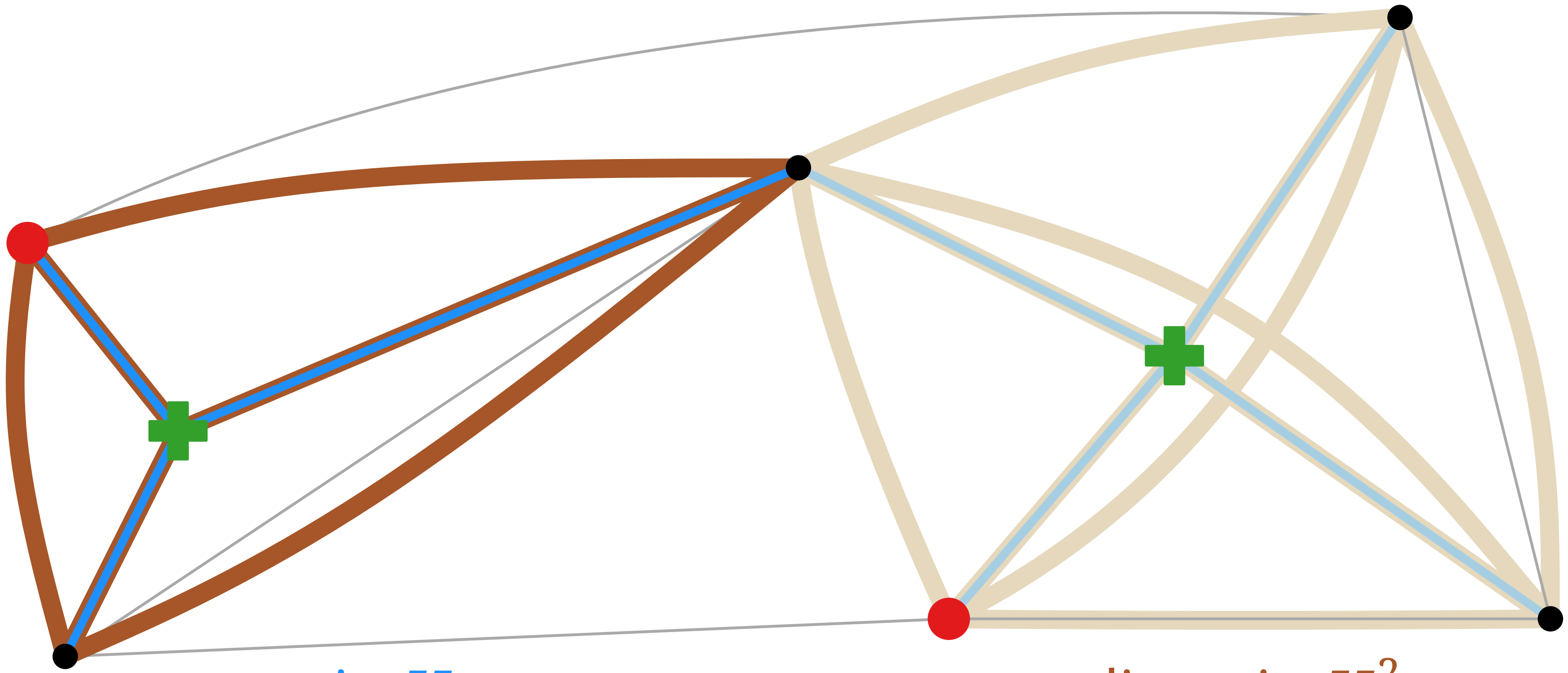

star in H $\Rightarrow$ clique in H^2

$\leq |I|$ stars $\Leftarrow$ ≤ 1 vertex of I per clique

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Metric-k-Center($G = (V, E; c), k$)

Sort the edges of G by cost: $c(e_1) \leq \ldots \leq c(e_m)$

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Factor-2 Approx. for Metric k-CENTER

Metric-k-CENTER($G = (V, E; c), k$)

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recall: parametric pruning:

- reduce to decision problem, work on pruned instance $I(t)$
- step 1: Use family of $I(t)$ to compute lower bound t^*
- step 2: find solution in $I(\alpha t^*)$

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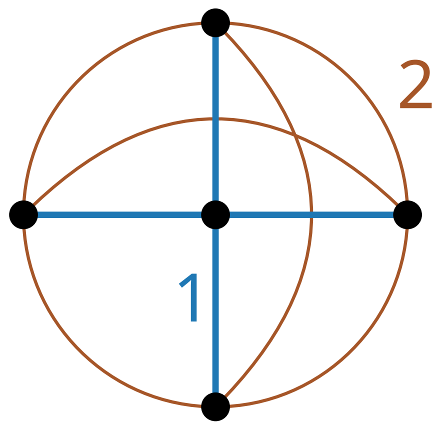

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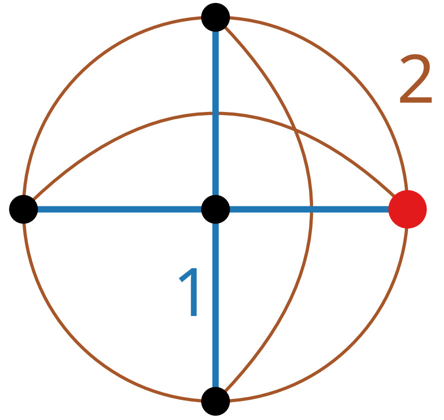

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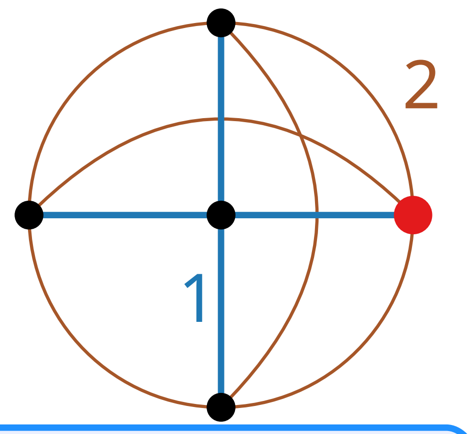

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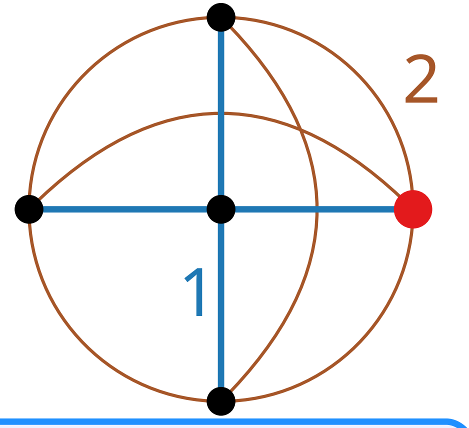

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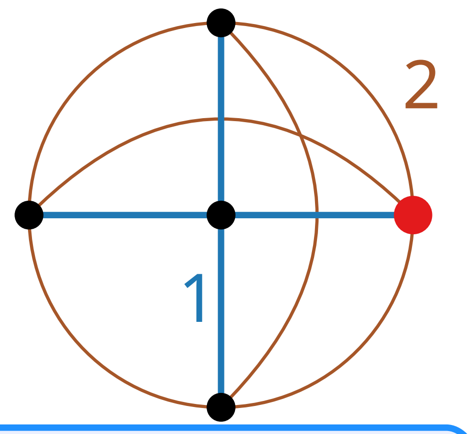

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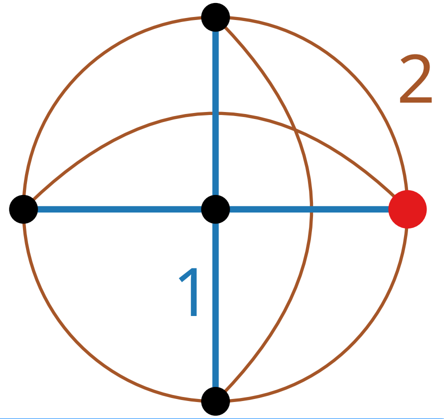

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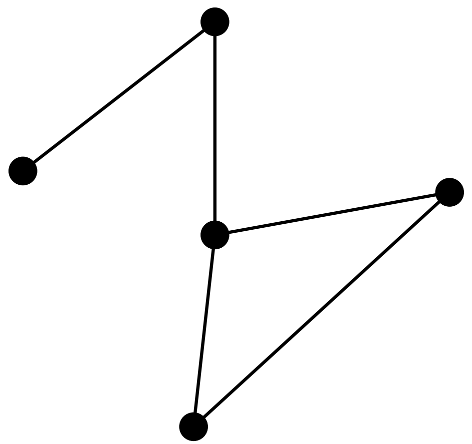

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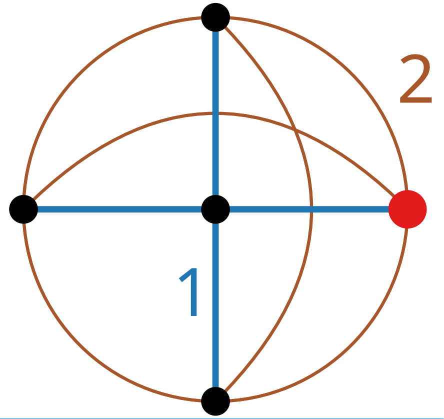

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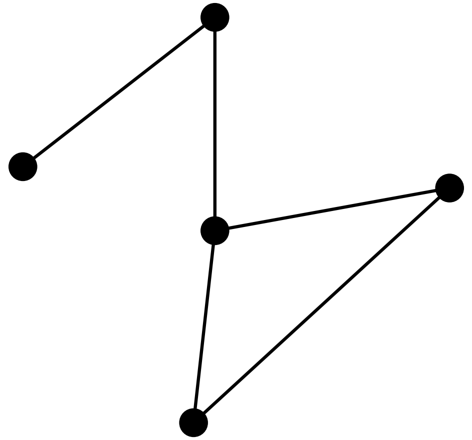

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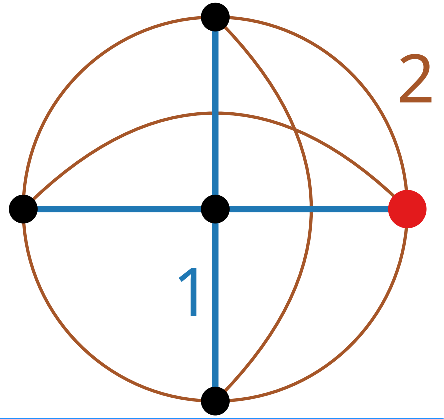

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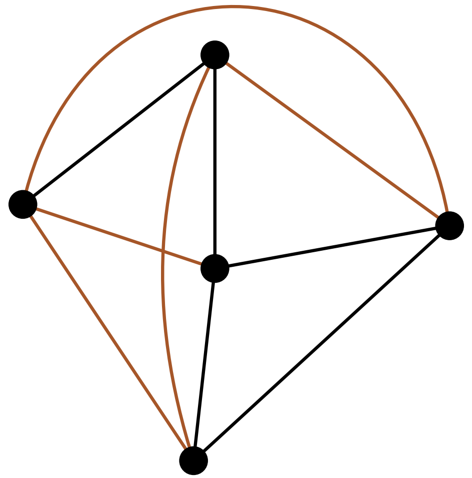

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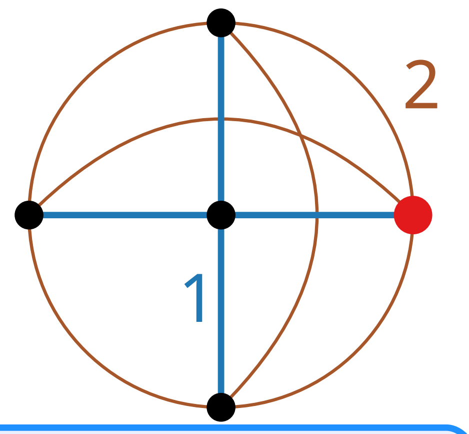

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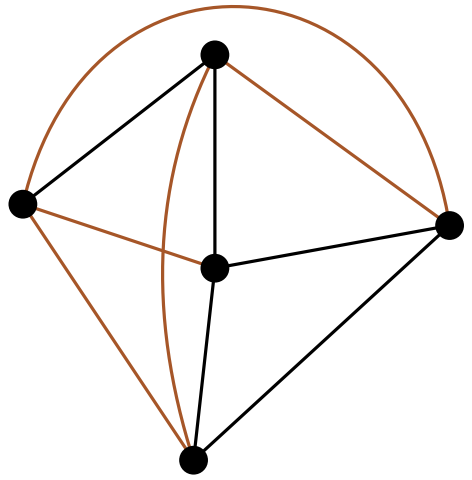

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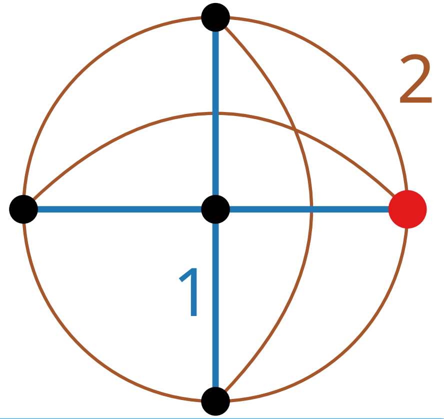

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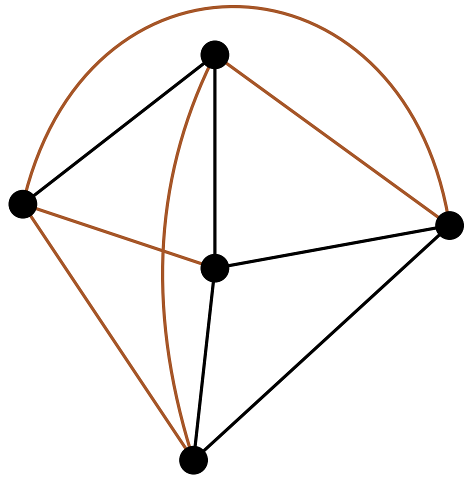

Let S be a metric k-center of G'.

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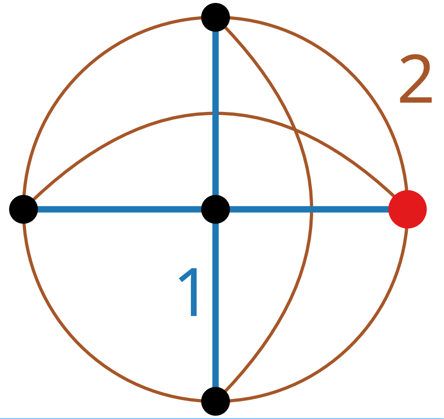

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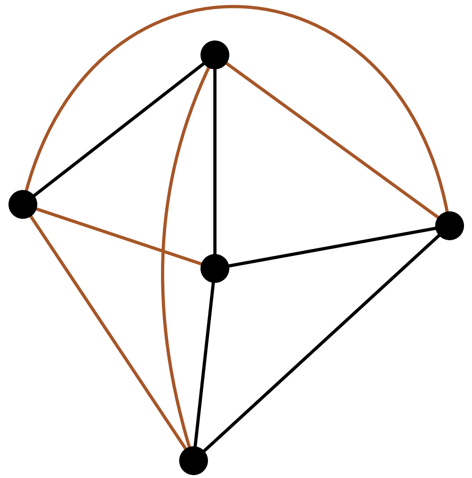

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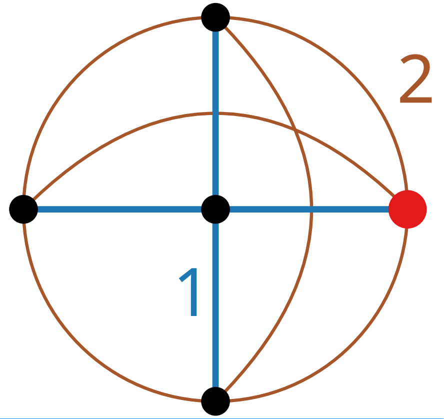

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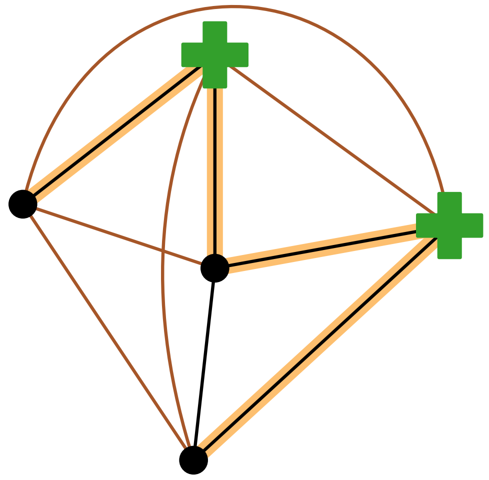

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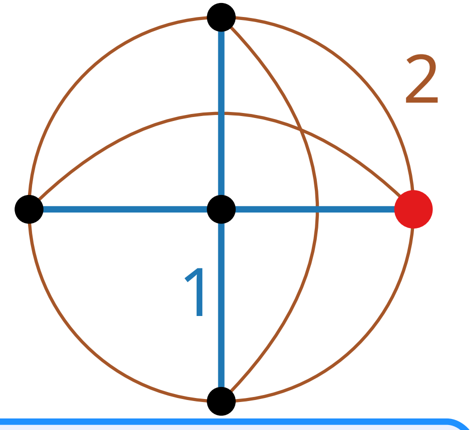

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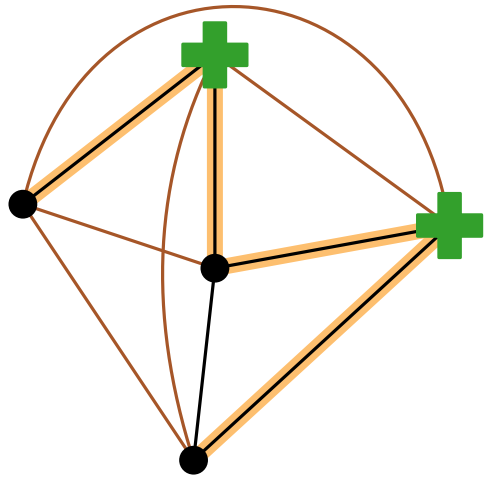

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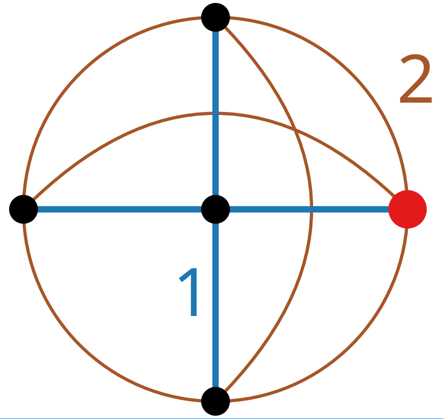

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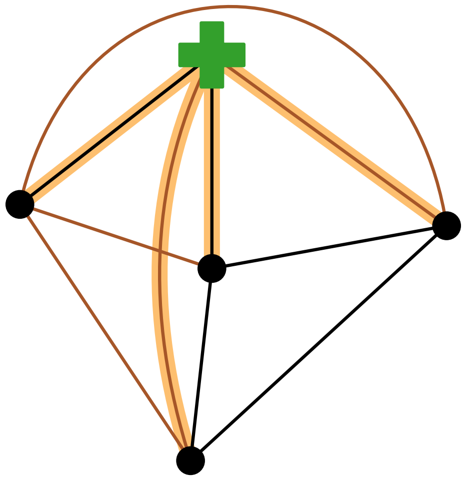

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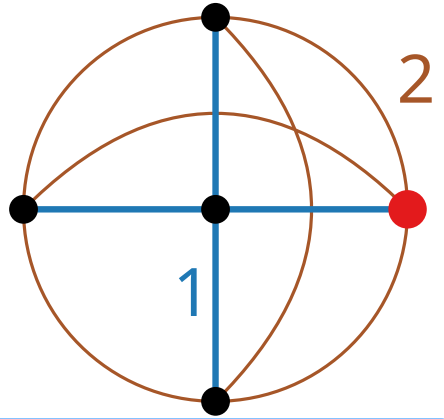

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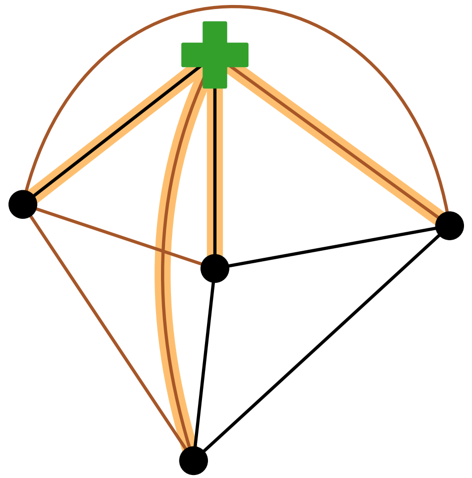

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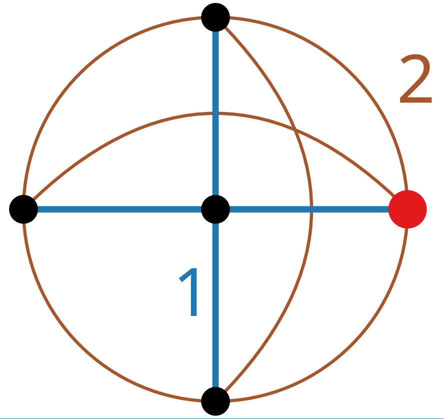

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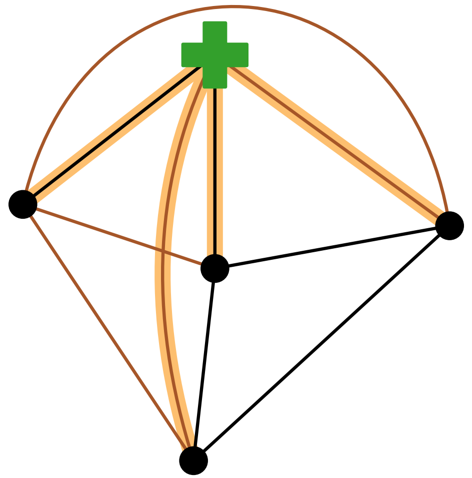

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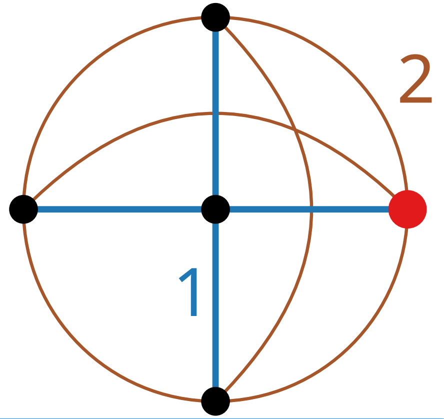

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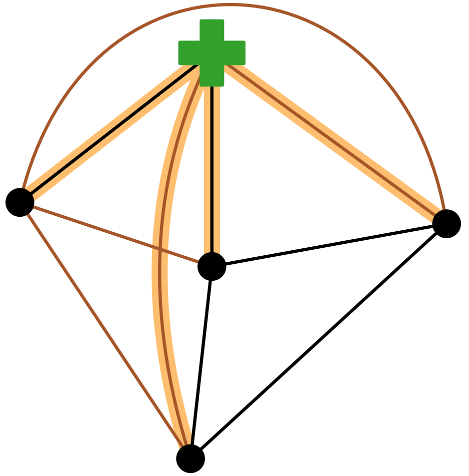

Why does this prove that, e.g., 1.5-approximation is NP-hard?

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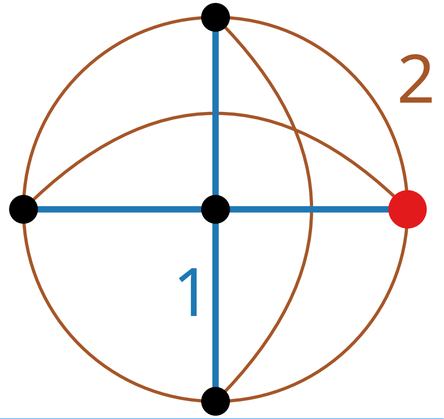

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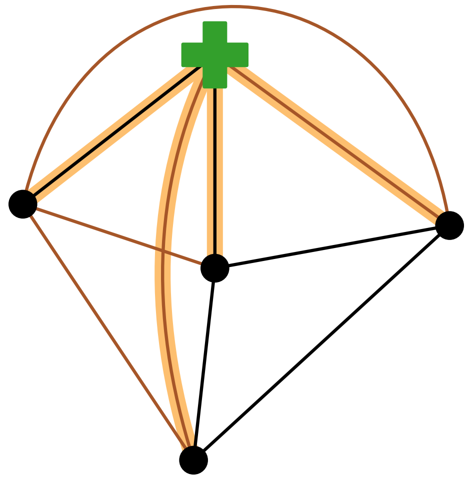

1.5-approx. allows to distinguish between 1 and 2, and these are the only possible answers.

3-approximation for METRIC-WEIGHTED-CENTER

Metric-k-Center

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

Find: A k-element vertex set S such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

METRIC-~~k~~-CENTER

WEIGHTED

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Metric-$\cancel{k}$-Center

Weighted

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \longrightarrow \mathbb{Q}_{\geq 0}$ and ~~a natural number $k \leq |V|$~~. , vertex weights $w\colon V \longrightarrow \mathbb{Q}_{\geq 0}$ and a budget $W \in \mathbb{Q}_{+}$

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Find: A ~~k-element vertex set S~~ such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.
vertex set S of weight at most W

Algorithm for the Weighted Version

Algorithm Metric- -Center

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

Construct G_j^2

Find a maximal independent set I_j in G_j^2

if $|I_j| \leq k$ **then**

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what about the weights?

if $|I_j| \leq k$ **then**

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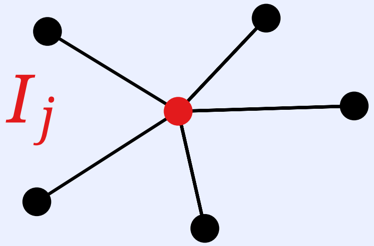

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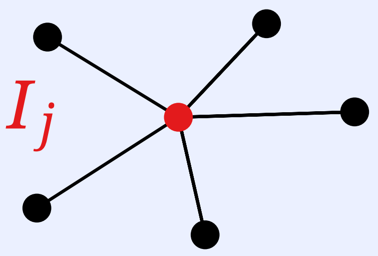


$s_j(u)$:= lightest node in $N_{G_j}(u) \cup \{u\}$

Algorithm for the Weighted Version

Algorithm Metric-Weighted-Center

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

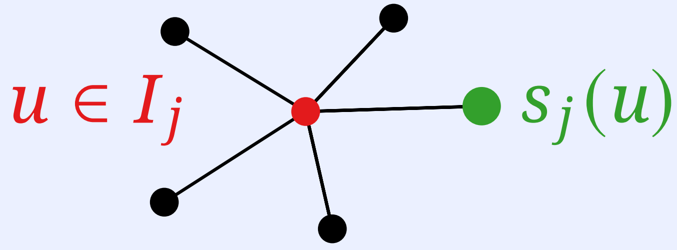
Construct G_j^2

Find a maximal independent set I_j in G_j^2

what about the weights?

if $|I_j| \leq k$ **then**

return I_j



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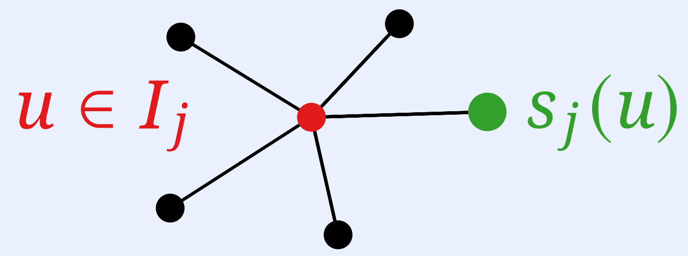

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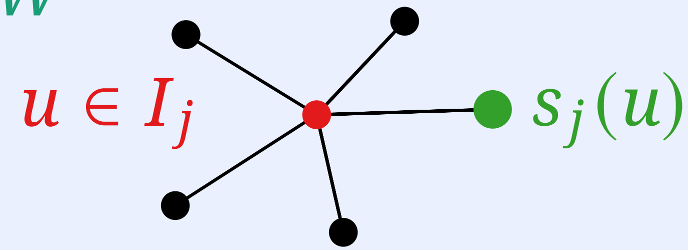

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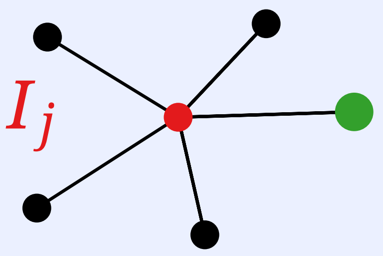
$u \in I_j$



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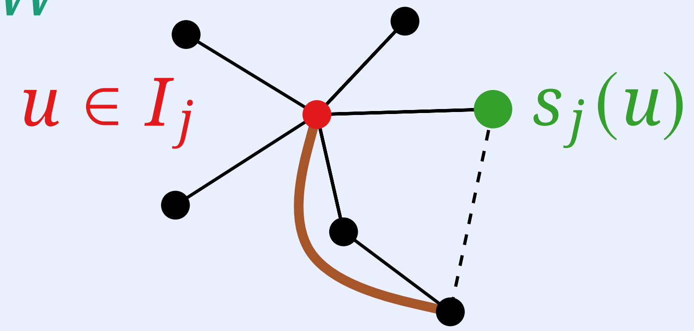

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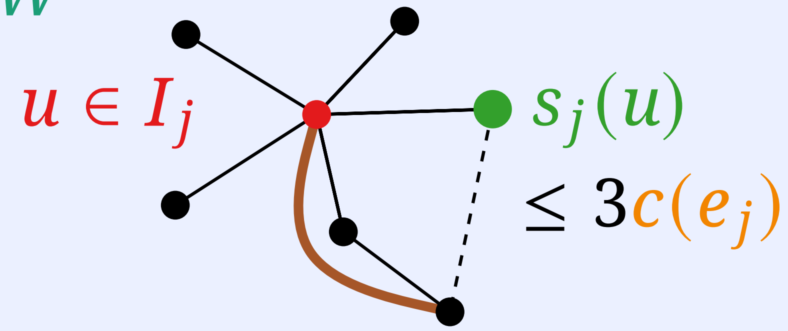

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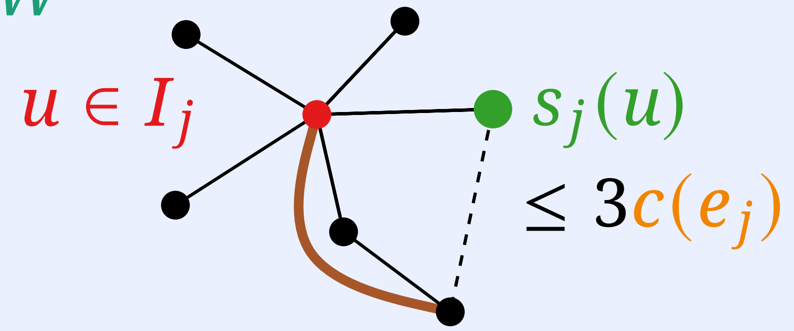

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Theorem. The above is a factor-3 approximation algorithm for METRIC-WEIGHTED-CENTER.

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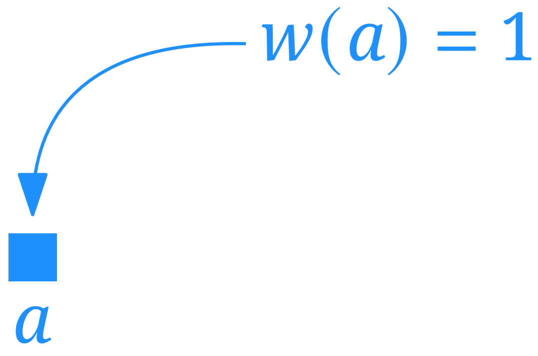

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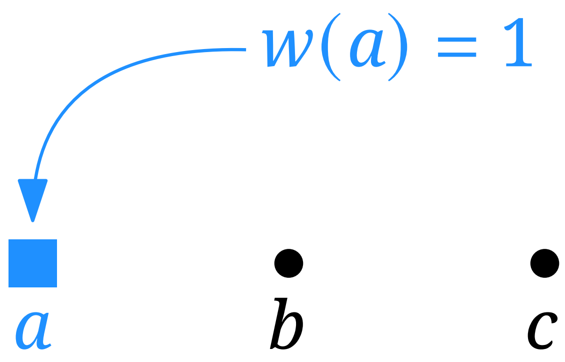

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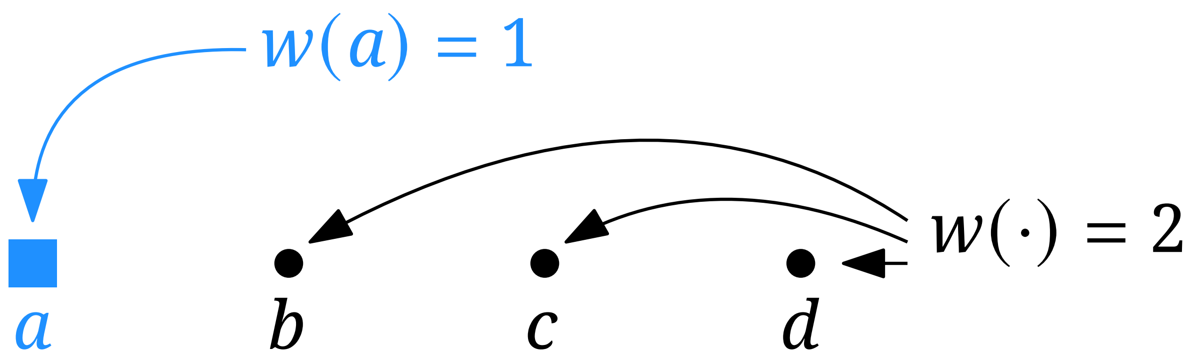

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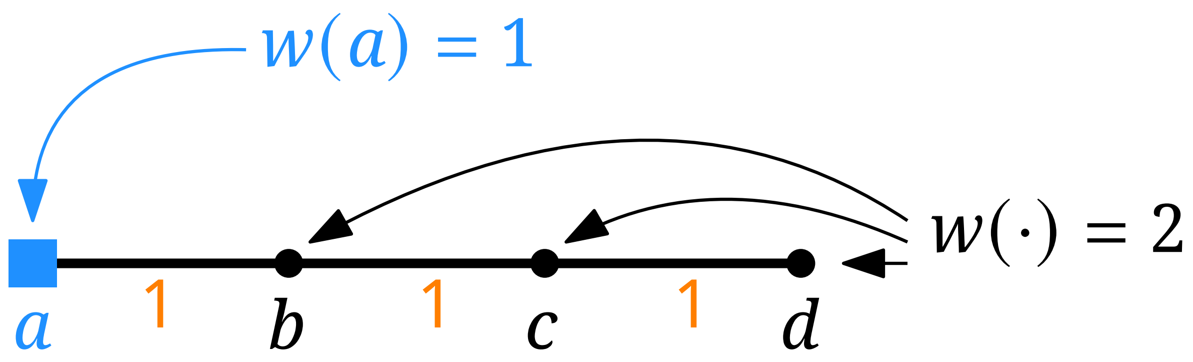

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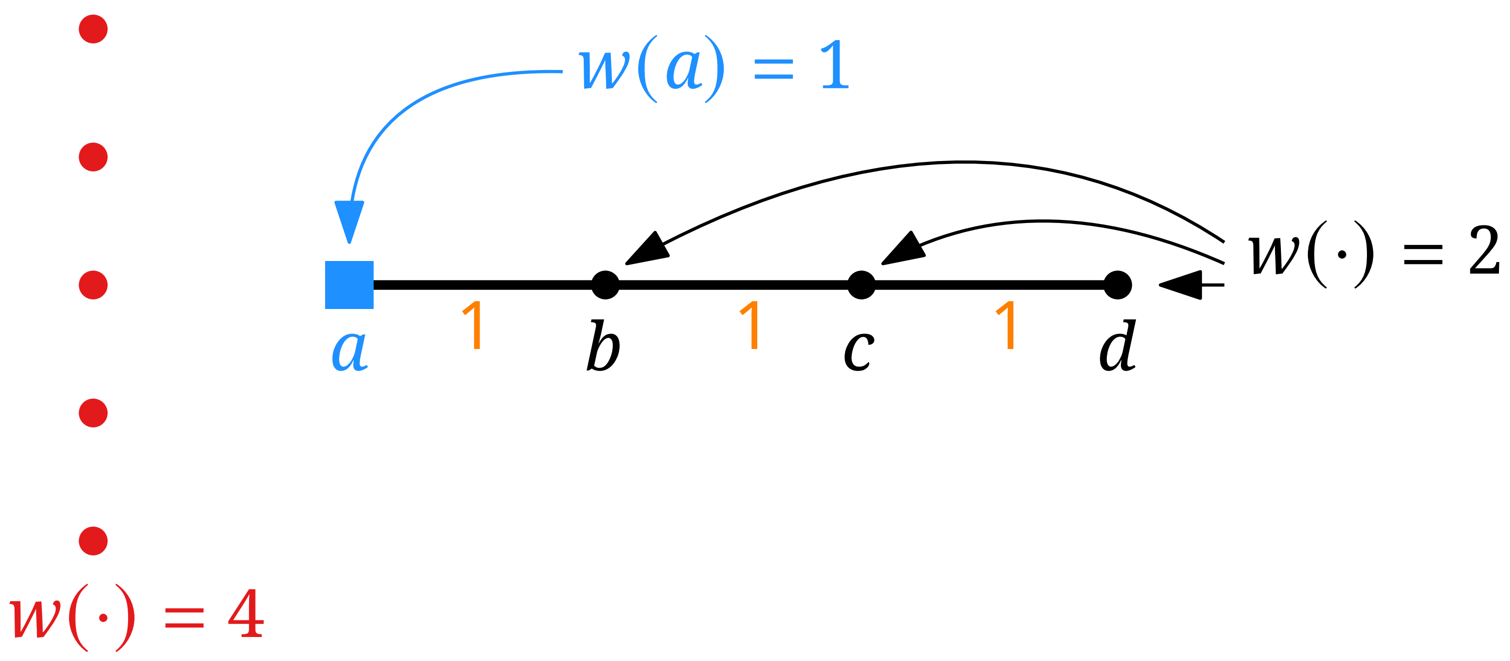

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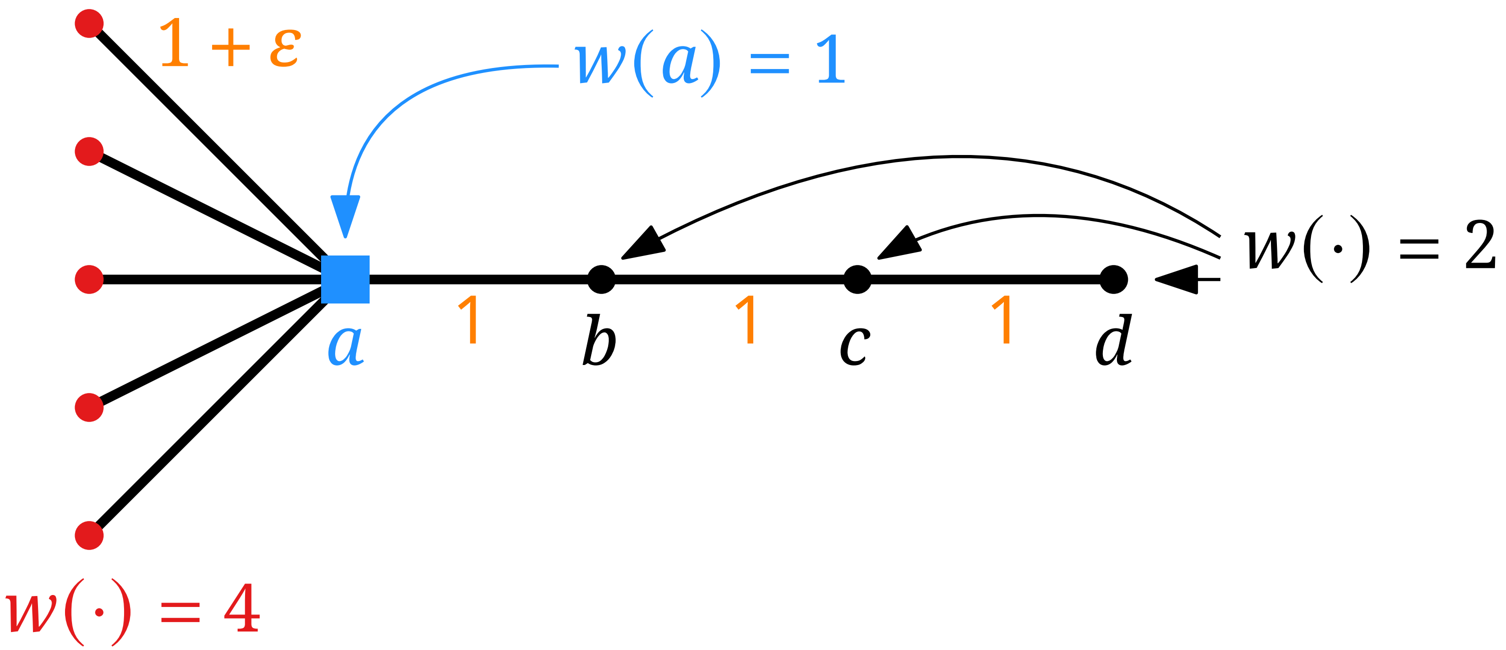

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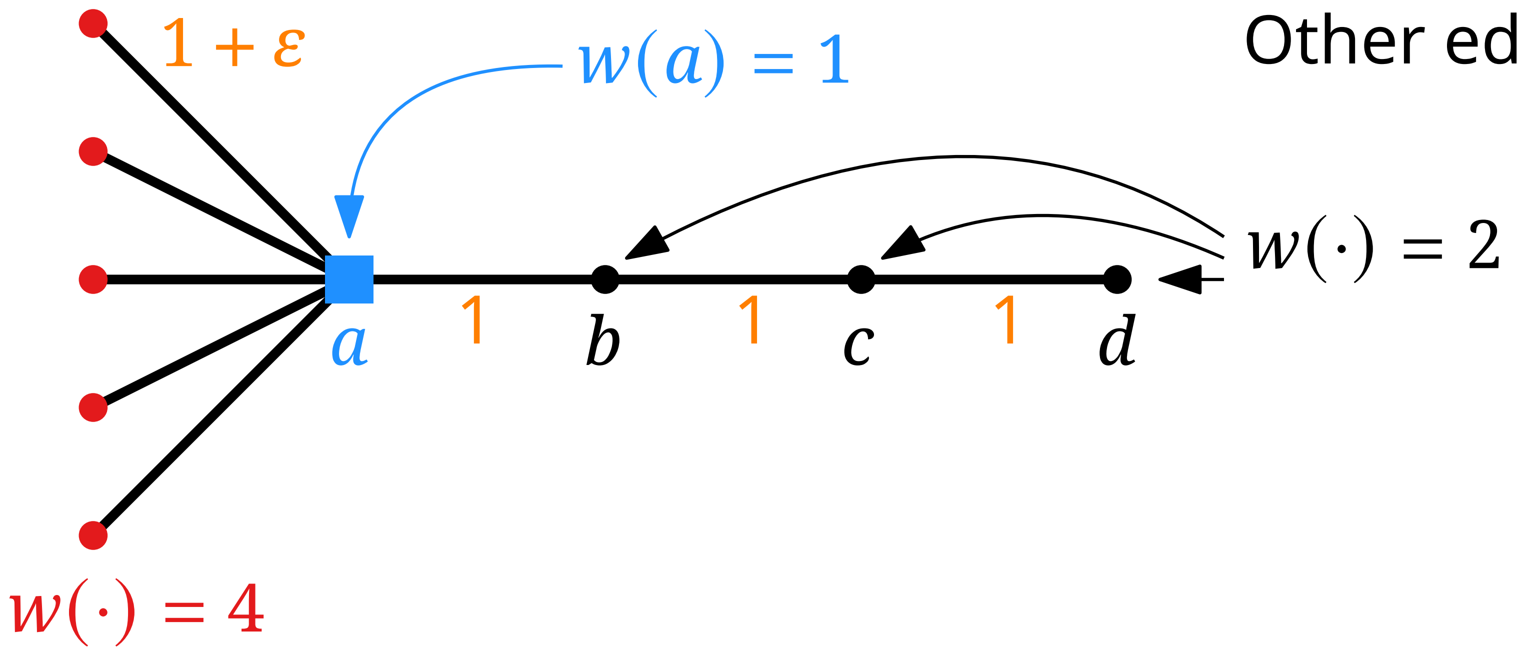

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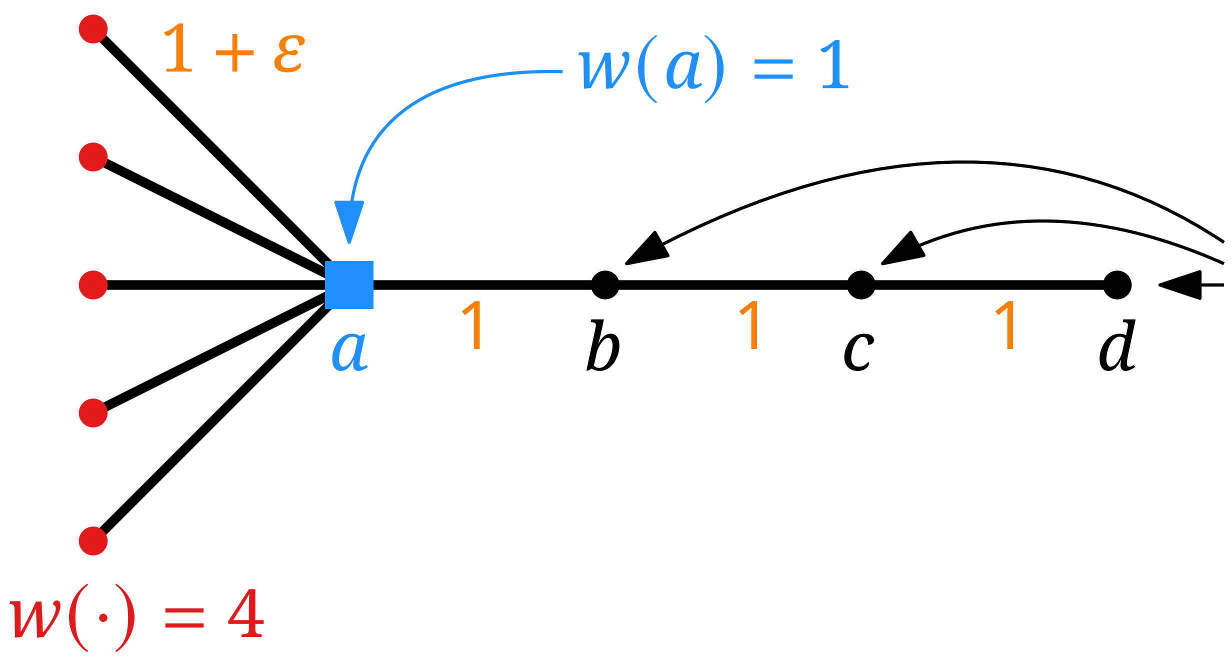

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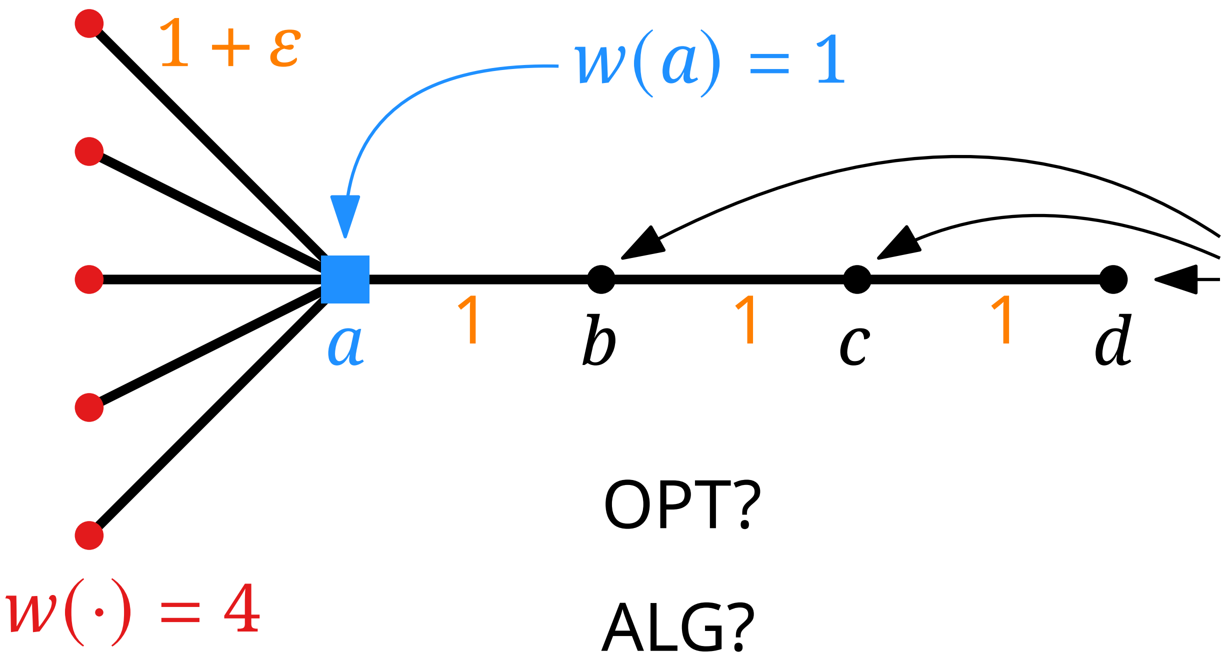

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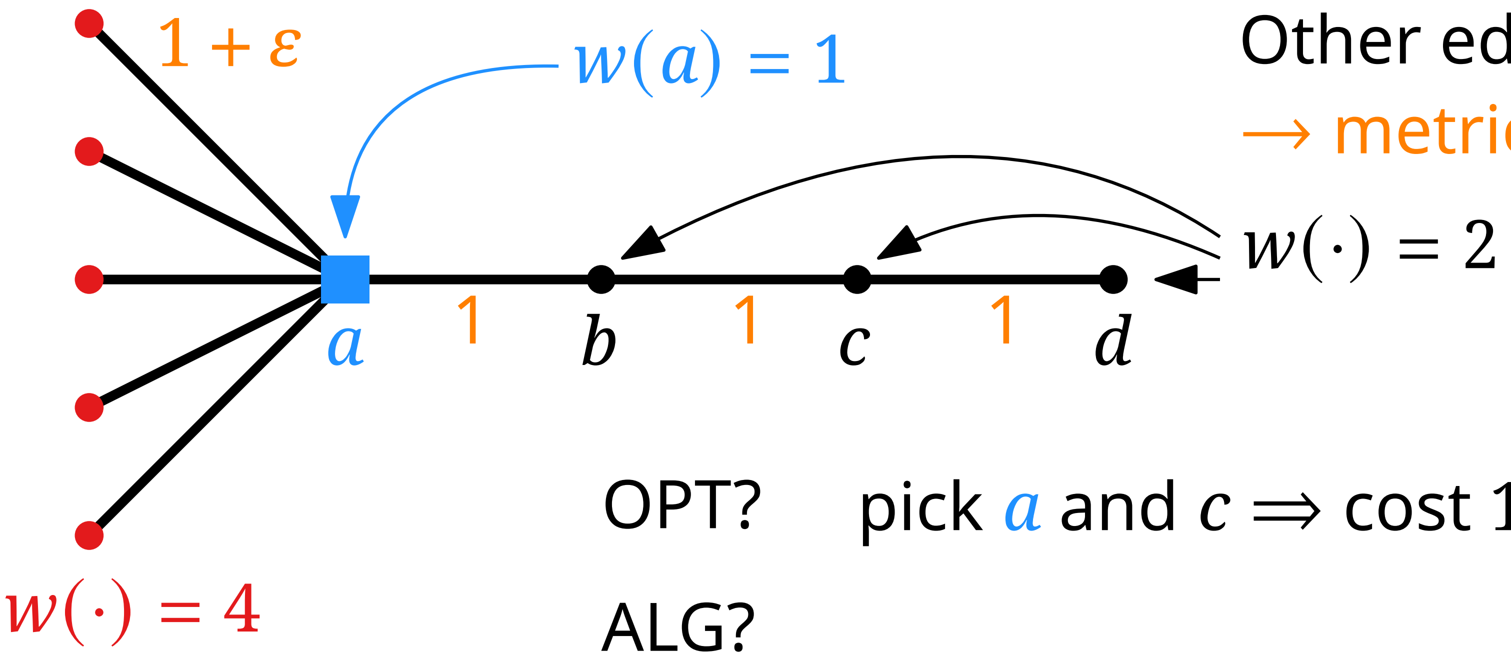

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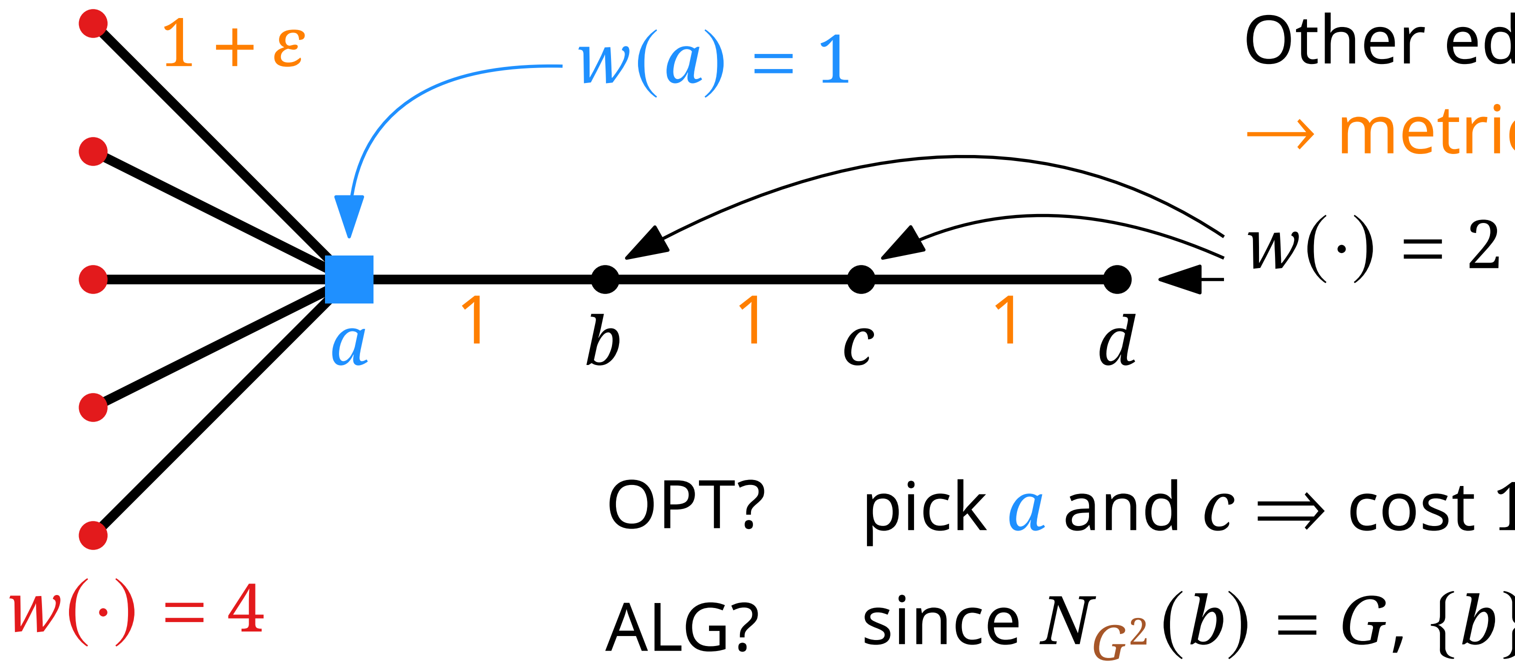

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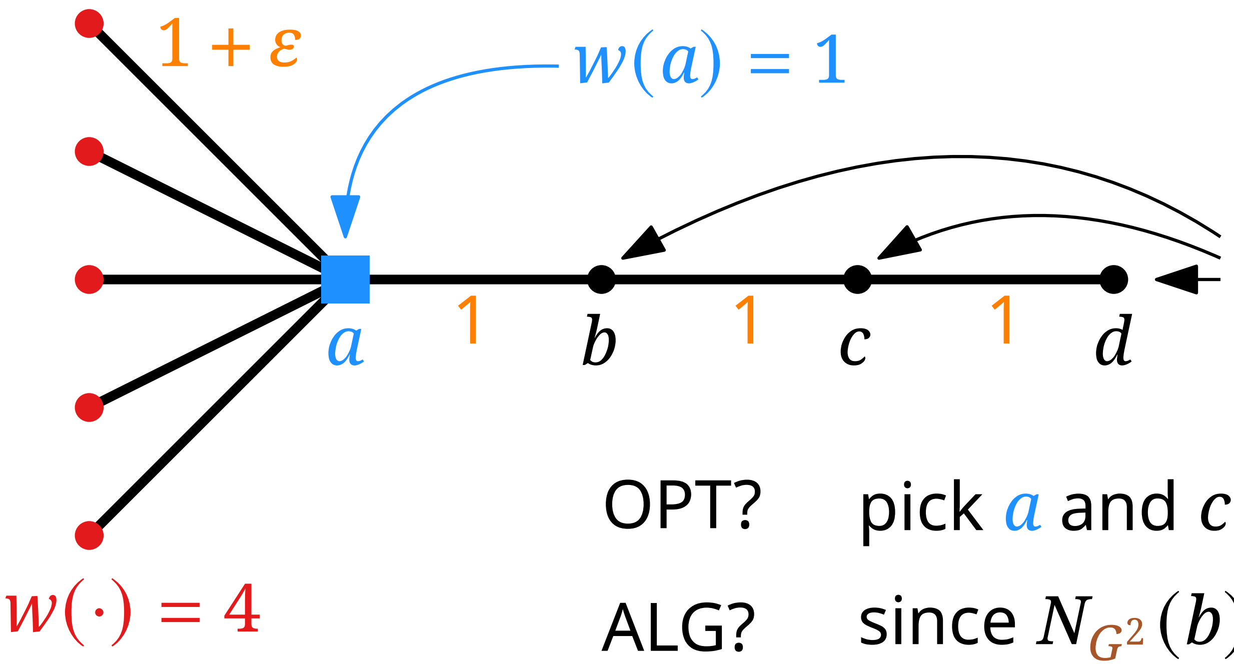

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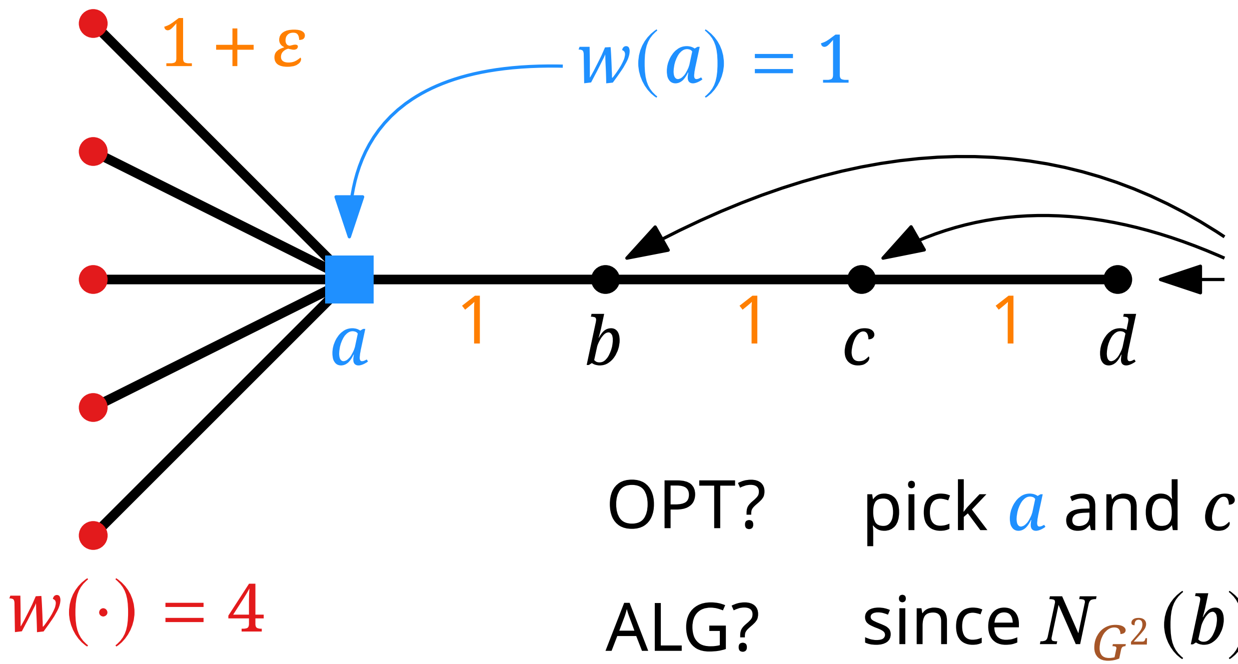

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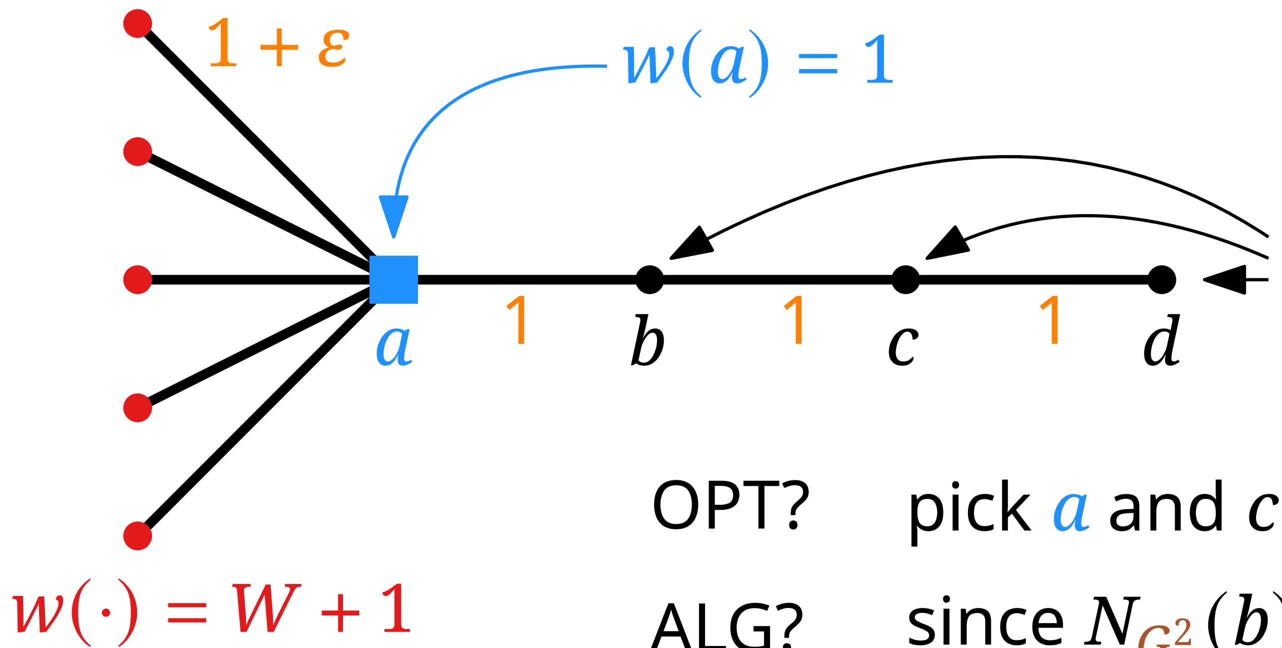

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Summary

parametric pruning: decision problem $\longrightarrow$ pruned instance $I(t)$
$\longrightarrow$ lower bound t^* opt OPT & find solution in $I(\alpha t^*)$

Metric k-Center: 2-approximation using pruned instances G_j, and finding a maximal independent set in G_j^2 (square of G_j)

Metric Weighted-Center: 3-approximation by same algorithm but taking for each v in independent set, lowest weight vertex in neighborhood

Note Alternative 2-approximation algorithm (Gonzalez 1985):
Pick an arbitrary vertex as first center, then greedily add the vertex with largest distance (cost) to centers selected, until k centers have been selected.