Duality in Linear Programming

Duality of linear programming

Maximize $2x_1 + 3x_2$

subject to:
$$\begin{aligned} 4x_1 + 8x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 4 \\ x_1, x_2, &\geq 0 \end{aligned}$$

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Can we infer an upper bound on the objective function from the constraints?

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Without computing the optimum, we can infer: $2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12$

by the nonnegative constraints

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Do you see an even better upper bound?

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Better: $2x_1 + 3x_2 \le \frac{1}{2}(4x_1 + 8x_2) \le \frac{1}{2}(12) = 6$

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Better: $2x_1 + 3x_2 \leq \frac{1}{3}(4x_1 + 8x_2 + 2x_1 + x_2) \leq \frac{1}{3}(12 + 3) = 5$

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How good of an upper bound can we get in this way?

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How good of an upper bound can we get in this way?

Derived from the constraints, we want an inequality

$$d_1 x_1 + d_2 x_2 \le h$$

with $d_1 \ge 2, d_2 \ge 3$ and h as small as possible.

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Use variables as coefficients for the inequalities!

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How do we get this?

$$y_1(4x_1 + 8x_2) + y_2(2x_1 + x_2) + y_3(3x_1 + 2x_2) \leq 12y_1 + 3y_2 + 4y_3$$

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$$= (4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_2$$

with $y_1, y_2, y_3 \ge 0$.

Thus $d_1 = 4y_1 + 2y_2 + 3y_3,\ d_2 = 8y_1 + y_2 + 2y_3,\ h = 12y_1 + 3y_2 + 4y_3$.

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Thus $d_1 = 4y_1 + 2y_2 + 3y_3,\ d_2 = 8y_1 + y_2 + 2y_3,\ h = 12y_1 + 3y_2 + 4y_3$.

To find the best y_1, y_2, y_3, we solve a dual linear program:

Minimize $12y_1 + 3y_2 + 4y_3$

subject to:
$$\begin{aligned} 4y_1 + 2y_2 + 3y_3 &\ge 2 \\ 8y_1 + y_2 + 2y_3 &\ge 3 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

Duality of linear programming

How well does a dual linear program bound the original? Perfectly!

Dual LP has optimum $(y_1, y_2, y_3) = (\frac{5}{16}, 0, \frac{1}{4})$ with value 4.75.

Primal LP has optimum $(x_1, x_2) = (\frac{1}{2}, \frac{5}{4})$ with value 4.75.

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Primal LP

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More generally, the dual of

maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$

is

minimize $b^T y$ subject to $A^T y \geq c$ and $y \geq 0$

Duality of linear programming

Weak duality theorem:

For feasible solutions x and y, we have

$$c^T x \leq b^T y.$$

If the primal is unbounded, then the dual is infeasible.

If the dual is unbounded (from below), then the primal is infeasible.

Duality of linear programming

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If the dual is unbounded (from below), then the primal is infeasible.

Proof: $c^T x \leq (A^T y)^T x$ (dual constraints: $A^T y \geq c$)

Duality of linear programming

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Proof:

$$\begin{aligned} c^T x &\leq (A^T y)^T x \\ &= y^T A x \\ &\leq y^T b \qquad \text{(primal constraints: } Ax \leq b\text{)} \end{aligned}$$

Duality of linear programming

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$$[y_1, \ldots, y_m] \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

$$= [b_1, \ldots, b_m] \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

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Strong duality theorem:

Optimal feasible solutions satisfy $c^T x = b^T y$.

Feasibilty vs Optimality (via Duality)

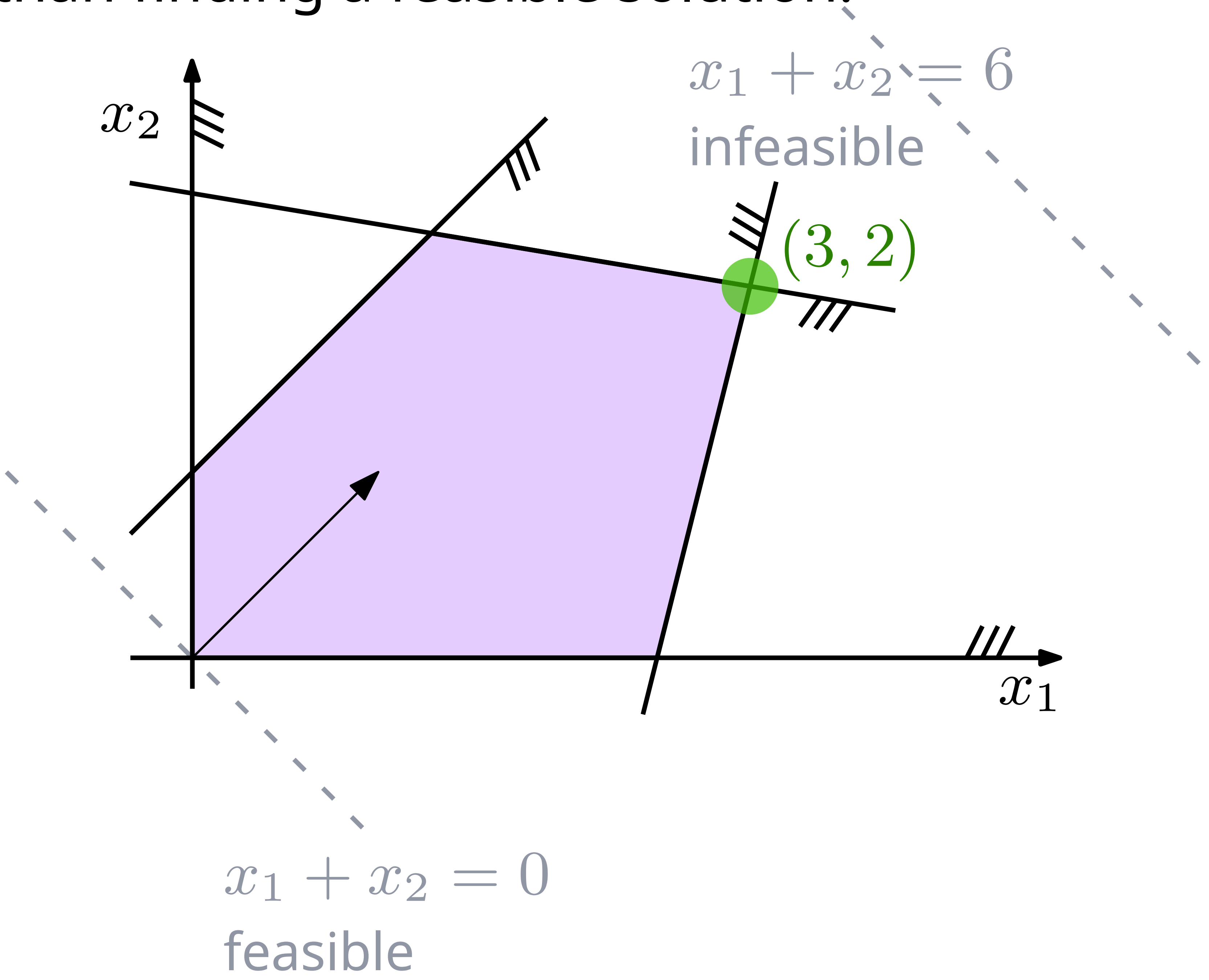
Feasibility vs. optimality

"Finding an optimal solution is no harder than finding a feasible solution."

First explanation: binary search

Example: Maximize $x_1 + x_2$

$$\begin{aligned} \text{subject to } -x_1 + x_2 &\leq 1 \\ x_1 + 6x_2 &\leq 15 \\ 4x_1 - x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$



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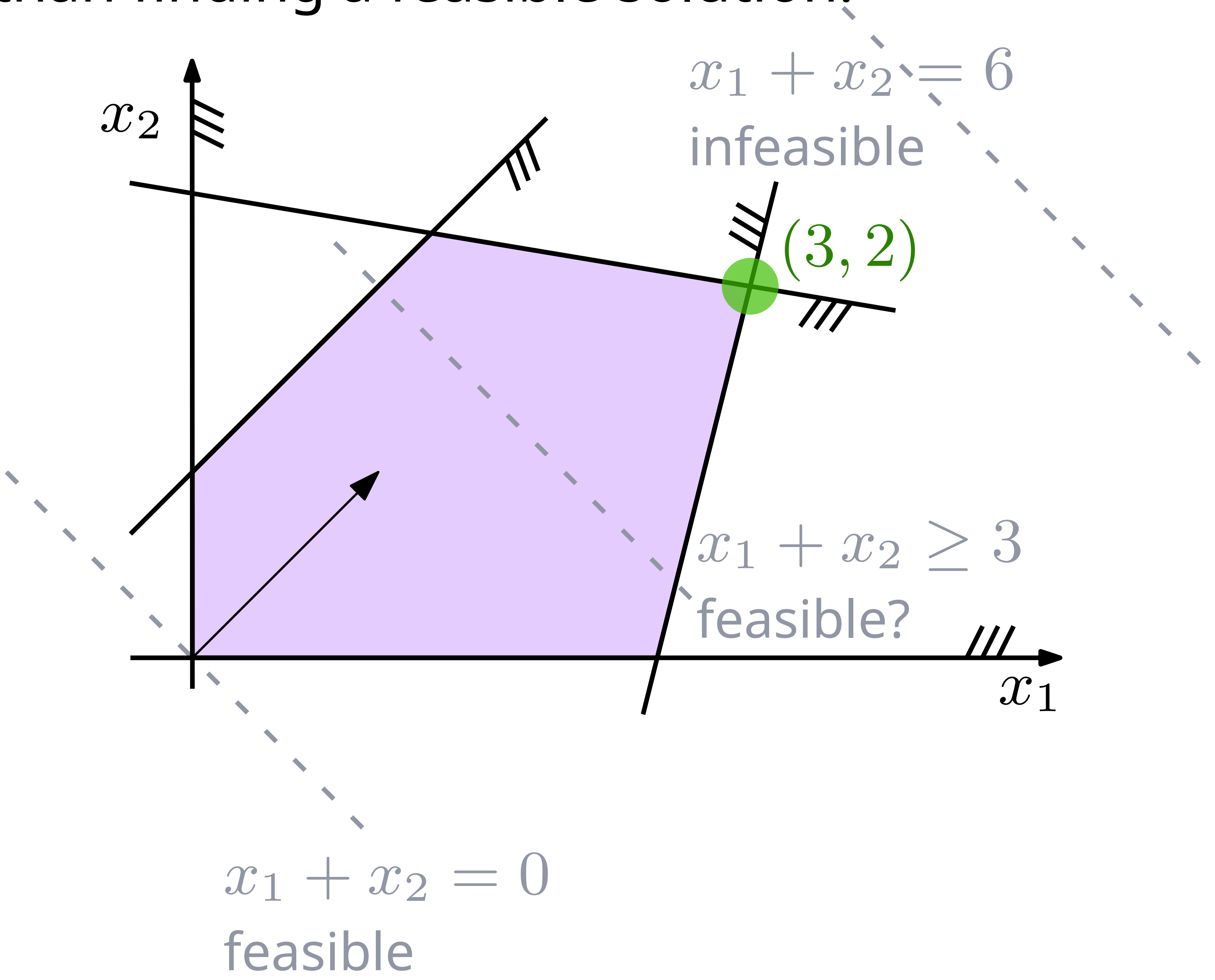

Feasibility vs. optimality

"Finding an optimal solution is no harder than finding a feasible solution."

Second explanation: Simplex method Phase 1 vs Phase 2

Maximize $x_1 + 2x_2$

subject to: $x_1 + 3x_2 + x_3 = 4$

$$2x_2 + x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Note: $(x_1, x_2, x_3) = (0, 0, 0)$
is not feasible.

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Note: $(x_1, x_2, x_3) = (0, 0, 0)$ is not feasible.

Auxilliary problem to find feasible solution via simplex method:

Maximize $-x_4 - x_5$

subject to: $x_1 + 3x_2 + x_3 + x_4 = 4$

$$2x_2 + x_3 + x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The objective value is $0 \iff$ there is a feasible solution to the original problem.

Feasibility vs. optimality

Third explanation, using duality:

Primal: maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$.

Dual: minimize $b^T y$ subject to $A^T y \geq c$ and $y \geq 0$.

Weak duality: $c^T x \leq b^T y$

Feasibility vs. optimality

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How can we combine this such that any feasible solution is optimal?

Feasibility vs. optimality

Third explanation, using duality:

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Weak duality: $c^T x \leq b^T y$

Finding an optimal solution to

Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$

is the same as finding a feasible solution to

~~Maximize $c^T x$~~

$$\begin{aligned} \text{subject to} \quad Ax &\leq b \\ A^T y &\geq c \\ c^T x &\geq b^T y \\ x &\geq 0, y \geq 0 \end{aligned}$$

Feasibility vs. optimality

Third explanation, using duality:

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$$\begin{aligned} \text{subject to} \quad Ax &\le b \\ A^T y &\ge c \\ c^T x &\ge b^T y \\ x \ge 0, y &\ge 0 \end{aligned}$$

We know $c^T x \le b^T y$ for any feasible solutions to the primal and the dual, so adding $c^T x \ge b^T y$ as a constraint implies optimality.

Duality recipe and physical interpretation

Dualization recipe

Primal:	Maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$.
Dual:	Minimize $b^T y$ subject to $A^T y \geq c$ and $y \geq 0$.

A is of size $m \times n$

Primal has n variables, m constraints

Dual has m variables, n constraints

Dualization recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	b	c
Objective function	$\max c^T x$	$\min b^T y$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

Dualization recipe

Primal: Maximize $c^T x$ subject to $Ax \leq b$.

(without nonnegative constraints for x_i)

Dual: Minimize $b^T y$ subject to ???

Dualization recipe

Primal: Maximize $c^T x$ subject to $Ax \leq b$.

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Dual: Minimize $b^T y$ subject to $A^T y = c$ and $y \geq 0$.

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Dual: Minimize $b^T y$ subject to $A^T y = c$ and $y \geq 0$.

Primal: Maximize $3x_1 + 2x_2 + 4x_3$

subject to

$$
\begin{aligned}
2x_1 - \quad x_3 &\geq 4 \\
x_1 + x_2 + 3x_3 &= 7 \\
x_1 \leq 0, x_3 &\geq 0
\end{aligned}
$$

Dual: Minimize

subject to

???

Dualization recipe

Primal: Maximize $c^T x$ subject to $Ax \leq b$.

(without nonnegative constraints for x_i)

Dual: Minimize $b^T y$ subject to $A^T y = c$ and $y \geq 0$.

Primal: Maximize $3x_1 + 2x_2 + 4x_3$

subject to

$$\begin{aligned} 2x_1 - \phantom{x_2 + {}} x_3 &\geq 4 \\ x_1 + x_2 + 3x_3 &= 7 \\ x_1 \leq 0, x_3 &\geq 0 \end{aligned}$$

Dual: Minimize $4y_1 + 7y_2$

subject to

$$\begin{aligned} 2y_1 + y_2 &\leq 3 \\ y_2 &= 2 \\ -y_1 + 3x_2 &\geq 4 \\ y_1 &\leq 0 \end{aligned}$$

A Physical Interpretation of Strong Duality

Primal: Maximize $c^T x$ subject to $Ax \leq b$ (no nonnegativity constraints).
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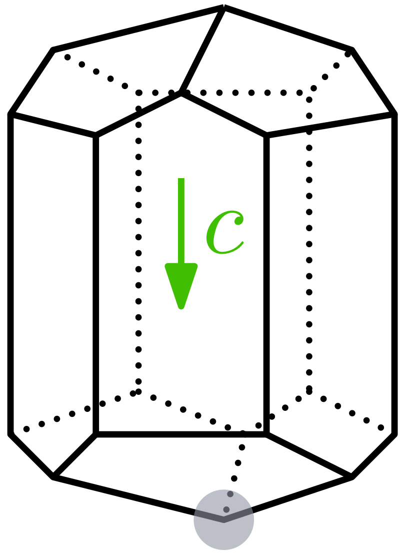

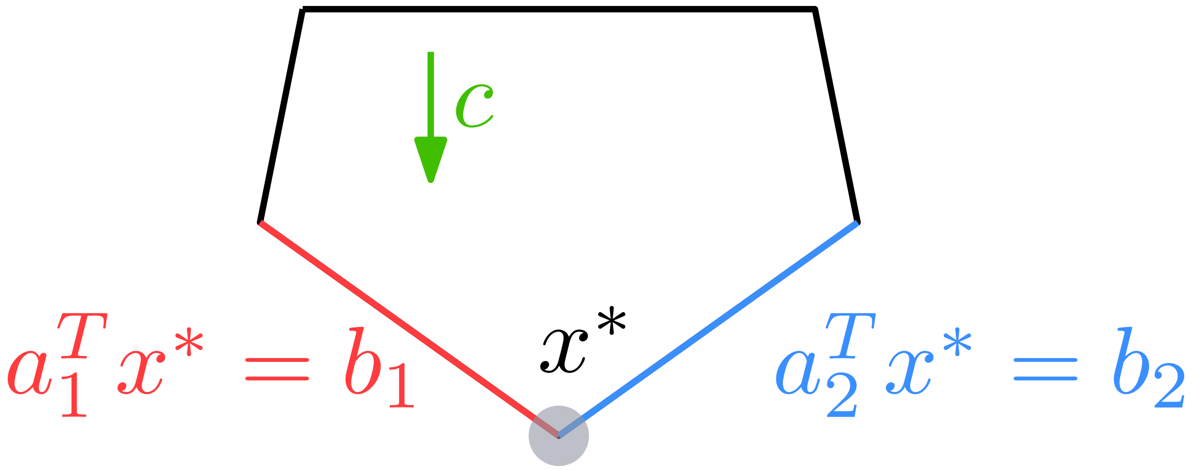

$$A = \begin{bmatrix} \cdots & a_1^T & \cdots \\ \cdots & a_2^T & \cdots \\ & \vdots & \\ \cdots & a_m^T & \cdots \end{bmatrix}$$

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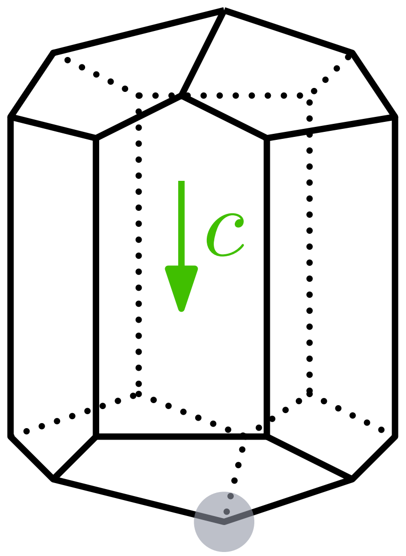
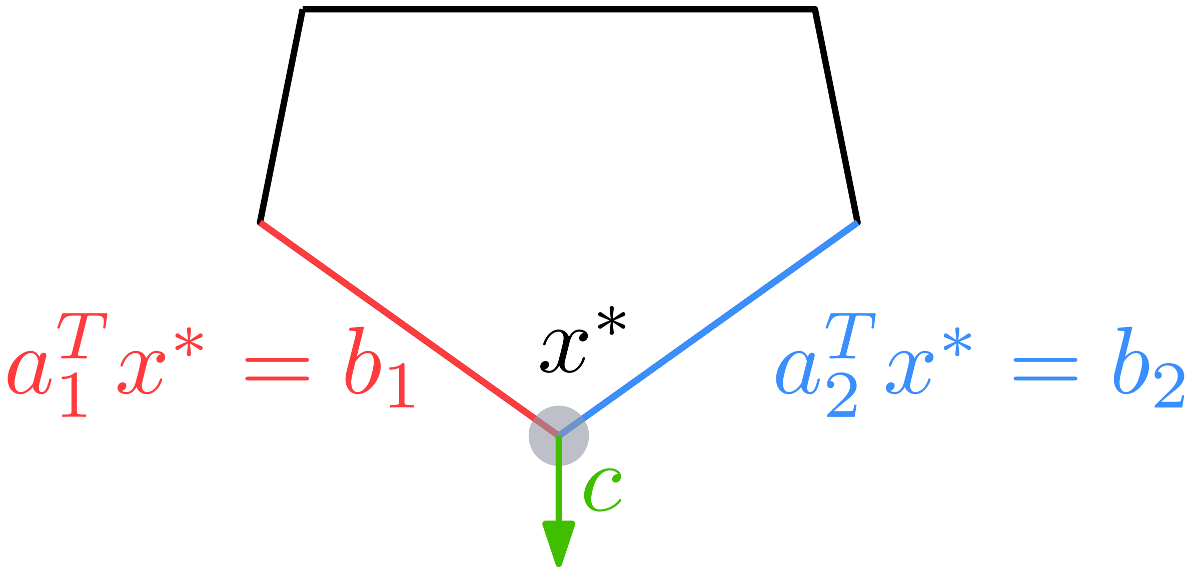


Imagine c points downwards $\approx$ gravity



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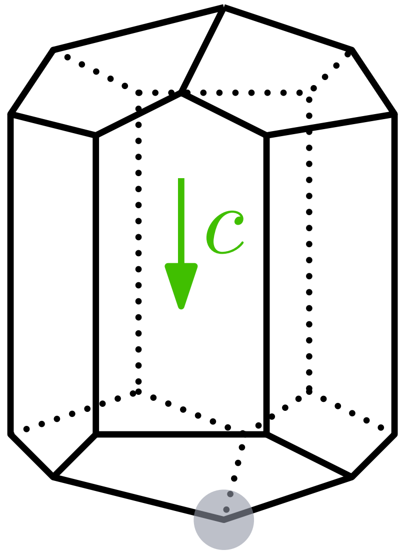

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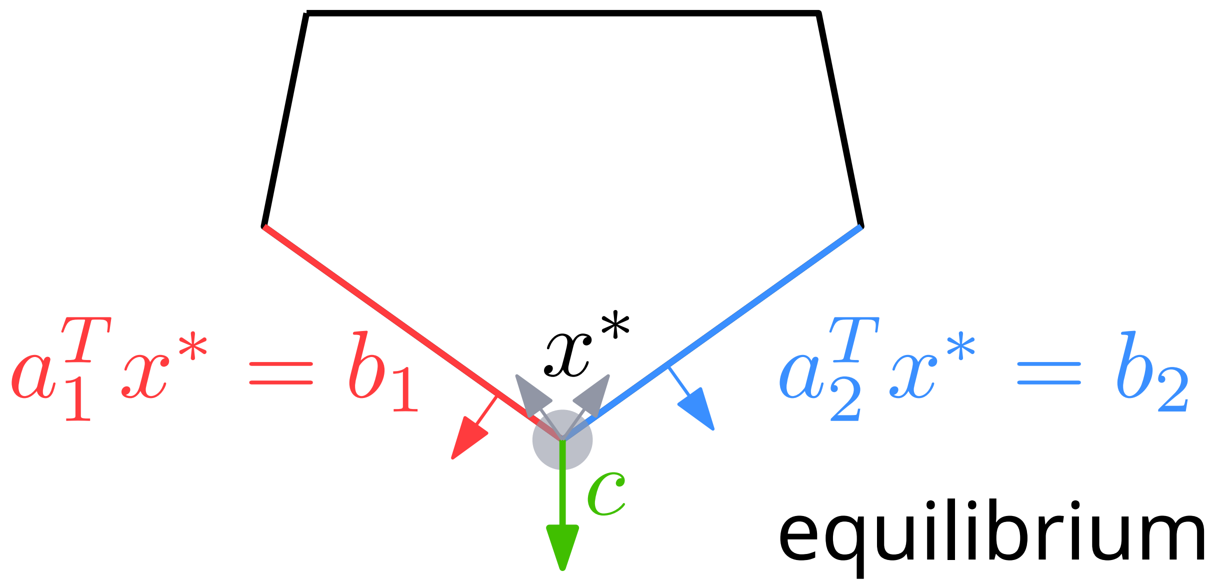

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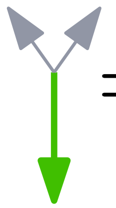
equilibrium of forces:  $= 0$

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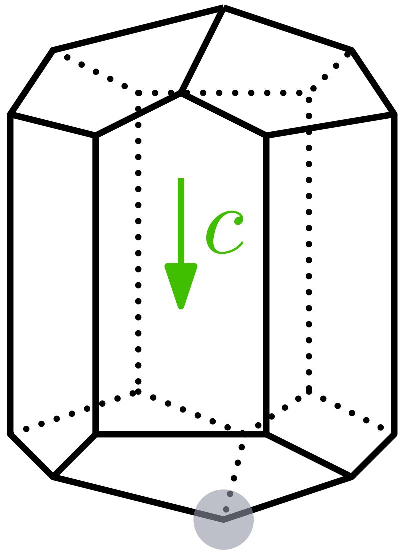

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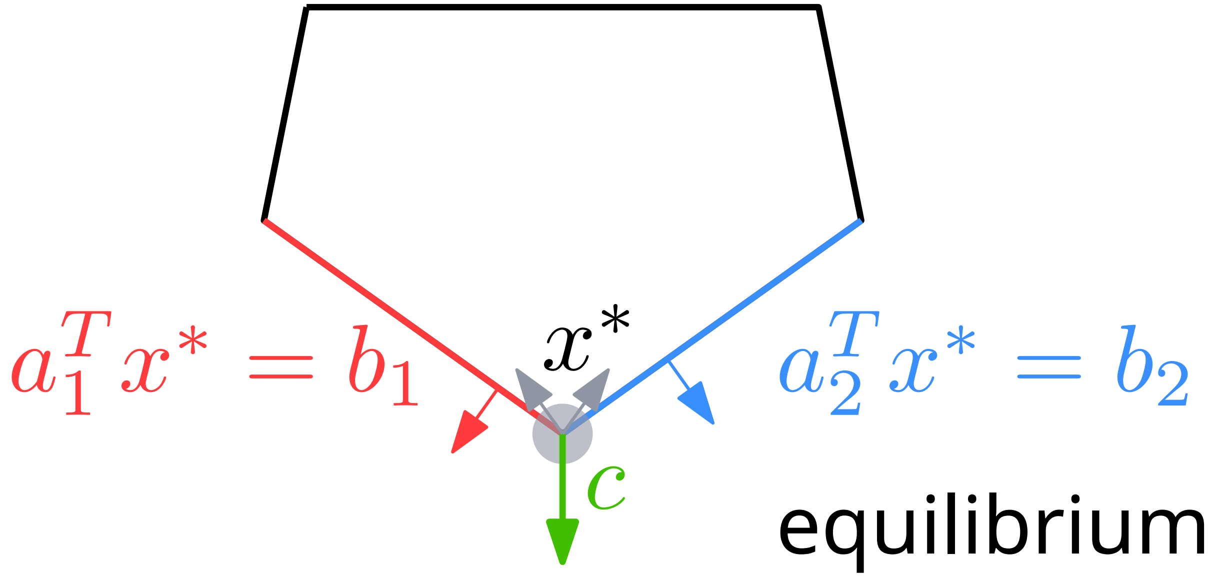
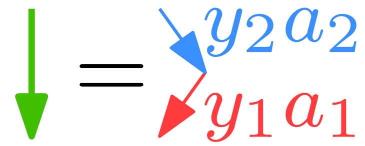


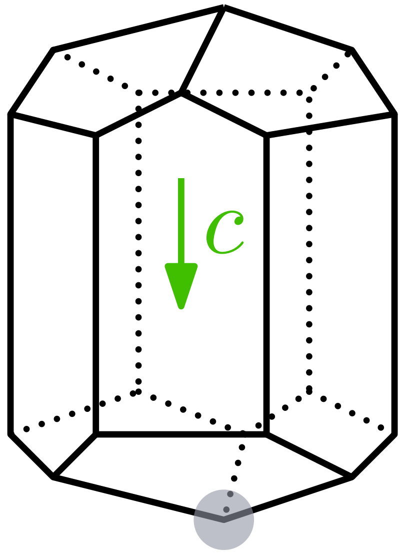
$$A = \begin{bmatrix} \dots & a_1^T & \dots \\ \dots & a_2^T & \dots \\ & \vdots & \\ \dots & a_m^T & \dots \end{bmatrix}$$

equilibrium of forces:

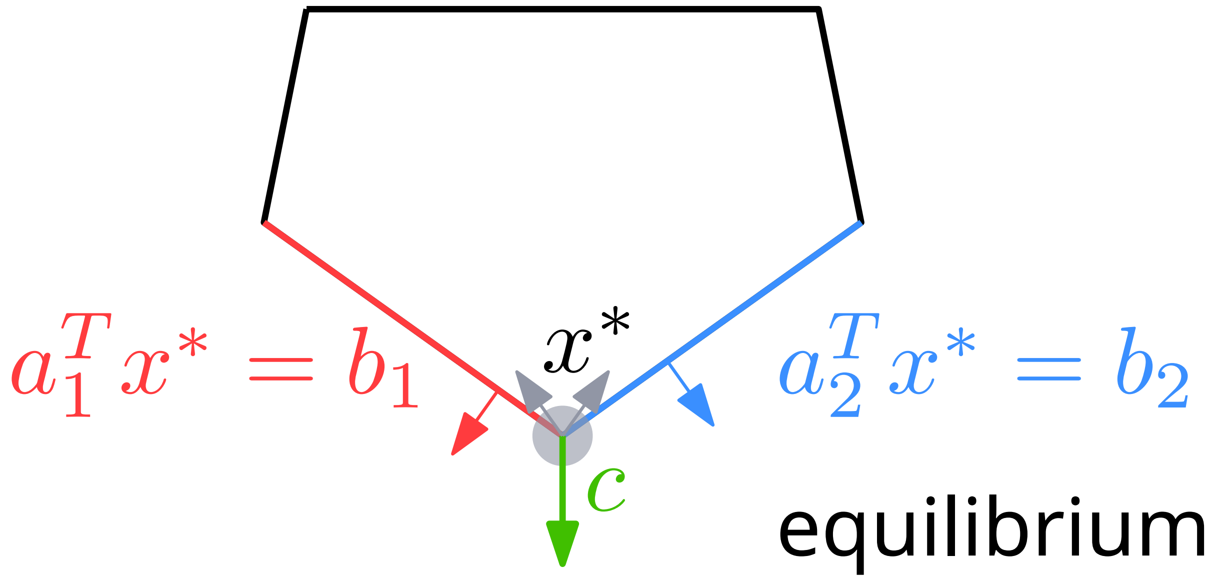


A Physical Interpretation of Strong Duality

Primal: Maximize $c^T x$ subject to $Ax \leq b$ (no nonnegativity constraints).
Dual: Minimize $b^T y$ subject to $A^T y = c$ and $y \geq 0$.



Imagine c points downwards $\approx$ gravity



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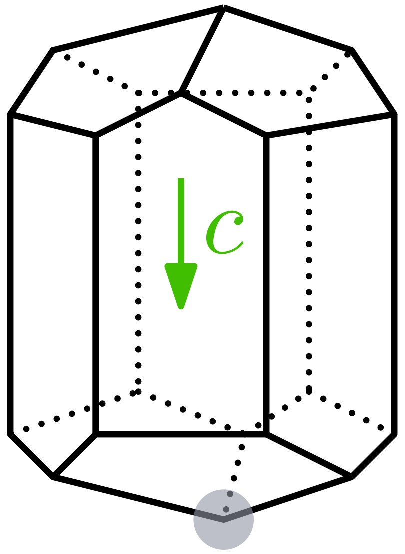

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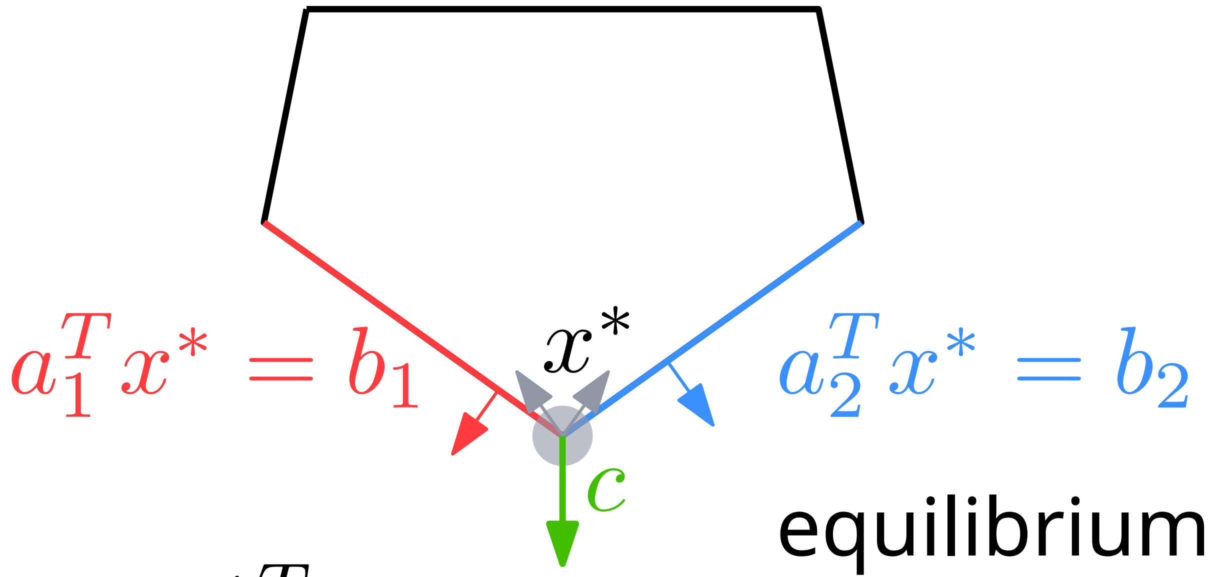

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We get $c = \sum_{i=1}^{m} y_i a_i = A^T y$.

(all other y_i's are 0)

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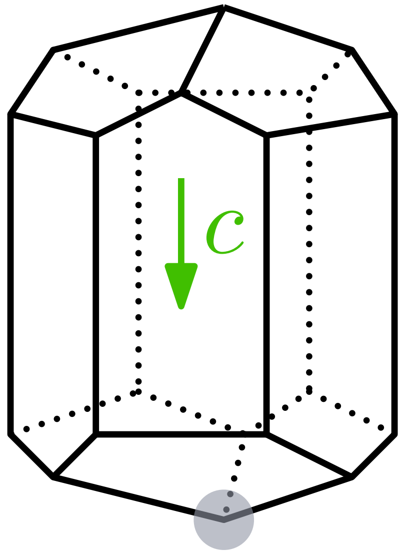

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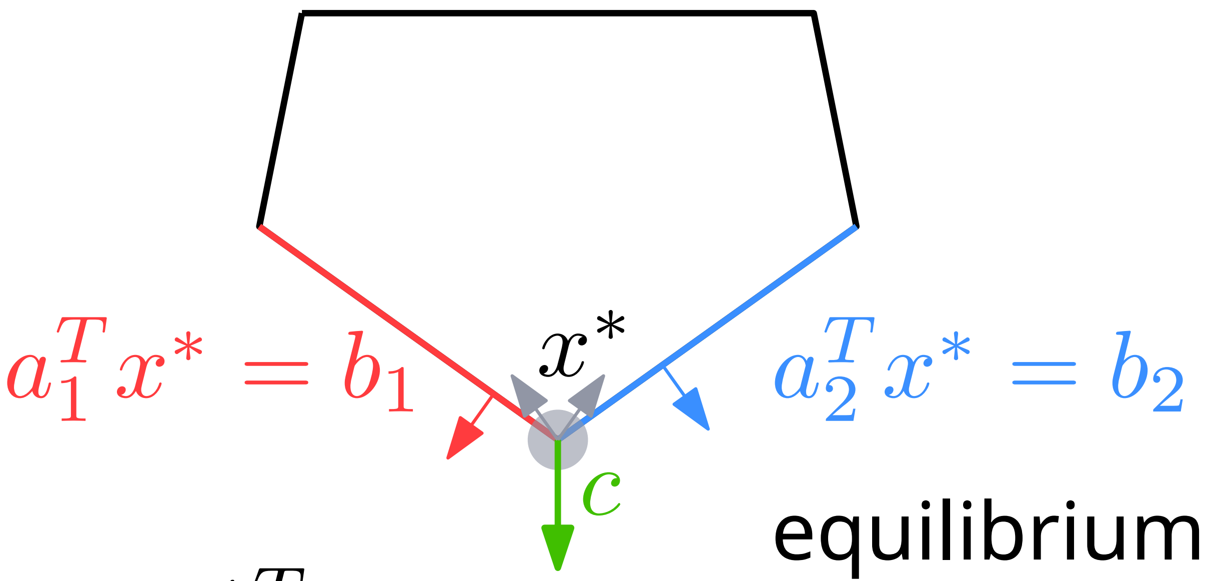

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equilibrium of forces: $c = A^T y$

We get $c = \sum_{i=1}^m y_i a_i = A^T y$.

(all other y_i's are 0)

So y is a feasible solution of the dual.

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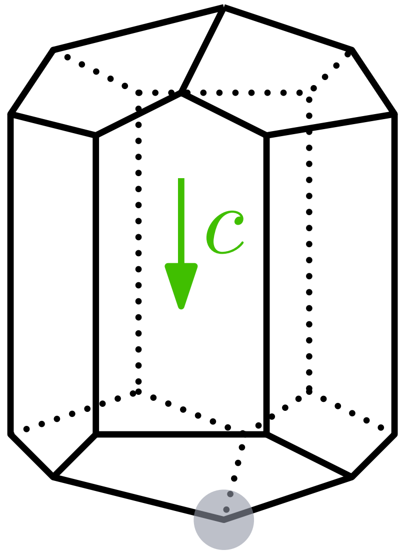

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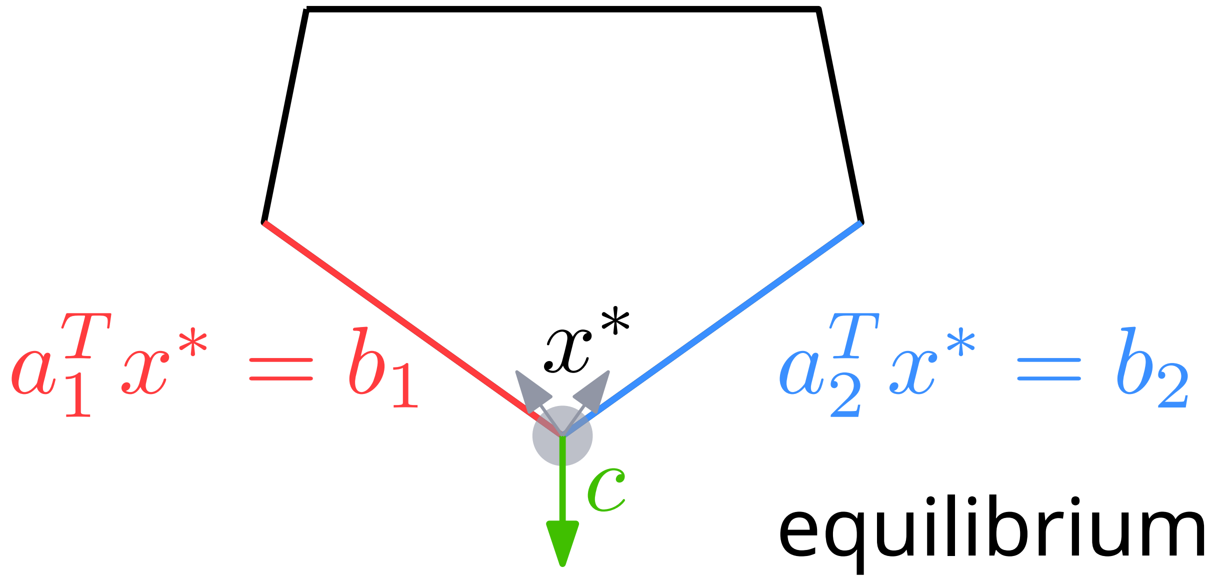

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equilibrium of forces: $c = A^T y$

Now, $y^T(Ax - b) = 0$, because

- $y_i = 0$ if the ith face is not supporting
- the ith component of $Ax - b$ is zero if the ith face is supporting.

A Physical Interpretation of Strong Duality

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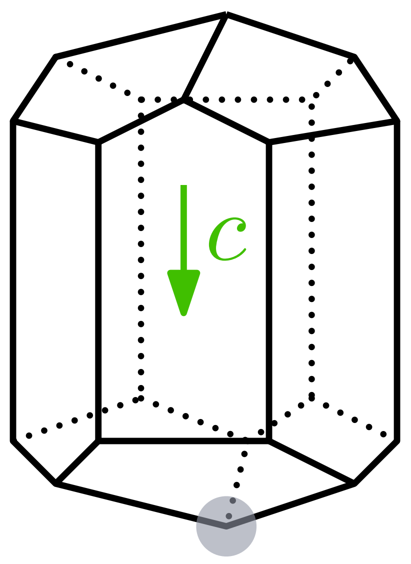

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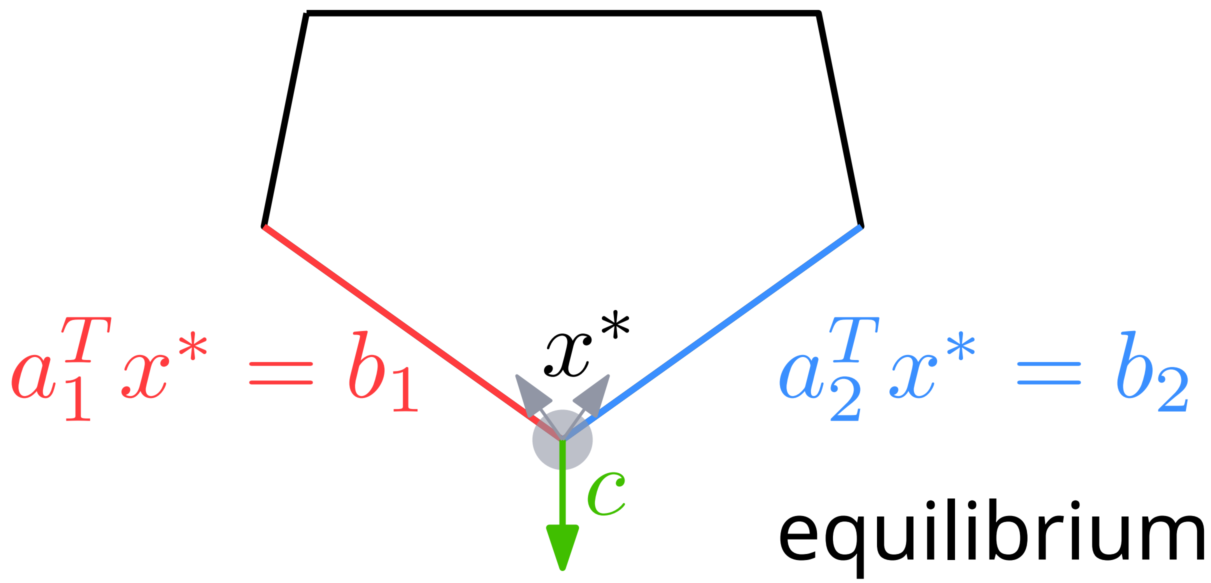

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$\Rightarrow b^T y = y^T b = y^T A x = c^T x$ (“physical proof” of strong duality)

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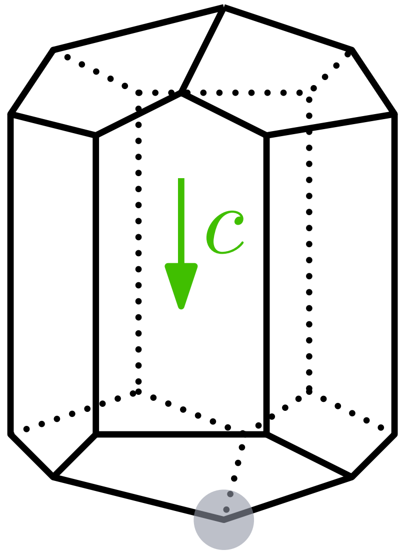

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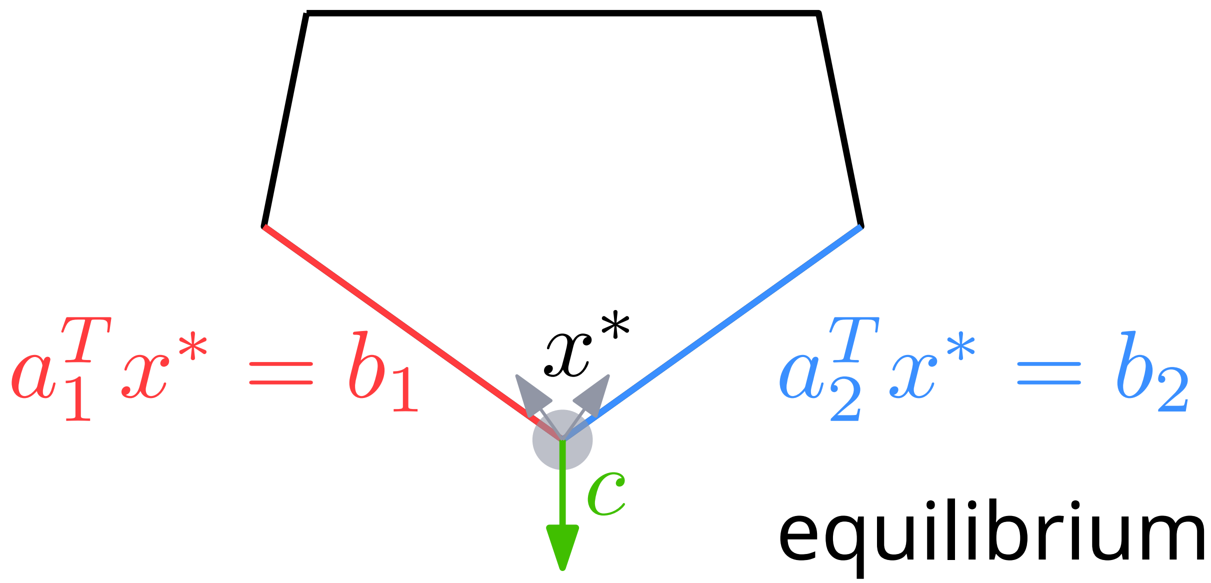

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Note: We will see a mathematical proof shortly

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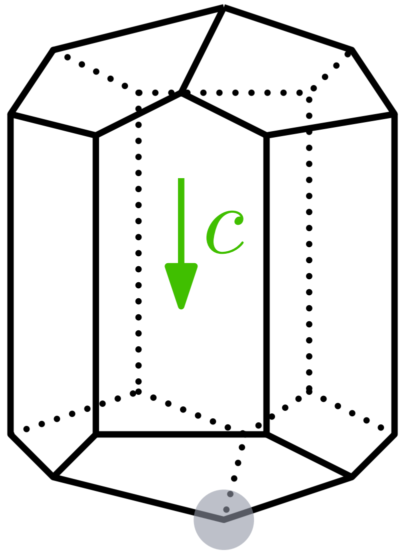

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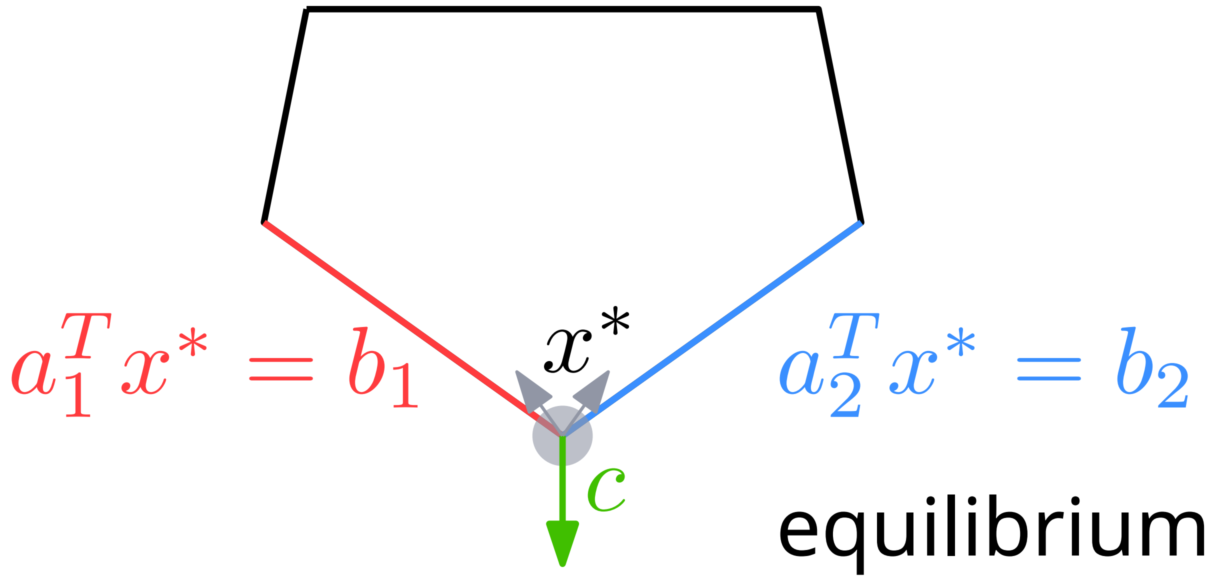

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Now, $y^T(Ax - b) = 0$, because

- $y_i = 0$ if the ith face is not supporting
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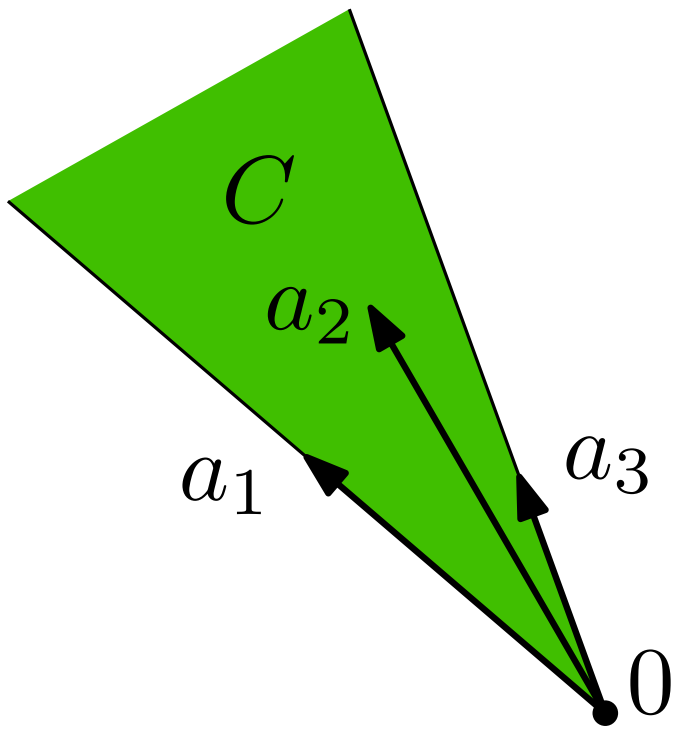
Remark: "$y_i > 0 \Rightarrow a_i^T x = b$" is called complementary slackness, and characterizes optimality here.

Farkas Lemma

Farkas Lemma

Def. The convex cone generated by $a_1, \ldots, a_n \in \mathbb{R}^m$ is

$\{x_1 a_1 + \ldots + x_n a_n \mid x_1, \ldots, x_n \geq 0\}$



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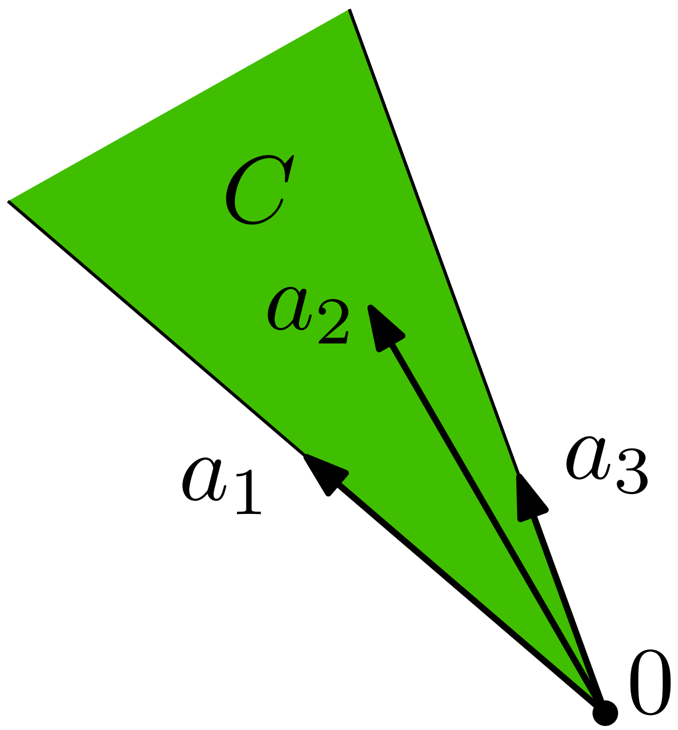
Note $C = \{Ax \mid x \geq 0\}$ is the convex cone generated by the columns of A.



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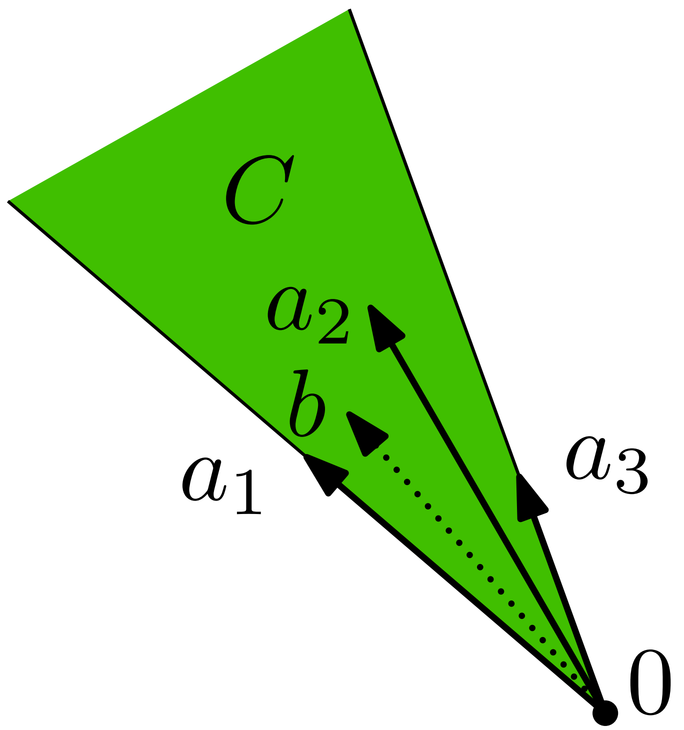

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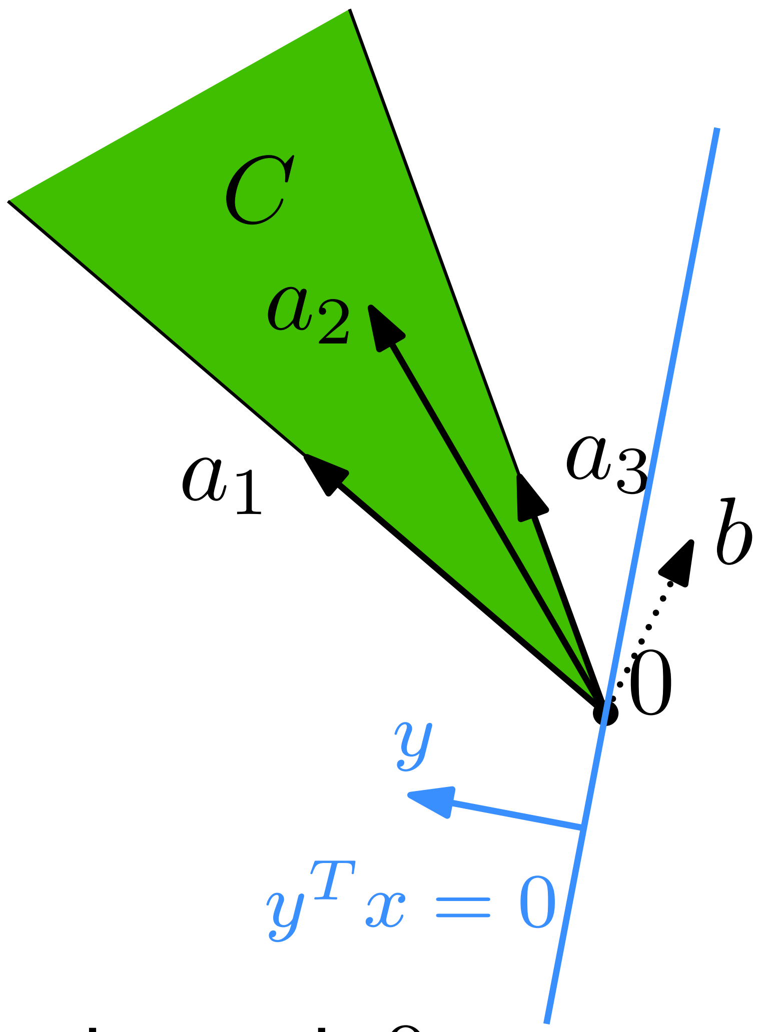

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Farkas Lemma (geometric): If $b \notin C$ then there is a hyperplane through 0 separating C and b.

Farkas Lemma

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Let A be an $m \times n$ matrix, and let $b \in \mathbb{R}^m$.

Then exactly one of the following two possibilities occurs.

(1) There exists $x \in \mathbb{R}^n$ with $Ax = b$ and $x \geq 0$

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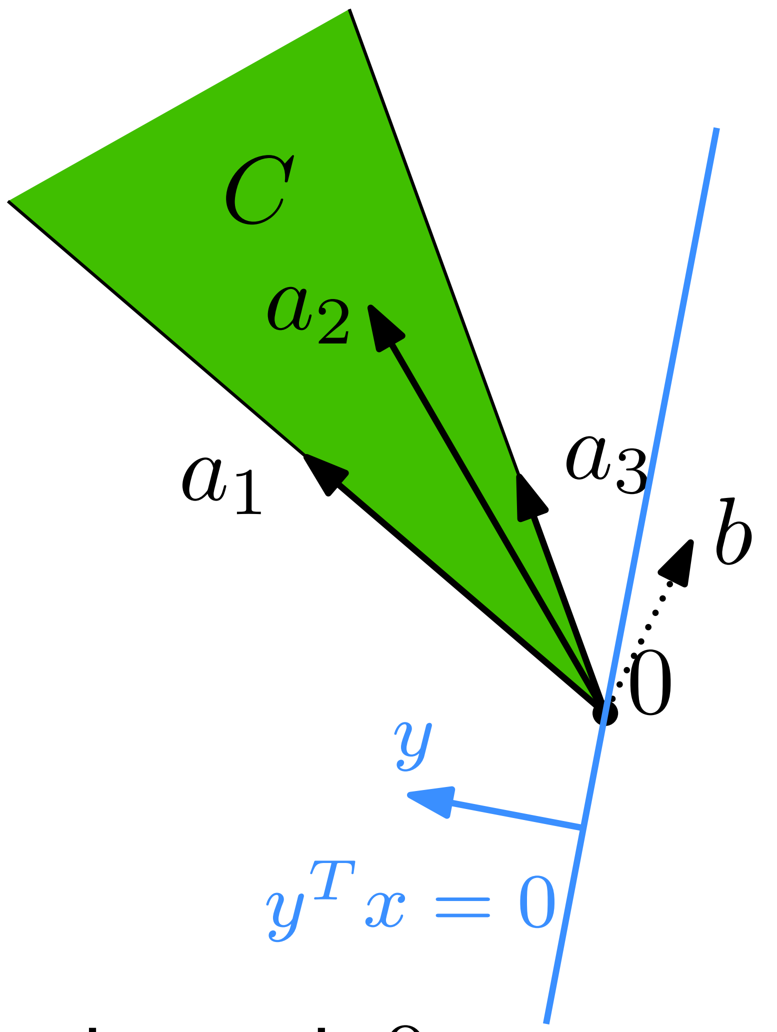

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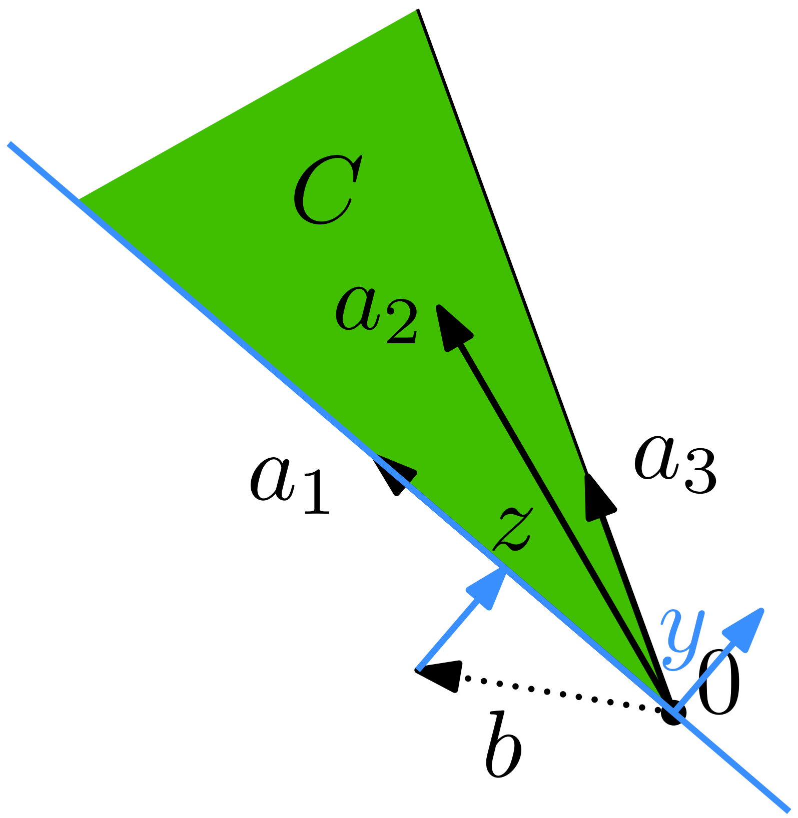

proof idea:

Let z be closest point in C to b. Choose $y = z - b$

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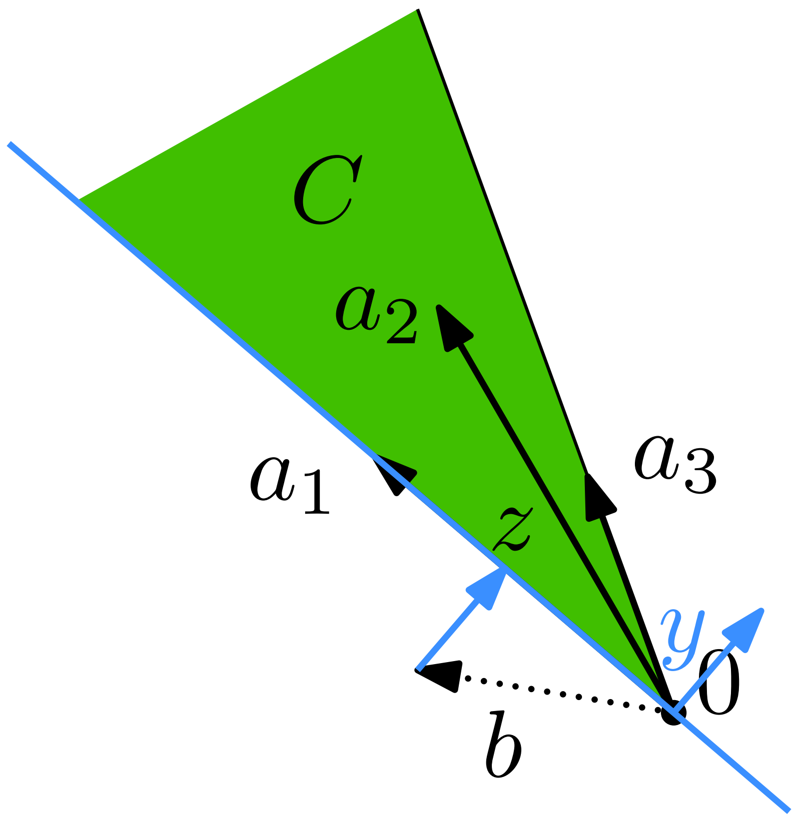

proof idea:

Let z be closest point in C to b. Choose $y = z - b$

would need to prove:

- z exists
- $y^T b < 0$
- $y^T x \geq 0$ for $x \in C$

A Variant of the Farkas Lemma

Farkas Lemma

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Farkas: When does a system of linear equalities have a nonnegative solution?

Variant of the Farkas Lemma

The system $Ax \leq b$ has a nonnegative solution $x \geq 0$ if and only if every nonnegative $y \in \mathbb{R}^m$ with $y^T A \geq 0^T$ also satisfies $y^T b \geq 0$.

Variant: When does a system of linear inequalities have a nonnegative solution?

A Variant of the Farkas Lemma – geometric

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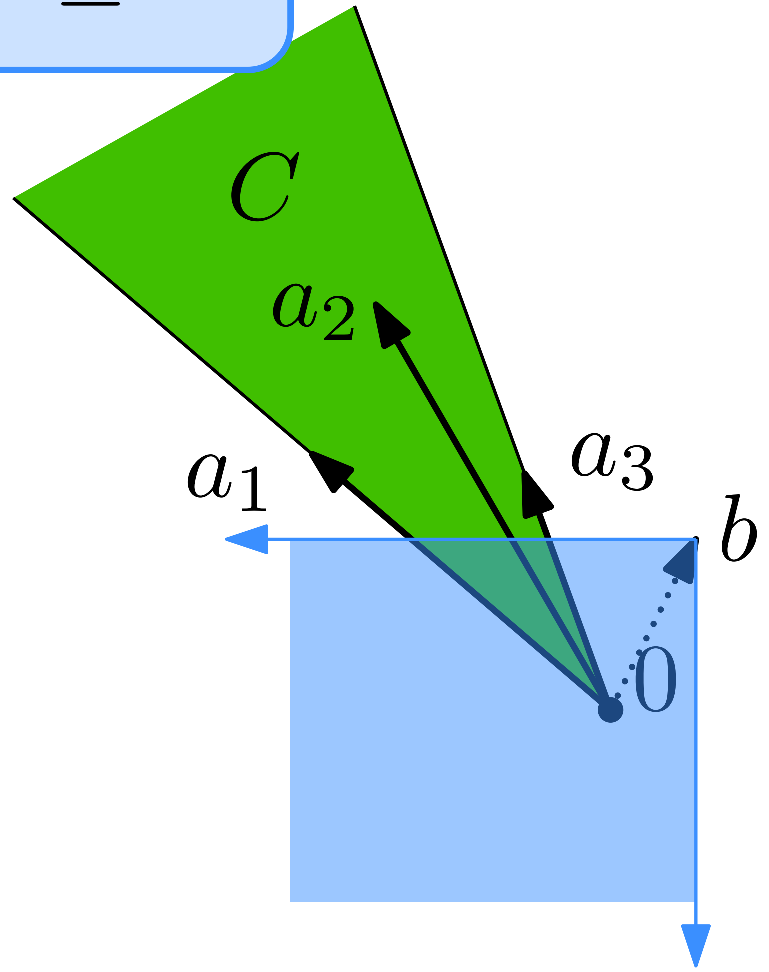

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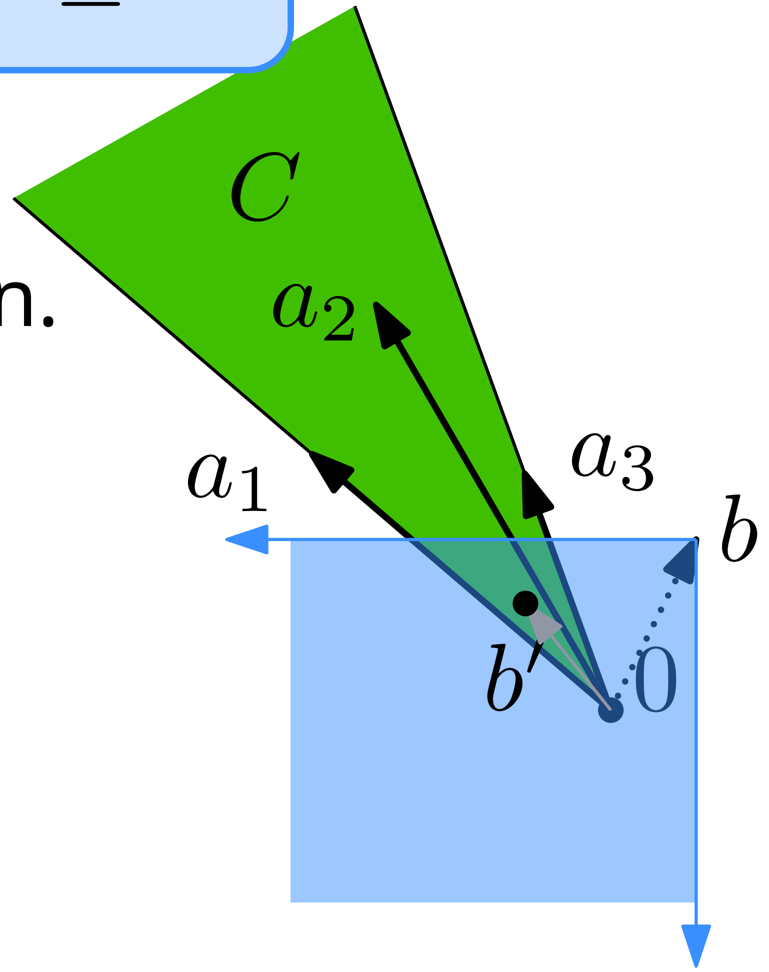
$\Leftrightarrow b' := Ax$ lies in C and the "negative octant" with b as origin.



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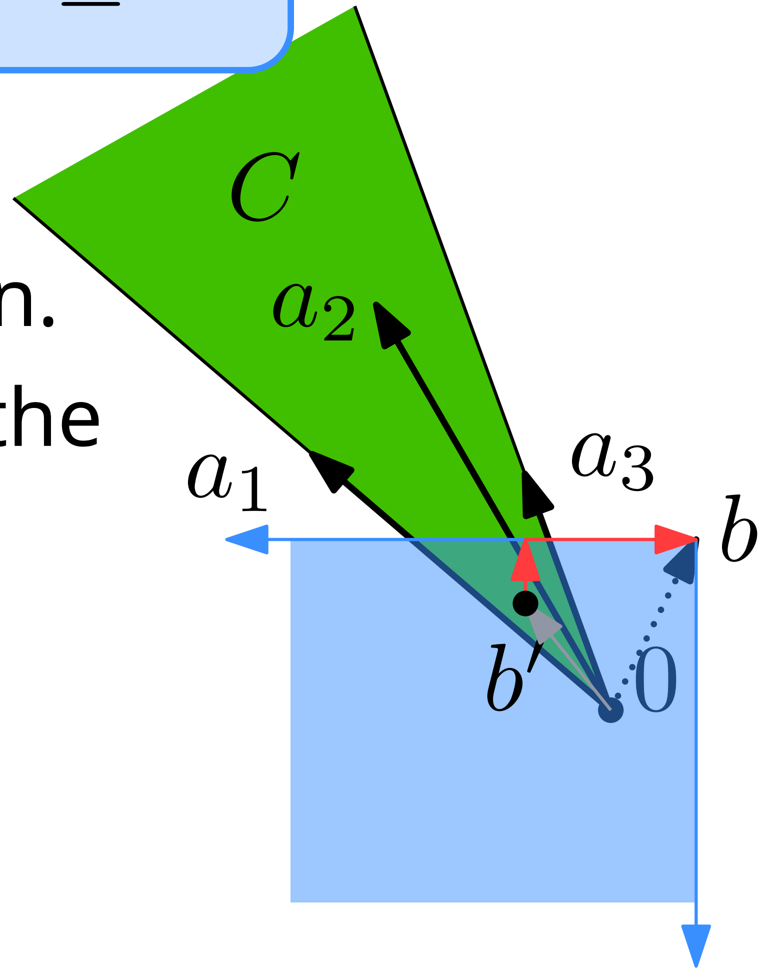
$\Leftrightarrow$ b lies in the convex cone C^+ spanned by $a_1, \ldots a_n$ and by the standard basis vectors $e_1, \ldots, e_m$



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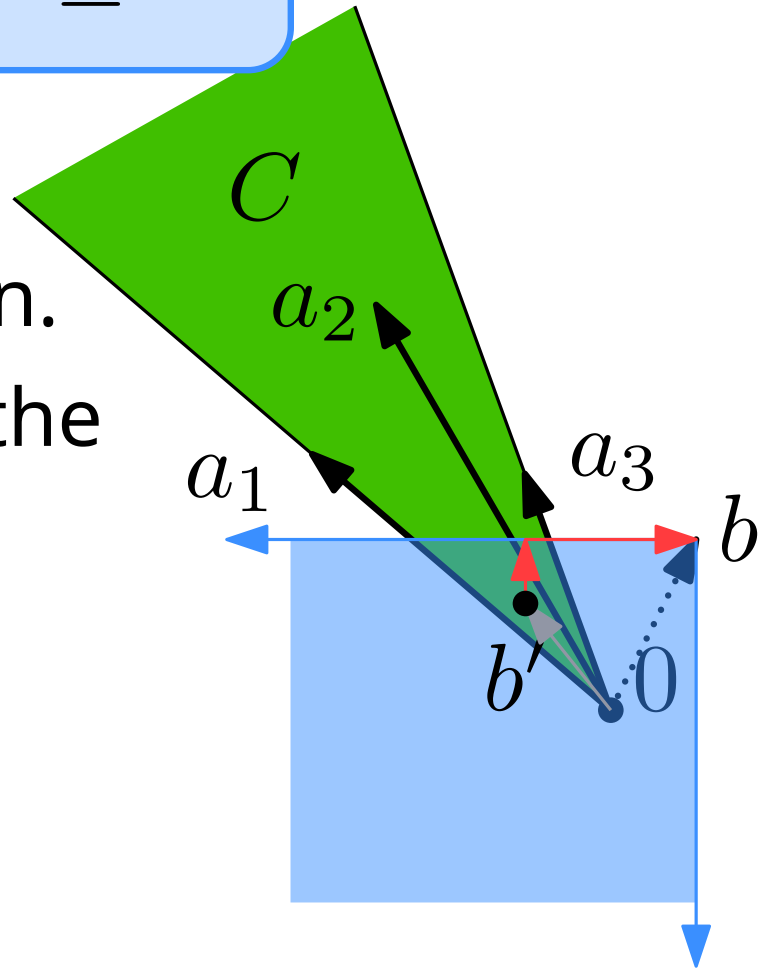

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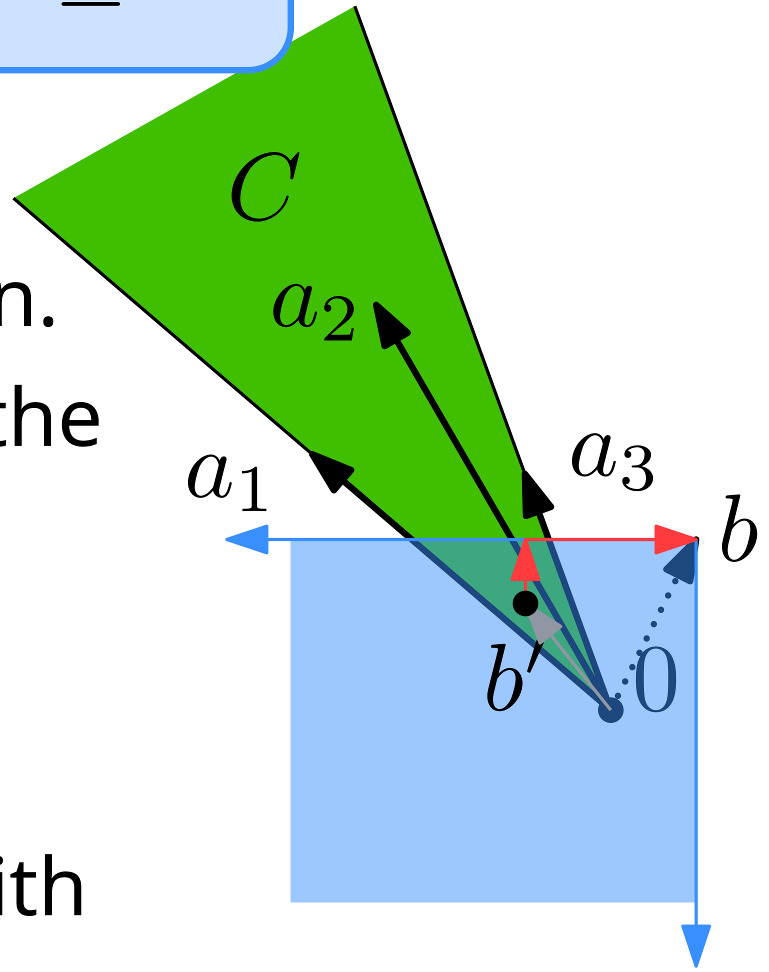
$\Leftrightarrow$ C and b cannot be separated by a hyperplane through 0 with $e_1, \ldots, e_n$ on the side of C



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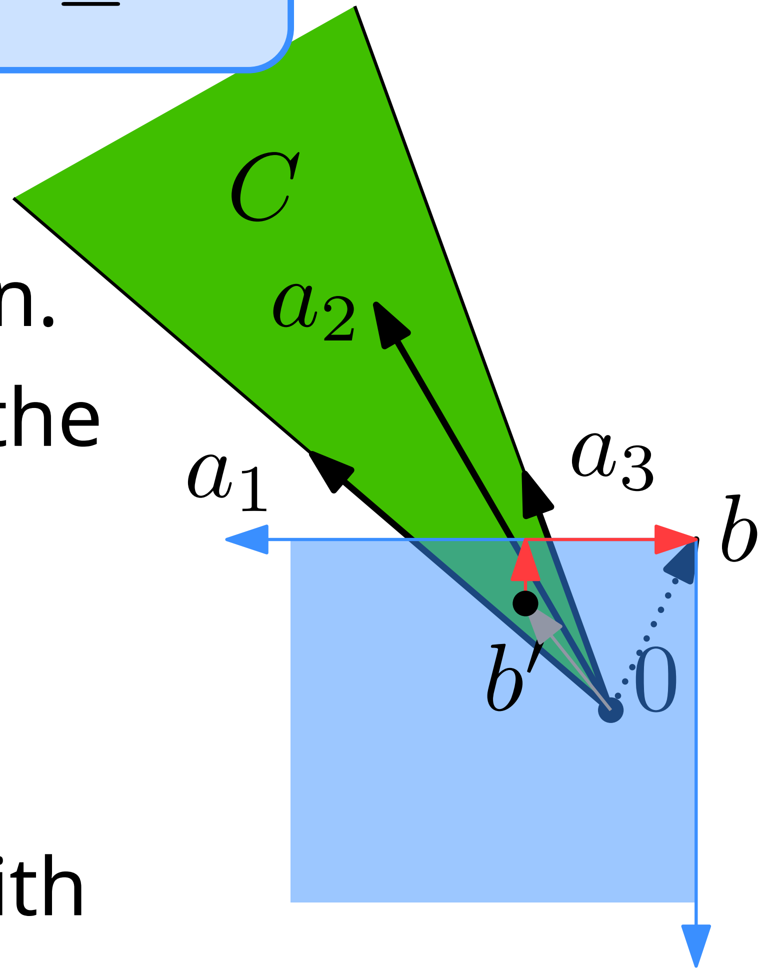

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$\rightsquigarrow$ Variant of Farkas Lemma

A Variant of the Farkas Lemma – algebraic

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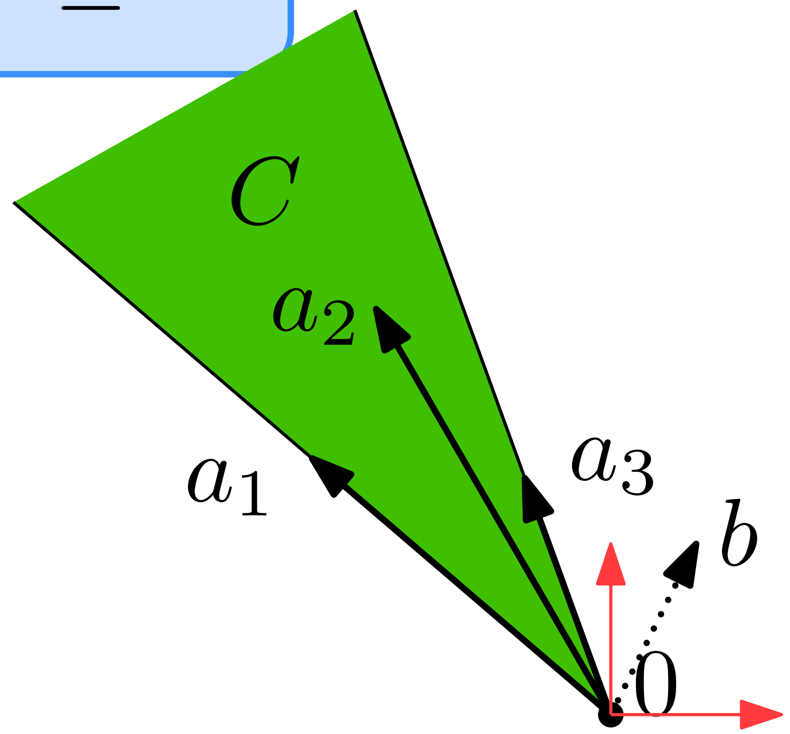

Proof

Bring $Ax \leq b$ into equational form using slack variables:

A Variant of the Farkas Lemma – algebraic

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Form the matrix $\bar{A} = (A \mid I_m)$.

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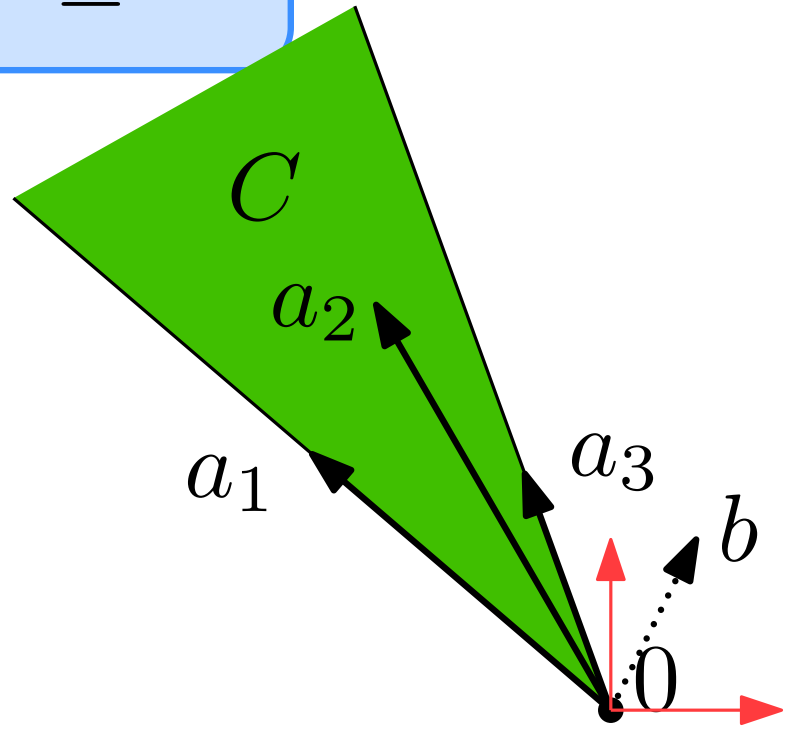

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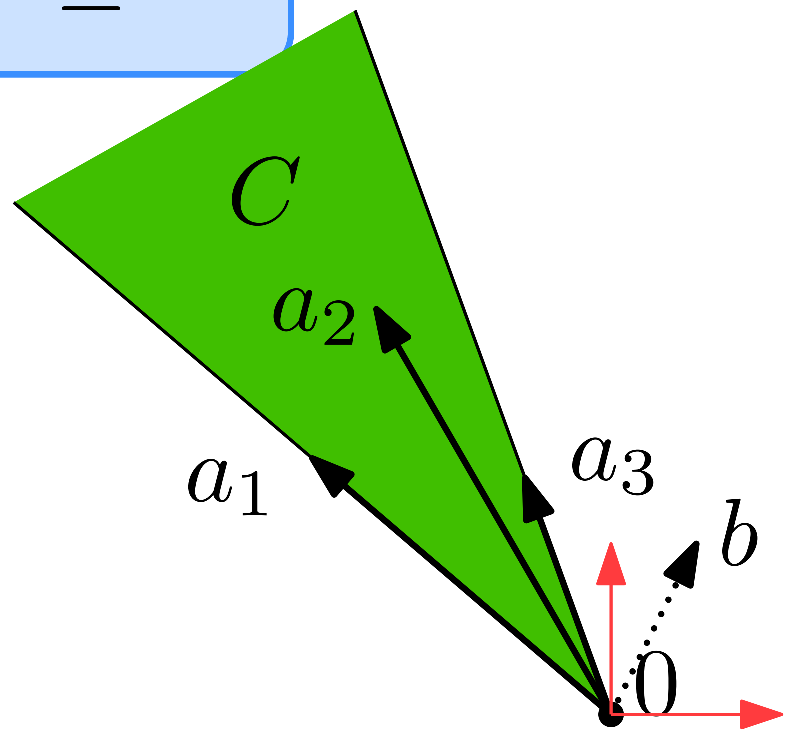

By the Farkas Lemma, this is if and only if every $y \in \mathbb{R}^m$ with $y^T \bar{A} \geq 0^T$ satifies $y^T b \geq 0$.

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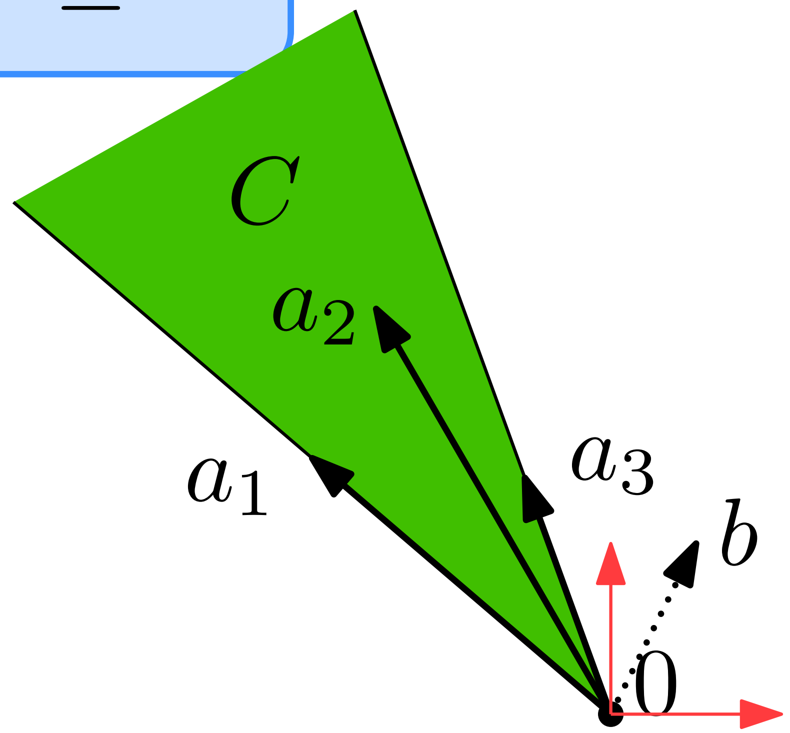

By the Farkas Lemma, this is if and only if every $y \in \mathbb{R}^m$ with $y^T \bar{A} \geq 0^T$ satifies $y^T b \geq 0$.

Means exactly $y^T A \geq 0^T$ and $y \geq 0$.

Proof of Strong Duality from the Farkas Lemma

Variant of the Farkas Lemma

The system $Ax \leq b$ has a nonnegative solution $x \geq 0$ if and only if every nonnegative $y \in \mathbb{R}^m$ with $y^T A \geq 0^T$ also satisfies $y^T b \geq 0$.

Primal: max $c^T x$ subject to $Ax \leq b$ and $x \geq 0$

Dual: min $b^T y$ subject to $A^T y \geq c$ and $y \geq 0$

want to prove:

Strong duality: Optimal solutions x^*, y^* satisfy $c^T x^* = b^T y^*$.

Proof of Strong Duality from the Farkas Lemma

Strong duality: Optimal solutions x^*, y^* satisfy $c^T x^* = b^T y^*$.

Proof

(1) $Ax \leq b$, $c^T x \geq c^T x^*$ has a solution $x \geq 0$.

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Proof of Strong Duality from the Farkas Lemma

Strong duality: Optimal solutions x^*, y^* satisfy $c^T x^* = b^T y^*$.

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Since (1) has a nonnegative solution, the same $\hat{y}$ satisfies $\hat{y}^T \hat{b}_0 \geq 0$

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$\Rightarrow b^T y^* = c^T x^*$, thus strong duality holds.

Complementary Slackness

Corollary Let $x^\star = (x_1^\star, \dots, x_n^\star)$ be a feasible solution of the linear program

$$\text{maximize } c^T x \text{ subject to } Ax \leq b \text{ and } x \geq 0, \quad \text{(P)}$$

and let $y^\star = (y_1^\star, \dots, y_m^\star)$ be a feasible solution of the dual linear program

$$\text{minimize } b^T y \text{ subject to } A^T y \geq c \text{ and } y \geq 0. \quad \text{(D)}$$

Then the following two statements are equivalent:

1. $x^\star$ is optimal for (P) and $y^\star$ is optimal for (D).
2. For all $i = 1, \dots, m$, $x^\star$ satisfies the ith constraint of (P) with equality or $y_i^\star = 0$; similarly, for all $j = 1, \dots, n$, $y^\star$ satisfies the jth constraint of (D) with equality or $x_j^\star = 0$.

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Proof: Follows from duality

$$\sum_{j=1}^n c_j x_j \ge \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i\right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j\right) y_i \ge \sum_{i=1}^m b_i y_i$$

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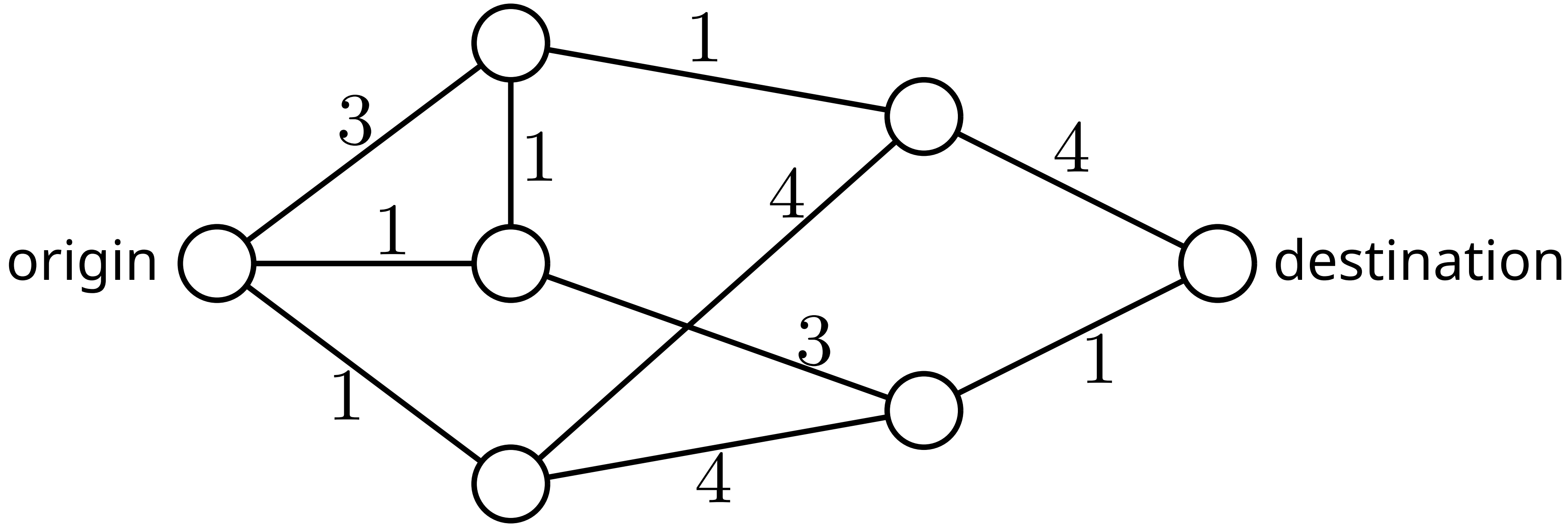
Duality shows Max Flow = Min Cut

Recall: Flow in a Network

How to send as much data as possible over a local network?



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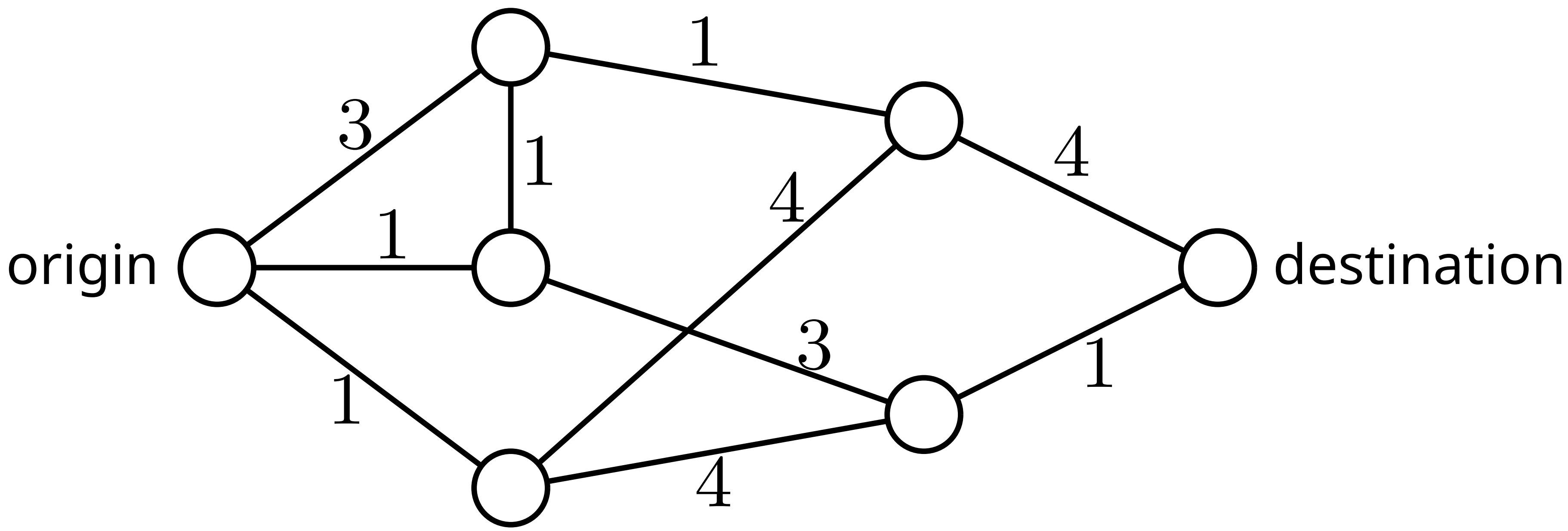

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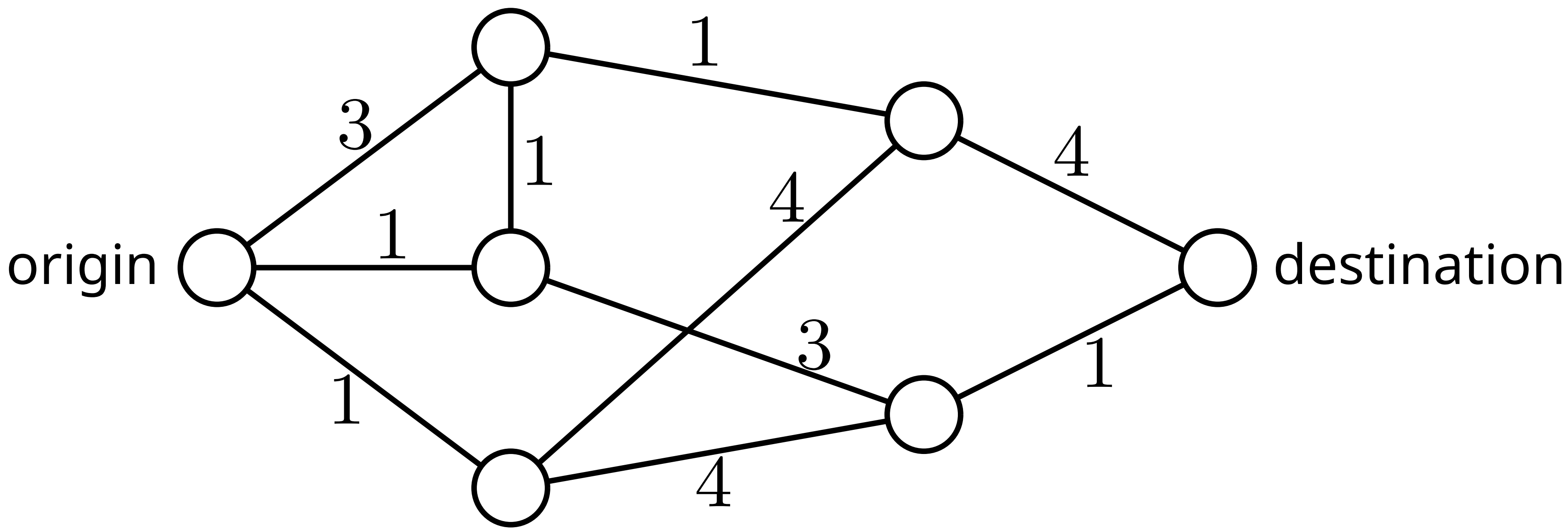

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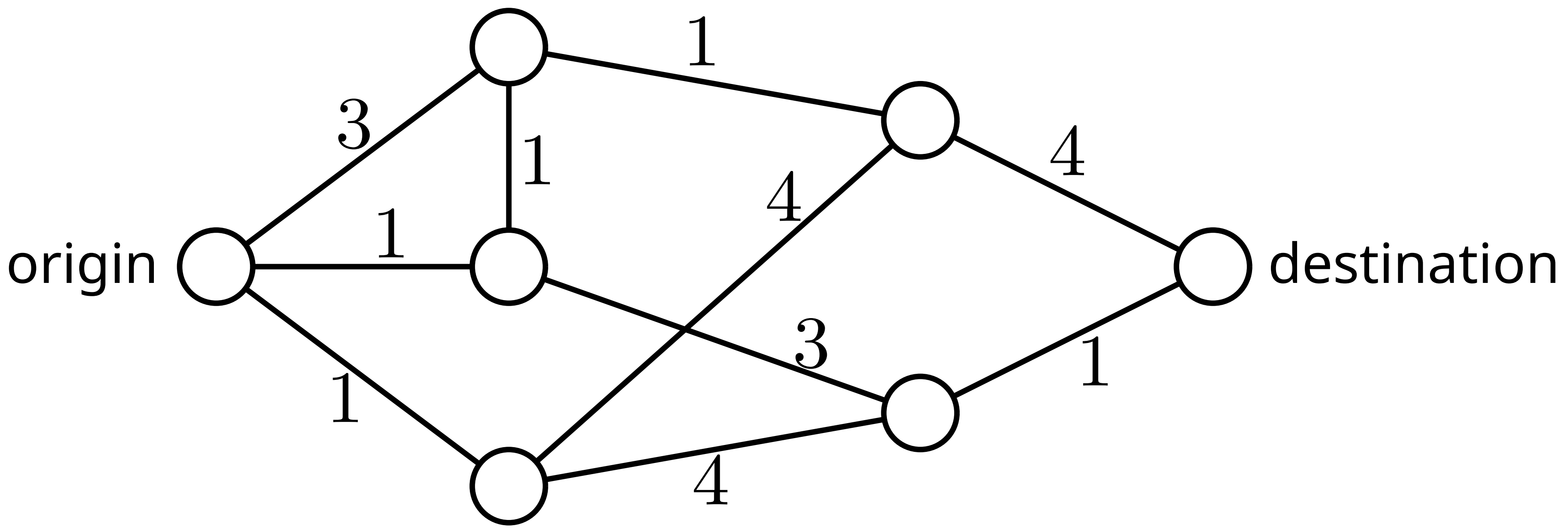

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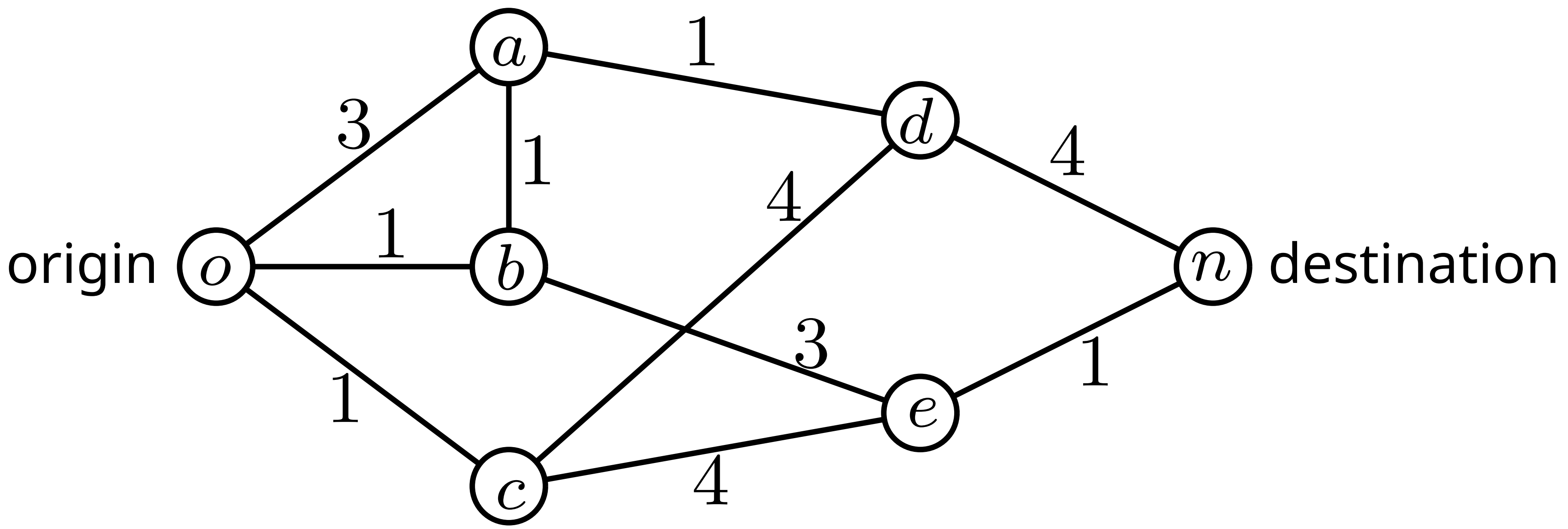

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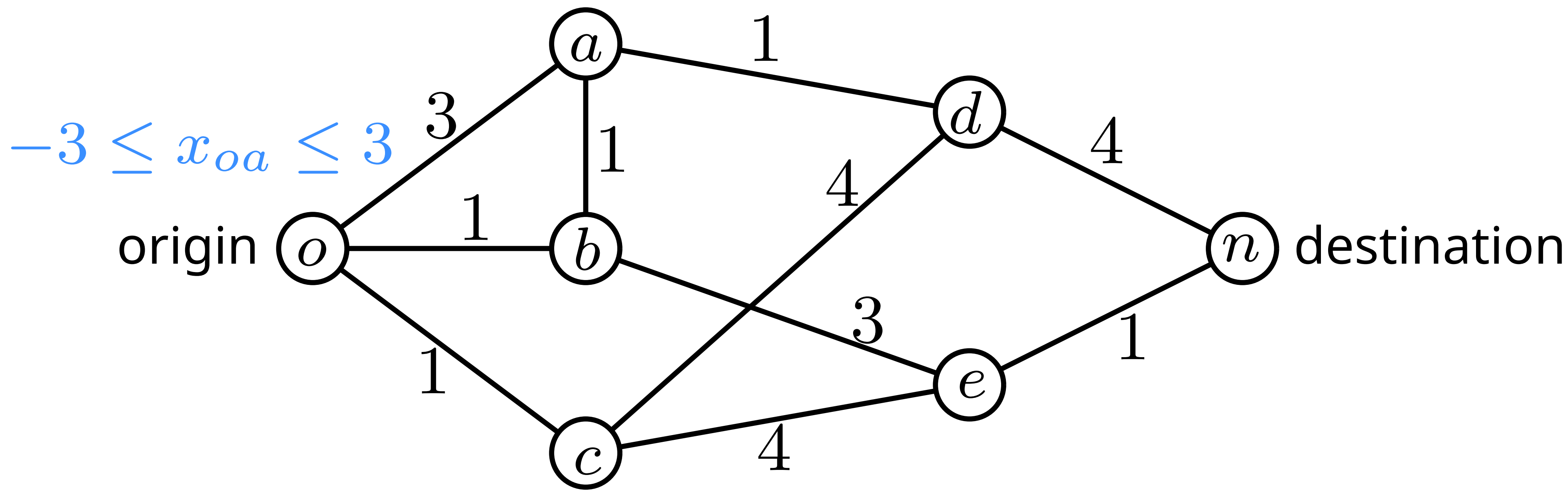

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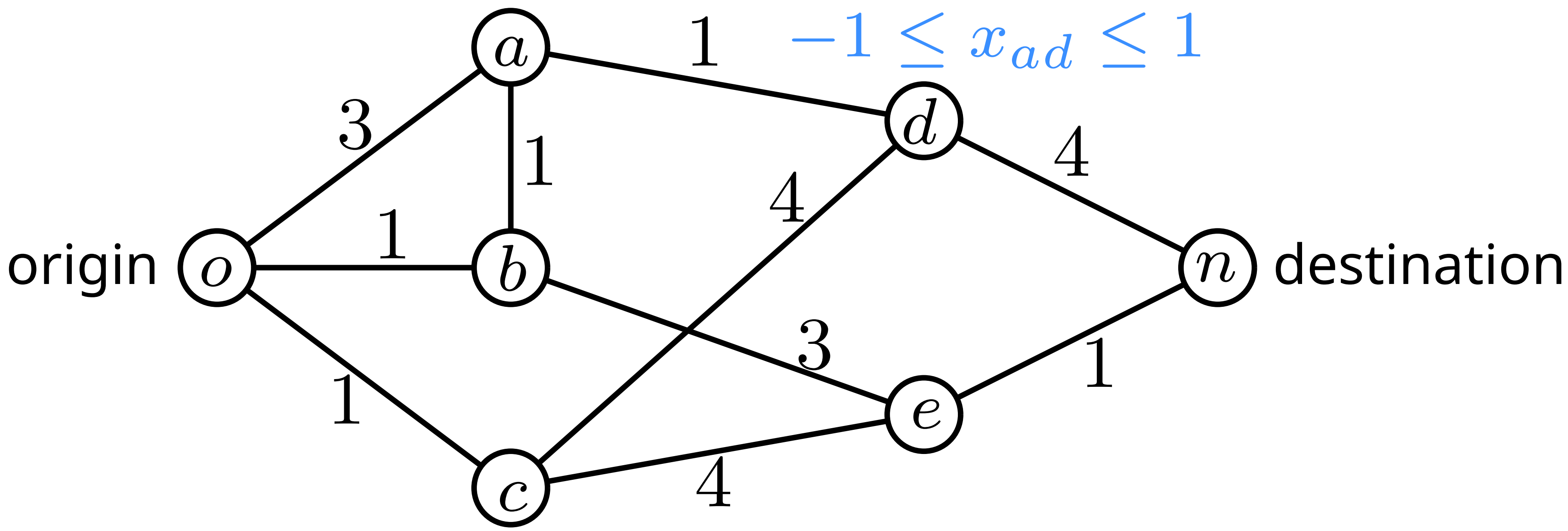

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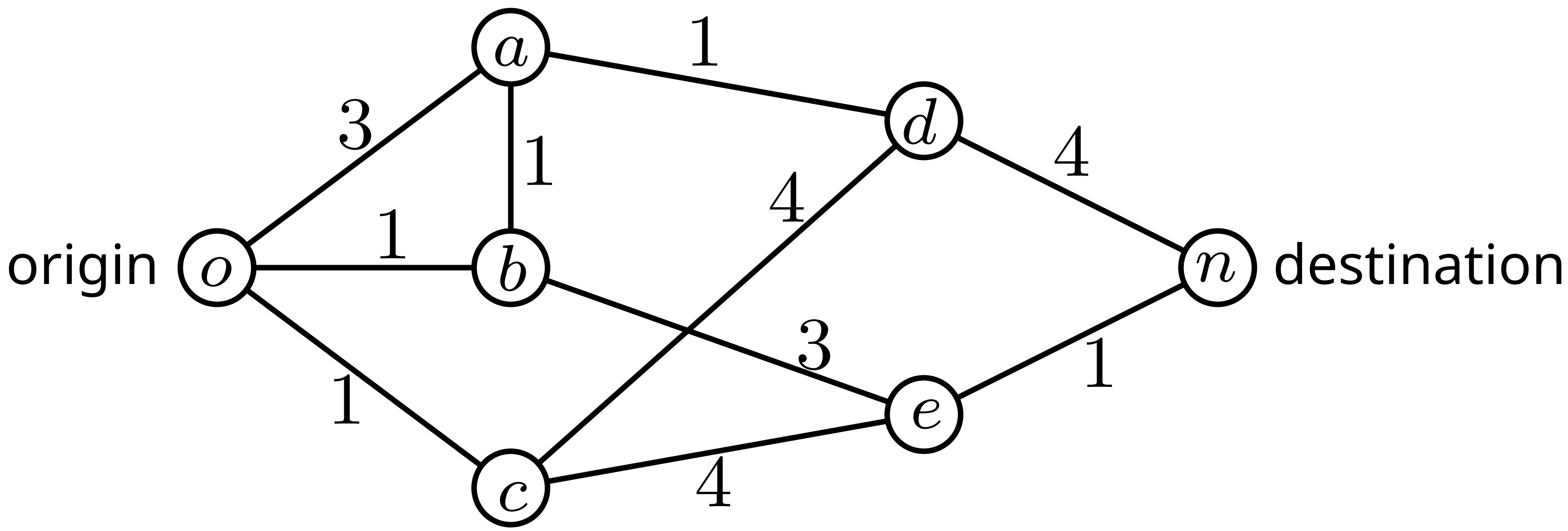

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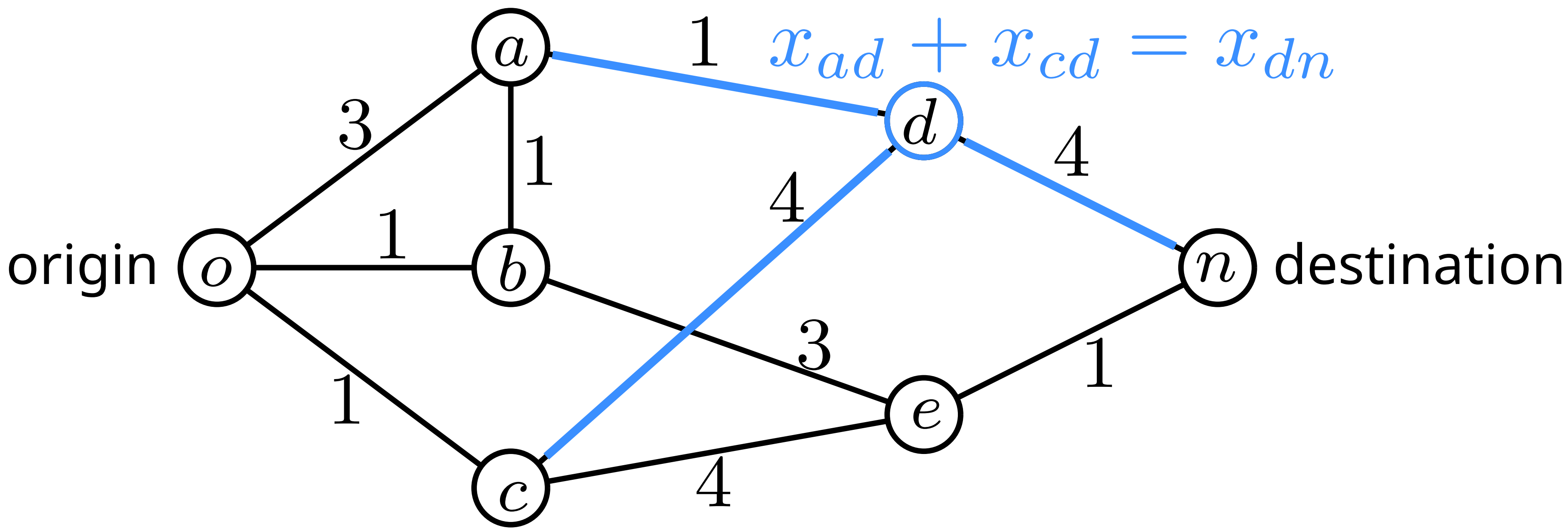

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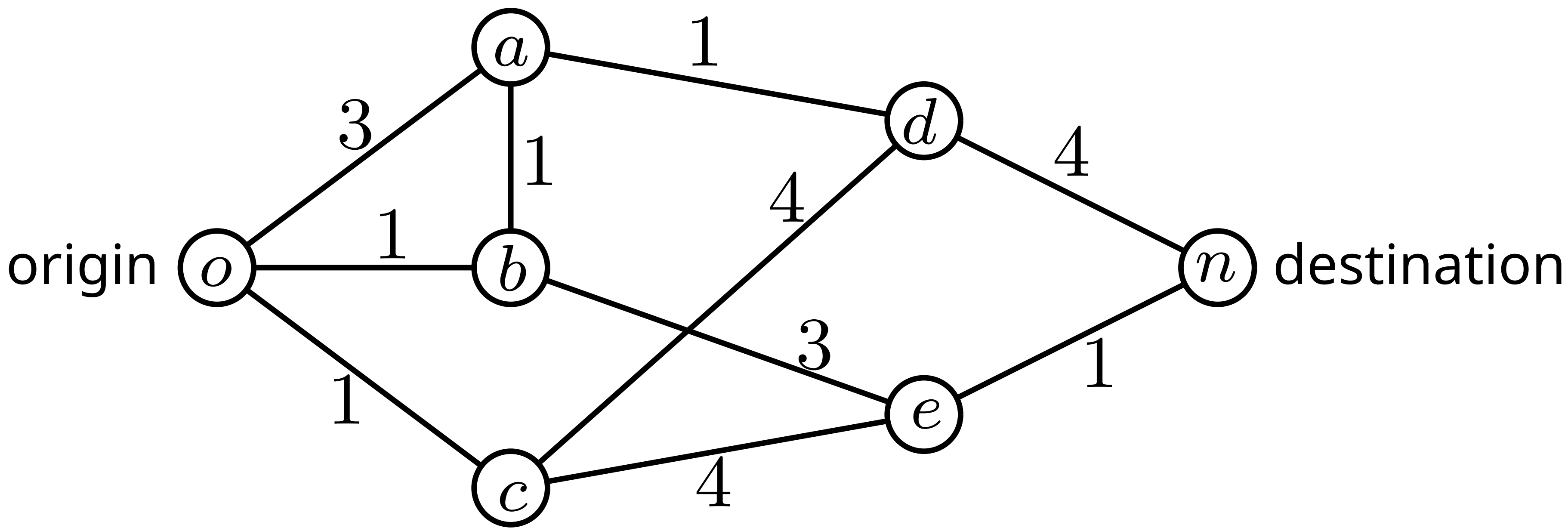

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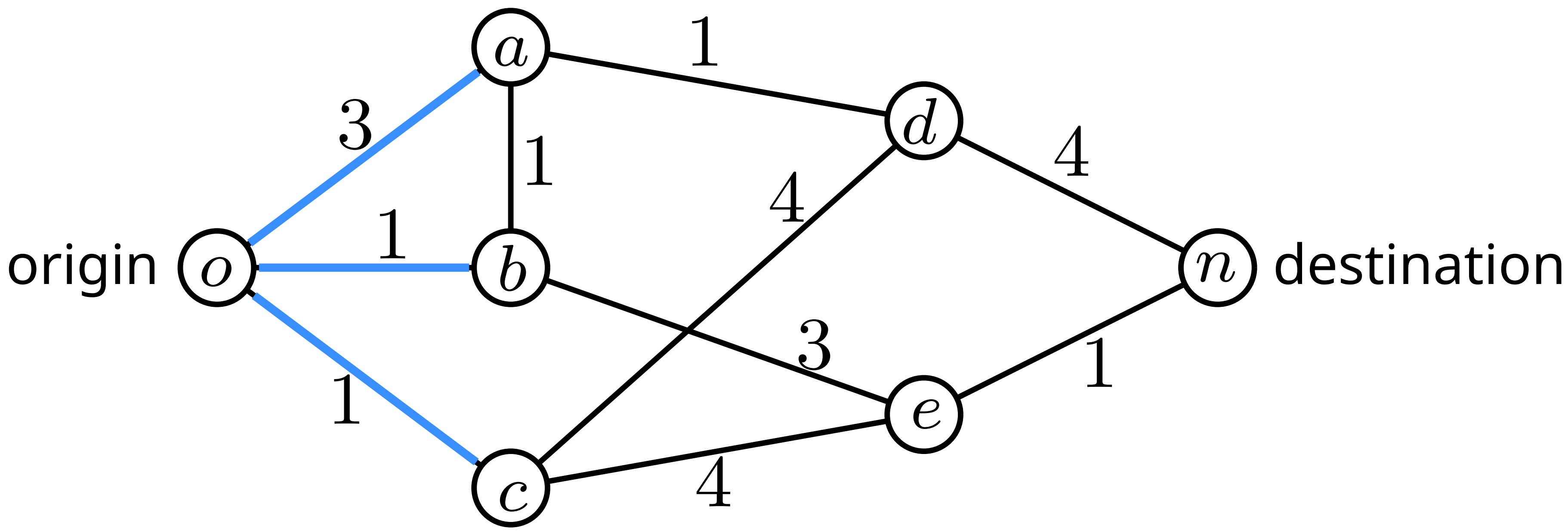

Maximize . . .?

$x_{oa} + x_{ob} + x_{oc}$

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Linear Program Formulation

maximize $x_{oa} + x_{ob} + x_{oc}$

subject to $-3 \leq x_{oa} \leq 3,\ -1 \leq x_{ob} \leq 1,\ -1 \leq x_{oc} \leq 1$

$-1 \leq x_{ab} \leq 1,\ -1 \leq x_{ad} \leq 1,\ -3 \leq x_{be} \leq 3$

$-4 \leq x_{cd} \leq 4,\ -4 \leq x_{ce} \leq 4,\ -4 \leq x_{dn} \leq 4$

$-1 \leq x_{en} \leq 1$

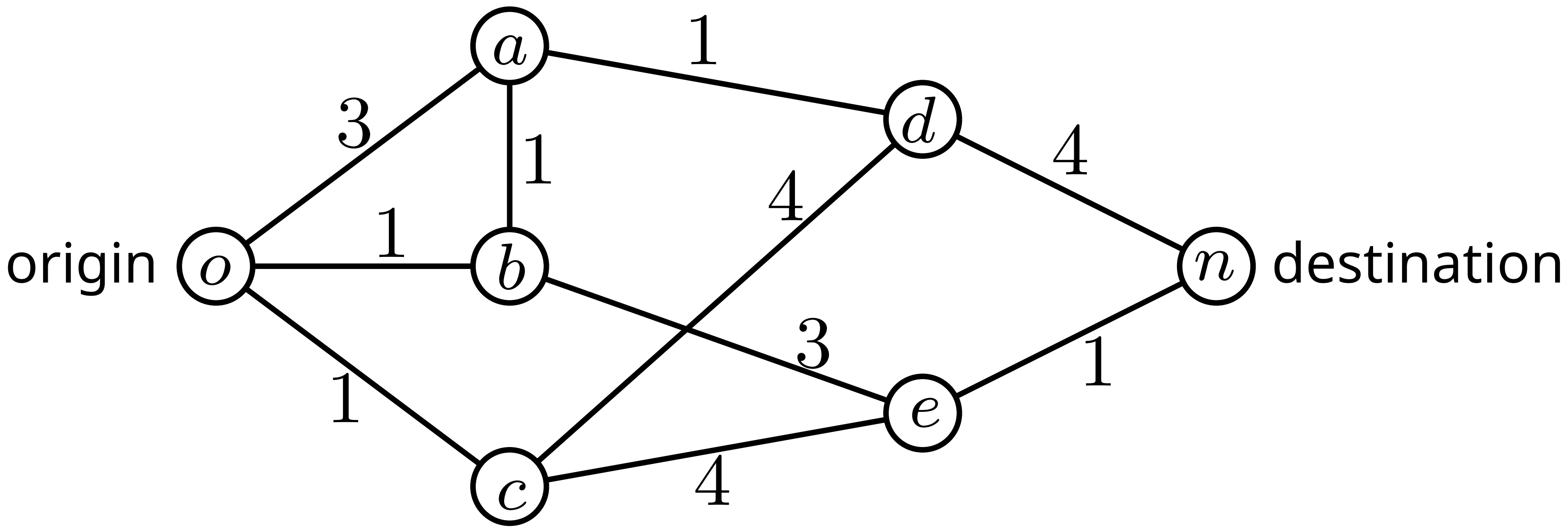
$x_{oa} = x_{ab} + x_{ad}$

$x_{ob} + x_{ab} = x_{be}$

$x_{oc} = x_{cd} + x_{ce}$

$x_{ad} + x_{cd} = x_{dn}$

$x_{be} + x_{ce} = x_{en}$



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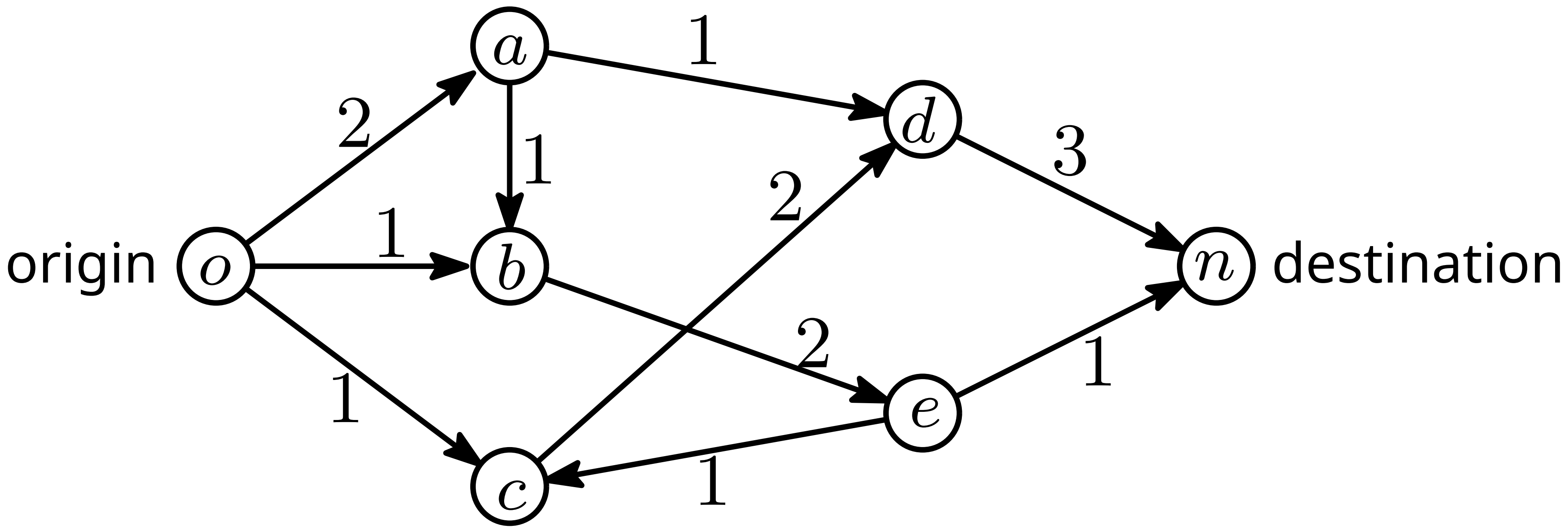
$$-3 \le x_{oa} \le 3,\ -1 \le x_{ob} \le 1,\ -1 \le x_{oc} \le 1$$
$$-1 \le x_{ab} \le 1,\ -1 \le x_{ad} \le 1,\ -3 \le x_{be} \le 3$$
$$-4 \le x_{cd} \le 4,\ -4 \le x_{ce} \le 4,\ -4 \le x_{dn} \le 4$$
$$-1 \le x_{en} \le 1$$
$$x_{oa} = x_{ab} + x_{ad}$$
$$x_{ob} + x_{ab} = x_{be}$$
$$x_{oc} = x_{cd} + x_{ce}$$
$$x_{ad} + x_{cd} = x_{dn}$$
$$x_{be} + x_{ce} = x_{en}$$

Optimal solution: 4

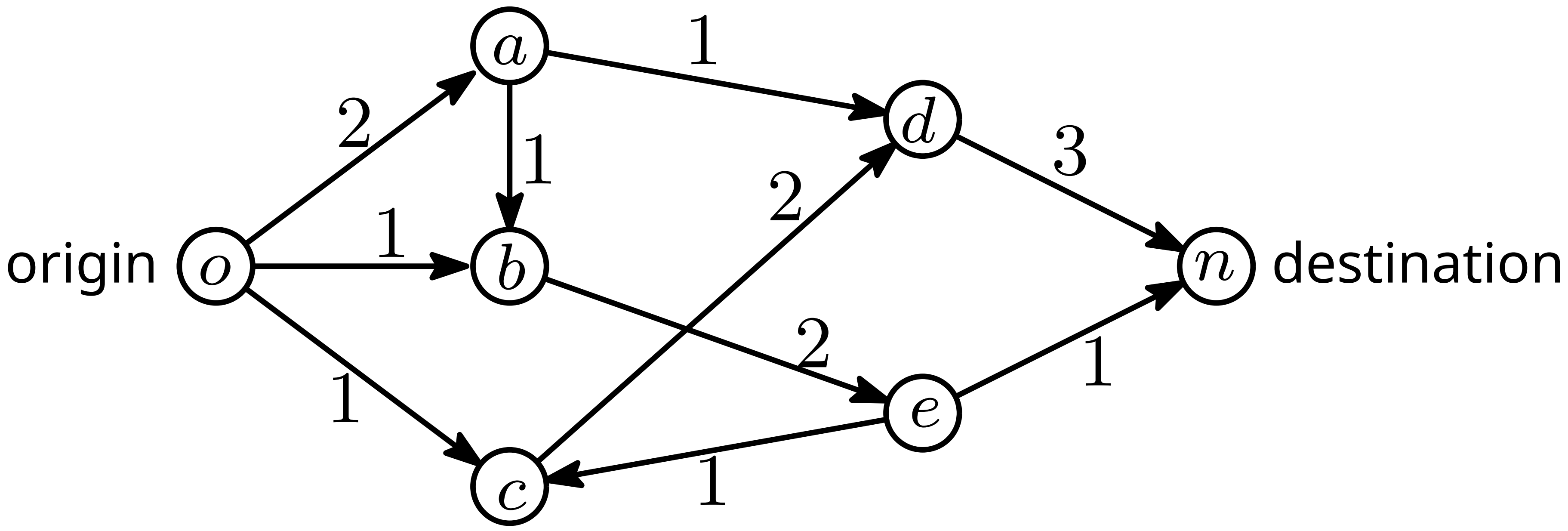


Recall: Flow in a Network

Linear Program Formulation

$$
\begin{aligned}
\text{maximize}\ & x_{oa} + x_{ob} + x_{oc} \\
\text{subject to}\ & -3 \le x_{oa} \le 3,\ -1 \le x_{ob} \le 1,\ -1 \le x_{oc} \le 1 \\
& -1 \le x_{ab} \le 1,\ -1 \le x_{ad} \le 1,\ -3 \le x_{be} \le 3 \\
& -4 \le x_{cd} \le 4,\ -4 \le x_{ce} \le 4,\ -4 \le x_{dn} \le 4 \\
& -1 \le x_{en} \le 1 \\
& x_{oa} = x_{ab} + x_{ad} \\
& x_{ob} + x_{ab} = x_{be} \\
& x_{oc} = x_{cd} + x_{ce} \\
& x_{ad} + x_{cd} = x_{dn} \\
& x_{be} + x_{ce} = x_{en}
\end{aligned}
$$

Optimal solution: 4



well-known “max flow = min cut” $\rightarrow$ now via LP-duality!

Recall: Flow in a Network

Linear Program Formulation

maximize $x_{oa} + x_{ob} + x_{oc}$

subject to $-3 \leq x_{oa} \leq 3,\ -1 \leq x_{ob}$

$-1 \leq x_{ab} \leq 1,\ -1 \leq x_{ad} \leq 1,\ -3 \leq x_{be} \leq 3$

$-4 \leq x_{cd} \leq 4,\ -4 \leq x_{ce} \leq 4,\ -4 \leq x_{dn} \leq 4$

$-1 \leq x_{en} \leq 1$

$x_{oa} = x_{ab} + x_{ad}$

$x_{ob} + x_{ab} = x_{be}$

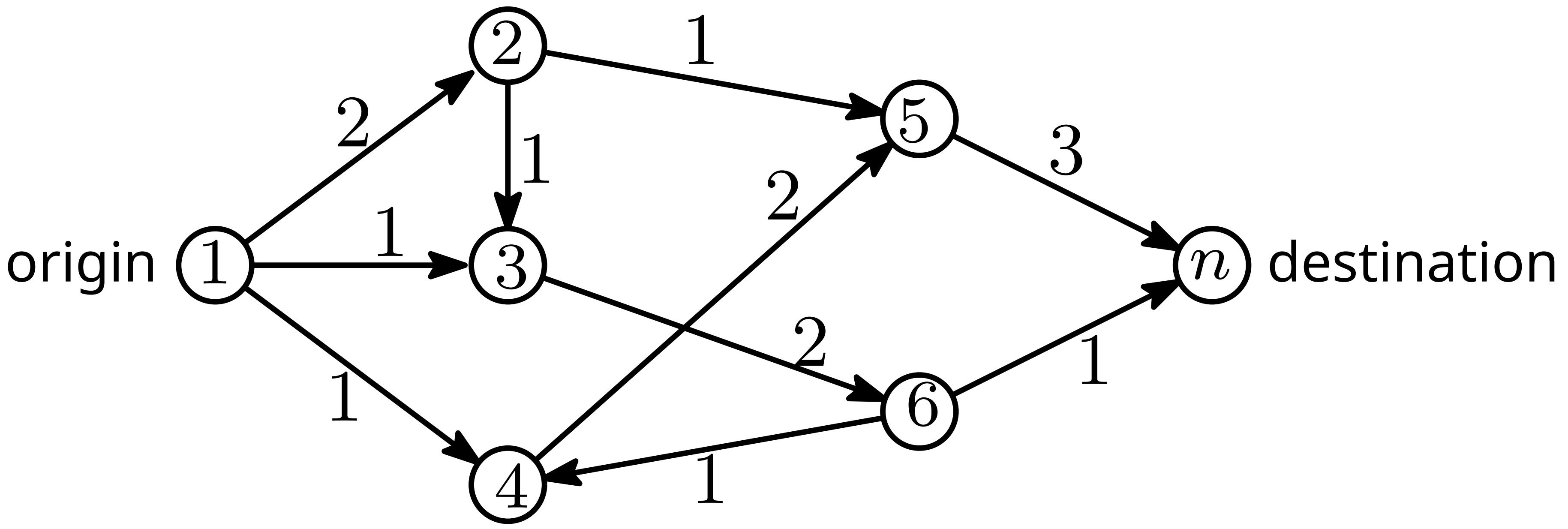
$x_{oc} = x_{cd} + x_{ce}$

$x_{ad} + x_{cd} = x_{dn}$

$x_{be} + x_{ce} = x_{en}$

first we formulate the LP more concisely:
let the vertices be numbered $1, \ldots, n$, let c_{ij} the capacity and x_{ij} the flow on directed edge (i, j), and let f be the max flow.

Optimal solution: 4



well-known "max flow = min cut" $\longrightarrow$ now via LP-duality!

Recall: Flow in a Network

Linear Program Formulation

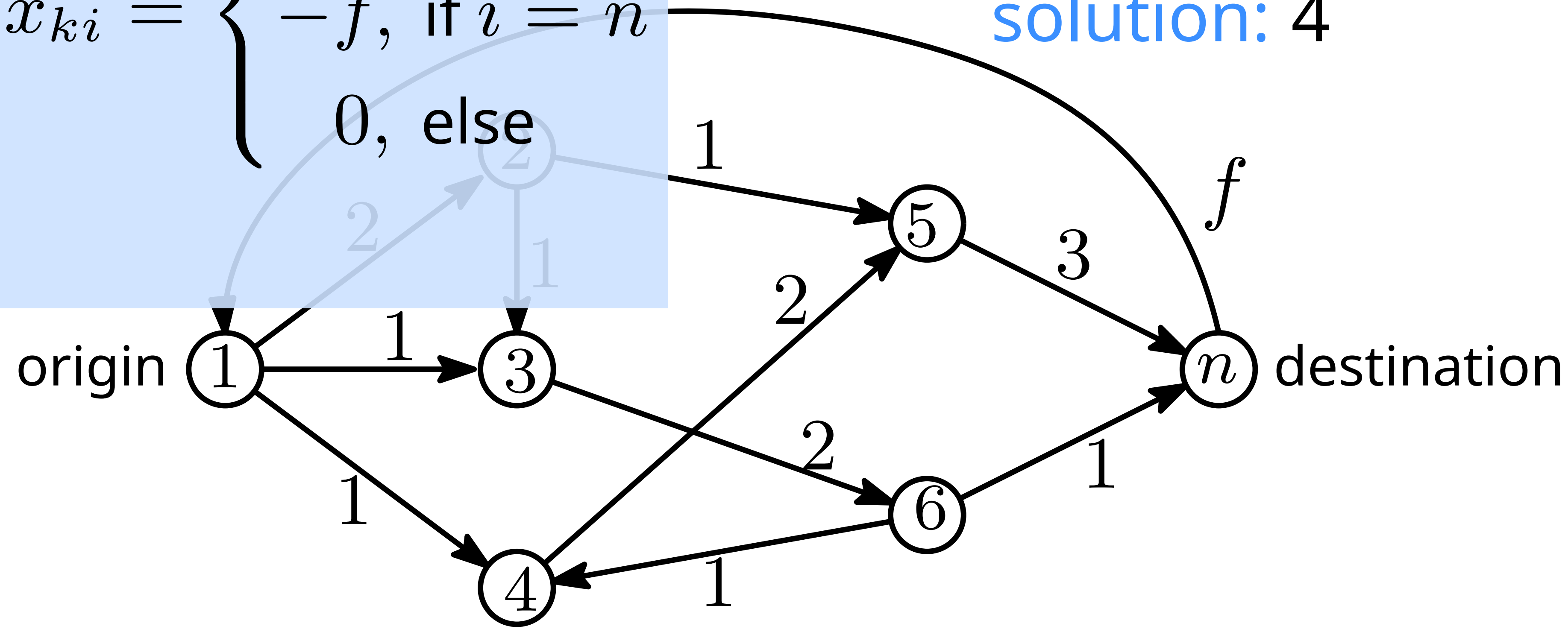
first we formulate the LP more concisely:
let the vertices be numbered $1, \dots, n$, let c_{ij} the capacity and x_{ij} the flow on directed edge (i, j), and let f be the max flow.

maximize f

subject to $$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} = \begin{cases} f, & \text{if } i = 1 \\ -f, & \text{if } i = n \\ 0, & \text{else} \end{cases}$$

$$0 \le x_{ij} \le c_{ij}$$

Optimal solution: 4



Recall: Flow in a Network

Linear Program Formulation

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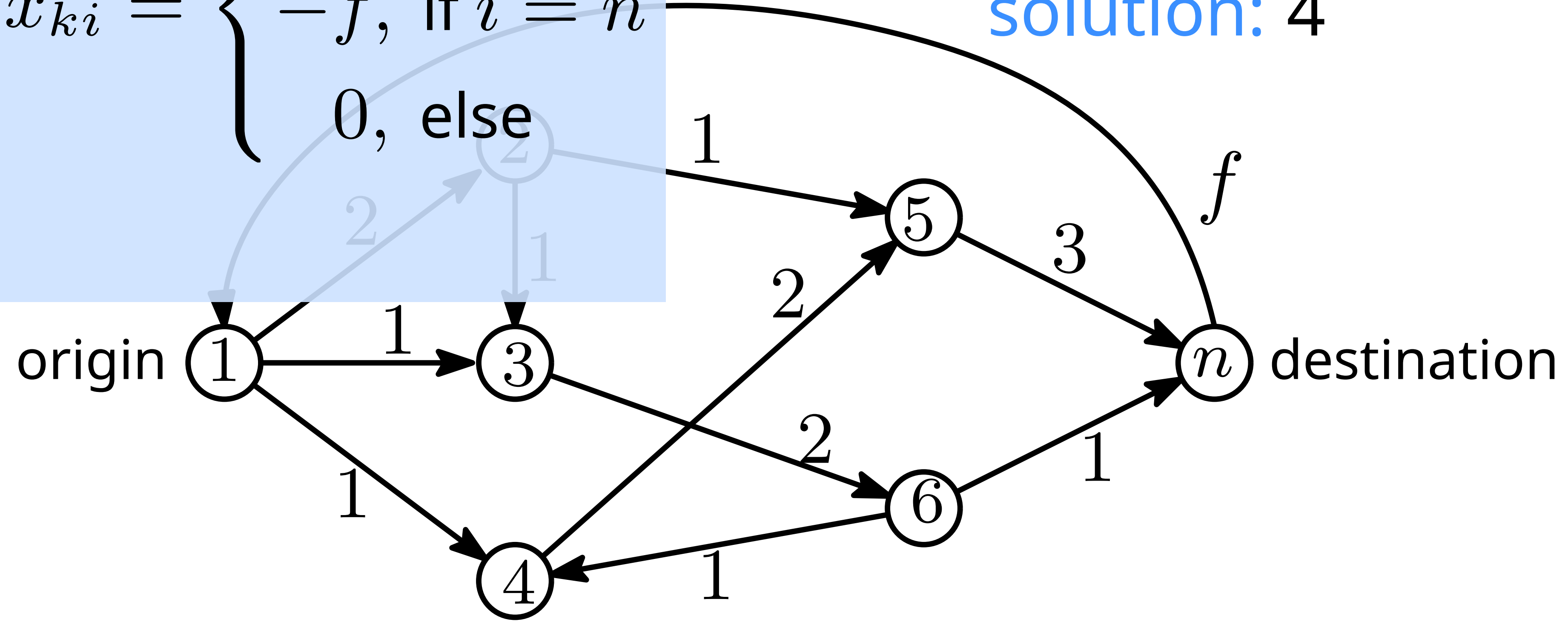
subject to $$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} \leq \begin{cases} f, & \text{if } i = 1 \\ -f, & \text{if } i = n \\ 0, & \text{else} \end{cases}$$

$$0 \leq x_{ij} \leq c_{ij}$$



Optimal solution: 4

actually we can relax the constraint without changing the optimum

Recall: Flow in a Network

Linear Program Formulation

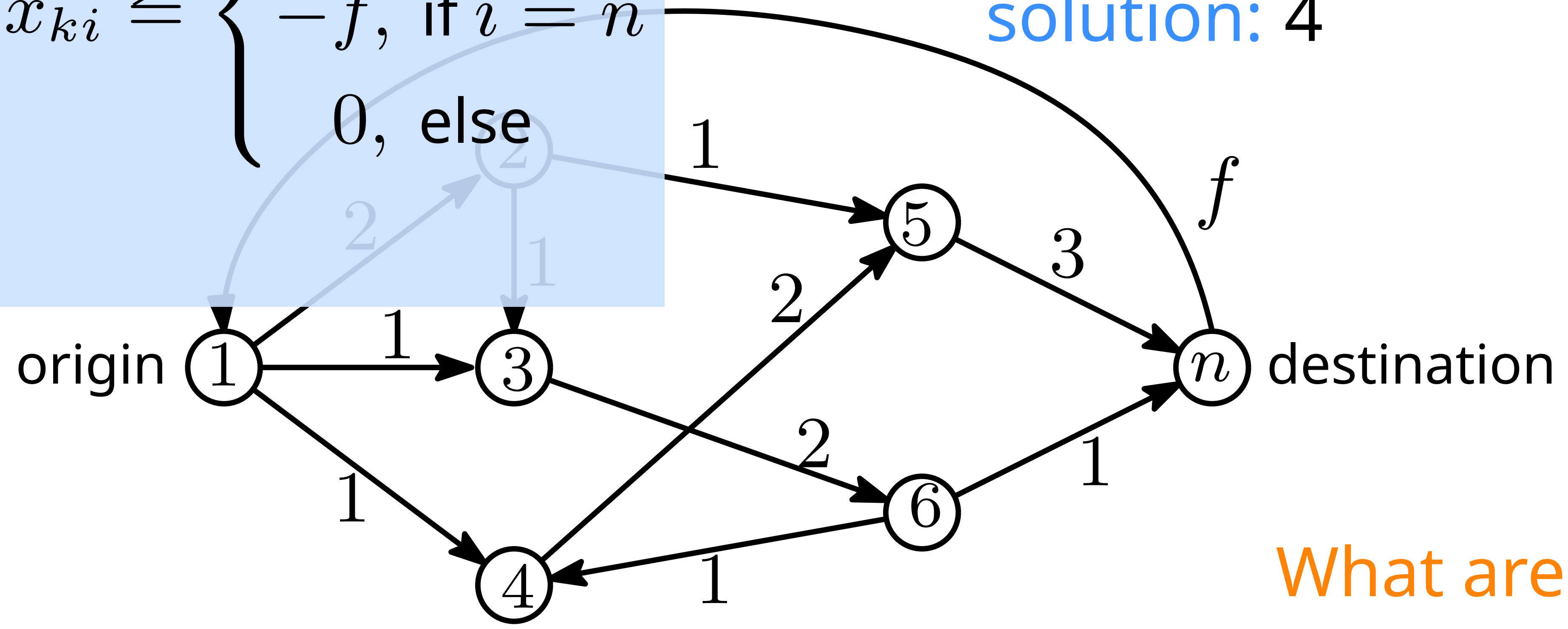
first we formulate the LP more concisely: let the vertices be numbered $1, \ldots, n$, let c_{ij} the capacity and x_{ij} the flow on directed edge (i, j), and let f be the max flow.

maximize f

subject to $$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} \leq \begin{cases} f, & \text{if } i = 1 \\ -f, & \text{if } i = n \\ 0, & \text{else} \end{cases}$$

$$0 \leq x_{ij} \leq c_{ij}$$

Optimal solution: 4



What are A, x, b, c?

Let's write this in Matrix form $\max f$ subject to $Ax \leq b, x \geq 0$.

Recall: Flow in a Network

Linear Program Formulation

in Matrix form $\max f$ subject to $Ax \leq b, x \geq 0$ where

$$x = \begin{bmatrix} f \\ x_{ij} \\ .. \\ .. \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ .. \\ .. \end{bmatrix} \quad A = \left[\begin{array}{c|cccc} -1 & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 1 & .. & .. & .. & .. \\ \hline 0 & 1 & 0 & .. & .. \\ .. & 0 & .. & .. & .. \\ .. & .. & .. & .. & 0 \\ 0 & .. & .. & 0 & 1 \end{array}\right] \quad b = \left[\begin{array}{c} 0 \\ .. \\ .. \\ 0 \\ \hline c_{ij} \\ .. \\ .. \\ .. \end{array}\right]$$

Recall: Flow in a Network

Linear Program Formulation

in Matrix form $\max f$ subject to $Ax \leq b, x \geq 0$ where

$$x = \begin{bmatrix} f \\ x_{ij} \\ .. \\ .. \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ .. \\ .. \end{bmatrix} \quad A = \left[\begin{array}{c|ccccc} -1 & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 1 & .. & .. & .. & .. \\ \hline 0 & 1 & 0 & .. & .. \\ .. & 0 & .. & .. & .. \\ .. & .. & .. & .. & 0 \\ 0 & .. & .. & 0 & 1 \end{array}\right] \quad b = \left[\begin{array}{c} 0 \\ .. \\ .. \\ 0 \\ \hline c_{ij} \\ .. \\ .. \\ .. \end{array}\right]$$

every column contains exactly one -1 and one 1

Recall: Flow in a Network

Linear Program Formulation

in Matrix form $\max f$ subject to $Ax \leq b, x \geq 0$ where

$$x = \begin{bmatrix} f \\ x_{ij} \\ .. \\ .. \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ .. \\ .. \end{bmatrix} \quad A = \left[\begin{array}{c|ccccc} -1 & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 1 & .. & .. & .. & .. \\ \hline 0 & 1 & 0 & .. & .. \\ .. & 0 & .. & .. & .. \\ .. & .. & .. & .. & 0 \\ 0 & .. & .. & 0 & 1 \end{array}\right] \quad b = \left[\begin{array}{c} 0 \\ .. \\ .. \\ 0 \\ \hline c_{ij} \\ .. \\ .. \\ .. \end{array}\right]$$

every column contains exactly one -1 and one 1

What is the dual?

Dual of the Max Flow LP

Linear Program Formulation

in Matrix form $\min \sum y_{ij}$ subject to $y^T A \geq c, x \geq 0$ where

every row contains exactly one -1 and one 1

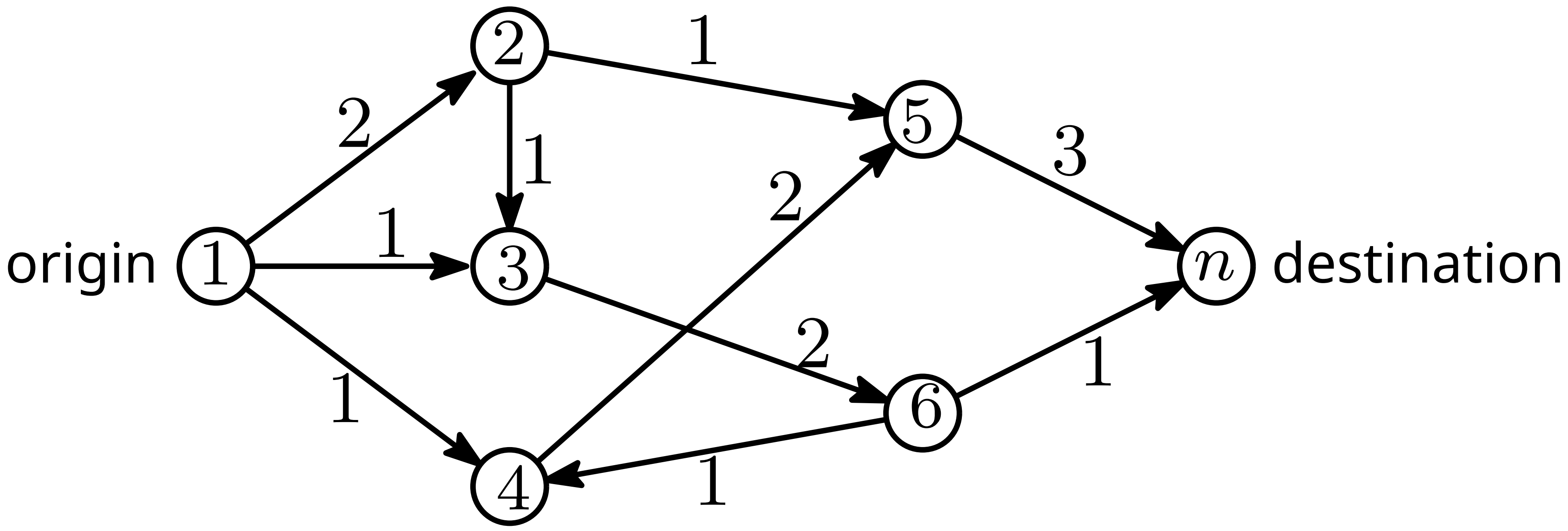
$$y = \begin{bmatrix} u_1 \\ .. \\ .. \\ u_n \\ y_{ij} \\ .. \\ .. \\ .. \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ .. \\ .. \\ 0 \\ c_{ij} \\ .. \\ .. \\ .. \end{bmatrix} \quad A^T = \left[\begin{array}{cccc|cccc} -1 & 0 & .. & 1 & 0 & .. & .. & 0 \\ \hline .. & .. & .. & .. & 1 & 0 & .. & .. \\ .. & .. & .. & .. & 0 & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & 0 & 1 \end{array}\right] \quad c = \begin{bmatrix} 1 \\ 0 \\ .. \\ .. \end{bmatrix}$$

Dual of the Max Flow LP

Linear Program Formulation

in constraint form

$$\begin{aligned}
\text{minimize} \quad & \sum c_{ij} y_{ij} \\
\text{subject to} \quad & -u_1 + u_n \geq 1 \\
& u_i - u_j + y_{ij} \geq 0 \\
& u_i \geq 0 \\
& y_{ij} \geq 0
\end{aligned}$$

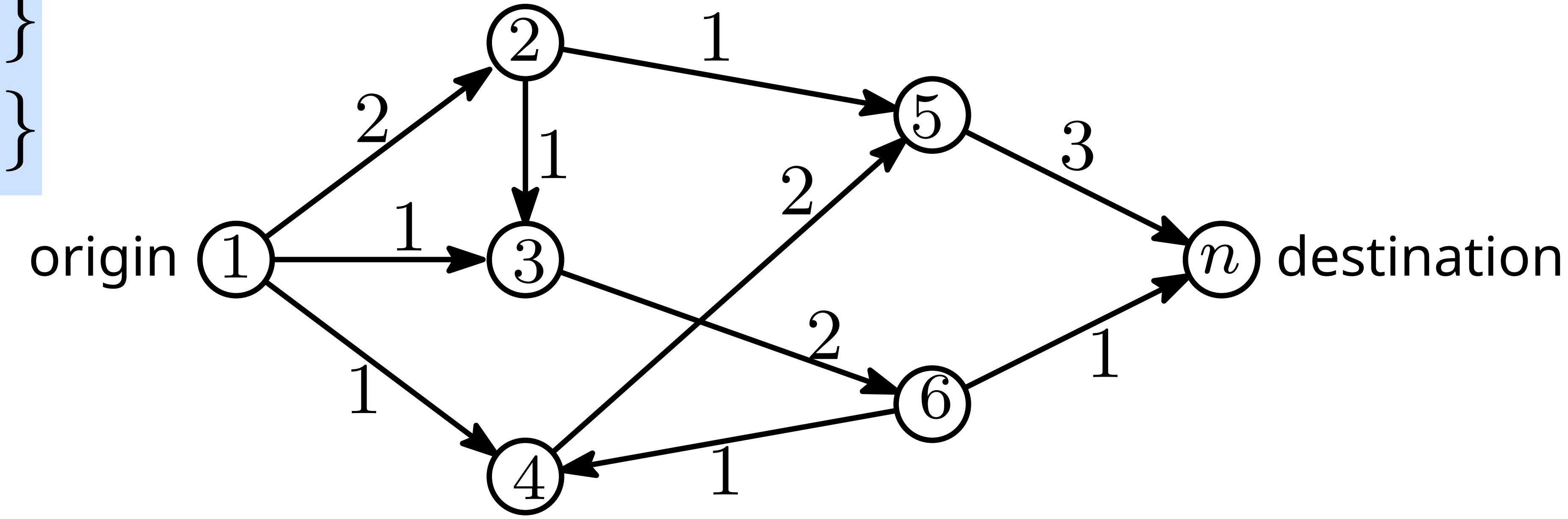


Dual of the Max Flow LP

Linear Program Formulation

in constraint form

$$\begin{aligned} \text{minimize} \quad & \sum c_{ij} y_{ij} \\ \text{subject to} \quad & -u_1 + u_n \geq 1 \\ & u_i - u_j + y_{ij} \geq 0 \\ & u_i \in \{0,1\} \\ & y_{ij} \in \{0,1\} \end{aligned}$$



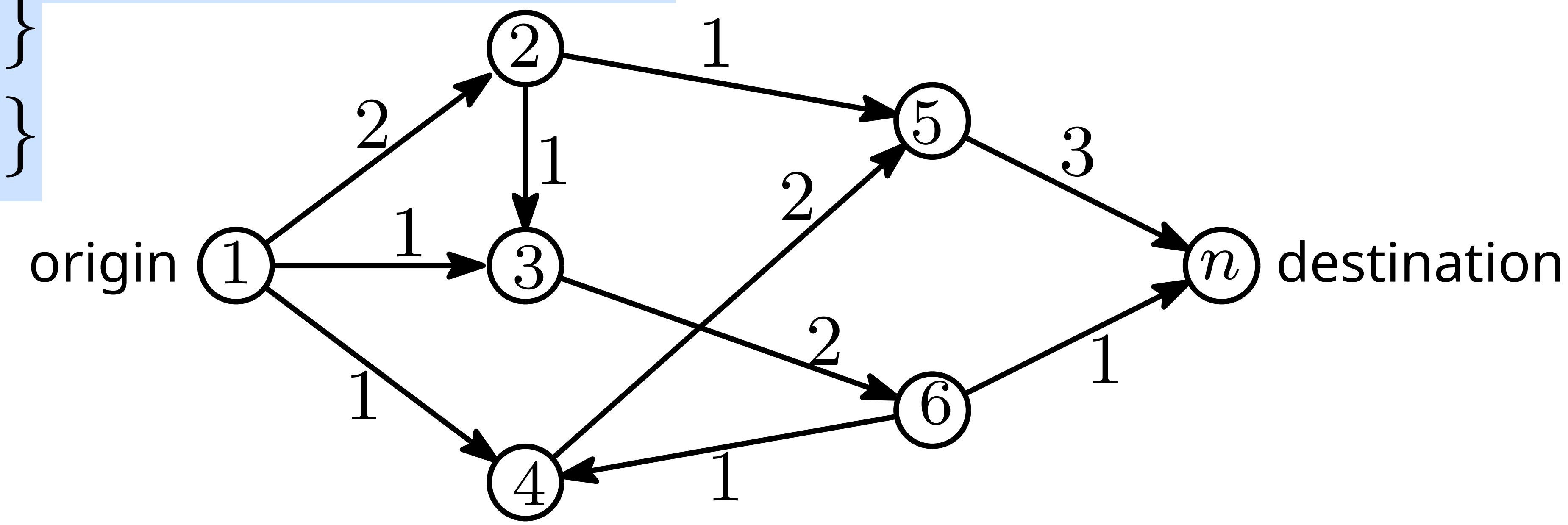
actually, we can restrict all variables to be integer, even 0-1

Dual of the Max Flow LP

Linear Program Formulation

in constraint form

$$\begin{aligned} \text{minimize} \quad & \sum c_{ij} y_{ij} \\ \text{subject to} \quad & -u_1 + u_n \geq 1 \\ & u_i - u_j + y_{ij} \geq 0 \\ & u_i \in \{0, 1\} \\ & y_{ij} \in \{0, 1\} \end{aligned}$$



actually, we can restrict all variables to be integer, even 0-1 $\rightarrow$ Min Cut