# Fibonacci Heaps – Practical Efficiency

# Fast Priority Queues - The Competitors

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sequence heaps: a heap designed for efficient memory access

$\ldots$

# Empirical Study of Priority Queues [Larkin et al., 2014]

lines of code

| Heap variant | Logical lines of code (lloc) |
|---|---|
| implicit simple | 184 |
| pairing | 186 |
| implicit | 194 |
| Fibonacci | 282 |
| binomial | 317 |
| explicit | 319 |
| rank-pairing | 376 |
| quake | 383 |
| violation | 481 |
| rank-relaxed weak | 638 |
| strict Fibonacci | 1009 |

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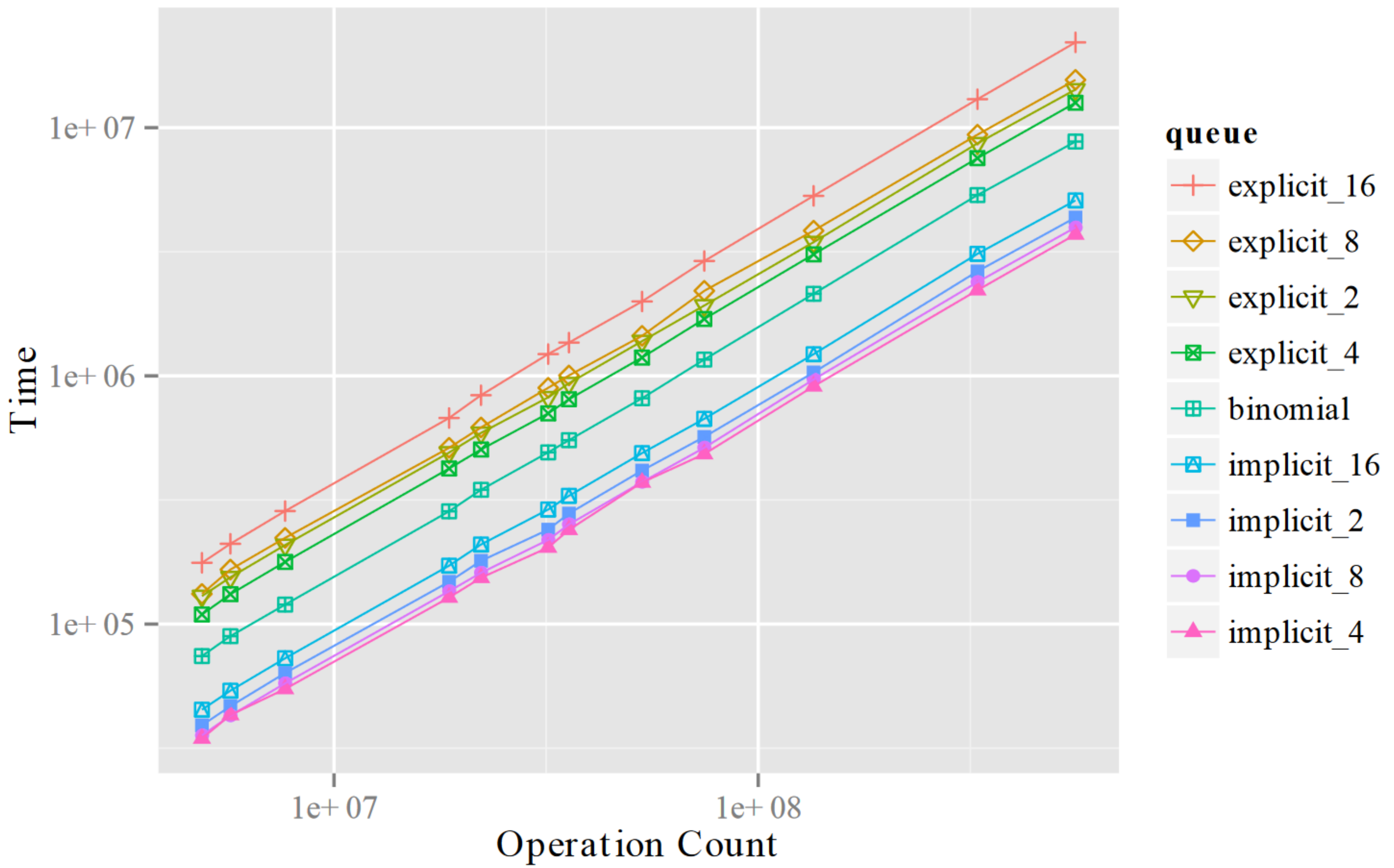
## Dijkstra on US street network, plot 1 of 2

Operation Count = number of Insert + DeleteMin + DecreaseKey operations



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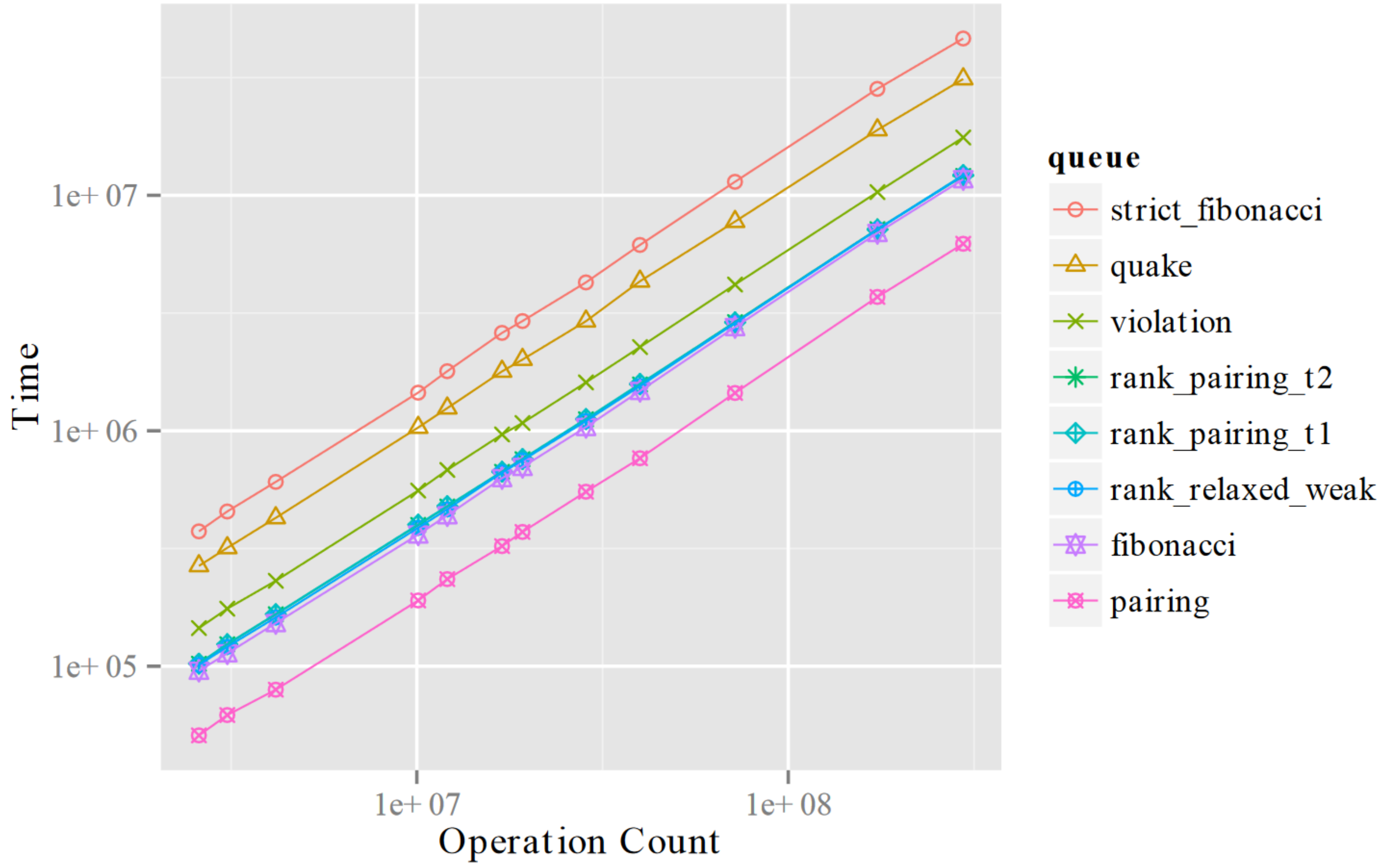

# Empirical Study of Priority Queues [Larkin et al., 2014]

## Dijkstra on US street network, table

Heap Size – max = 4200, average = 2489
Ratio of Operations – INSERT : DELETEMIN : DECREASEKEY = 13.98 : 13.98 : 1.00

| queue | time | inst | l1_rd | l1_wr | l2_rd | l2_wr | br | l1_m | l2_m | br_m |
|---|---|---|---|---|---|---|---|---|---|---|
| implicit_4 | **1.00** | **1.00** | **1.00** | 1.10 | 1.35 | 1.06 | **1.00** | **1.00** | 1.00 | **1.00** |
| implicit_8 | 1.07 | 1.12 | 1.12 | 1.03 | 1.61 | 1.18 | 1.01 | 1.07 | 1.20 | 1.01 |
| implicit_2 | 1.17 | 1.10 | 1.01 | 1.27 | 1.35 | **1.00** | 1.33 | 1.05 | **1.00** | 1.33 |
| implicit_16 | 1.37 | 1.42 | 1.38 | **1.00** | 2.20 | 1.35 | 1.21 | 1.24 | 1.63 | 1.21 |
| pairing | 1.68 | 1.09 | 1.12 | 2.95 | 1.71 | 28.57 | 1.39 | 1.60 | 1.75 | 1.39 |
| binomial | 2.37 | 1.49 | 1.83 | 3.49 | 1.30 | 34.57 | 1.49 | 2.24 | 1.56 | 1.49 |
| fibonacci | 3.15 | 2.00 | 2.09 | 5.03 | 1.73 | 79.53 | 2.91 | 2.85 | 2.67 | 2.91 |
| violation | 4.74 | 2.85 | 2.67 | 3.92 | 2.60 | 4.24 | 4.38 | 2.95 | 1.97 | 4.38 |
| explicit_16 | 5.94 | 3.94 | 4.56 | 6.81 | 8.02 | 276.59 | 7.13 | 5.06 | 10.76 | 7.13 |
| quake | 8.40 | 5.84 | 6.82 | 10.69 | 3.45 | 137.91 | 6.90 | 7.72 | 4.97 | 6.90 |
| strict_fibonacci | 12.49 | 9.47 | 12.50 | 22.07 | 6.96 | 84.51 | 11.47 | 14.83 | 6.58 | 11.47 |

column titles have been abbreviated: `time` is wallclock time, `inst` is the dynamic instruction count, `l1_rd` and `l1_wr` are the number of L1 reads and writes respectively, `l2_rd` and `l2_wr` are the L2 reads and writes respectively, `br` is the number of dynamic branches, and `l1_m`, `l2_m` and `br_m` are the number of L1 misses, L2 misses, and branch mispredictions.

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| fibonacci | 3.15 | 2.00 | 2.09 | 5.03 | 1.73 | 79.53 | 2.91 | 2.85 | 2.67 | 2.91 |
| rank_pairing_t2 | 3.26 | 1.98 | 2.16 | 2.85 | 1.34 | 35.46 | 3.19 | 2.29 | 1.61 | 3.19 |
| rank_relaxed_weak | 3.27 | 2.21 | 2.72 | 3.62 | 2.34 | 10.01 | 3.08 | 2.90 | 1.89 | 3.08 |
| rank_pairing_t1 | 3.29 | 1.98 | 2.16 | 2.85 | 1.33 | 35.35 | 3.19 | 2.29 | 1.60 | 3.19 |
| explicit_4 | 3.39 | 2.69 | 2.83 | 4.11 | 1.97 | 104.57 | 4.22 | 3.11 | 3.29 | 4.22 |
| explicit_2 | 3.84 | 3.35 | 3.39 | 4.84 | **1.00** | 74.61 | 5.01 | 3.71 | 2.05 | 5.01 |
| explicit_8 | 4.20 | 3.01 | 3.32 | 5.00 | 4.50 | 168.91 | 5.04 | 3.70 | 6.28 | 5.04 |
| violation | 4.74 | 2.85 | 2.67 | 3.92 | 2.60 | 4.24 | 4.38 | 2.95 | 1.97 | 4.38 |
| explicit_16 | 5.94 | 3.94 | 4.56 | 6.81 | 8.02 | 276.59 | 7.13 | 5.06 | 10.76 | 7.13 |
| quake | 8.40 | 5.84 | 6.82 | 10.69 | 3.45 | 137.91 | 6.90 | 7.72 | 4.97 | 6.90 |
| strict_fibonacci | 12.49 | 9.47 | 12.50 | 22.07 | 6.96 | 84.51 | 11.47 | 14.83 | 6.58 | 11.47 |

# Empirical Study of Priority Queues [Larkin et al., 2014]

$n$ insertions, then $n$ times: insert + decreaseKey with new minimum + deleteMin

Table 6: Randomized DecreaseKey – Min, $c = 1$, $k = 1$
Heap Size – max = 8388609, average = 7340032
Ratio of Operations – Insert : DeleteMin : DecreaseKey = 2.00 : 1.00 : 1.00

| queue | time | inst | l1_rd | l1_wr | l2_rd | l2_wr | br | l1_m | l2_m | br_m |
|---|---|---|---|---|---|---|---|---|---|---|
| pairing | **1.00** | **1.00** | **1.00** | **1.00** | **1.00** | 1.65 | **1.00** | **1.00** | **1.00** | **1.00** |
| fibonacci | 3.04 | 2.62 | 2.21 | 2.58 | 3.00 | 3.52 | 4.91 | 2.37 | 2.43 | 4.91 |
| rank_relaxed_weak | 3.65 | 3.16 | 3.05 | 2.18 | 4.66 | 2.01 | 5.89 | 2.68 | 2.40 | 5.89 |
| rank_pairing_t2 | 5.43 | 3.42 | 2.57 | 1.96 | 5.94 | 1.58 | 7.54 | 2.32 | 2.67 | 7.54 |
| rank_pairing_t1 | 5.95 | 3.39 | 2.56 | 1.95 | 5.94 | 1.57 | 7.46 | 2.30 | 2.67 | 7.46 |
| violation | 6.32 | 4.54 | 3.32 | 2.40 | 5.93 | 1.04 | 10.47 | 2.93 | 2.46 | 10.47 |
| quake | 6.55 | 6.59 | 4.37 | 3.53 | 5.28 | 2.61 | 13.31 | 4.02 | 2.86 | 13.31 |
| implicit_8 | 6.98 | 3.58 | 2.42 | 1.24 | 40.06 | 1.02 | 5.55 | 1.92 | 14.19 | 5.55 |
| implicit_4 | 7.25 | 3.47 | 2.25 | 1.41 | 33.17 | **1.00** | 6.20 | 1.90 | 11.81 | 6.20 |
| strict_fibonacci | 7.39 | 8.44 | 8.38 | 8.31 | 9.76 | 4.81 | 13.05 | 8.35 | 5.27 | 13.05 |
| implicit_16 | 9.40 | 4.35 | 2.94 | 1.14 | 56.29 | 1.05 | 6.02 | 2.18 | 19.79 | 6.02 |
| implicit_2 | 10.17 | 4.48 | 2.59 | 1.93 | 38.66 | 1.01 | 9.67 | 2.31 | 13.70 | 9.67 |
| binomial | 12.14 | 5.90 | 5.55 | 6.54 | 30.10 | 14.21 | 10.45 | 5.97 | 16.01 | 10.45 |
| explicit_4 | 14.95 | 12.84 | 9.64 | 8.33 | 24.42 | 18.16 | 30.31 | 9.09 | 15.63 | 30.31 |
| explicit_2 | 16.24 | 17.12 | 12.53 | 10.60 | 24.32 | 12.53 | 38.42 | 11.71 | 13.36 | 38.42 |
| explicit_8 | 21.07 | 13.80 | 10.88 | 9.74 | 39.17 | 28.94 | 34.46 | 10.40 | 25.00 | 34.46 |
| explicit_16 | 31.01 | 17.68 | 14.40 | 12.85 | 60.51 | 48.34 | 47.08 | 13.74 | 40.06 | 47.08 |

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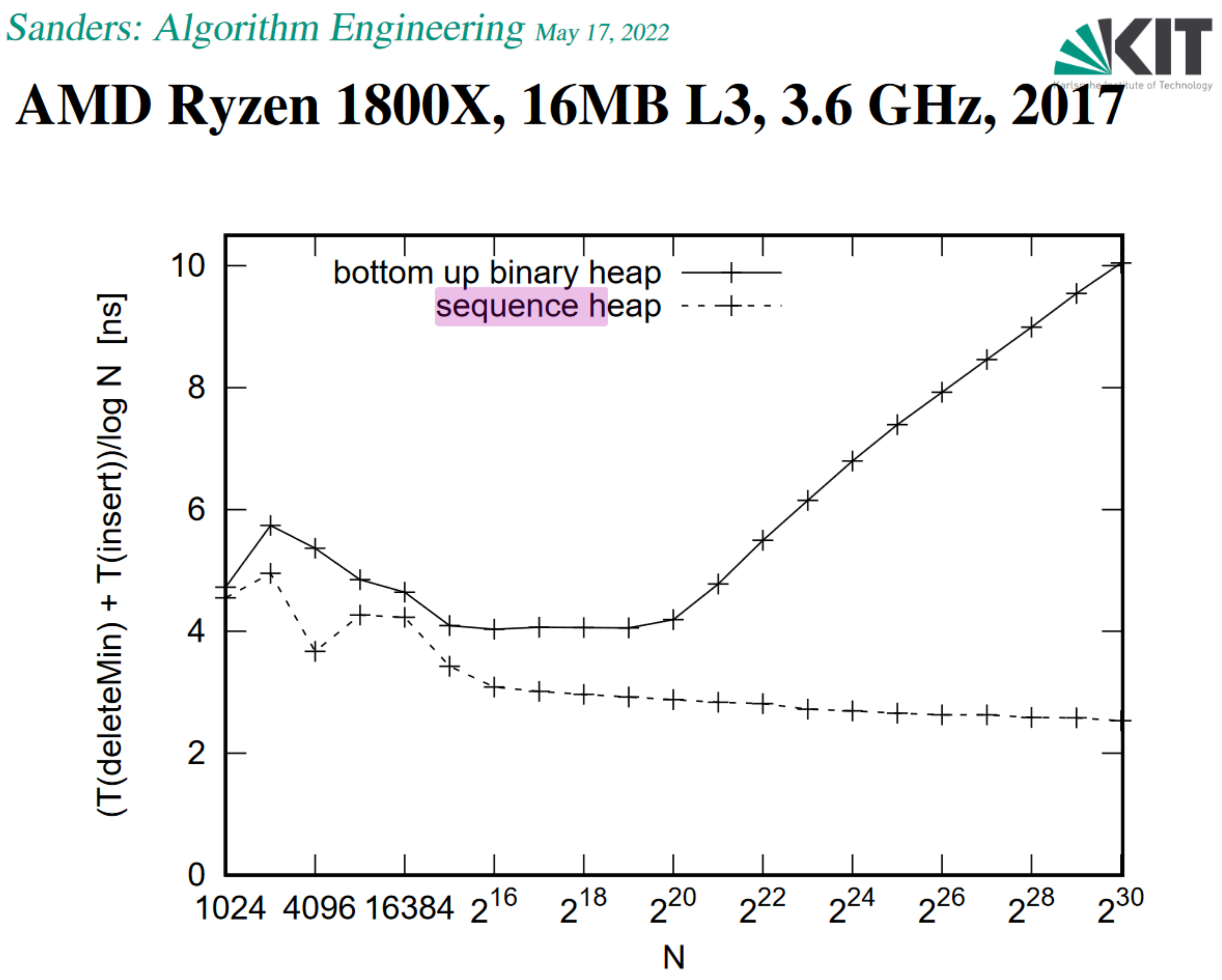

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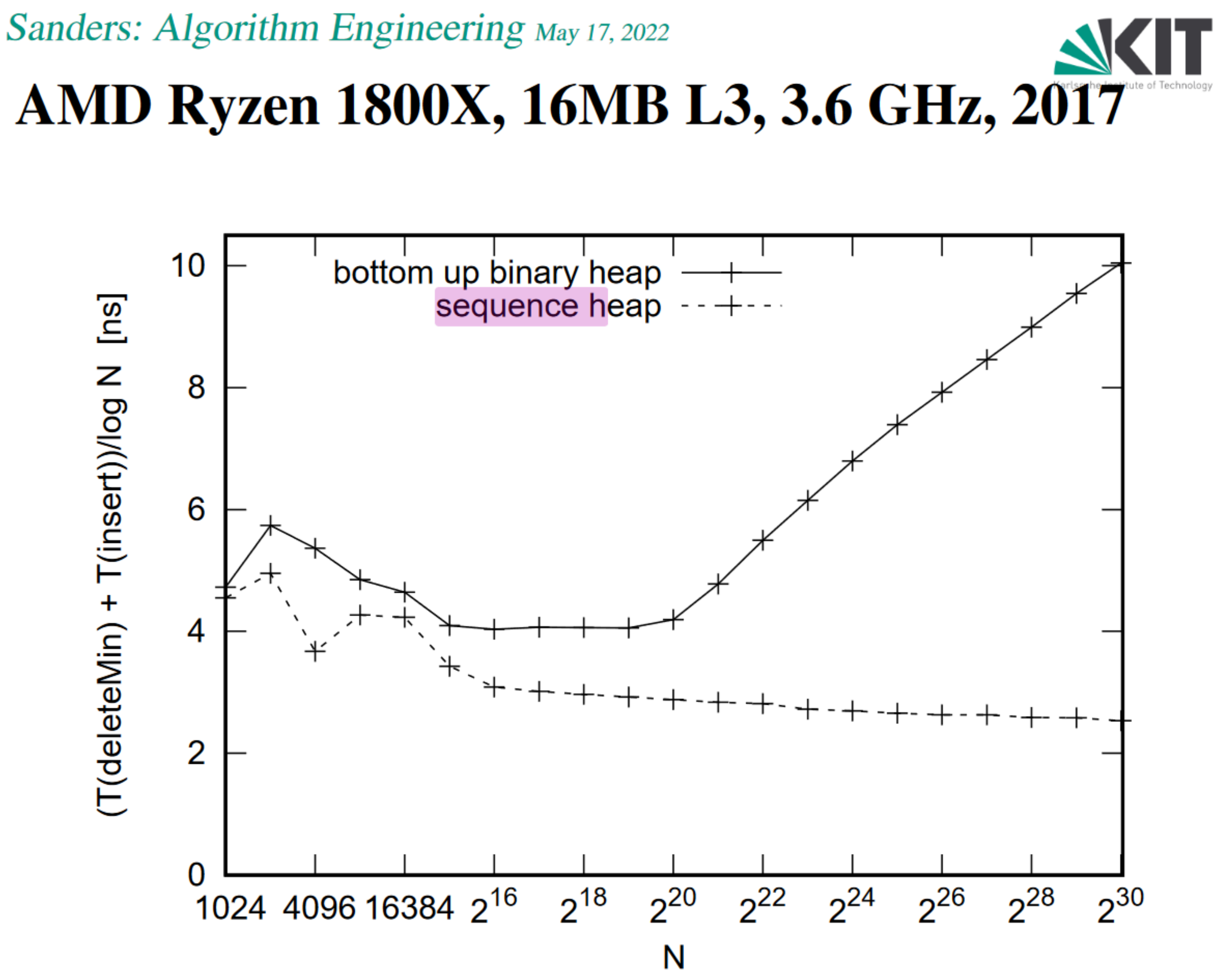


## Fibonacci heaps in libraries

- Fibonacci, Binomial, Pairing, $k$-ary heaps are included in boost (C++)
- Pairing heaps in GNU C++ Library

# Practical Heaps: References


Larkin, Daniel H., Siddhartha Sen, and Robert E. Tarjan. "A back-to-basics empirical study of priority queues." Proceedings of the Sixteenth Workshop on Algorithm Engineering and Experiments (ALENEX), 2014.

Sanders, Peter. "Fast priority queues for cached memory." Journal of Experimental Algorithmics (JEA) 5 (2000): 7-es. (plot from Peter Sanders *Algorithm Engineering* lecture)

Fredman, Michael L., et al. "The pairing heap: A new form of self-adjusting heap." Algorithmica 1.1-4 (1986): 111-129.

Mehlhorn, Kurt, Peter Sanders, and Peter Sanders. Algorithms and data structures: The basic toolbox. Vol. 55. Berlin: Springer, 2008. (for reading about pairing heaps)


boost: `https://www.boost.org/`

GNU C++ library: `https://gcc.gnu.org/onlinedocs/libstdc++/`