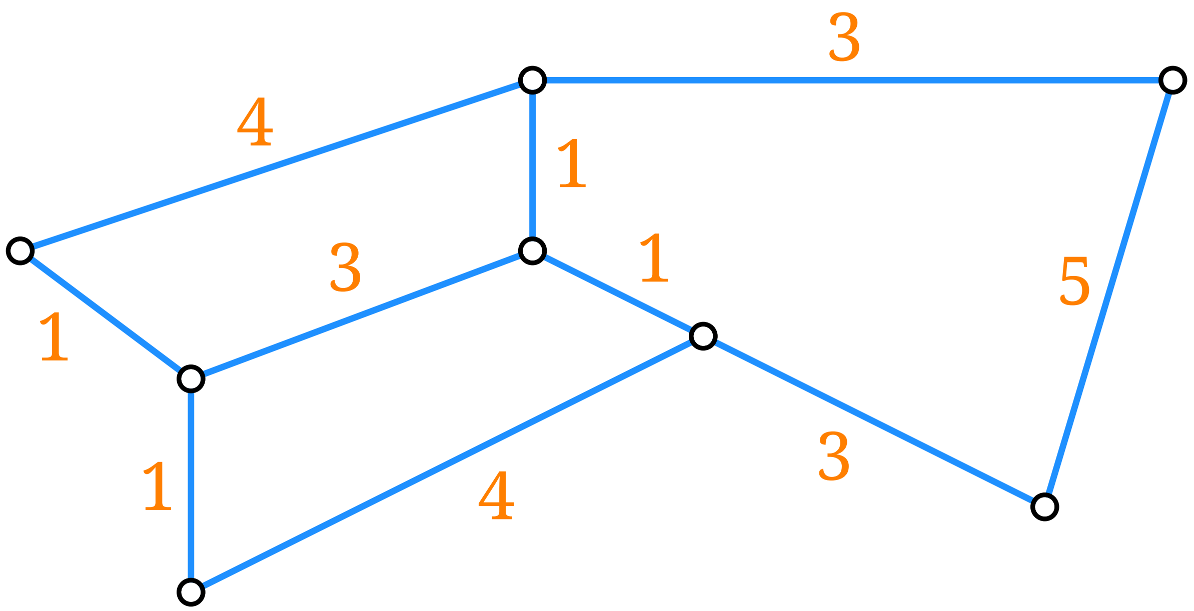
SteinerForest via Primal–Dual

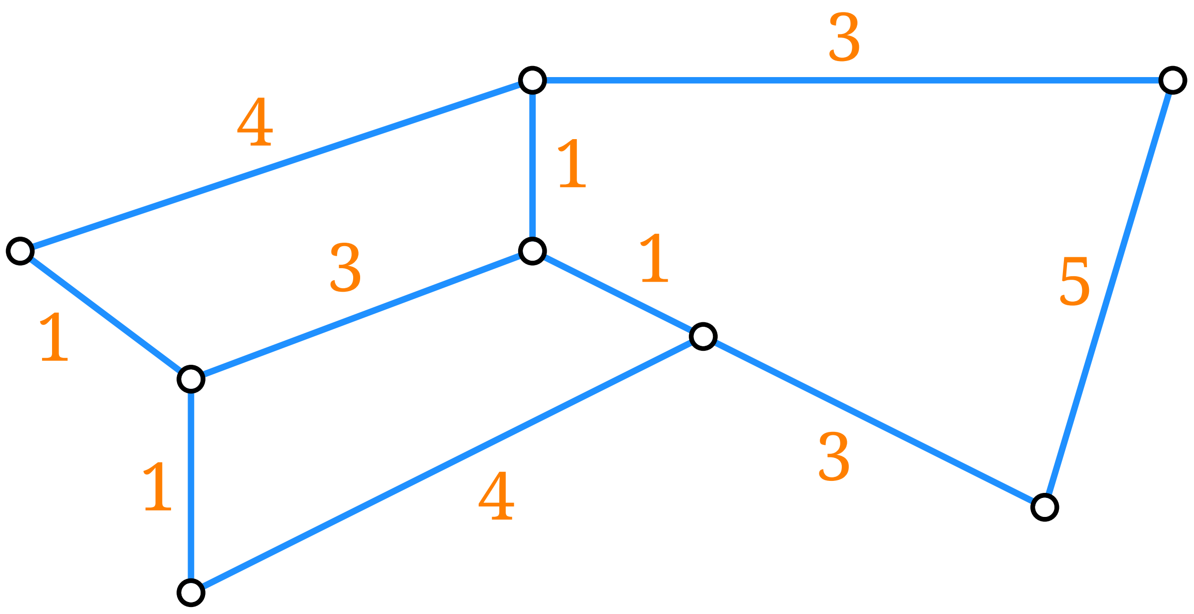
SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a



SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of k vertex pairs.



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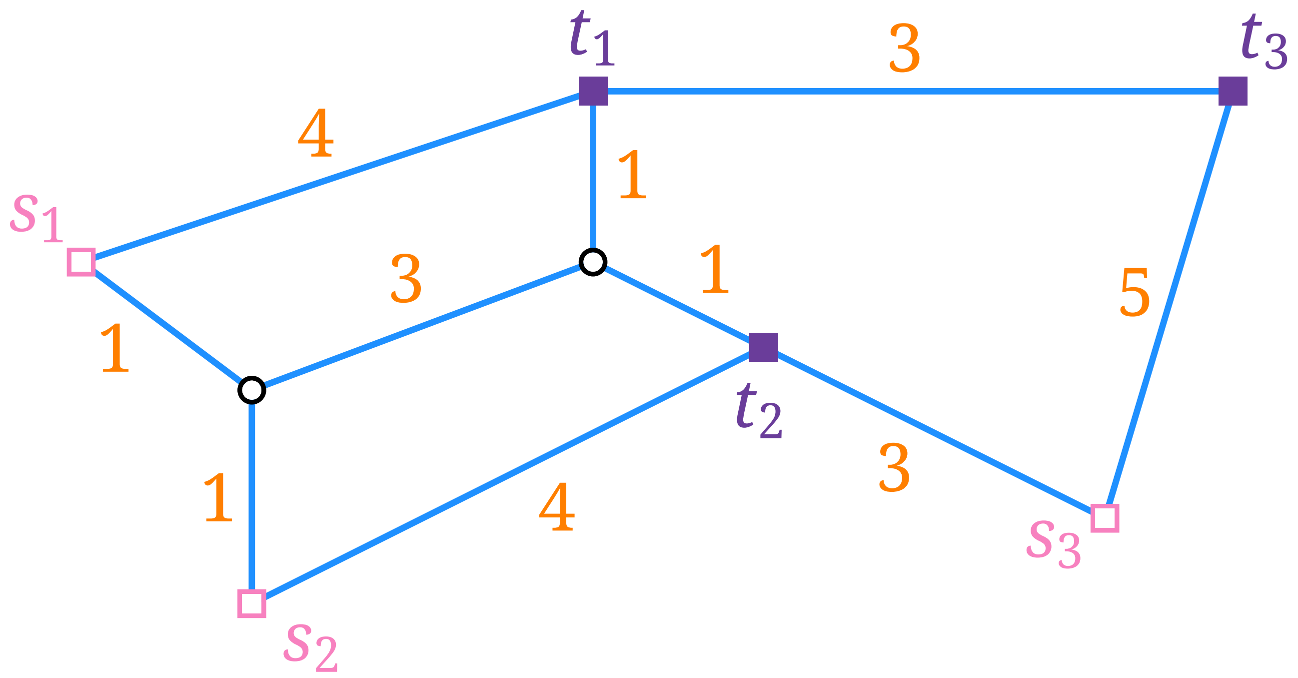

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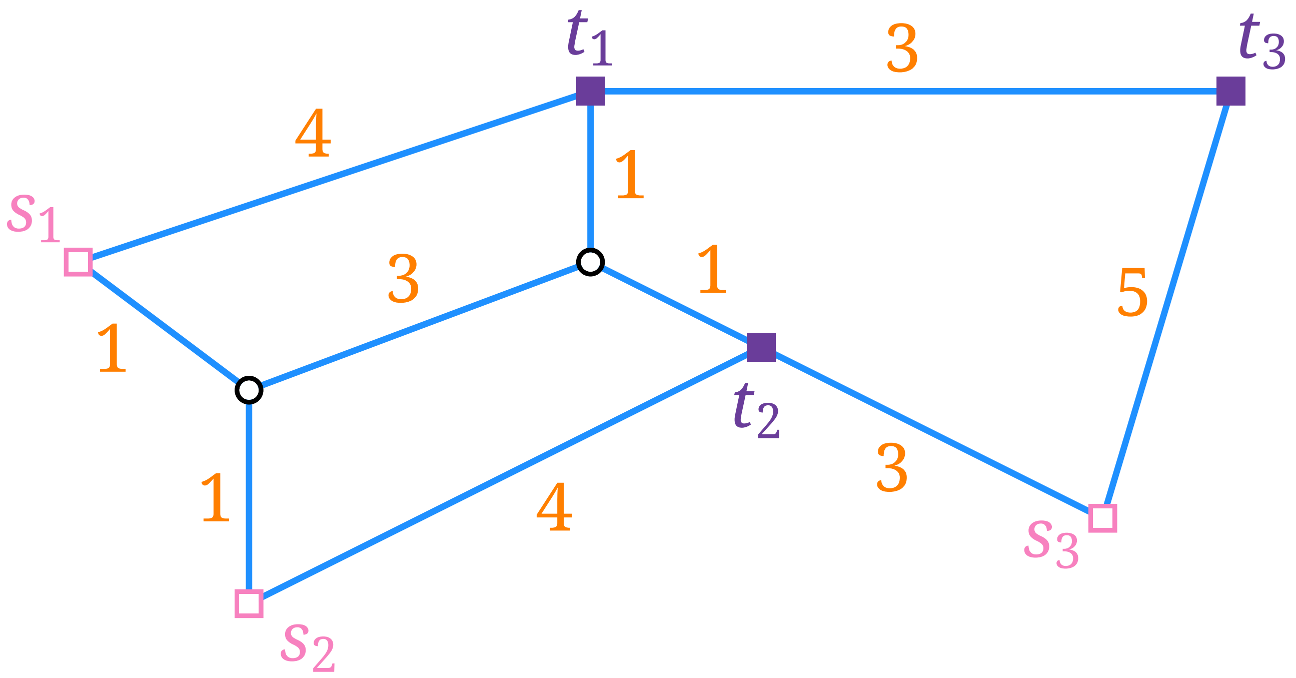
Task: Find an edge set $F \subseteq E$ of minimum total cost $c(F)$



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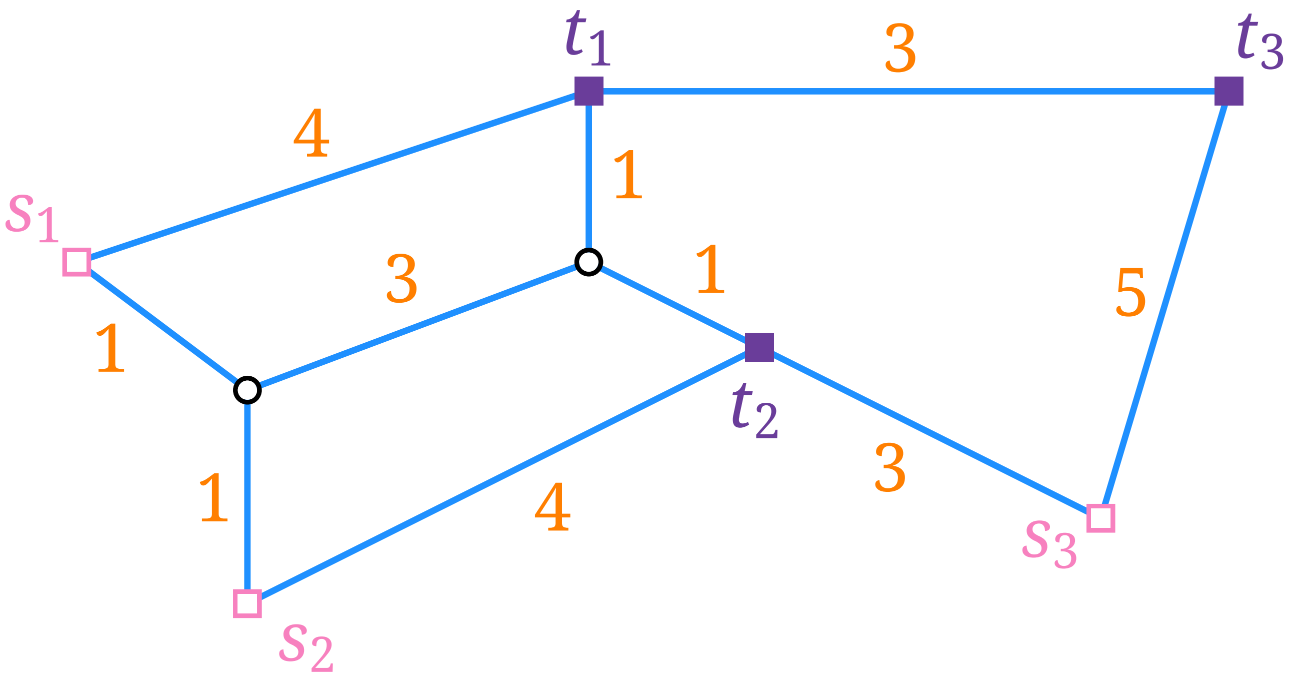
Task: Find an edge set $F \subseteq E$ of minimum total cost $c(F)$ such that the subgraph (V, F) connects every pair (s_i, t_i), $i = 1, \ldots, k$.



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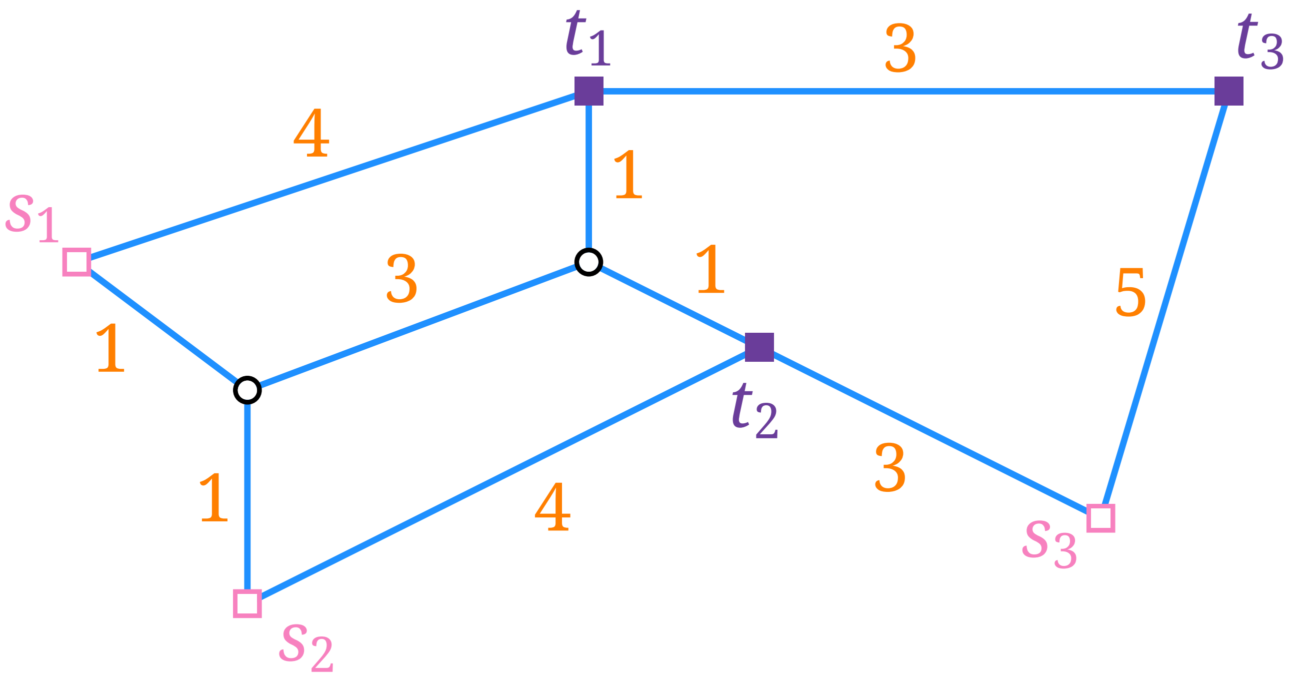

What is a minimal SteinerForest here?

SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \dots, (s_k, t_k)\}$ of k vertex pairs.

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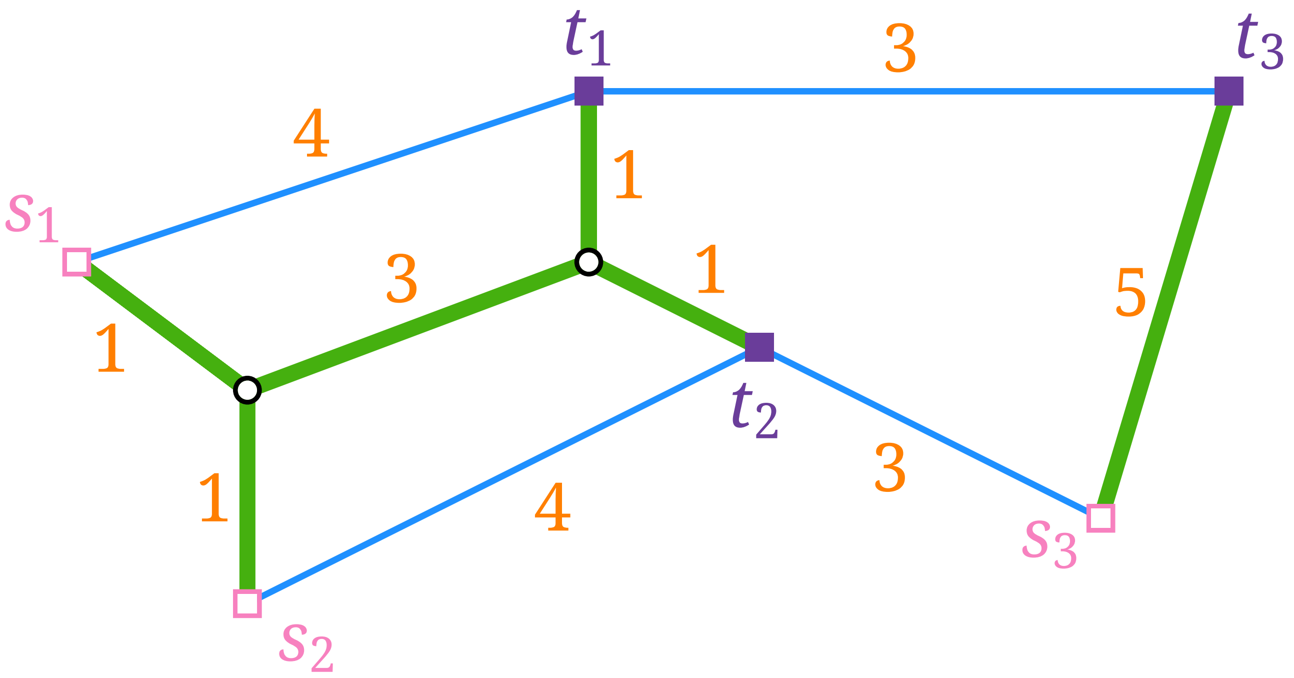

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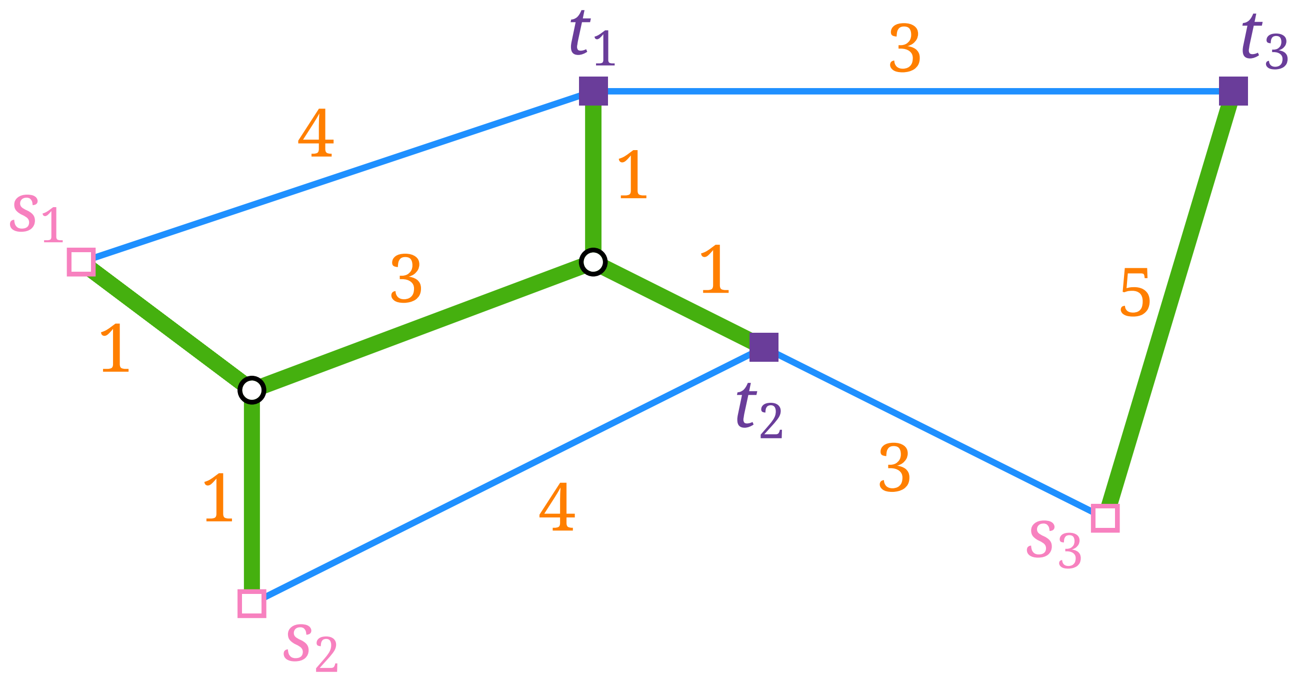

Special cases?

SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \dots, (s_k, t_k)\}$ of k vertex pairs.

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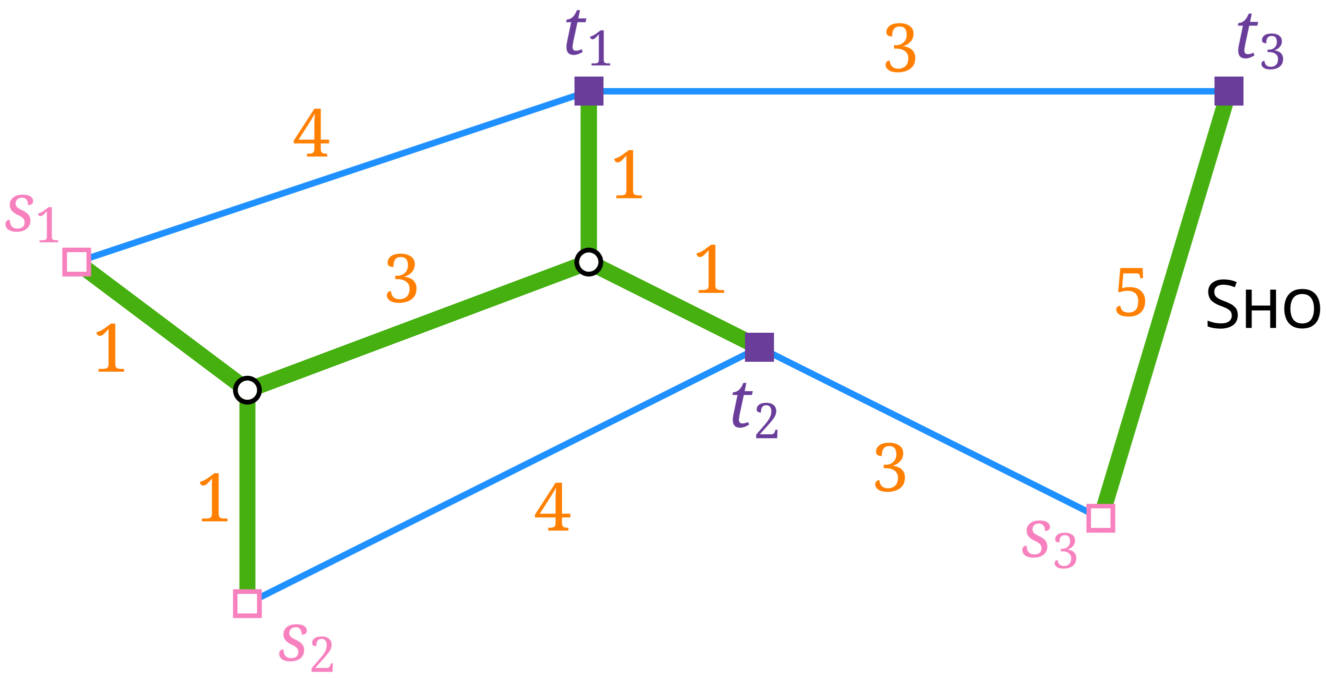

Special cases?

ShortestPath ($R = \{s, t\}$)

SteinerForest

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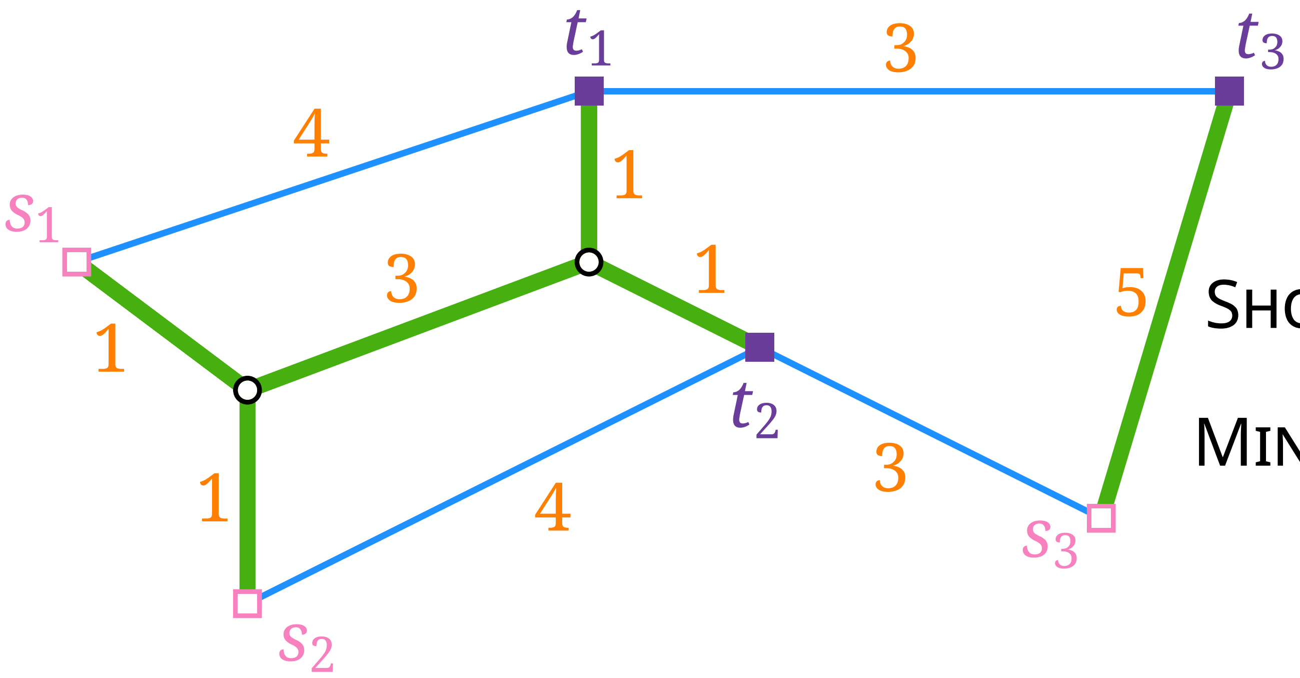

Special cases?

ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = V \times V$)

SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of k vertex pairs.

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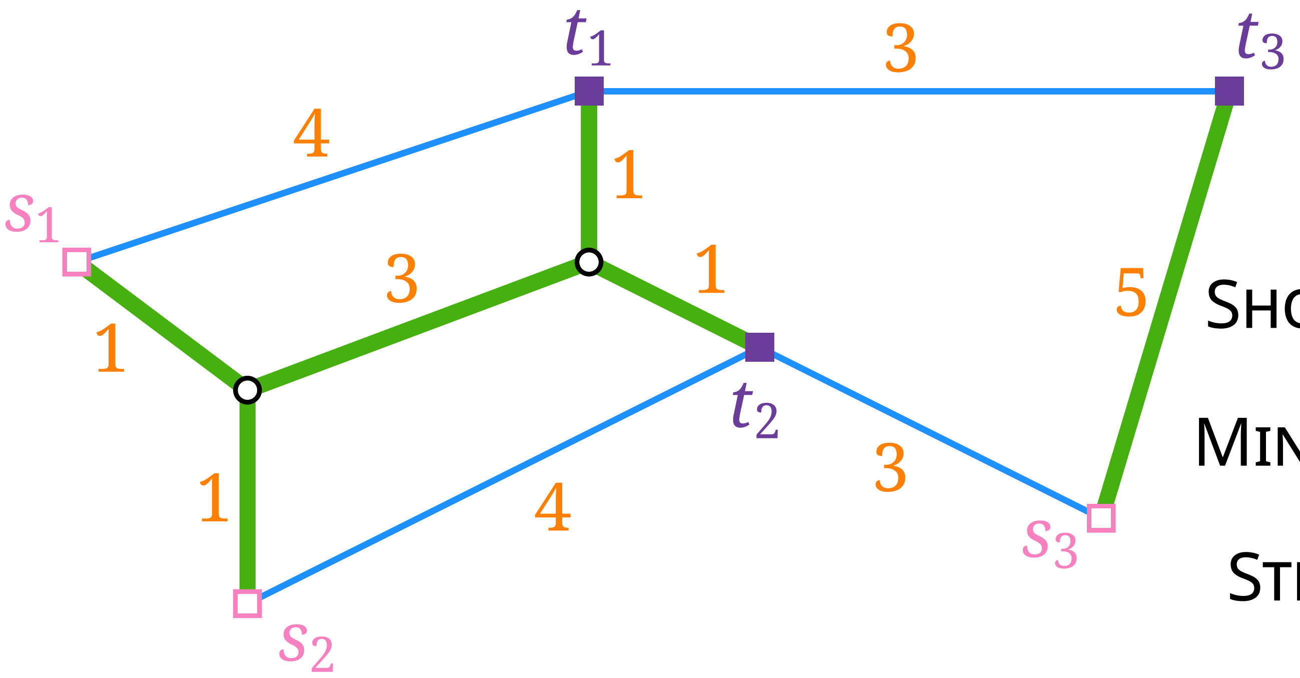


Special cases?

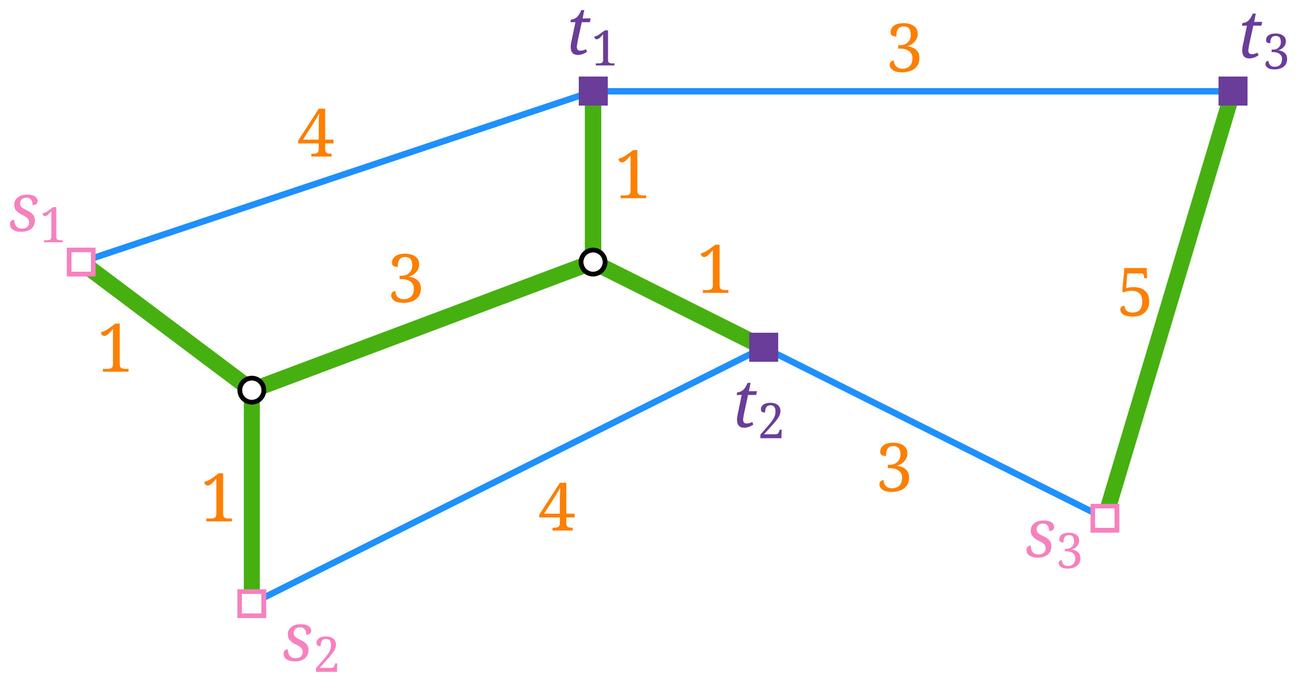
ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = V \times V$)

SteinerTree ($R = T \times T$)

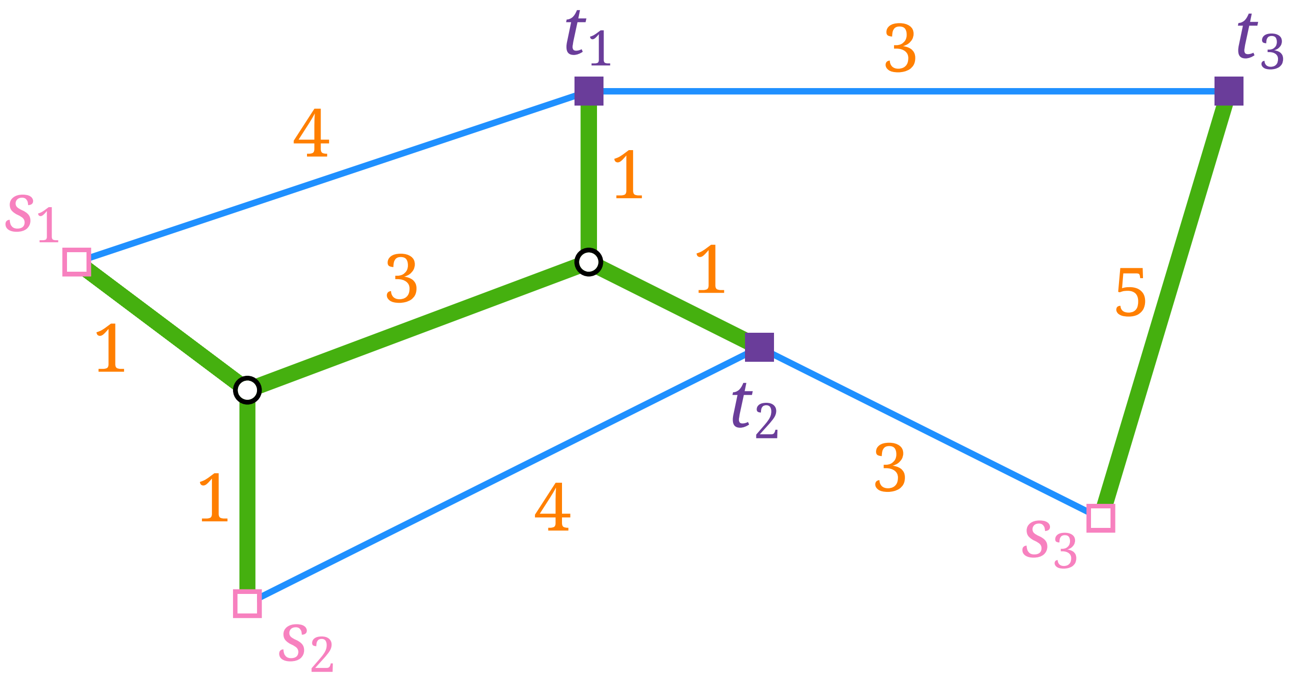
Approaches?

- Merge k shortest s_i–t_i paths



Approaches?

- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals



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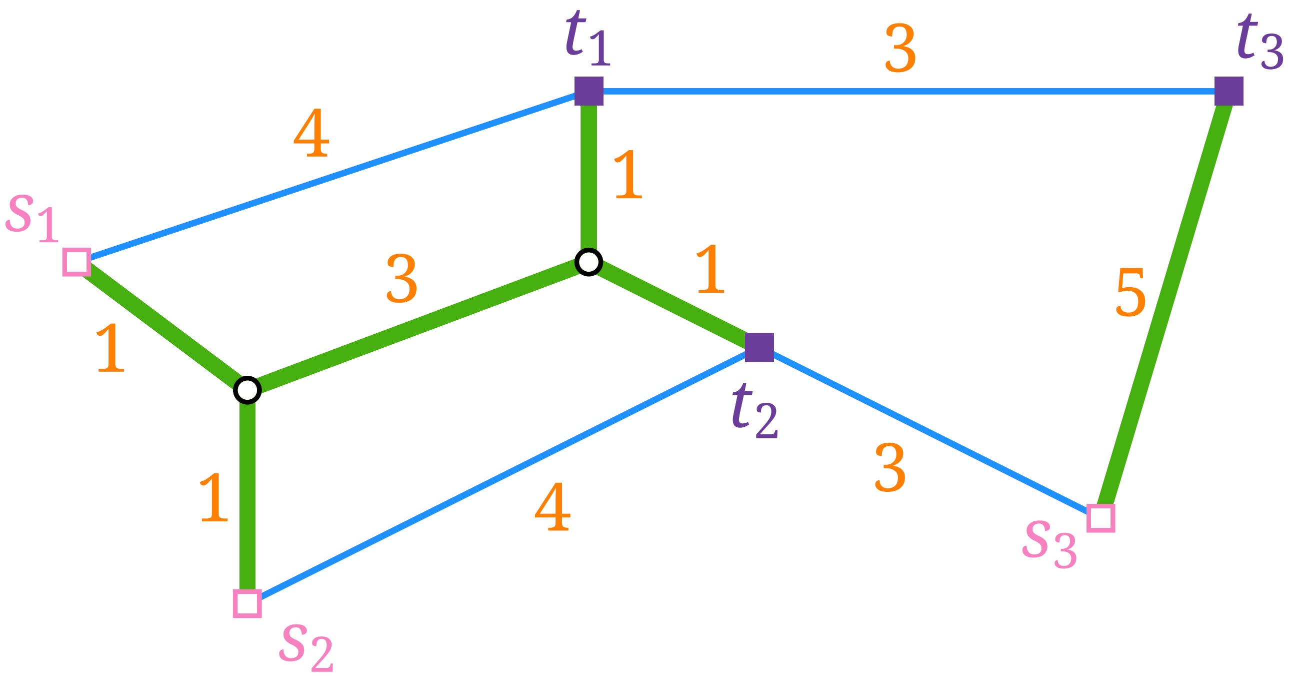
- SteinerTree on the set of terminals

Above approaches perform poorly :-(

Why?



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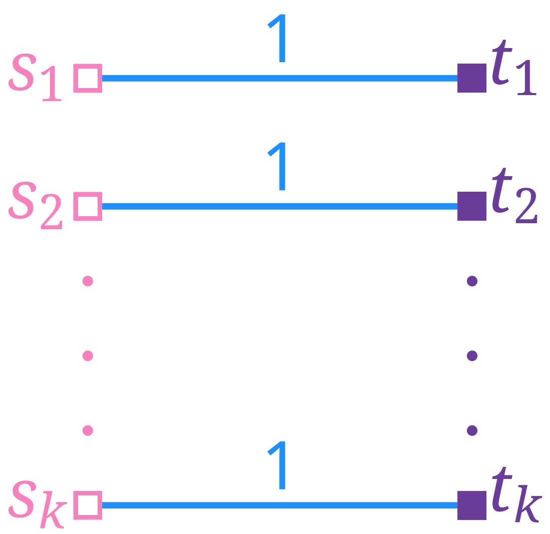

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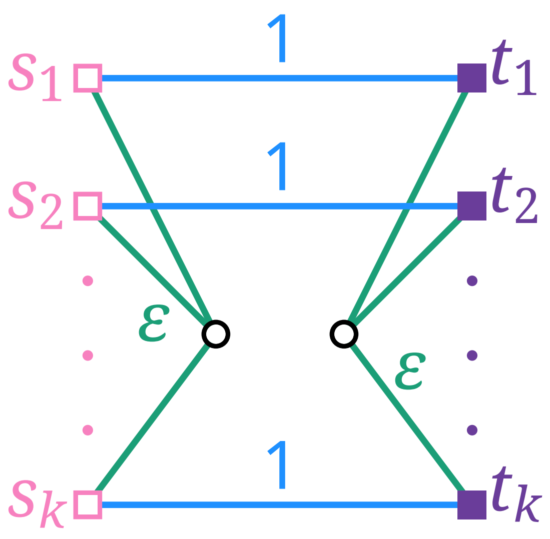

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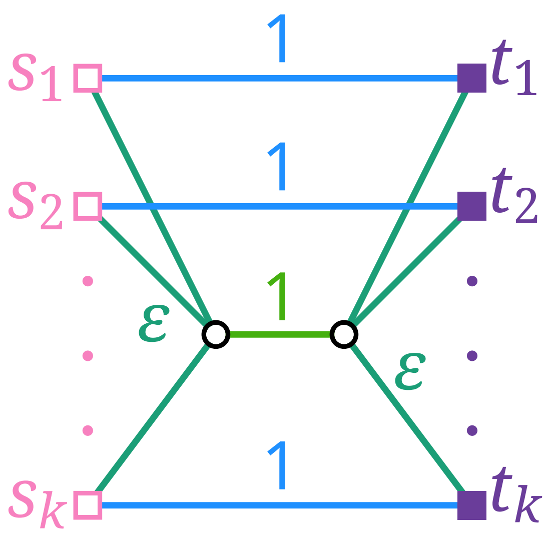

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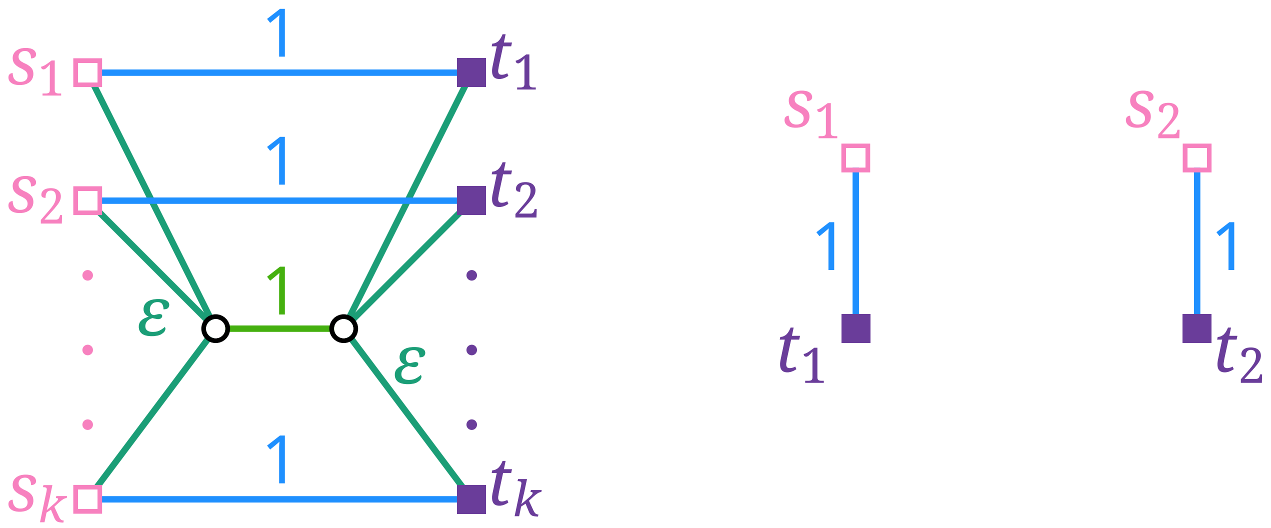

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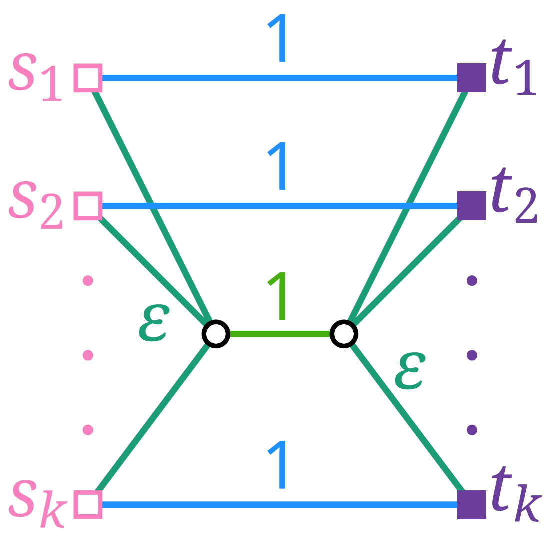

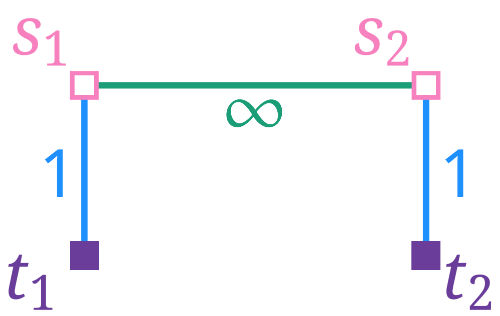

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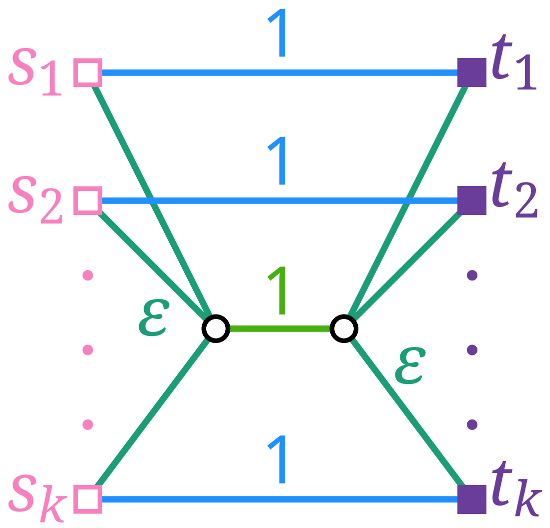
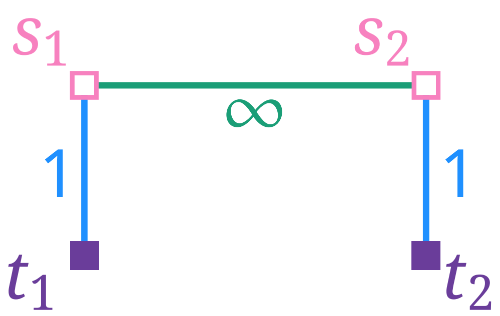
Difficulty:
Which terminals belong to the same tree of the forest?





Primal and Dual LP

ILP for SteinerForest

minimize

subject to

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minimize

subject to

$$x_e \in \{0,1\} \qquad e \in E$$

ILP for SteinerForest

minimize $\sum_{e \in E} c_e x_e$

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$x_e \in \{0, 1\} \qquad e \in E$

ILP for SteinerForest

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How to ensure connectedness of all pairs in R?

ILP for SteinerForest

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{e \in E} c_e x_e & \\
\textbf{subject to} & & \\
& x_e \in \{0, 1\} & e \in E
\end{array}
$$

How to ensure connectedness of all pairs in R?

Using cuts?

ILP for SteinerForest

minimize $\sum_{e \in E} c_e x_e$

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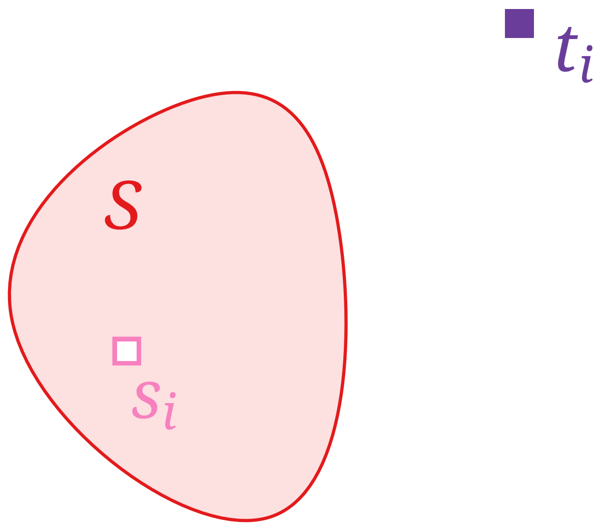

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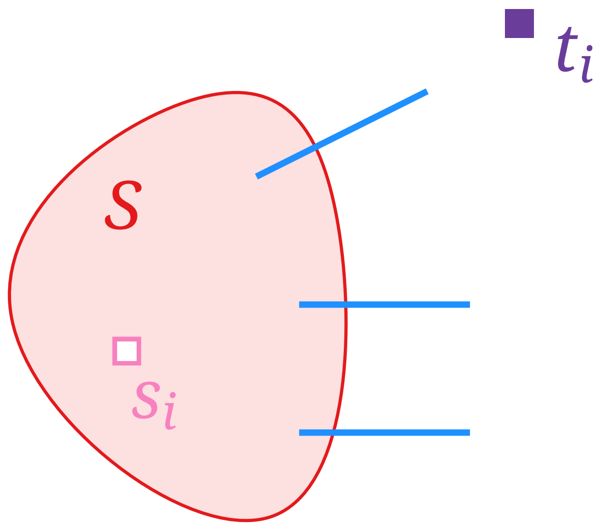

ILP for SteinerForest

minimize	$\sum_{e \in E} c_e x_e$
subject to	
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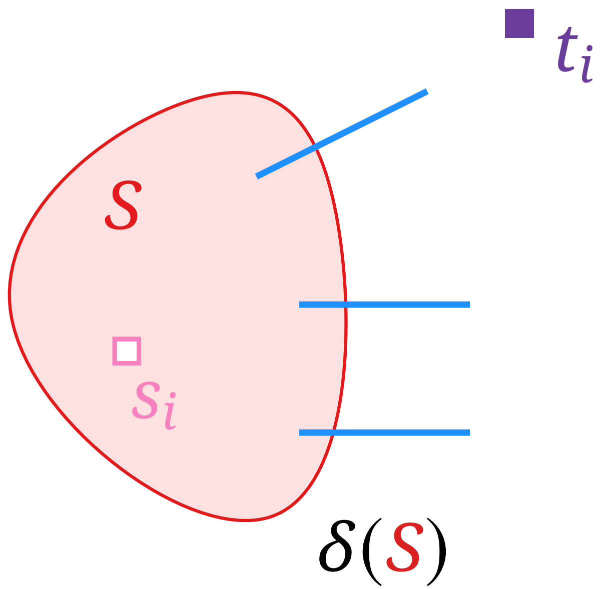

ILP for SteinerForest

minimize	$\sum_{e \in E} c_e x_e$
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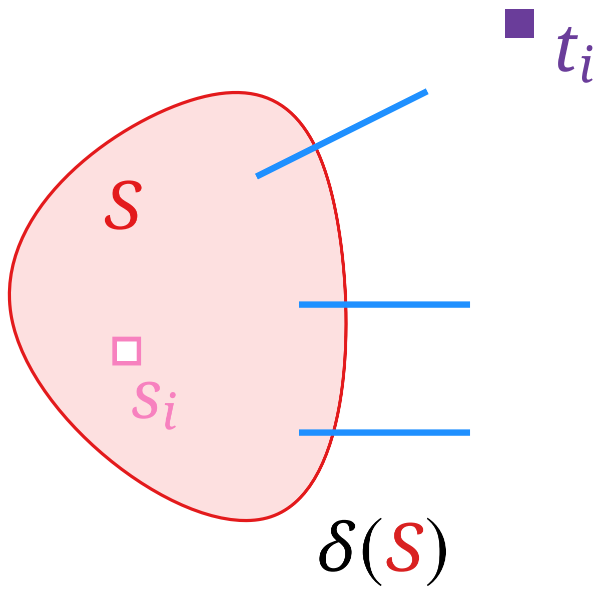
$\delta(S) := \{(u,v) \in E : u \in S \text{ and } v \notin S\}$



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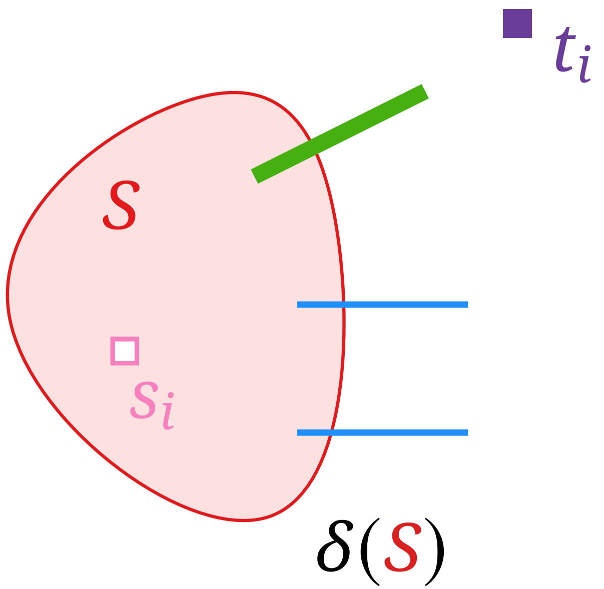

ILP for SteinerForest

$$\begin{array}{ll} \textbf{minimize} & \displaystyle\sum_{e \in E} c_e x_e \\ \textbf{subject to} & \displaystyle\sum_{e \in \delta(S)} x_e \geq 1 \\ & x_e \in \{0,1\} \qquad e \in E \end{array}$$

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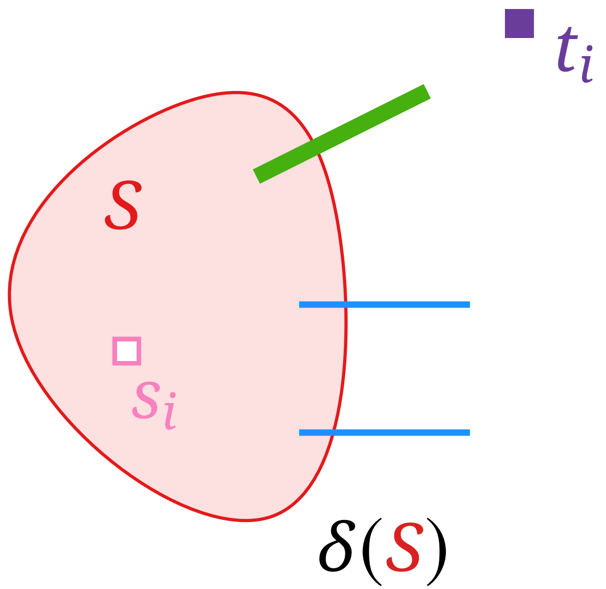

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\end{array}
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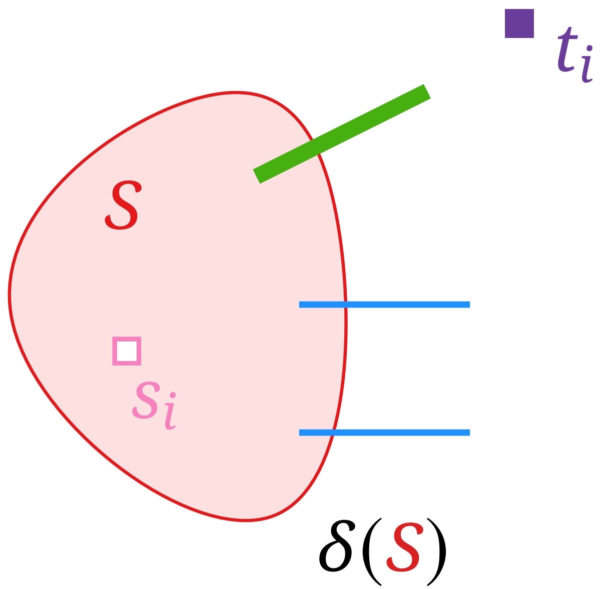

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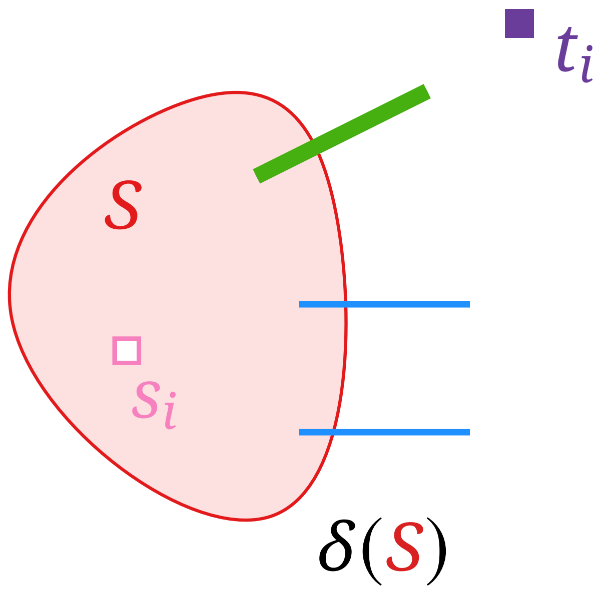
where $\mathcal{S}_i := \{S \subseteq V : |S \cap \{s_i, t_i\}| = 1\}$

cuts separating s_i and t_i

and $\delta(S) := \{(u, v) \in E : u \in S \text{ and } v \notin S\}$

cut edges



ILP for SteinerForest

$$\begin{aligned} &\textbf{minimize} && \sum_{e \in E} c_e x_e && \\ &\textbf{subject to} && \sum_{e \in \delta(S)} x_e \geq 1 && S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\} \\ & && x_e \in \{0, 1\} && e \in E \end{aligned}$$

Why does this enforce connectedness?

where $\mathcal{S}_i := \{S \subseteq V : |S \cap \{s_i, t_i\}| = 1\}$

cuts separating s_i and t_i

and $\delta(S) := \{(u, v) \in E : u \in S \text{ and } v \notin S\}$

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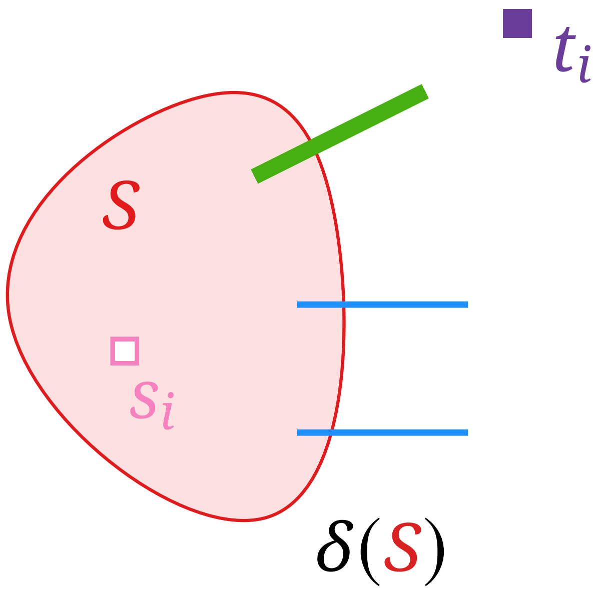

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Why does this enforce connectedness?

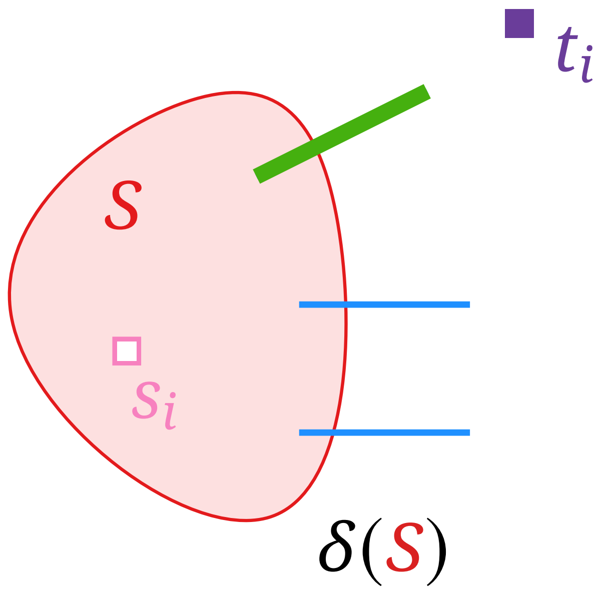
incrementally grow a path from s_i to t_i

where $\mathcal{S}_i := \{S \subseteq V : |S \cap \{s_i, t_i\}| = 1\}$

cuts separating s_i and t_i

and $\delta(S) := \{(u, v) \in E : u \in S \text{ and } v \notin S\}$

cut edges



ILP for SteinerForest

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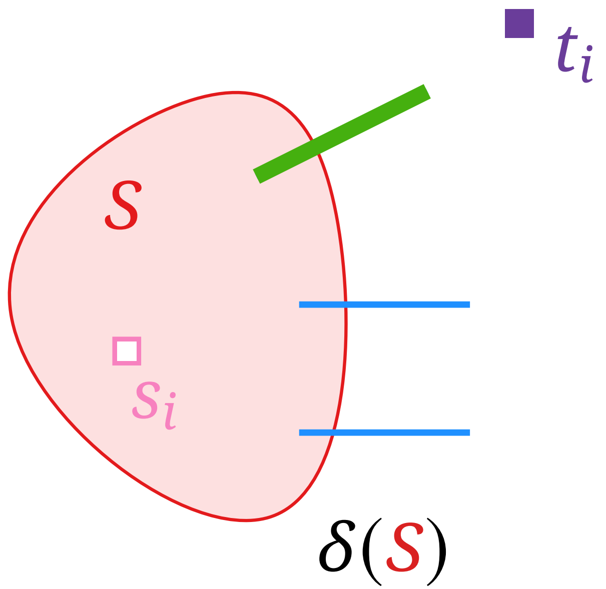
cut edges

$\rightsquigarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

minimize	$\sum_{e \in E} c_e x_e$	
subject to	$\sum_{e \in \delta(S)} x_e \geq 1$	$S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}$
	$x_e \geq 0$	$e \in E$

LP-Relaxation and Dual LP

minimize	$\sum_{e \in E} c_e x_e$	
subject to	$\sum_{e \in \delta(S)} x_e \geq 1$	$S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}$ (y_S)
	$x_e \geq 0$	$e \in E$

LP-Relaxation and Dual LP

minimize	$\sum_{e \in E} c_e x_e$	
subject to	$\sum_{e \in \delta(S)} x_e \geq 1$	$S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}\ (y_S)$
	$x_e \geq 0$	$e \in E$

maximize		
subject to		
	$y_S \geq 0$	$S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}$

LP-Relaxation and Dual LP

$$\begin{array}{lll} \textbf{minimize} & \sum_{e \in E} c_e x_e & \\ \textbf{subject to} & \sum_{e \in \delta(S)} x_e \geq 1 & S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}\ (y_S) \\ & x_e \geq 0 & e \in E \end{array}$$

$$\begin{array}{lll} \textbf{maximize} & \sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S & \\ \textbf{subject to} & & \\ & y_S \geq 0 & S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\} \end{array}$$

LP-Relaxation and Dual LP

minimize $\displaystyle\sum_{e \in E} c_e x_e$

subject to $\displaystyle\sum_{e \in \delta(S)} x_e \geq 1 \qquad S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}\ (y_S)$

$x_e \geq 0 \qquad e \in E$

maximize $\displaystyle\sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S$

subject to $\displaystyle\sum_{S:\ e \in \delta(S)} y_S \leq c_e \qquad e \in E$

$y_S \geq 0 \qquad S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}$

LP-Relaxation and Dual LP

covering LP

minimize	$\sum_{e\in E} c_e x_e$	
subject to	$\sum_{e\in\delta(S)} x_e \geq 1$	$S \in \mathcal{S}_i,\ i \in \{1,\ldots,k\}\ (y_S)$
	$x_e \geq 0$	$e \in E$

packing LP

maximize	$\sum_{\substack{S\in\mathcal{S}_i\\ i\in\{1,\ldots,k\}}} y_S$	
subject to	$\sum_{S:\ e\in\delta(S)} y_S \leq c_e$	$e \in E$
	$y_S \geq 0$	$S \in \mathcal{S}_i,\ i \in \{1,\ldots,k\}$

Intuition for the Dual

$$
\begin{array}{lll}
\textbf{maximize} & \displaystyle\sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S & \\
\textbf{subject to} & \displaystyle\sum_{S:\, e \in \delta(S)} y_S \le c_e & e \in E \\
& y_S \ge 0 & S \in \mathcal{S}_i,\ i \in \{1,\ldots,k\}
\end{array}
$$

Intuition for the Dual

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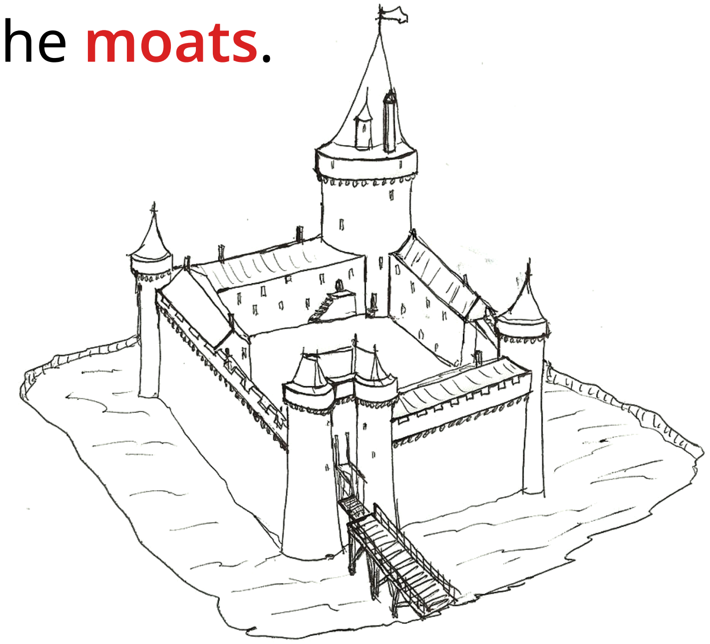
The graph is a network of **bridges**, spanning the **moats**.



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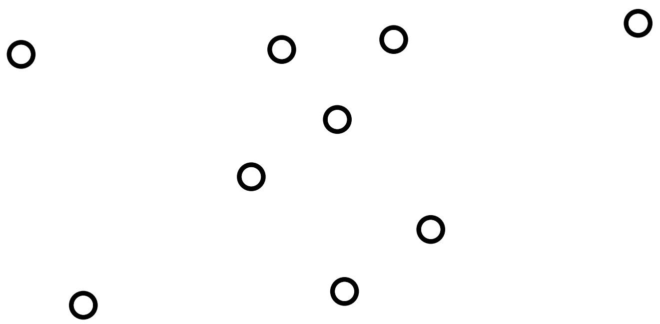

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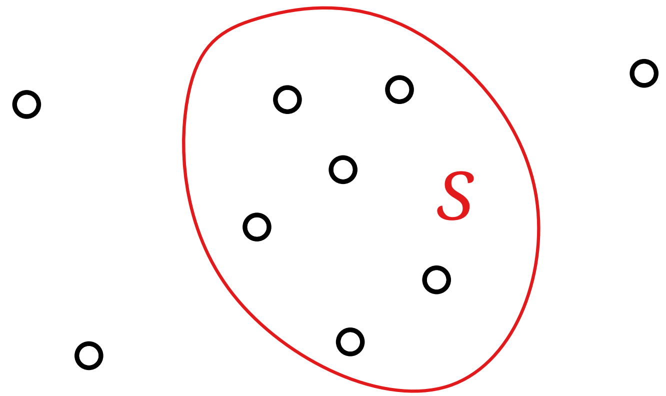

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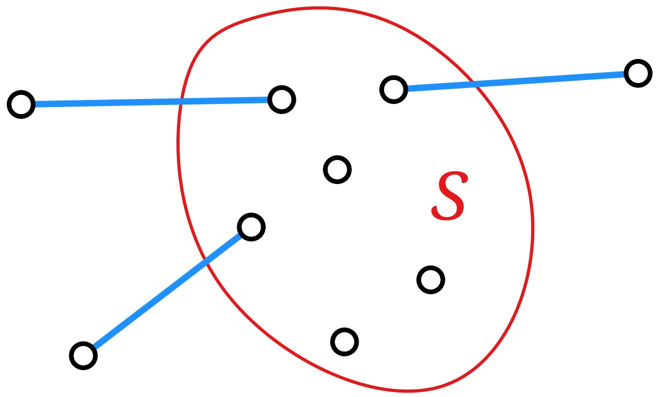

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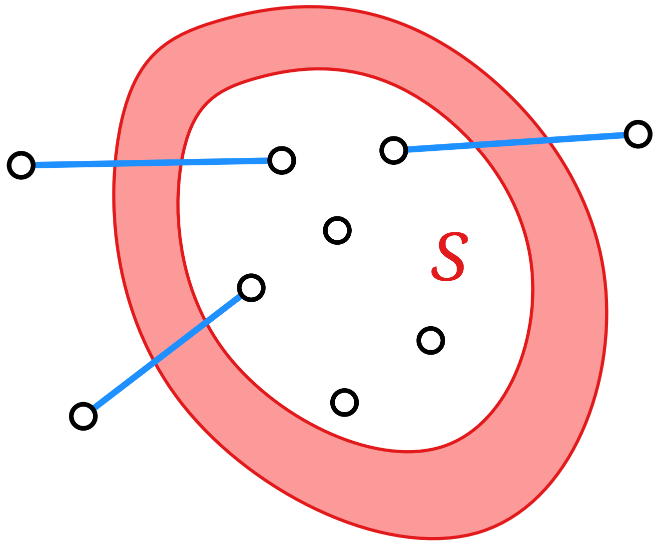

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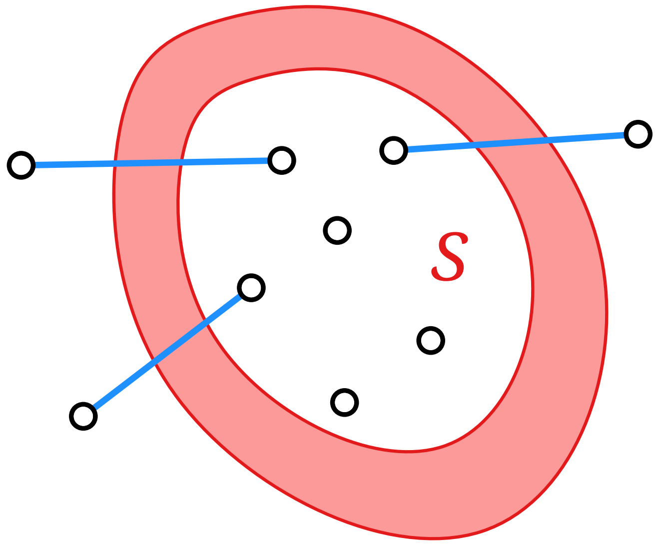

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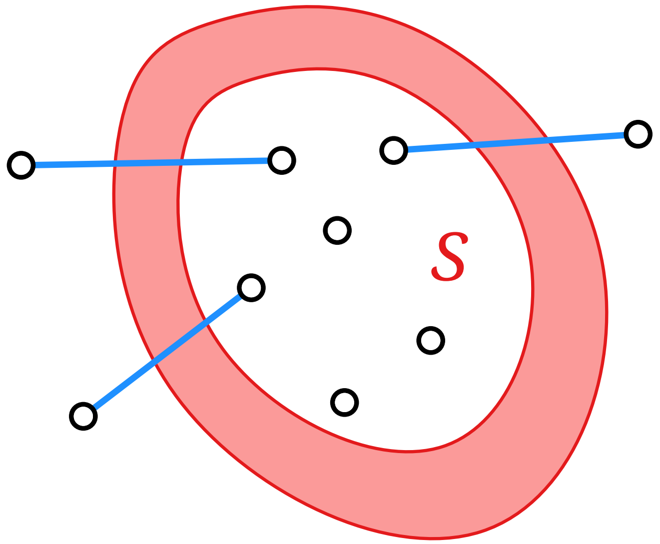

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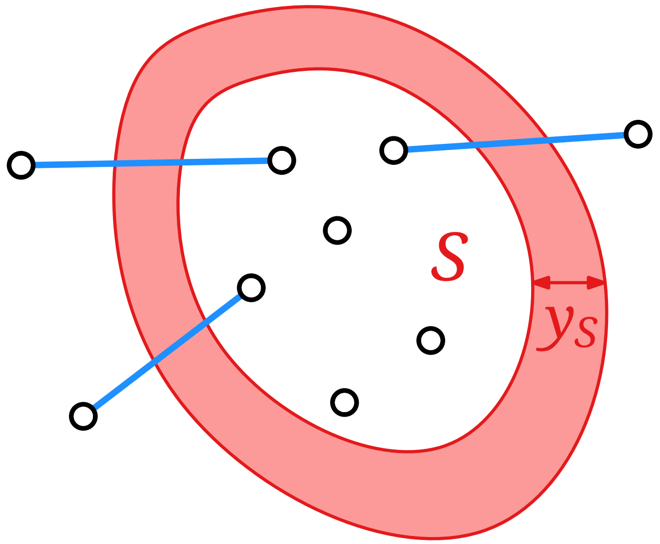

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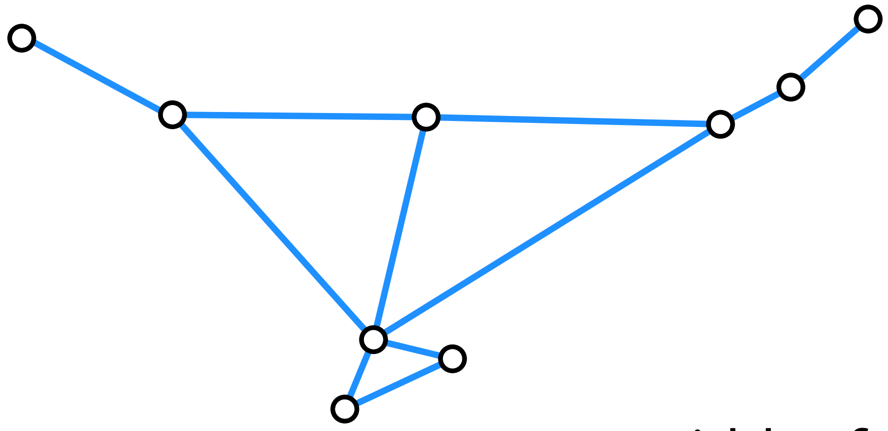

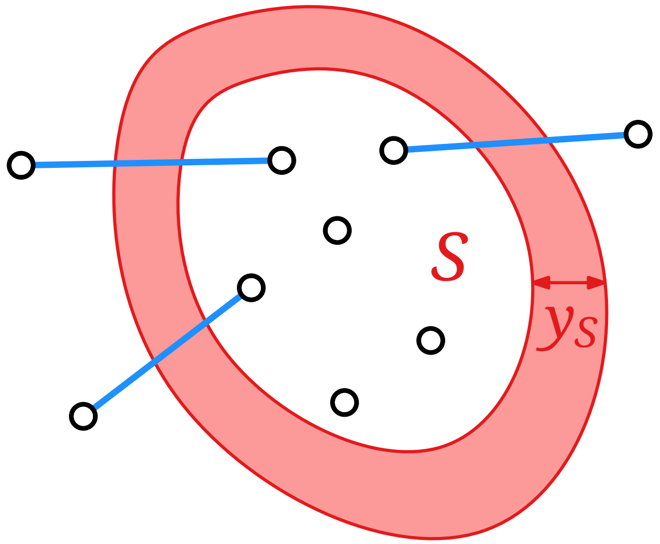

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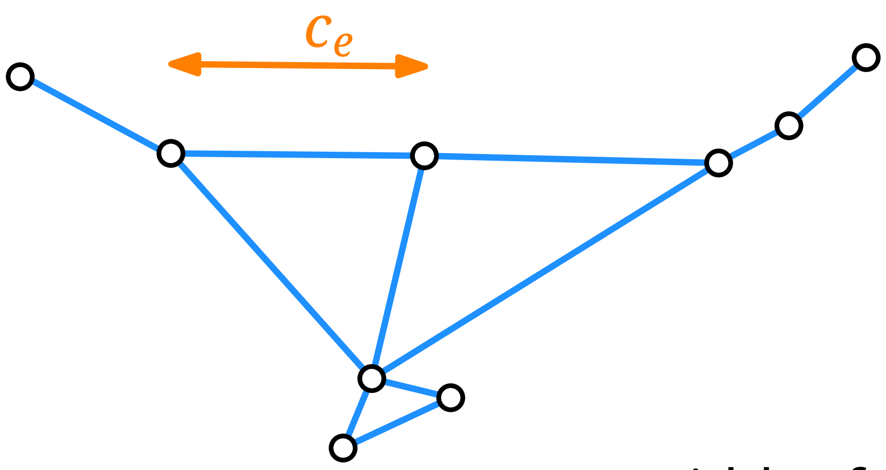

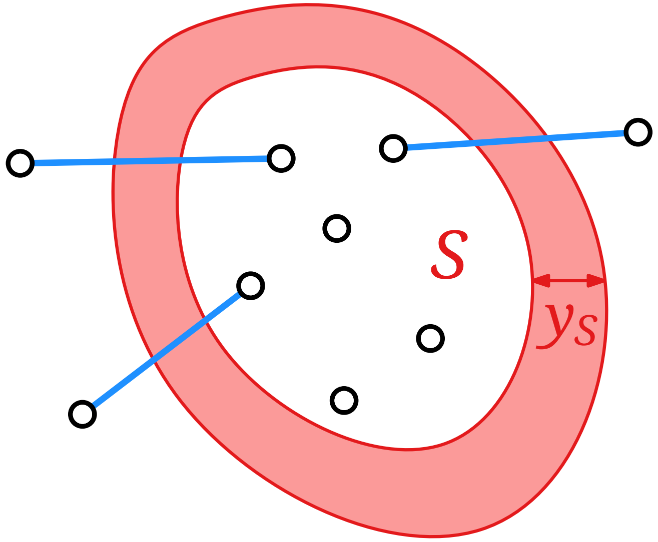

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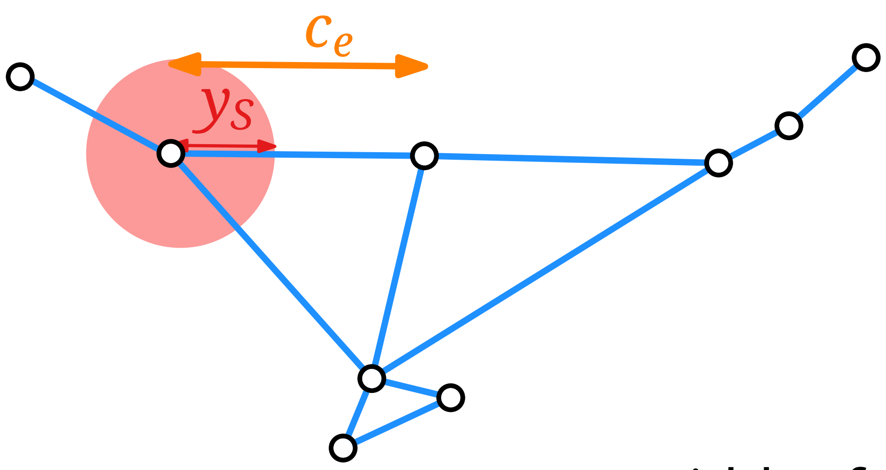

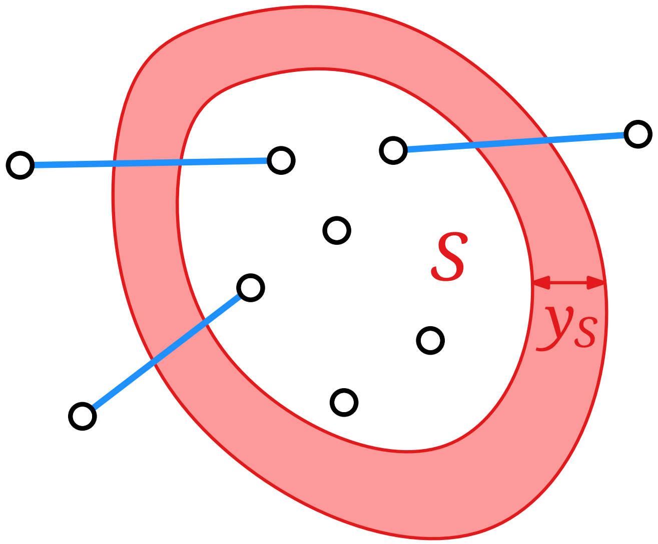

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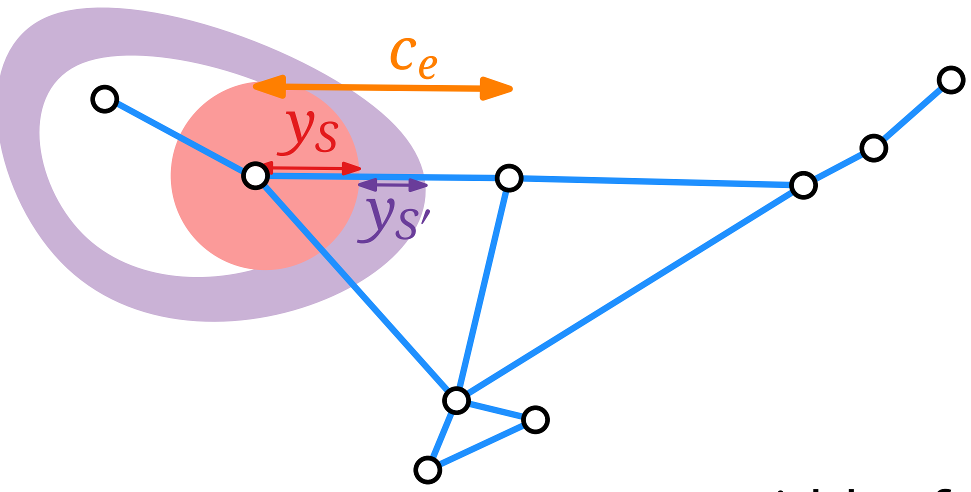

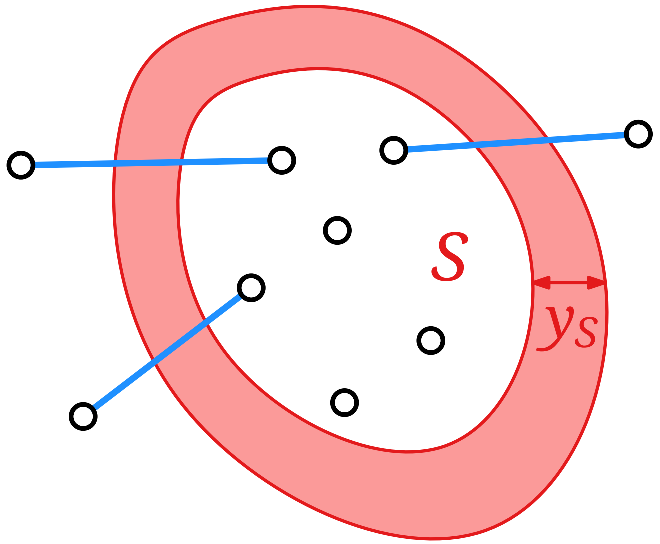

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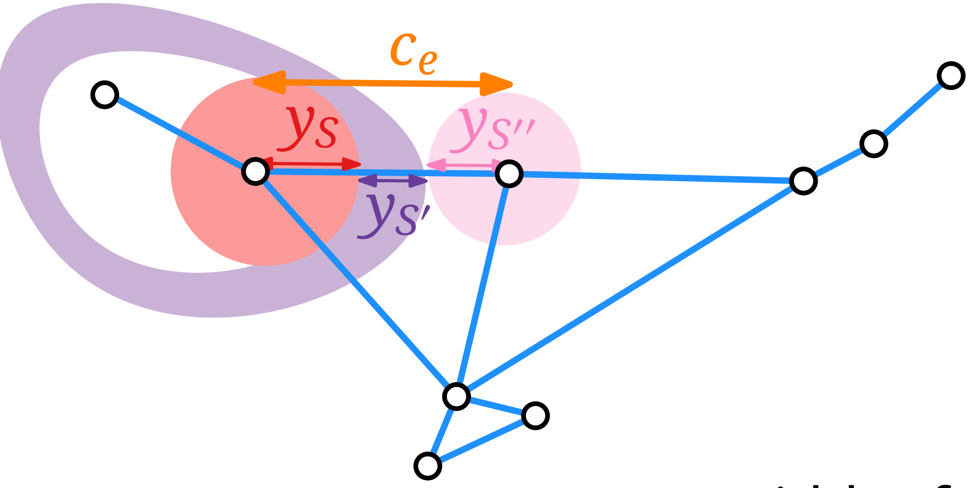

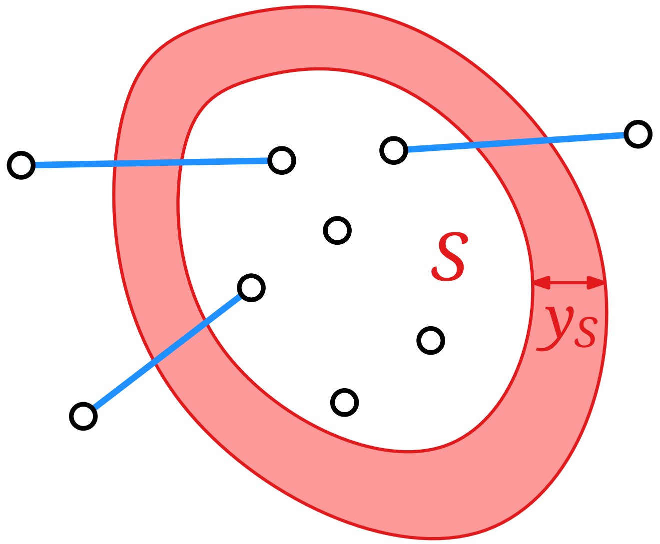

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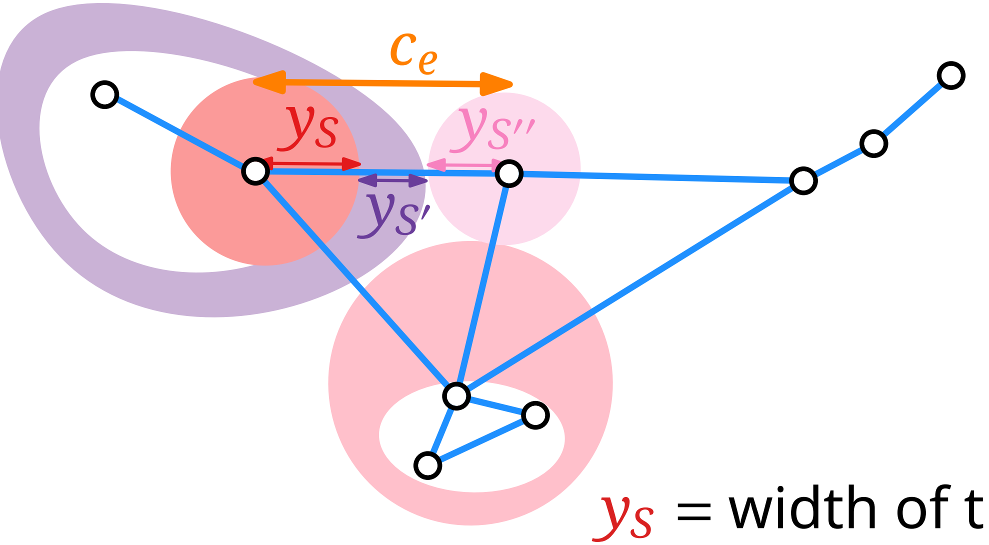

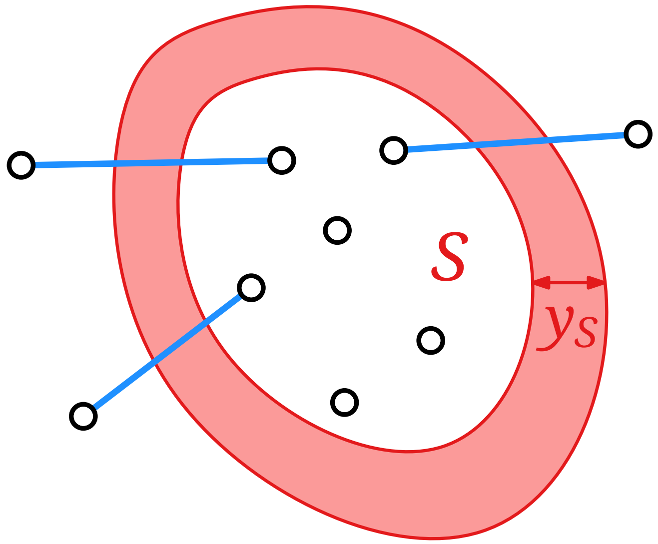

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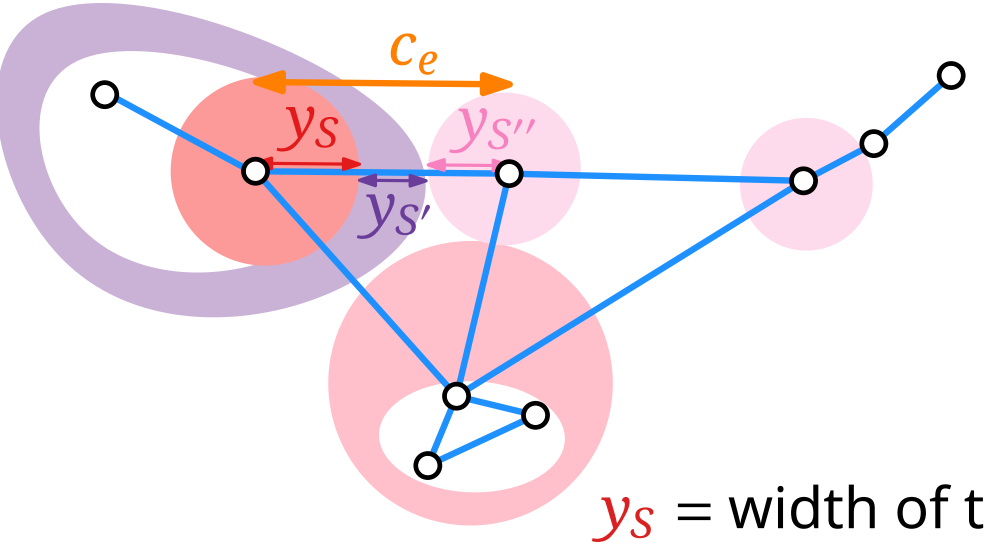

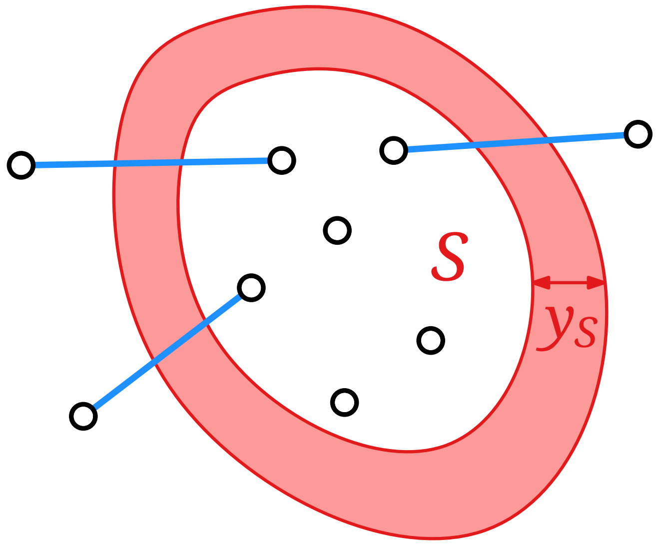

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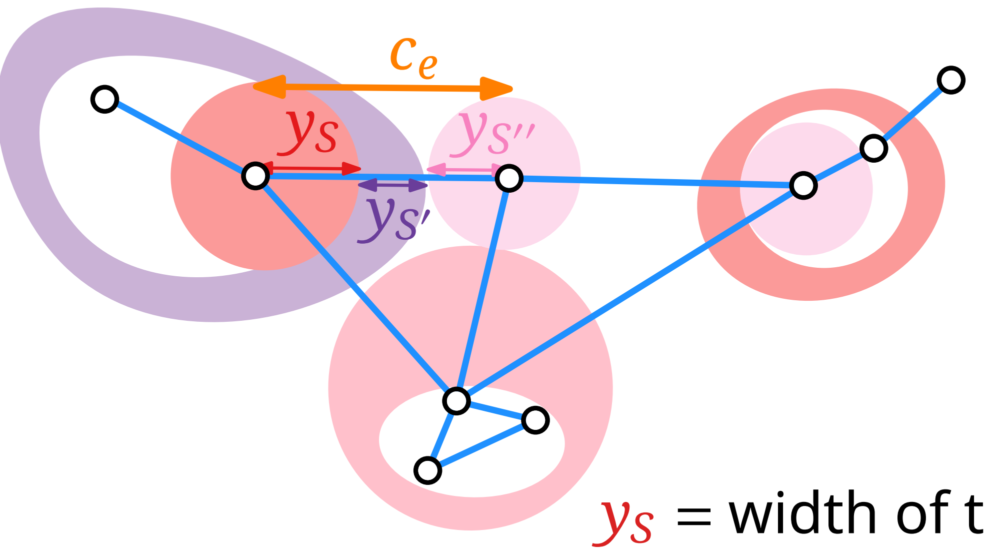

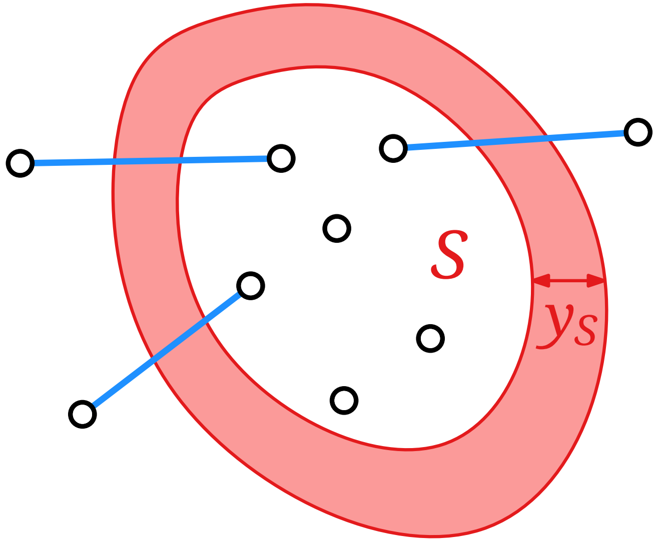

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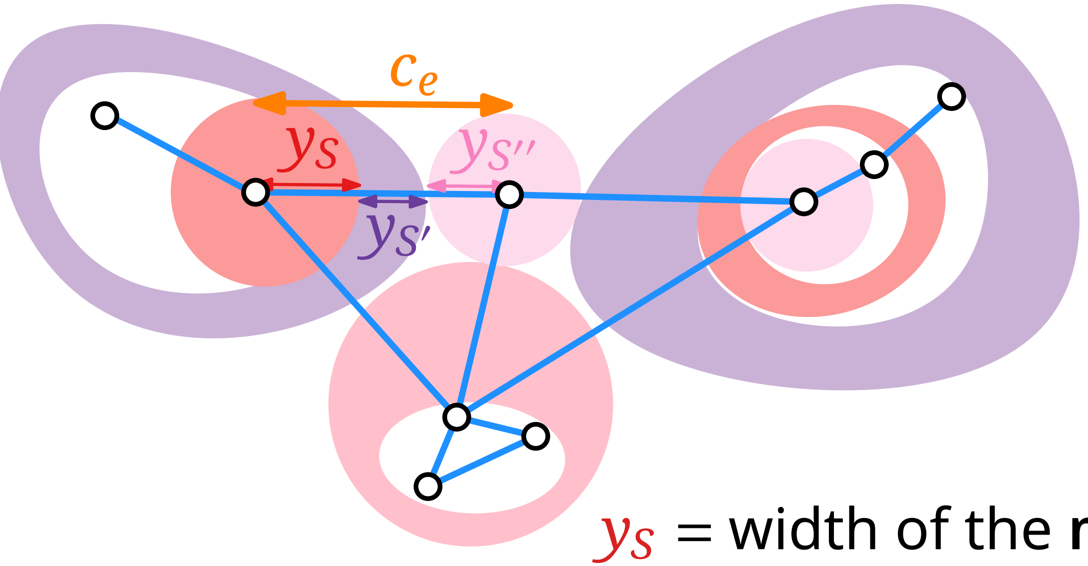

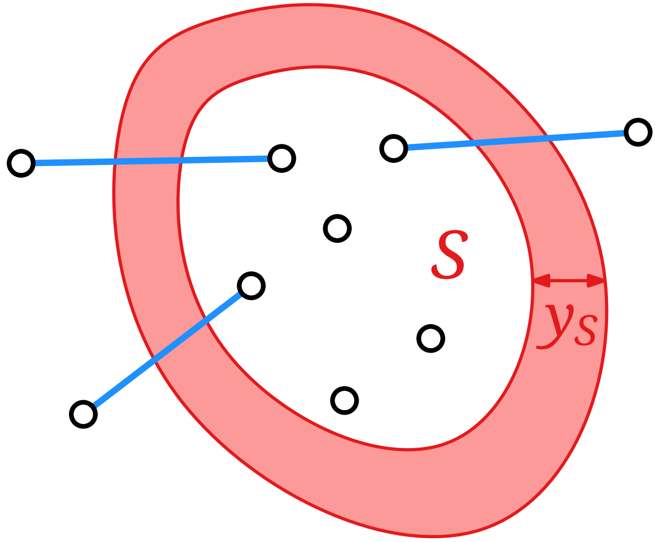

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A First Primal–Dual Approach

Reminder: Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

Reminder: Complementary Slackness

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Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

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How to find a violated primal constraint? $\left(\sum_{e\in\delta(S)} x_e < 1\right)$

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How do we iteratively improve the dual solution?

$\rightsquigarrow$ Increase y_C (until some edge in $\delta(C)$ becomes critical)!

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PrimalDualSteinerForestNaive(G, c, R)

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Trick: Handle all y_S with $y_S = 0$ implicitly

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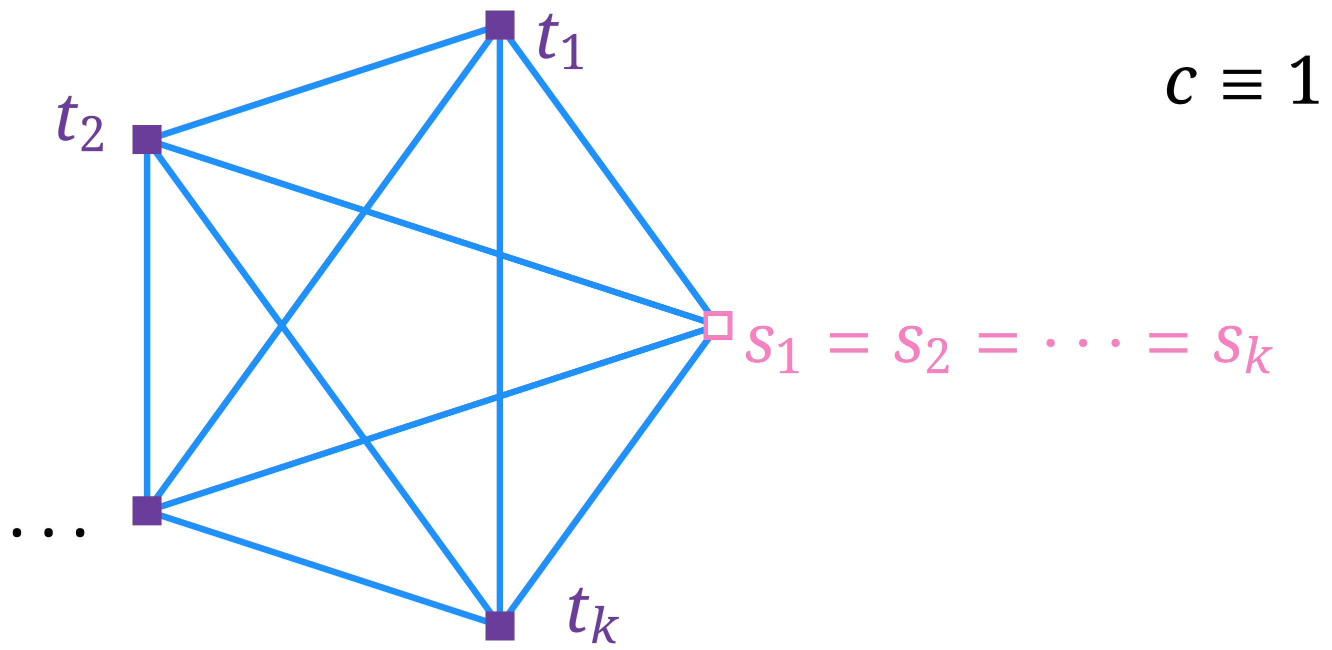

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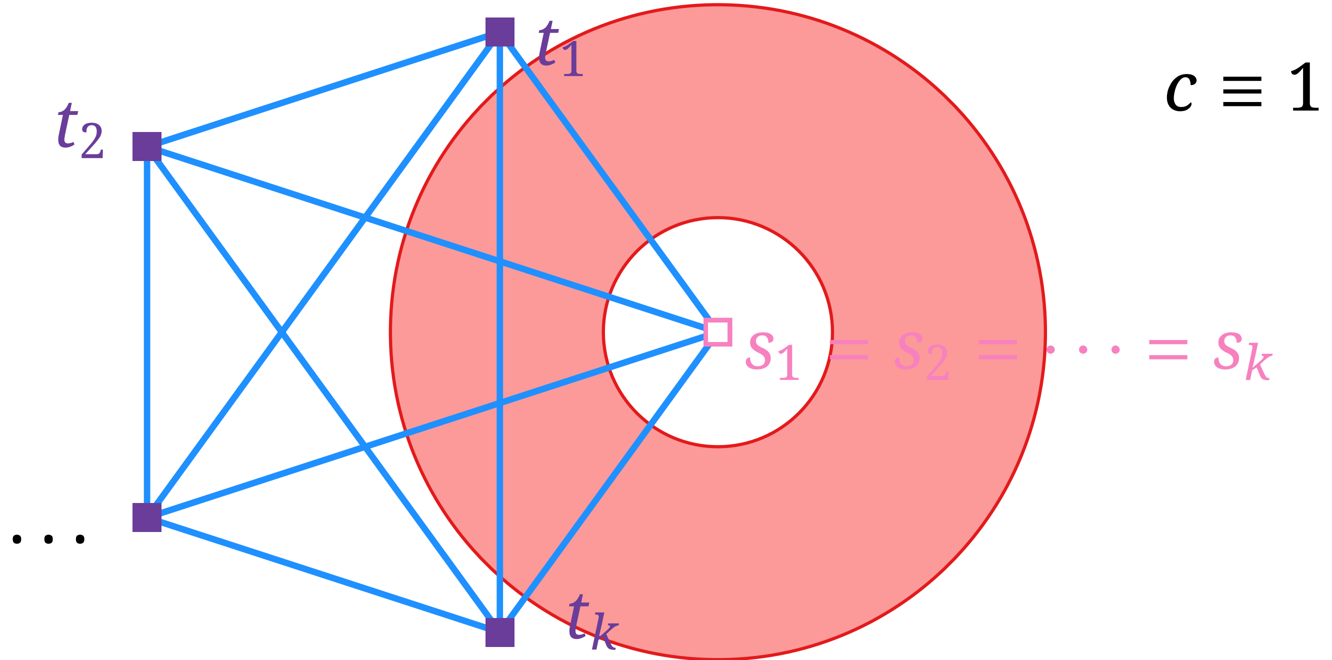

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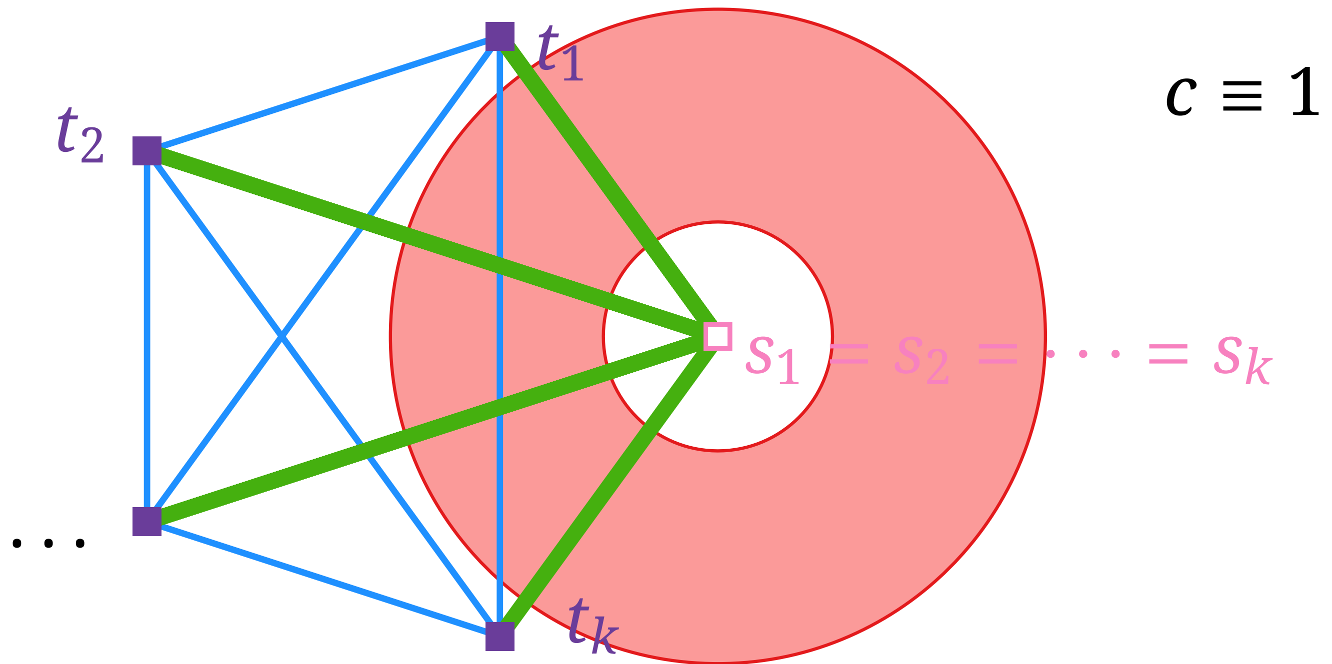

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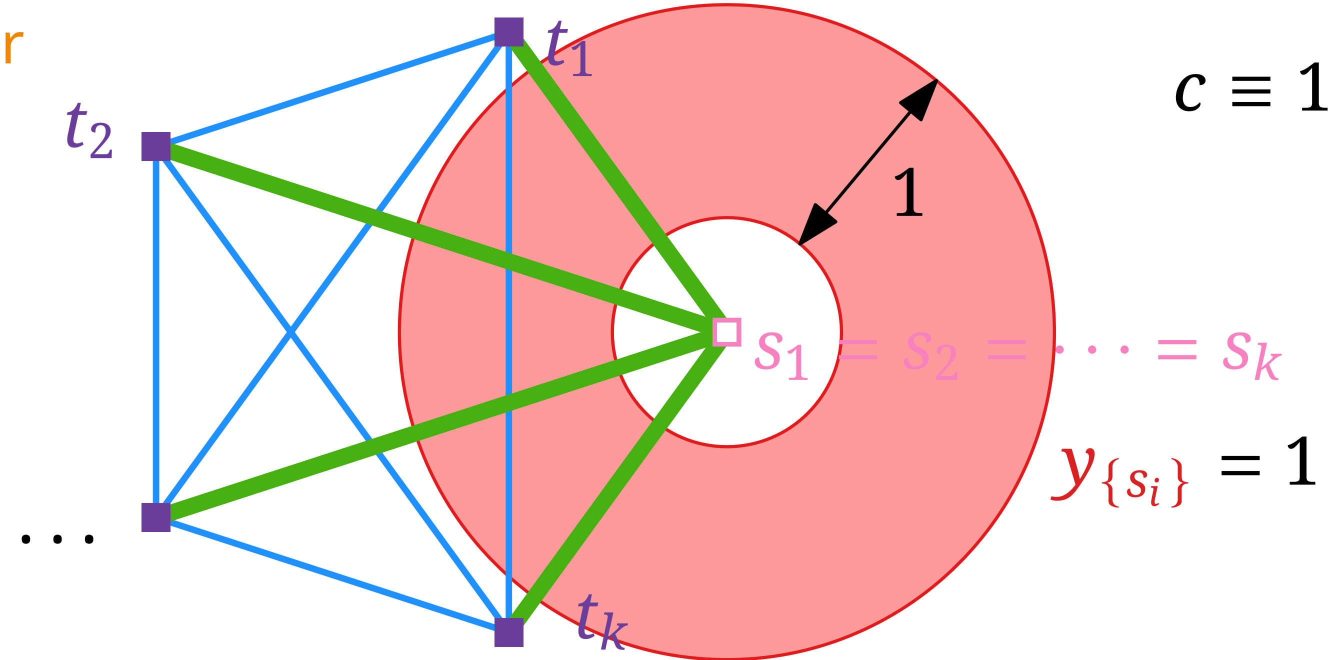

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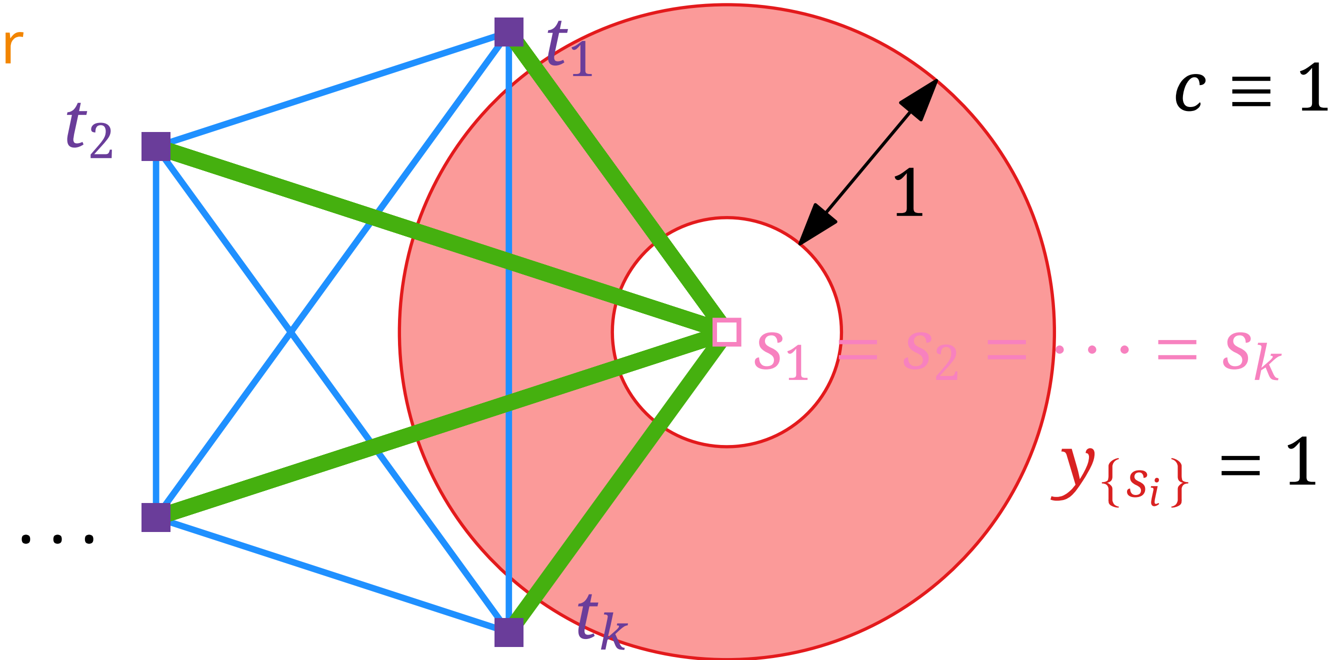

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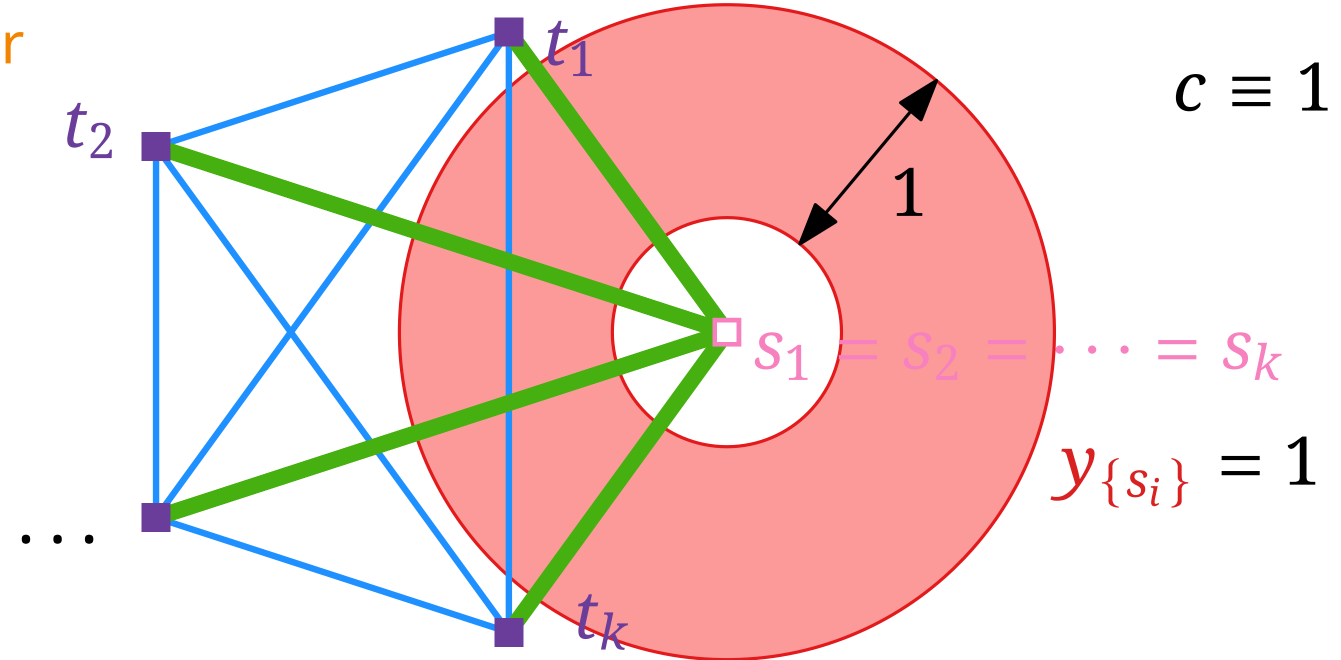

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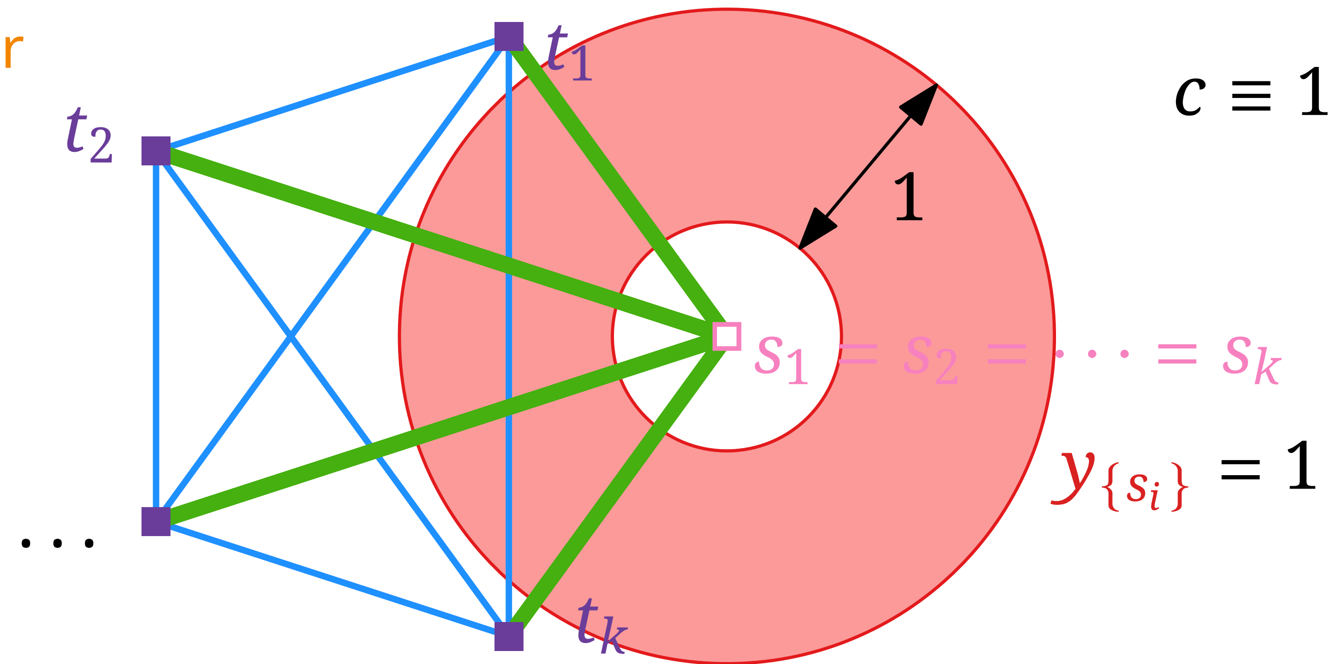

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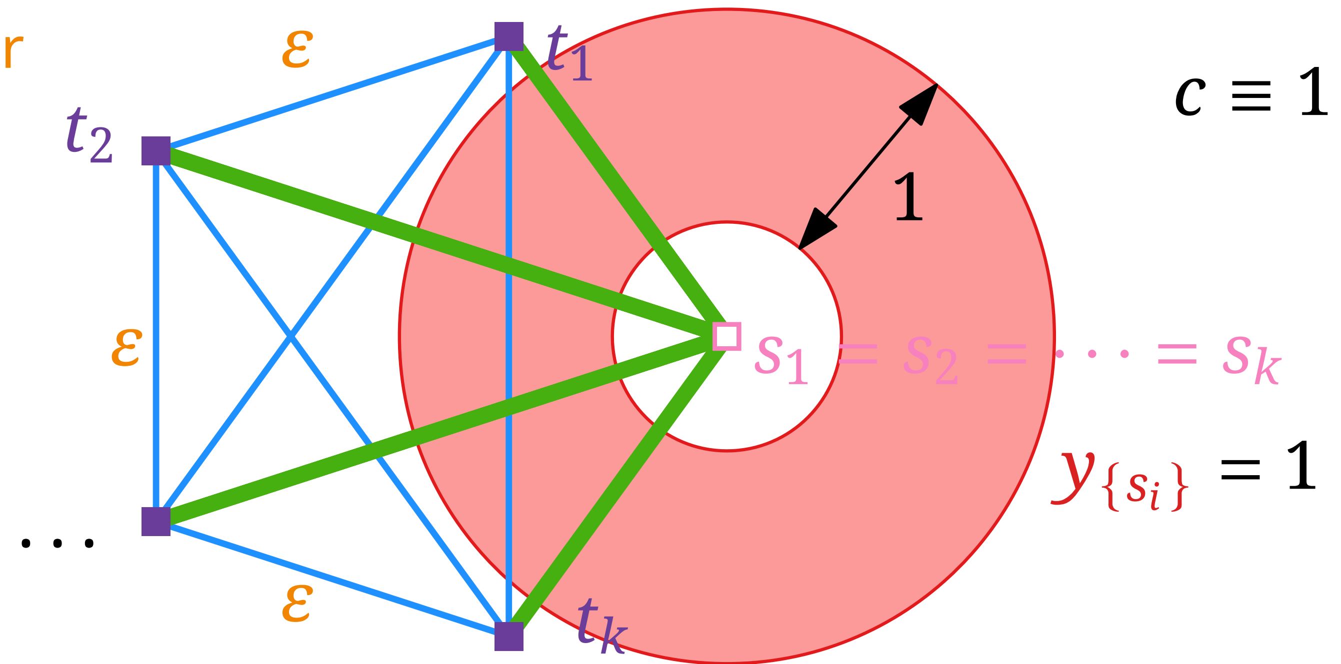

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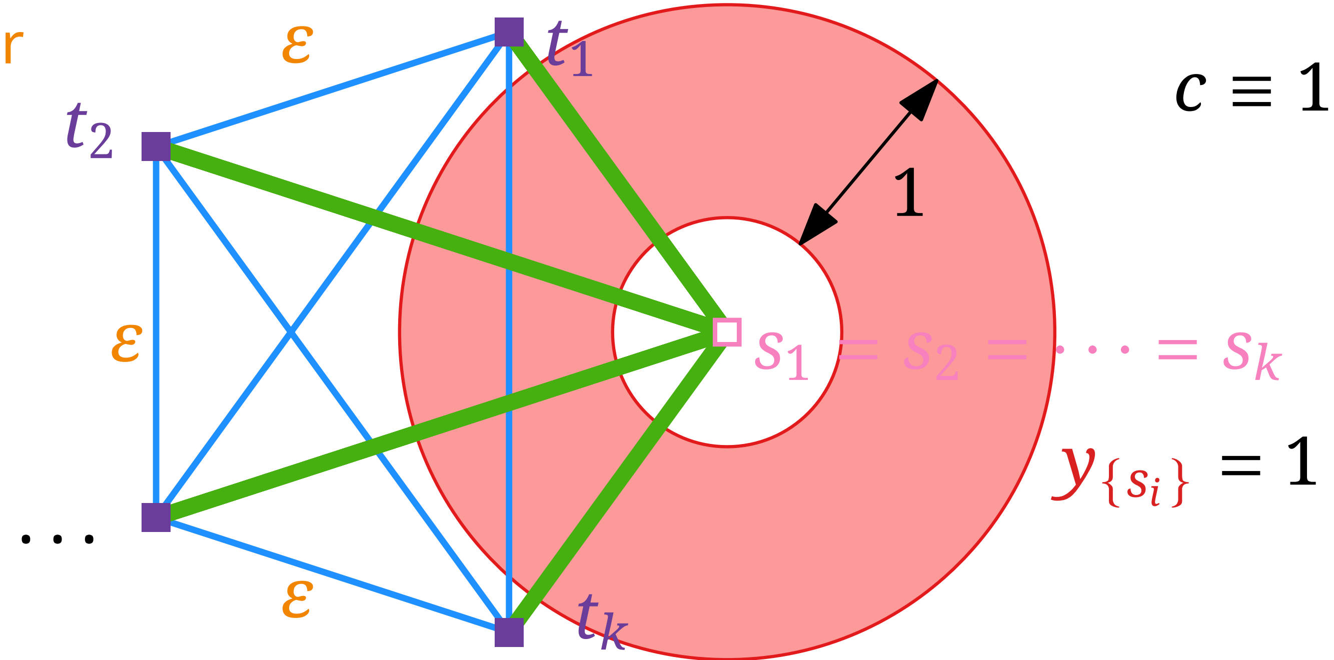

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but again, simultaneous increase helps

Primal–Dual with Synchronized Increases

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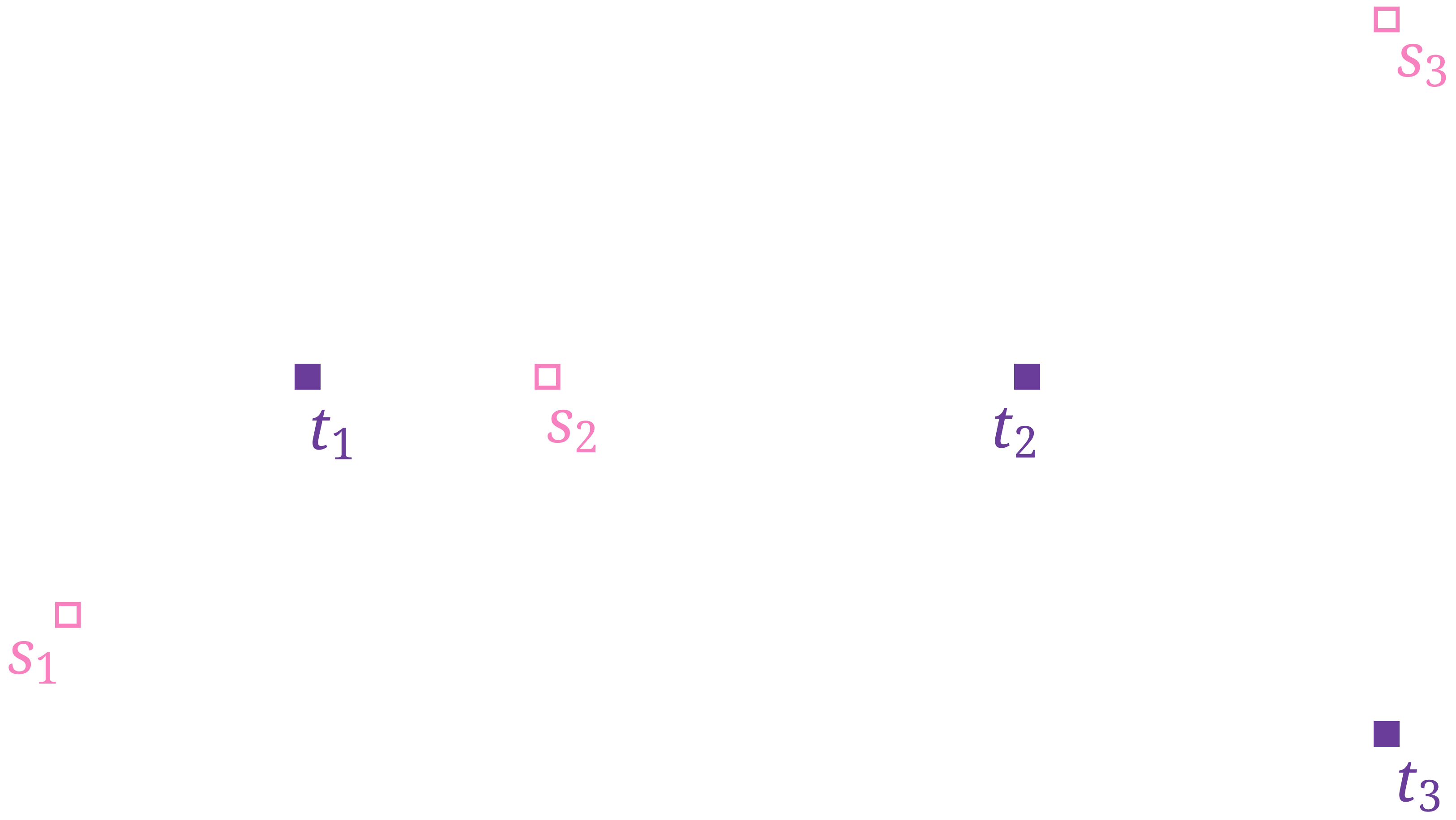
$G = K_6$ with Euclidean edge costs



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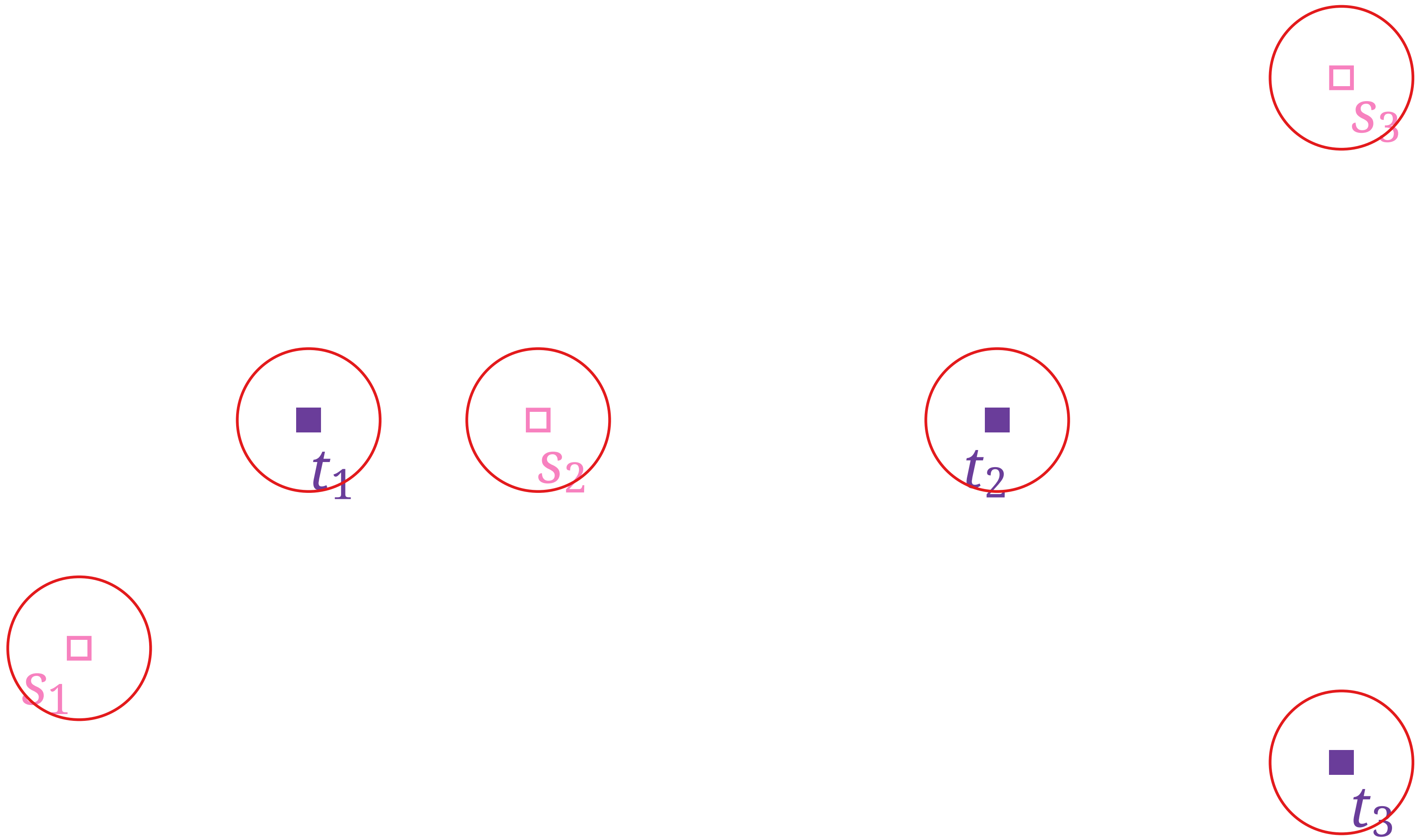

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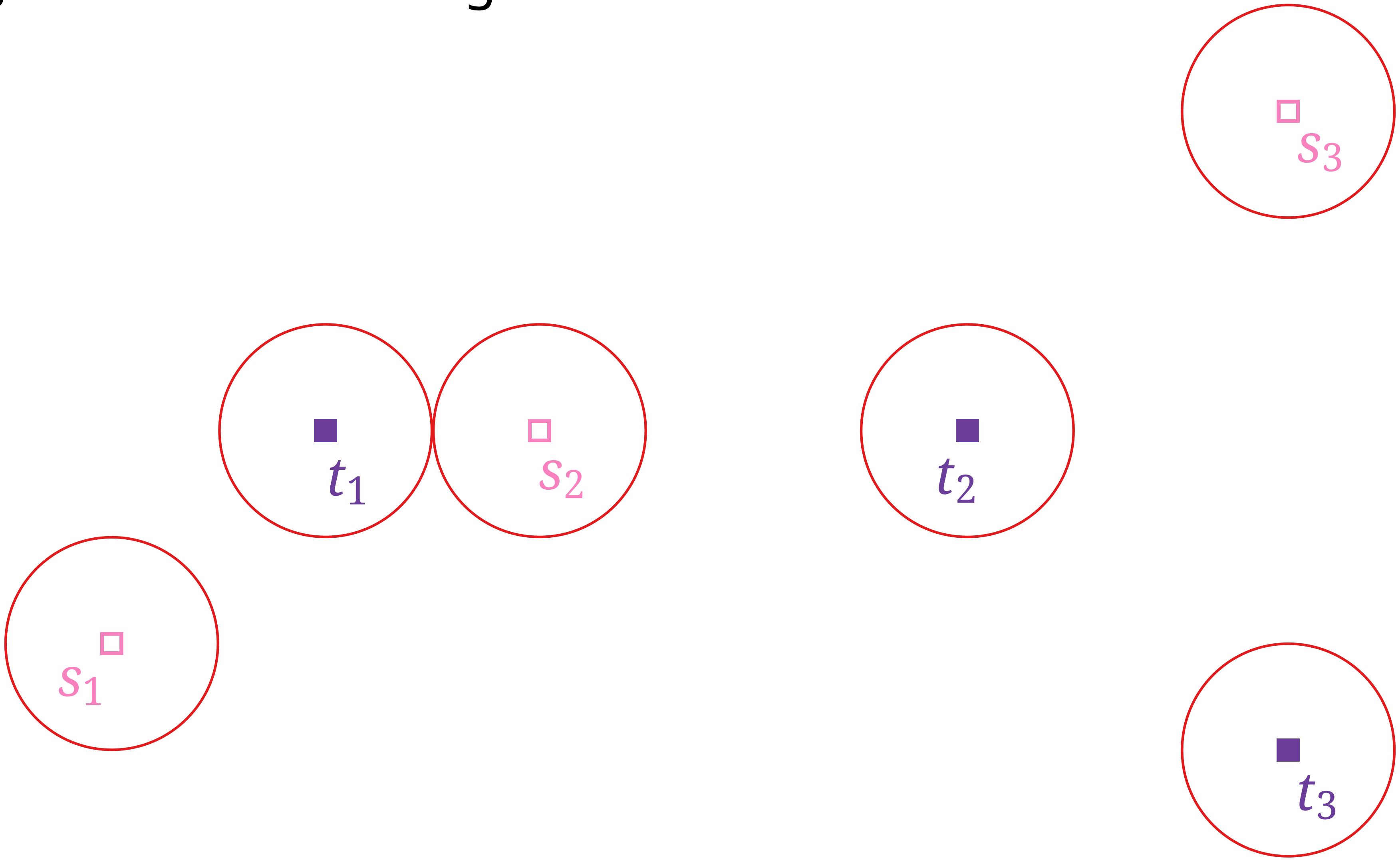

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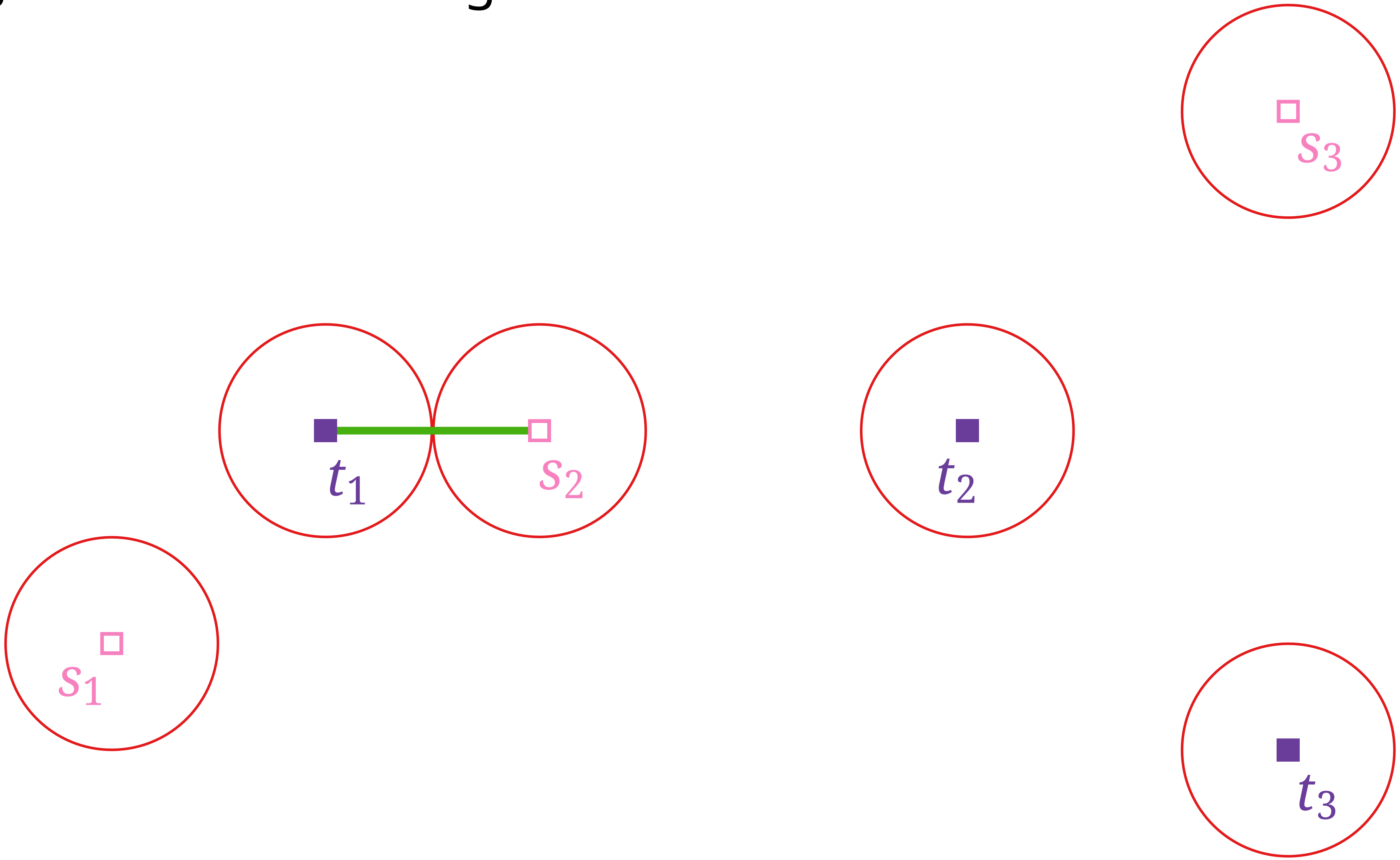

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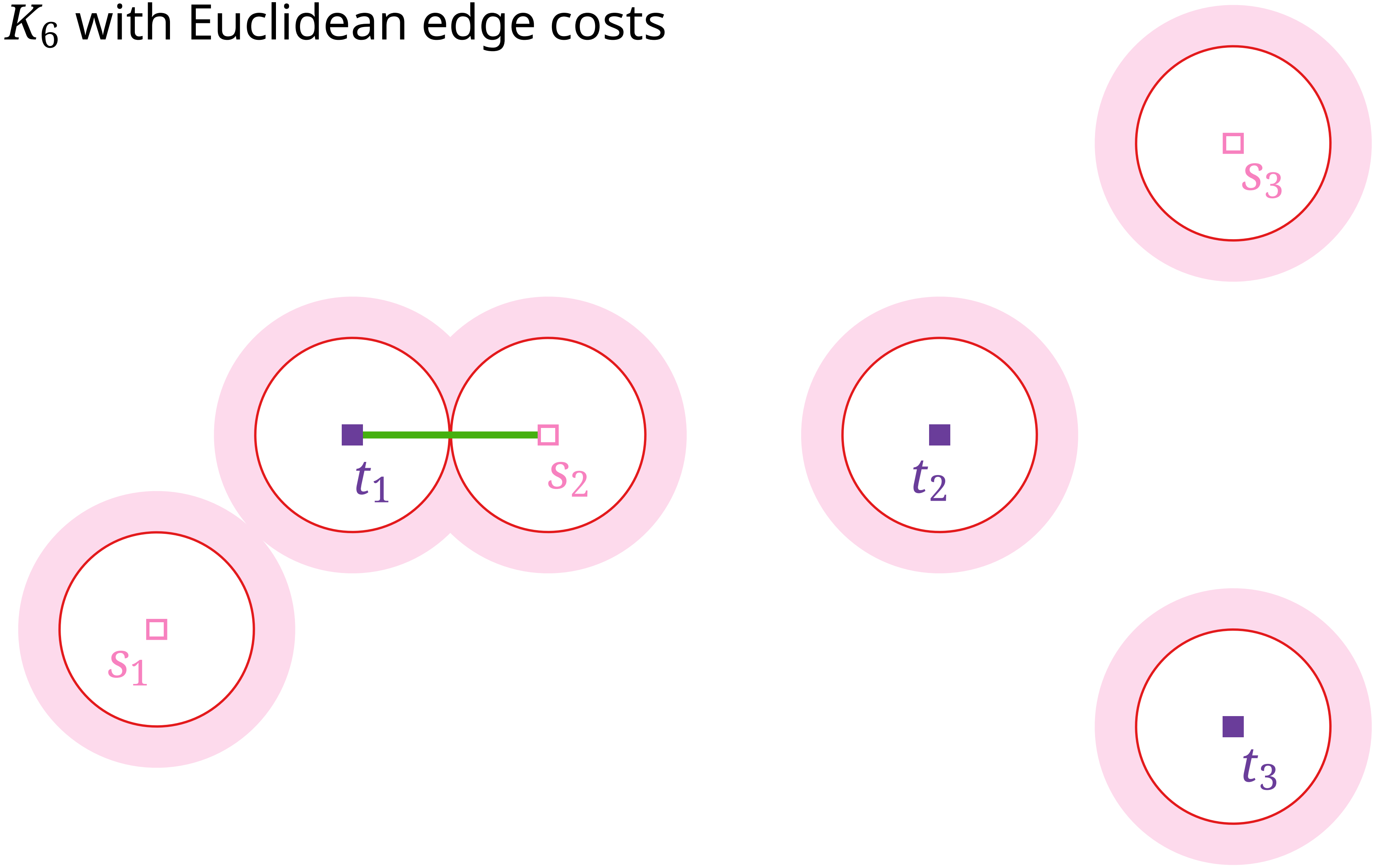

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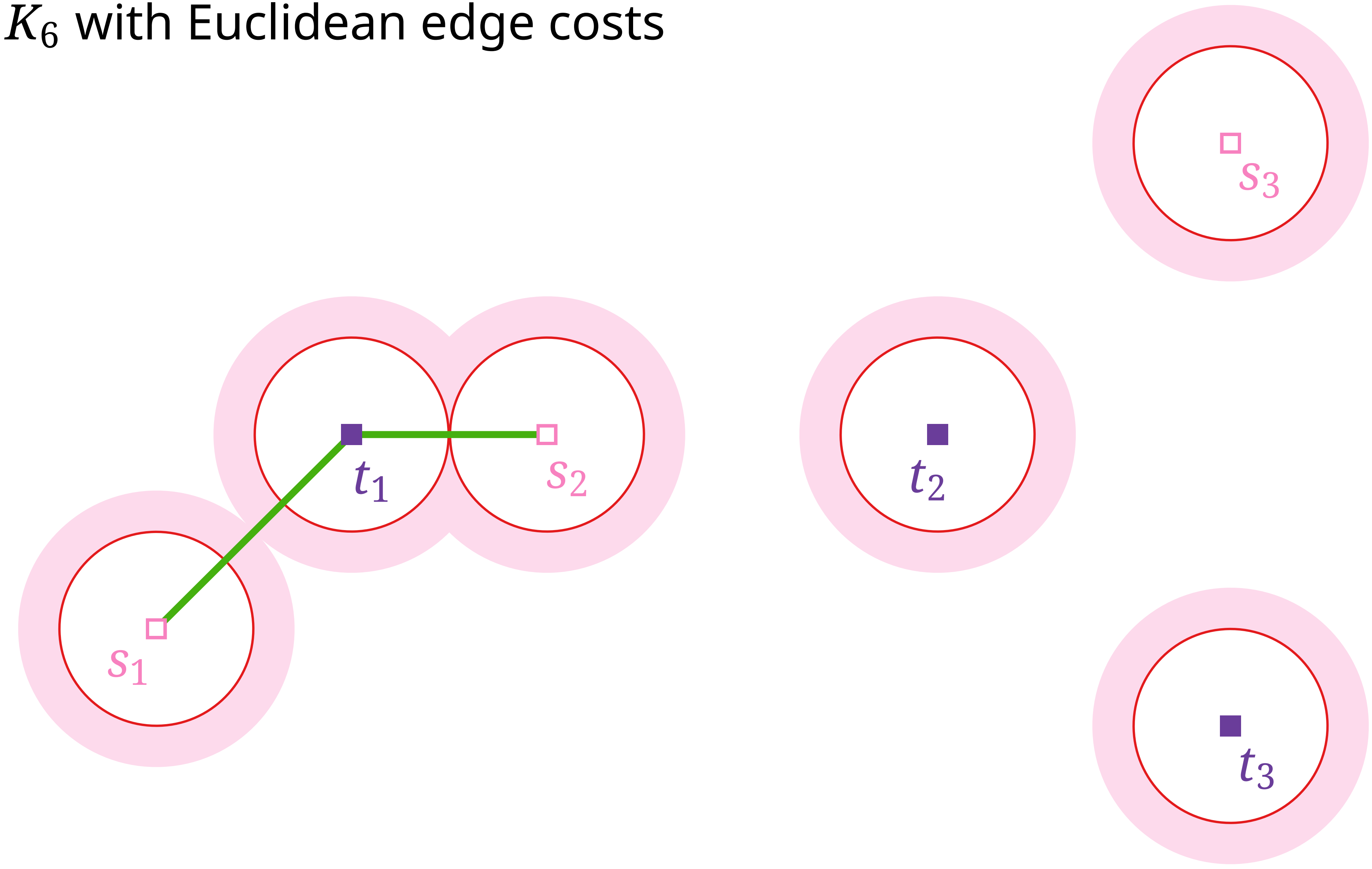

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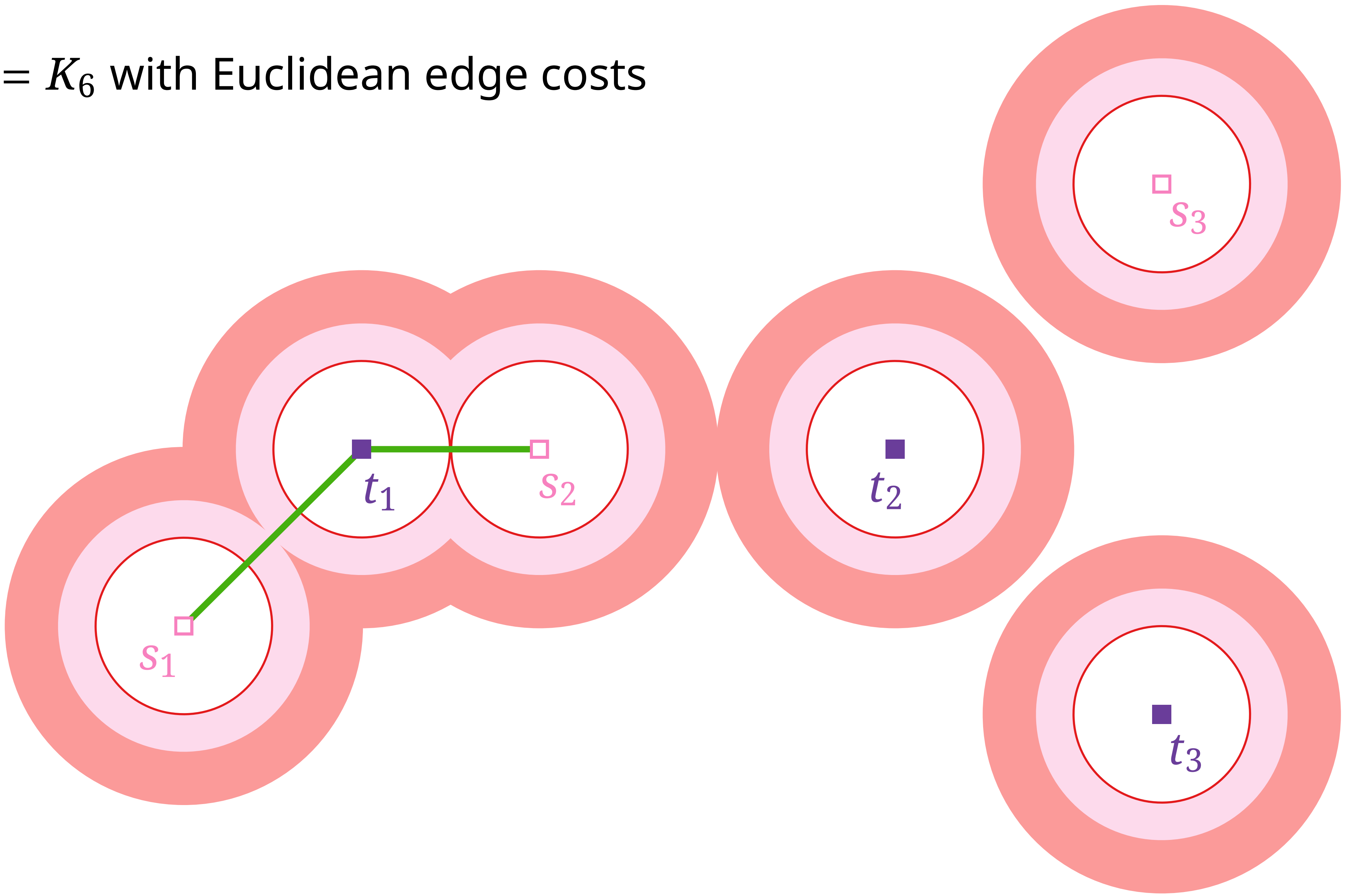

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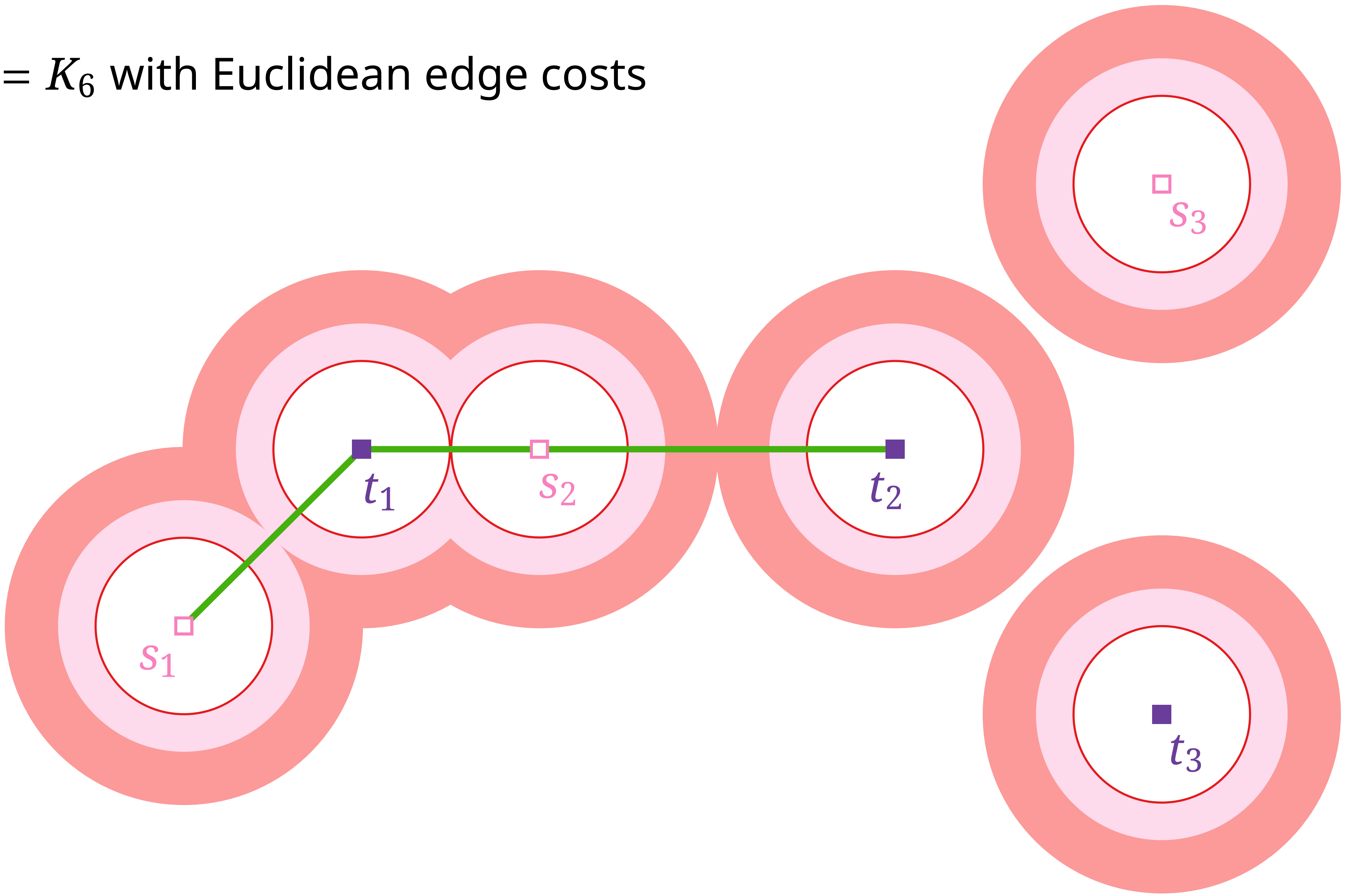

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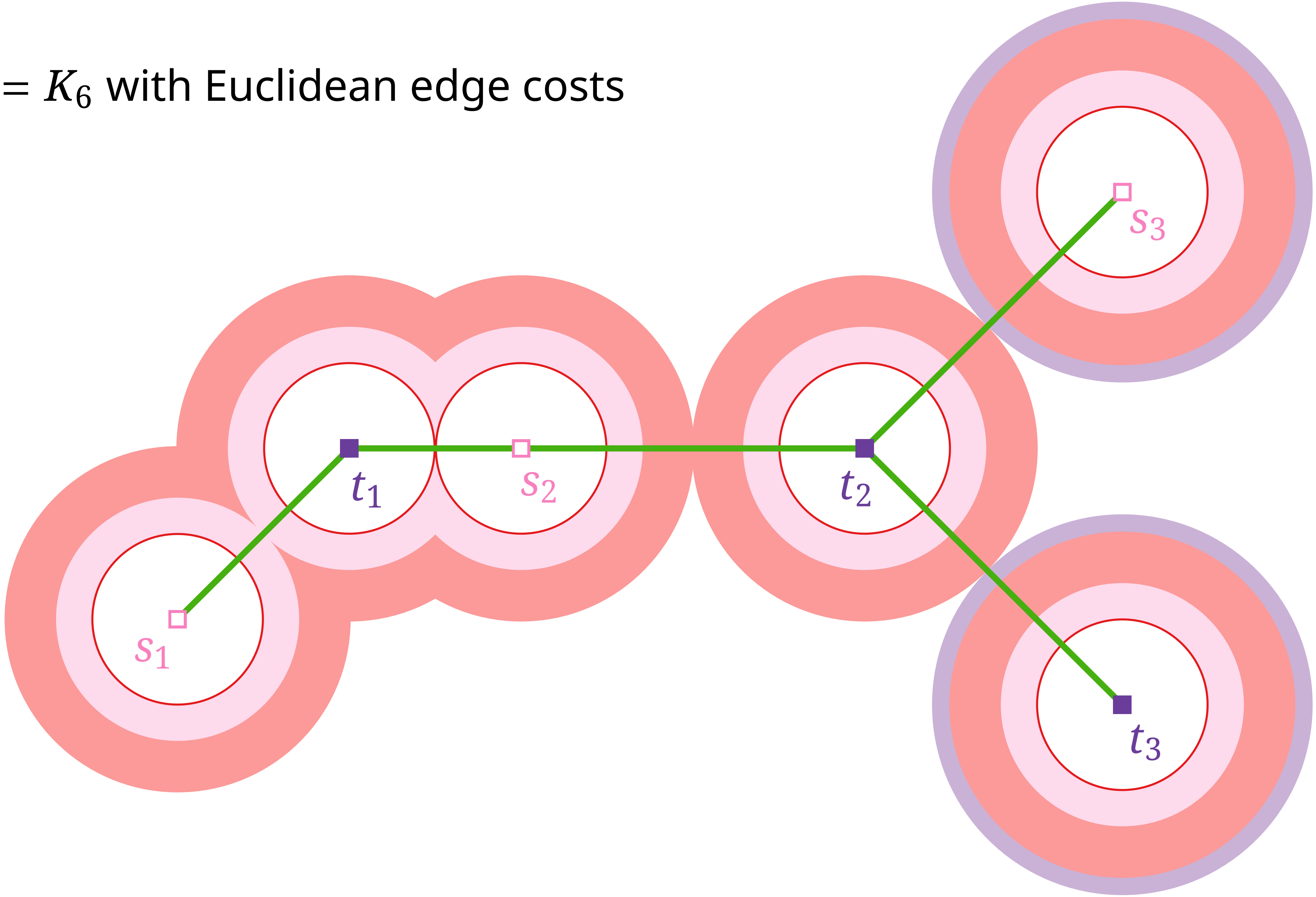

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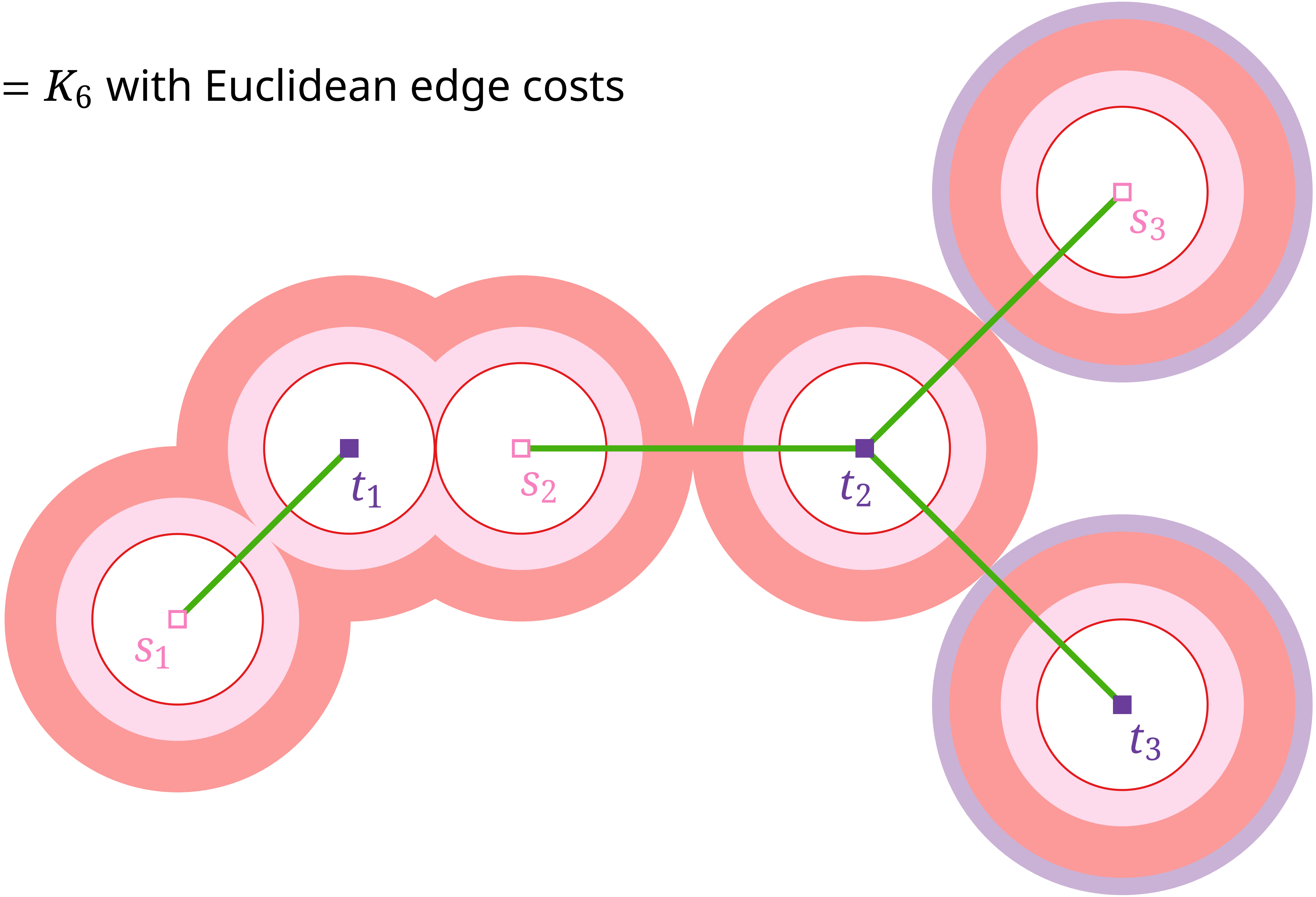

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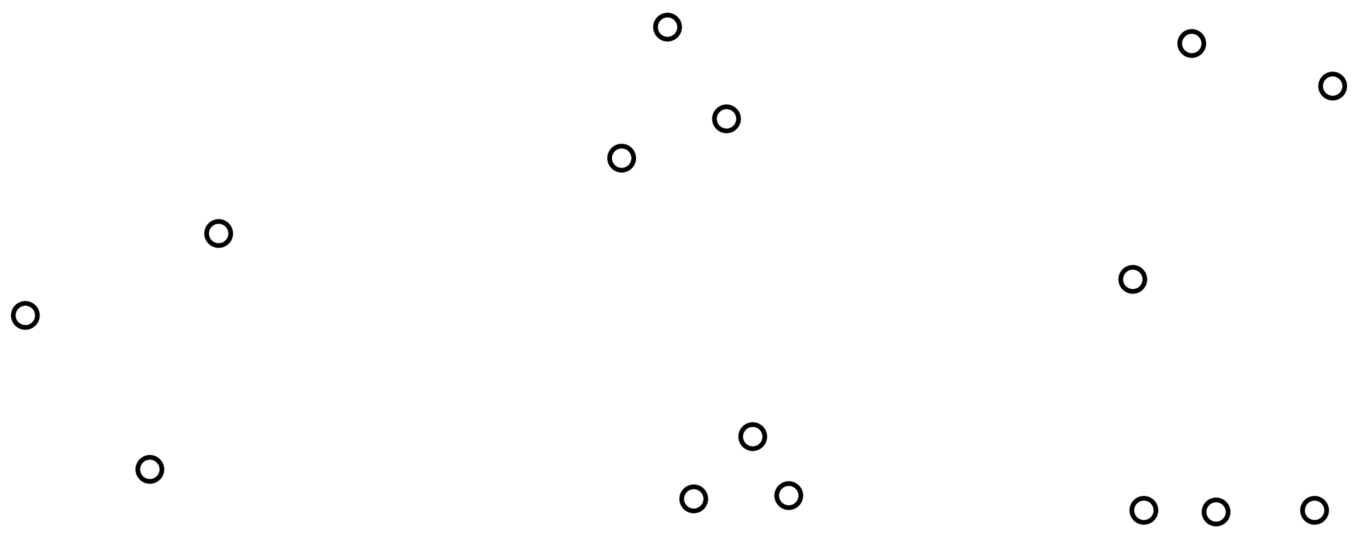

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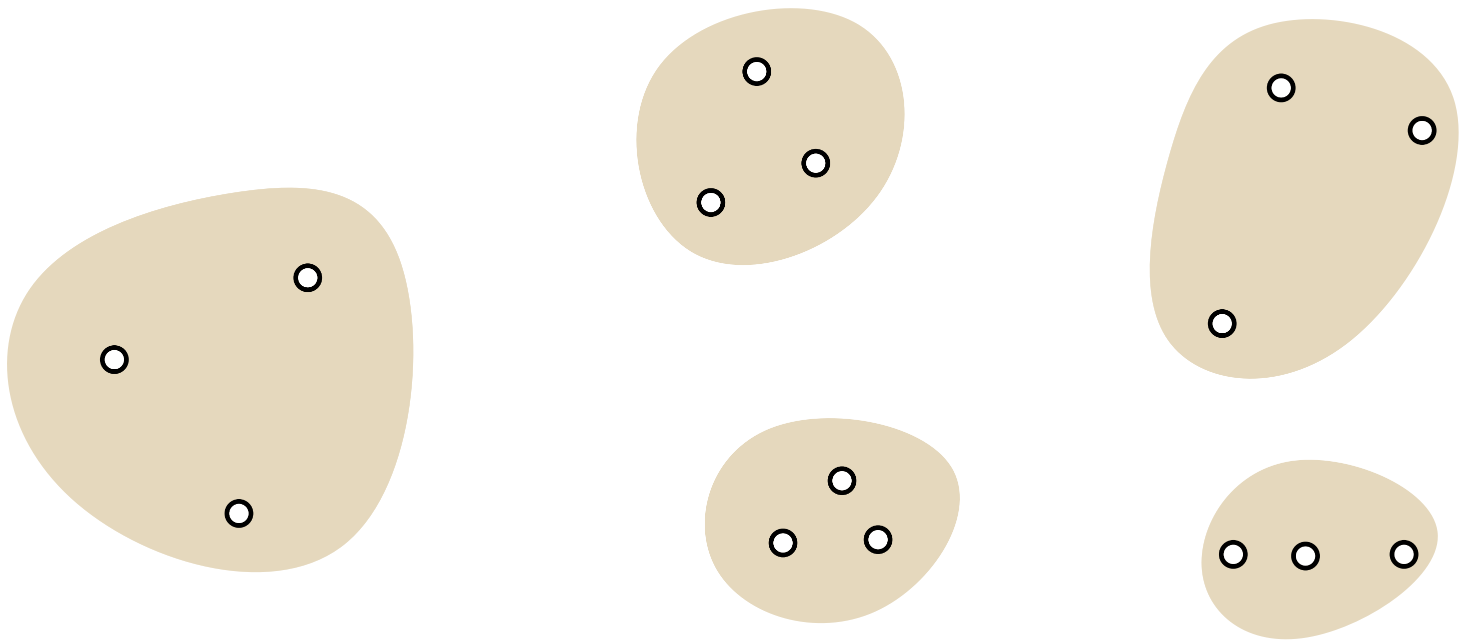

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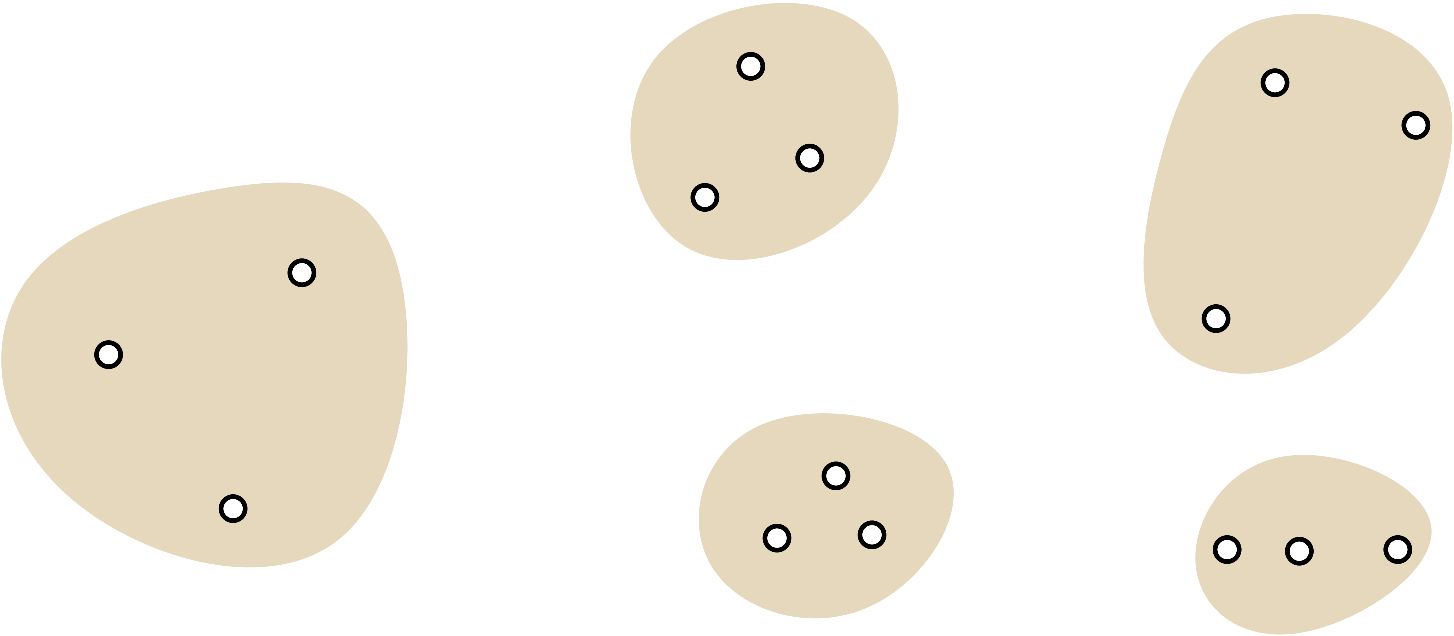

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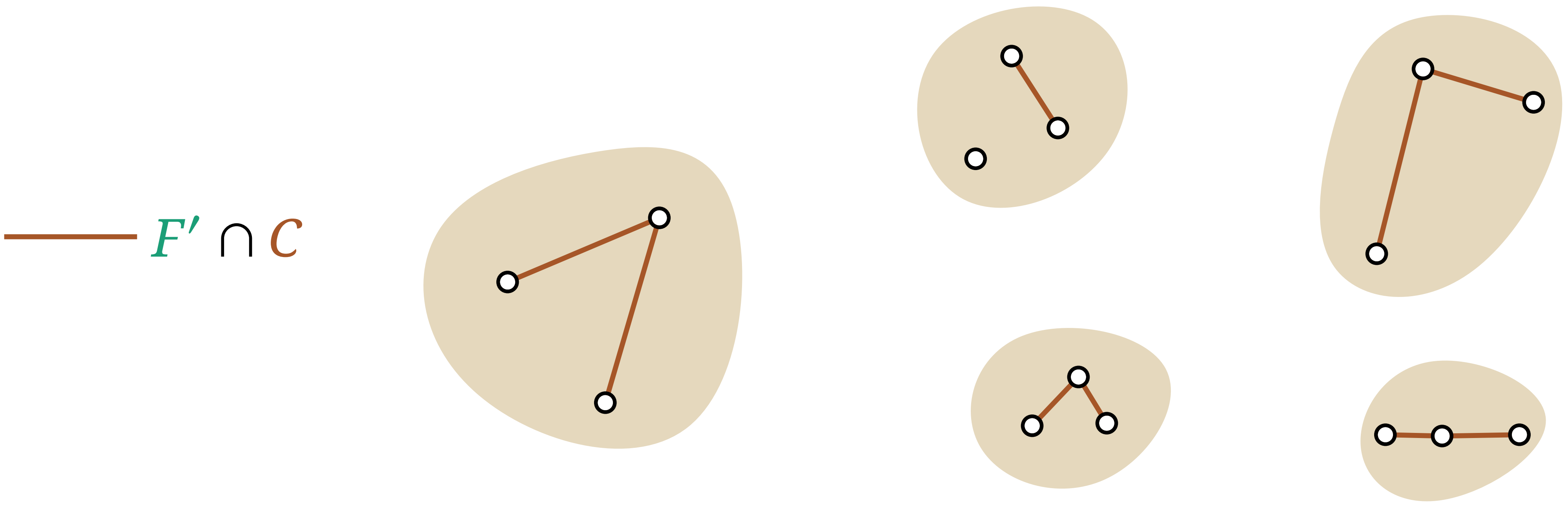

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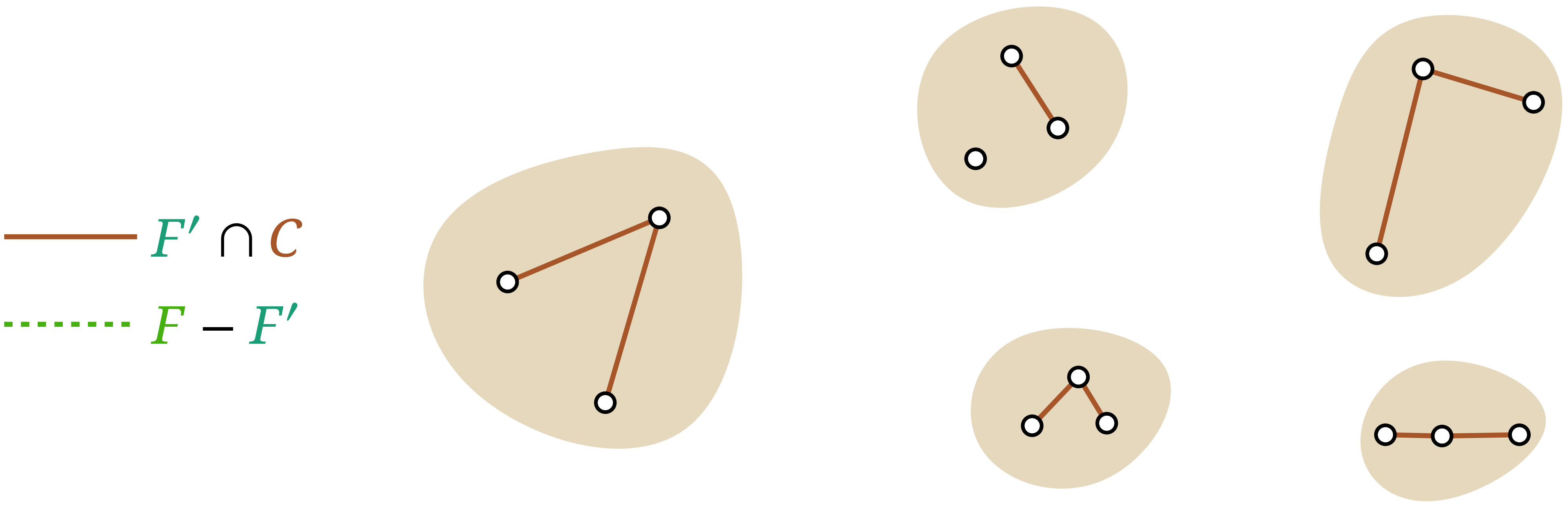

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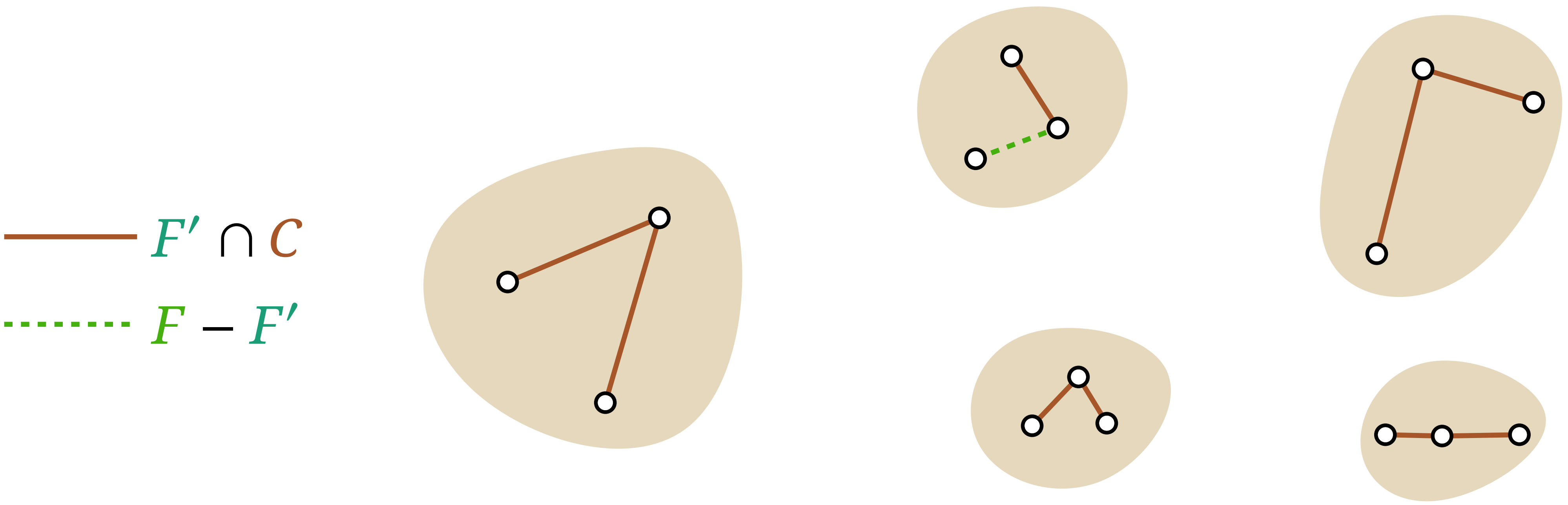

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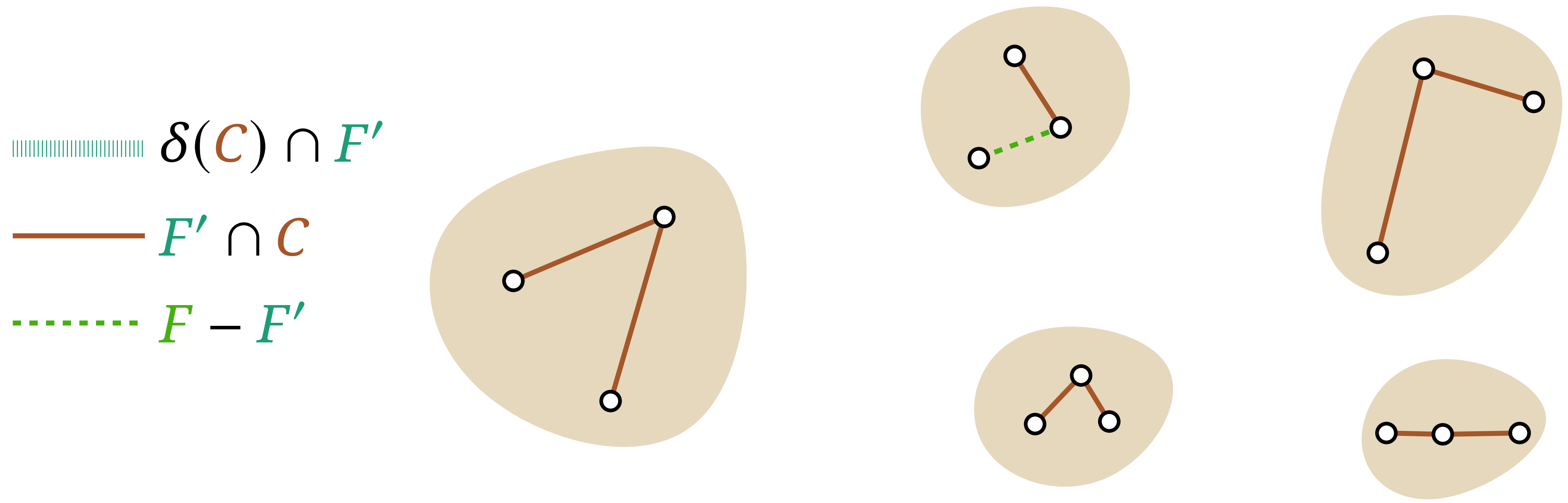

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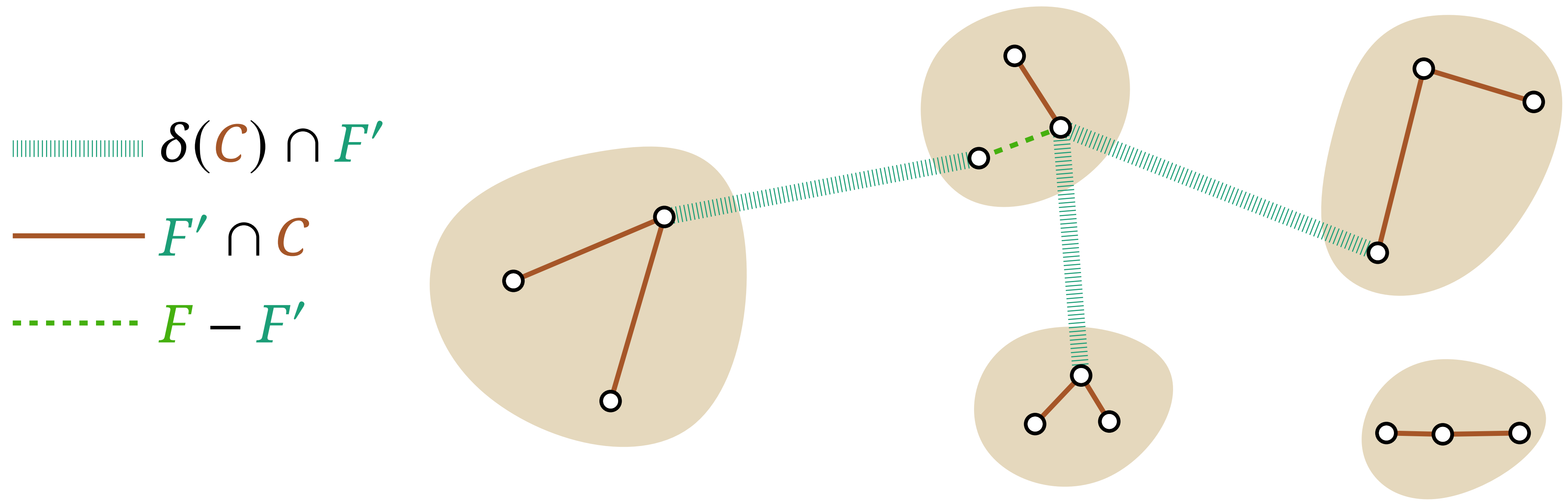

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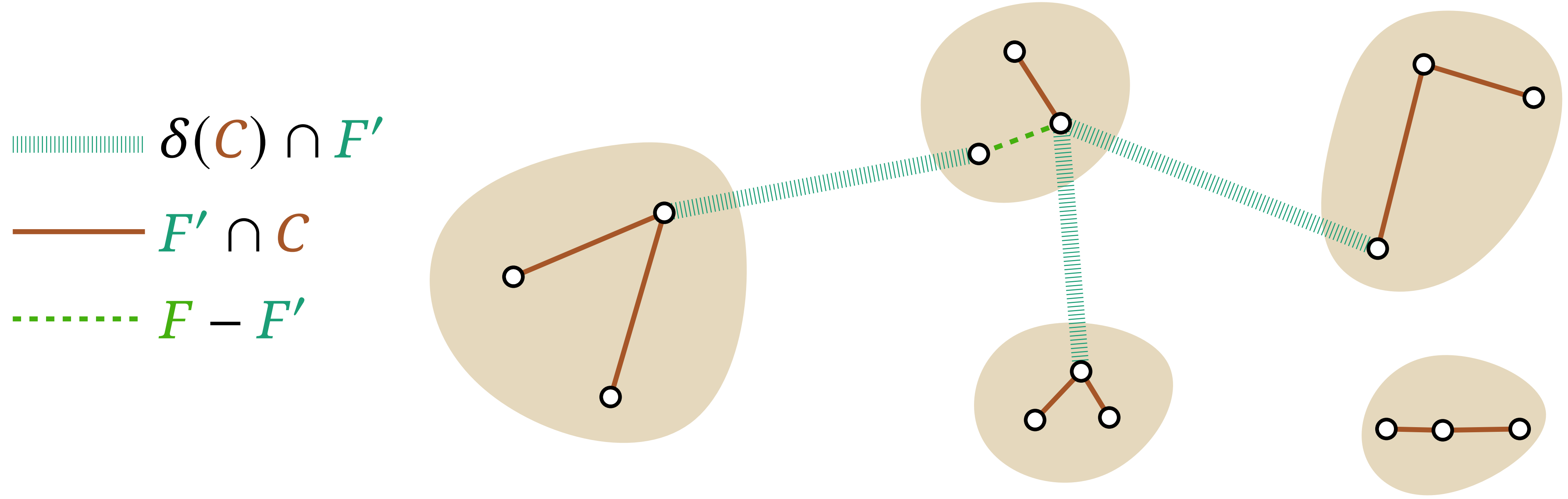

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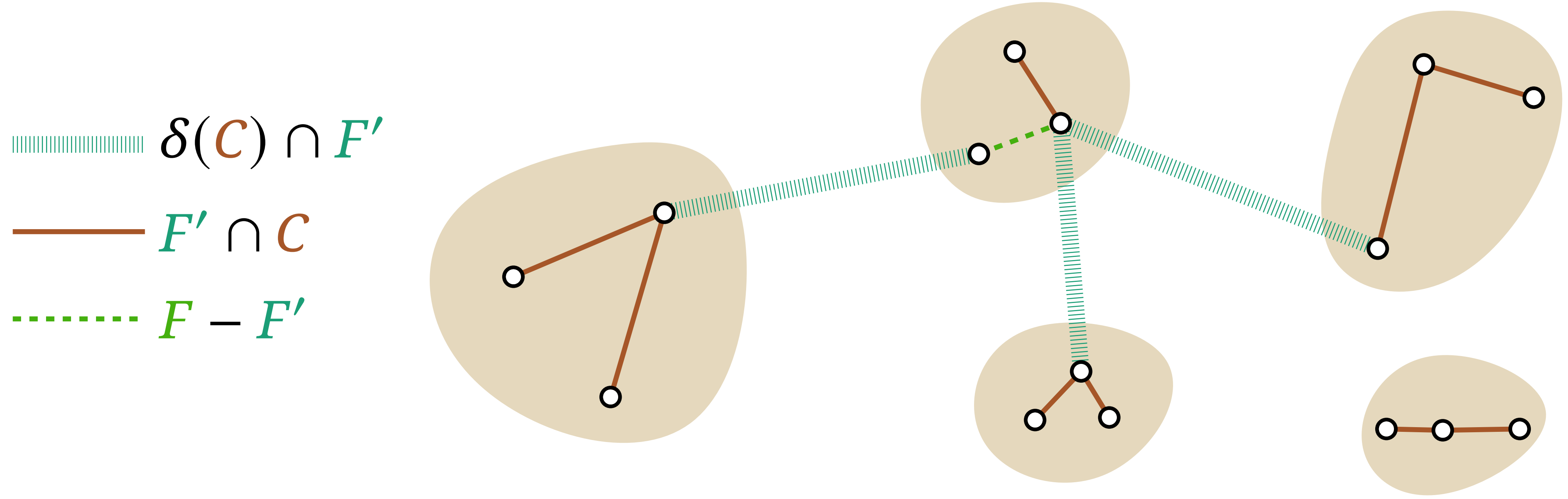

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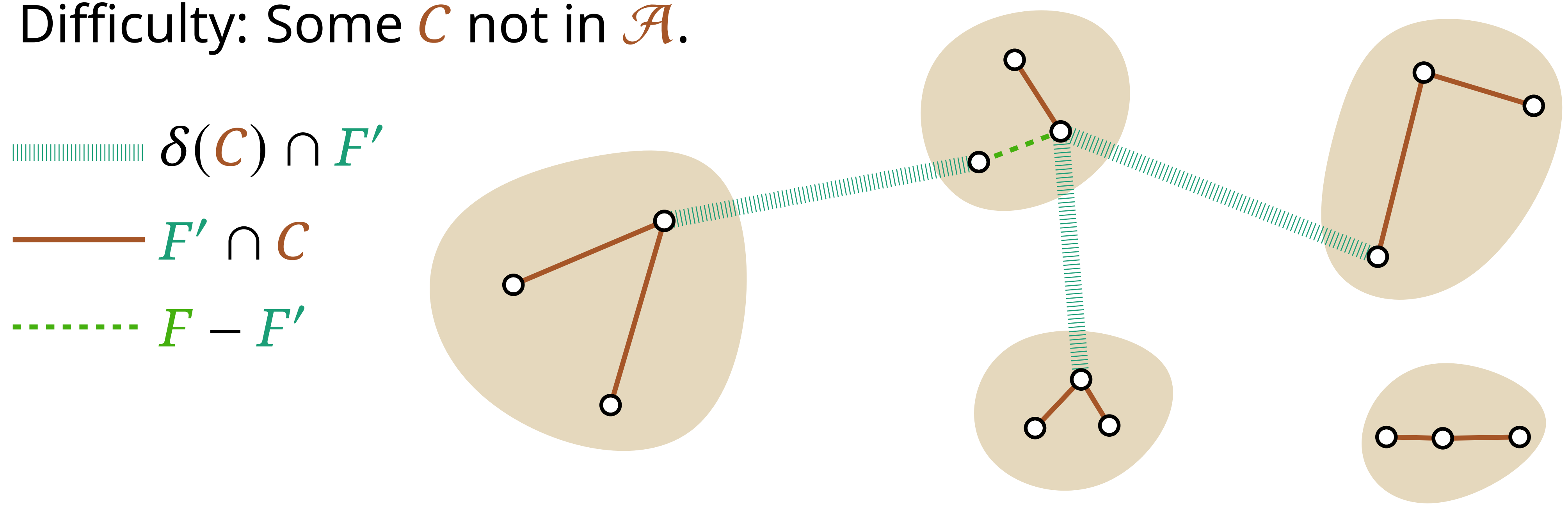

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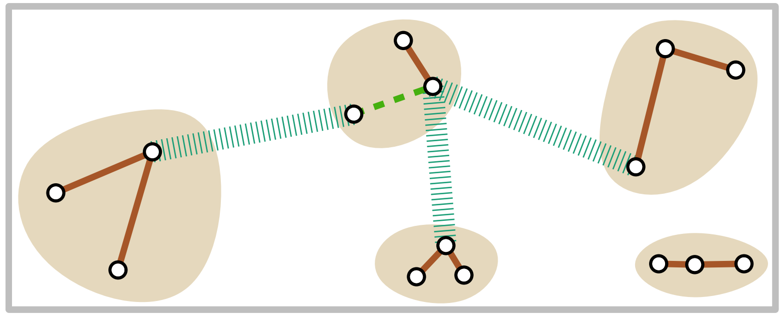

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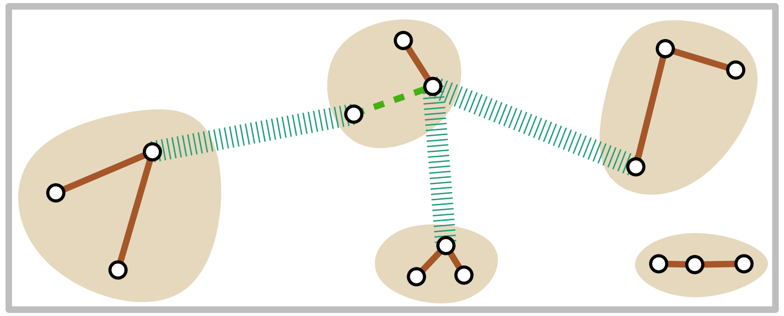

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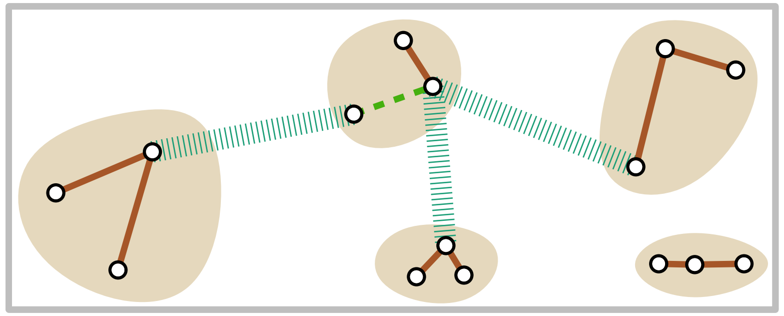

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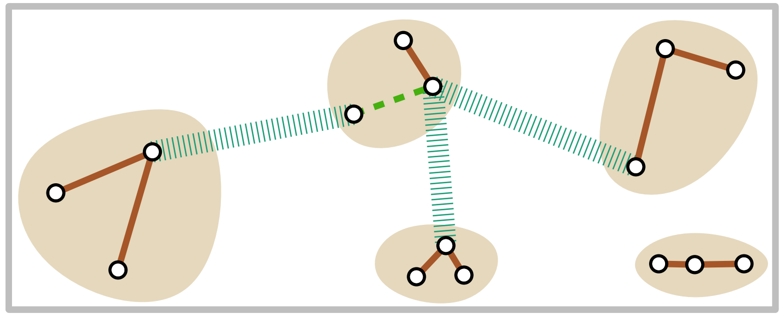

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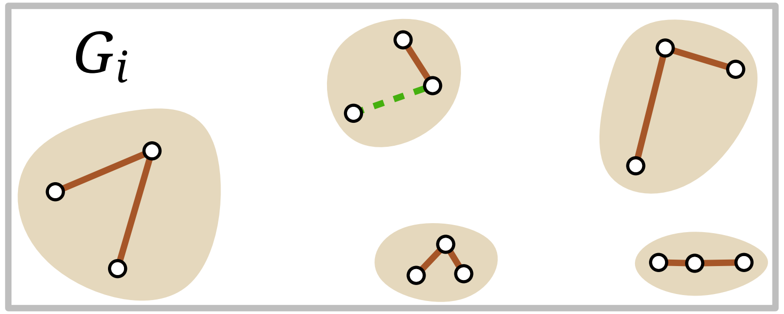

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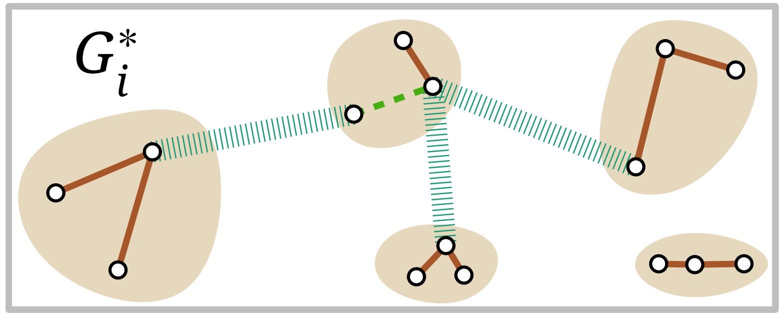

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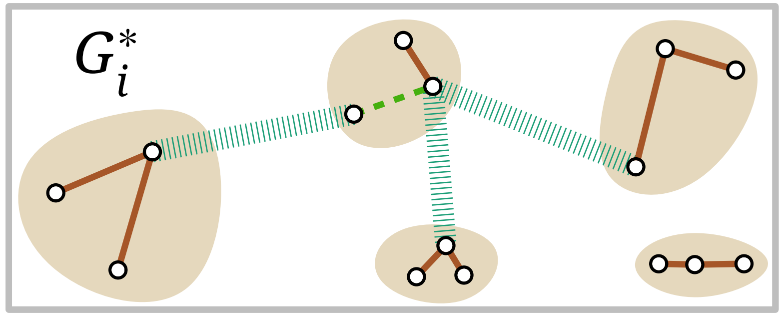

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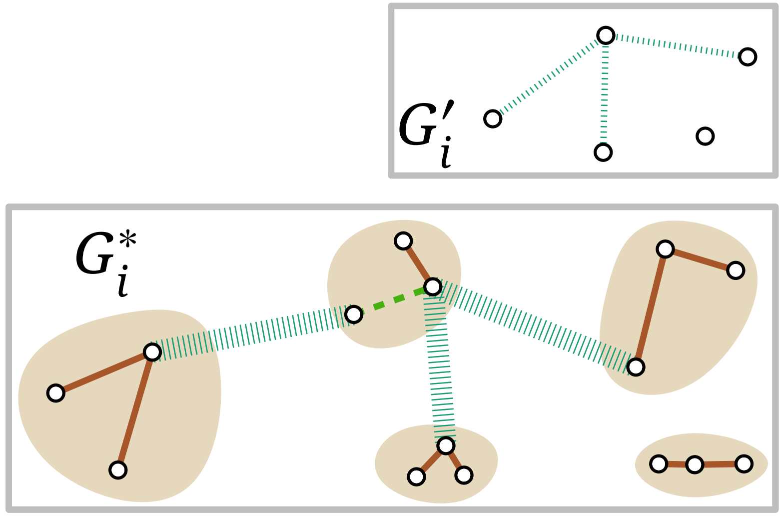

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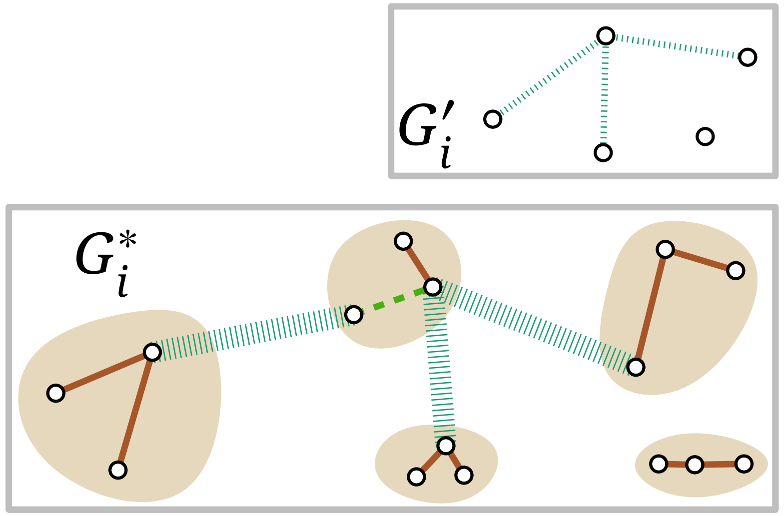

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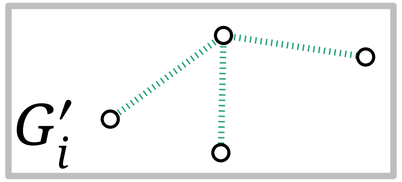

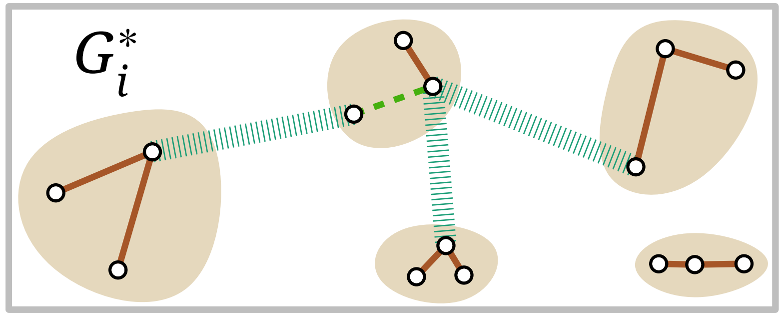

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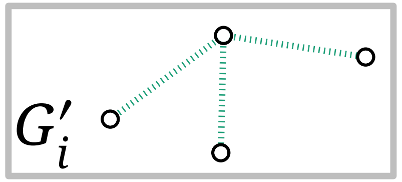

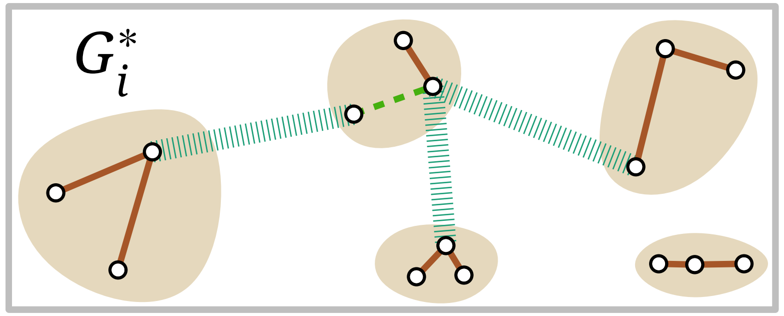

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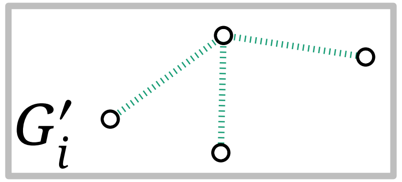

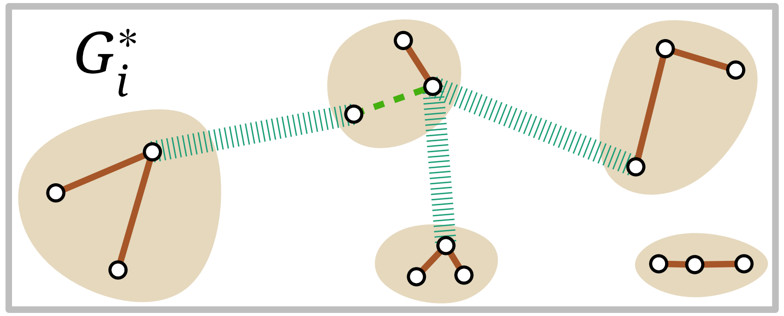

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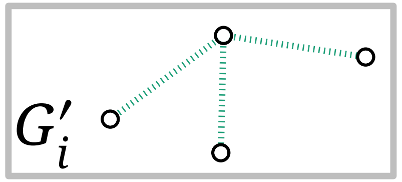

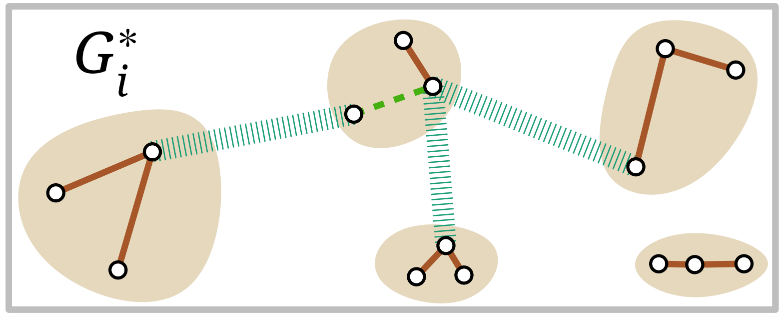

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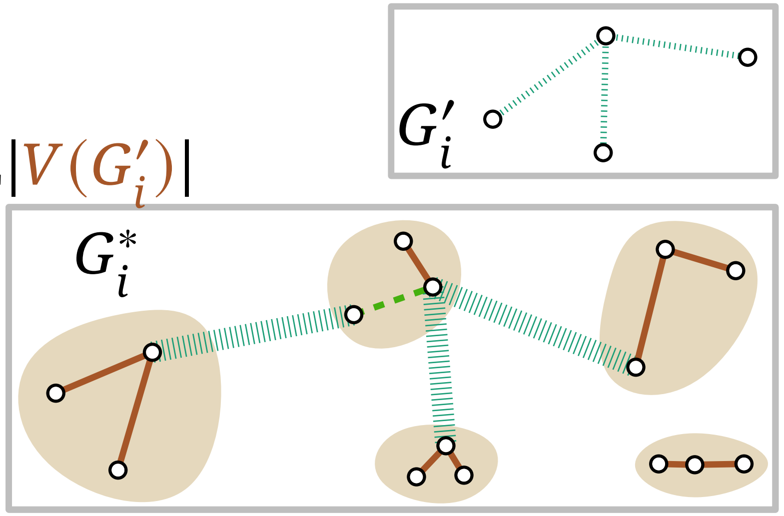

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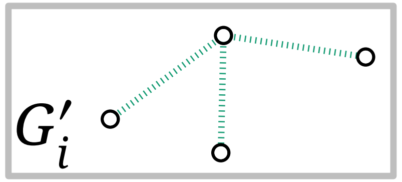

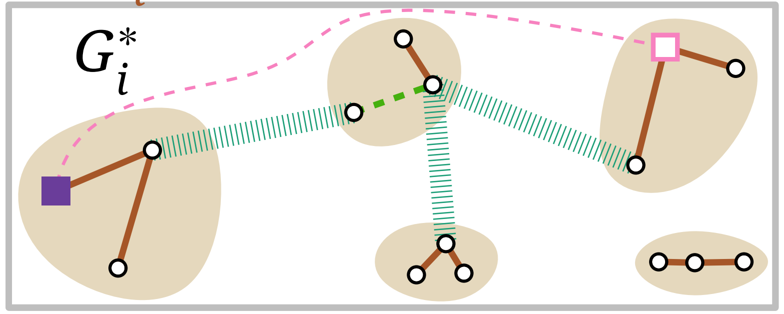

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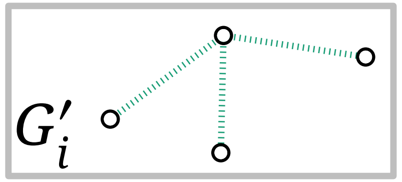

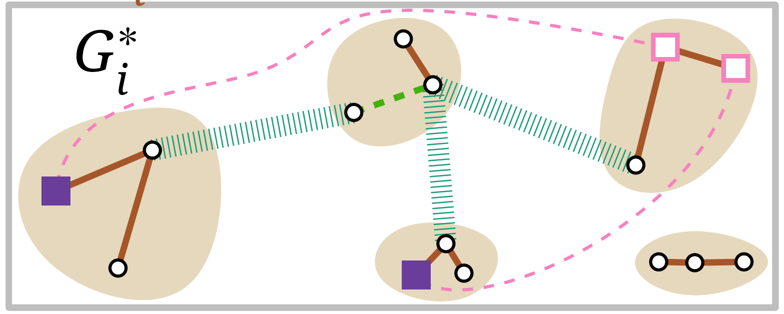

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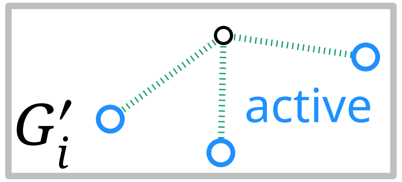

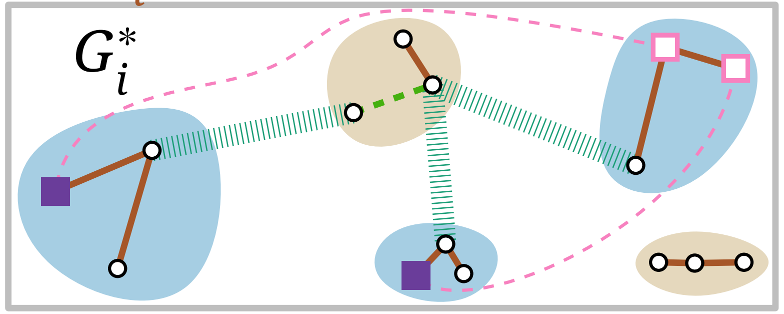

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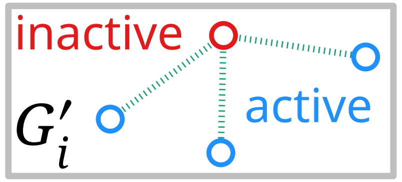

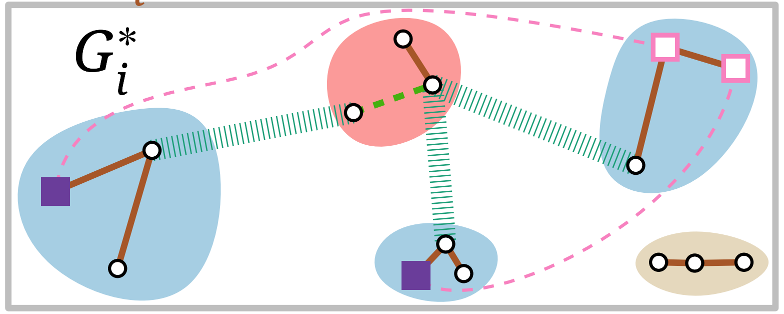

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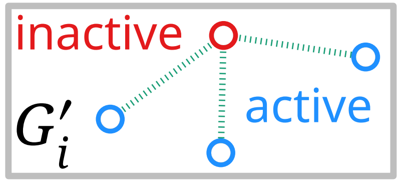

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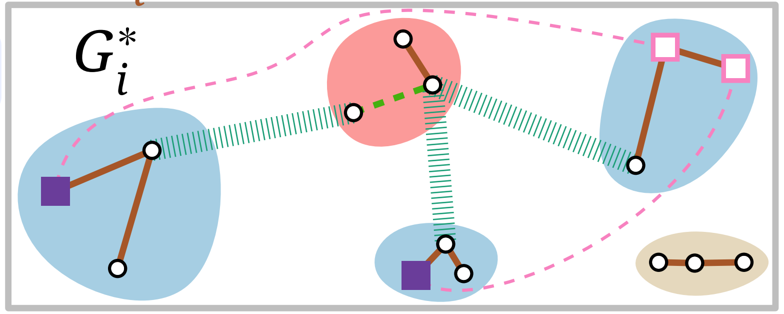

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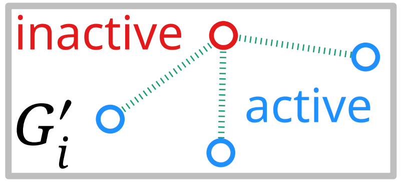

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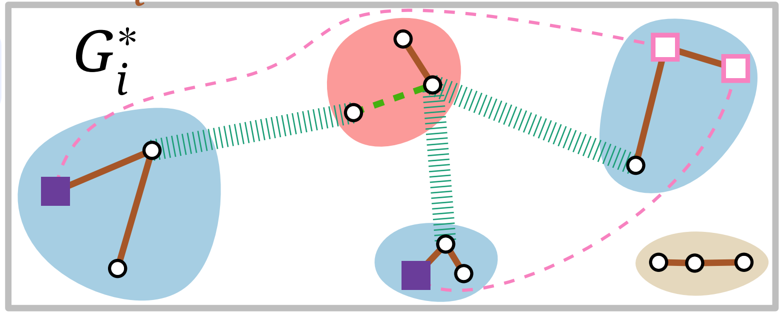

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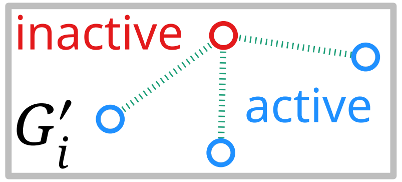

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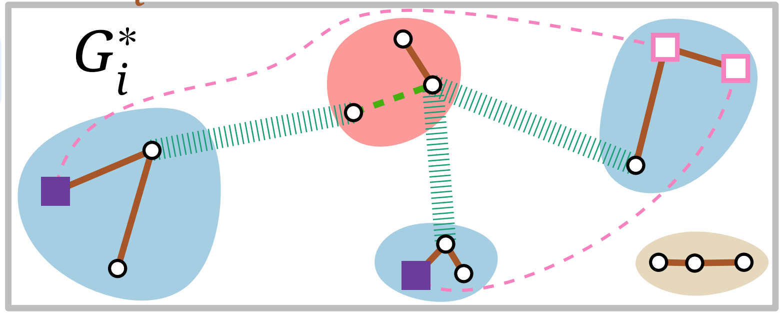

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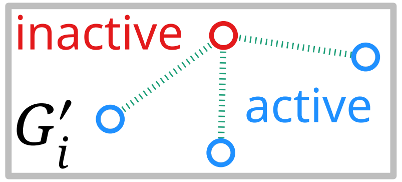

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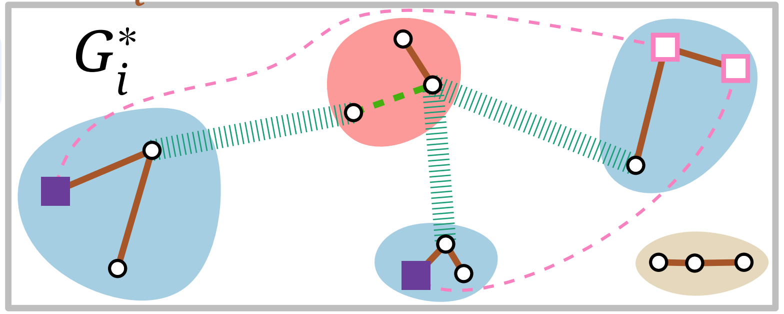

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As mentioned before,

$$\sum_{e \in F'} c_e \stackrel{\text{CS}}{=} \sum_{e \in F'} \sum_{S\colon e \in \delta(S)} y_S = \sum_{S} |\delta(S) \cap F'| \cdot y_S.$$

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Structure lemma $\Rightarrow (*)$ also holds after the current iteration.

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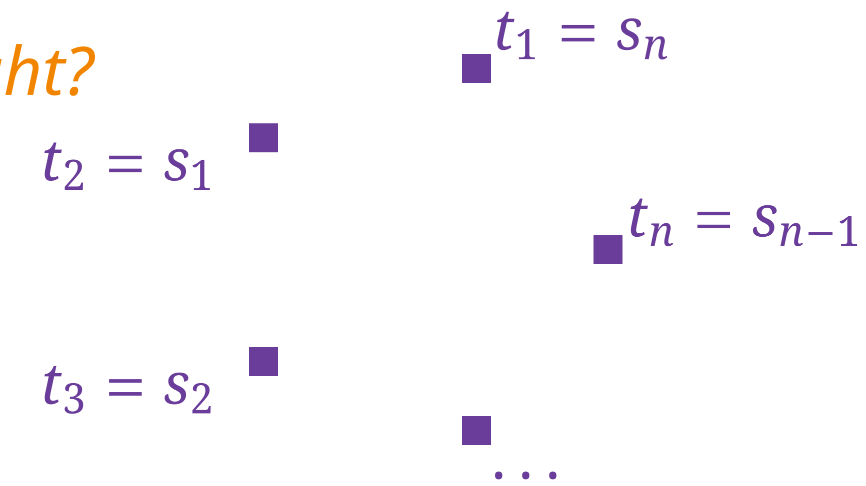

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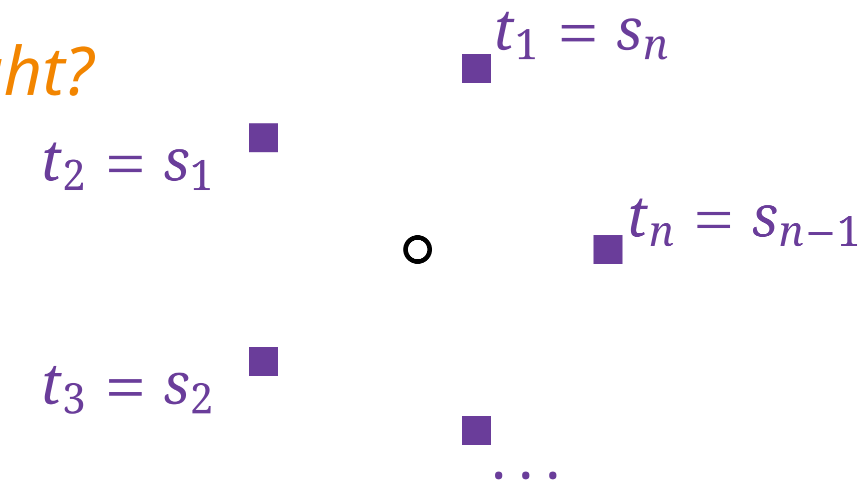

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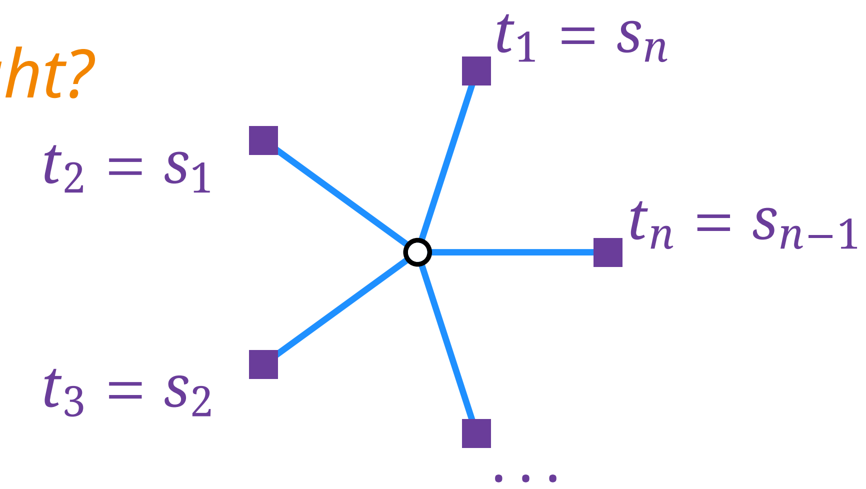

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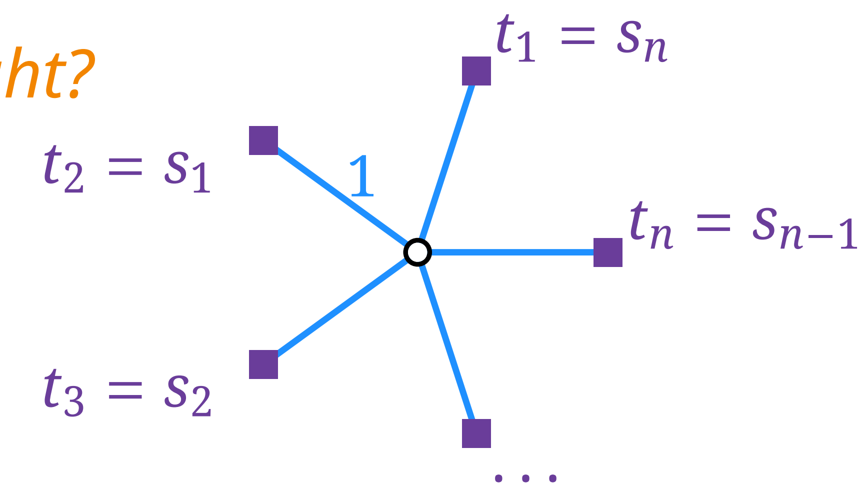

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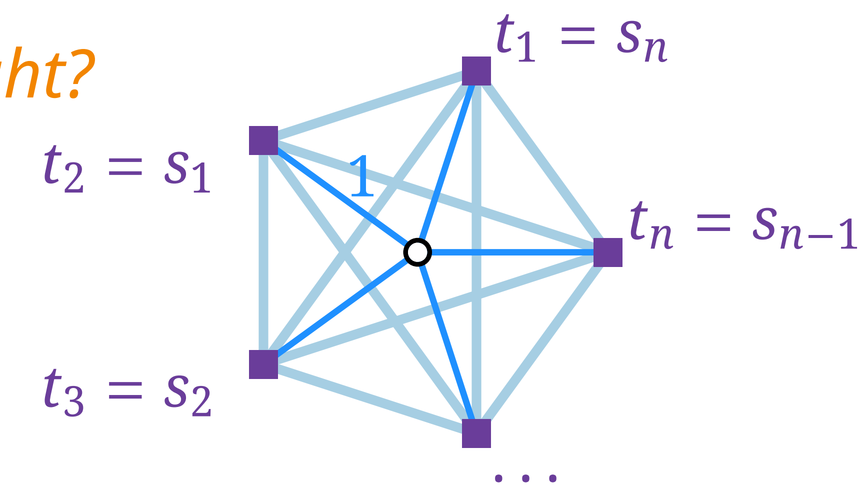

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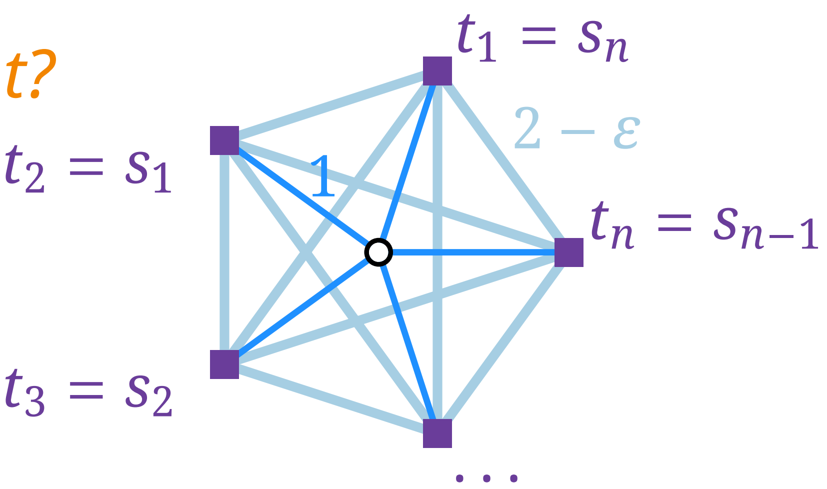

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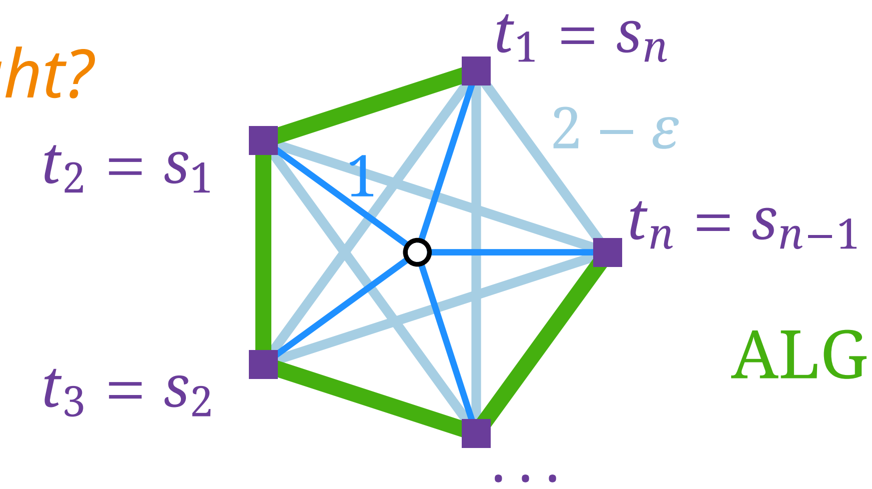

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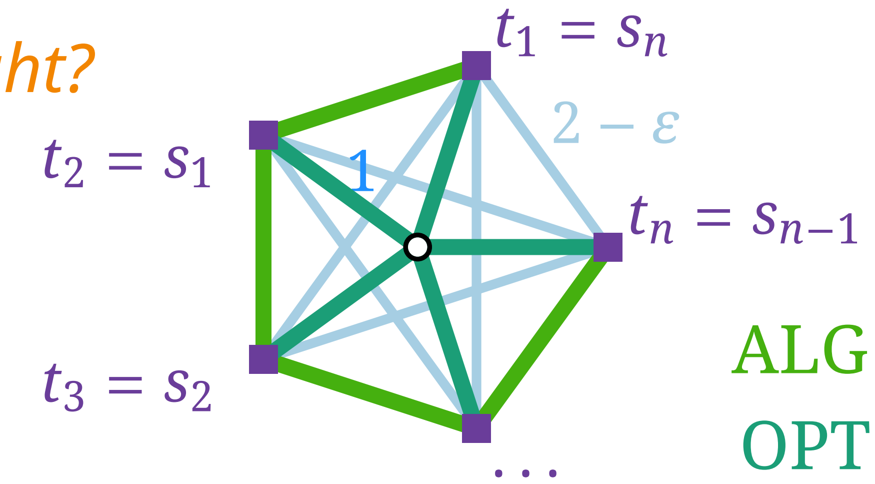

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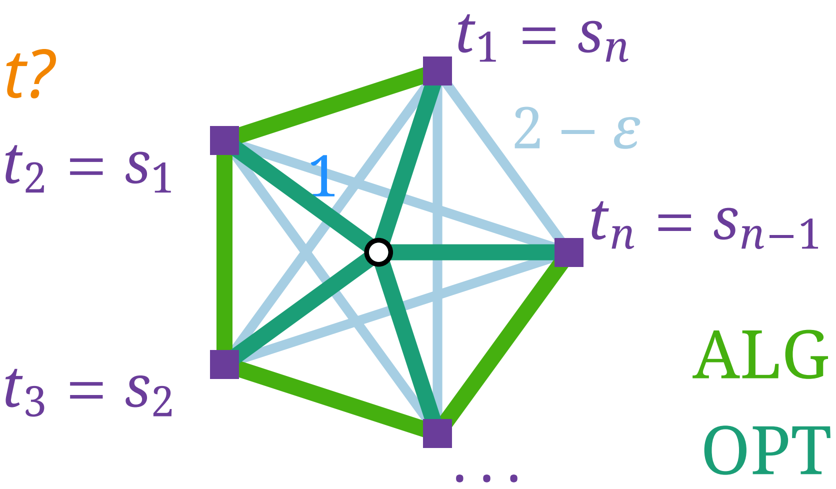

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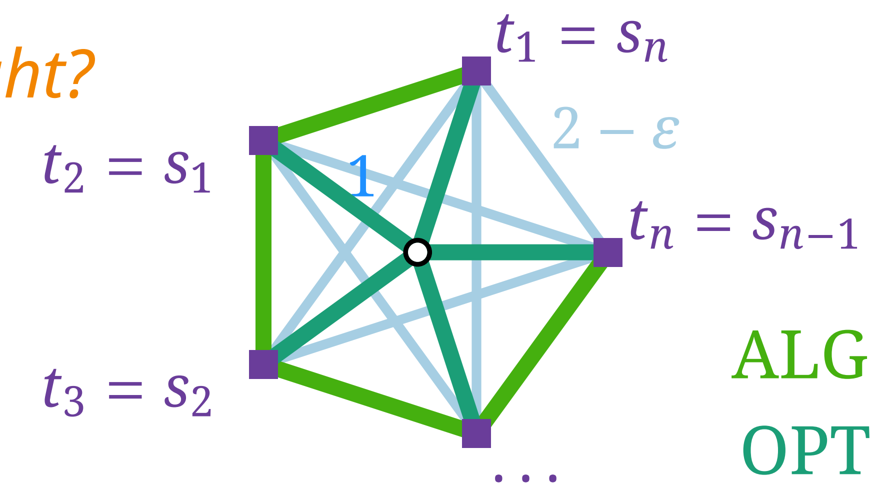

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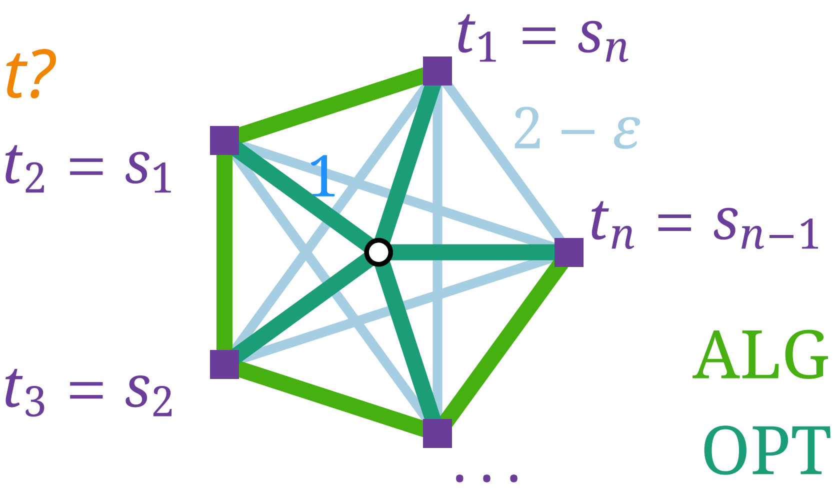

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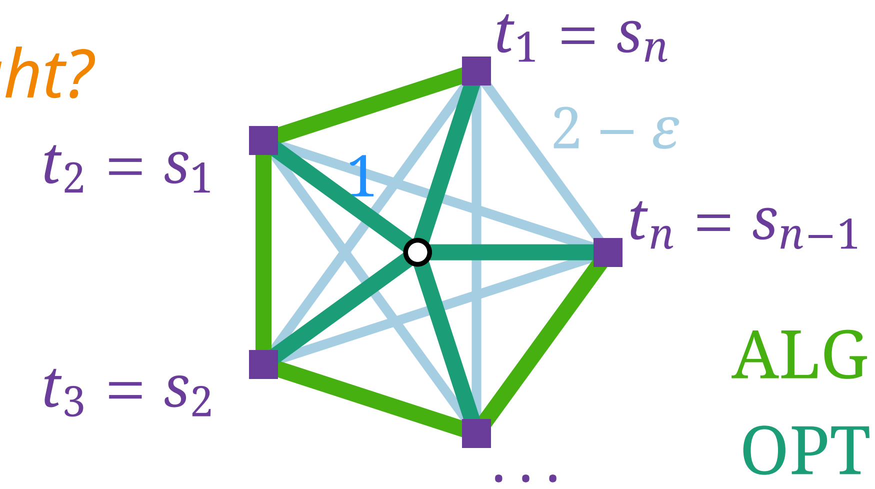

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SteinerForest (as SteinerTree) cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P = NP). [Chlebík, Chlebíková '08]