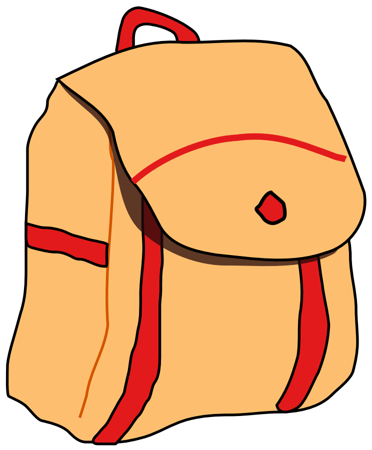
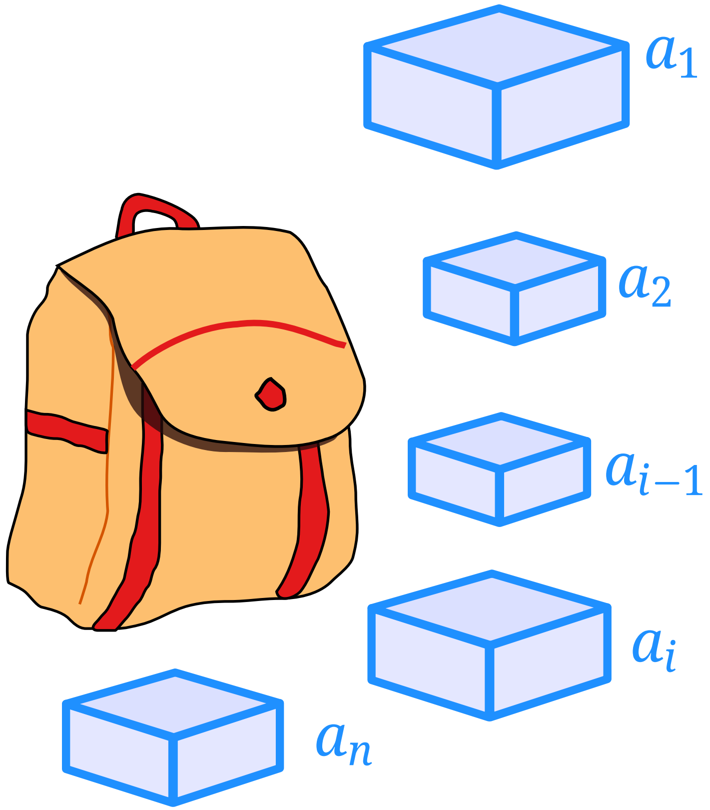
Approximation Schemes and the KNAPSACK Problem



KNAPSACK

Given: ■ A set $S = \{a_1, \ldots, a_n\}$ of **objects**.



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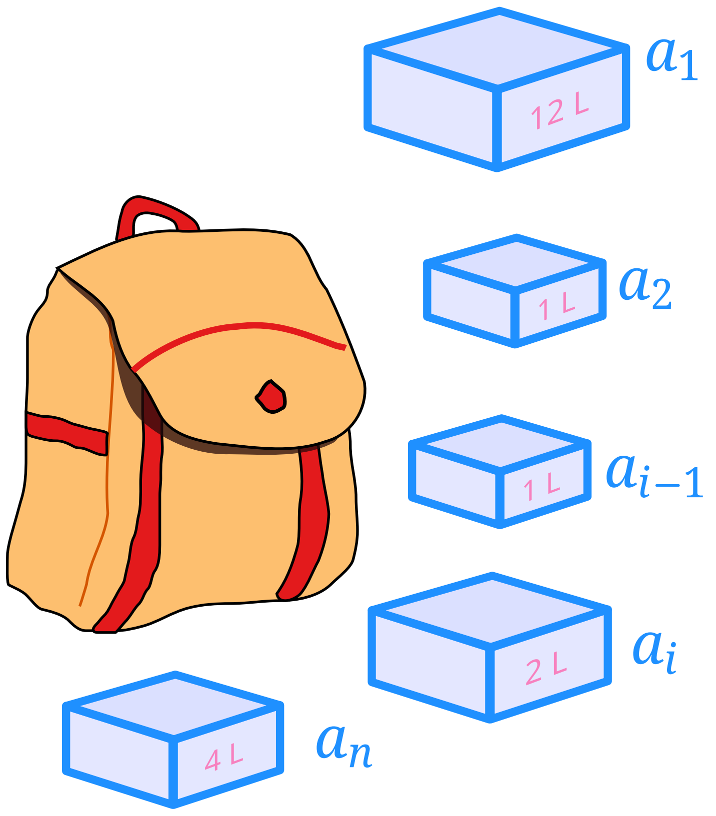

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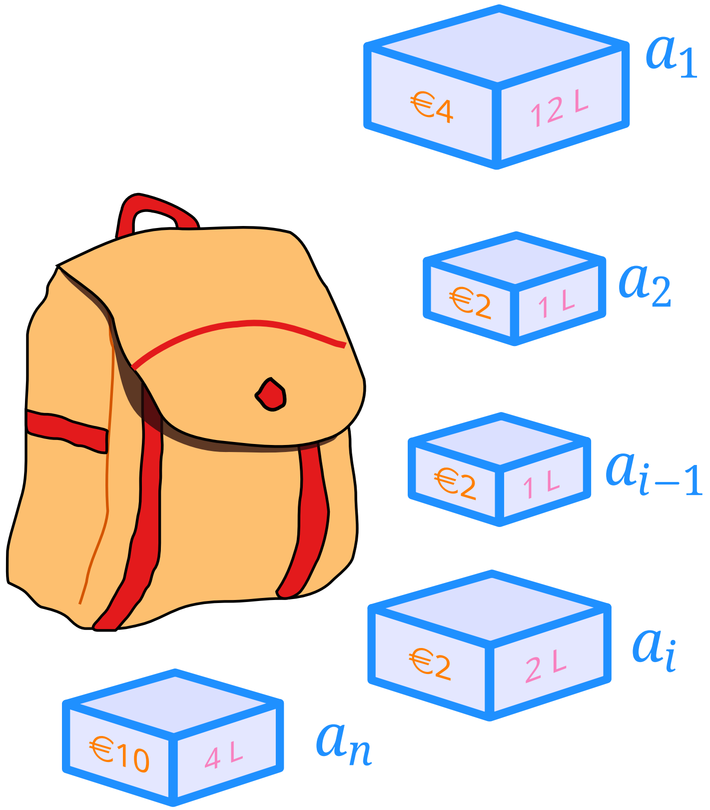

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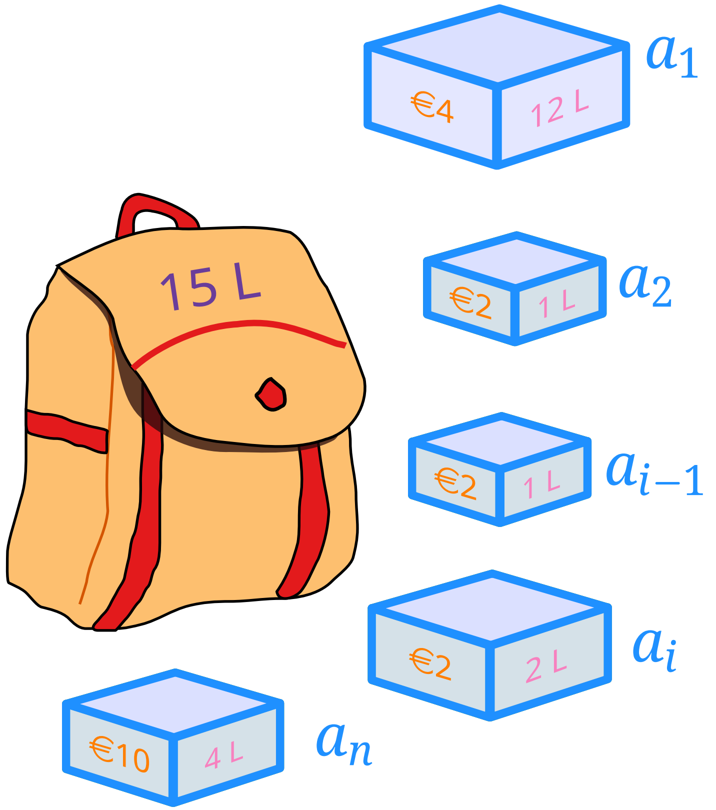

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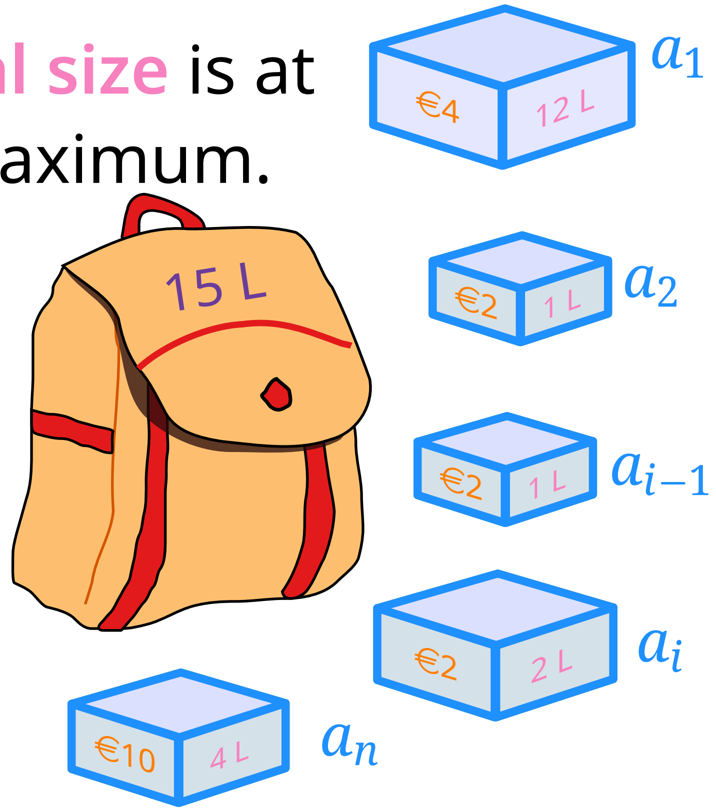

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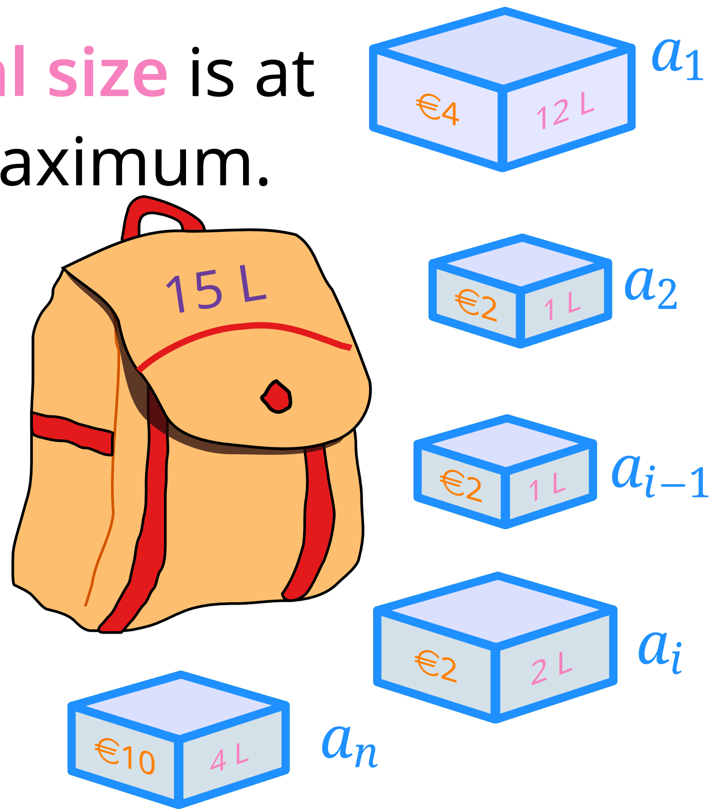

Which subset is optimal in the example?

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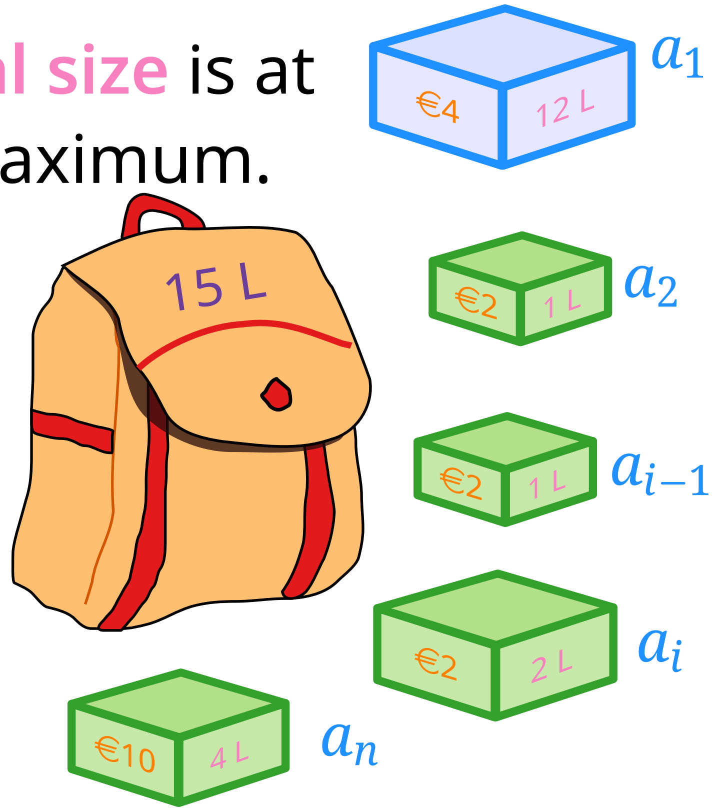

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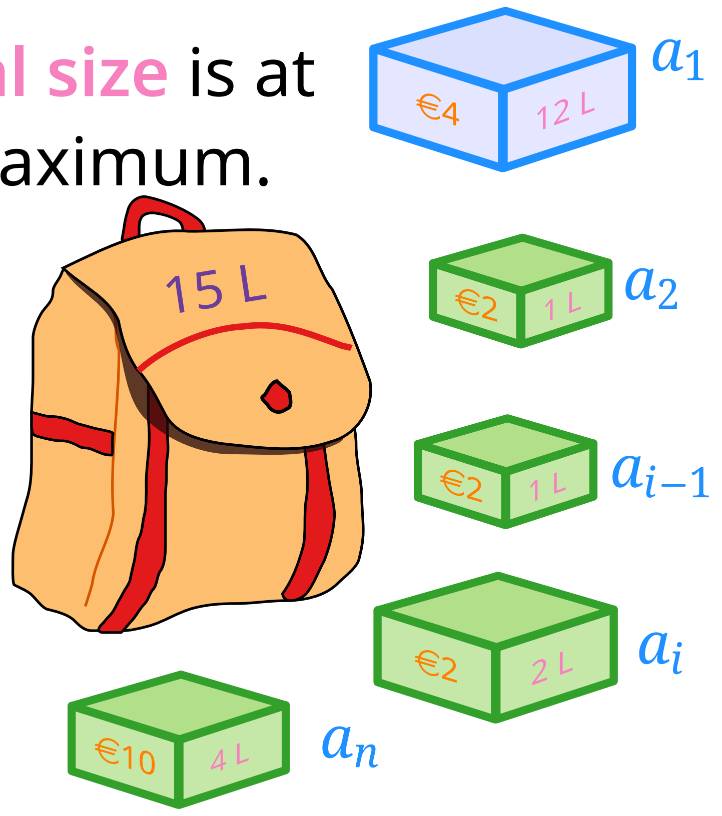
NP-hard



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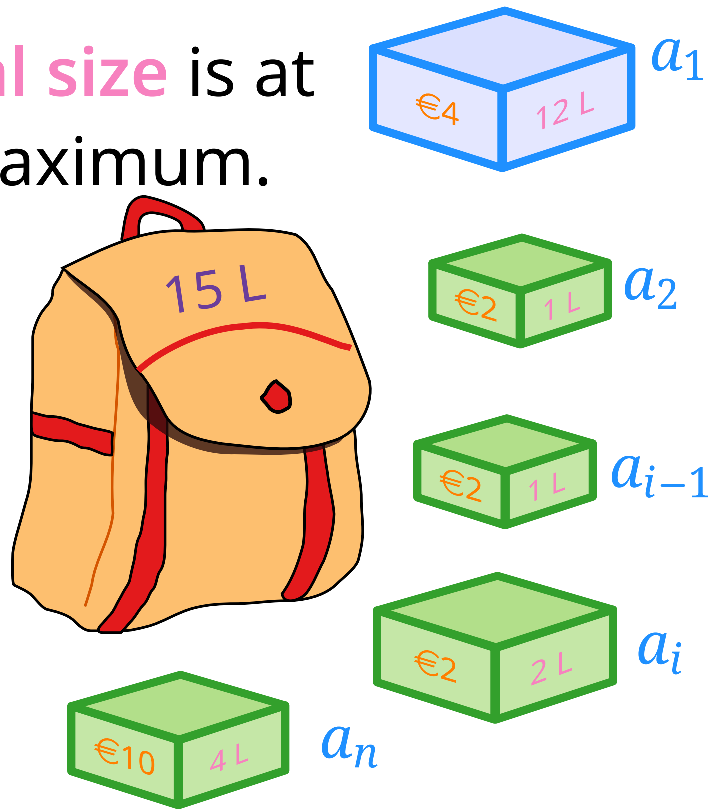

Natural greedy approach:

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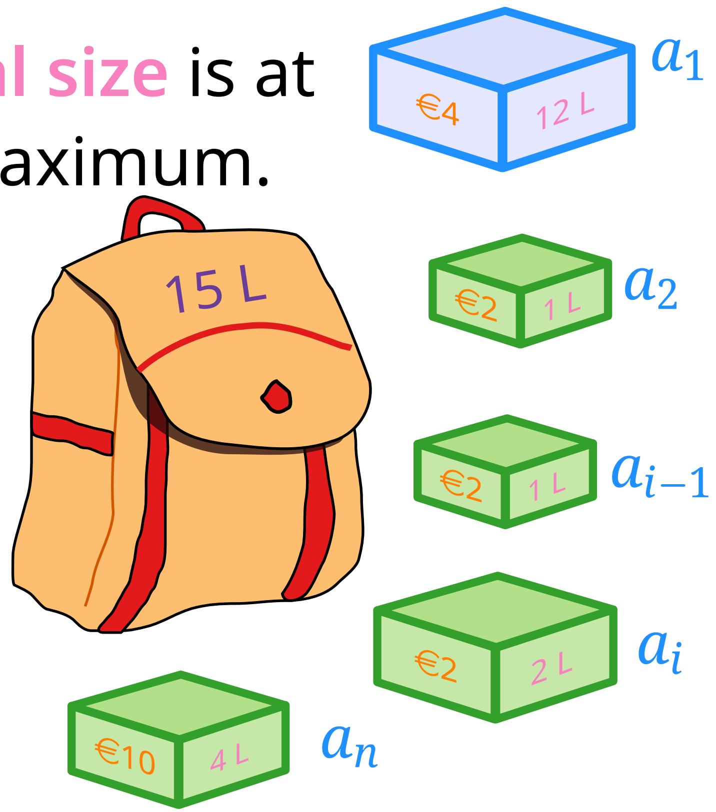

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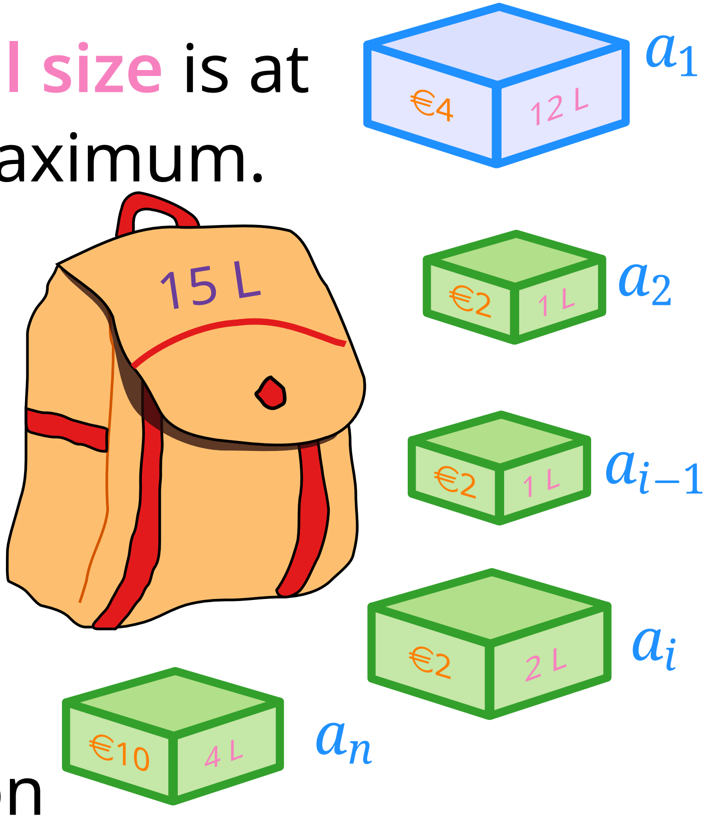

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How well does this do?

arbitrarily bad!
but picking the max of this and the first element not picked gives a 2-approximation

Pseudo-Polynomial Algorithms

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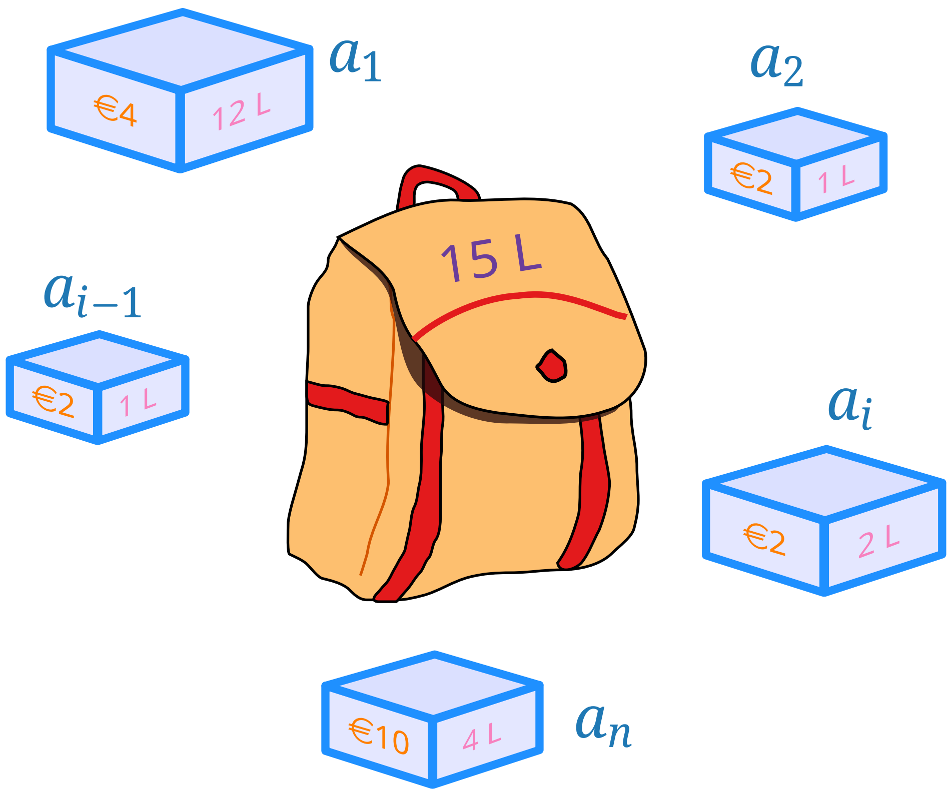

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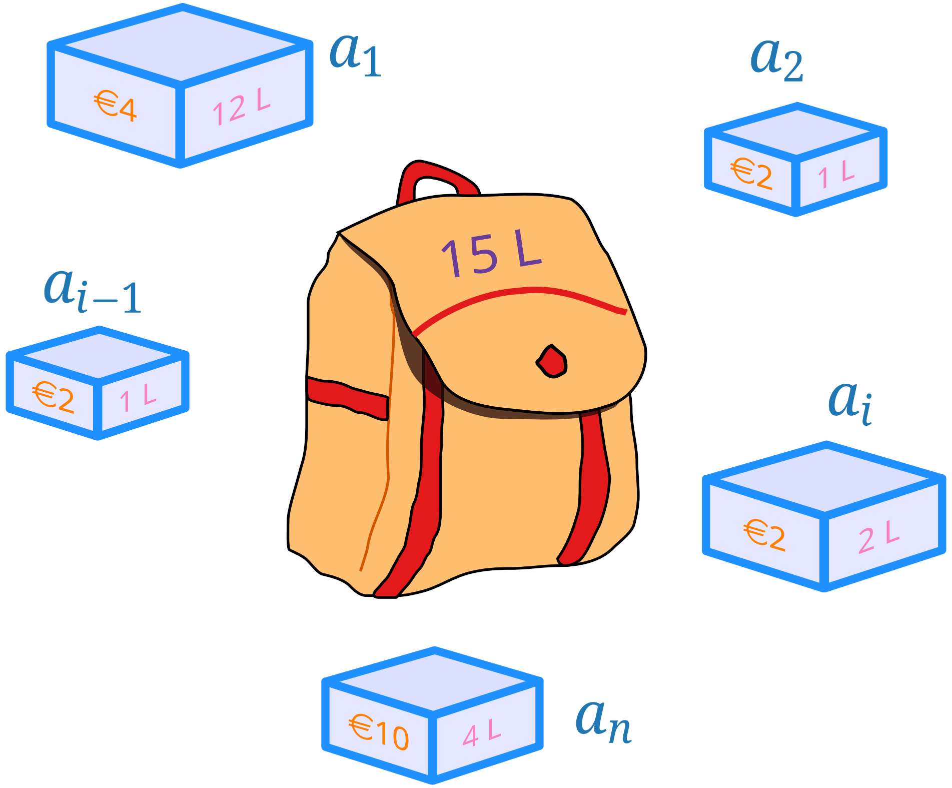

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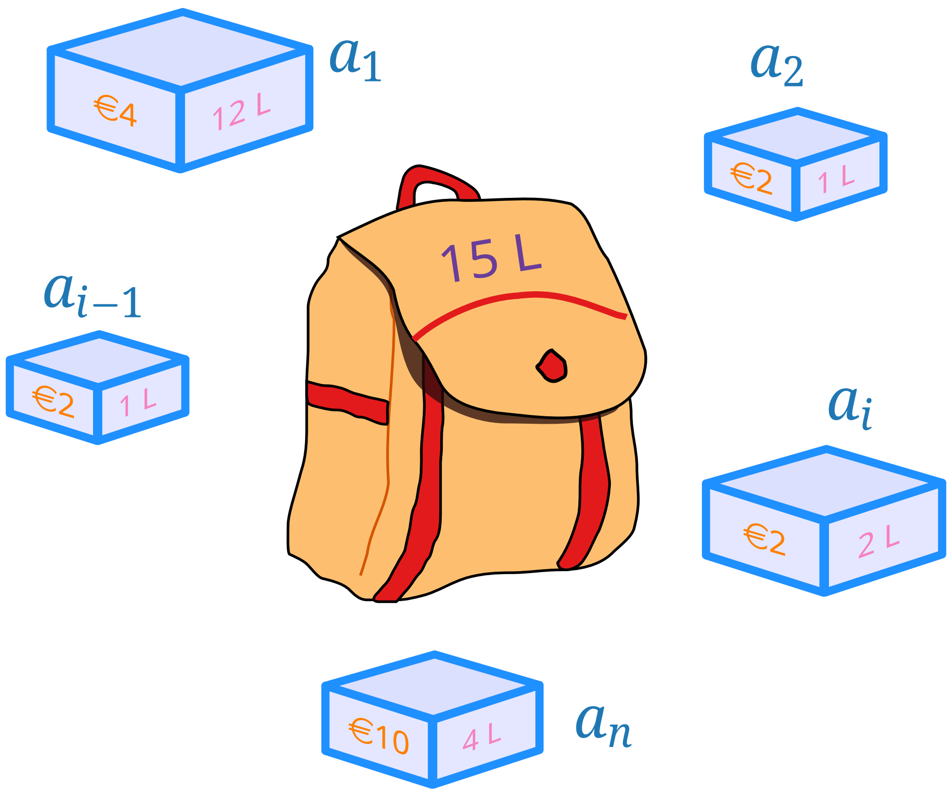

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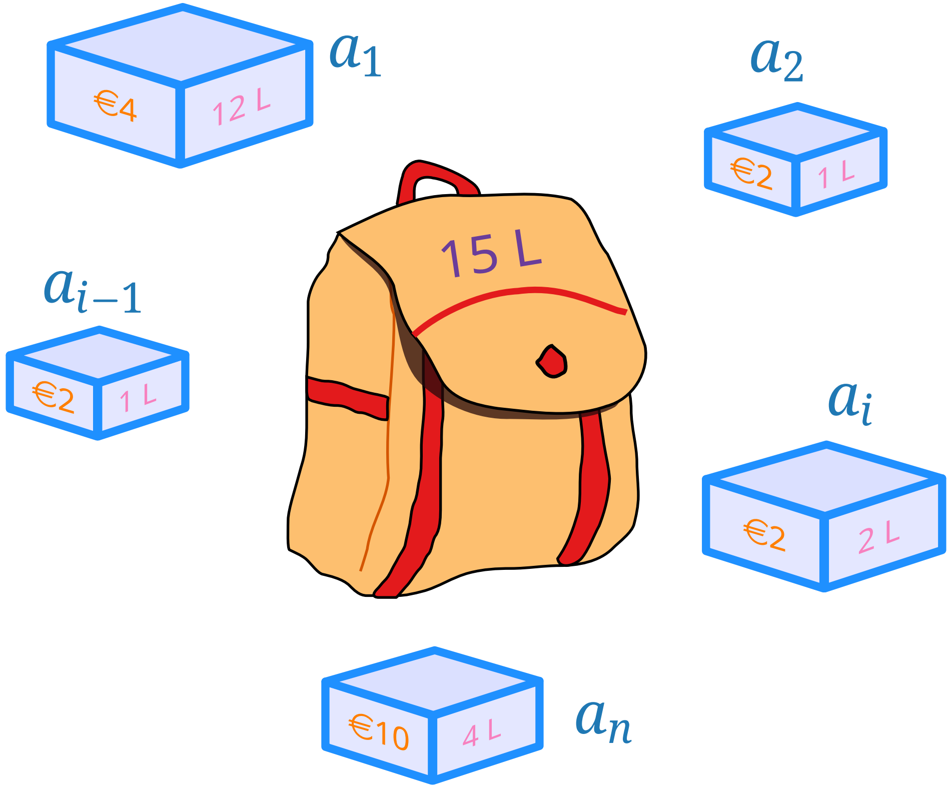

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Examples:

any NP-hard problem without numeric input, such as SAT, Hamilton circuit, ...

3-partition, bin packing, ...

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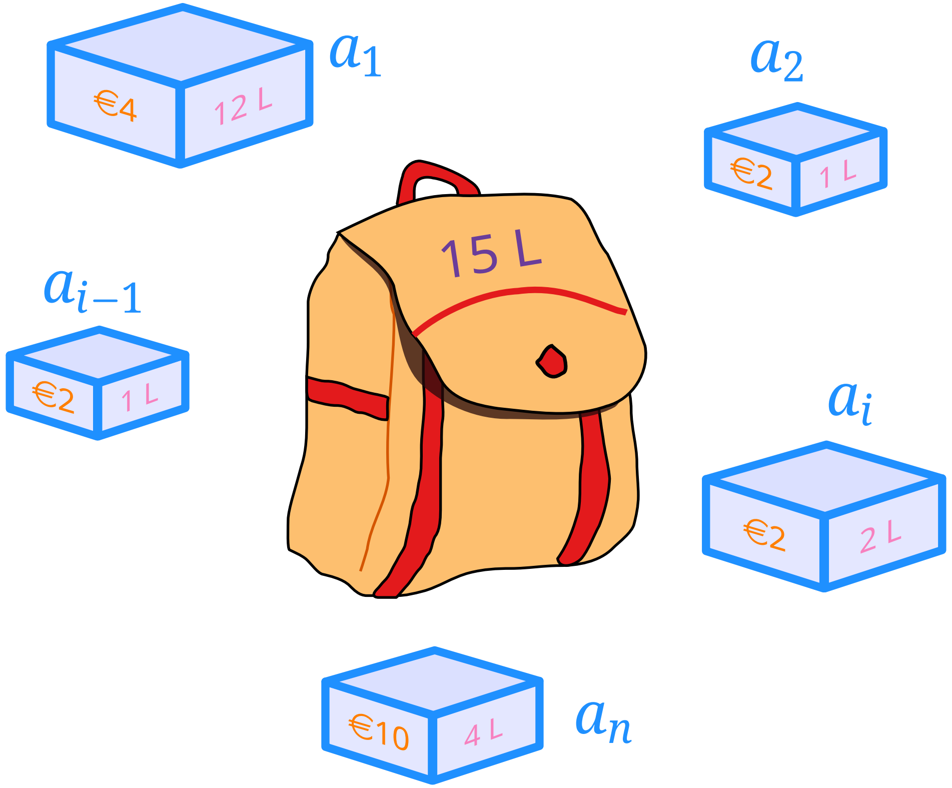

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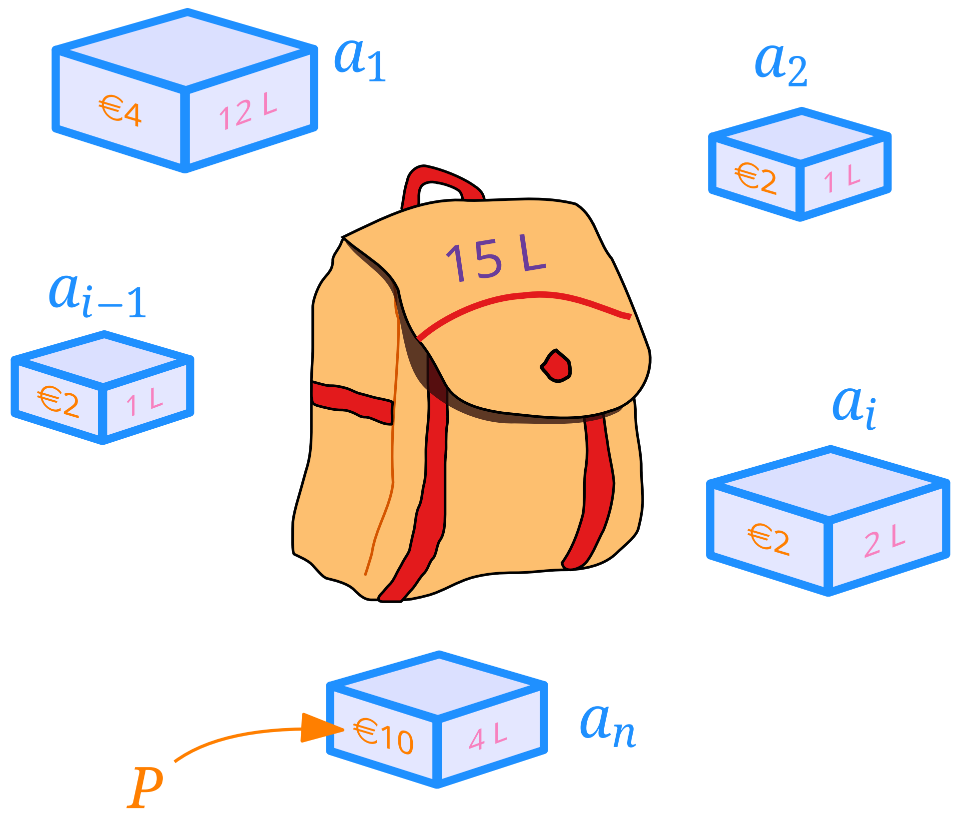

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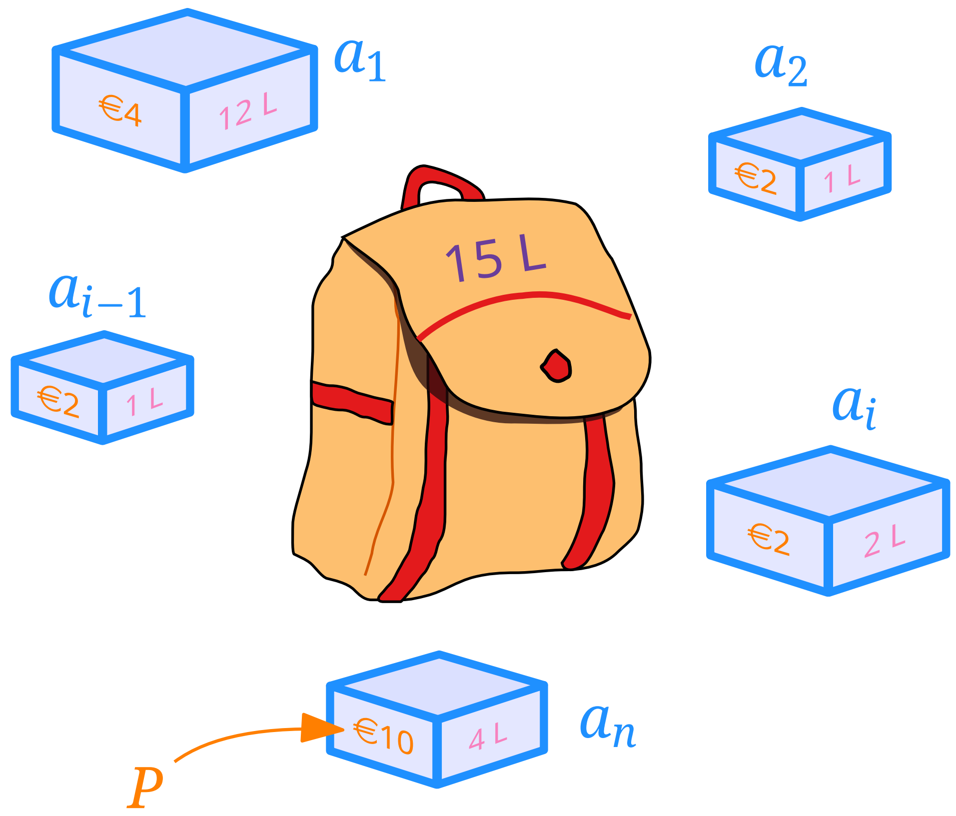

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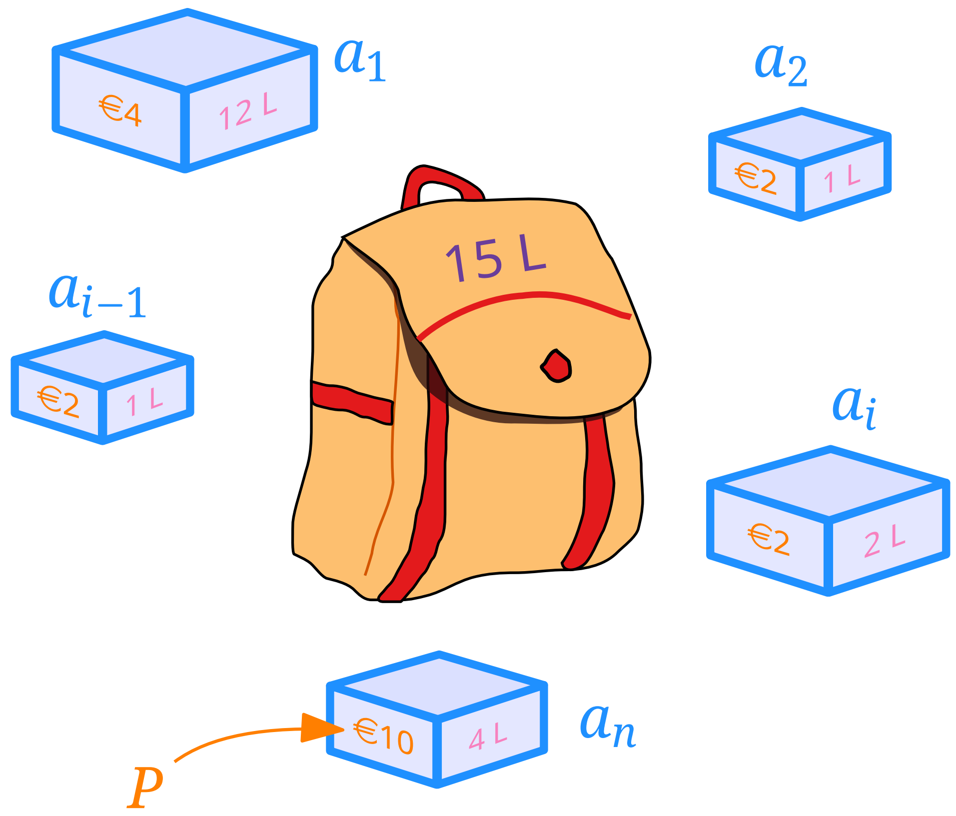

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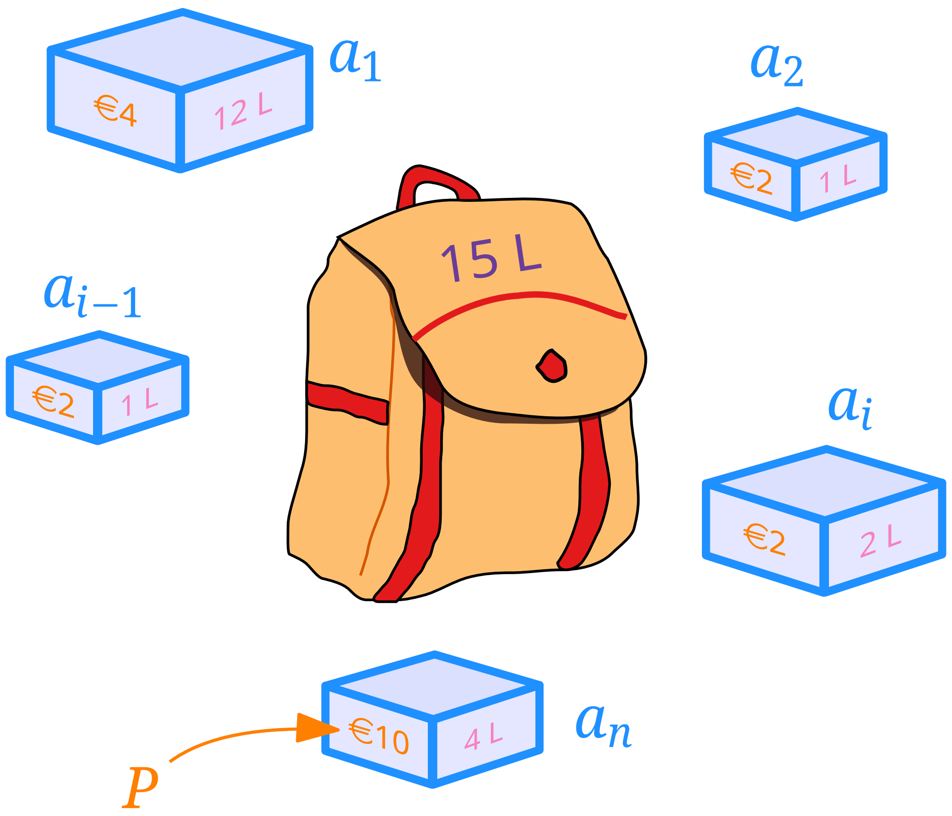

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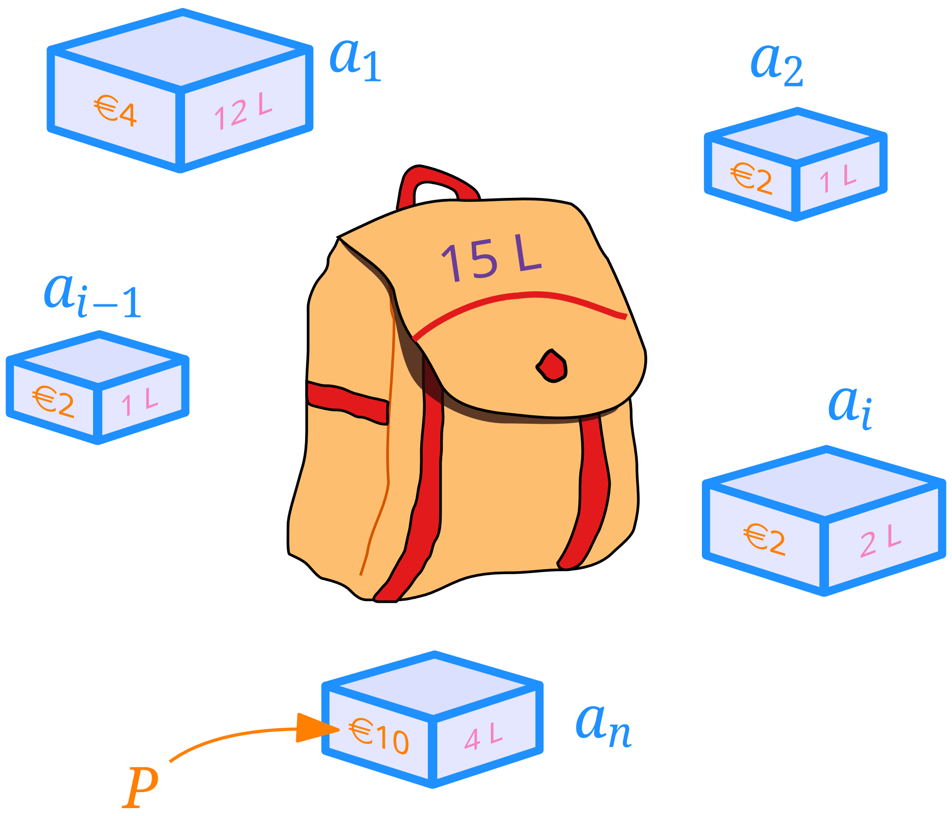

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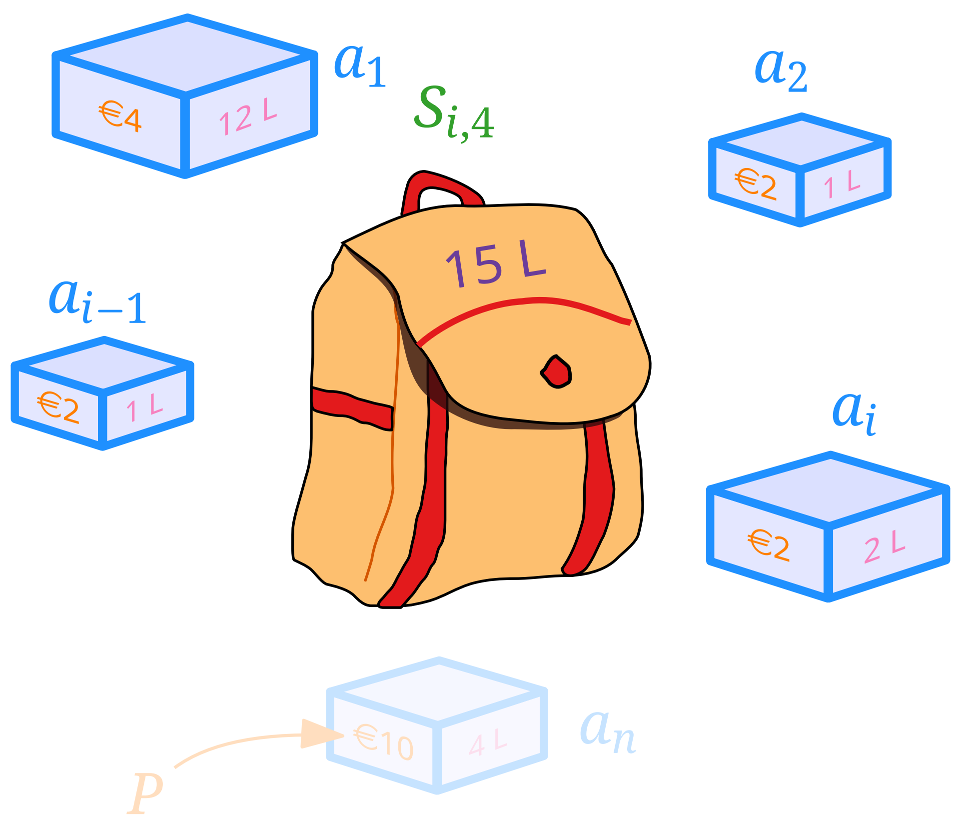

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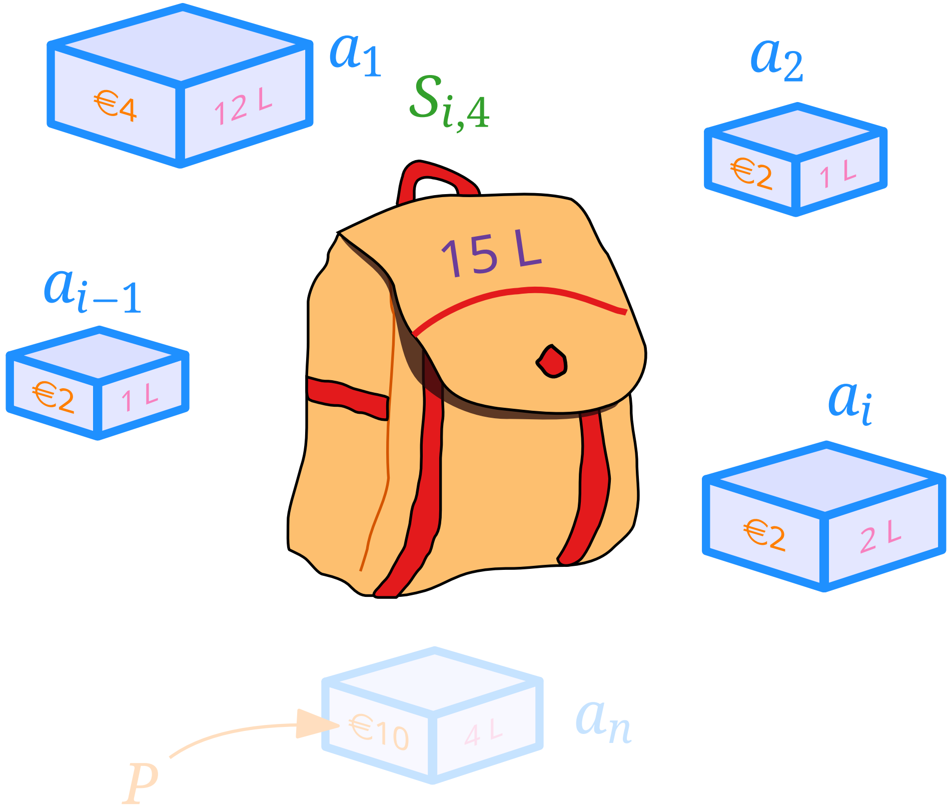

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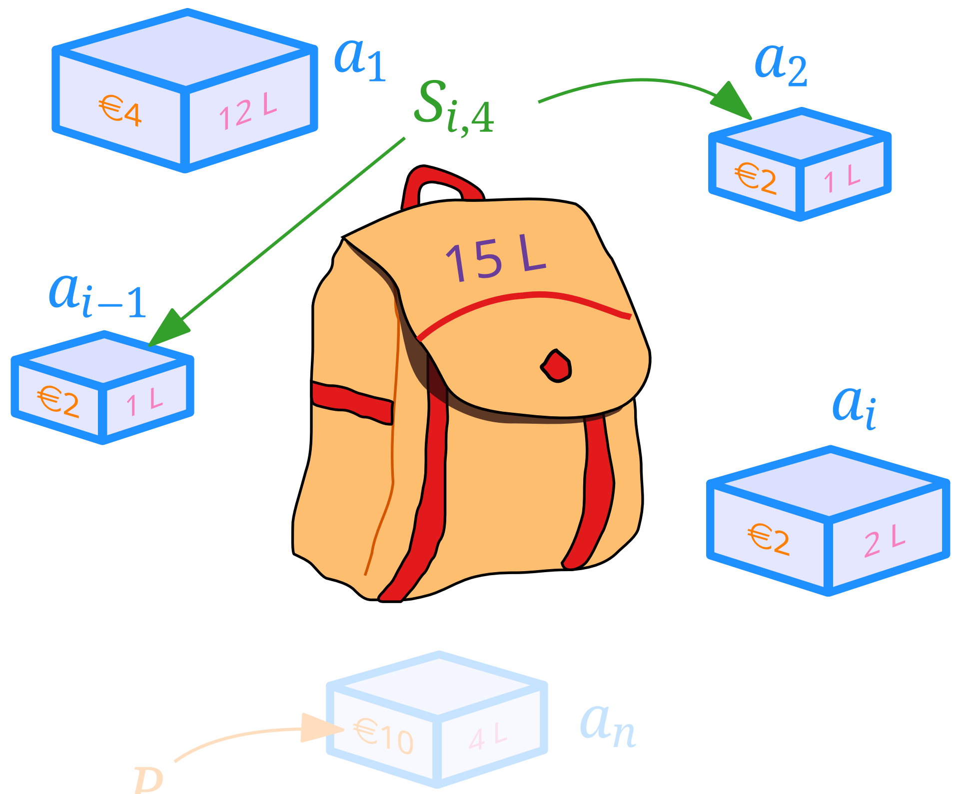

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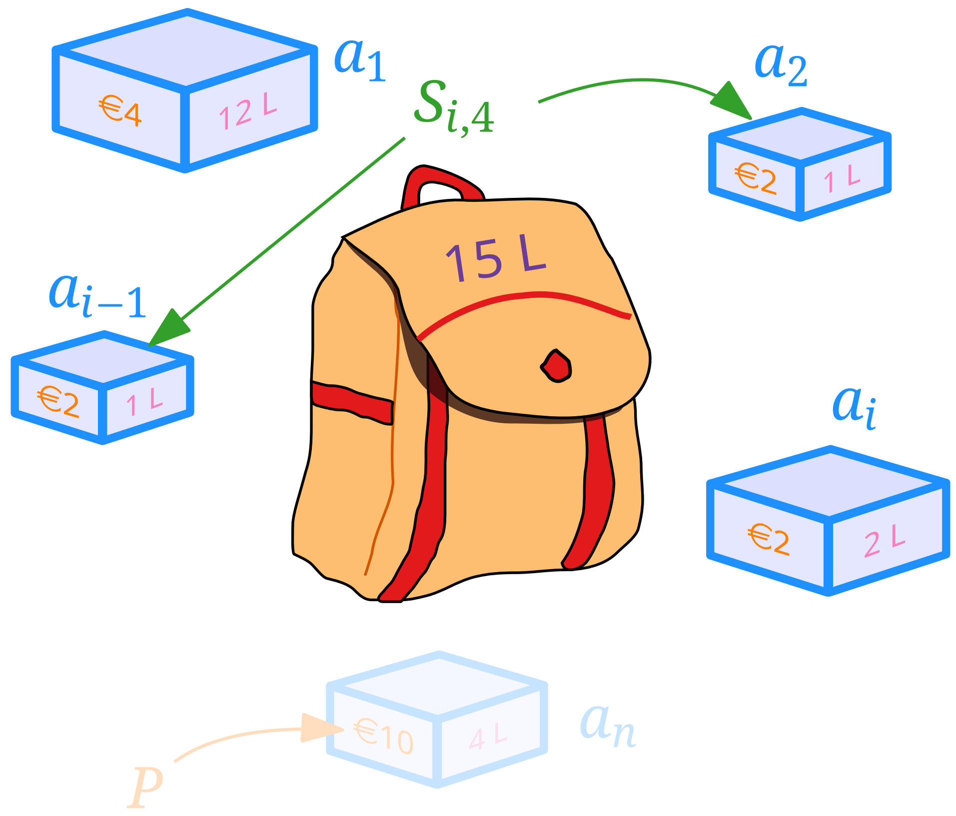

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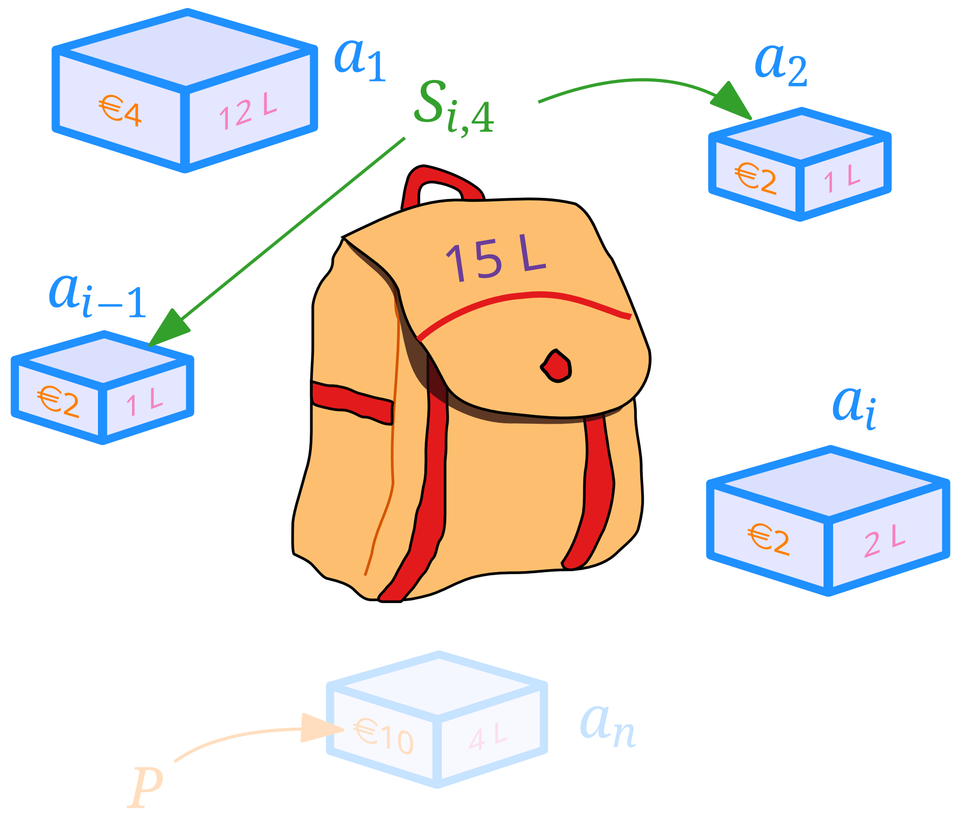

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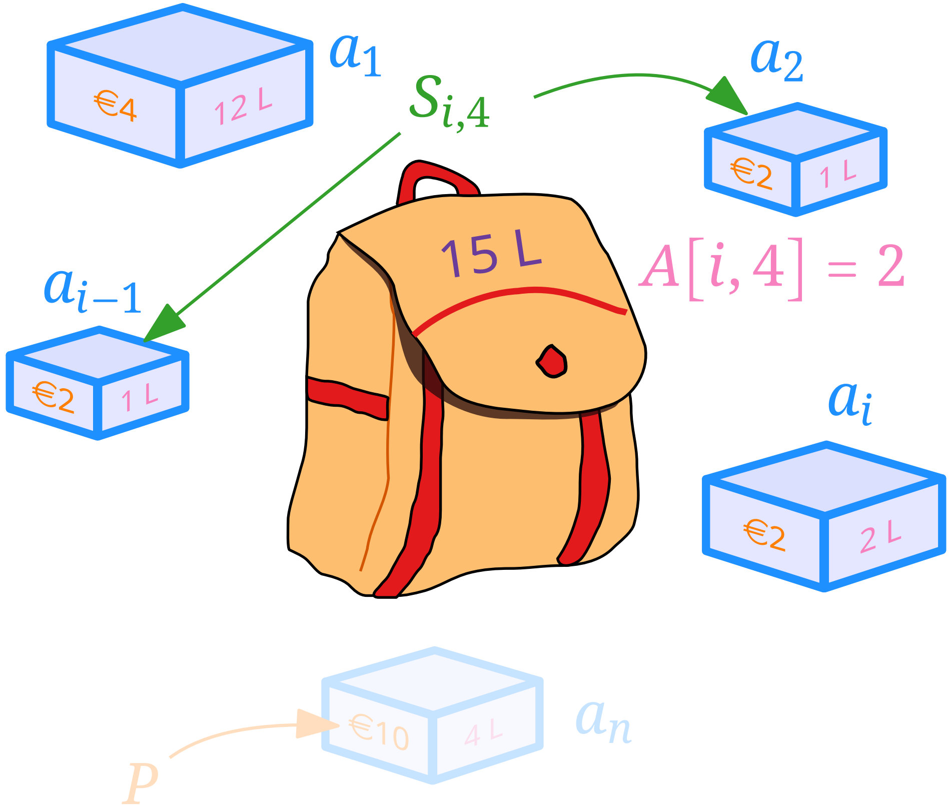

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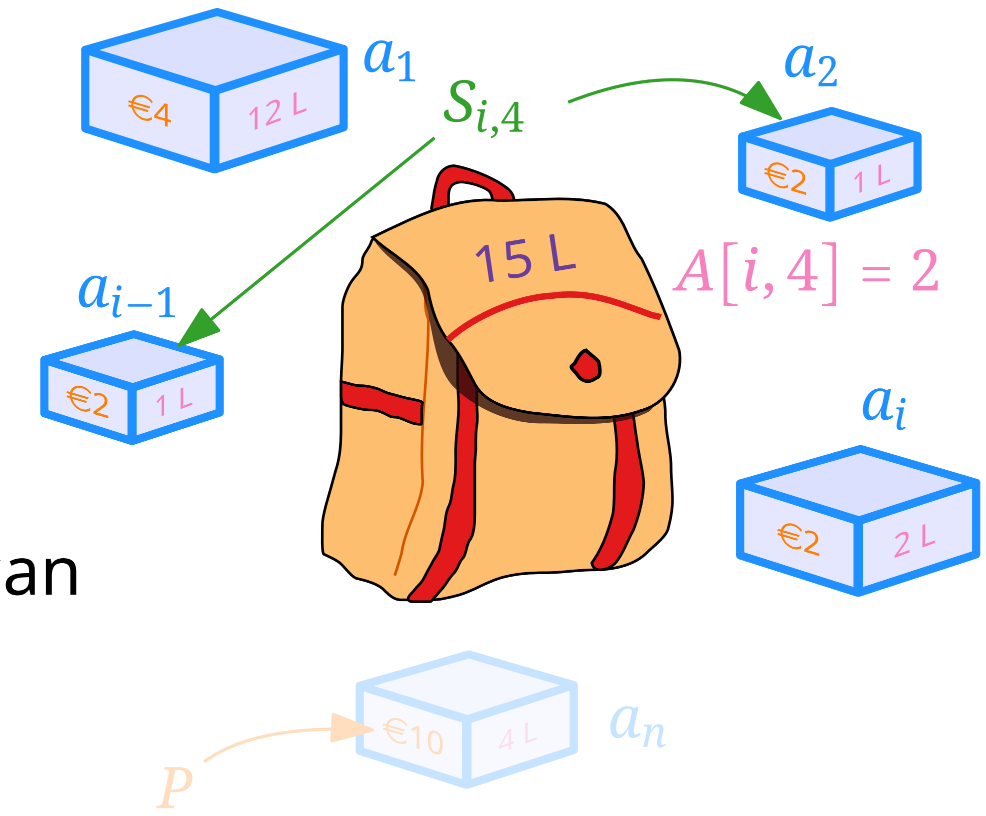

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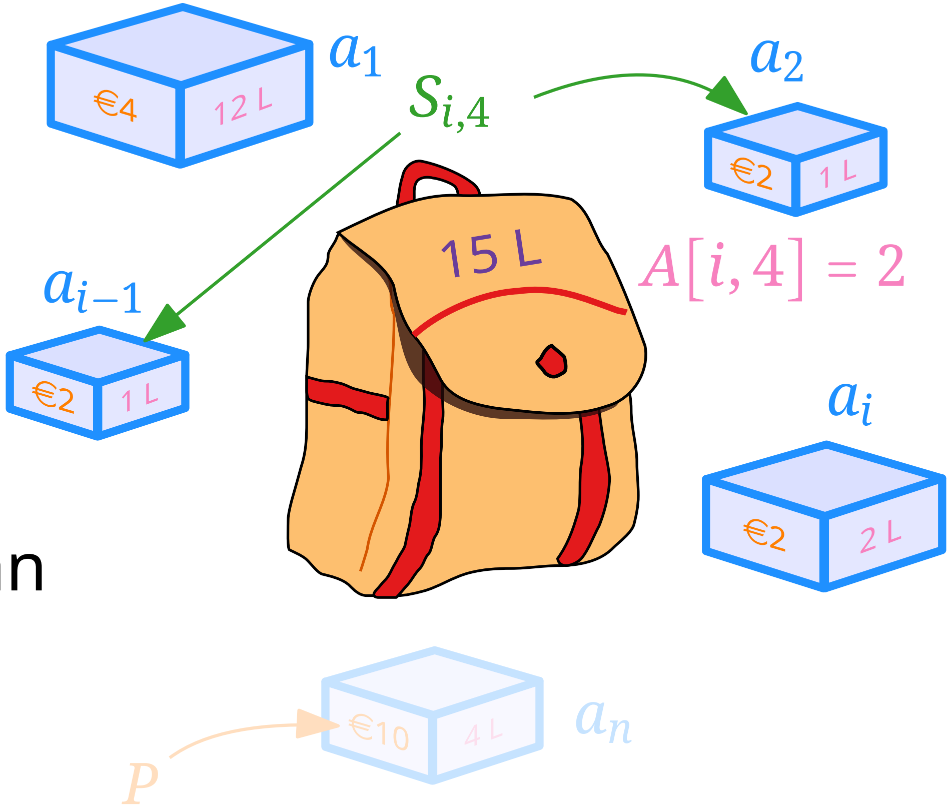

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$\mathrm{OPT} = \max\{\, p \mid A[n, p] \leq B \,\}$.

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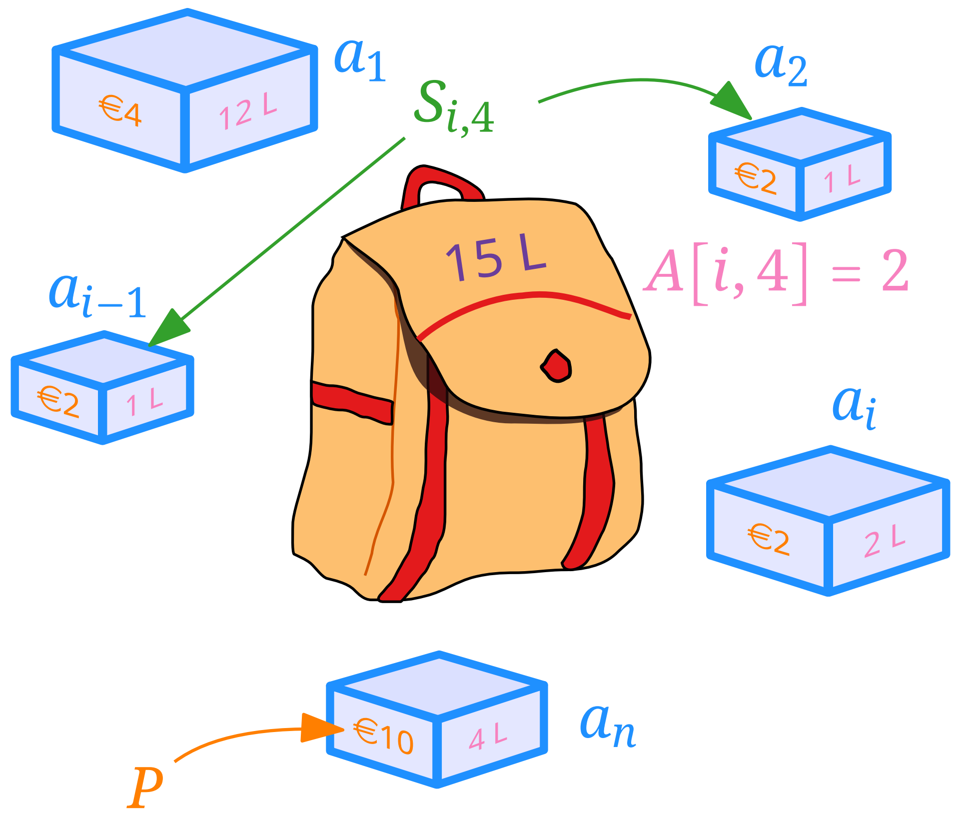

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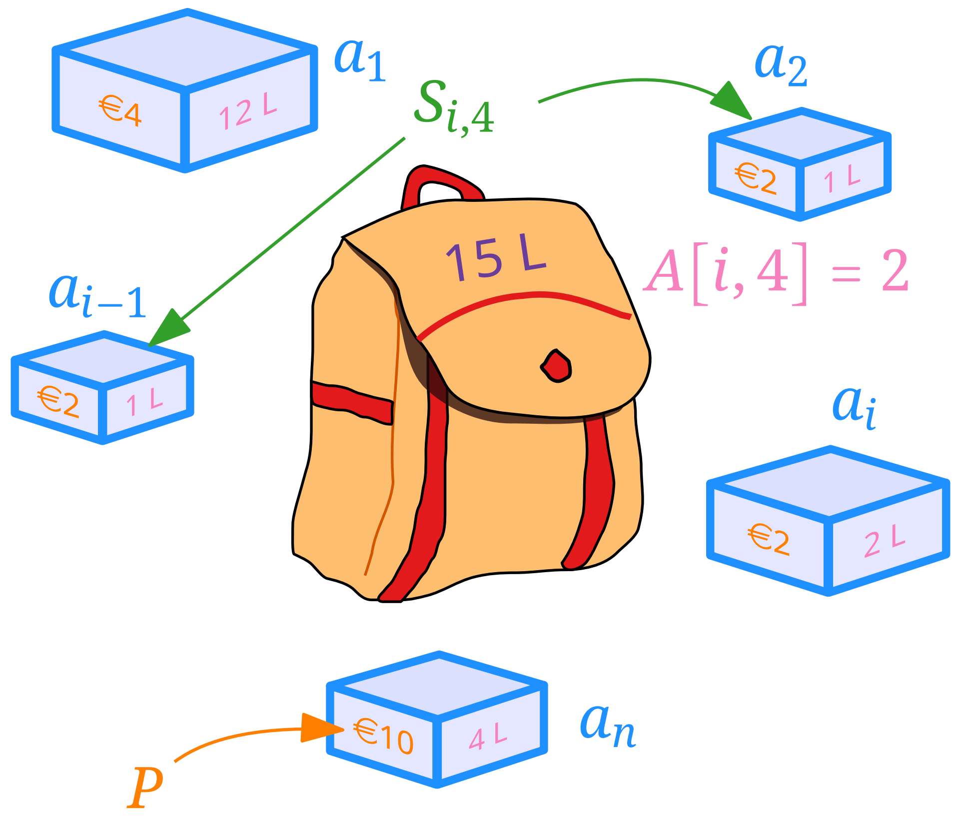

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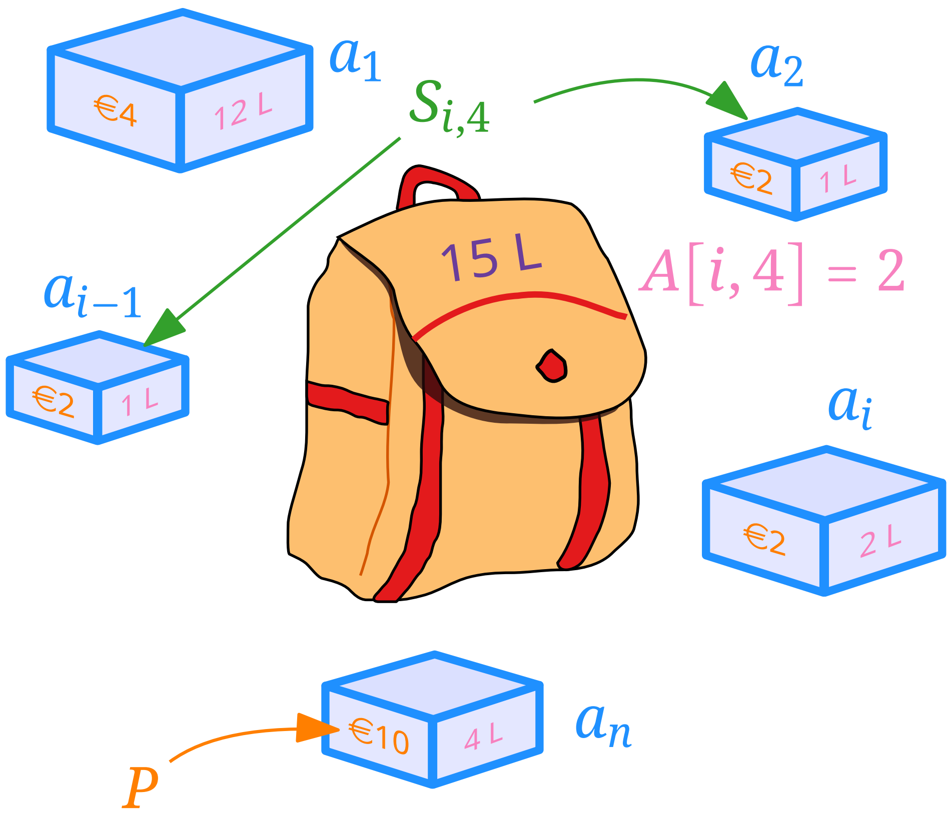

Pseudo-Polynomial Alg. for KNAPSACK

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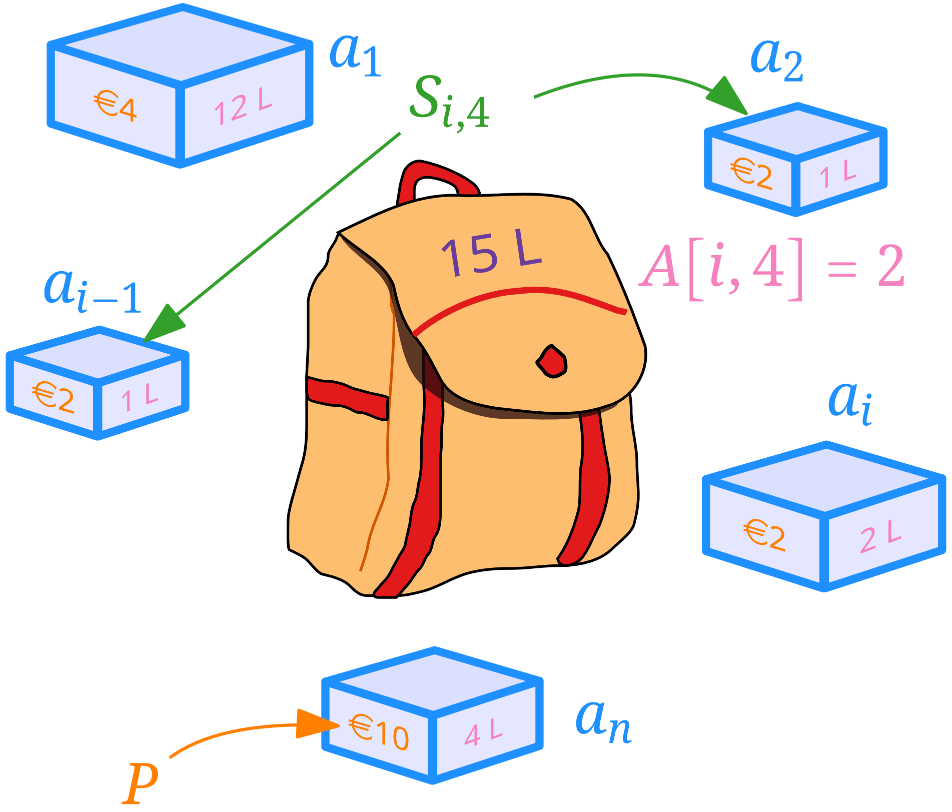

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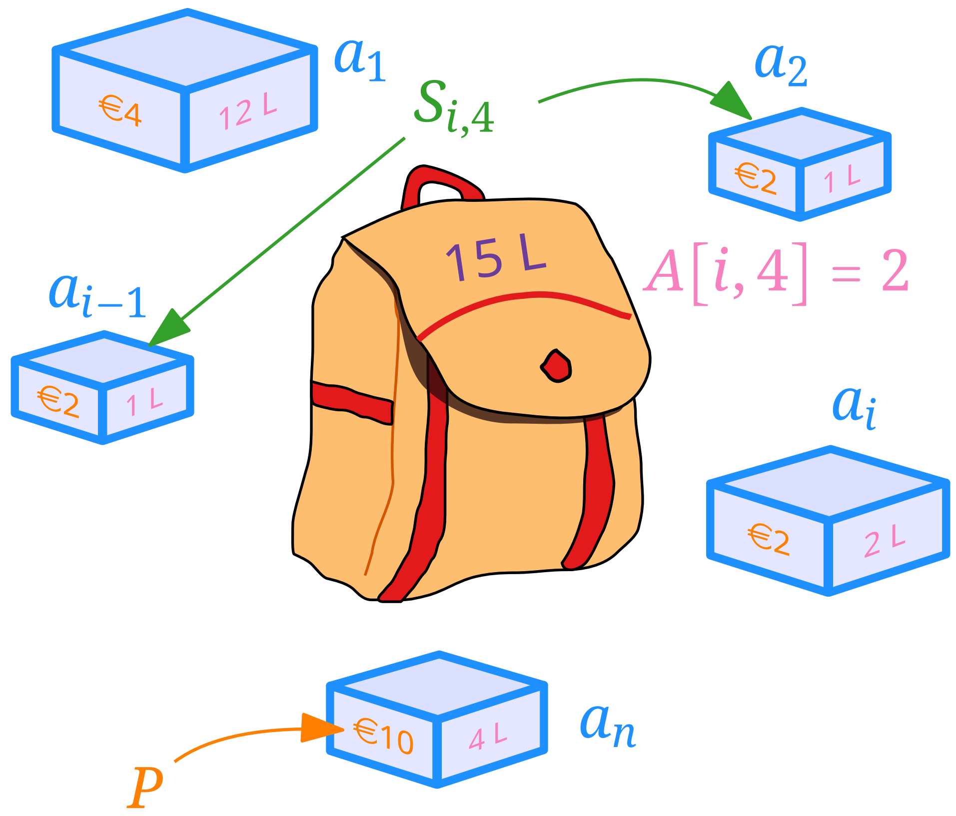

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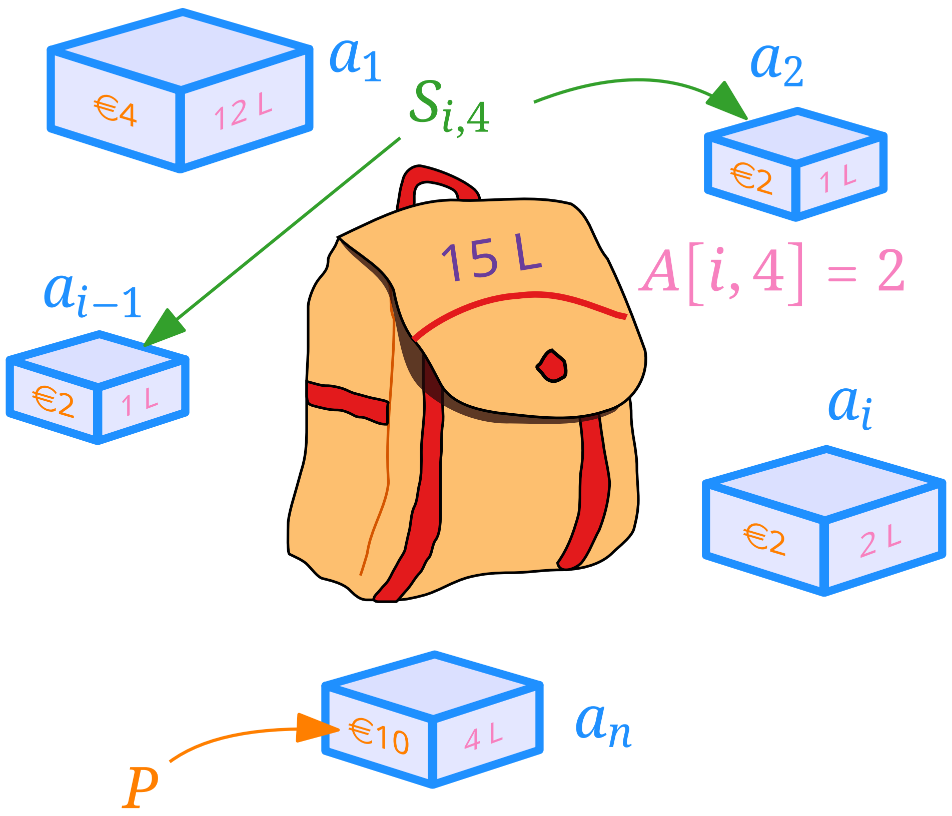

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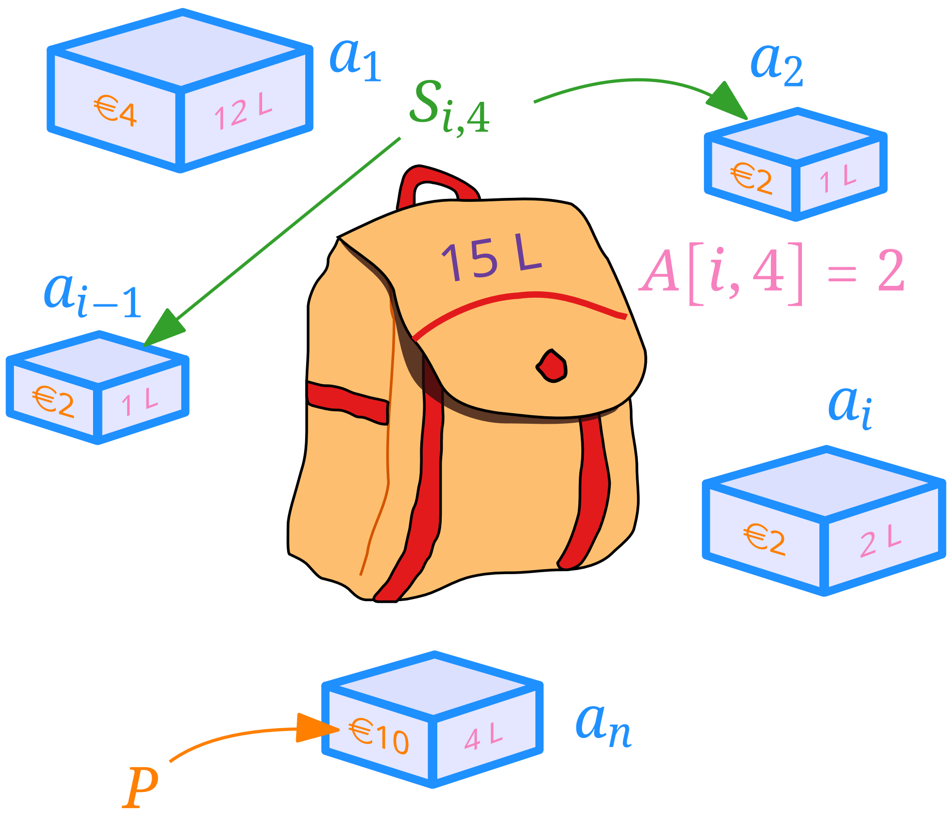

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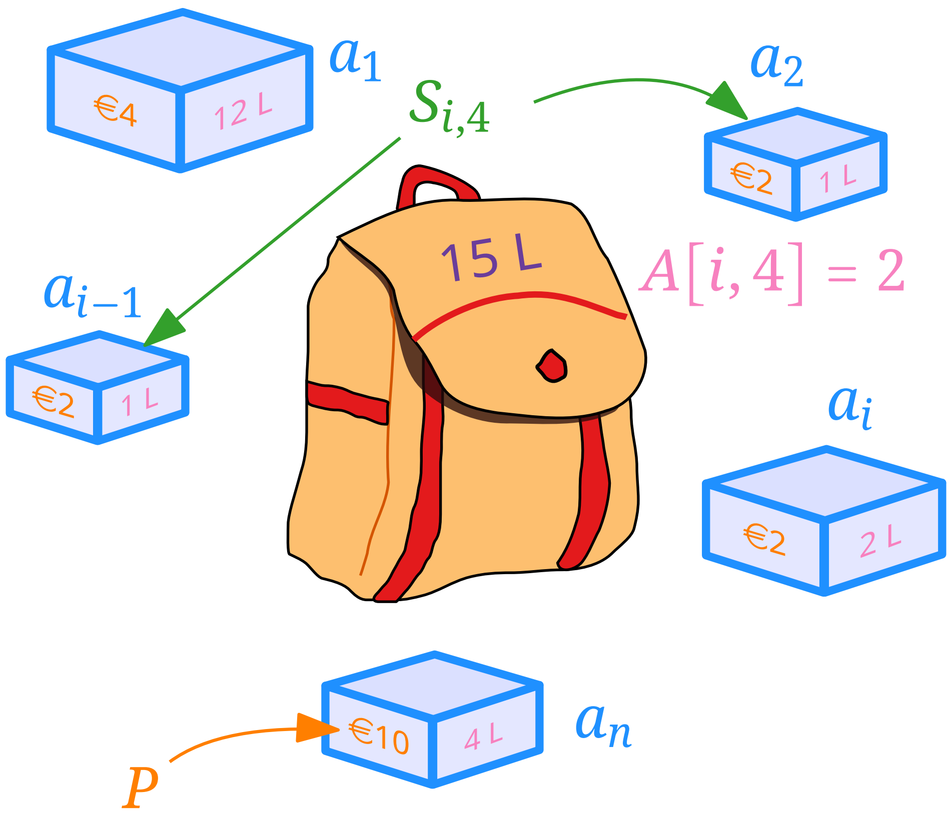

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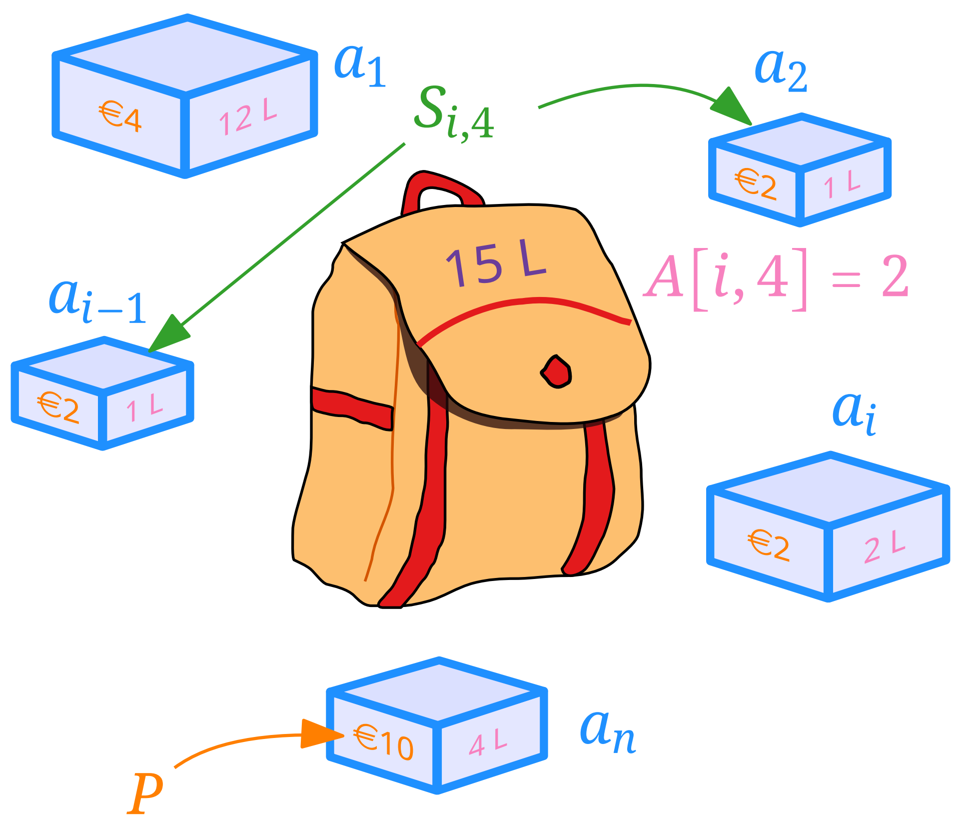

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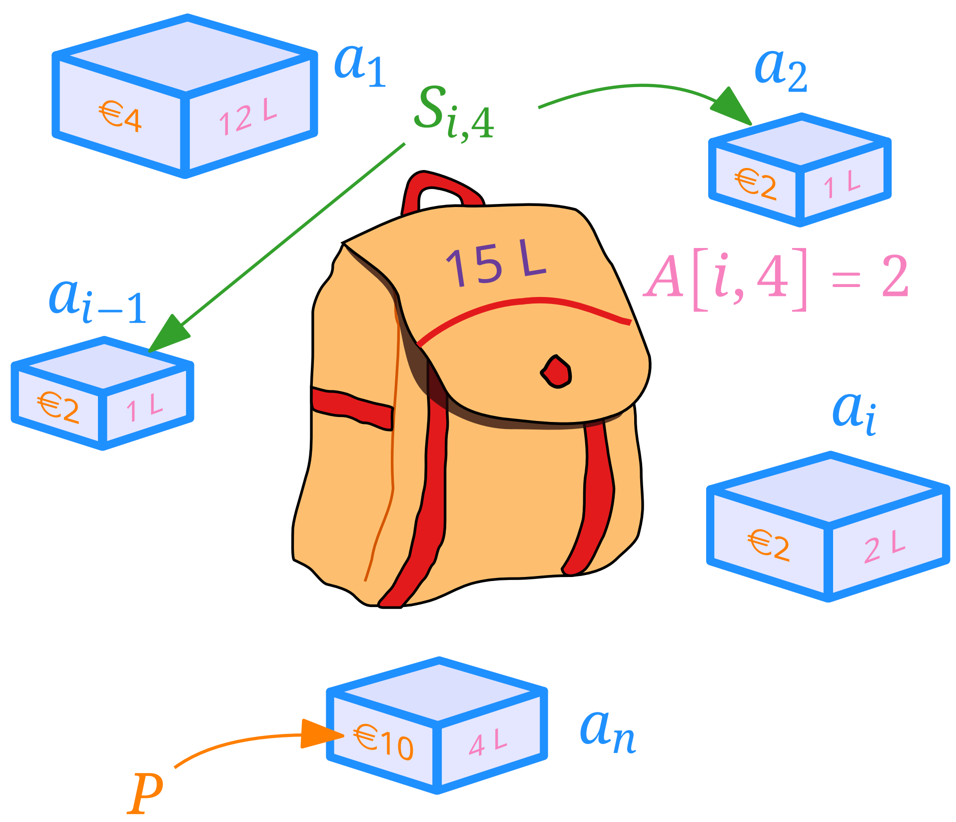

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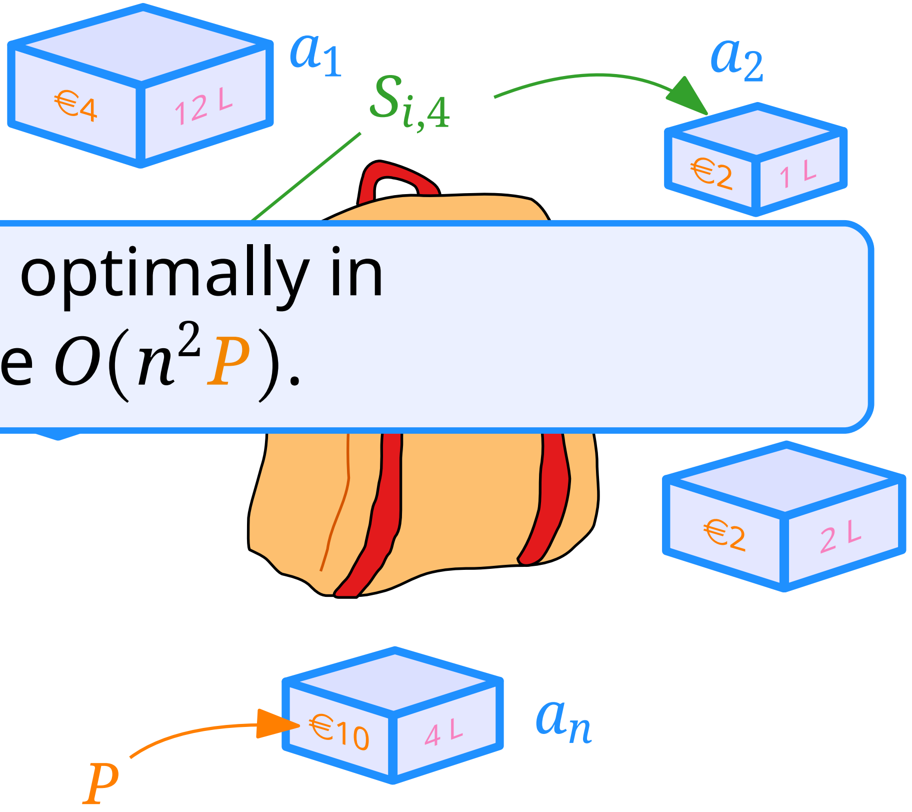

Theorem. KNAPSACK can be solved optimally in pseudo-polynomial time $O(n^2P)$.

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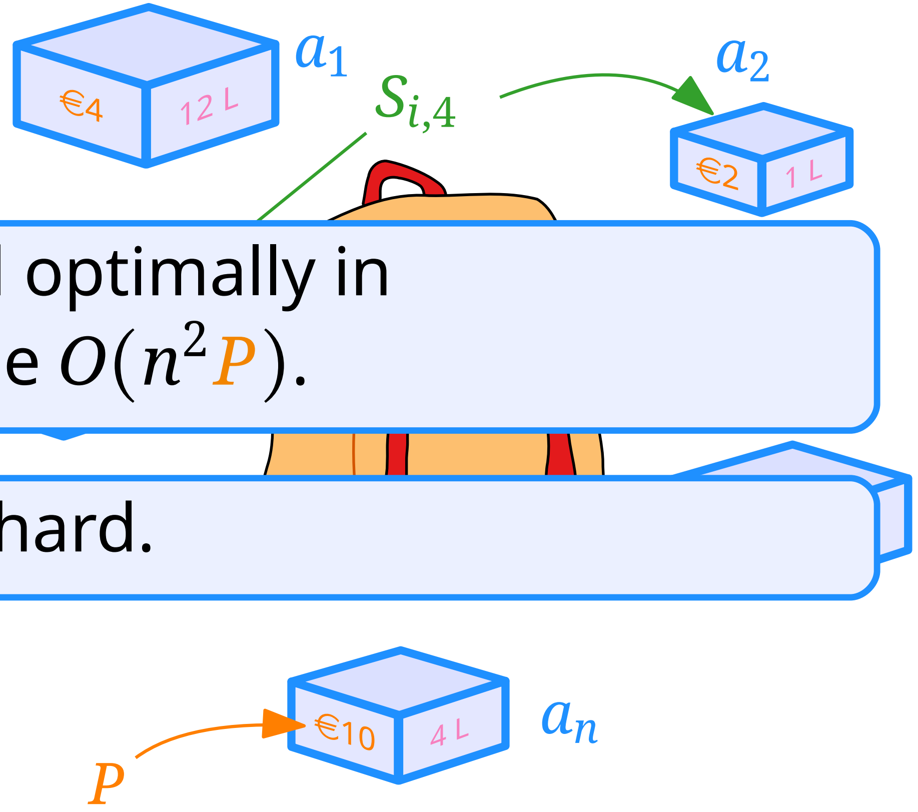

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Corollary. KNAPSACK is weakly NP-hard.

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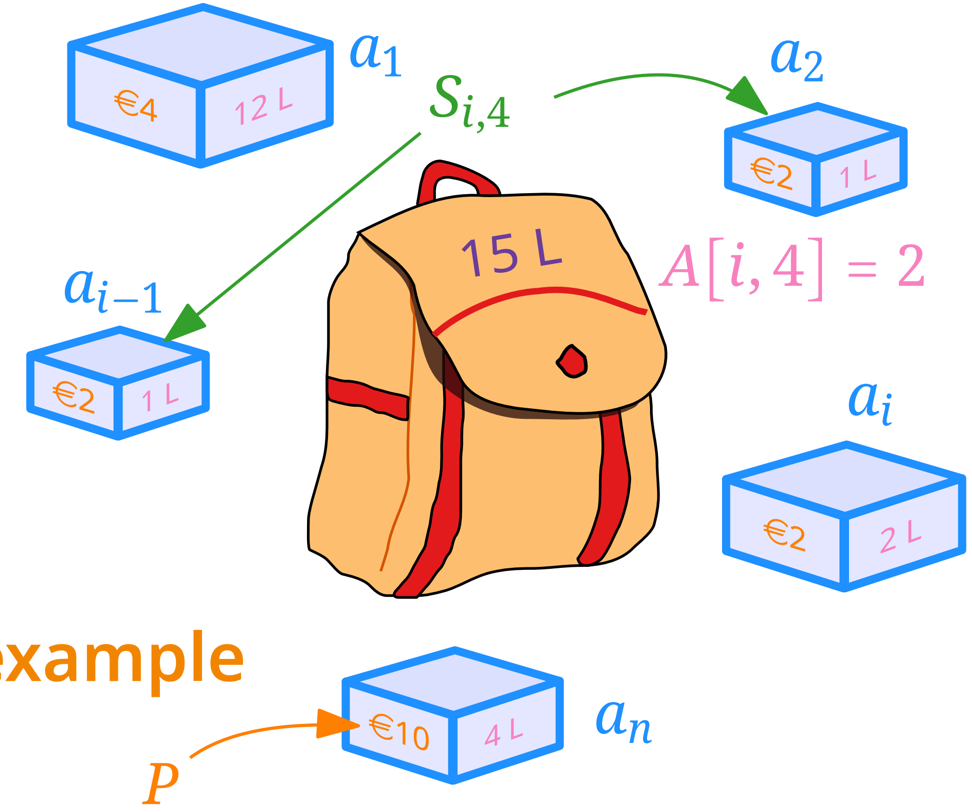
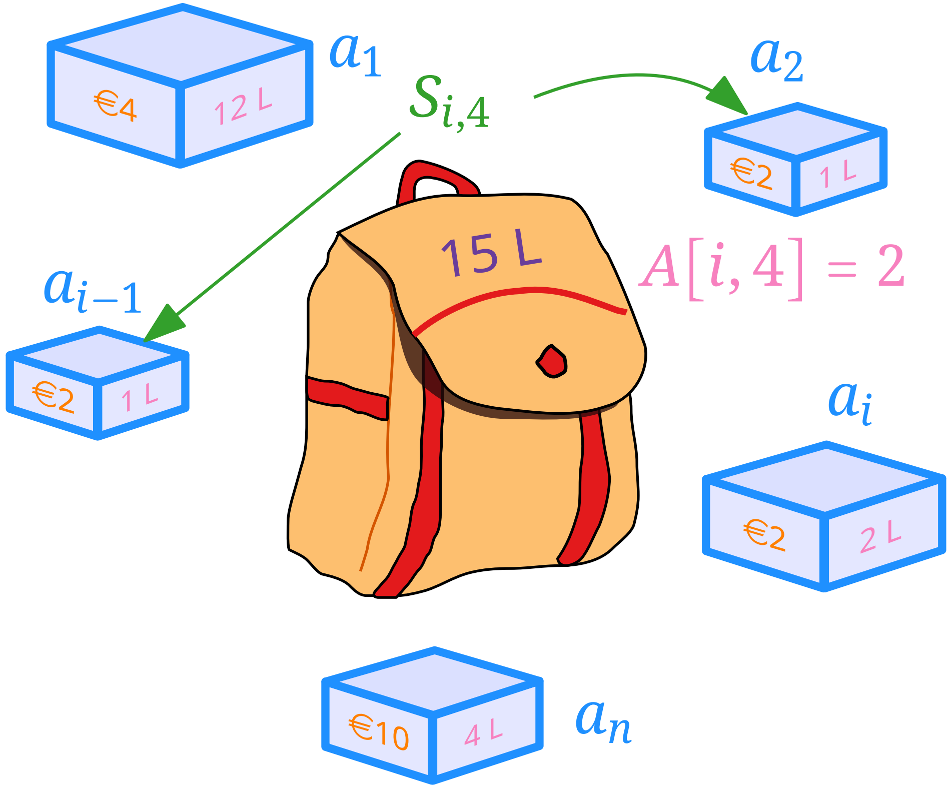


Exercise: Execute algorithm on example

Solution to exercise

$i \setminus p$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	∞	∞	∞	12	∞	...												
2	0	∞	1	∞	12	∞	13	∞	...										
3	0	∞	1	∞	2	∞	13	∞	14	∞	...								
4	0	∞	1	∞	2	∞	4	∞	14	∞	(16)	∞	...						
5	0	∞	1	∞	2	∞	4	∞	14	∞	4	∞	5	∞	6	∞	8	∞	(18)



Pseudo-Polynomial Alg. for KNAPSACK

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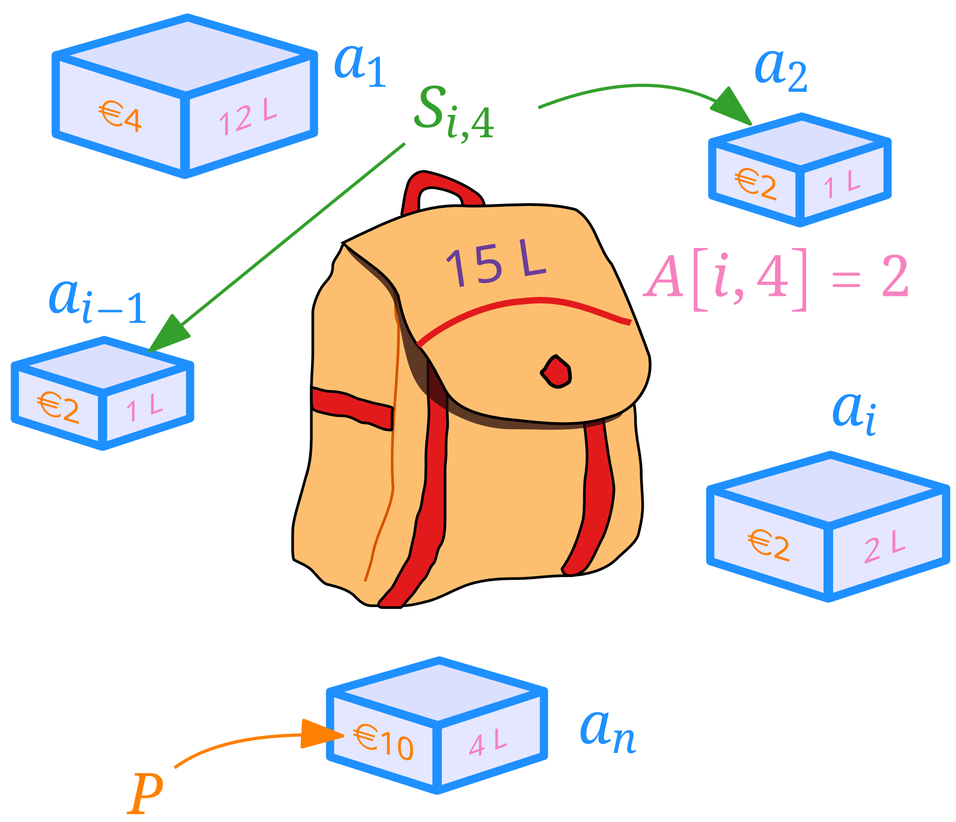
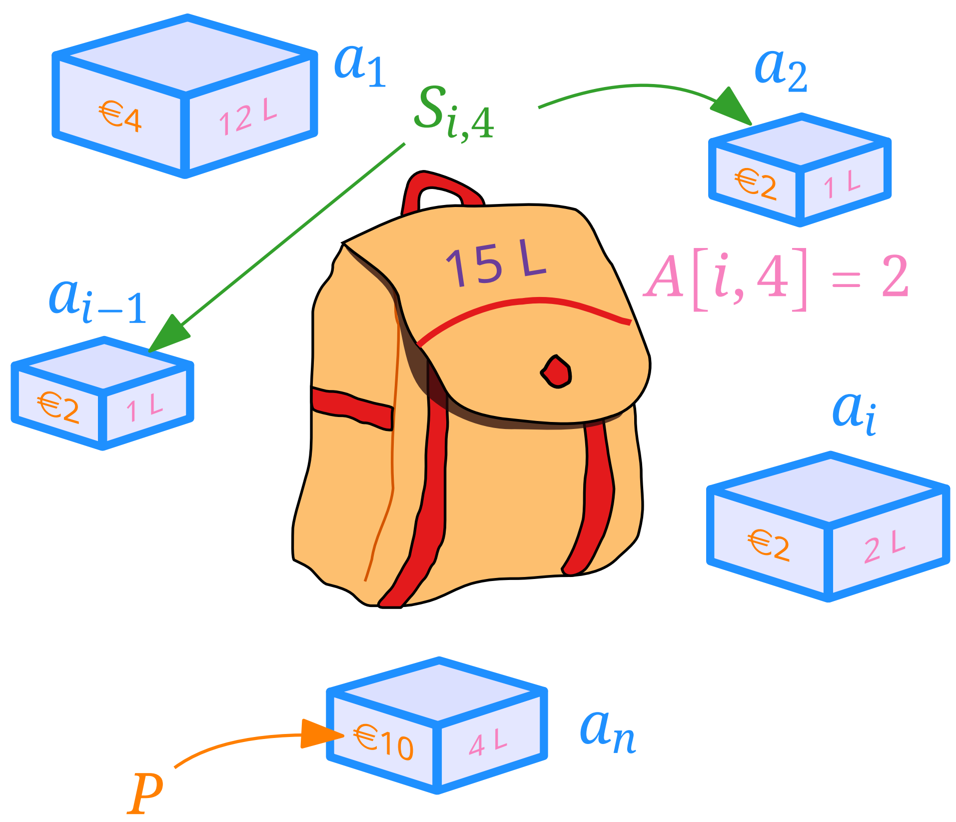


Pseudo-Polynomial Alg. for KNAPSACK

Theorem. KNAPSACK can be solved optimally in pseudo-polynomial time $O(n^2 P)$.

Observe. The running time $O(n^2 P)$ is polynomial in n if P is polynomial in n.



Approximation Schemes

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FPTAS for KNAPSACK

An FPTAS for KNAPSACK via Scaling

FPTAS idea: **Scale** profits to polynomial size (as required by the error parameter ε)...

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An FPTAS for KNAPSACK via Scaling

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FPTAS vs strong NP-hardness

FPTAS and Pseudo-Poly. Algorithms

Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem

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Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless $\mathrm{P} = \mathrm{NP}$.

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Corollary. Let Π be an NP-hard optimization problem that fulfills the restrictions above. If Π is strongly NP-hard, then there is no FPTAS for Π (unless $\mathrm{P} = \mathrm{NP}$).