Line Segment Intersection

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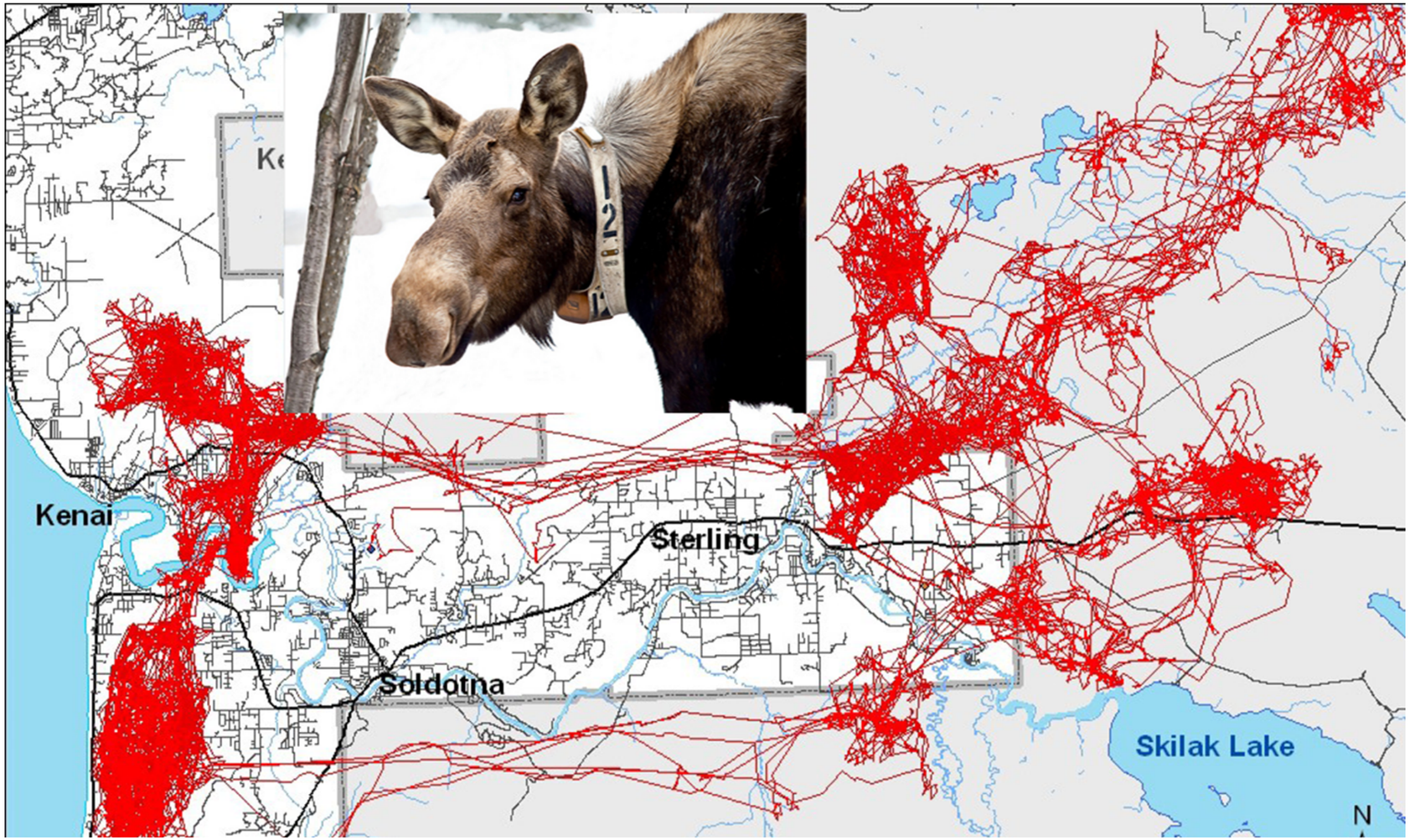
Algorithmic Problem: Computing intersections in a set of line segments

Algorithmic Technique: Plane Sweep

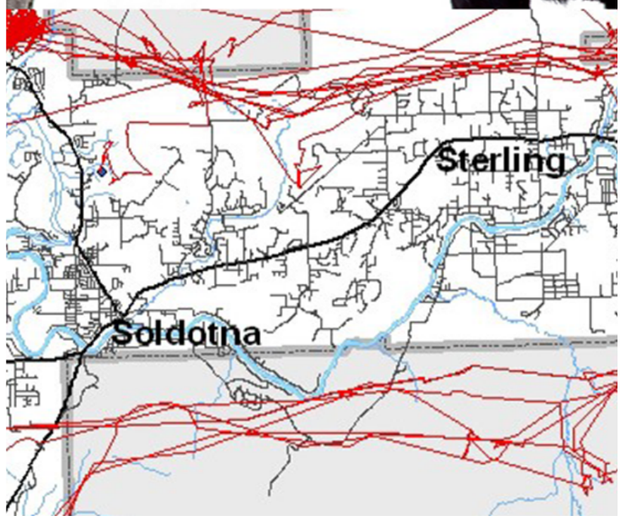
Data Structure: How to store a planar subdivision

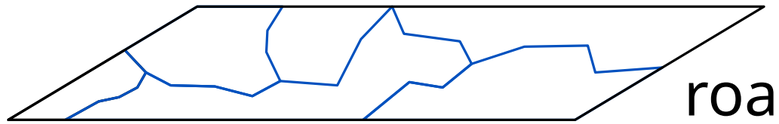
Motivation

Where did the moose cross the road?



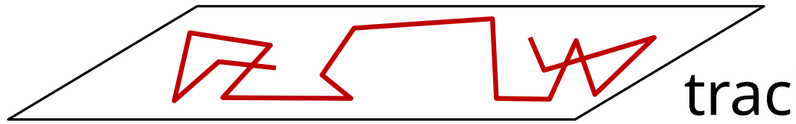
Motivation: Map overlay





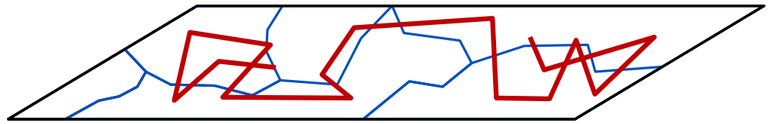
road network

+



track

=



combination of layers

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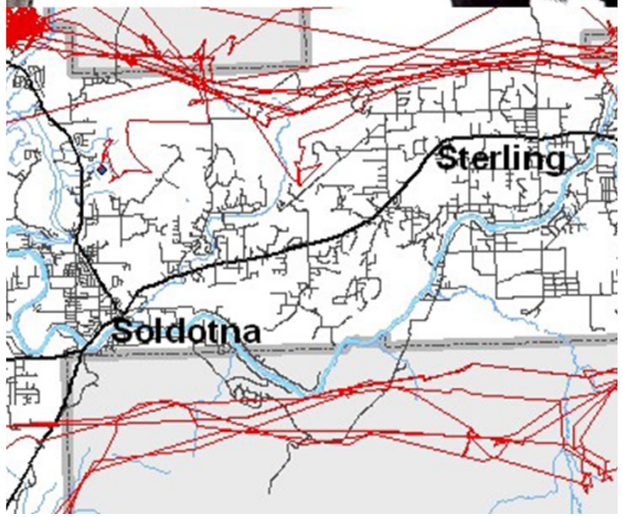

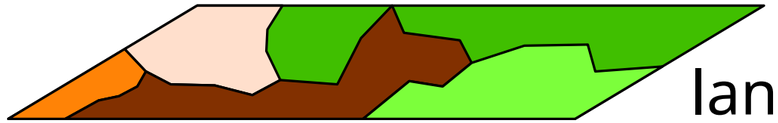

land use

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rain fall

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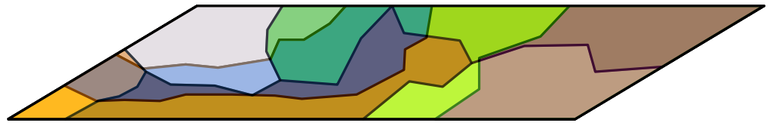
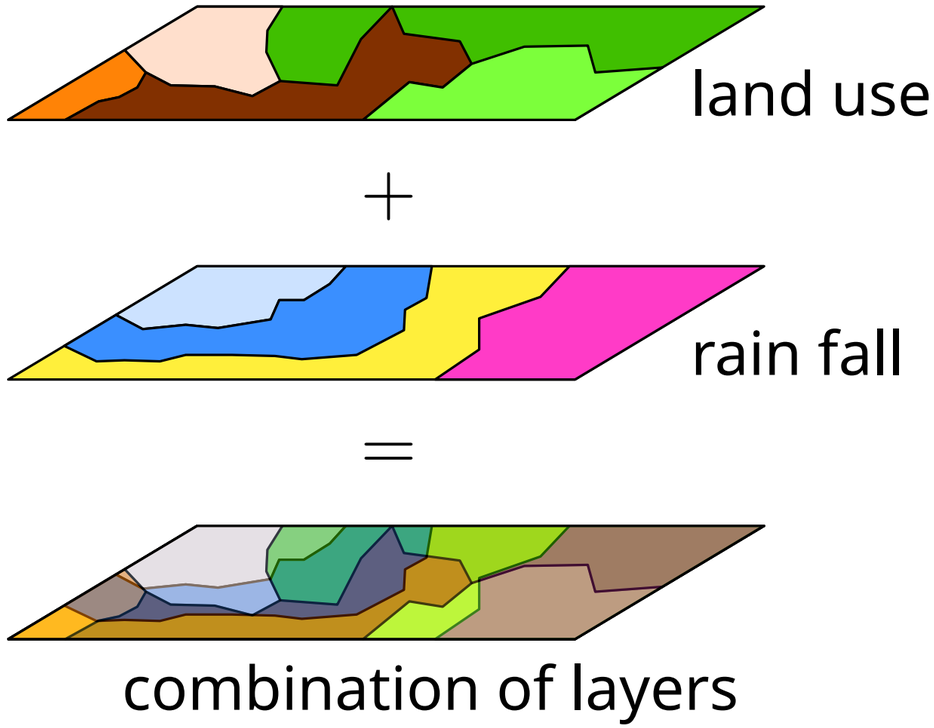


combination of layers

Motivation: Map overlay

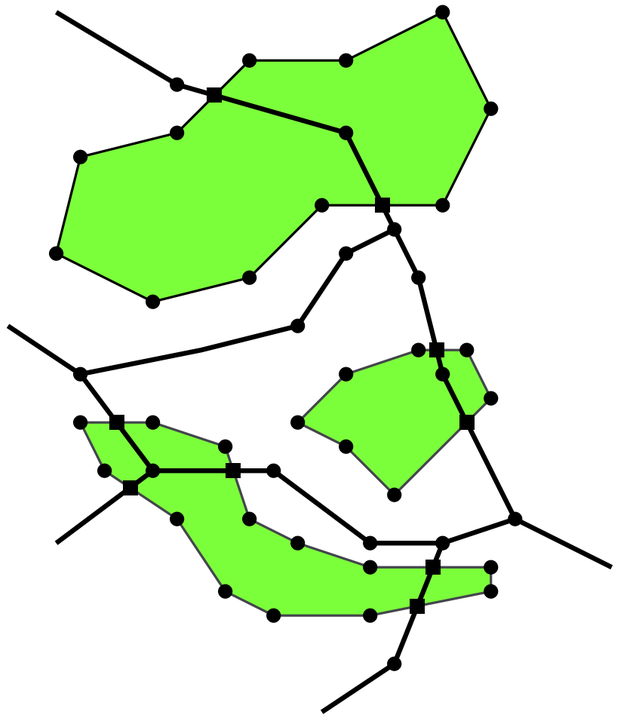
In a geographic information system (GIS) data is stored in separate layers

A layer stores information about particular theme, like land cover, road network, municipality boundaries, …



Motivation: Map overlay

Map overlay: combination of two (or more) map layers



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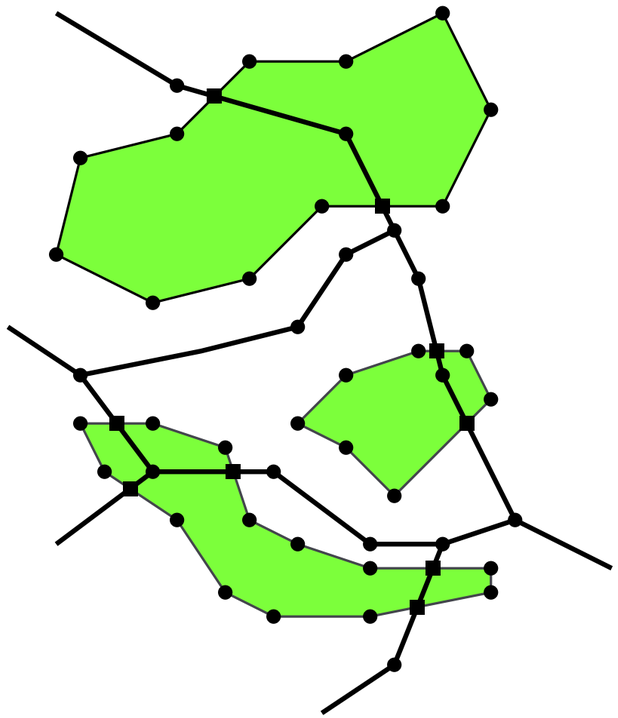
Map overlay: combination of two (or more) map layers

Example:

What is the total length of roads through forests?



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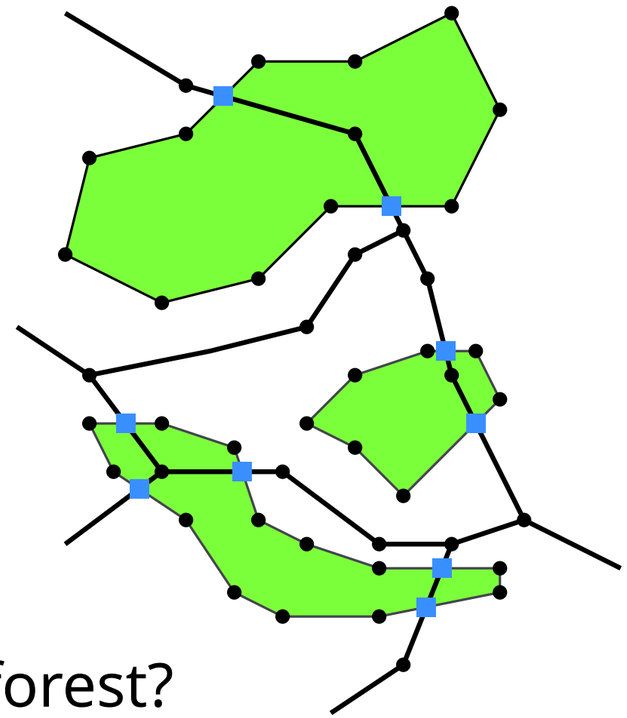

Example:

What is the total length of roads through forests?

First step:

Where does the road enter/exit the forest?

$\rightarrow$ intersections of a set of line segments

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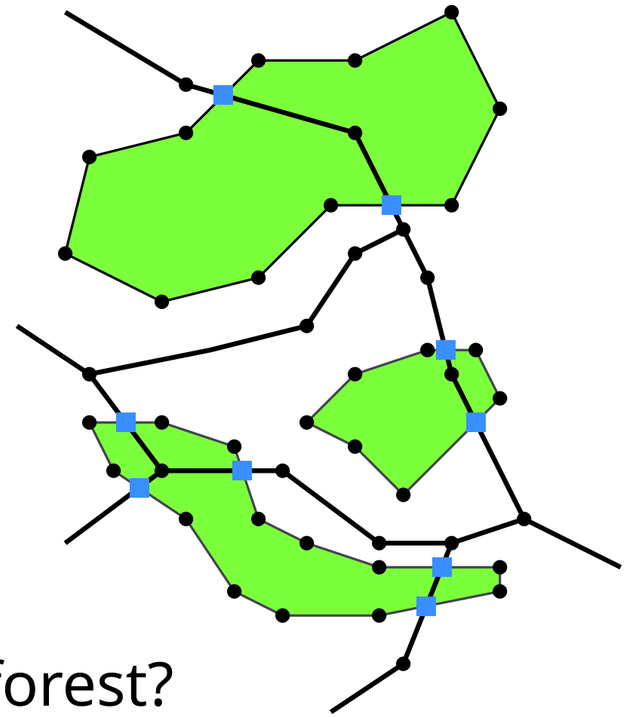

First step:

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→ intersections of a set of line segments

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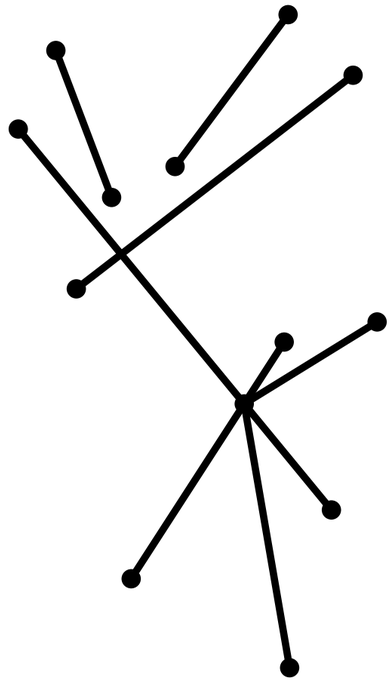
Given: Set $S = \{s_1, \ldots, s_n\}$ of line segments in the plane



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Find:
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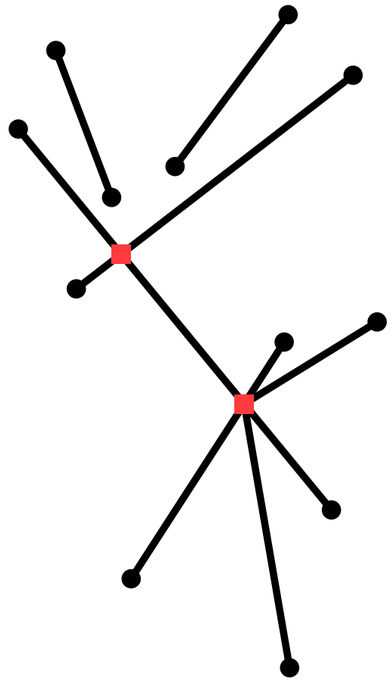

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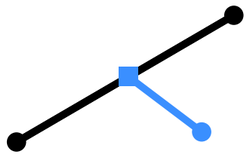
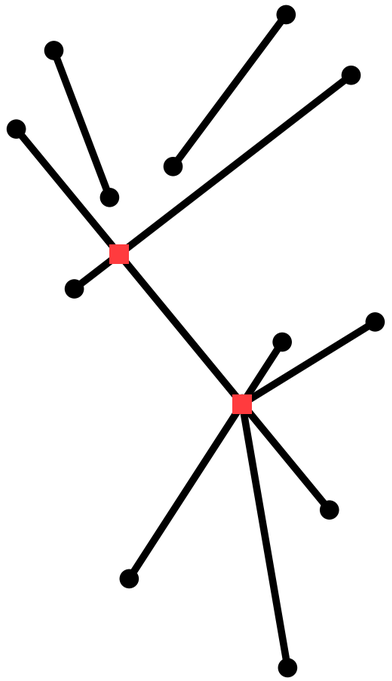
Def: line segments are **closed** sets





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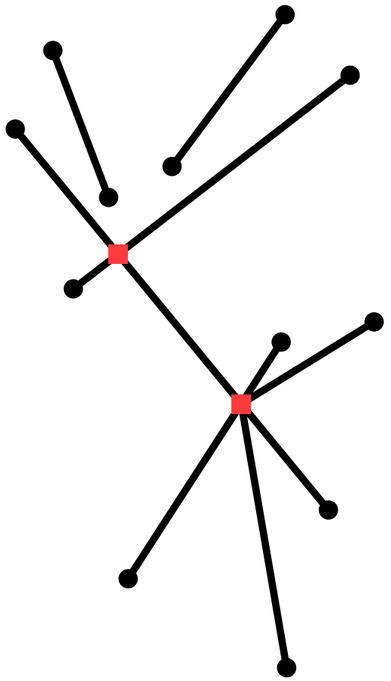

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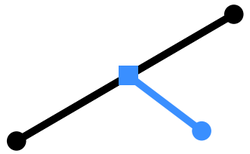

Discussion:
- simple solution?
- worst-case optimality?

A simple, optimal algorithm?

Algorithm FINDINTERSECTIONS(S)

Input: set S of line segments in the plane

Output: set of intersection points among the segments in S

```
1: for each pair of line segments e_i, e_j ∈ S do
2:     if e_i and e_j intersect then
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running time: $\Theta(n^2)$

correctness: all possibilities checked

Quiz

Is this algorithm wost-case optimal?

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A: yes

B: no

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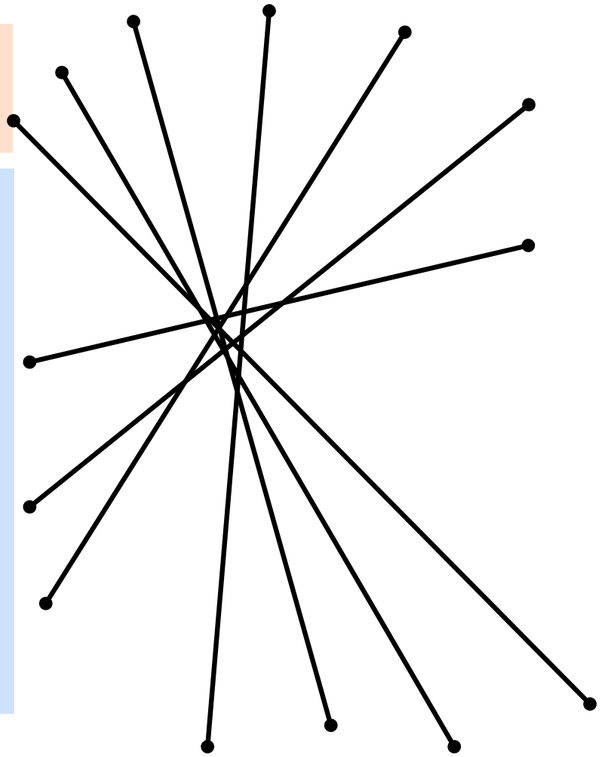
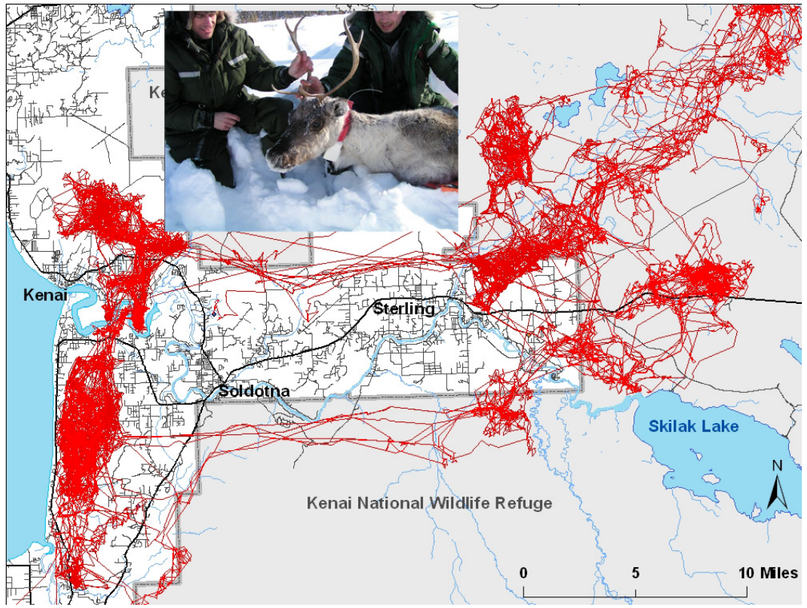


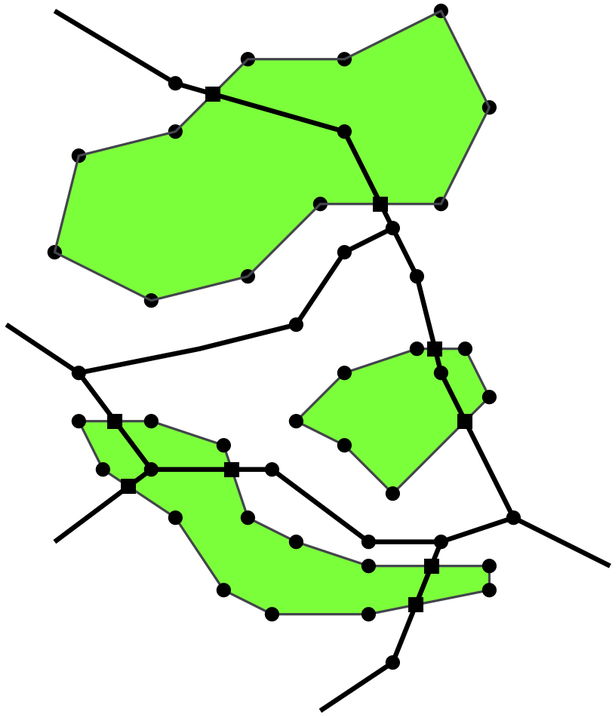
A: yes

B: no

Intersection points in practice

How many intersection points do we typically expect in these applications?





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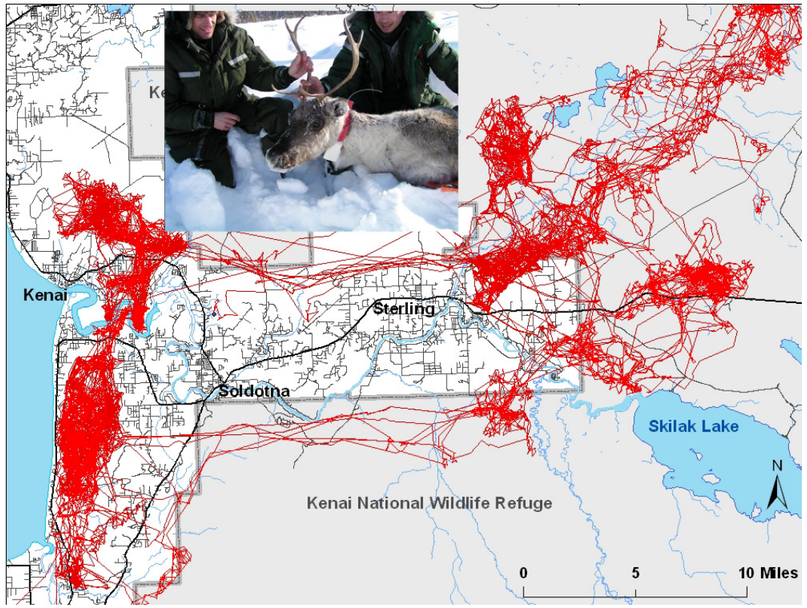
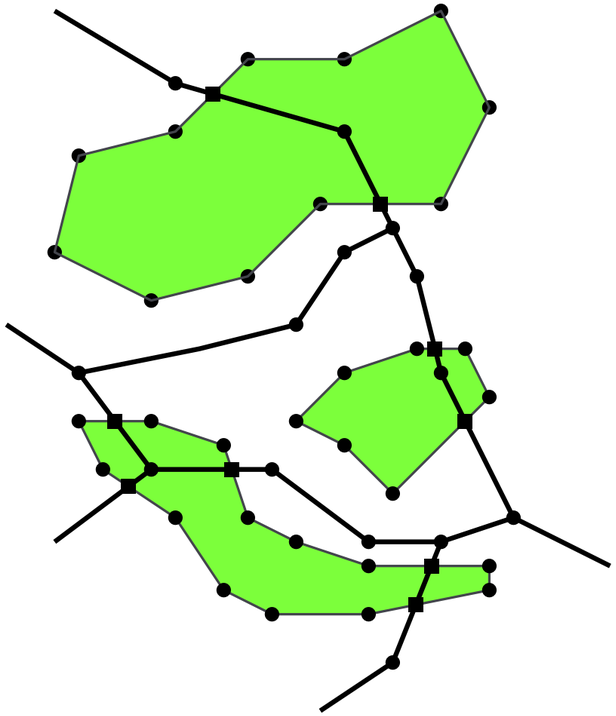
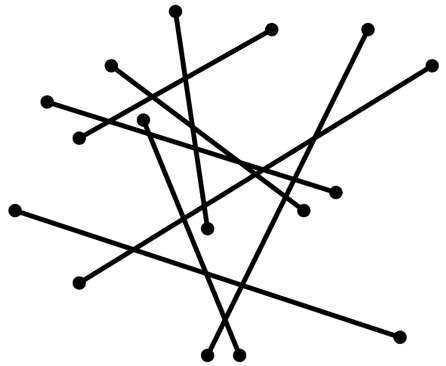




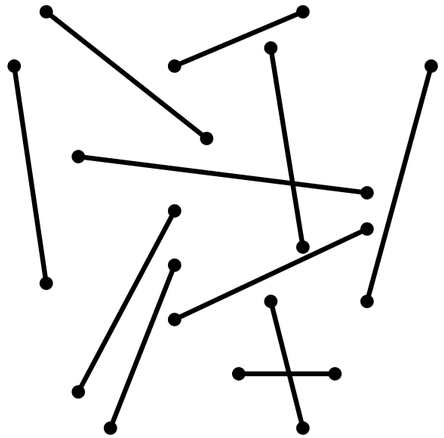
If this number is k, and if $k = O(n)$, it would be nice if the algorithm runs in $O(n \log n)$ time

Output-sensitive algorithm

asymptotic running time is always input-sensitive: depends on n



We may also want output-sensitive: algorithm faster if output size k is small



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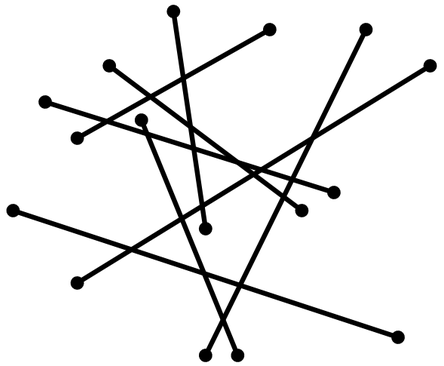

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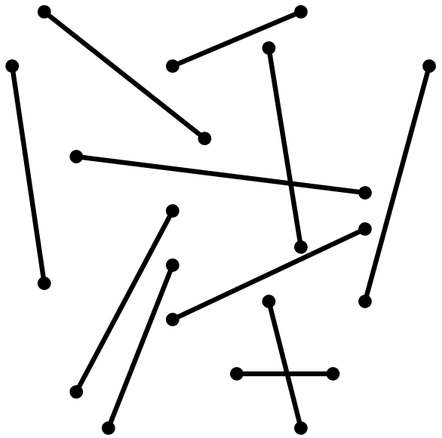


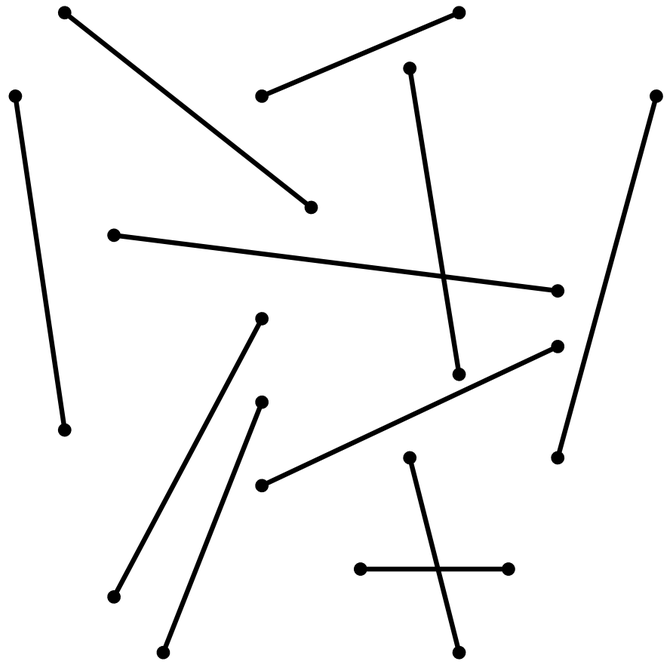
next: designing an output-sensitive algorithm for line segment intersection

Plane Sweep Algorithm for Line Segment Intersection

observations and concepts

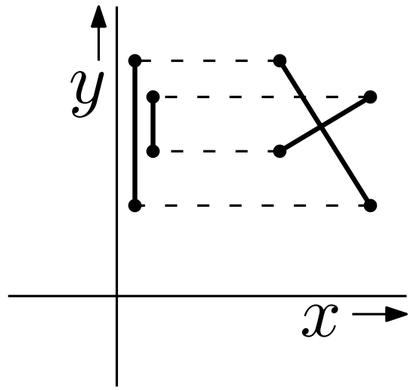
Developing Geometric Algorithms

To develop an algorithm, find useful properties, make various observations, draw many sketches to gain insight.



First attempt

Observation: Two line segments can only intersect if their y-spans have an overlap



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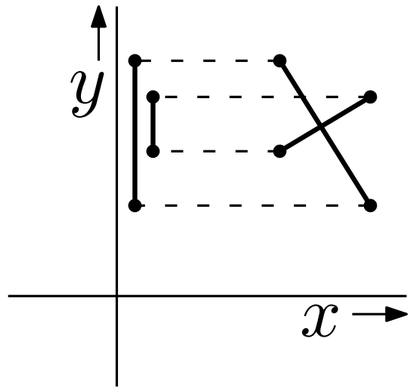
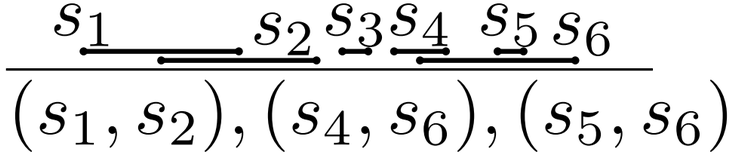


1D problem: Given a set of intervals on the real line, find all partly overlapping pairs



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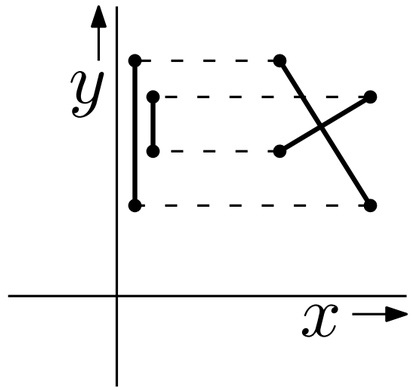
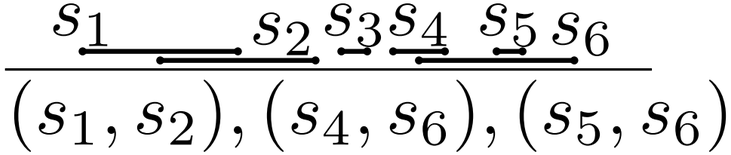
Only test all pairs of line segments with overlapping y-coordinates for intersections



A: yes, but it is not correct



B: no, but testing segments with overlapping x- and overlapping y-coordinates would be.

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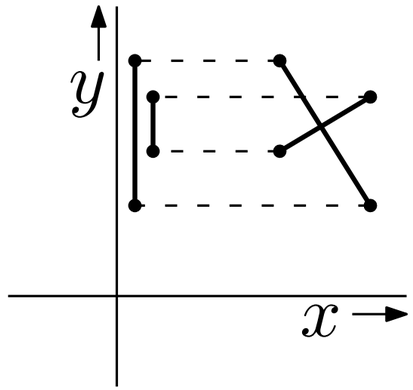

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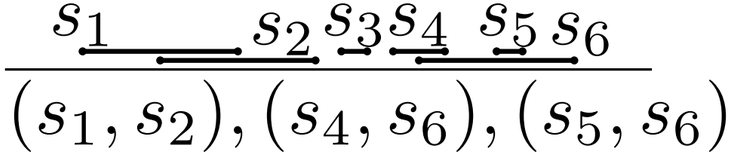

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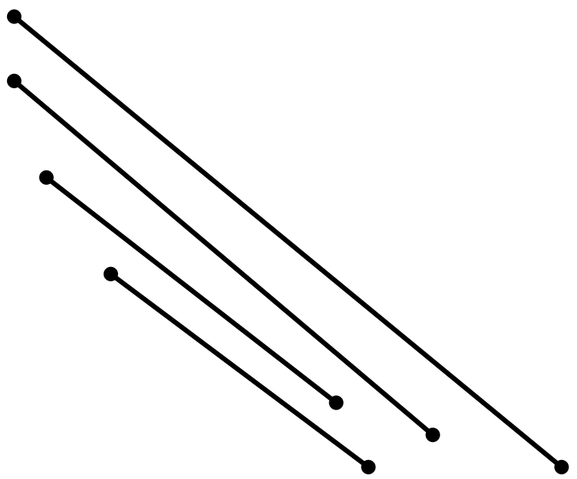

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Second attempt

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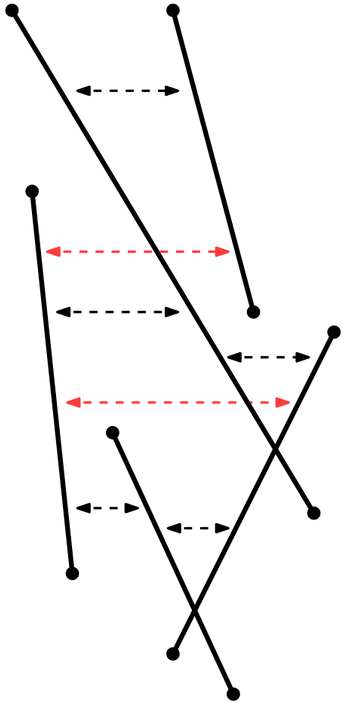

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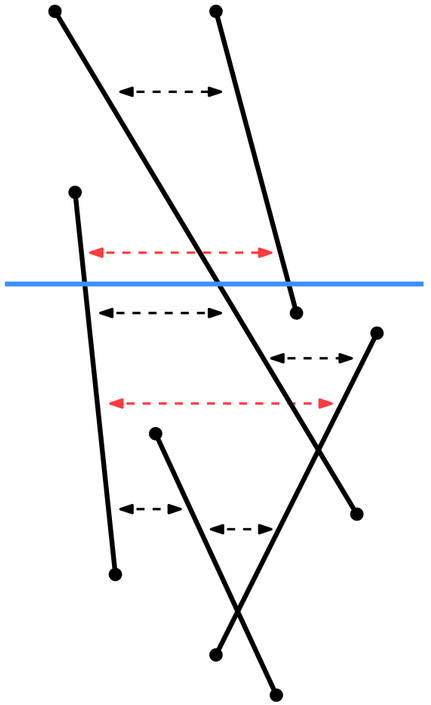
For any y-value: one-dimensional problem



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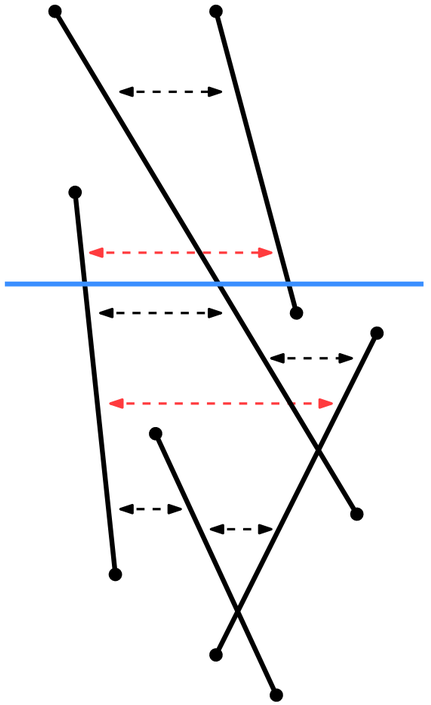

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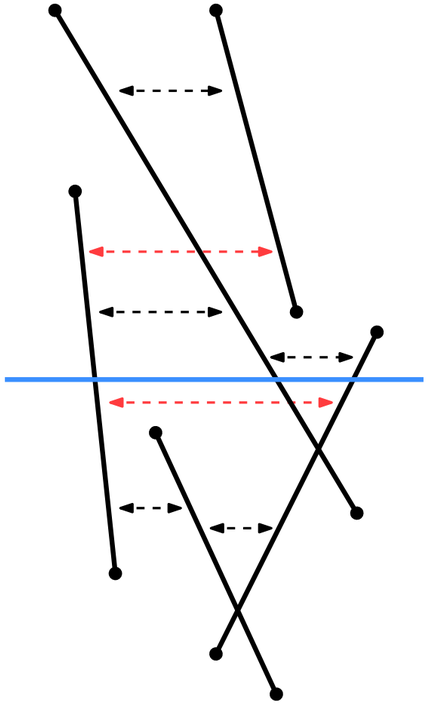

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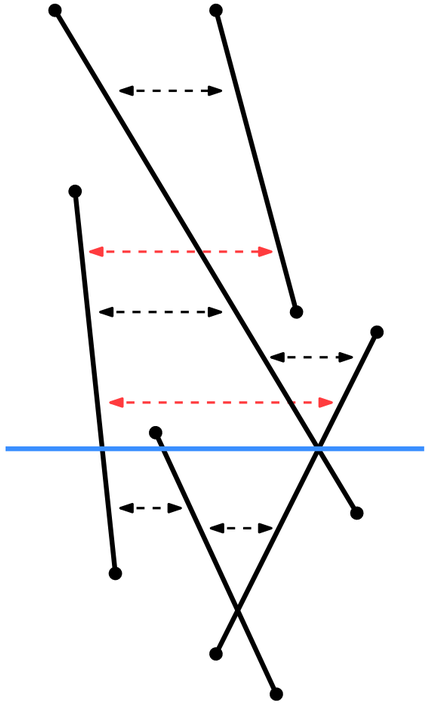

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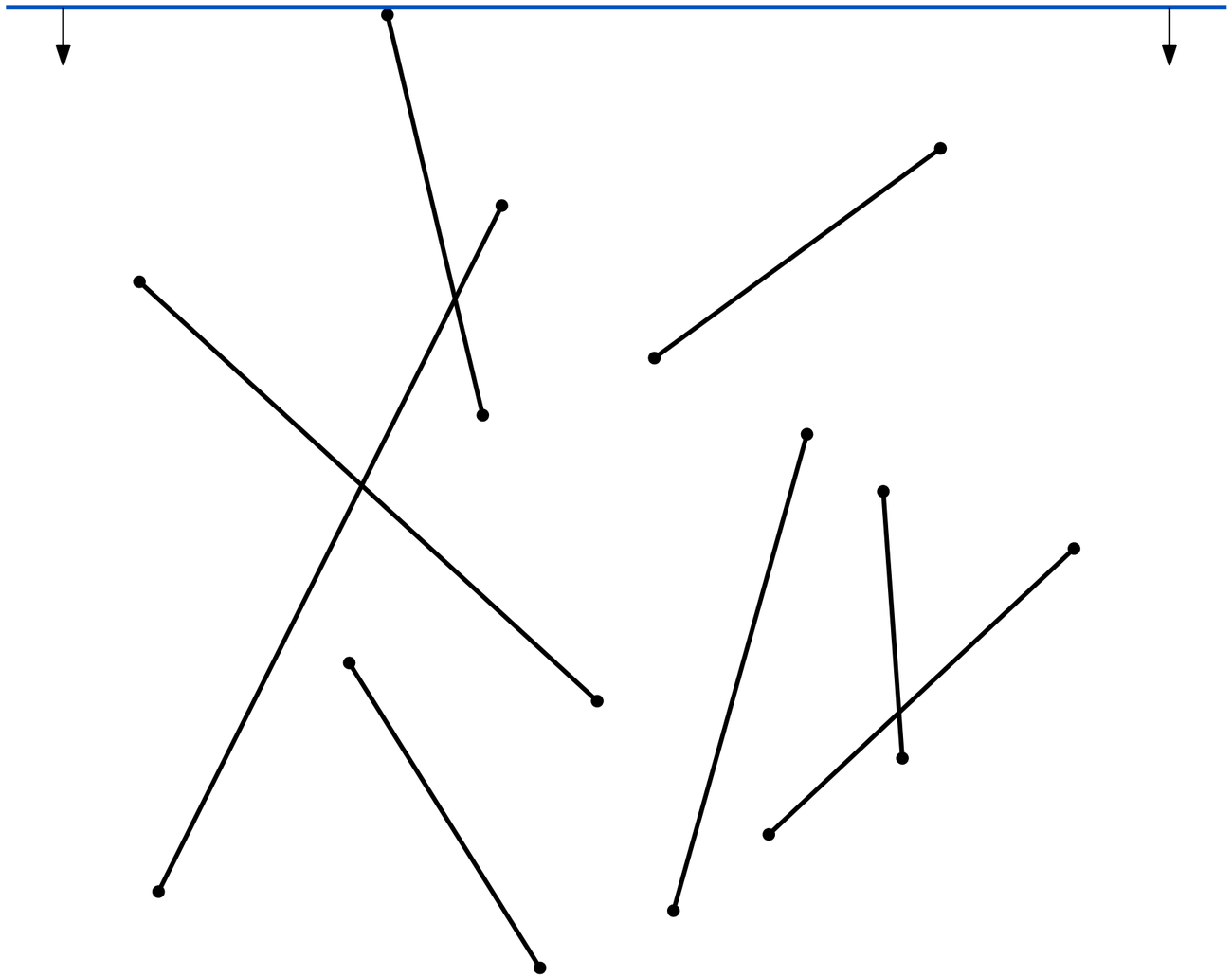
Sweep



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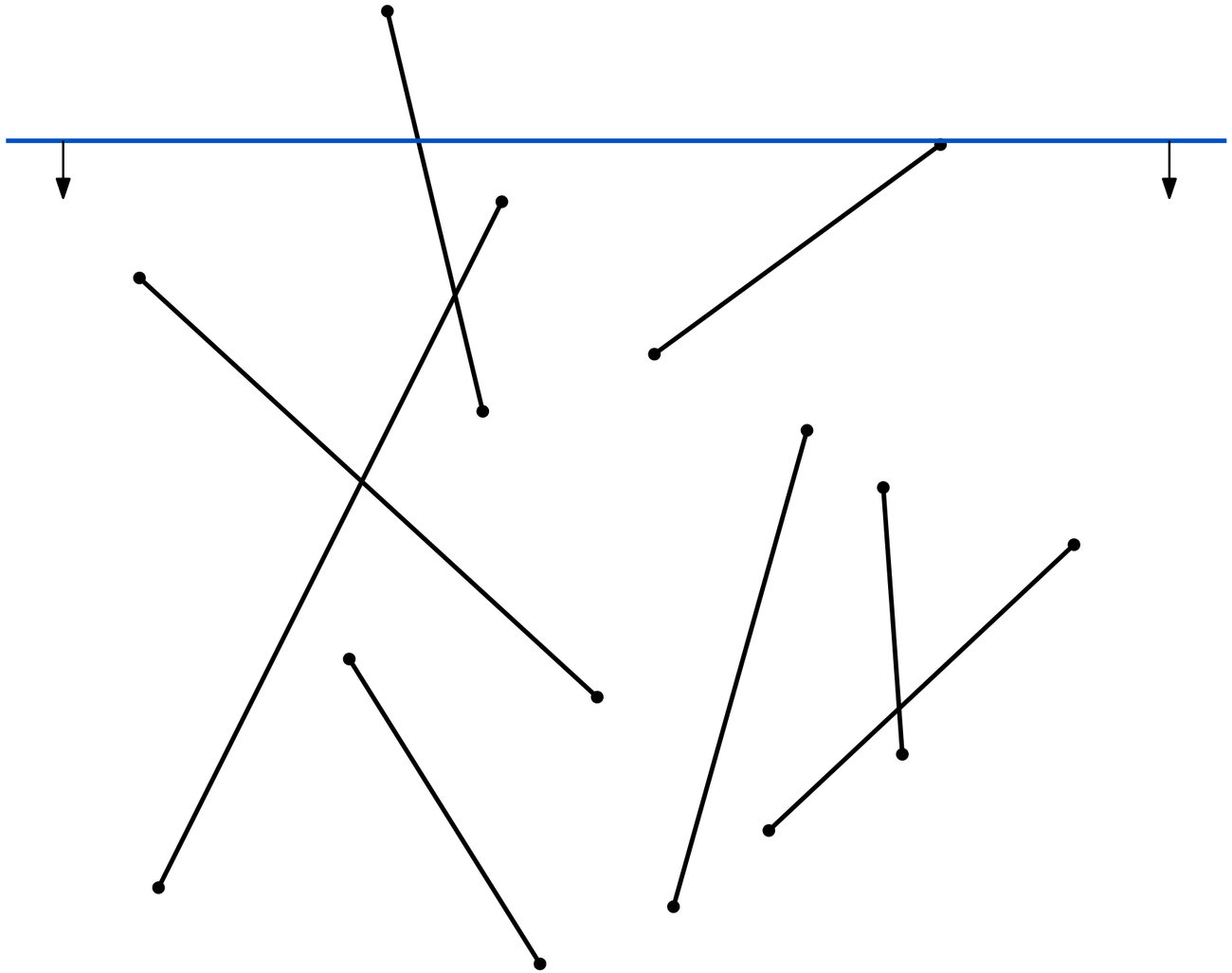

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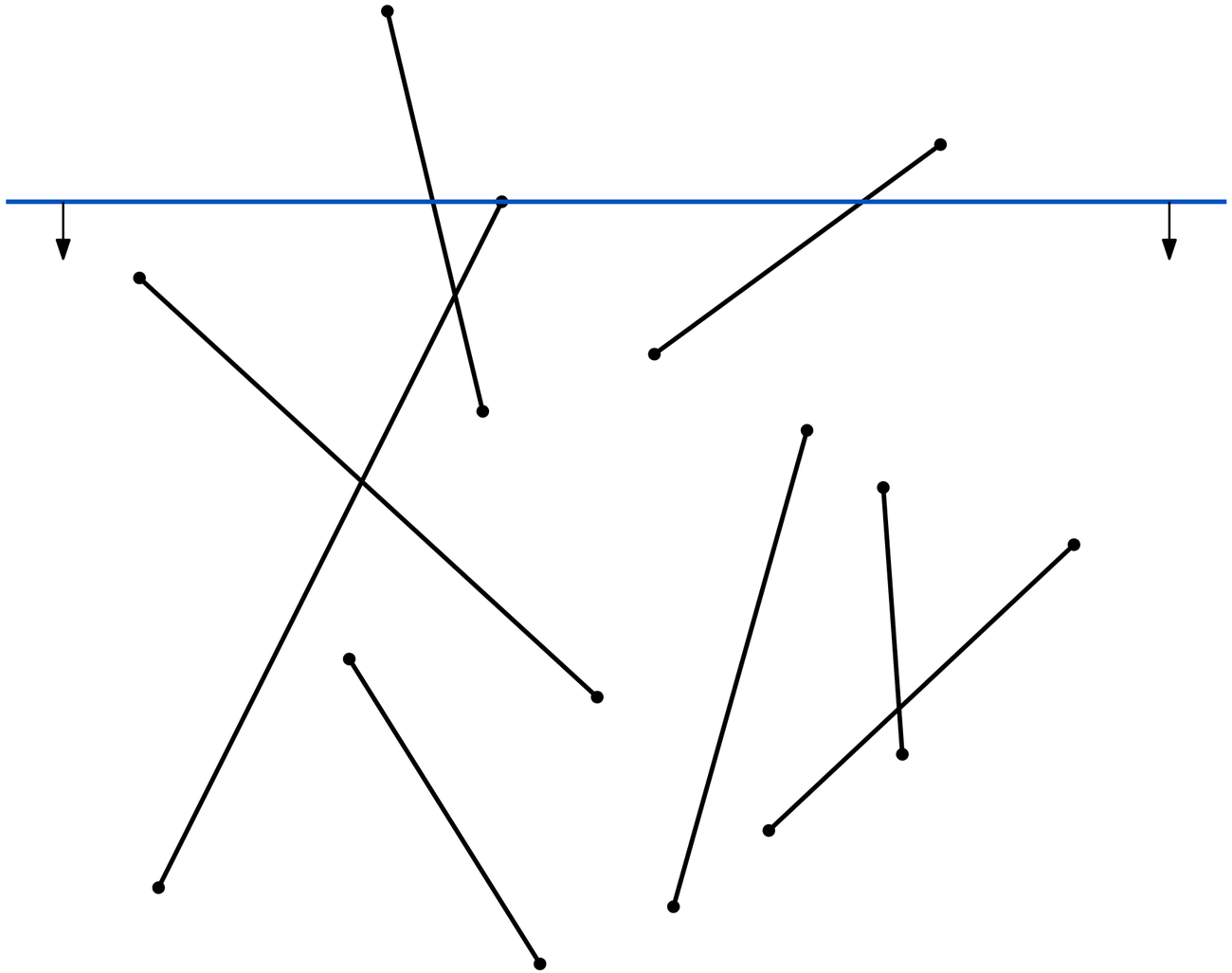

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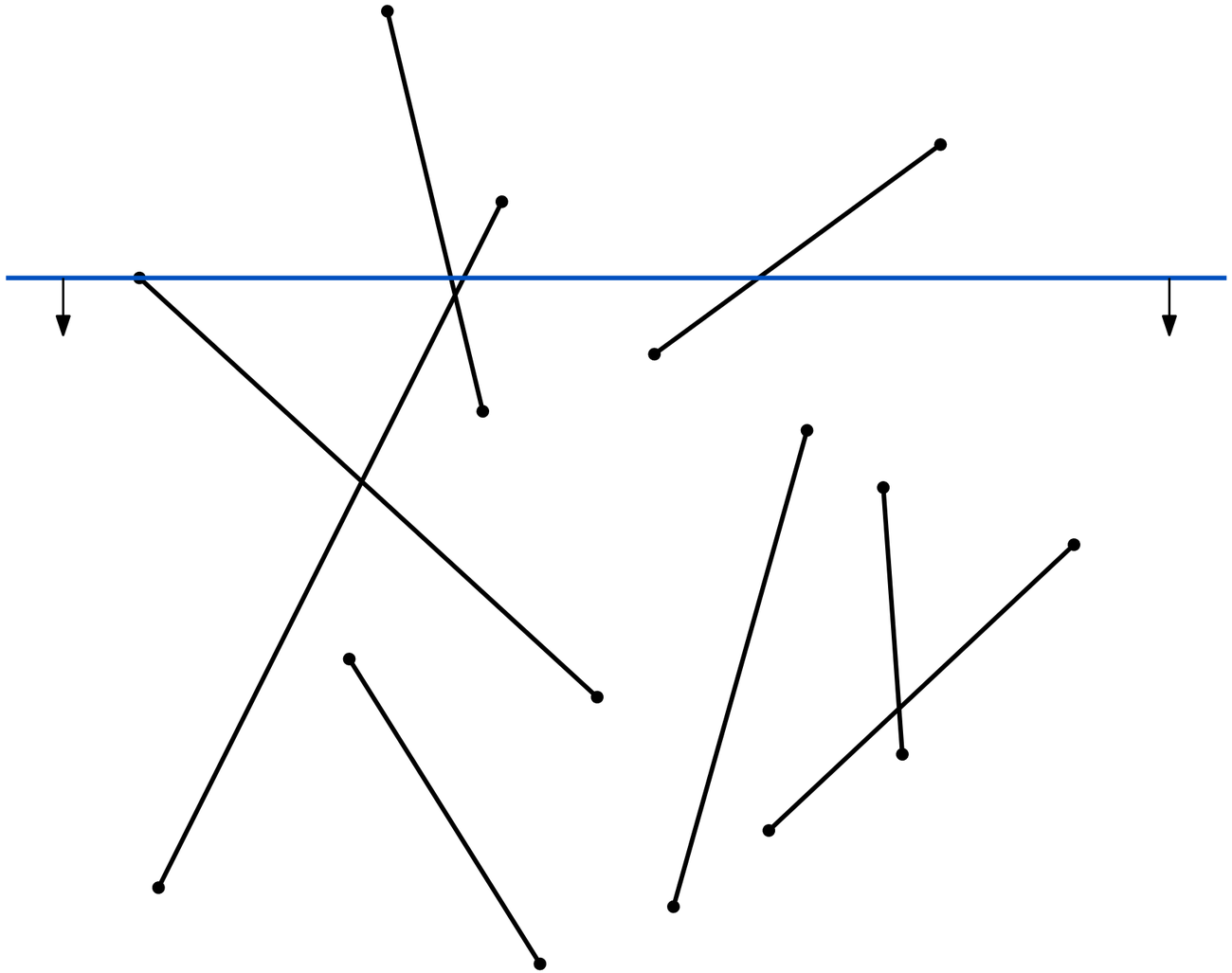

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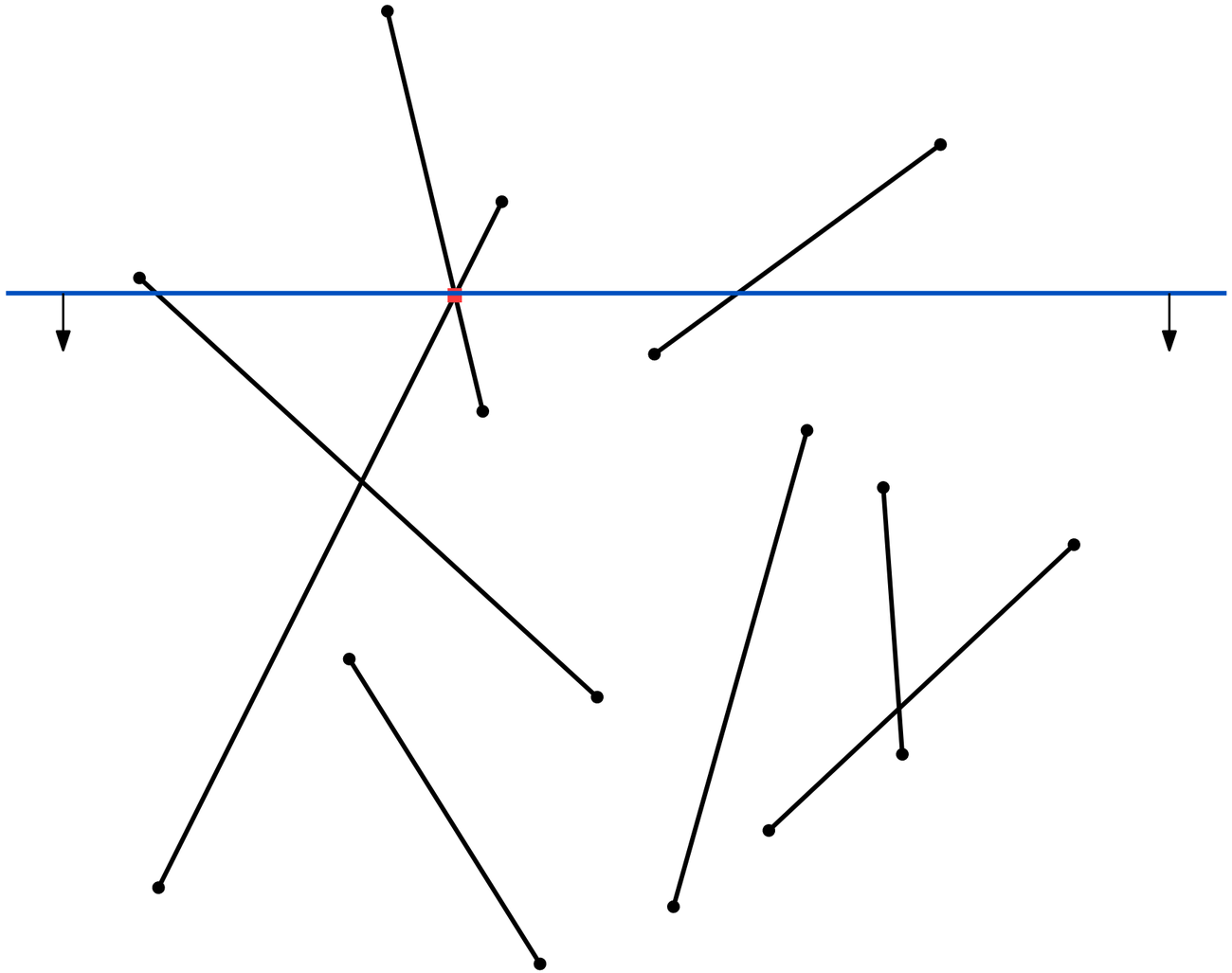

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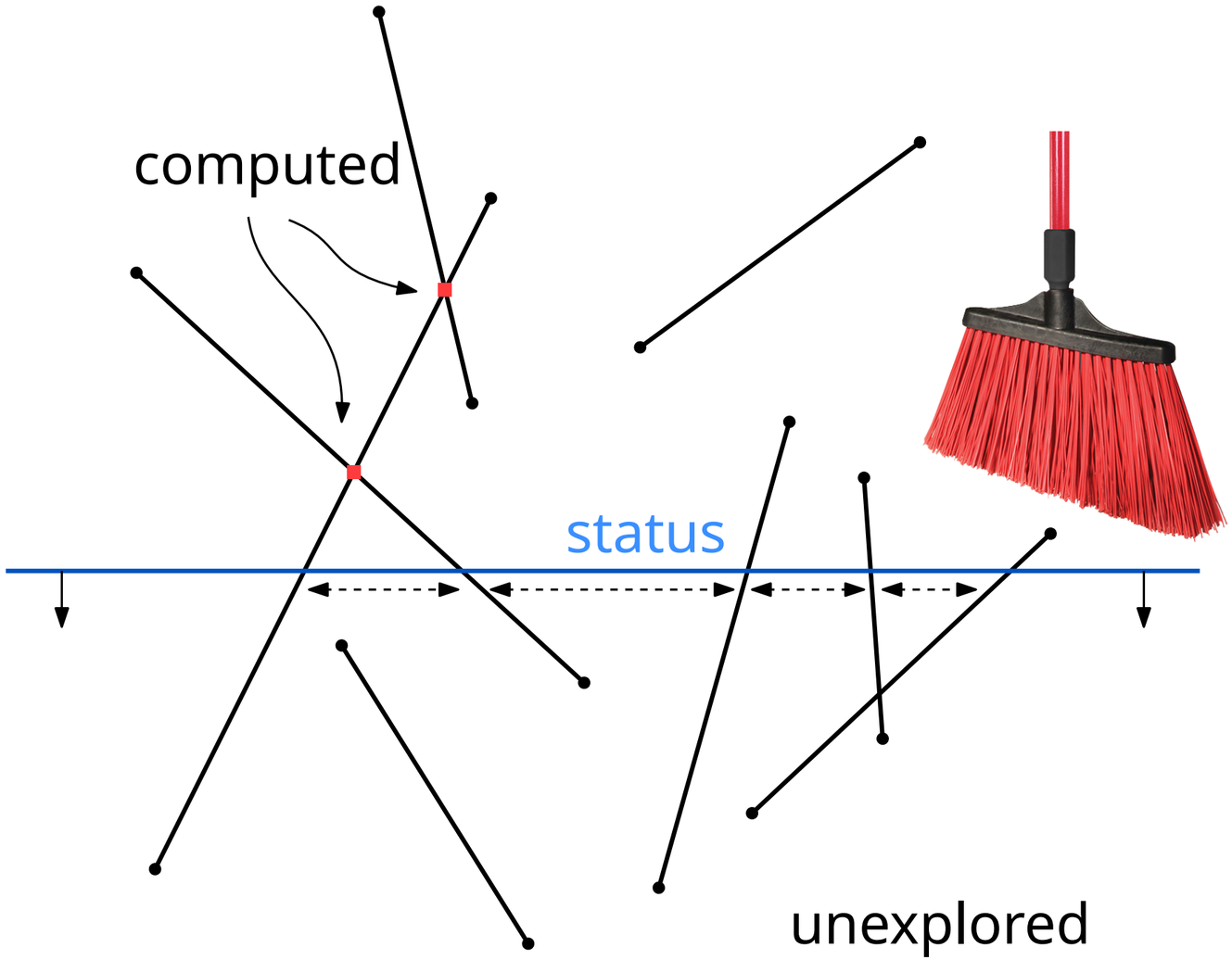
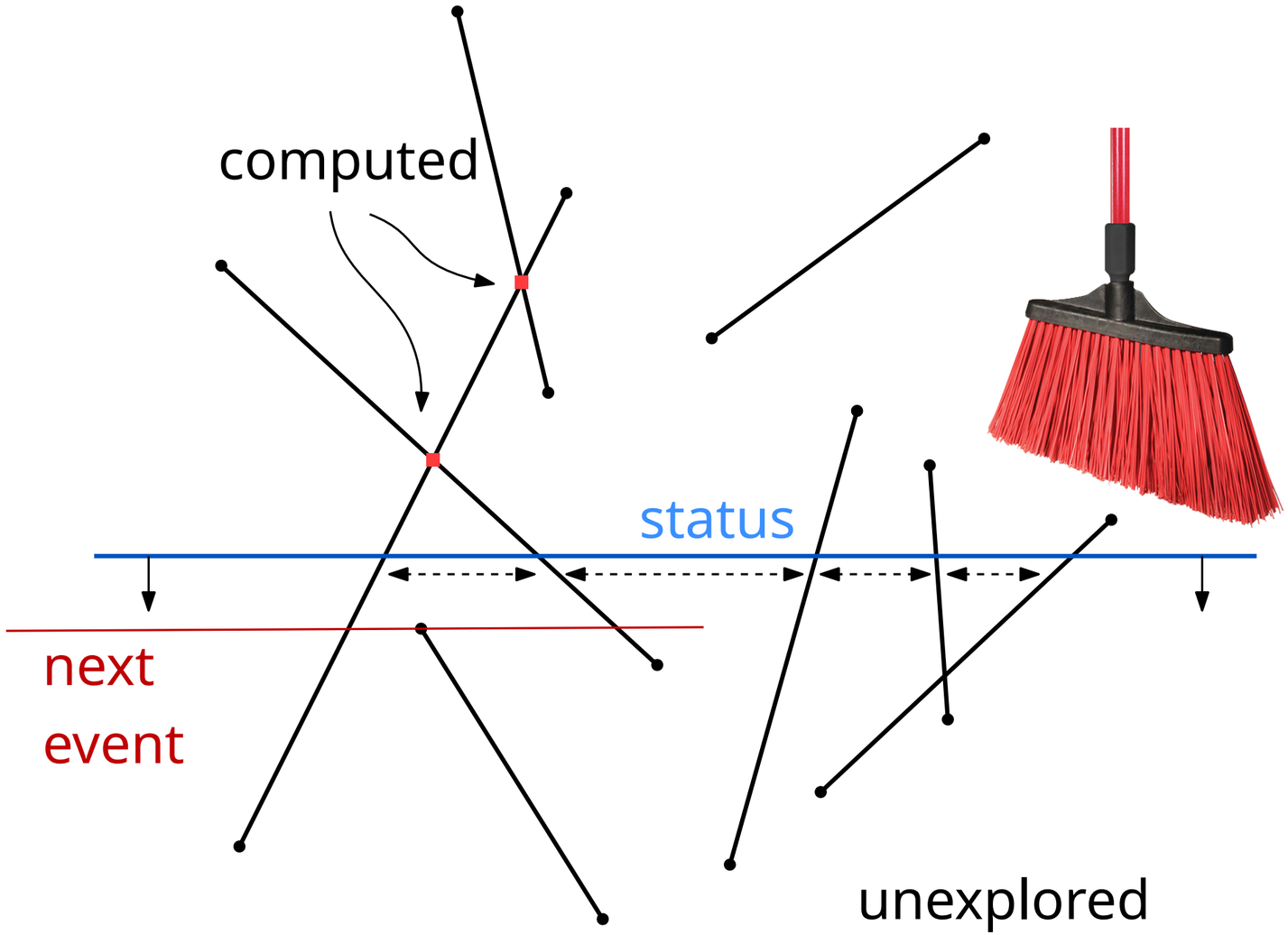
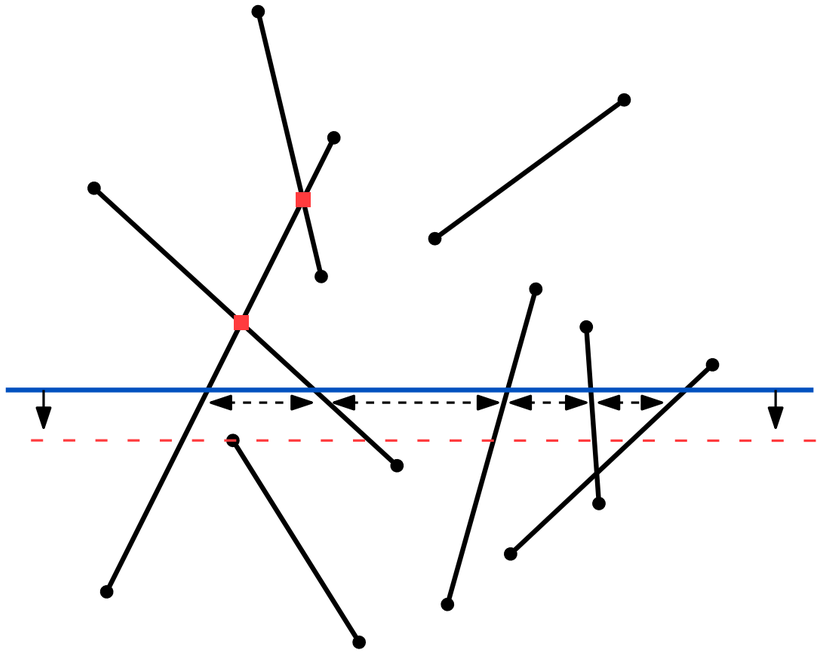


Sweep



Plane sweep

The plane sweep technique: Imagine a horizontal line passing over the plane from top to bottom, solving the problem as it moves



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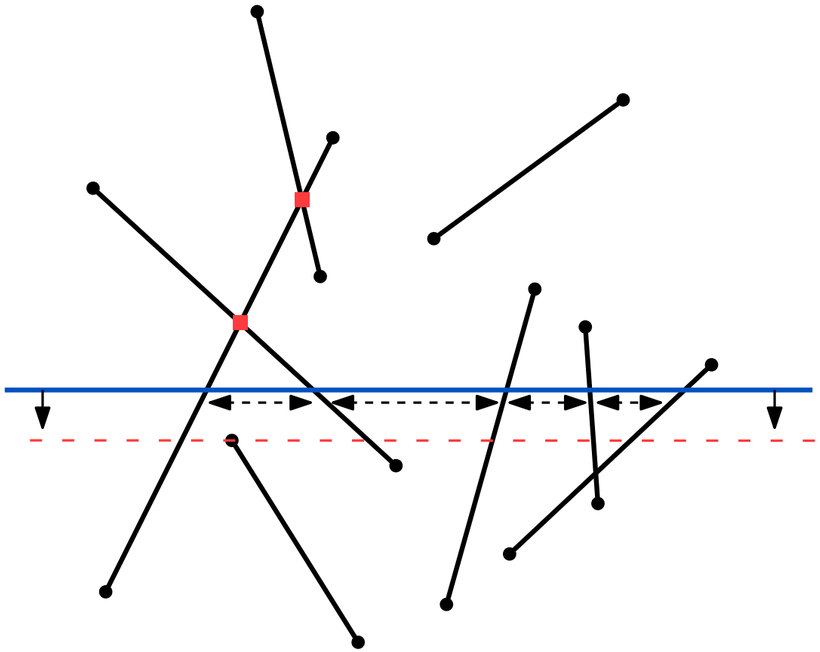

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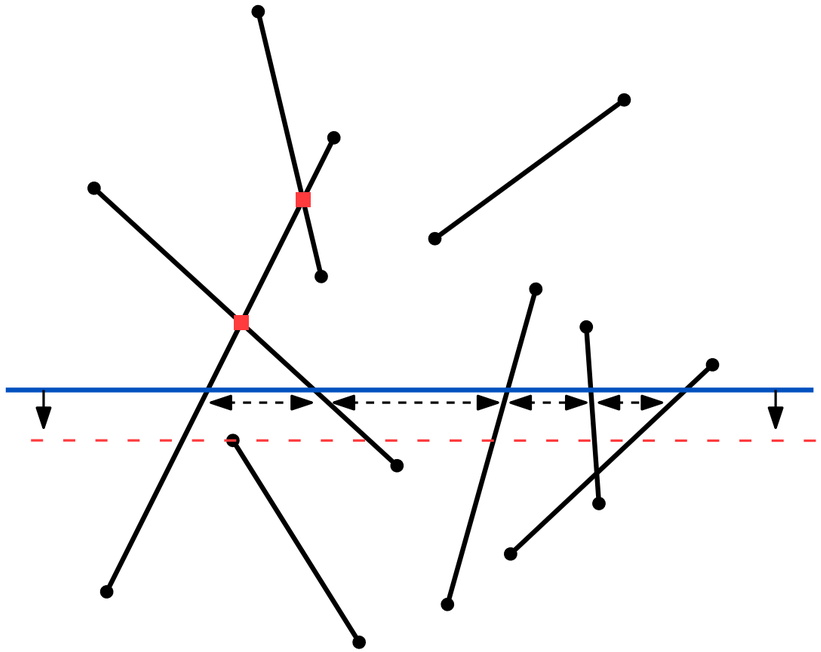
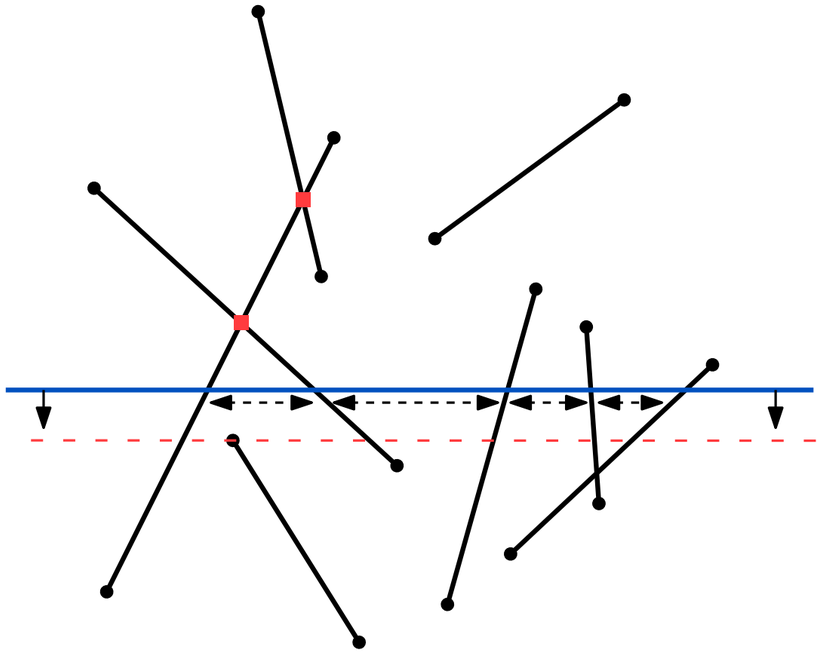


Plane sweep

The plane sweep technique: Imagine a horizontal line passing over the plane from top to bottom, solving the problem as it moves

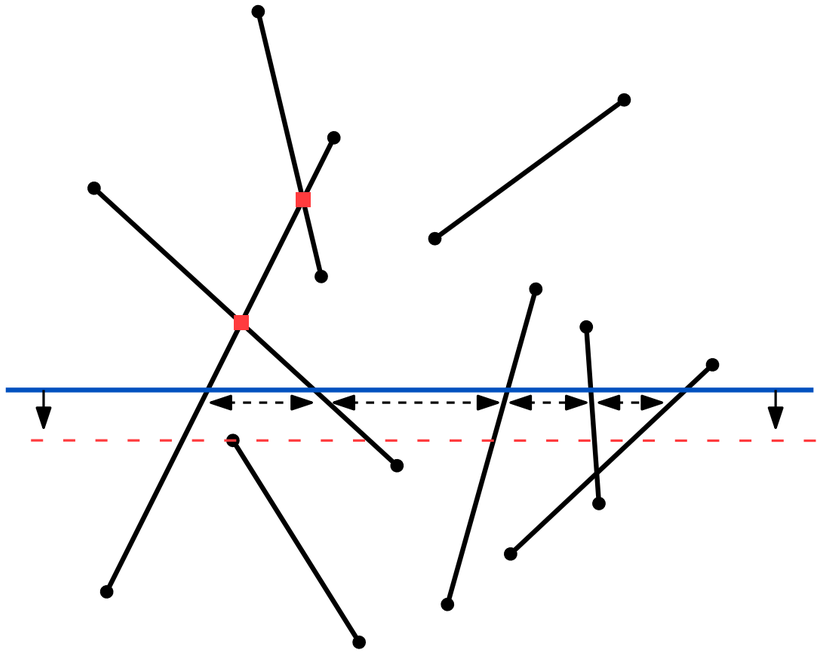
- The algorithm stores the relevant situation at the current position of the sweep line $\Rightarrow$ status
- The sweep line stops and the algorithm computes at certain positions $\Rightarrow$ events

- The algorithm knows everything it needs to know above the sweep line, and found all intersection points



Status and events for line segment intersection

status: set of line segments intersecting the sweep line, ordered from left to right



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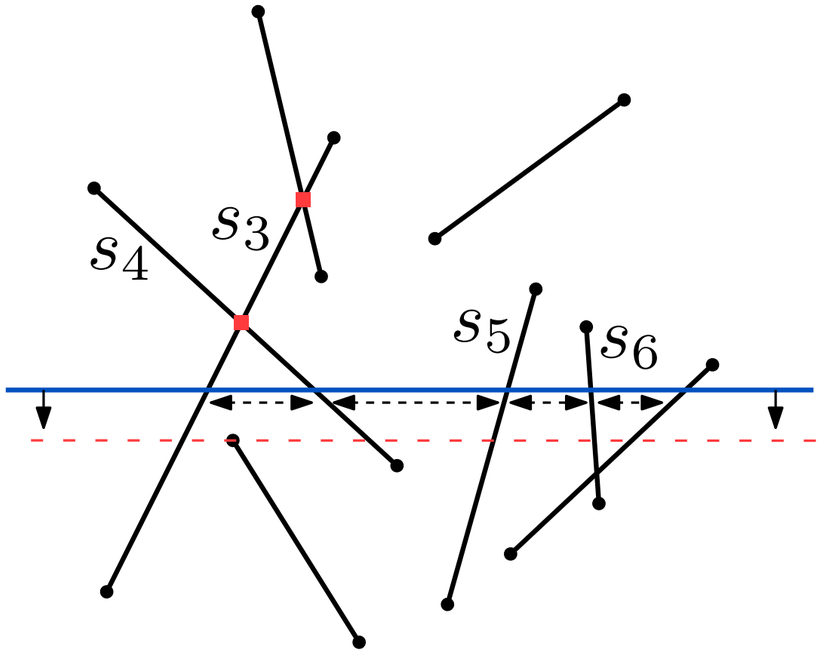
s_3, s_4, s_5, s_6



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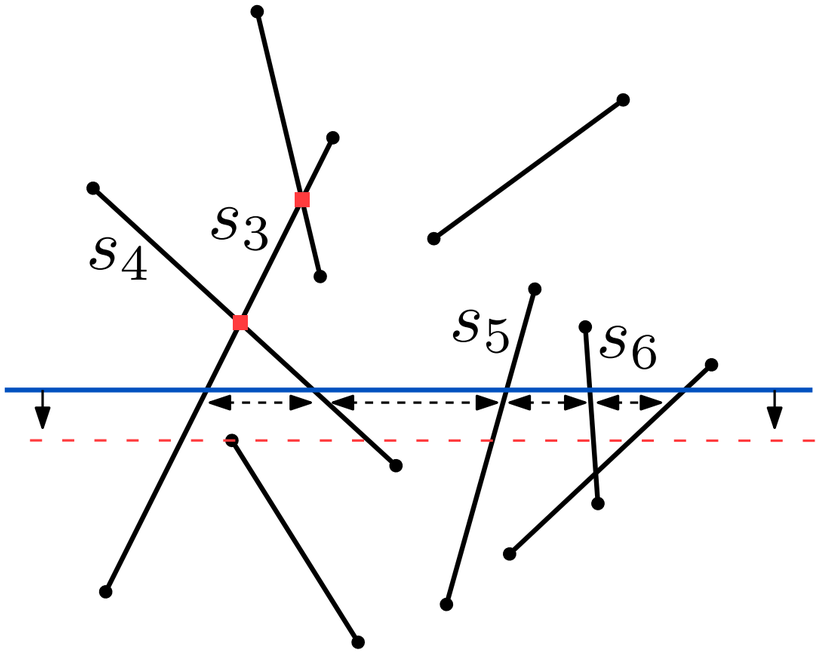
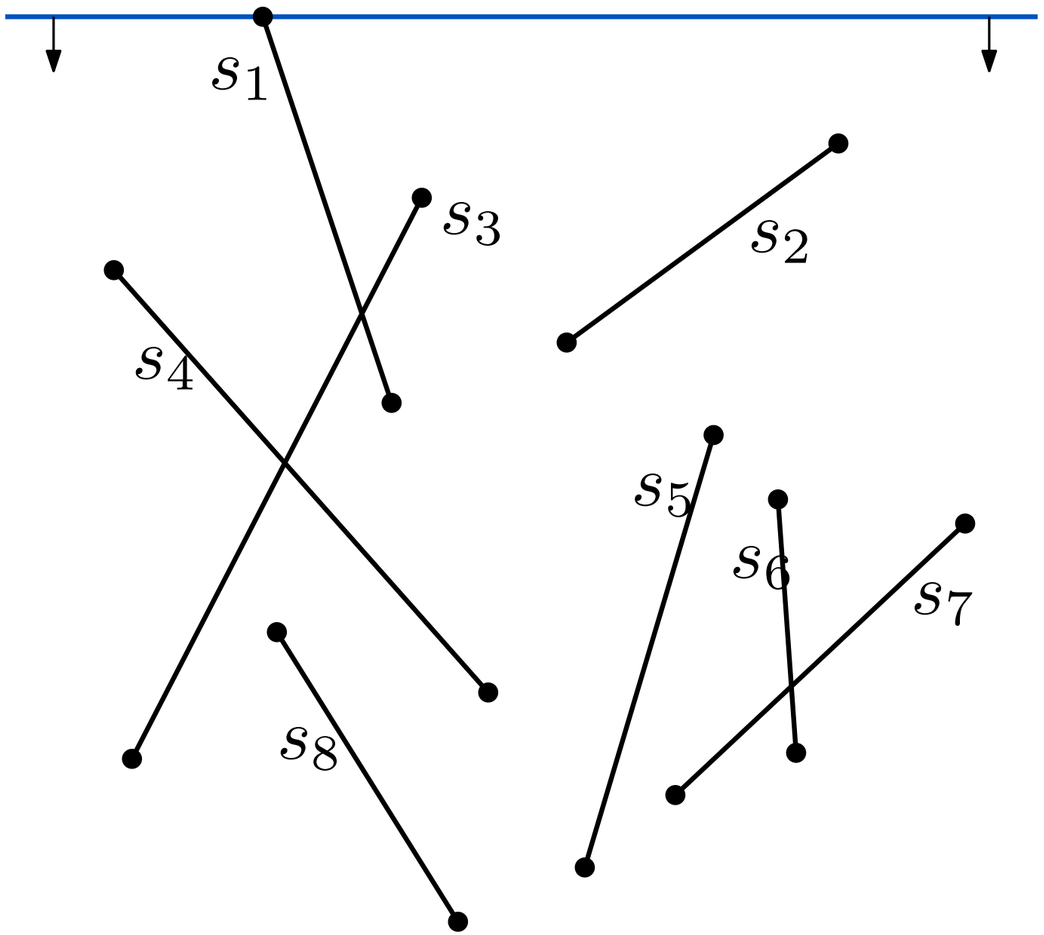


events occur when the *status changes*, and when *output is generated*

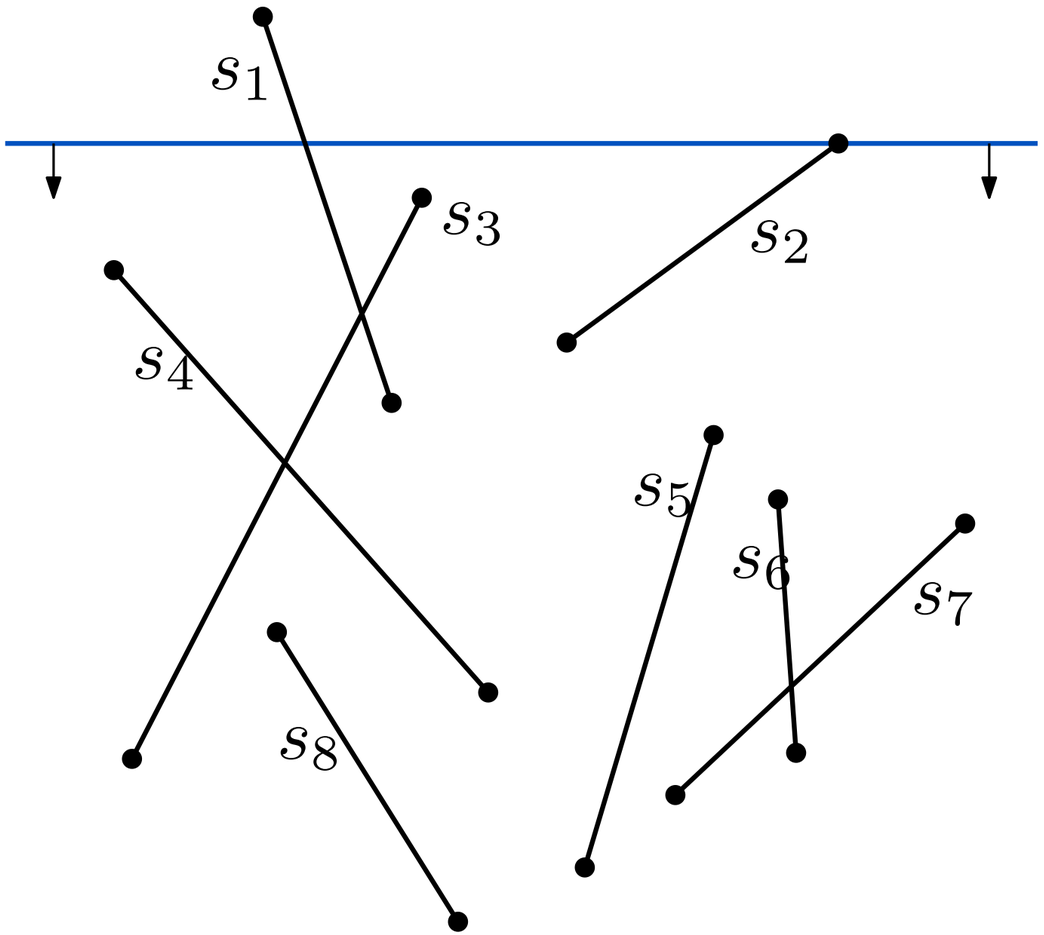
event $\approx$ interesting y-coordinate

Line sweep example



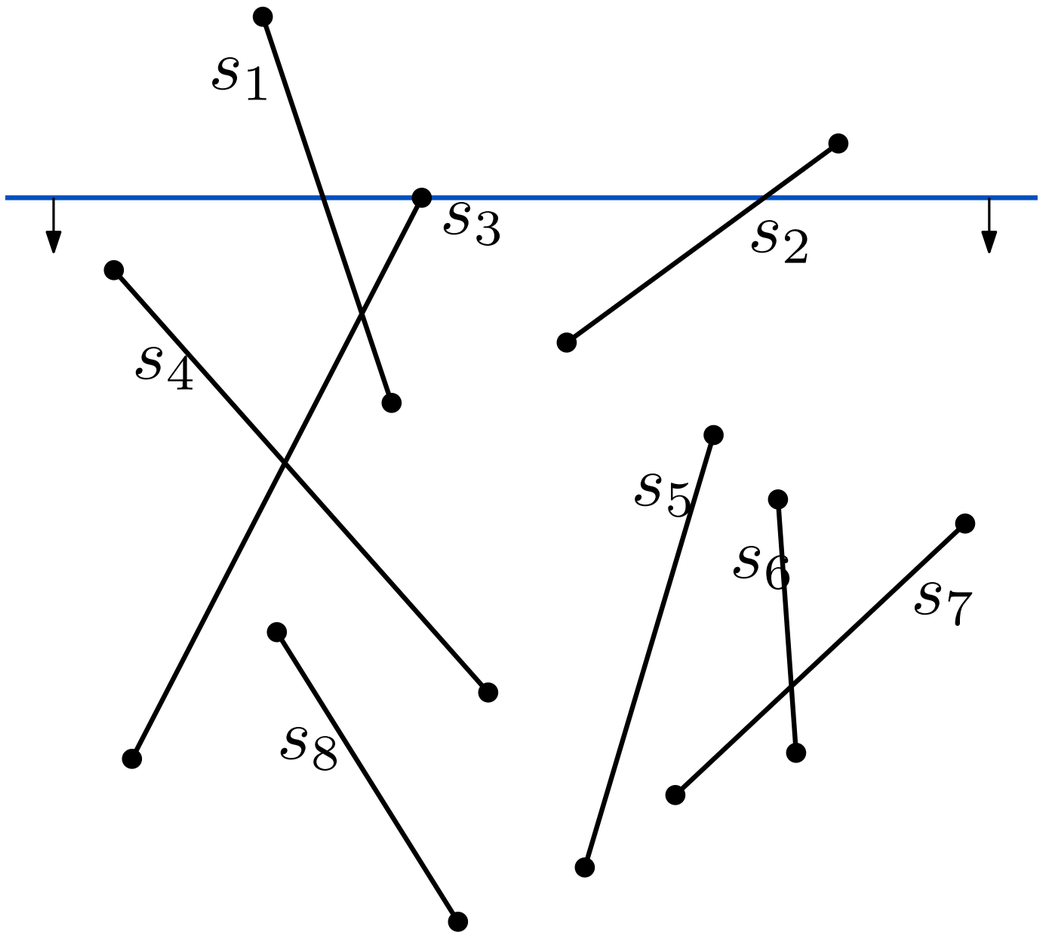
add s_1

Line sweep example



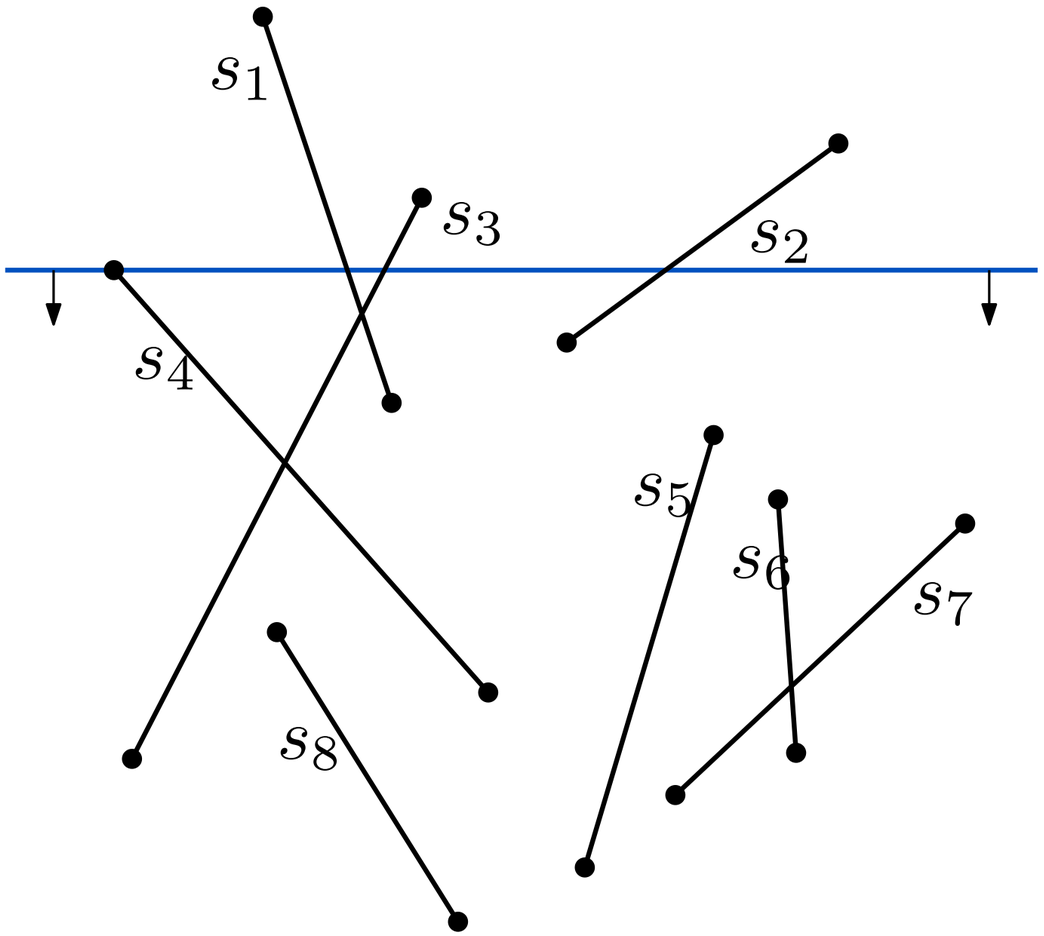
add s_2 after s_1

Line sweep example



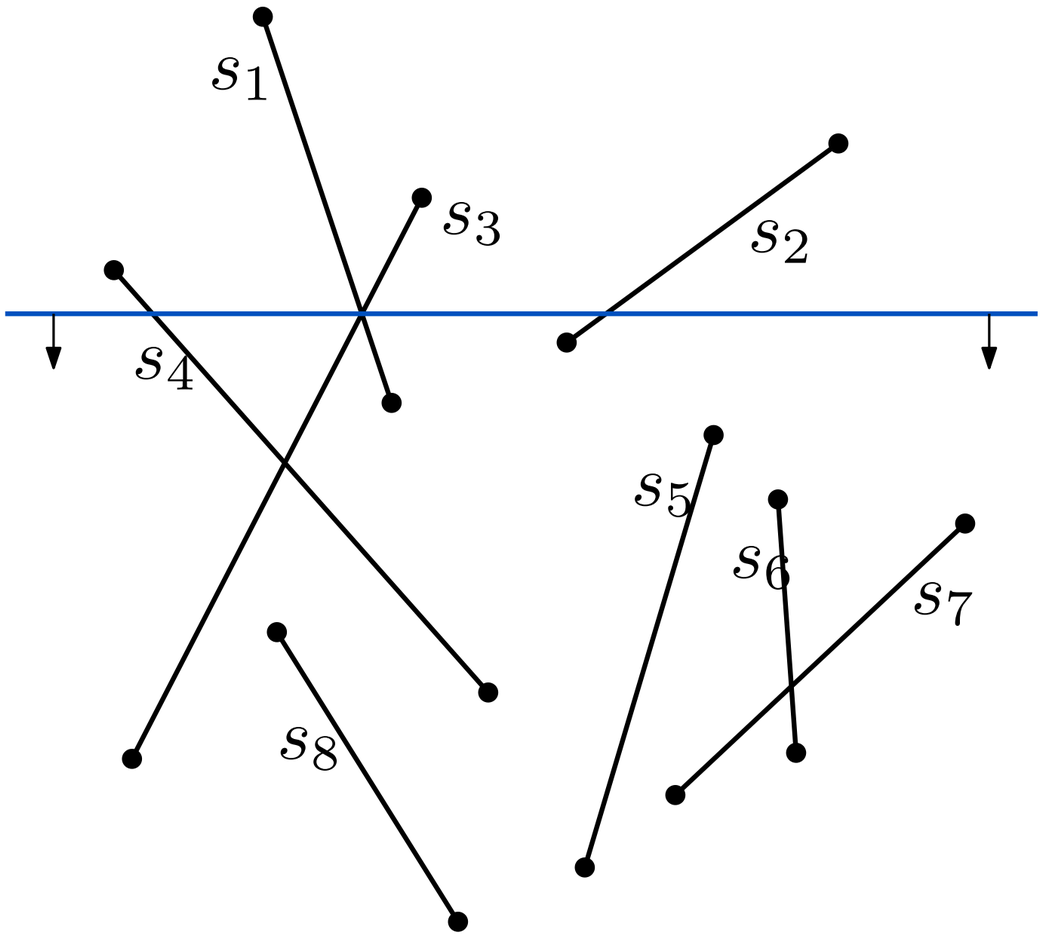
add s_3 between s_1 and s_2

Line sweep example



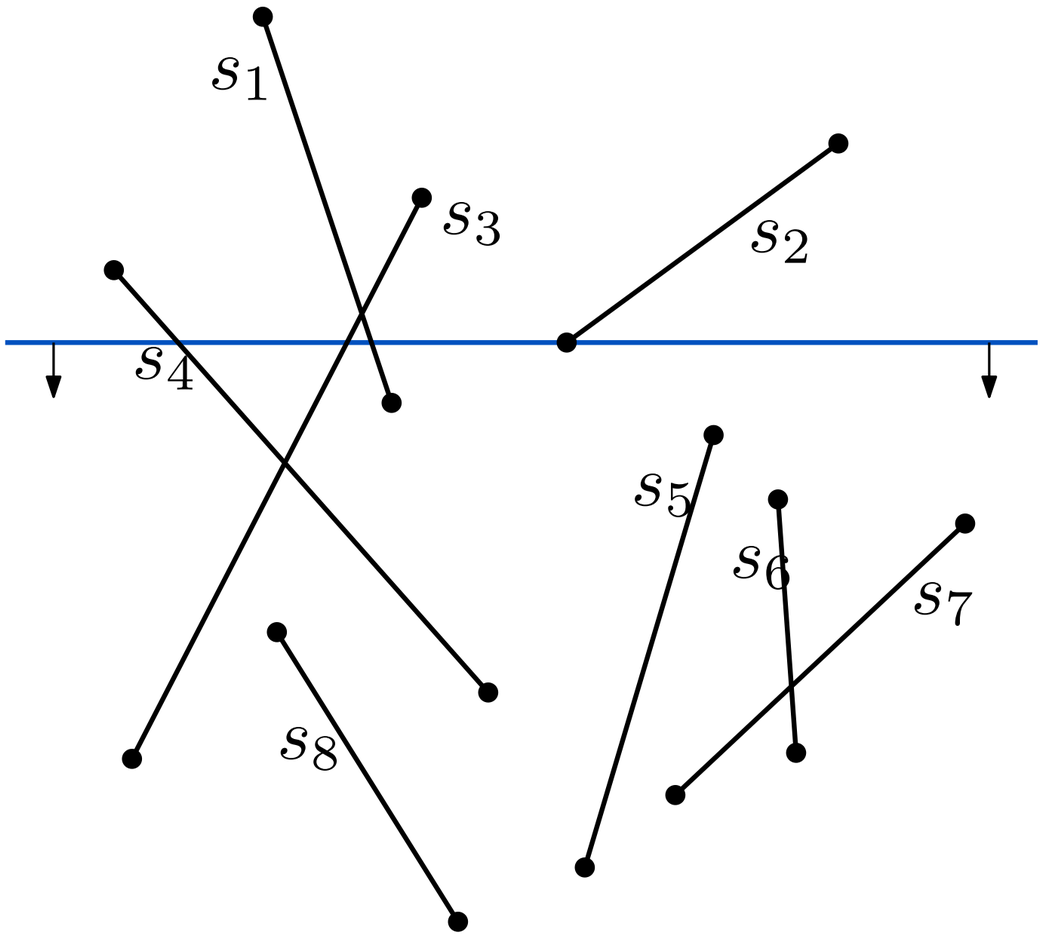
add s_4 before s_1

Line sweep example



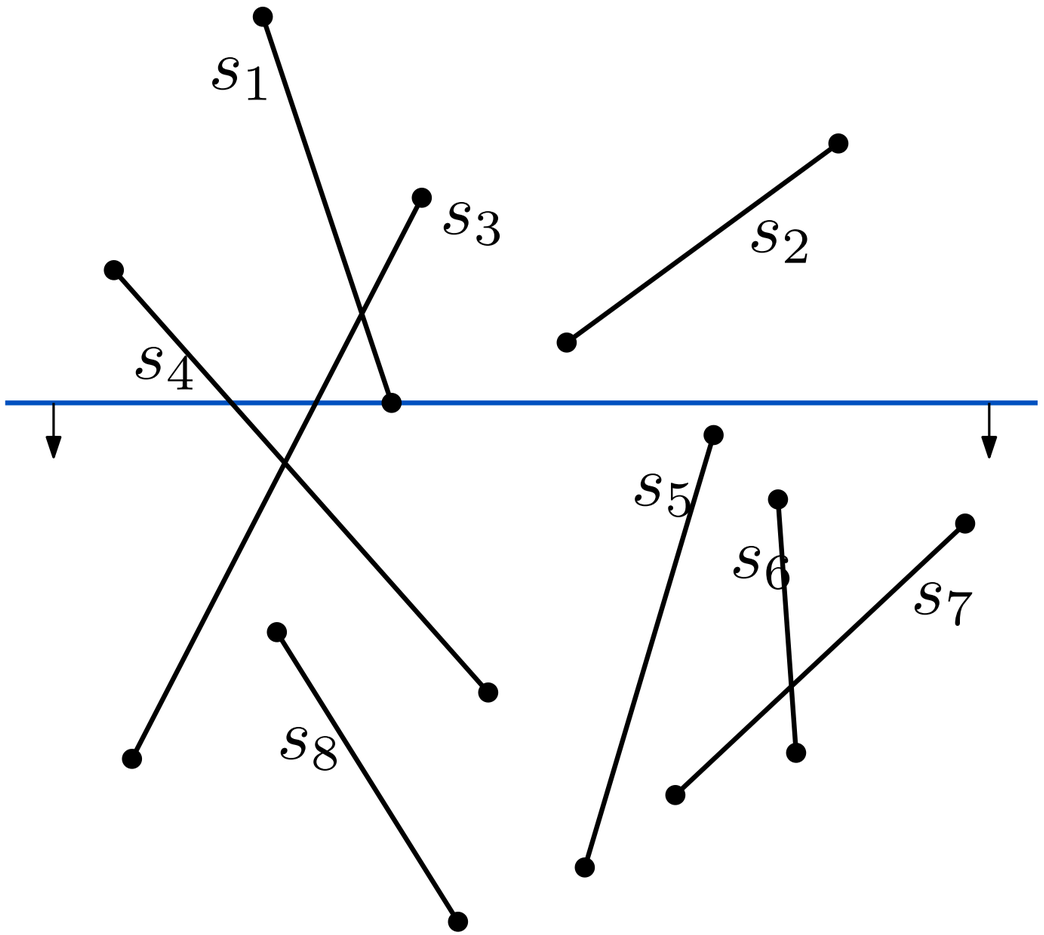
report intersection (s_1, s_3);
swap s_1 and s_3

Line sweep example



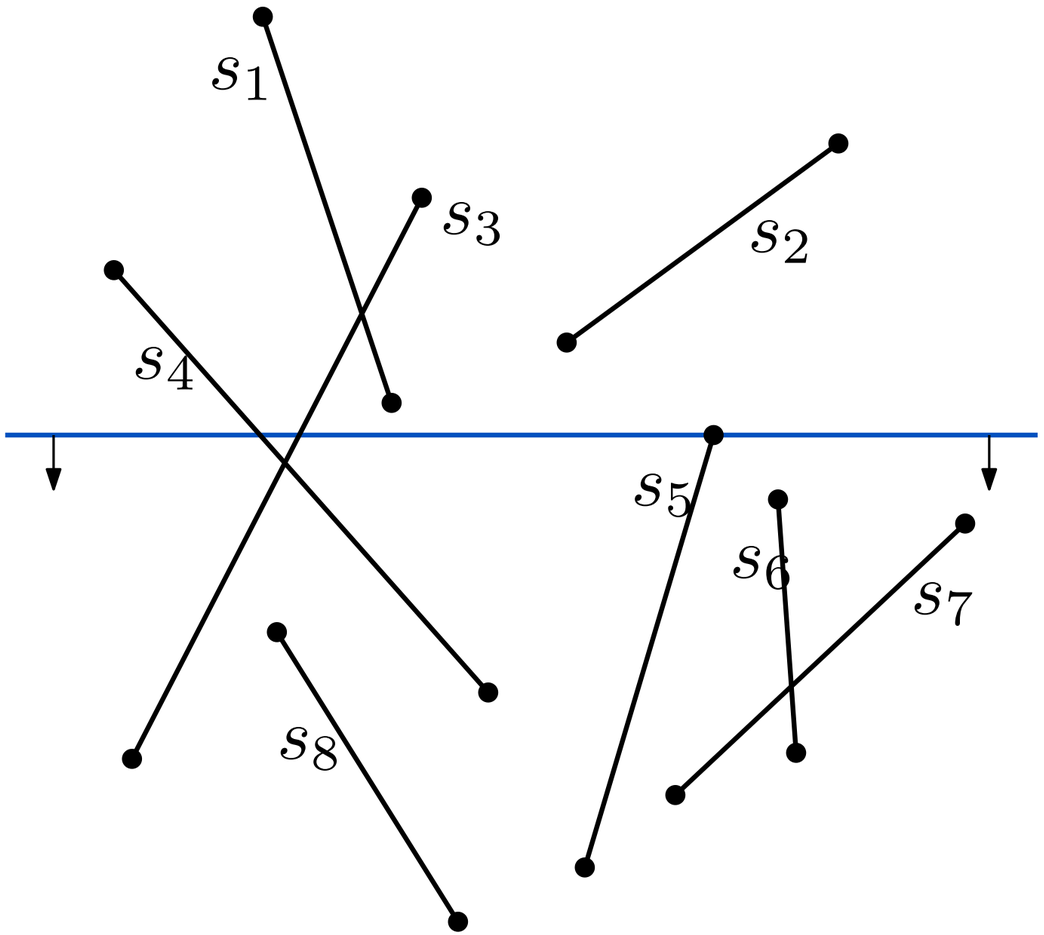
remove s_2

Line sweep example



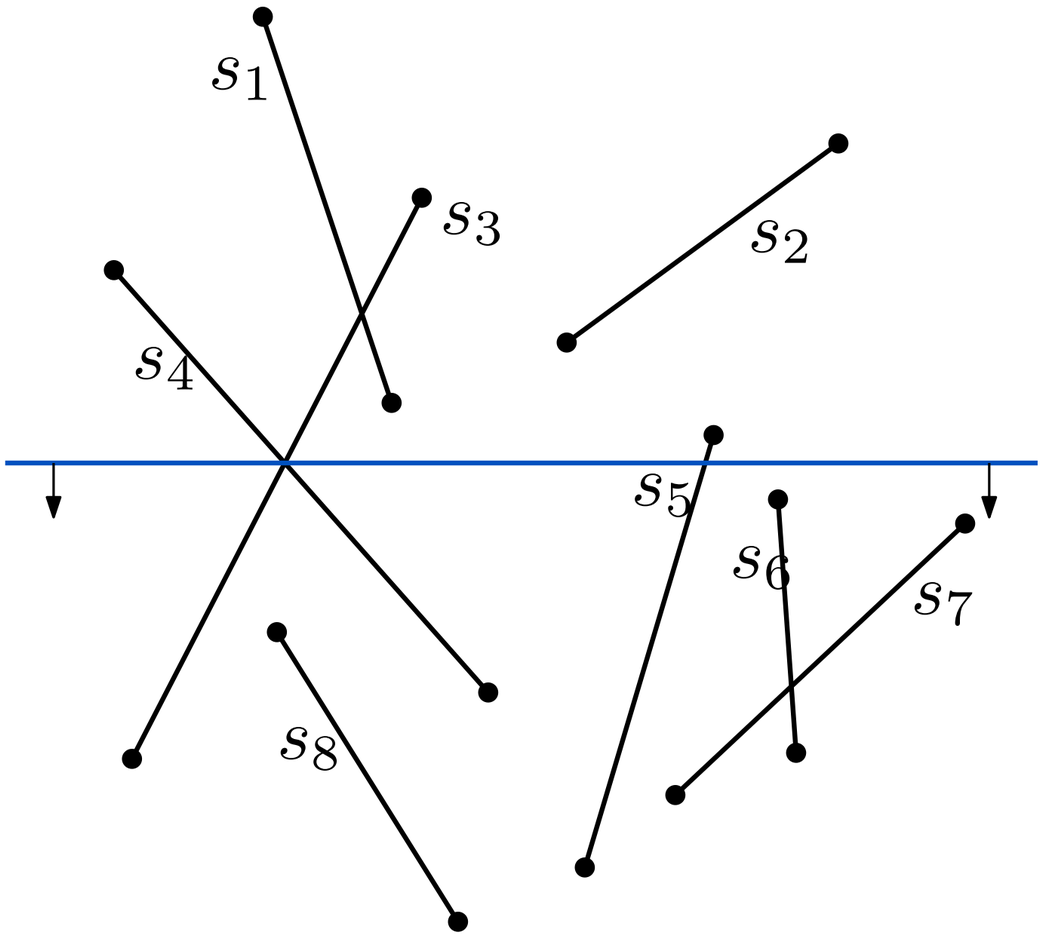
remove s_1

Line sweep example



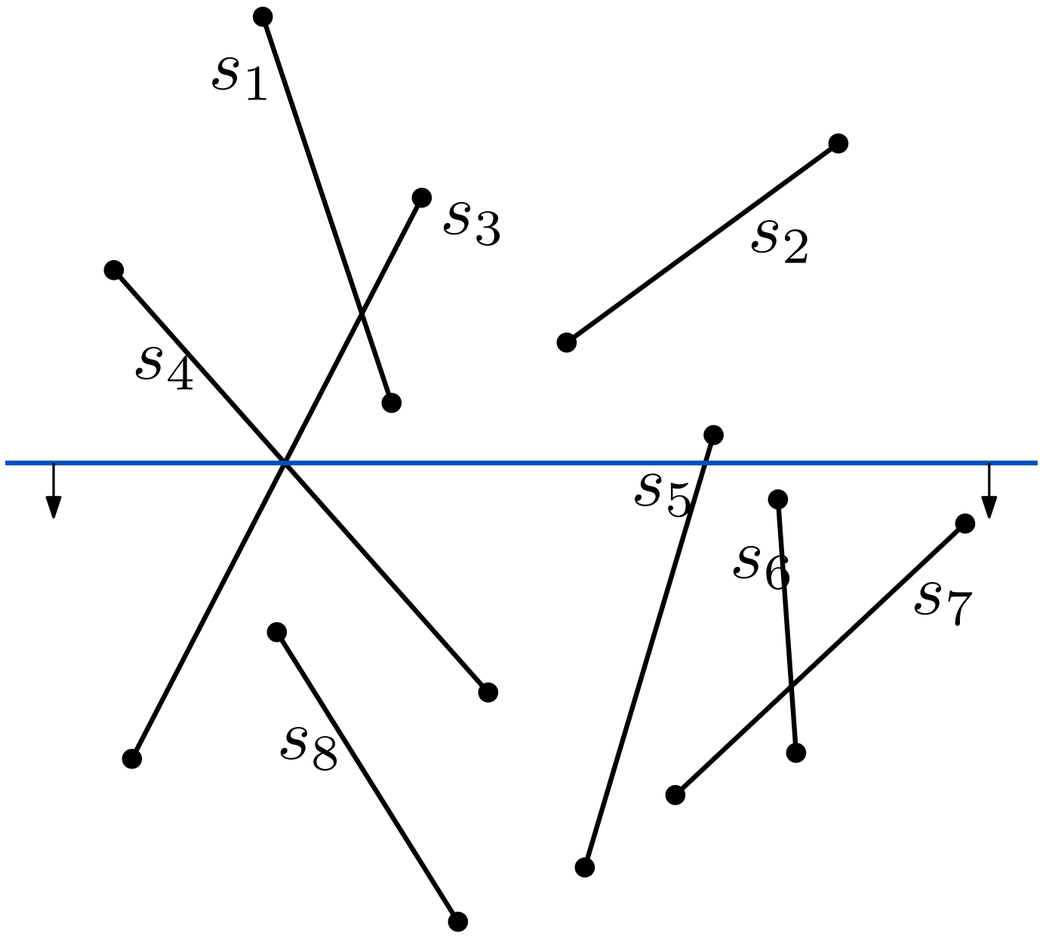
add s_5

Line sweep example



report intersection (s_3, s_4);
swap s_3 and s_4

Line sweep example

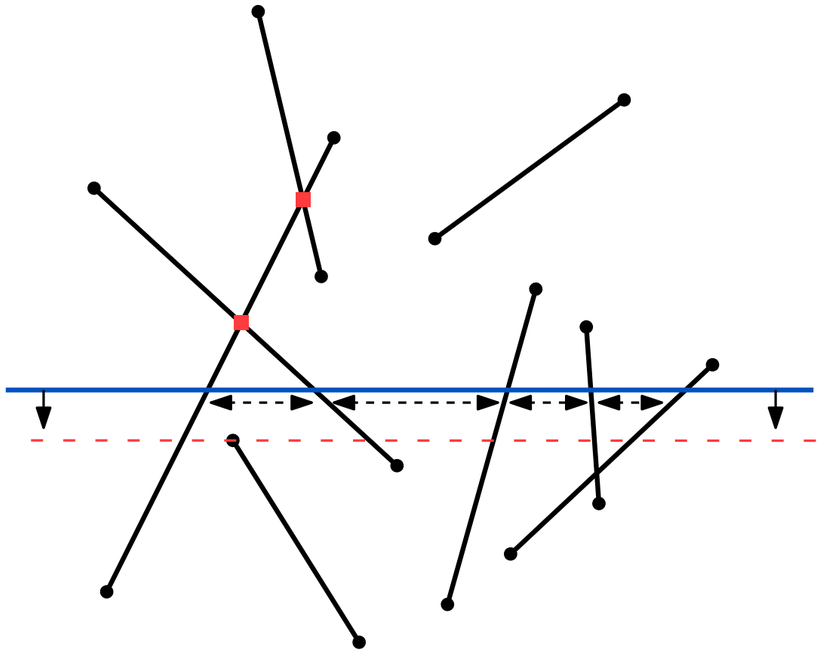


report intersection (s_3, s_4);
swap s_3 and s_4

…and so on …

The events

When do the events happen?

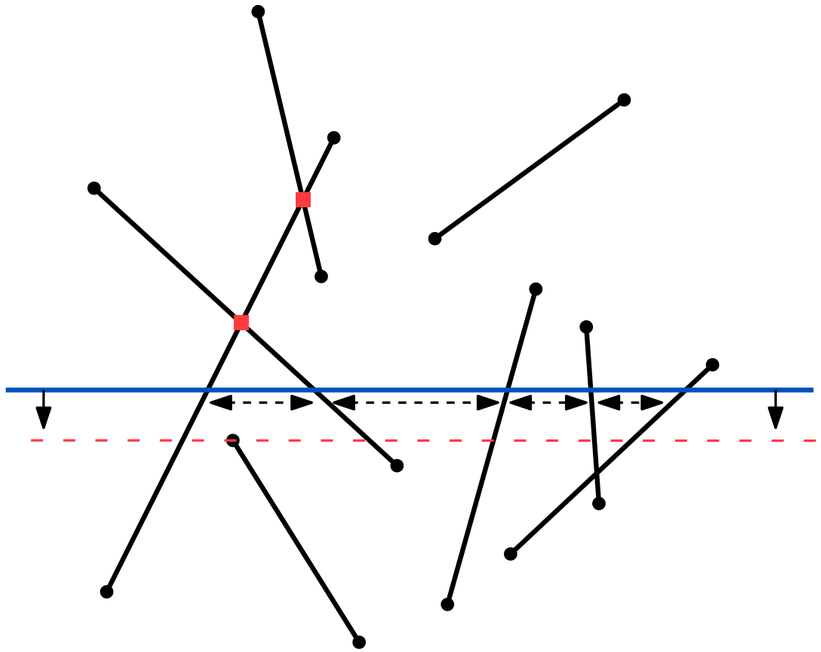


The events

When do the events happen?

When the sweep line is

- at an upper endpoint of a line segment
- at a lower endpoint of a line segment
- at an intersection point of line segments



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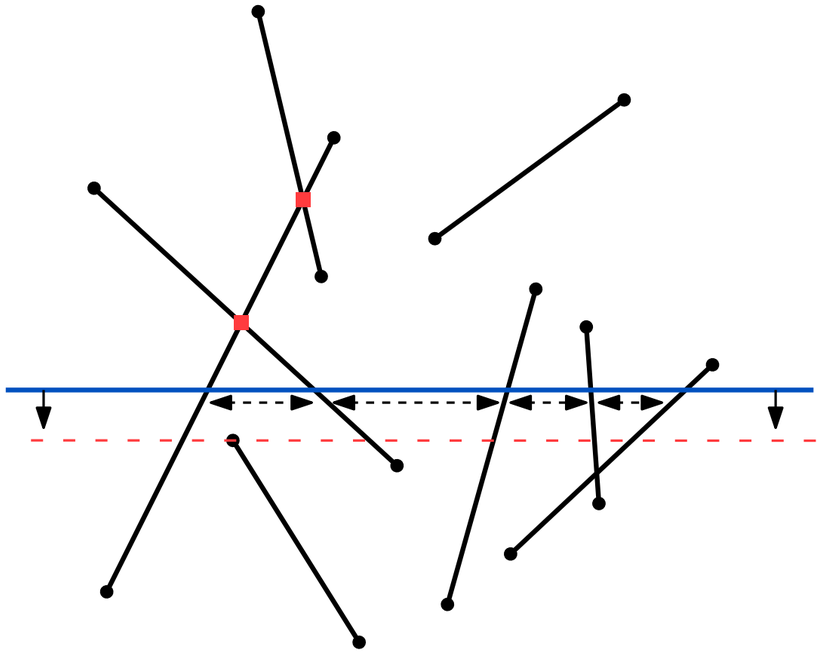

At each type, the status changes; at the third type output is found too

At no other y-coordinates the status changes

Assume no degenerate cases

We will at first exclude degenerate cases:

- No two events (endpoints/intersection points) have the same y-coordinate

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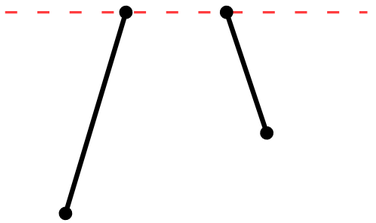
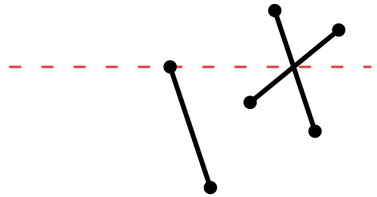
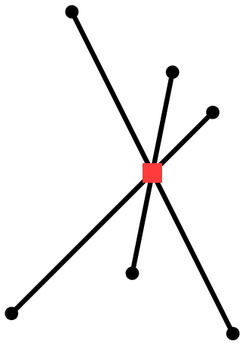




- No more than two line segments intersect in a point



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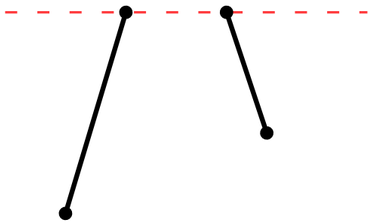

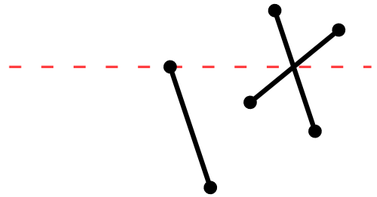

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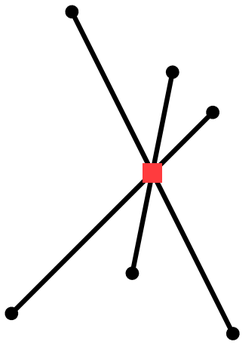

Question: Are there more degenerate cases?

Assume no degenerate cases

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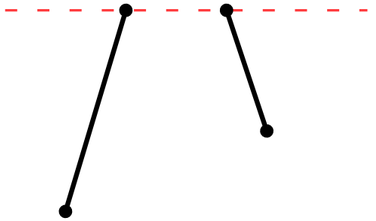

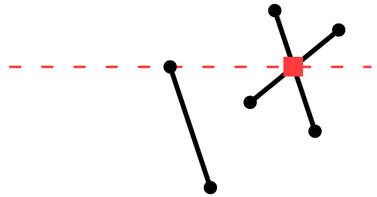

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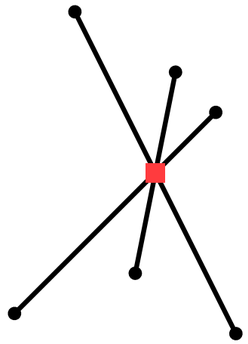
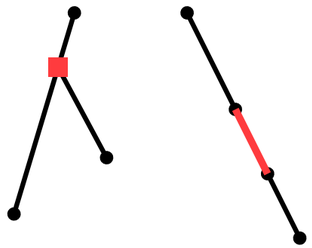


- No intersections include endpoints



Quiz

Assume we want to compute the intersection of two convex polygons using a plane sweep algorithm. What is the complexity of the status?

A: $O(n)$

B: $O(1)$

C: $O(n^2)$

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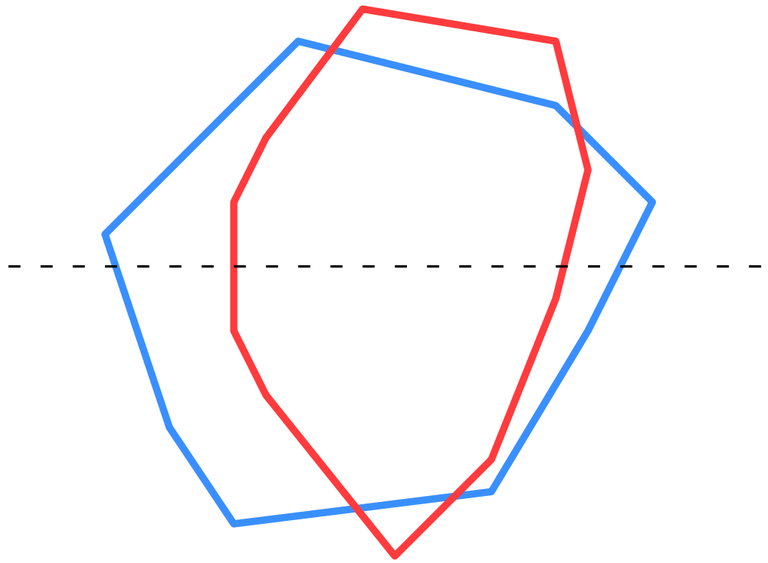
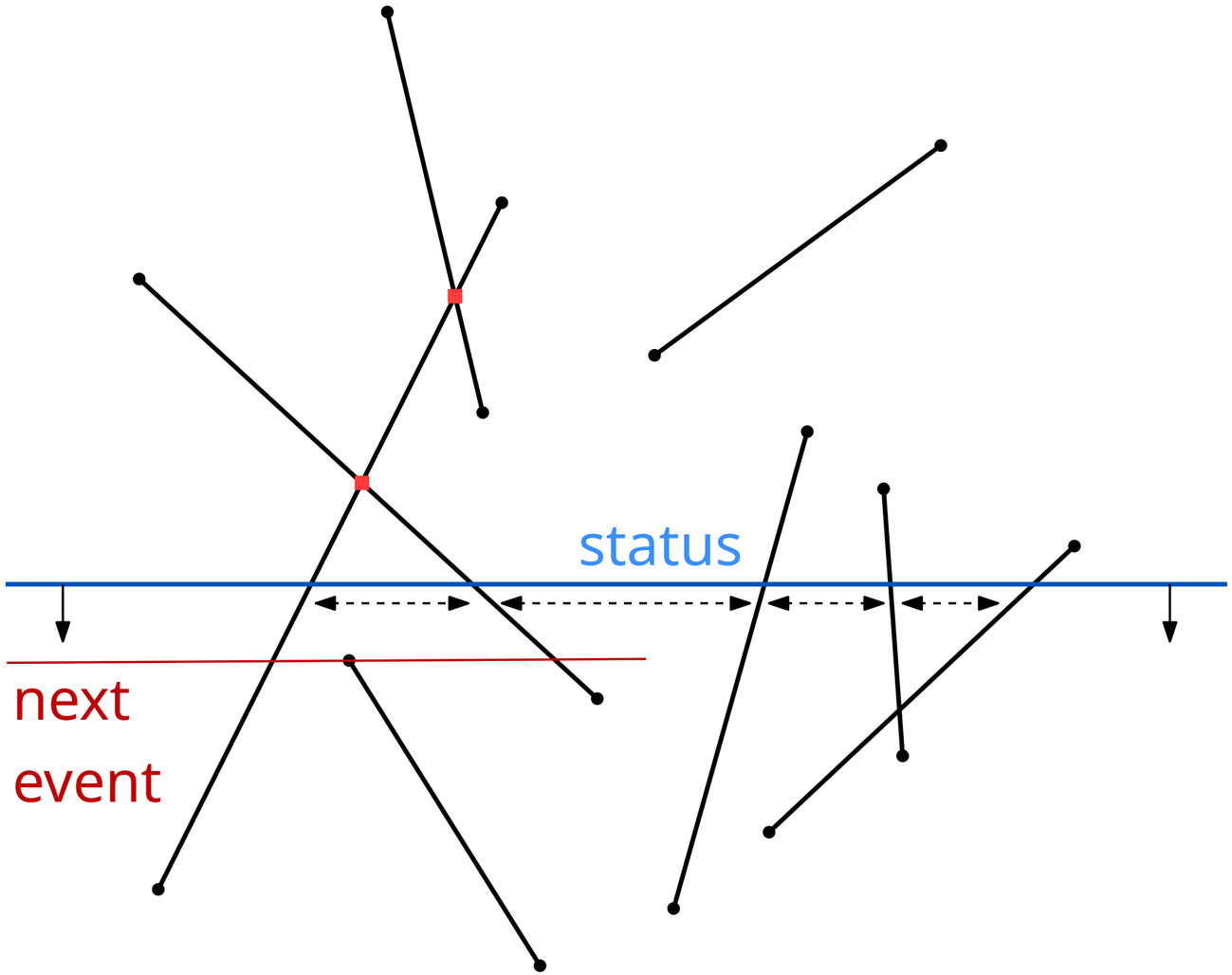


The status stores the currently intersected boundaries: ≤ 4

Summary



Ingredients for

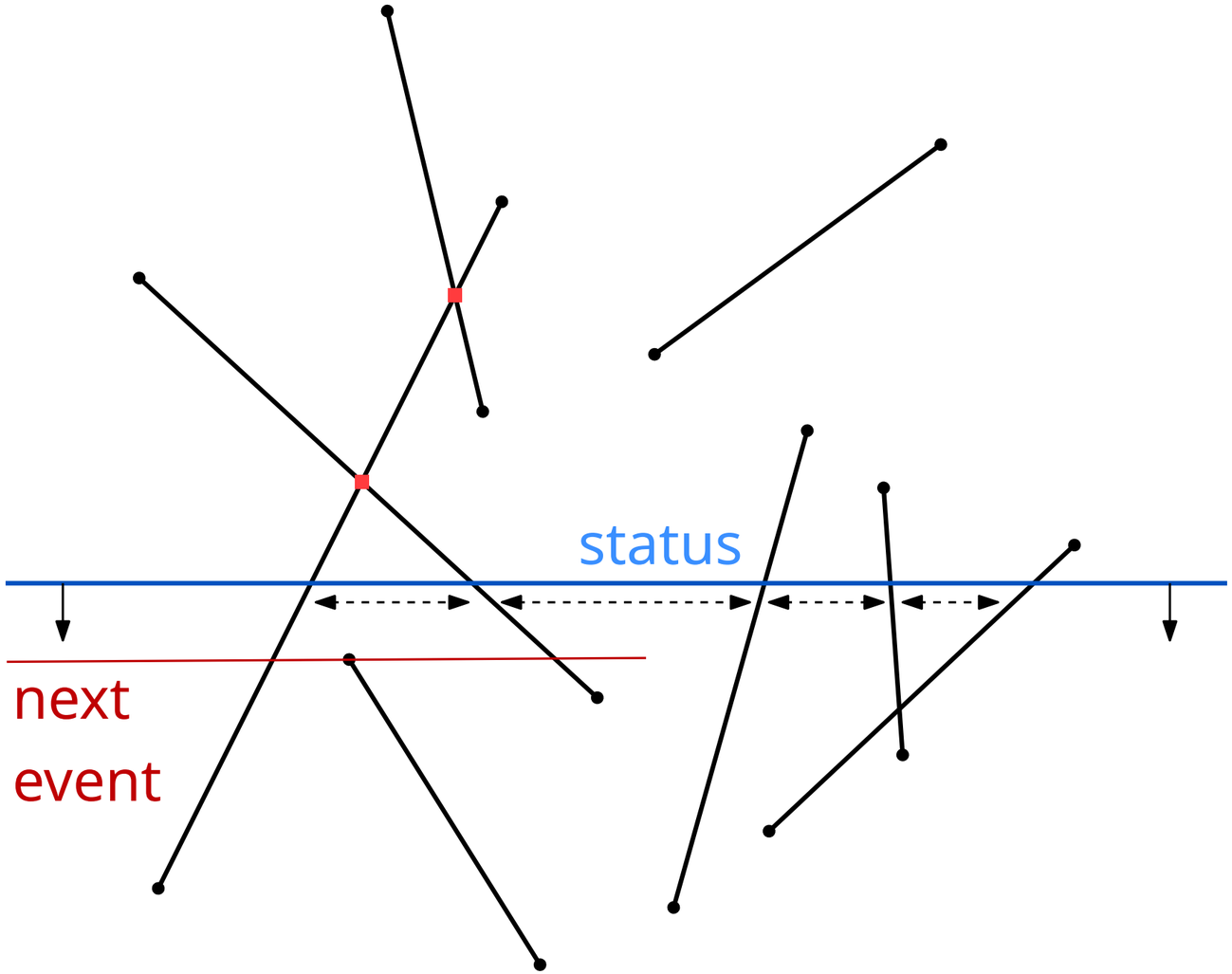
Plane Sweep technique:

- status
- events

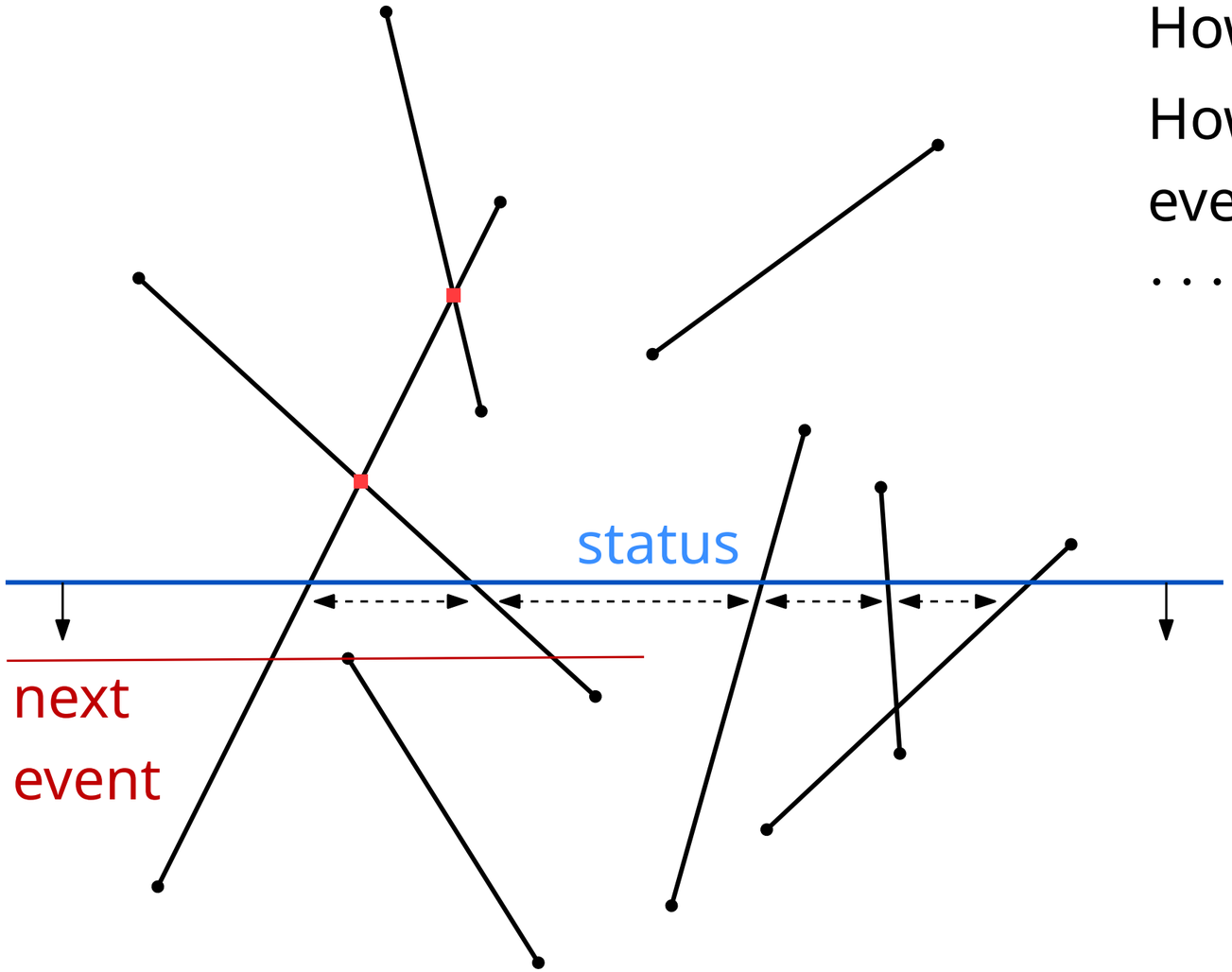
Plane Sweep Algorithm for Line Segment Intersection

data structures

Status and Events



Status and Events

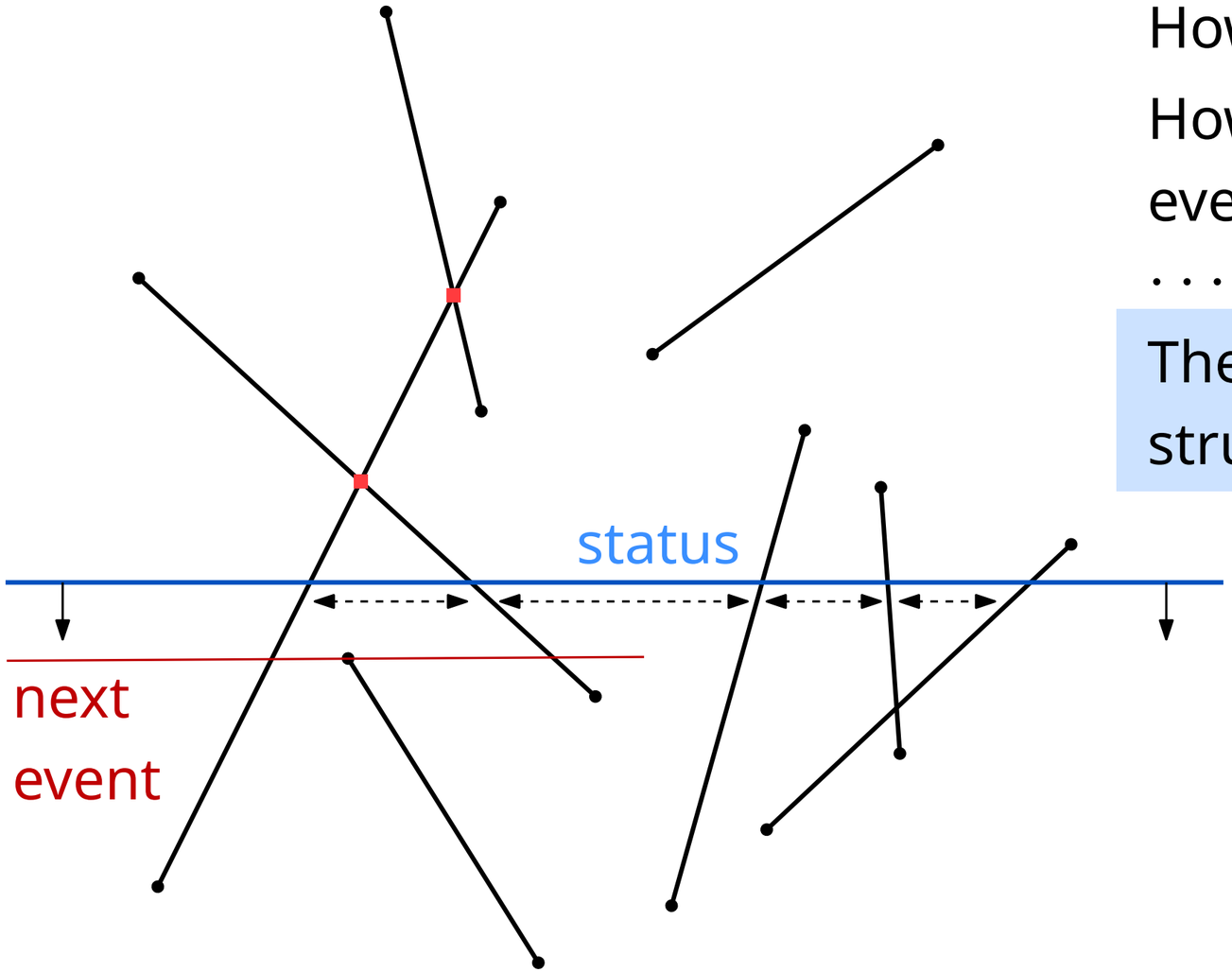


How do we find the next event?

How do we update the status at the event?

. . .

Status and Events



How do we find the next event?

How do we update the status at the event?

. . .

These operations require suitable data structures!

Event list and status structure

The event list is an abstract data structure that stores all events in the order in which they occur

The status structure is an abstract data structure that maintains the current status

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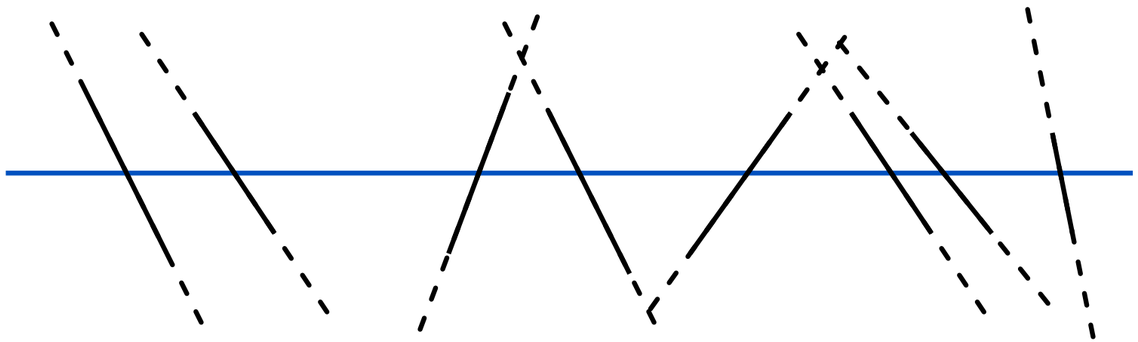

Here: The status is the subset of currently intersected line segments in the order of intersection by the sweep line

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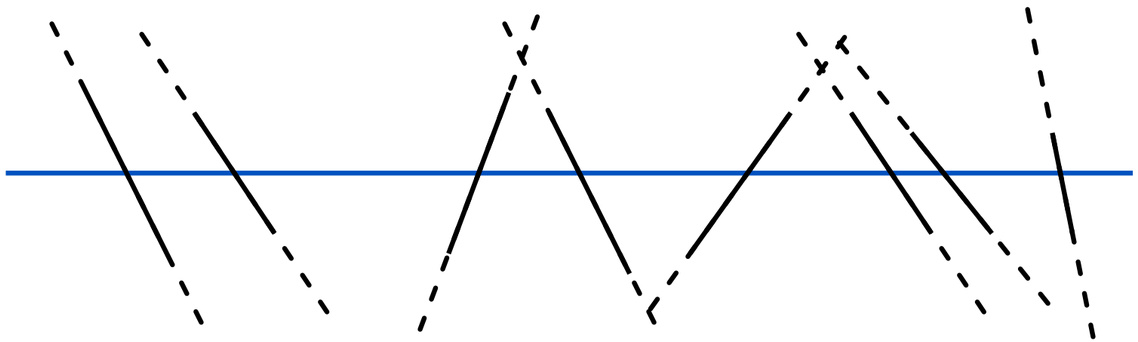
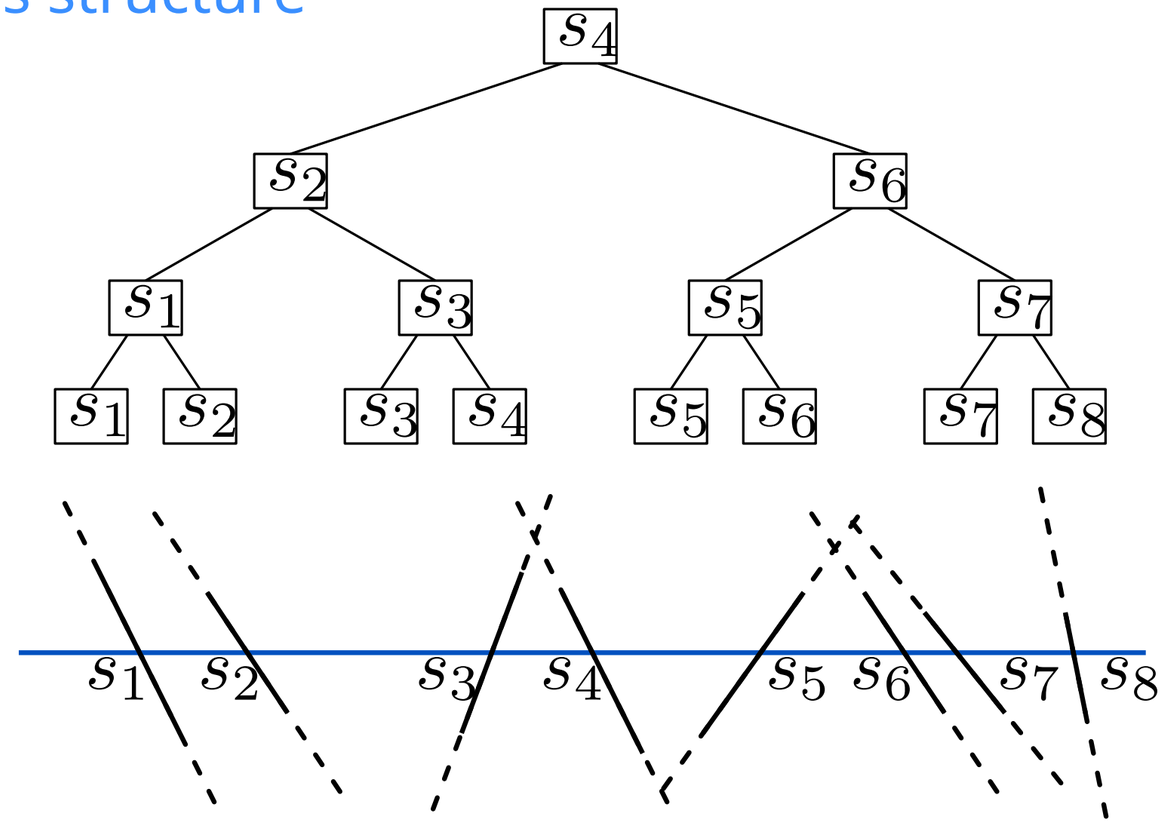


Here: The status is the subset of currently intersected line segments in the order of intersection by the sweep line

Question: Data structure for status?

Status structure

We use a balanced binary search tree with the line segments in the leaves as the status structure



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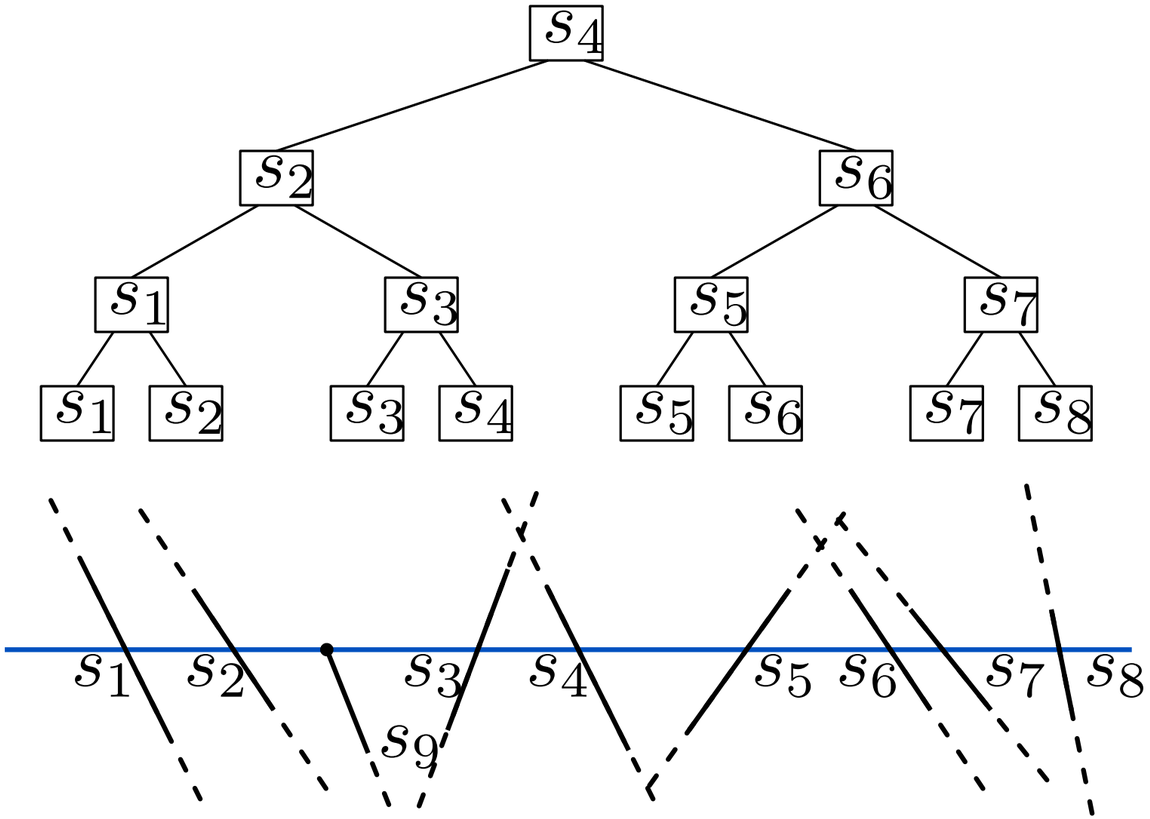
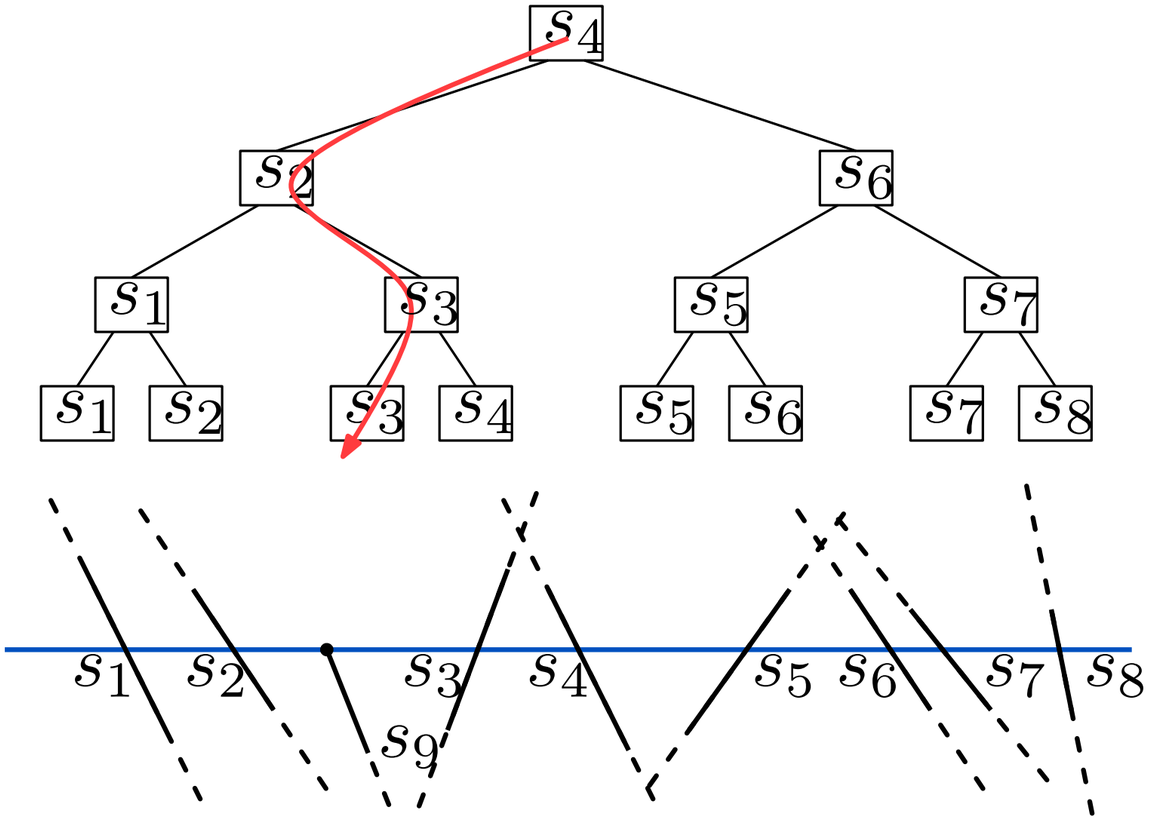


Upper endpoint: search, and insert

Status structure



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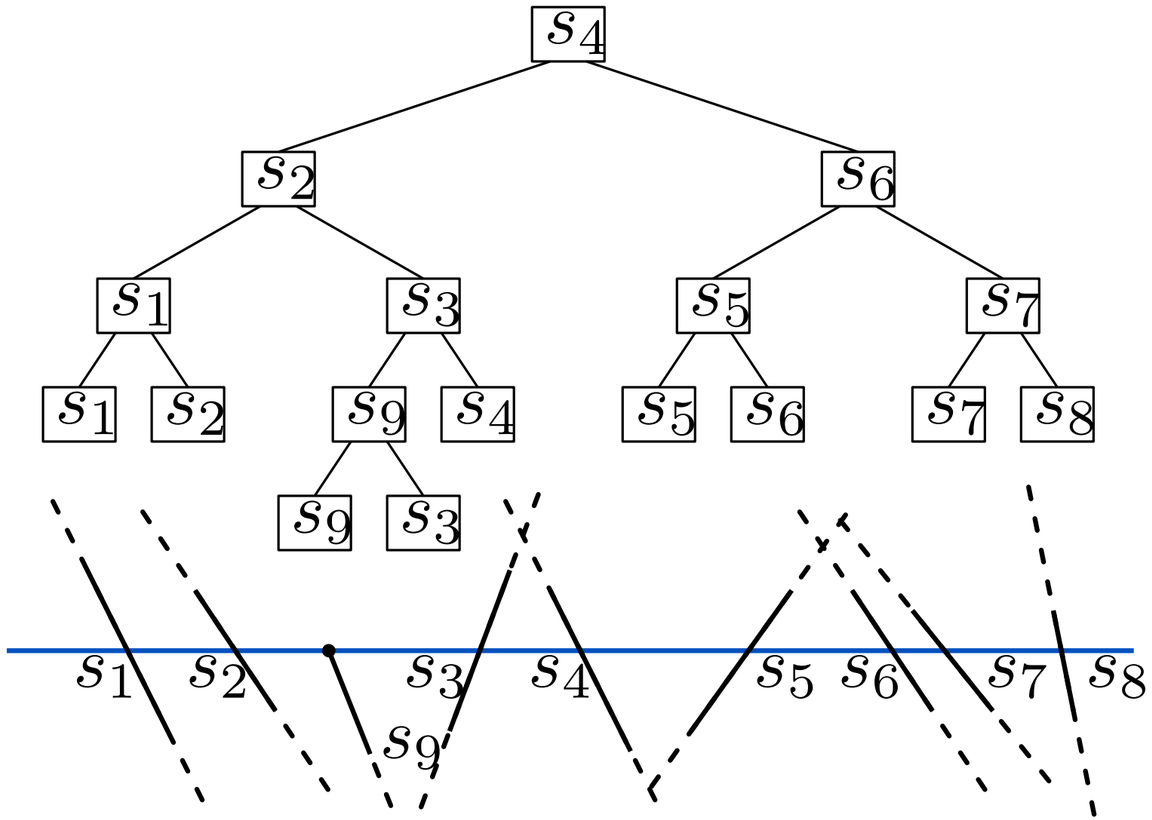

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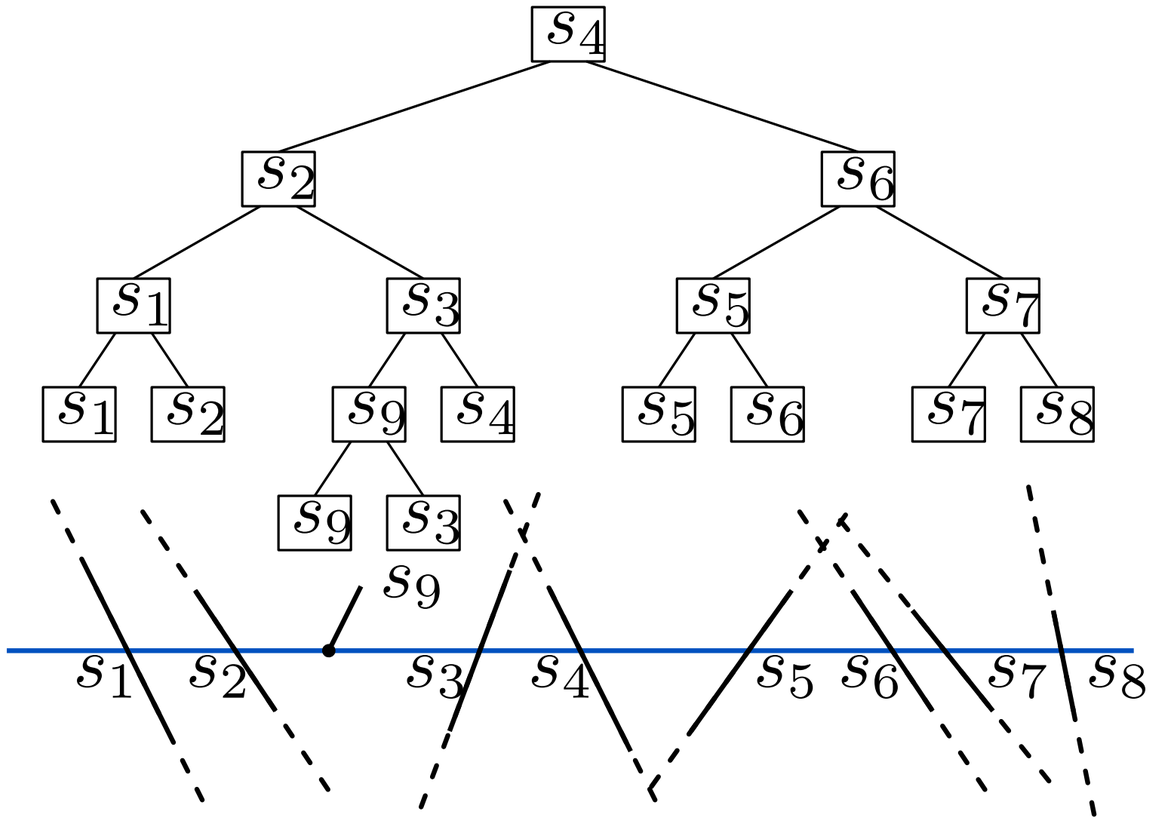

Lower endpoint: delete

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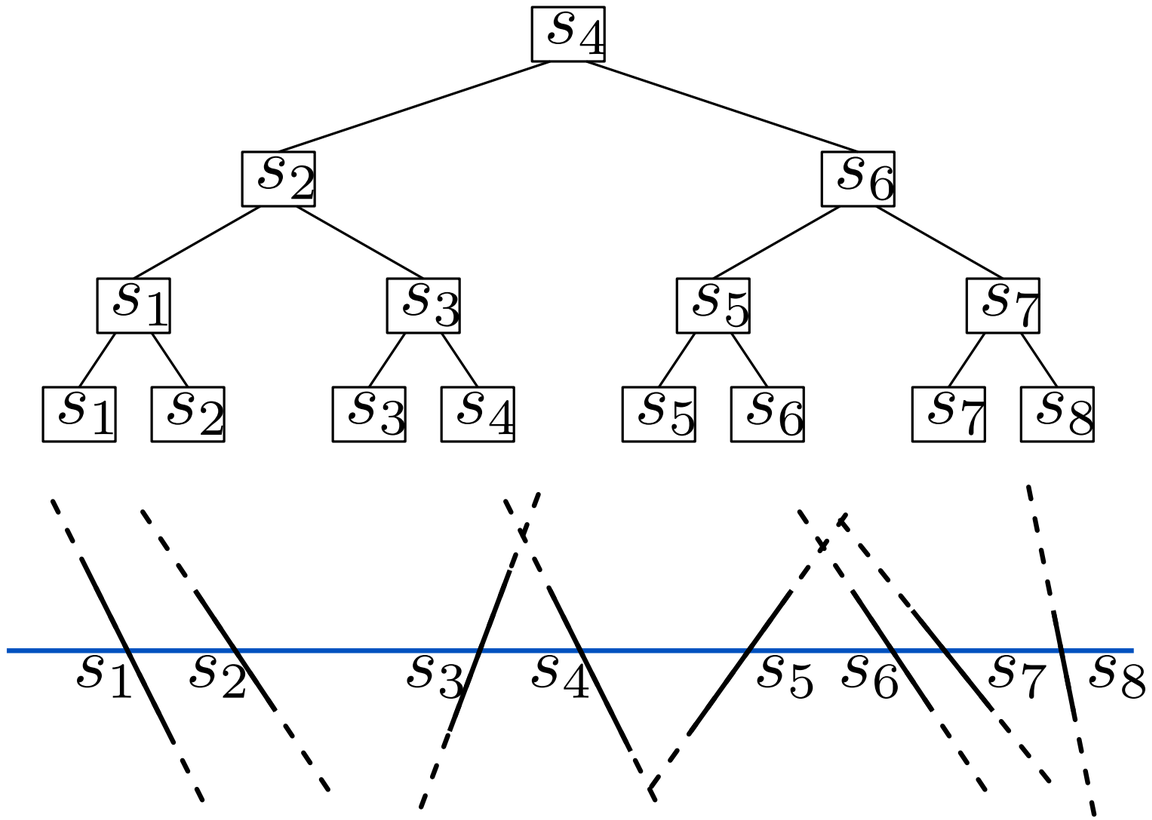

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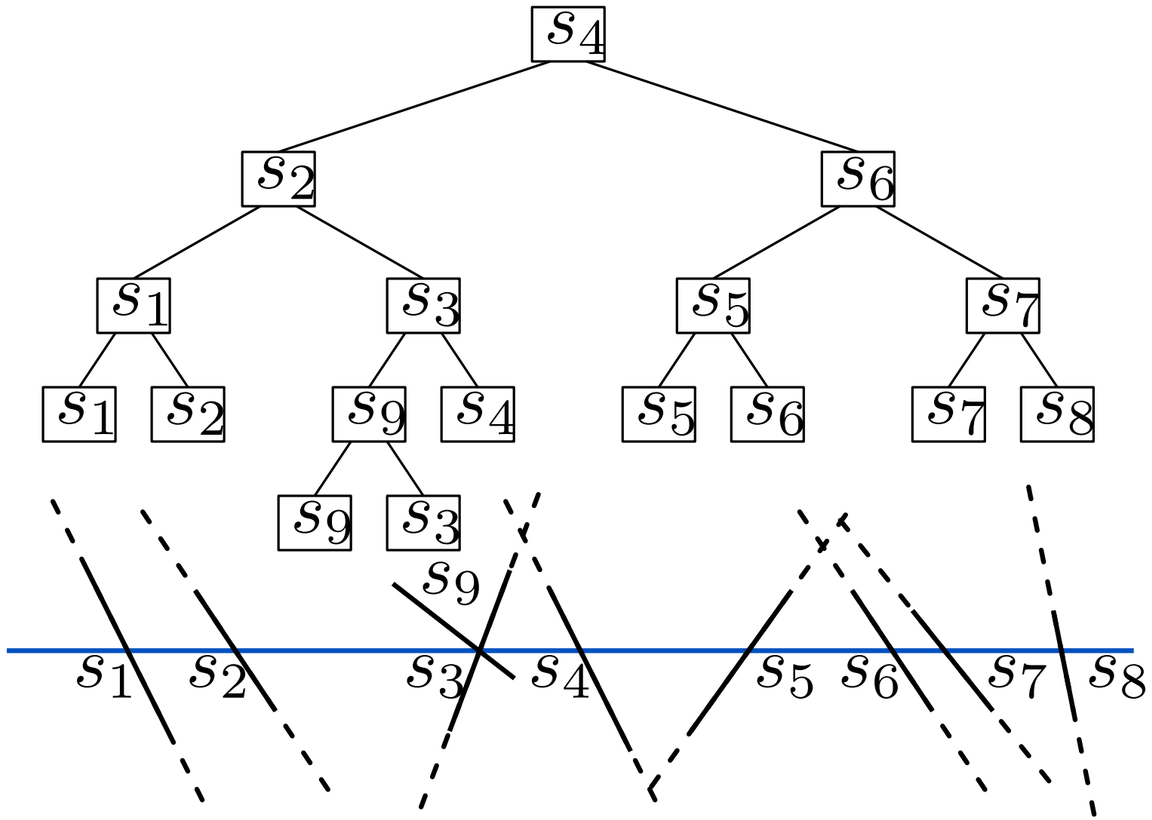

Intersection point: swap leaves and update information in search path

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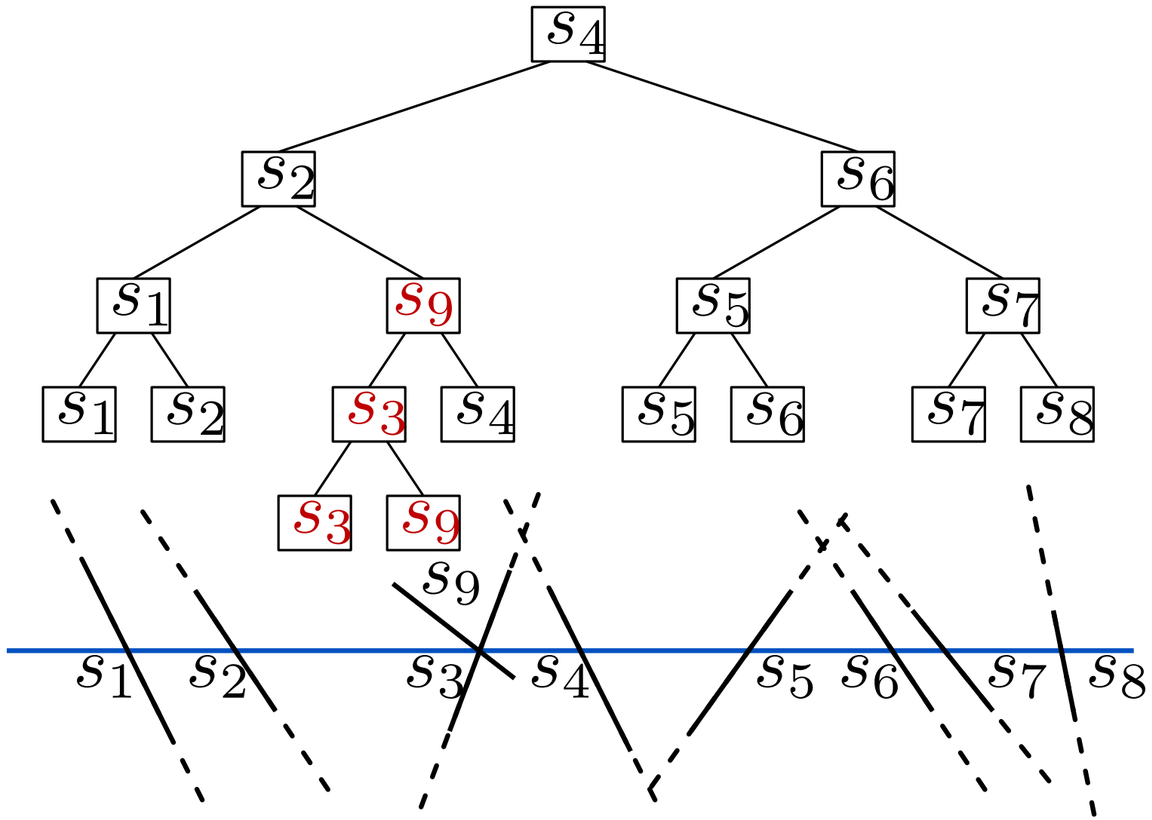

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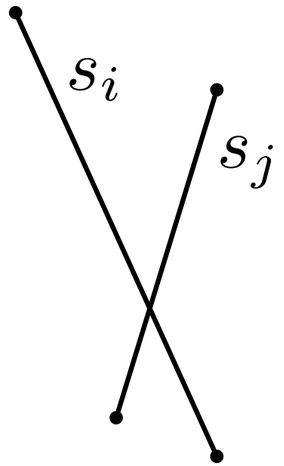

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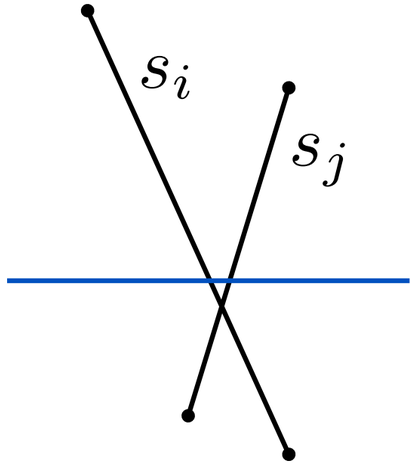

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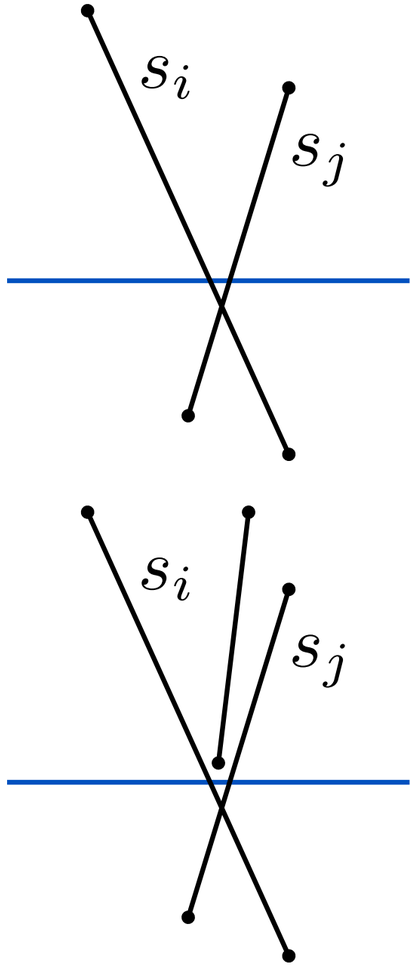
Recall: Two line segments can only intersect if they are horizontal neighbors

Finding events

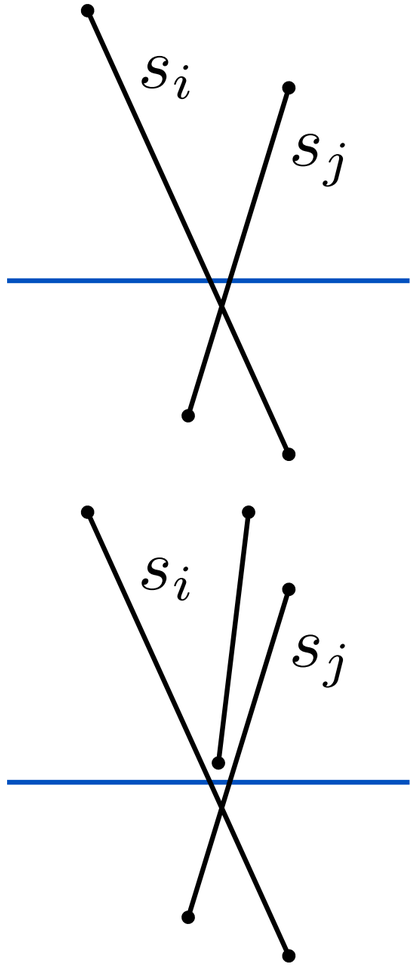
Lemma: Two line segments s_i and s_j can only intersect after (= below) they have become horizontal neighbors



Finding events

Lemma: Two line segments s_i and s_j can only intersect after (= below) they have become horizontal neighbors

Proof idea: Just imagine that the sweep line is ever so slightly above the intersection point of s_i and s_j, but below any other event □



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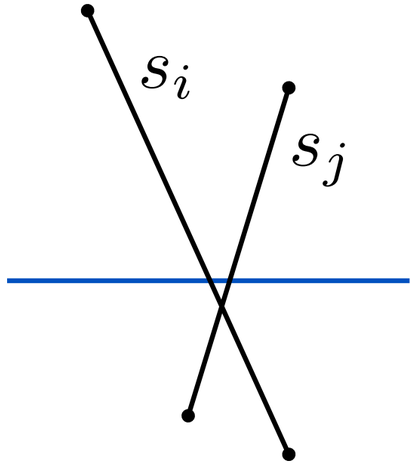
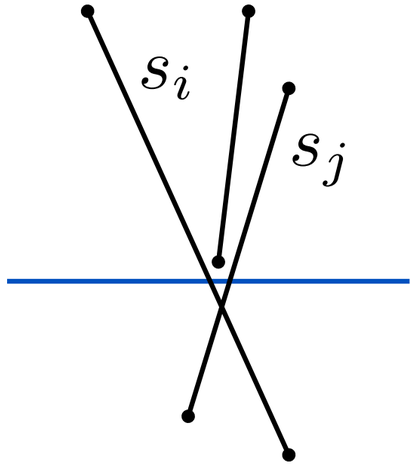


Also: some earlier (= higher) event made s_i and s_j horizontally adjacent



Quiz

Which data structure should we use as event queue if we want to avoid inserting the same event several times?

A: array

B: min-heap

C: balanced binary search tree

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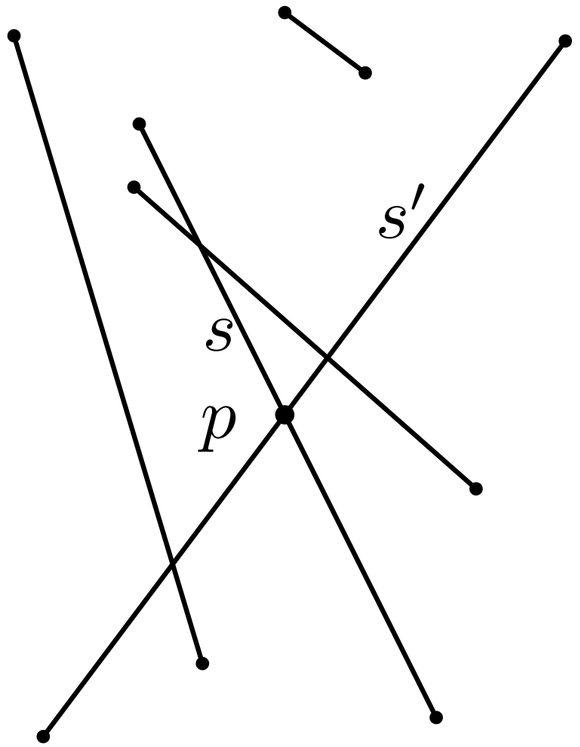

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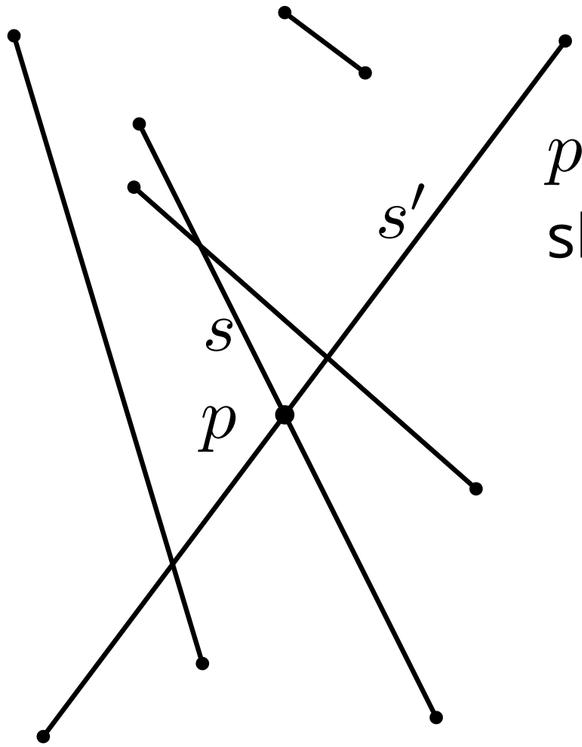

p is detected twice, but should be inserted once

Summary

The event list is an abstract data structure that stores all events in the order in which they occur

Here: binary search tree (ordered by decreasing y coordinate)

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The event list is an abstract data structure that stores all events in the order in which they occur

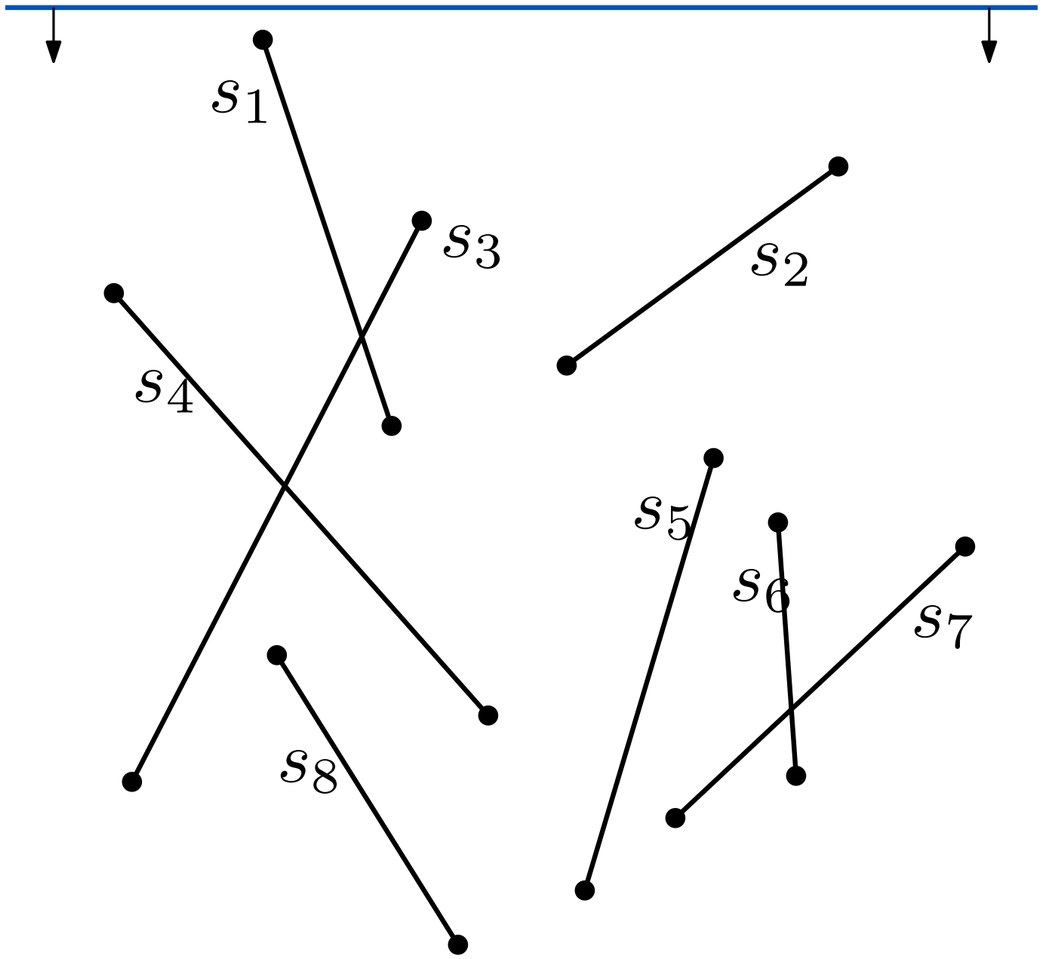
Here: binary search tree (ordered by decreasing y coordinate)

The status structure is an abstract data structure that maintains the current status

Here: binary search tree (ordered by x at current status)

Plane Sweep Algorithm

Structure of sweep algorithm



Structure of sweep algorithm

Algorithm FINDINTERSECTIONS(S)

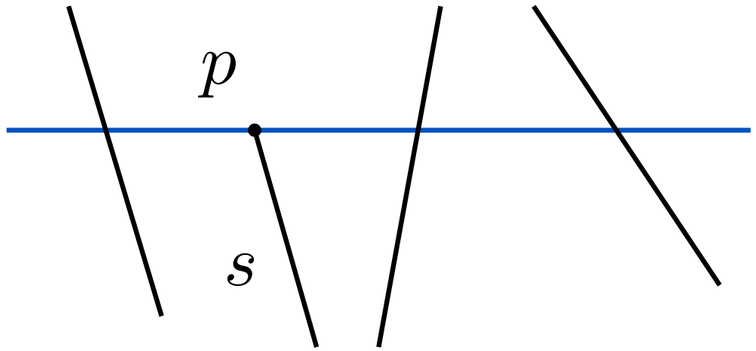
Input: set S of line segments in the plane
Output: intersection points of the segments in S, with a pair of segments containing it for each

1: initialize an empty event queue Q
2: insert the segment endpoints into Q; when an upper endpoint is inserted, the corresponding segment should be stored with it
3: initialize an empty status structure T
4: **while** Q is not empty **do**
5: determine next event point p in Q and delete it
6: HANDLEEVENTPOINT(p)

Event handling

If the event is an upper endpoint event, and s is the line segment that starts at p:

1. search with p in T, and
   insert s



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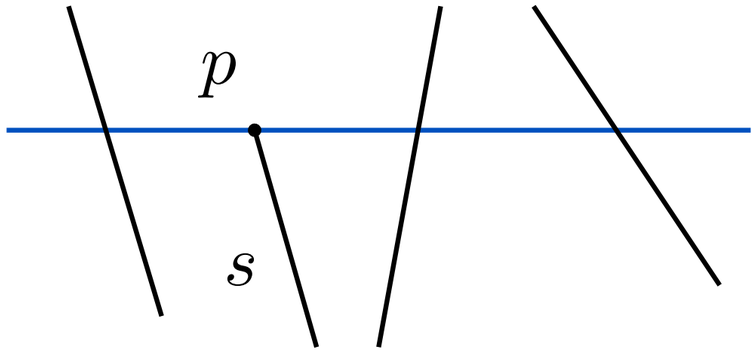

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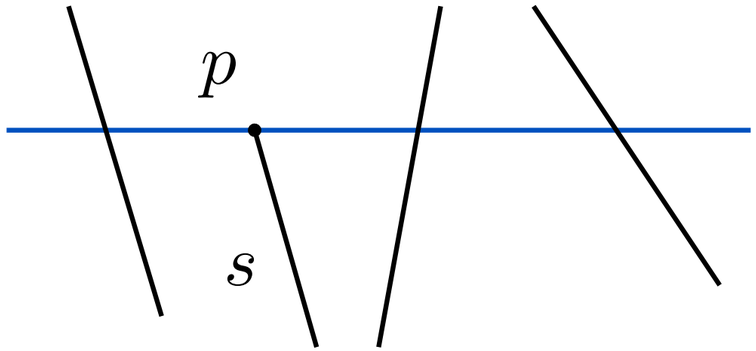
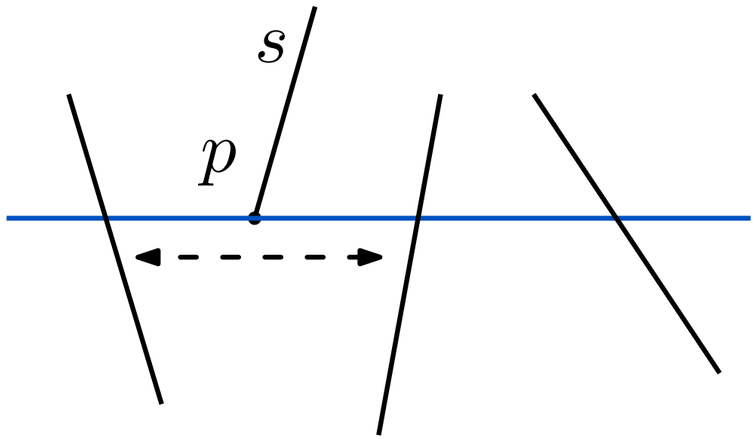


Event handling

If the event is an lower endpoint event, and s is the line segment that ends at p:

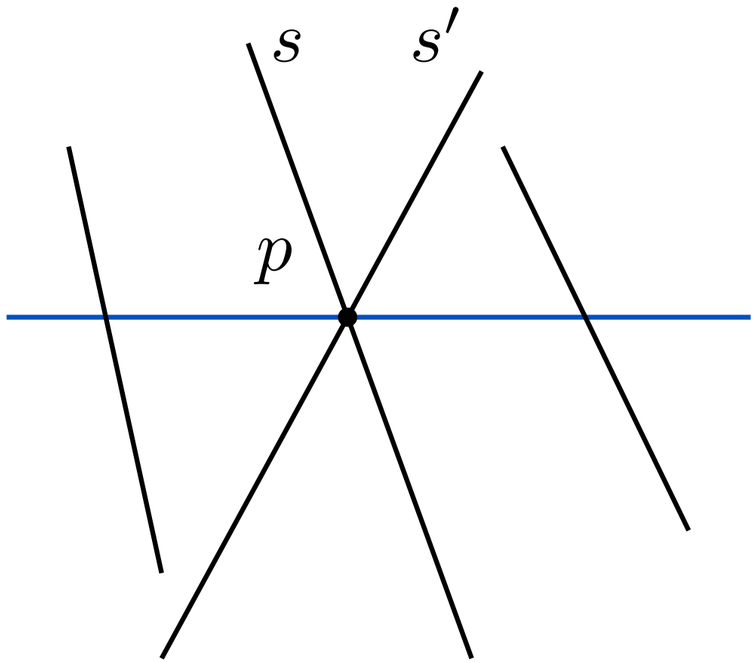
1. search with p in T, and delete s
2. let s_ℓ and s_r be the left and right neighbors of s in T (before deletion). If they intersect below the sweep line, then insert their intersection point as an event in Q



Event handling

If the event is an intersection point event, and s and s' intersect at p:

Question: steps?



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If the event is an intersection point event, and s and s' intersect at p:

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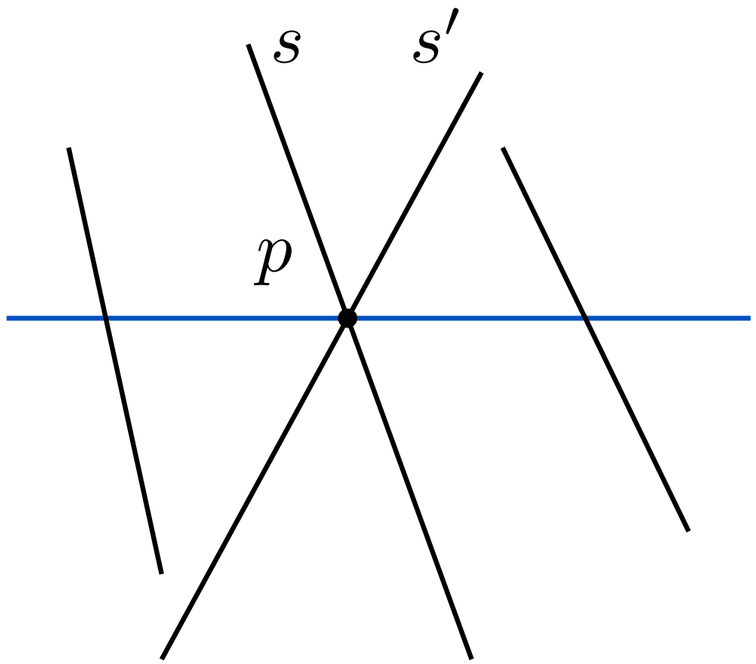

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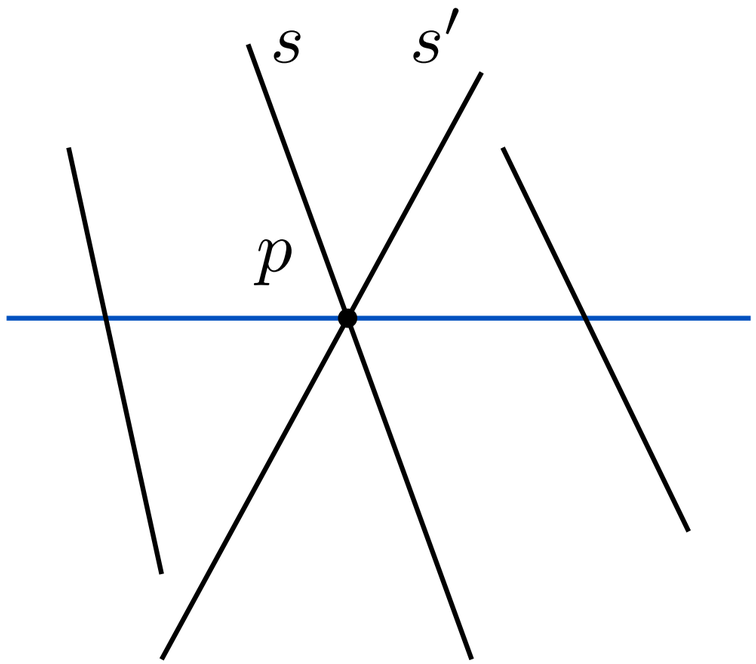

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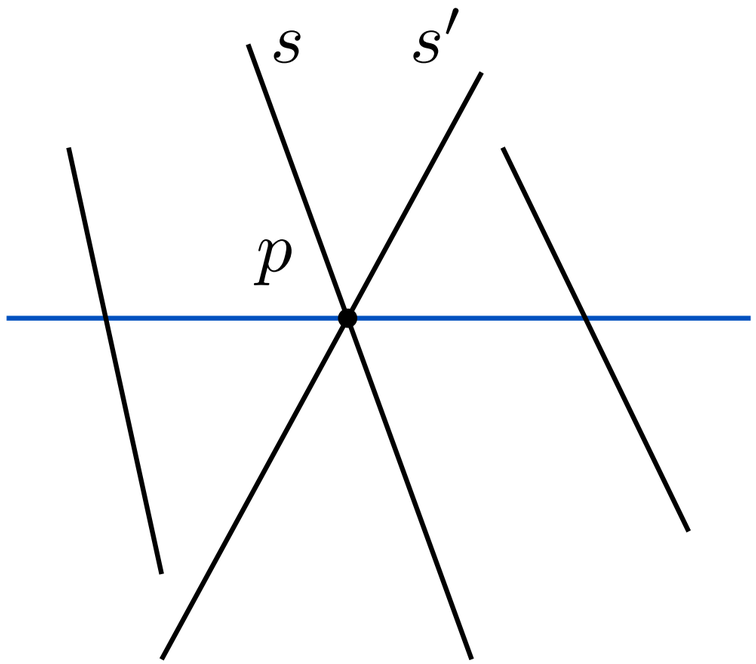

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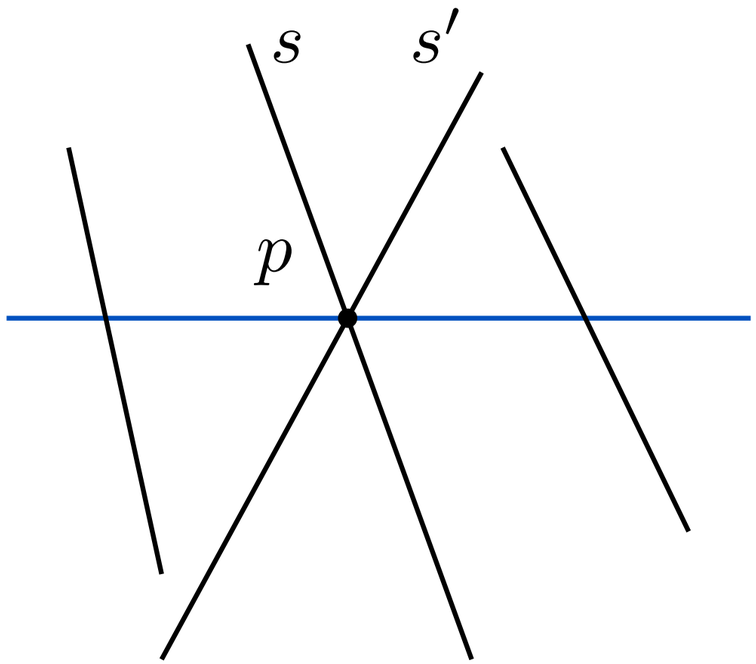

Quiz

What is the overall running time of the algorithm for n line segments with k intersections?

A: $O(n \log n + k)$

B: $O((n + k) \log n)$

C: $O(nk)$

Algorithm FINDINTERSECTIONS(S)

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At most one deletion from and two insertions in Q

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Since T and Q are balanced binary search trees, handling an event takes only $O(\log n)$ time

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- $2n$ for the upper and lower endpoints
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In total: $O(n + k)$ events

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Notes:

if k is really large, the brute force $O(n^2)$ time algorithm is more efficient

Faster algorithm exists: $O(n \log n + k)$

Efficiency

Question: How much storage does the algorithm take?

Efficiency

Question: Given that the event list is a binary tree that may store $O(k) = O(n^2)$ events, is the efficiency in jeopardy?

Efficiency

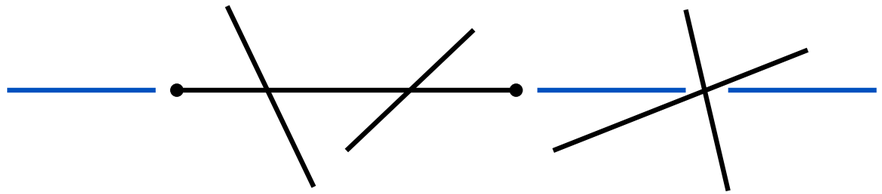
Solution:

Only store intersection points of currently adjacent segments.

Degenerate cases

How do we deal with degenerate cases?

For two different events with the same y-coordinate, we treat them from left to right $\Rightarrow$ the “upper” endpoint of a horizontal line segment is its left endpoint



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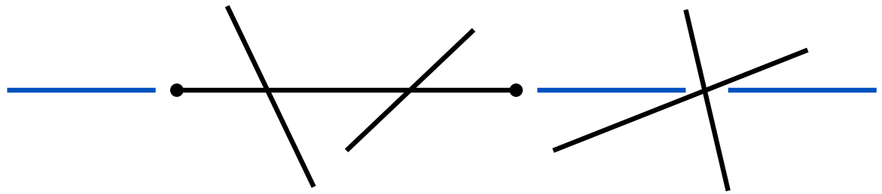
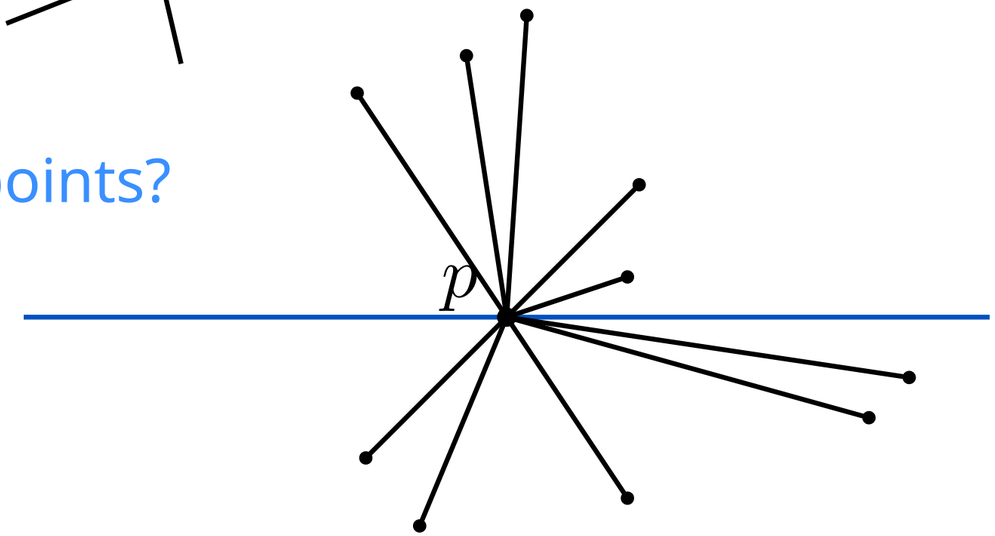


How about multiply coinciding event points?



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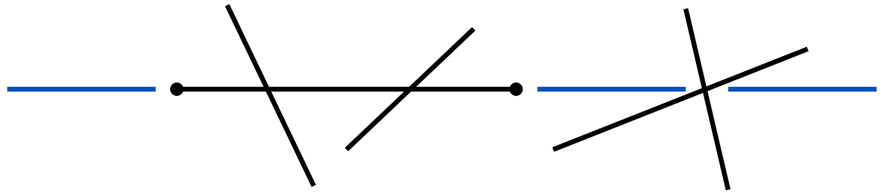

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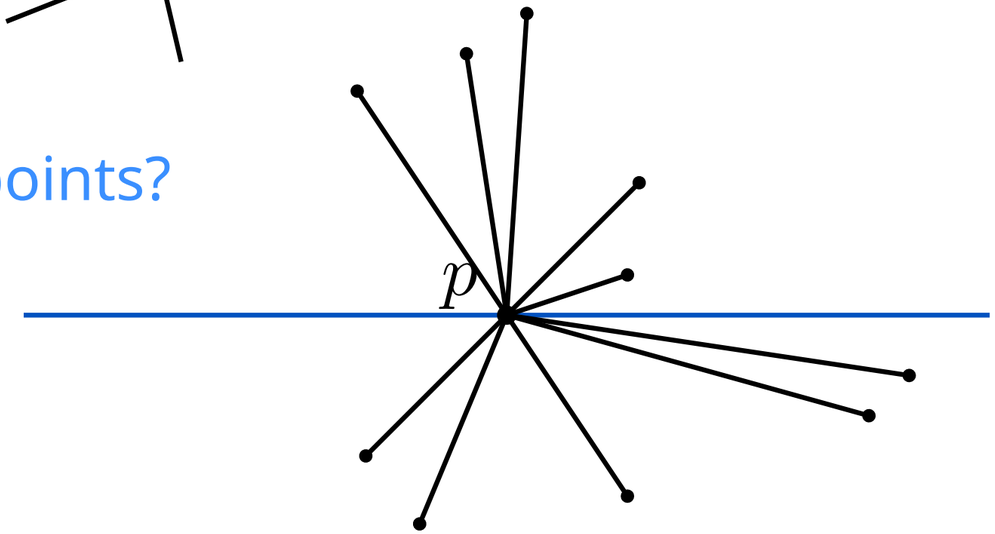

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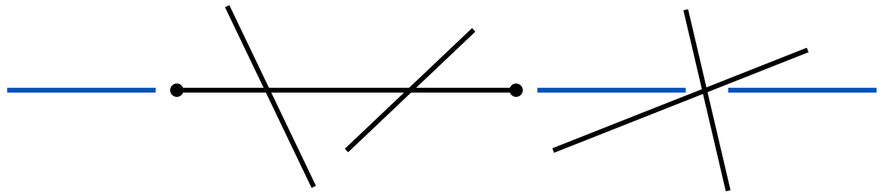

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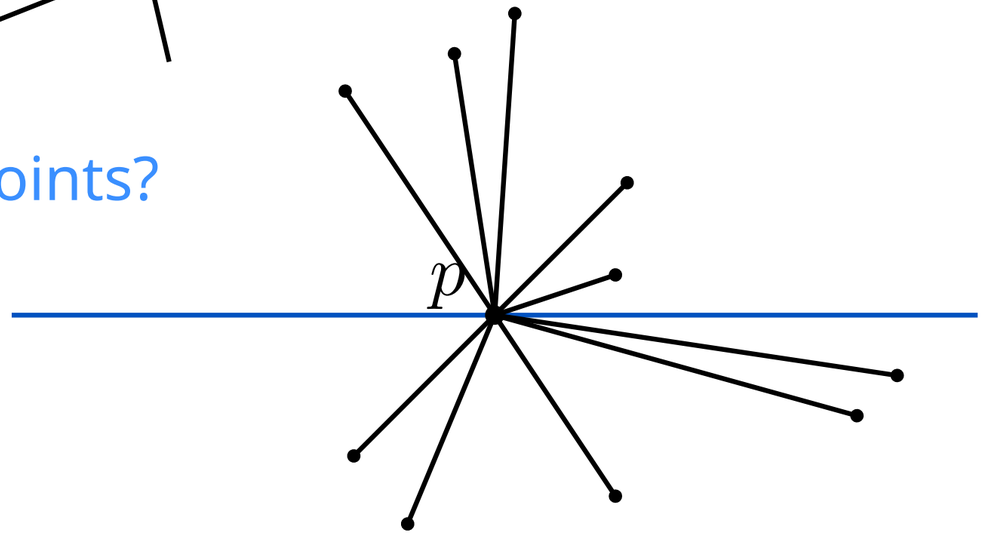
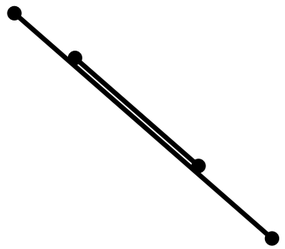


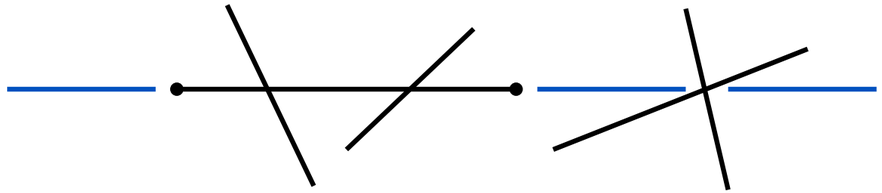
How do we handle overlap?



Degenerate cases

How do we deal with degenerate cases?

For two different events with the same y-coordinate, we treat them from left to right $\Rightarrow$ the “upper” endpoint of a horizontal line segment is its left endpoint



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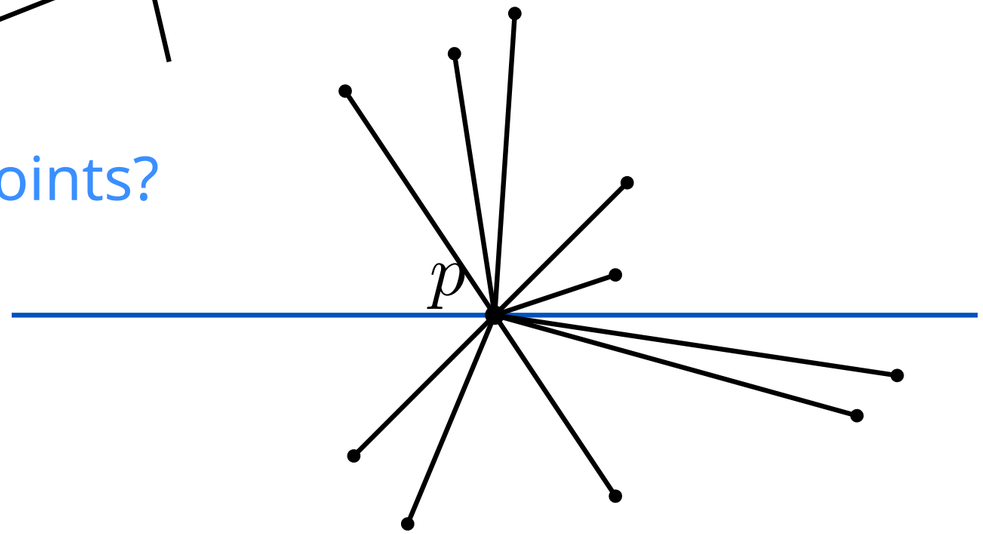

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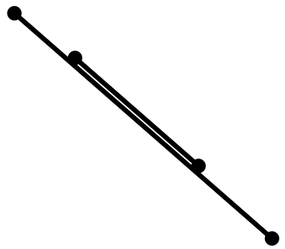

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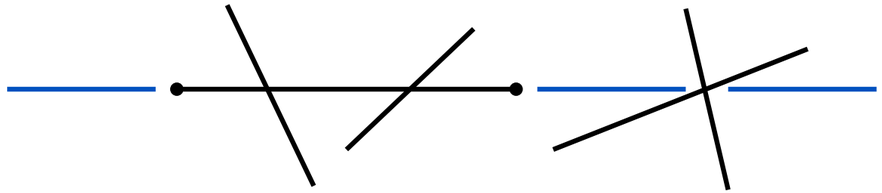

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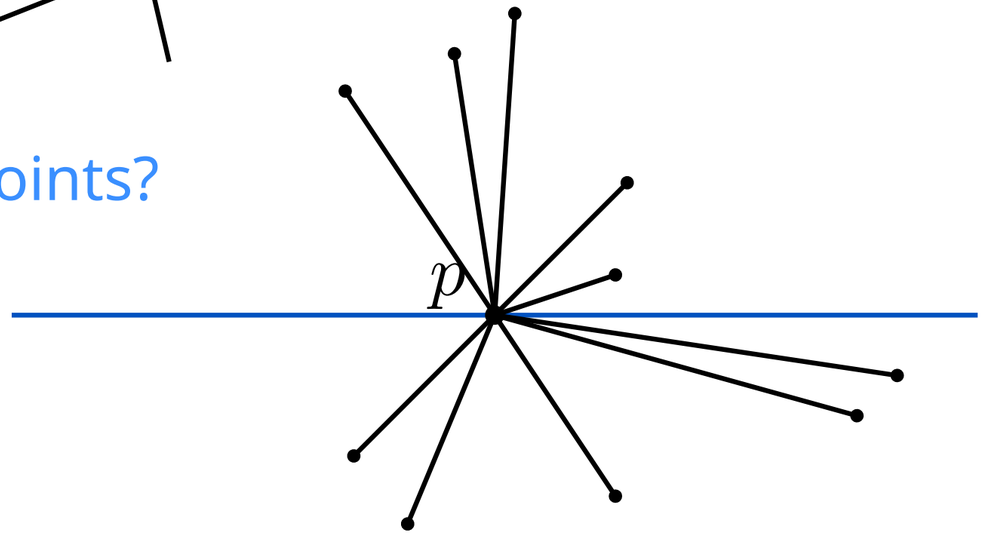

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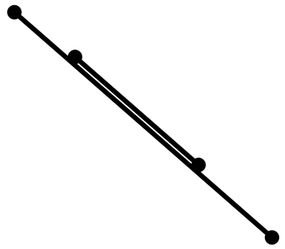
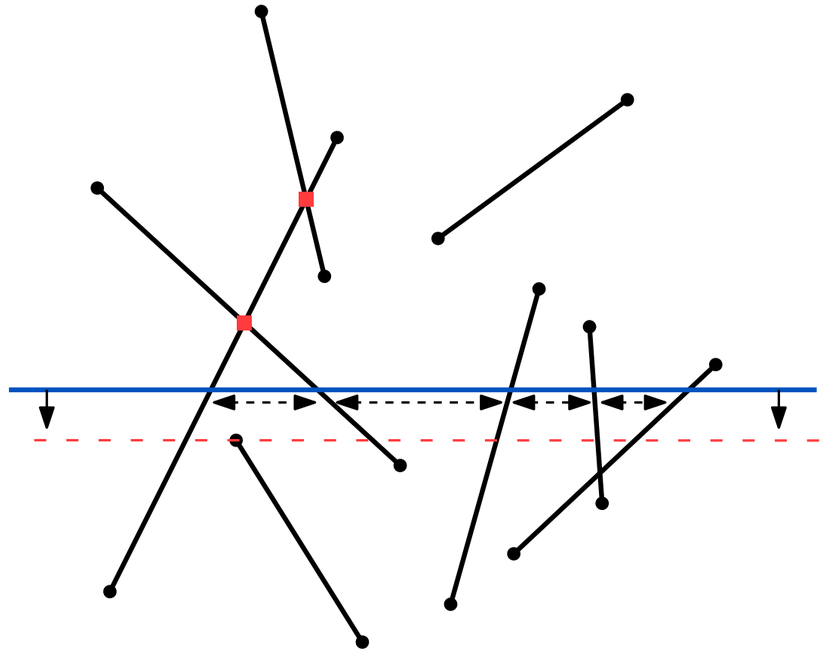


more difficult than it might seem

Plane Sweep – Conclusion

For every sweep algorithm:



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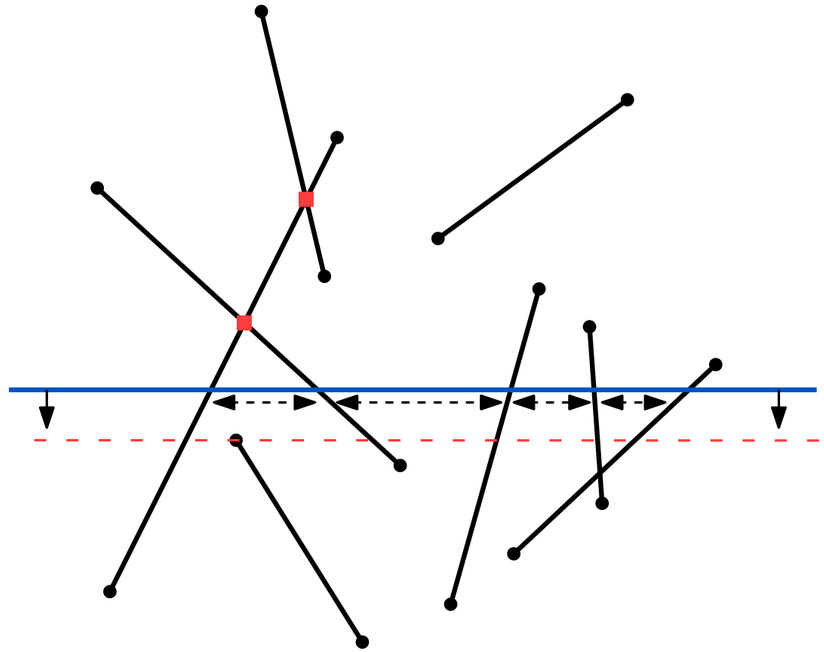
For every sweep algorithm:

- define the status, determine events



Plane Sweep – Conclusion

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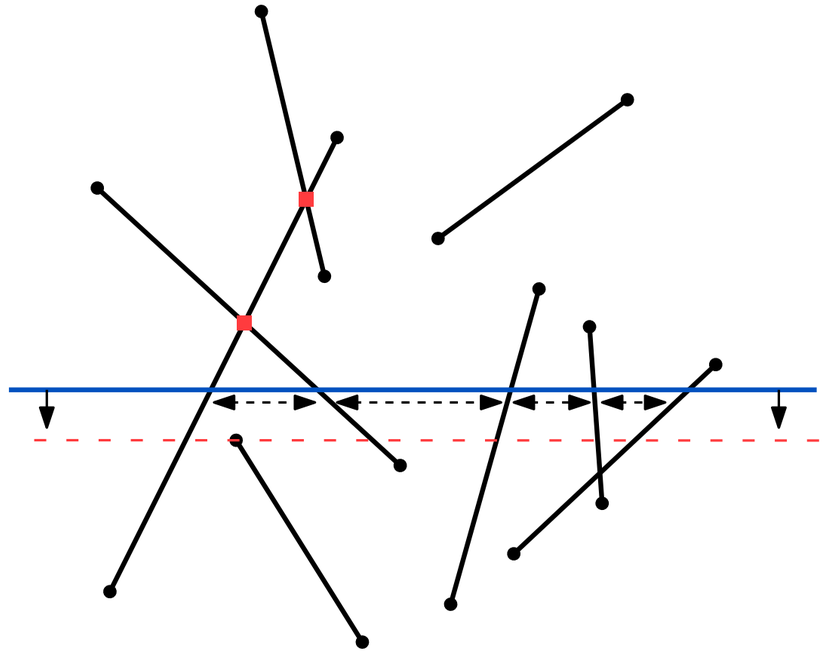

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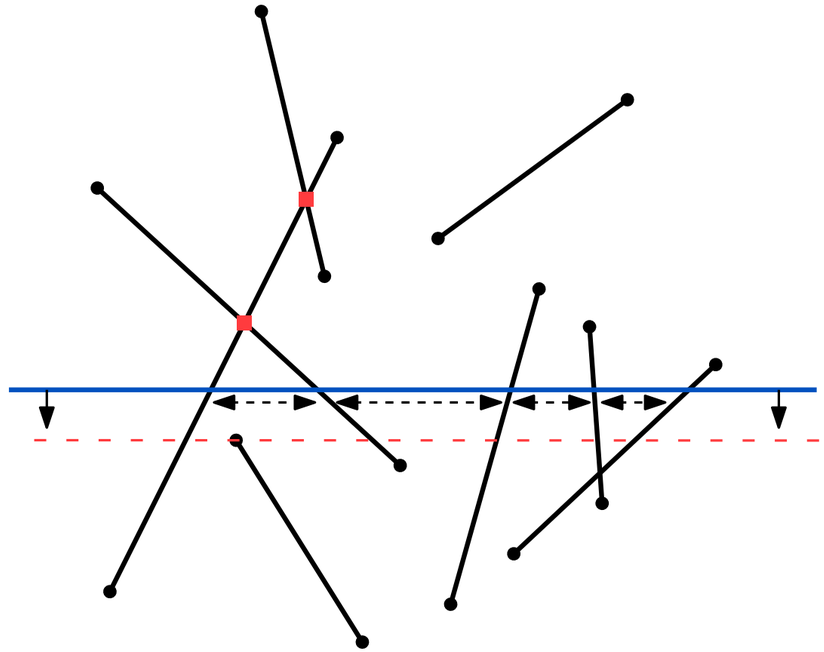

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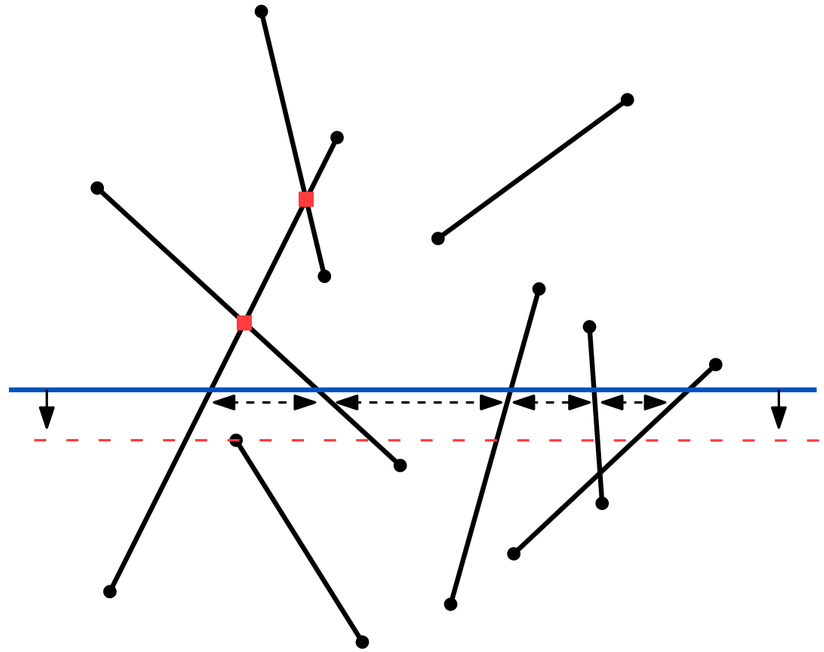
- to analyze, determine the number of events and how much time they take



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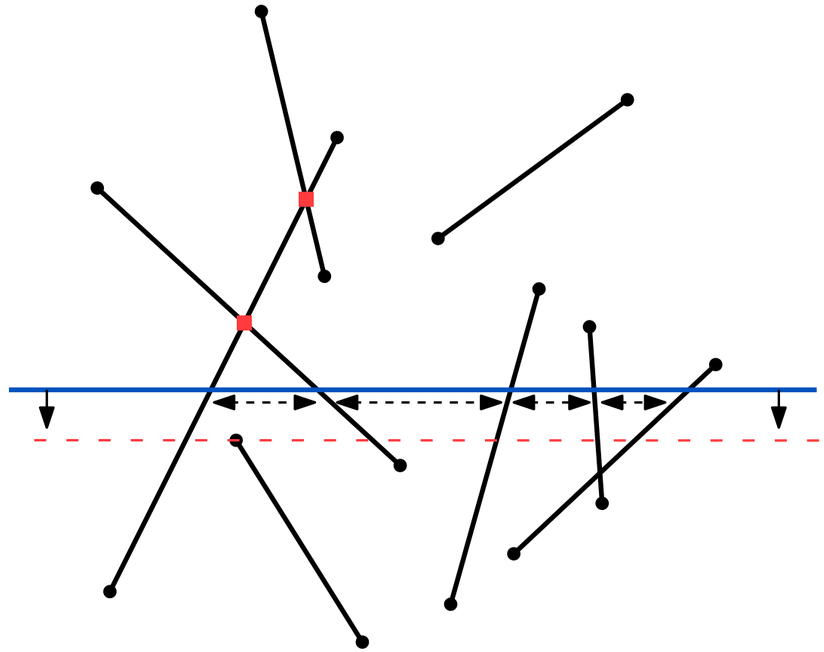
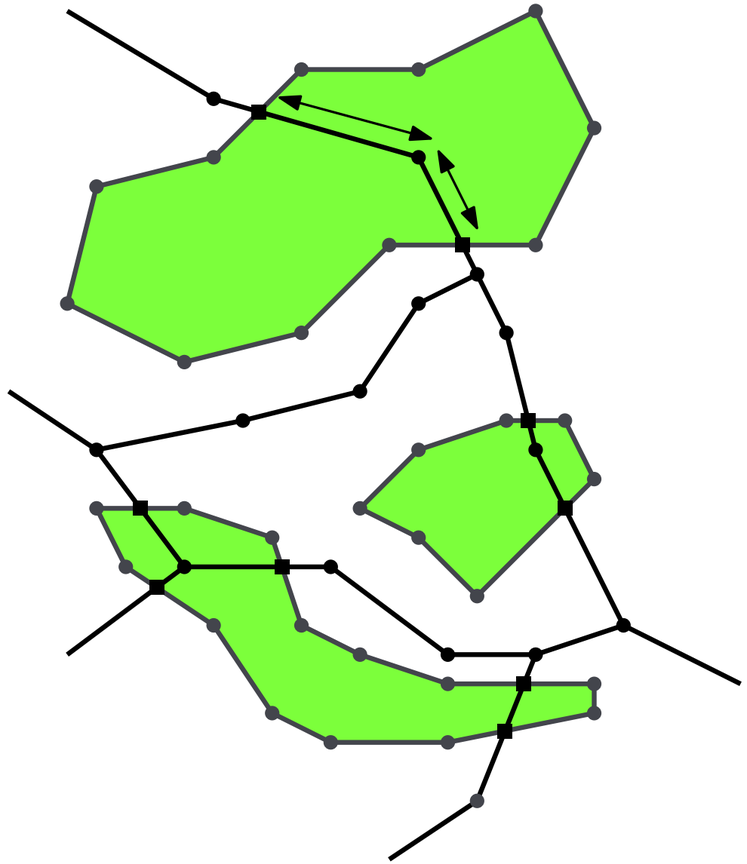


Then deal with degeneracies

Planar Subdivisions and Map Overlay

Map overlay

To solve map overlay questions, we need (at the least) intersection points from two sets of line segments.



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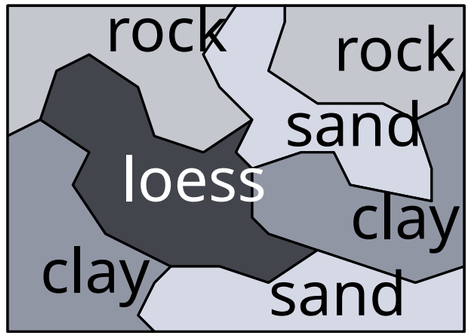
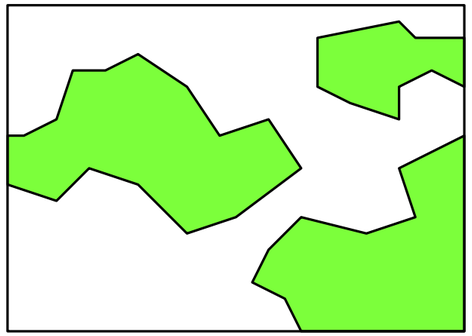
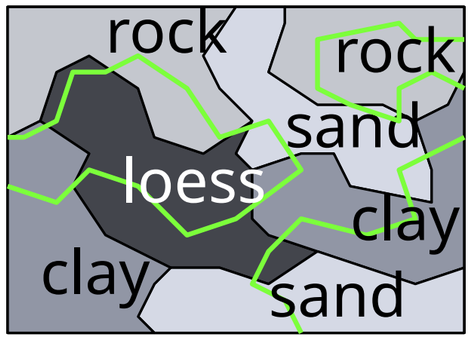


To solve map overlay questions, we also need to be able to represent subdivisions





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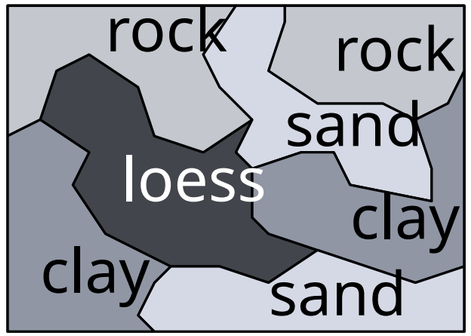

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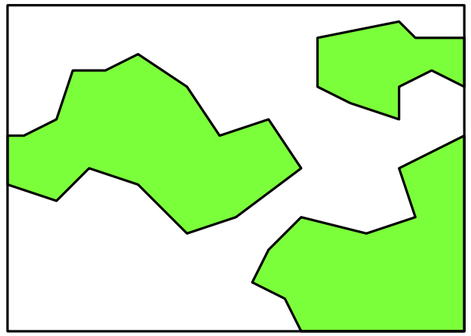
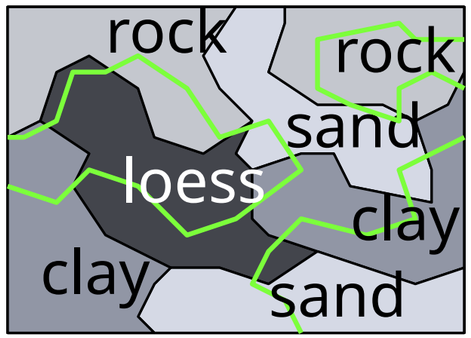


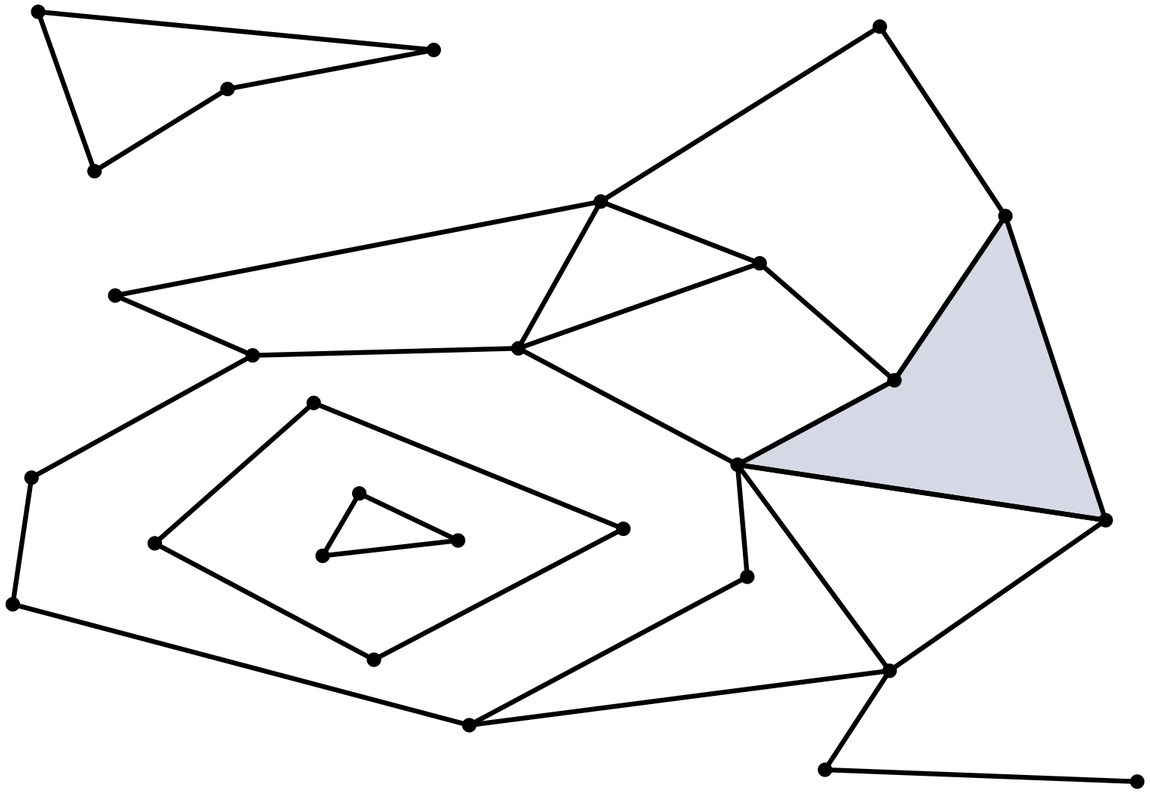
now

a data structure for planar subdivisions:
Doubly-connected edge list (DCEL)



Subdivisions

A planar subdivision is a structure induced by a set of line segments in the plane that can only intersect at common endpoints.



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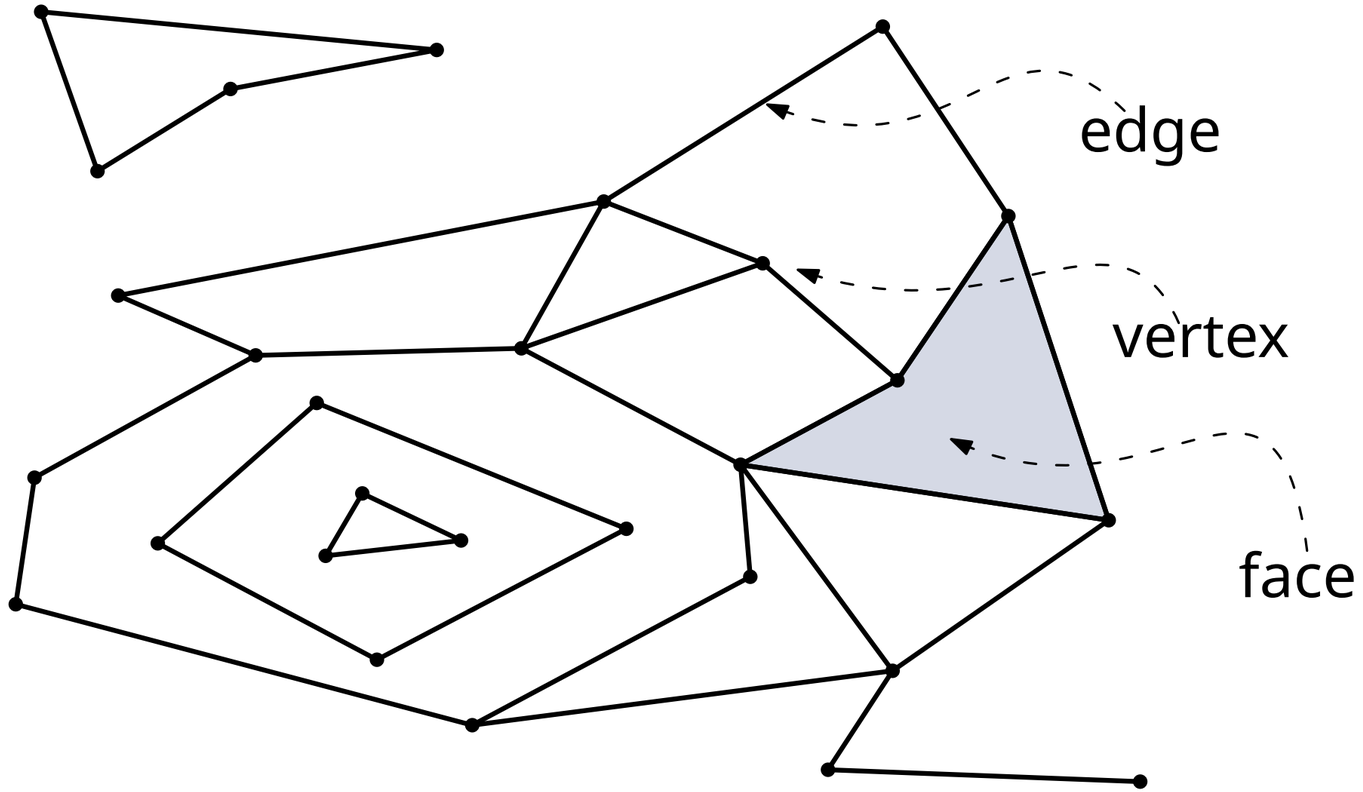


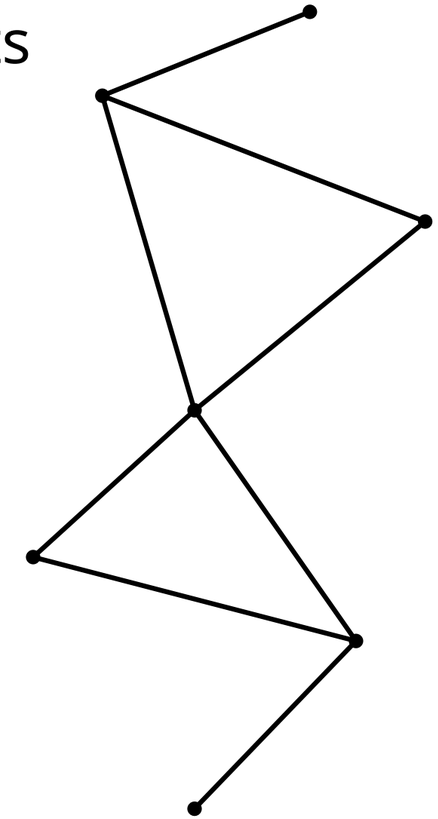
It consists of vertices, edges, and faces

Subdivisions

Vertices are the endpoints of the line segments

Edges are the interiors of the line segments

Faces are the interiors of connected two-dimensional regions that do not contain any point of any line segment



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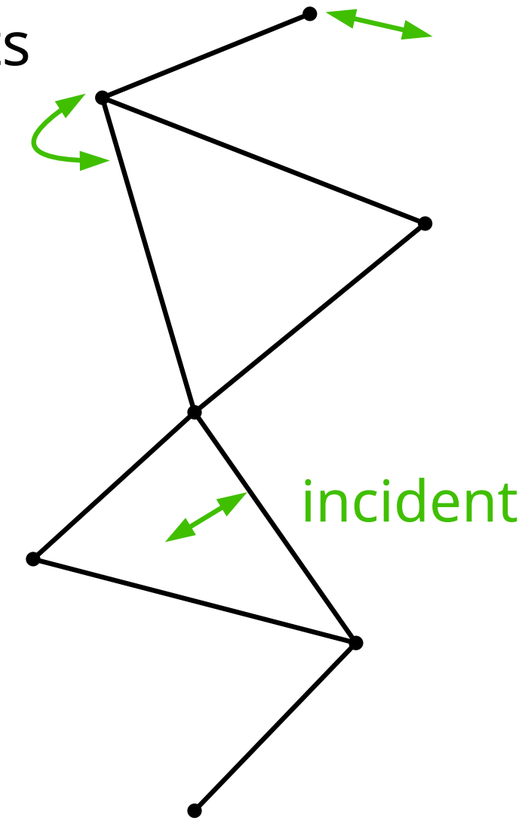

Objects of *different* dimensionality are incident or not

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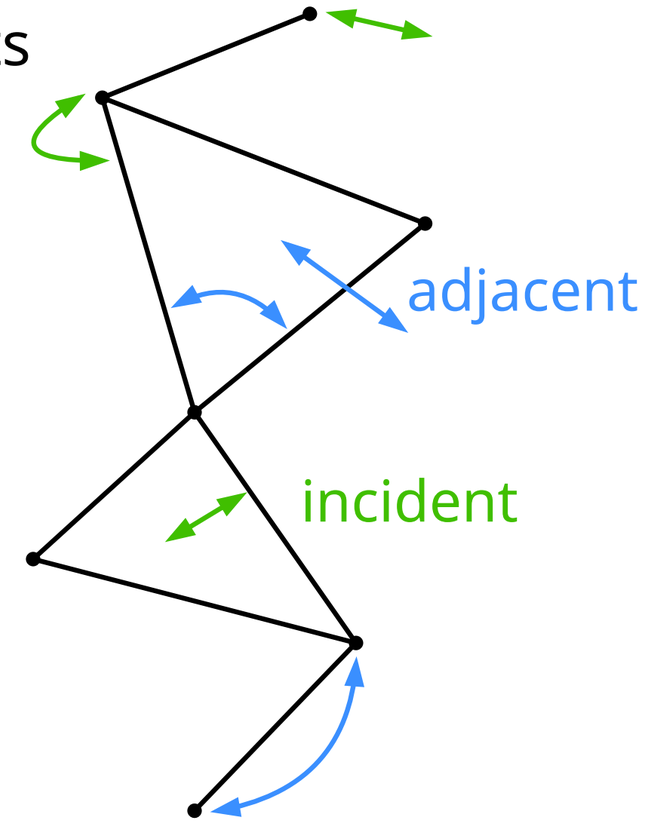
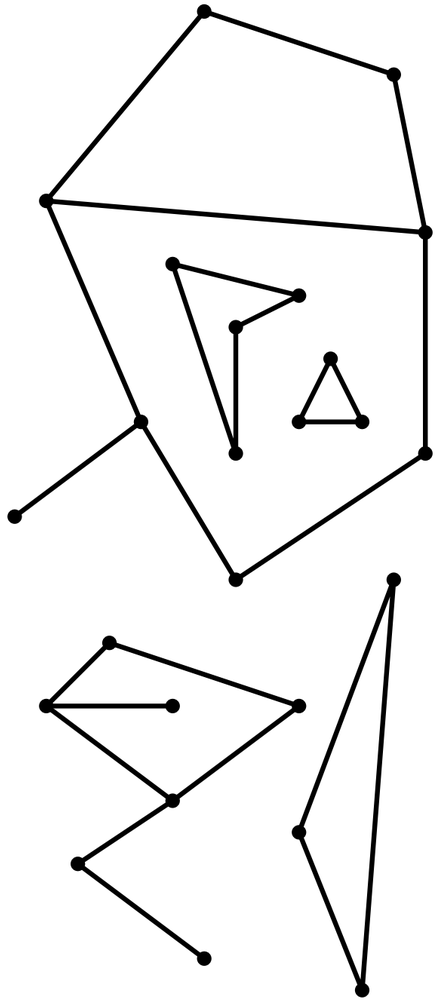


Objects of *different* dimensionality are incident or not

Objects of *the same* dimensionality are adjacent or not

Subdivisions

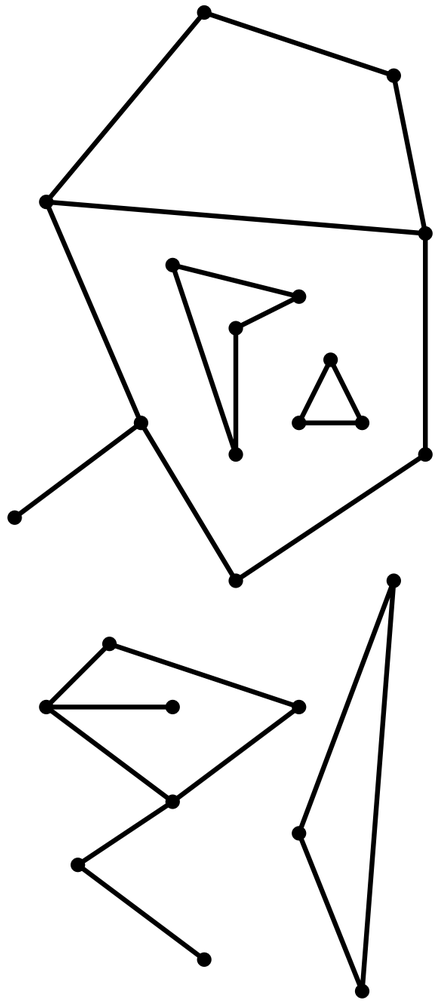
Exactly one face is unbounded, the outer face



Subdivisions

Exactly one face is unbounded, the outer face

Every other face is bounded and has an outer boundary consisting of vertices and edges



Subdivisions

Exactly one face is unbounded, the outer face

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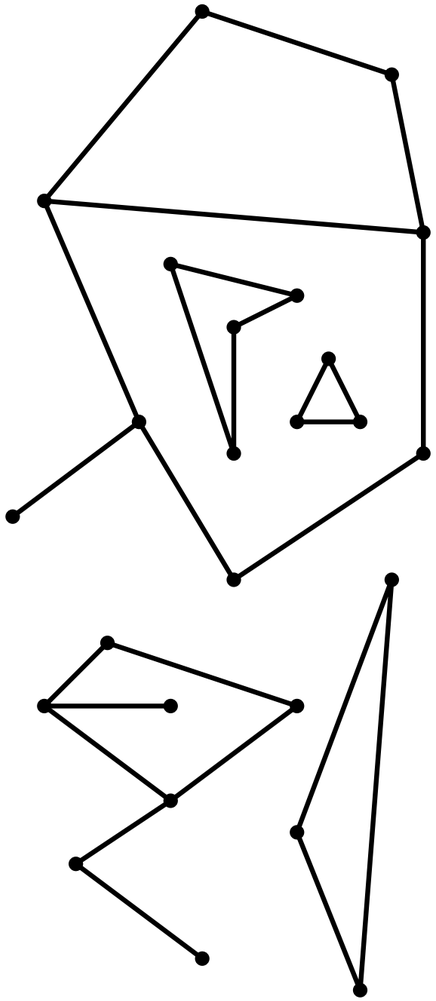
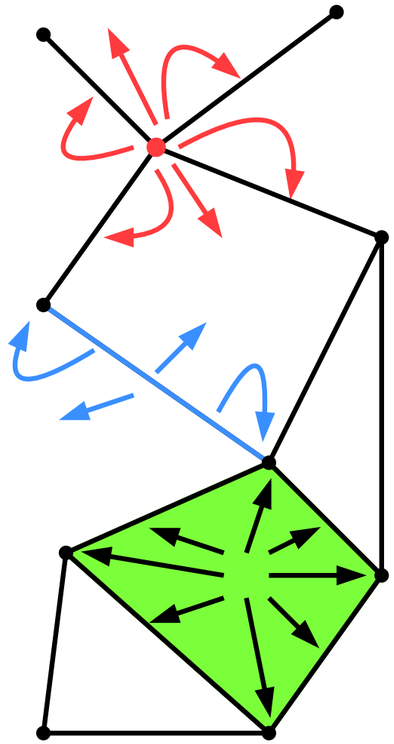
Any face has zero or more inner boundaries



Representing subdivisions

A subdivision representation has a vertex-object class, an edge-object class, and a face-object class

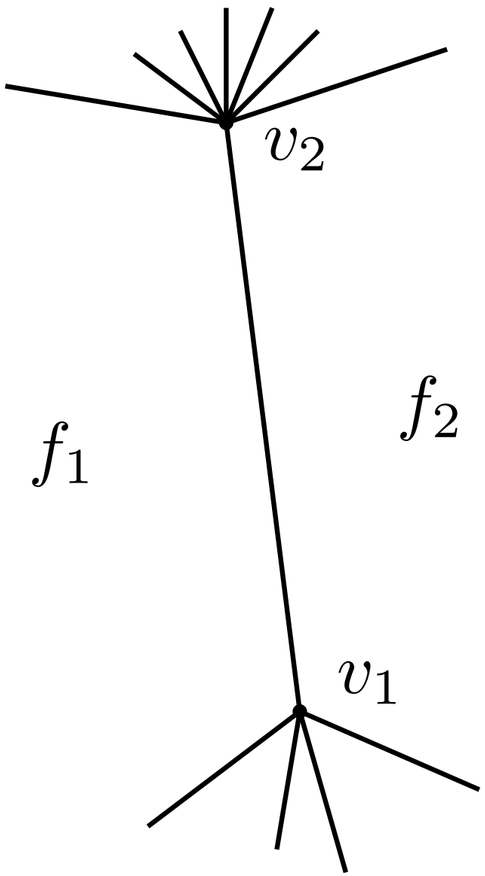


It is a pointer structure where objects can reach incident (or adjacent) objects easily

Representing subdivisions

Use the edge as the central object

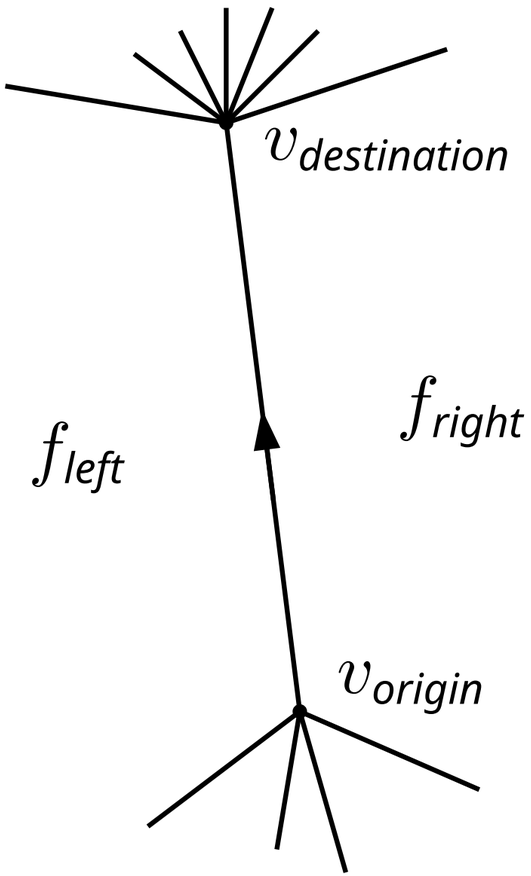
For any edge, exactly two vertices are incident, up to two faces are incident, and zero or more other edges are adjacent



Representing subdivisions

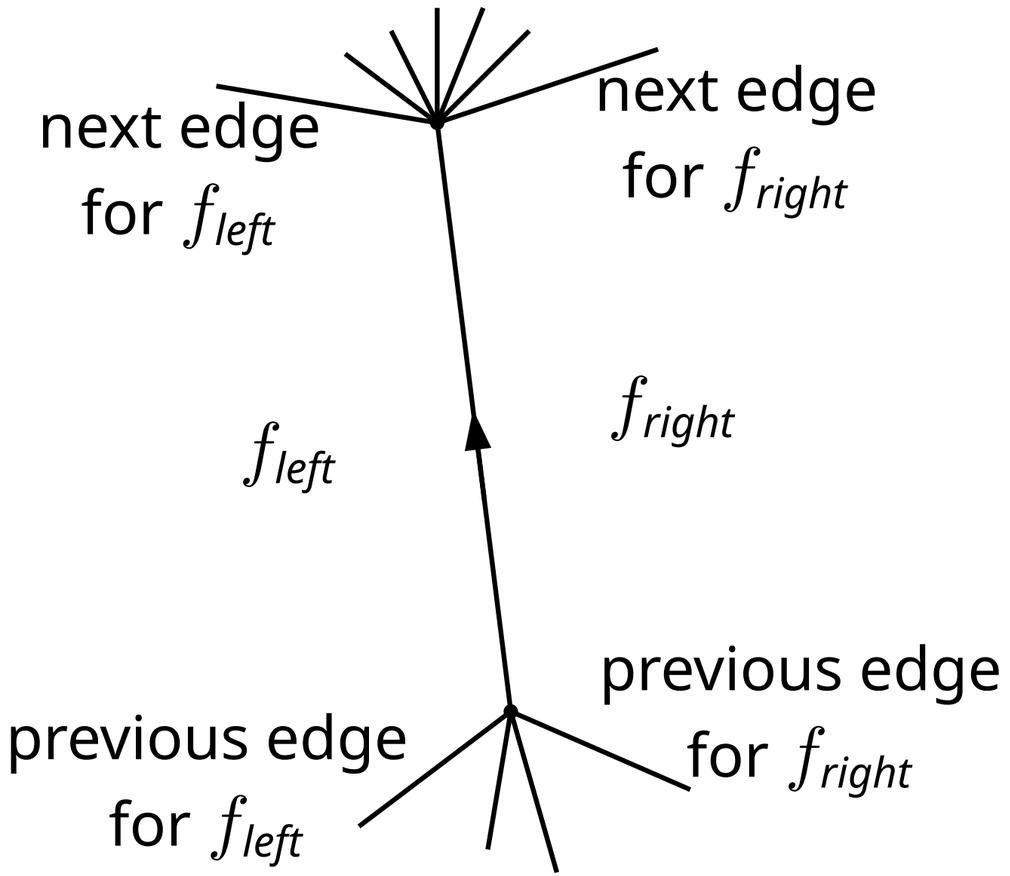
Use the edge as the central object,
and give it a direction

Now we can speak of origin, destination,
left face, and right face



Representing subdivisions

Four edges are of special interest



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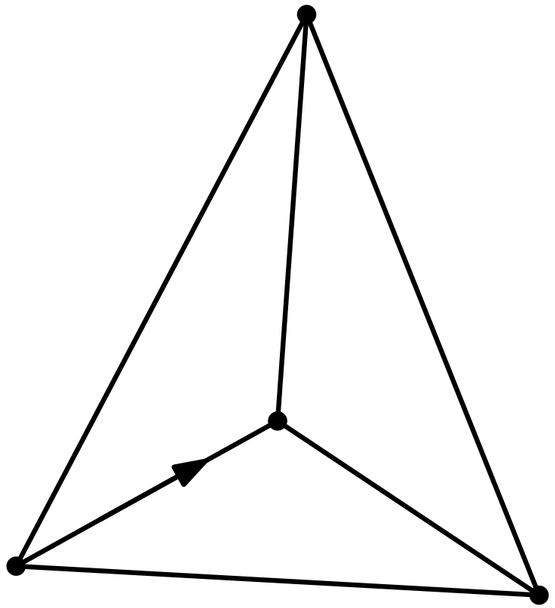
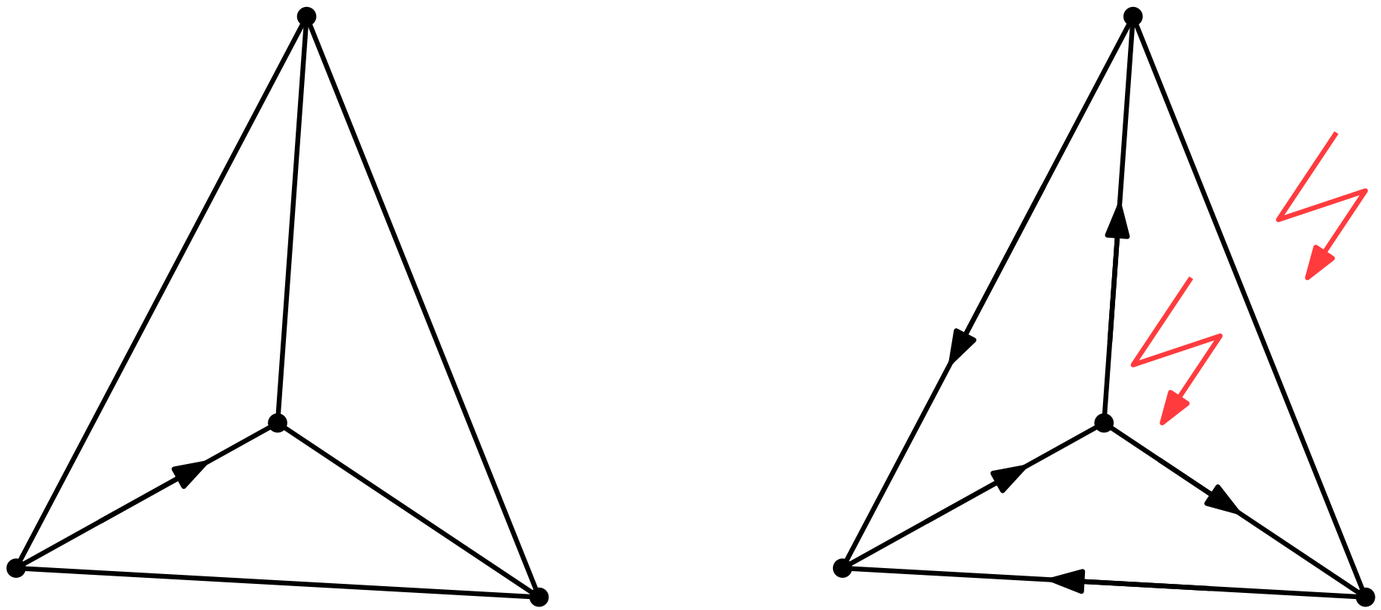


It would be nice if we could traverse a boundary cycle by continuously following the next edge for f_{left} or f_{right}

Representing subdivisions

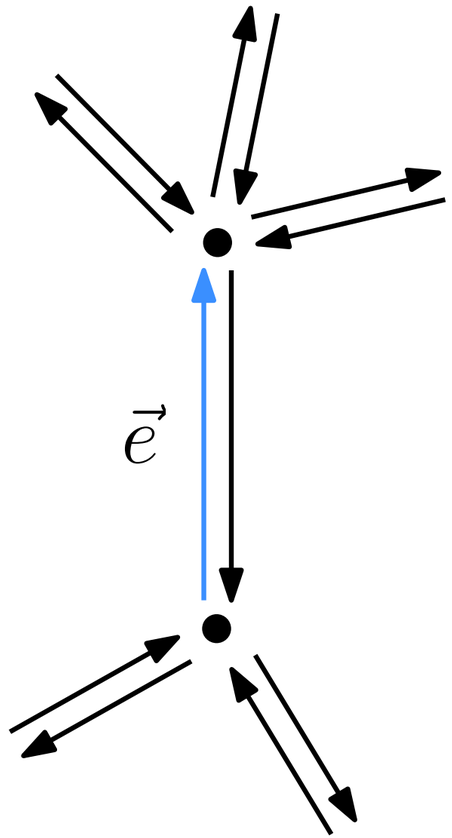


It would be nice if we could traverse a boundary cycle by continuously following the next edge for f_{left} or f_{right}

. . . but, no consistent edge orientation needs to exist

Representing subdivisions

We apply a trick/hack/impossibility: split every edge *length-wise*(!) into two half-edges



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Every half-edge:

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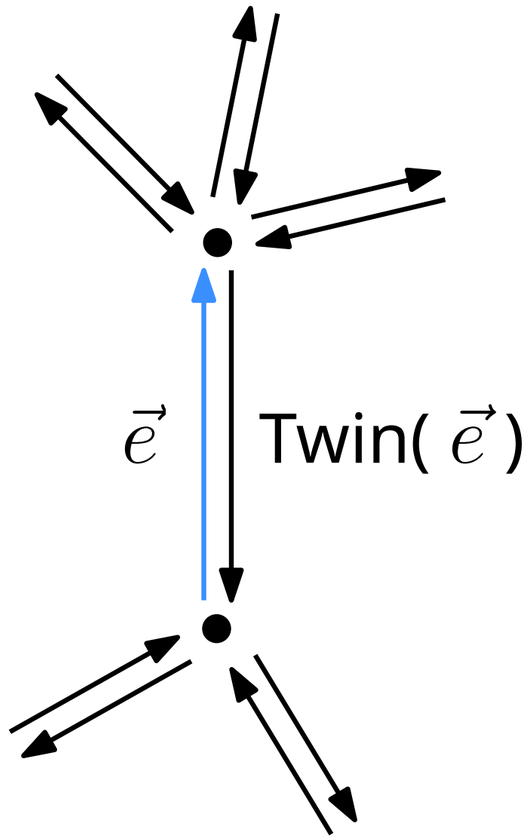
- is directed opposite to its Twin



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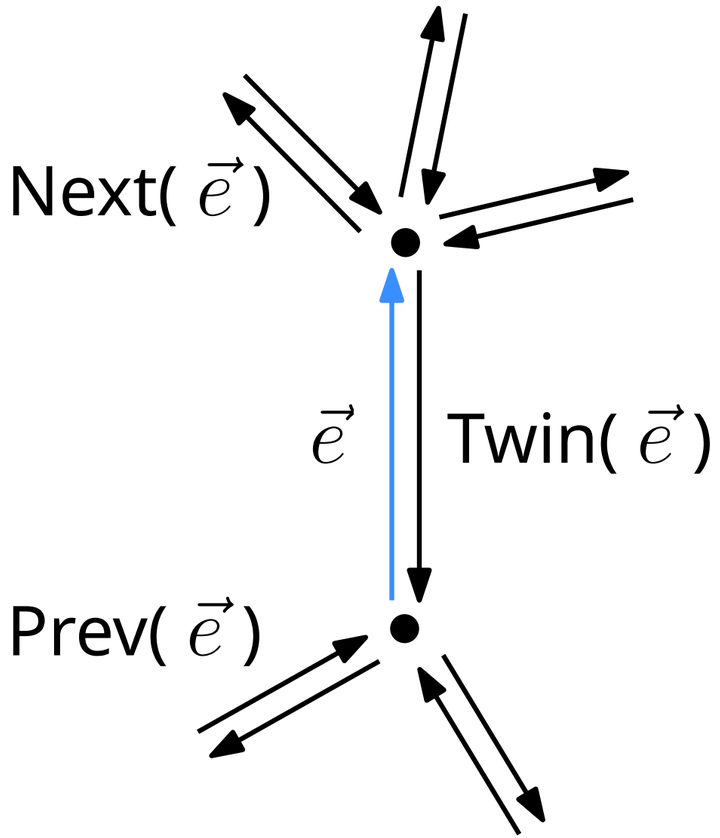
- is incident to only one face (left)



The doubly-connected edge list

The doubly-connected edge list is a subdivision representation structure with an object for every vertex, every half-edge, and every face

A vertex object stores:

- Coordinates
- **IncidentEdge** (some half-edge leaving it)

A half-edge object stores:

- **Origin** (vertex)
- **Twin** (half-edge)
- **IncidentFace** (face)
- **Next** (half-edge in cycle of the incident face)
- **Prev** (half-edge in cycle of the incident face)

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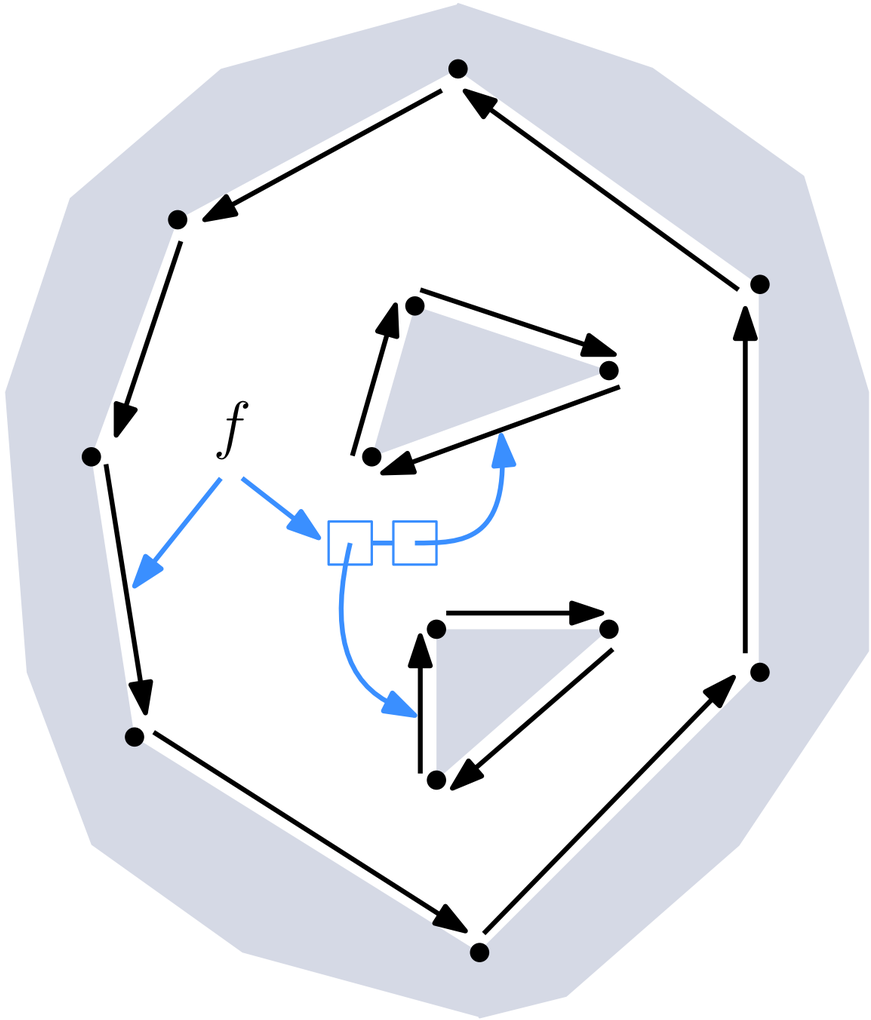
Question: How can we report all incident edges at a vertex?

The doubly-connected edge list

A face object stores:

- **OuterComponent** (half-edge of outer cycle)
- **InnerComponents** (list of half-edges for the inner cycles bounding the face)



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Quiz

Which of the following equalities is not always true?

A: next(previous(e)) = e

B: IncidentFace(e) = IncidentFace(Next(e))

C: Twin(Prev(Twin(e))) = Next(e)

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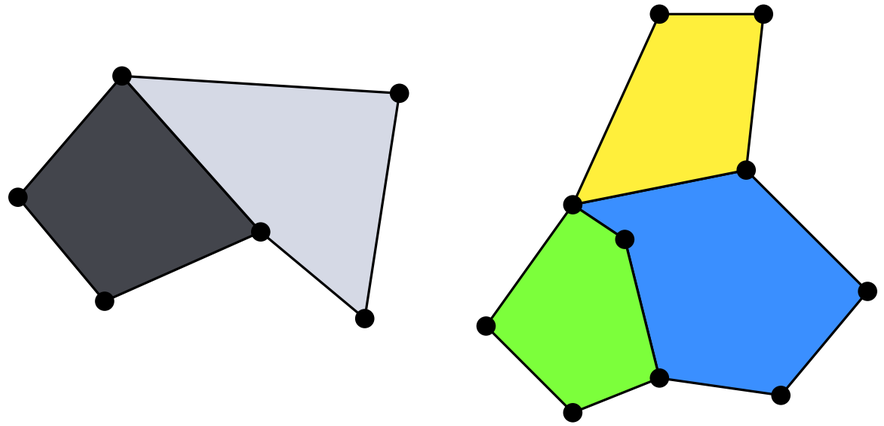
Map overlay problem

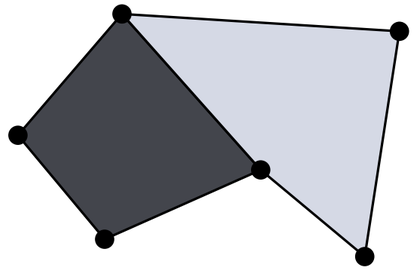
The map overlay problem for two subdivisions S_1 and S_2 is to compute a subdivision S that is the overlay of S_1 and S_2

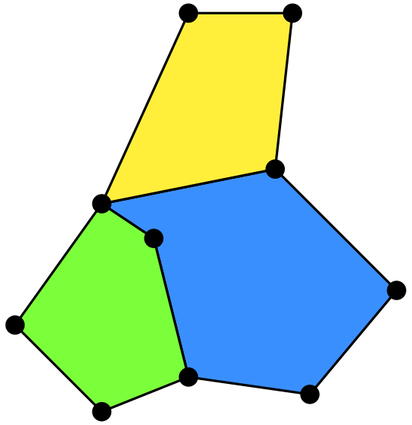


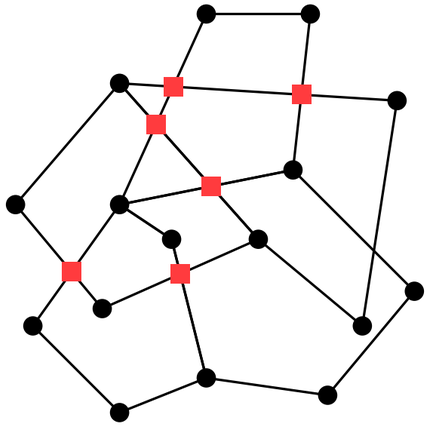
Map overlay problem

The map overlay problem for two subdivisions S_1 and S_2 is to compute a subdivision S that is the overlay of S_1 and S_2

All edges of S are (parts of) edges from S_1 and S_2. All vertices of S are also in S_1 or S_2, or intersections of edges from S_1 and S_2

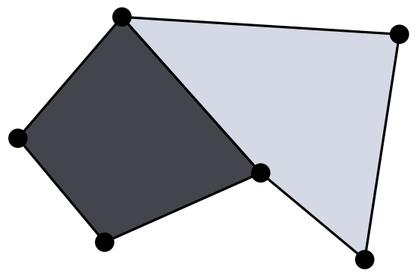


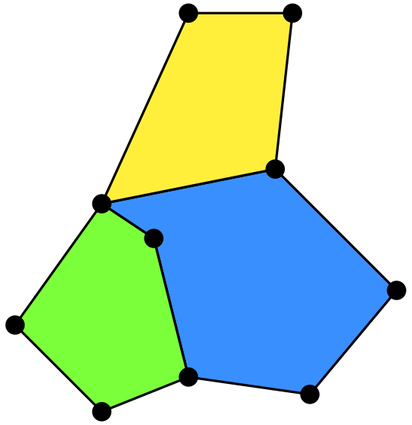




Map overlay problem

The map overlay problem for two subdivisions S_1 and S_2 is to compute a subdivision S that is the overlay of S_1 and S_2





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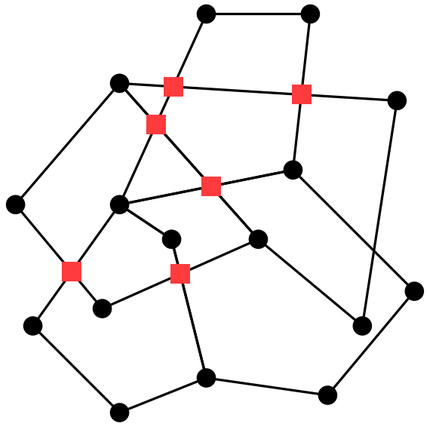

The line-segment intersection algorithm can be extended to this problem using doubly-connected edge lists.
[Chapter 2.3]

Summary

Application: map overlay

Geometric problem: line segment intersection

Algorithmic technique: Plane sweep

Data structure: Doubly-connected edge list