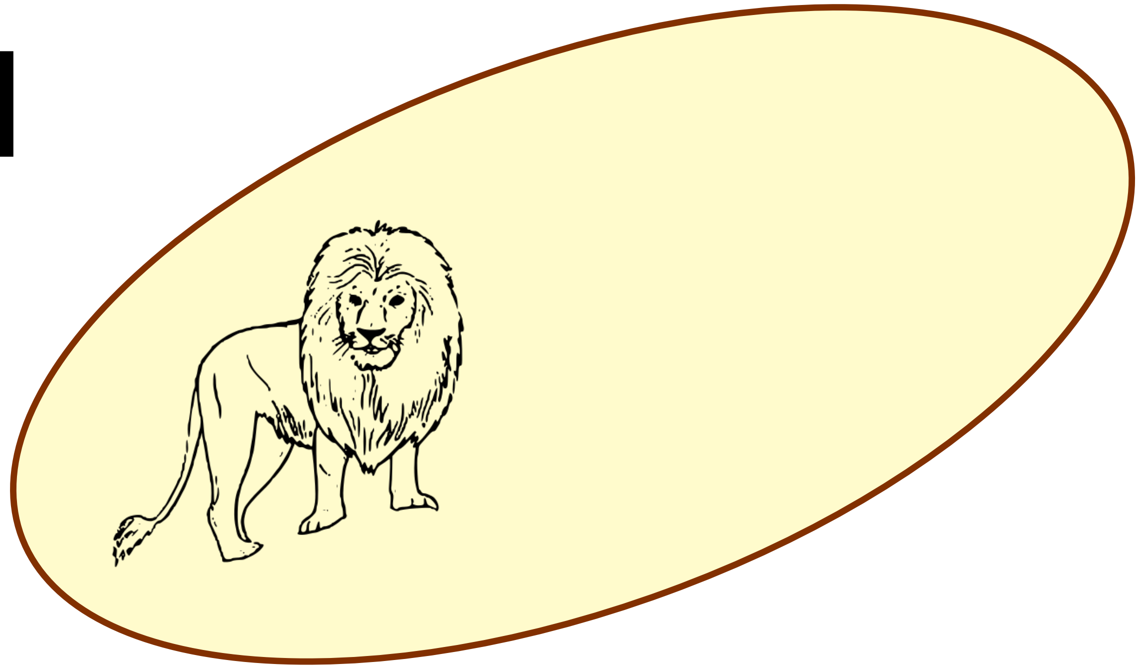
Other algorithms besides the simplex method



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today: sketch of polynomial-time algorithms

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The New York Times

Similar Transportation Bond Issues Are Passed in New York and Jersey

CARTER SAID TO ASK FOR LOYALTY CHECK OF AIDES ON POLITICS

IRAN'S CIVIL GOVERNMENT OUT; HOSTAGES FACE DEATH THREAT; OIL EXPORTS BELIEVED HALTED

F.A.A. Seeks Fine of $1.5 Million For 'Unairworthy' Braniff Flights

Main Iran Oil Port Reported Closed And Prices of Spot Petroleum Soar

Energy Dept. Accuses 9 Refiners Of $1.18 Billion Oil Overcharges

A Soviet Discovery Rocks World of Mathematics

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From the archive, 29 October 1979: Russian way with the mathematical travelling salesman

The Guardian

Unknown Soviet mathematician LG Khachian offers a solution to a problem that has the computing world baffled

A 1970s mainframe computer: LG Khachian offered a solution to the 'travelling salesman' problem of computer processing. Photograph: ClassicStock/Alamy

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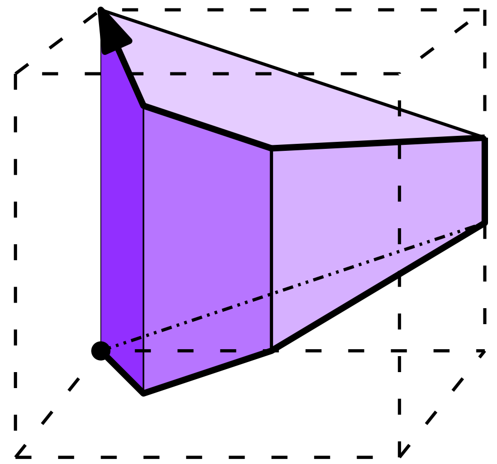

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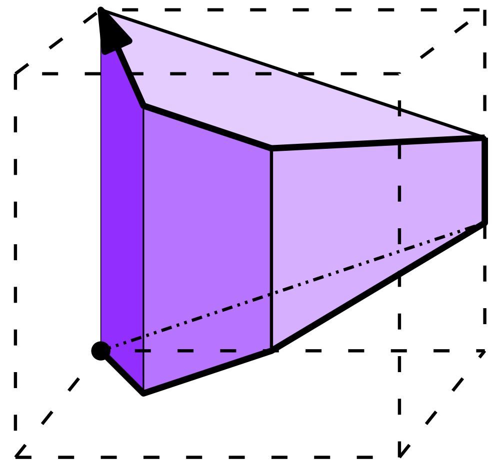

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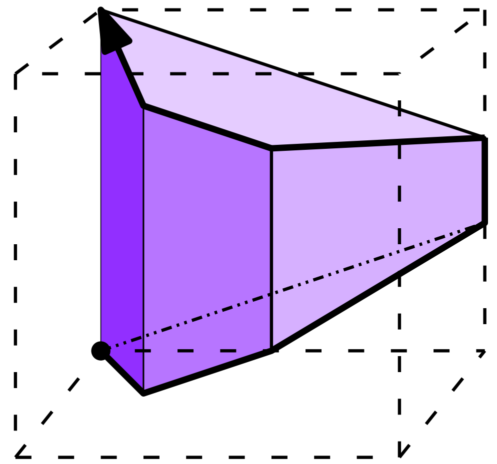

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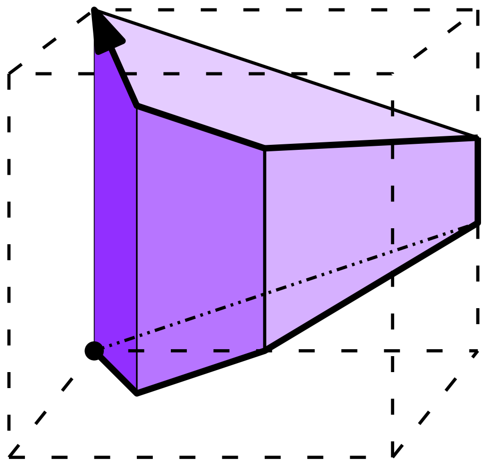

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	Most problems	Worst case
Simplex method	Extremely fast	Exponentially bad
Ellipsoid method	High degree polynomial	High degree polynomial

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For $r = p/q$ rational, we define $\langle r \rangle = \langle p \rangle + \langle q \rangle$.

→ Similar for vectors and matrices.

Polynomial and strongly polynomial algorithms

A linear program Maximize $c^T x$ subject to $Ax \leq b$ with rational entries has size $\langle A \rangle + \langle b \rangle + \langle c \rangle$.

Example: $A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 7 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $c = \begin{bmatrix} 1,000,521 \\ 3,012,554 \\ 19,728,213 \end{bmatrix}$

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Steps are in terms of bit operations — adding two k bit integers requires at least k steps!

Digression: Gaussian Elimination

Solve $Ax = b$, A size $n \times n$, rational entries.

→ At most Cn^3 arithmetic operations.

$$\begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix}$$

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Naive implementations can produce entries requiring 2^n bits, giving an exponential algorithm!

But smarter implementations are indeed polynomial.

Ellipsoid and interior point methods are polynomial, simplex method is not.

Strongly polynomial algorithms

Strongly polynomial algorithms are polynomial in the sense defined before,
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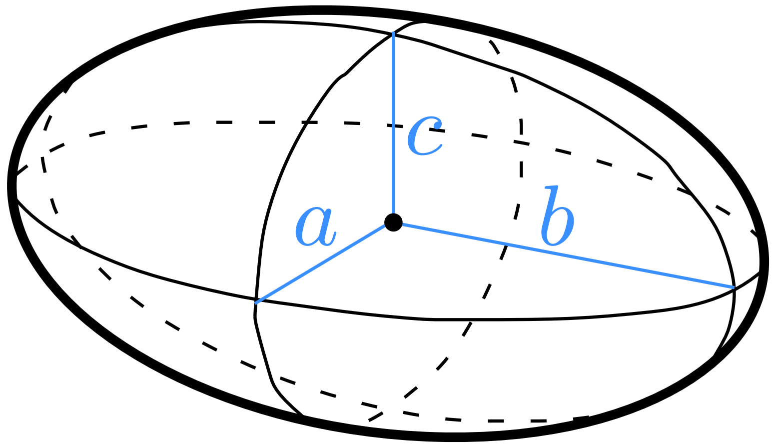
Ellipsoid method is not: For all $K \in \mathbb{N}$, we can find a LP with 2 variables, 2 constraints, and ellipsoid method requires $\geq K$ arithmetic steps!

The bit sizes in these examples go to ∞ with K: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\langle A \rangle \to \infty$ as $K \to \infty$

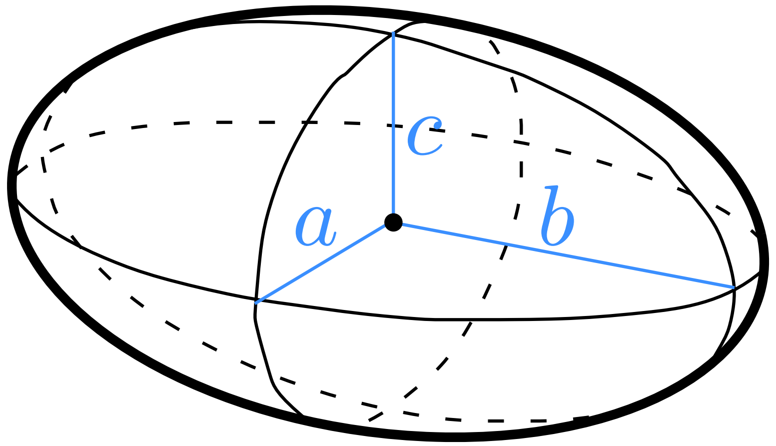
The Ellipsoid Method

- First polynomial time algorithm for linear programming
- Slow in practice



The Ellipsoid Method

- First polynomial time algorithm for linear programming
- Slow in practice
- We will show how to find a feasible solution to $Ax \leq b$



Recall:

In some sense finding an optimal solution is no harder than finding a feasible solution.

Finding an optimal solution to

Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$

is the same as finding a feasible solution to

~~Maximize $c^T x$~~

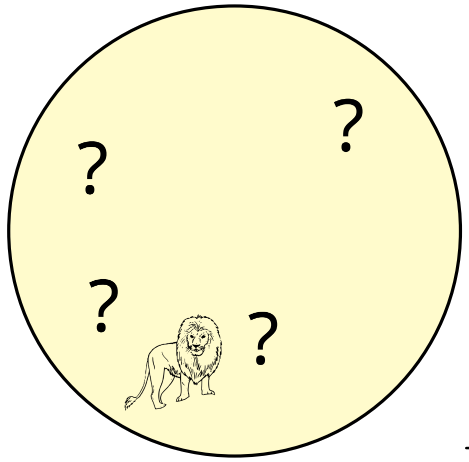
subject to

$$
\begin{aligned}
Ax &\leq b \\
A^T y &\geq c \\
c^T x &\geq b^T y \\
x &\geq 0, y \geq 0
\end{aligned}
$$

The Ellipsoid Method

Analogy: Find feasible solution to $Ax \leq b$

Find lion in Sahara



$B(\mathbf{0}, R)$

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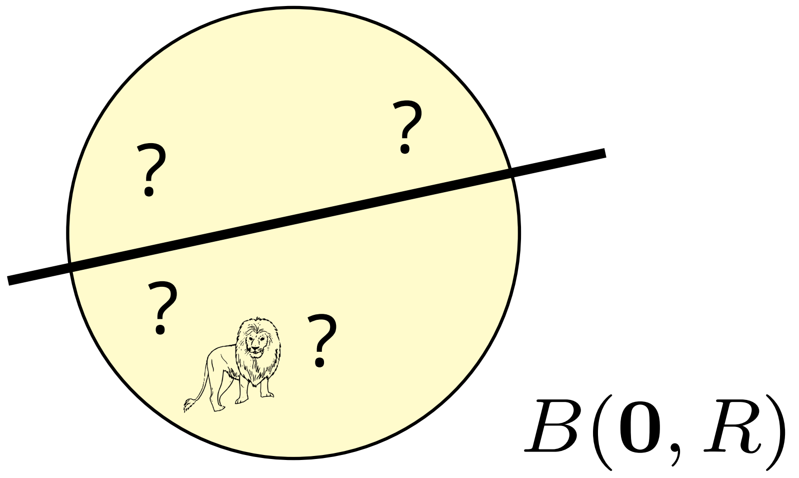

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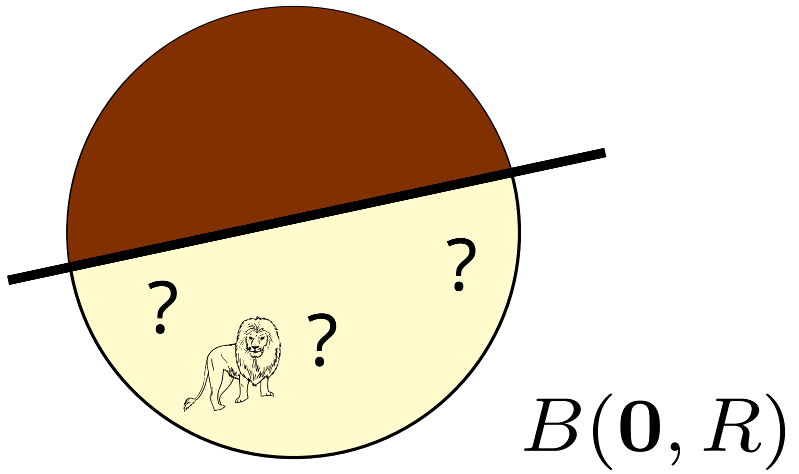

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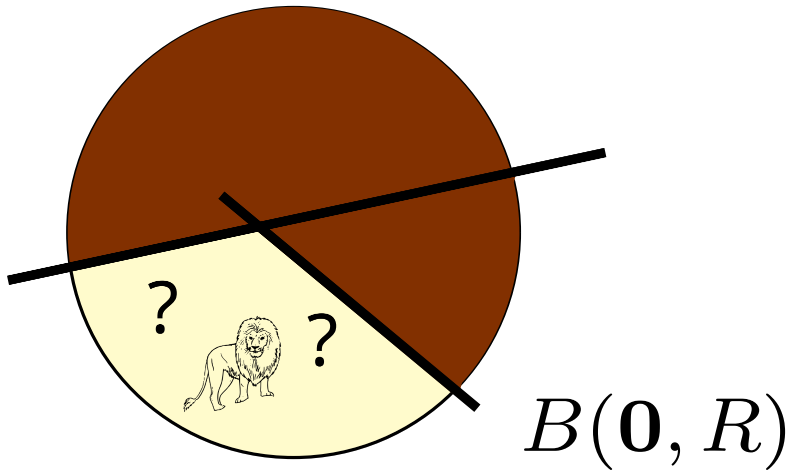

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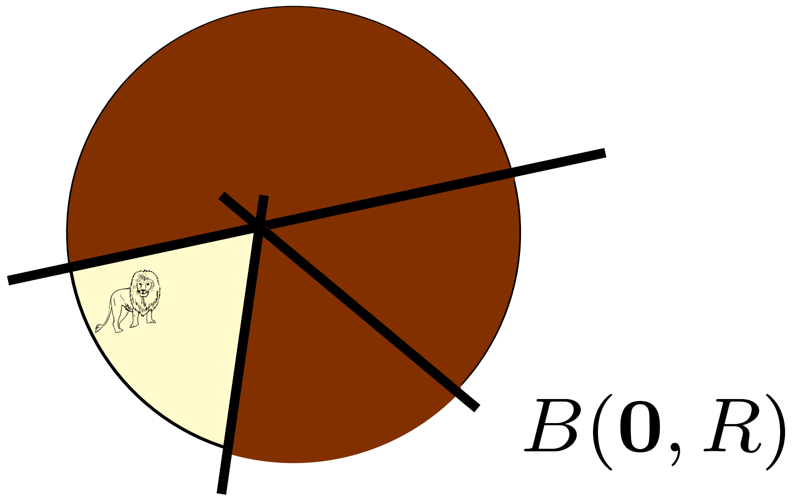

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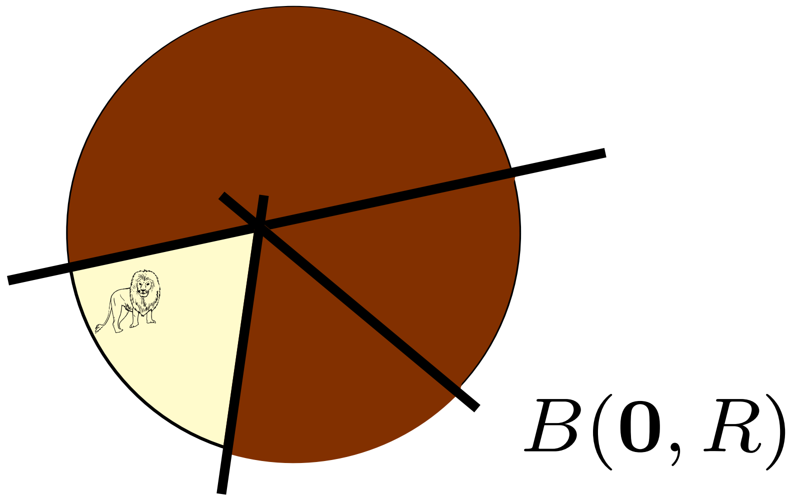

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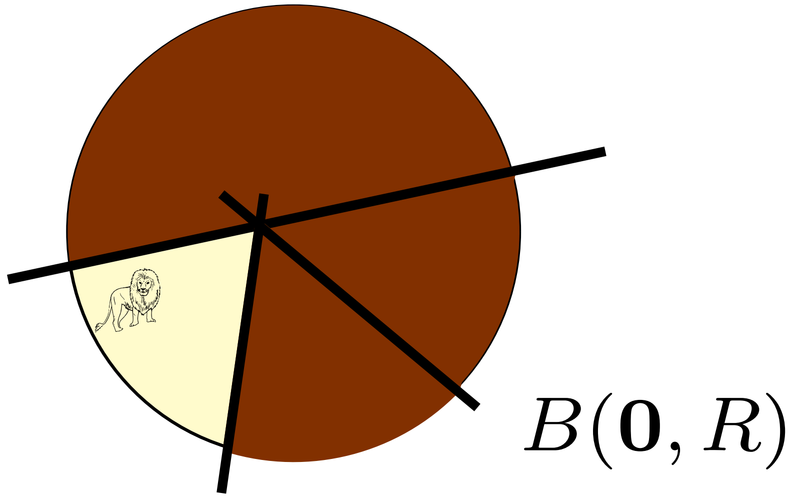
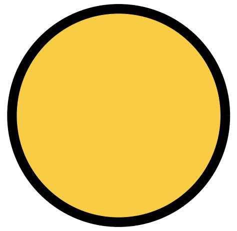


Cannot find arbitrary small lions

Looking for a lion of size at least an ε-ball



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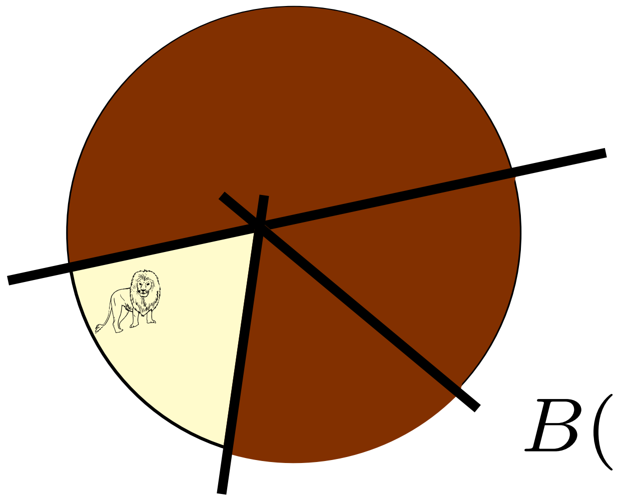

$B(\mathbf{0}, R)$

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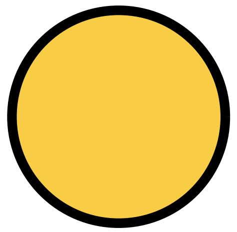

Let $\varphi = \langle A \rangle + \langle b \rangle$ be the input size of $Ax \leq b$.

Let $\eta = 2^{-5\varphi}, \varepsilon = 2^{-6\varphi}$. Then:

$Ax \leq b$ has a solution $\iff$ $Ax \leq b + \eta$ has a solution (indeed a full ε-ball).

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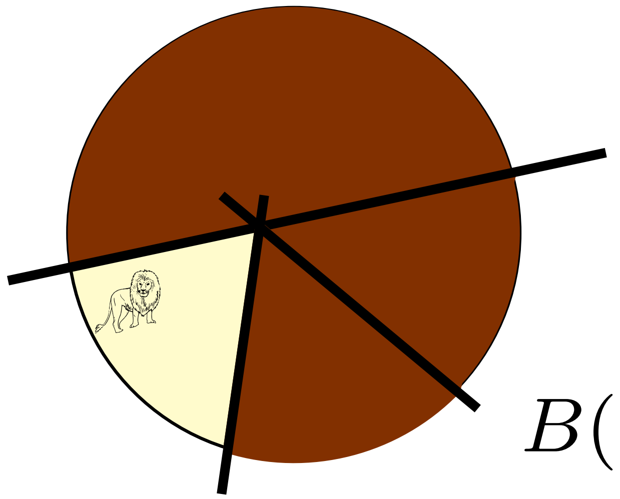

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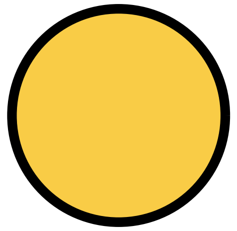

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Aside: Also, $Ax \leq b$ has a solution $\iff$

$$Ax \leq b, -2^{\varphi} \leq x_1 \leq 2^{\varphi}, \ldots, -2^{\varphi} \leq x_n \leq 2^{\varphi} \text{ does.}$$

All solutions to the latter live in $B(\mathbf{0}, 2^{\varphi}\sqrt{n})$.

Ellipsoid Algorithm

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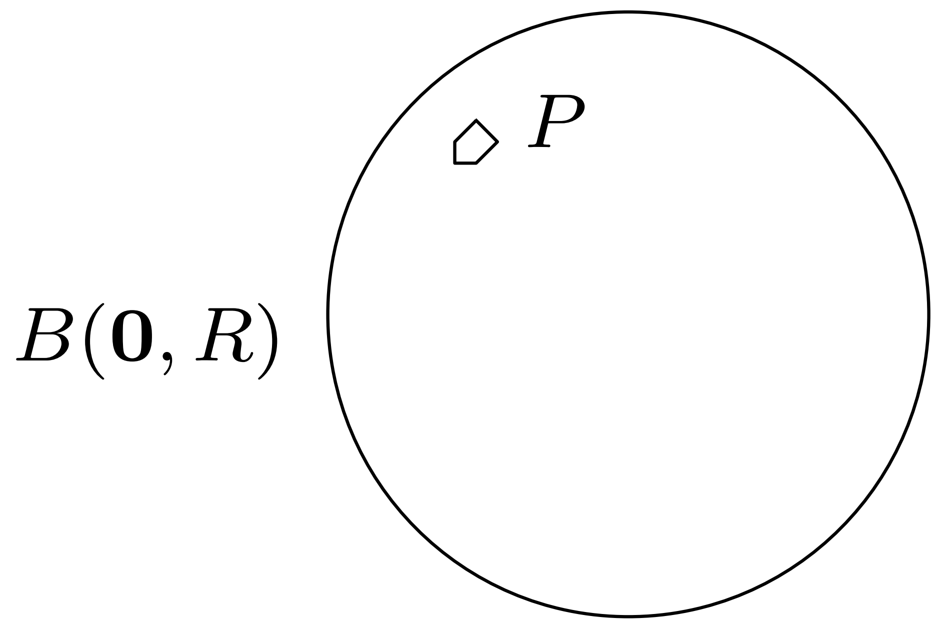
rational numbers $R > \varepsilon > 0$ with feasible region
$P = \{x \in \mathbb{R}^n | Ax \leq b\}$ contained in $B(\mathbf{0}, R)$.



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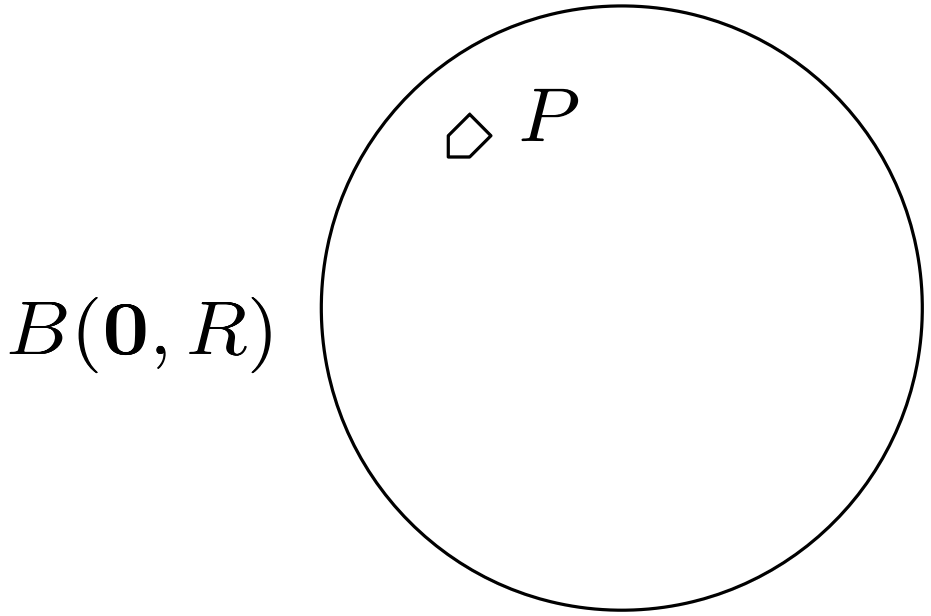

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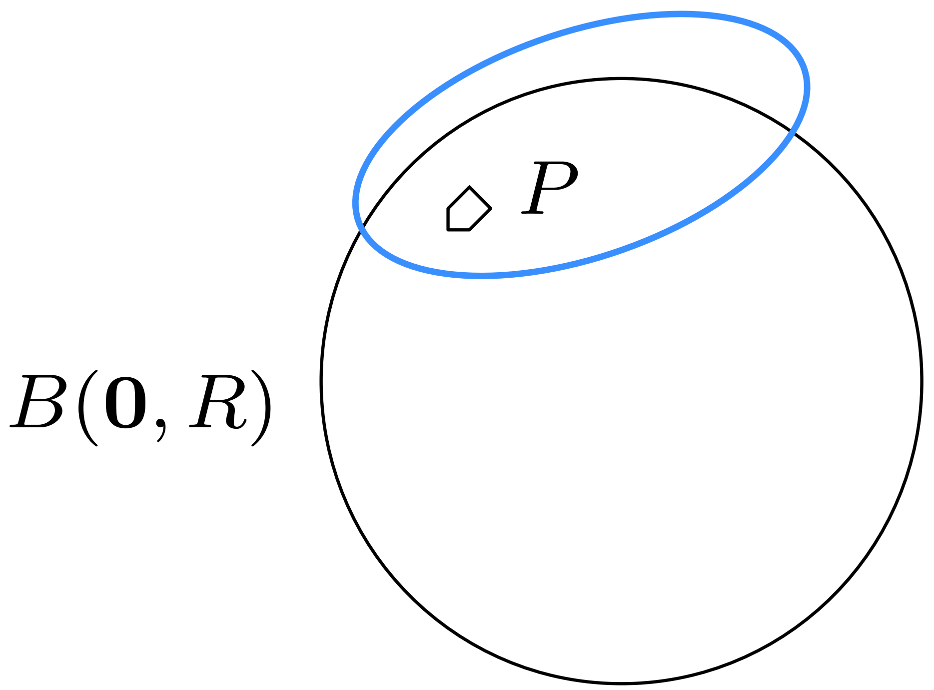

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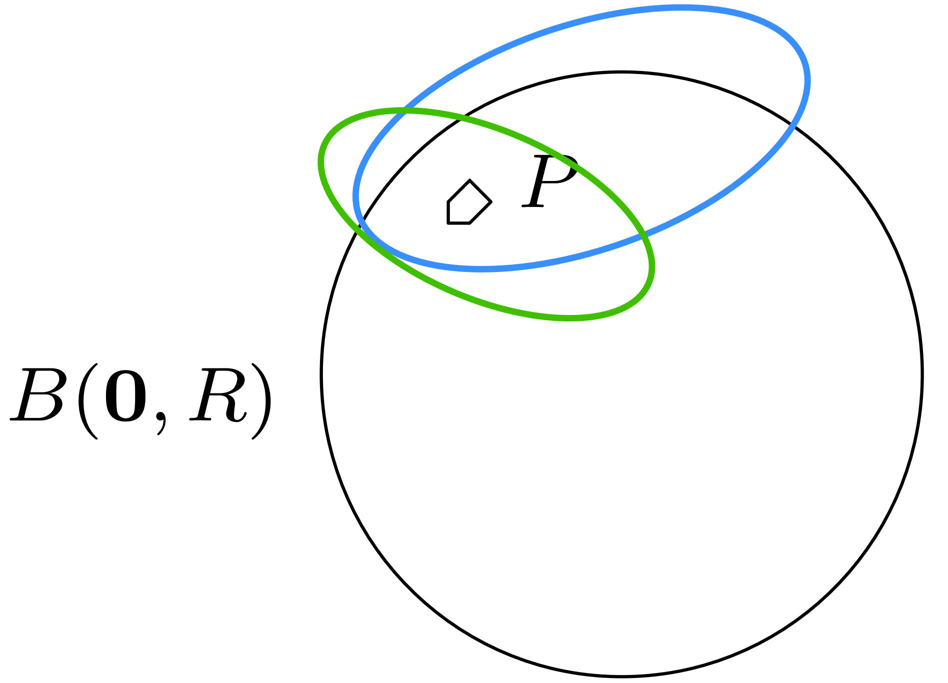

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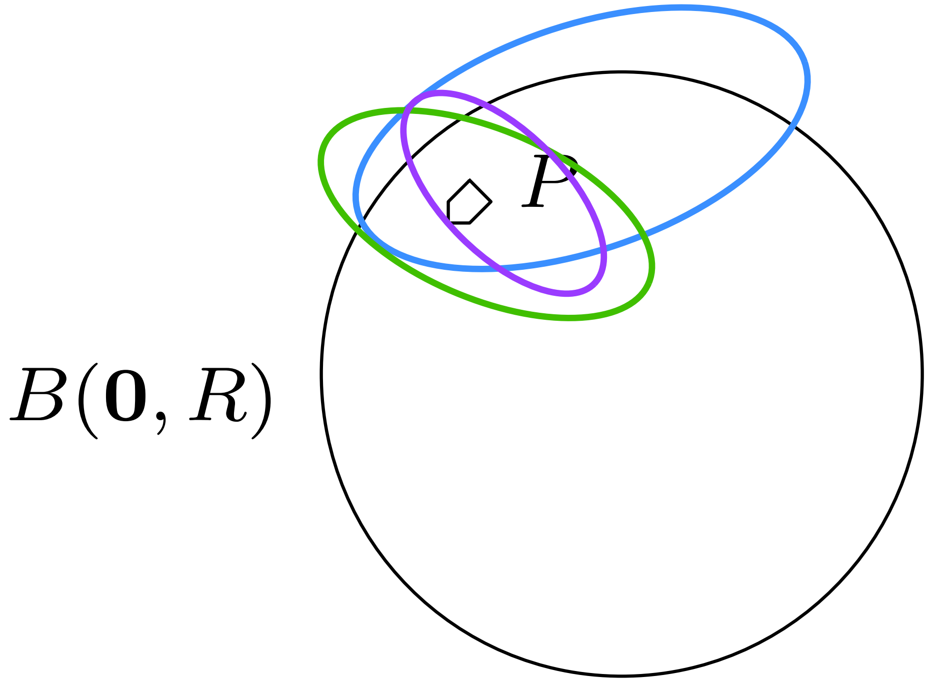

Ellipsoid Algorithm

Iterative step:

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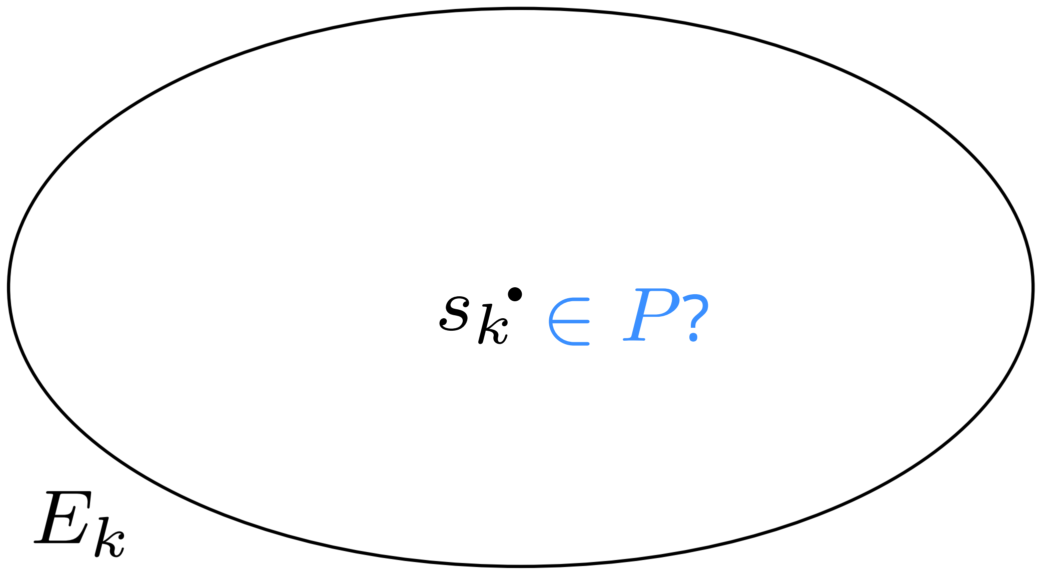
Else, find E_{k+1} from E_k as follows:



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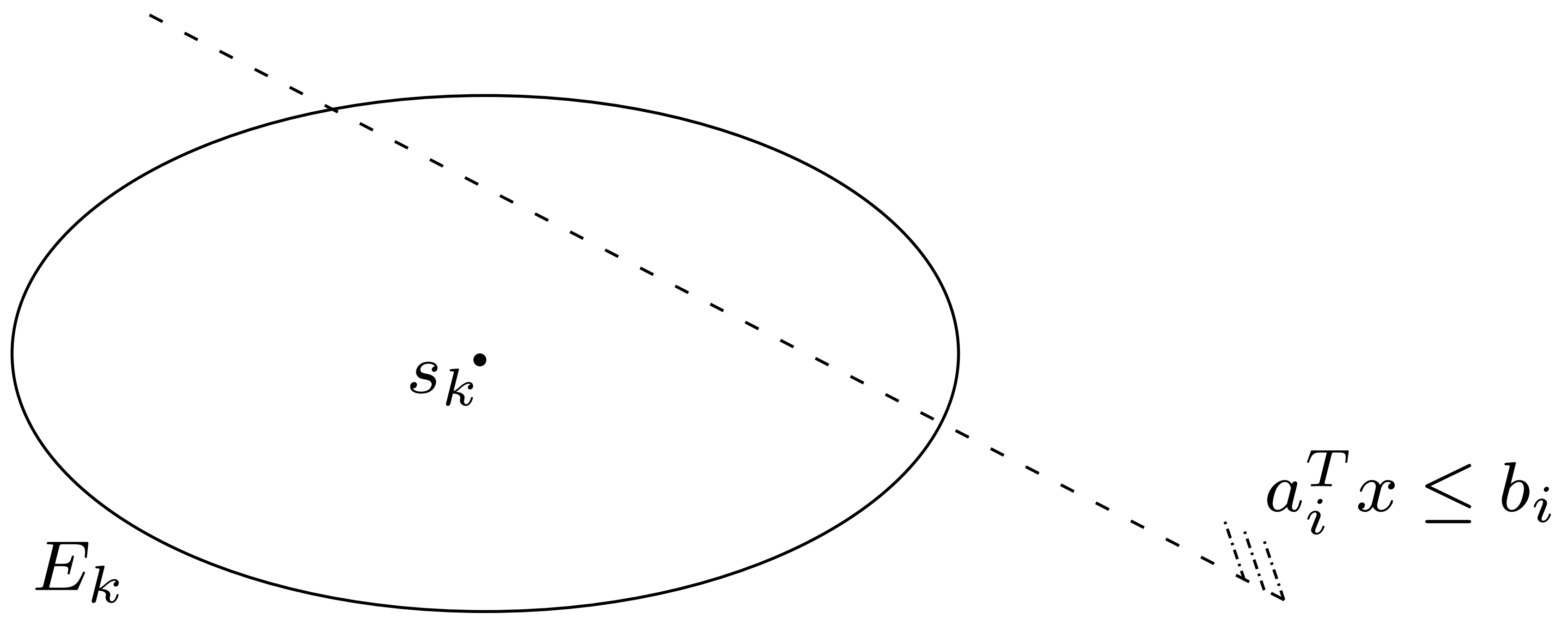

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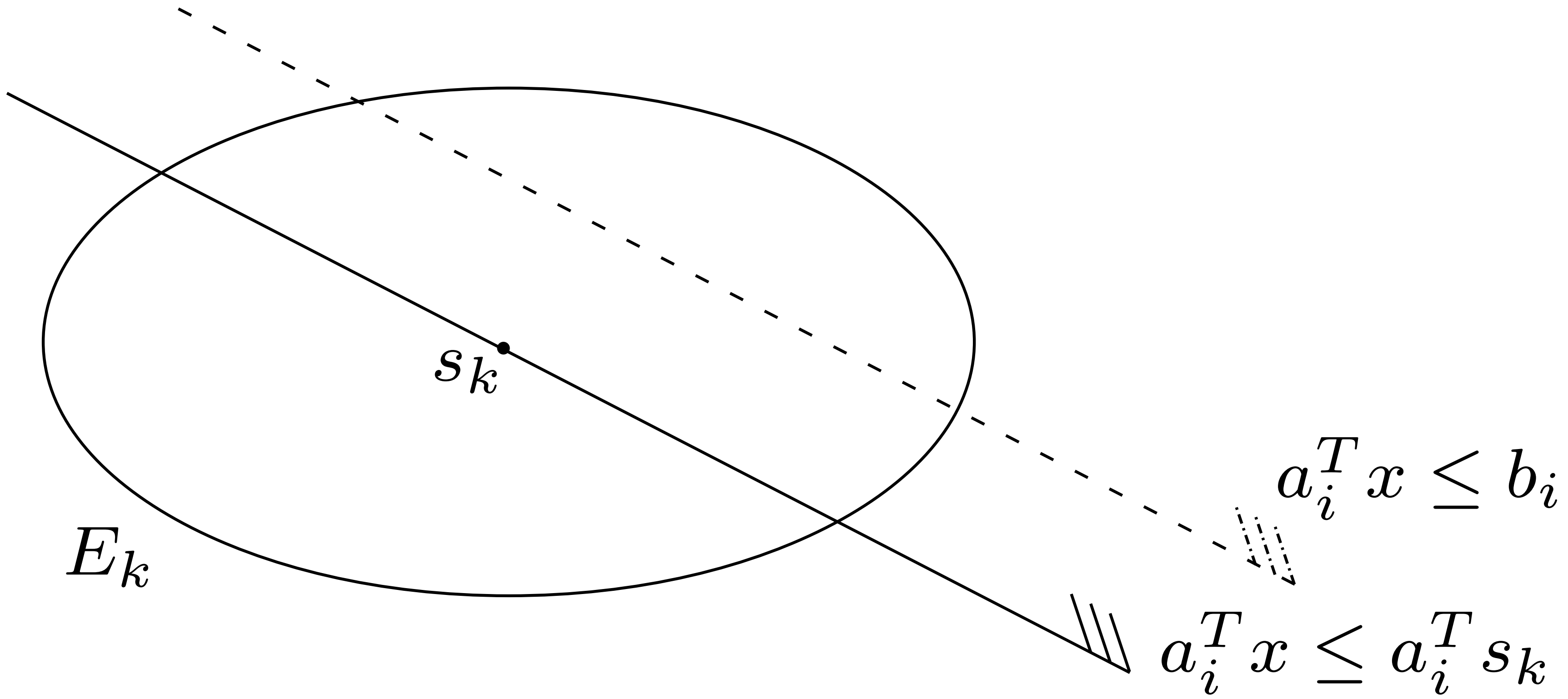

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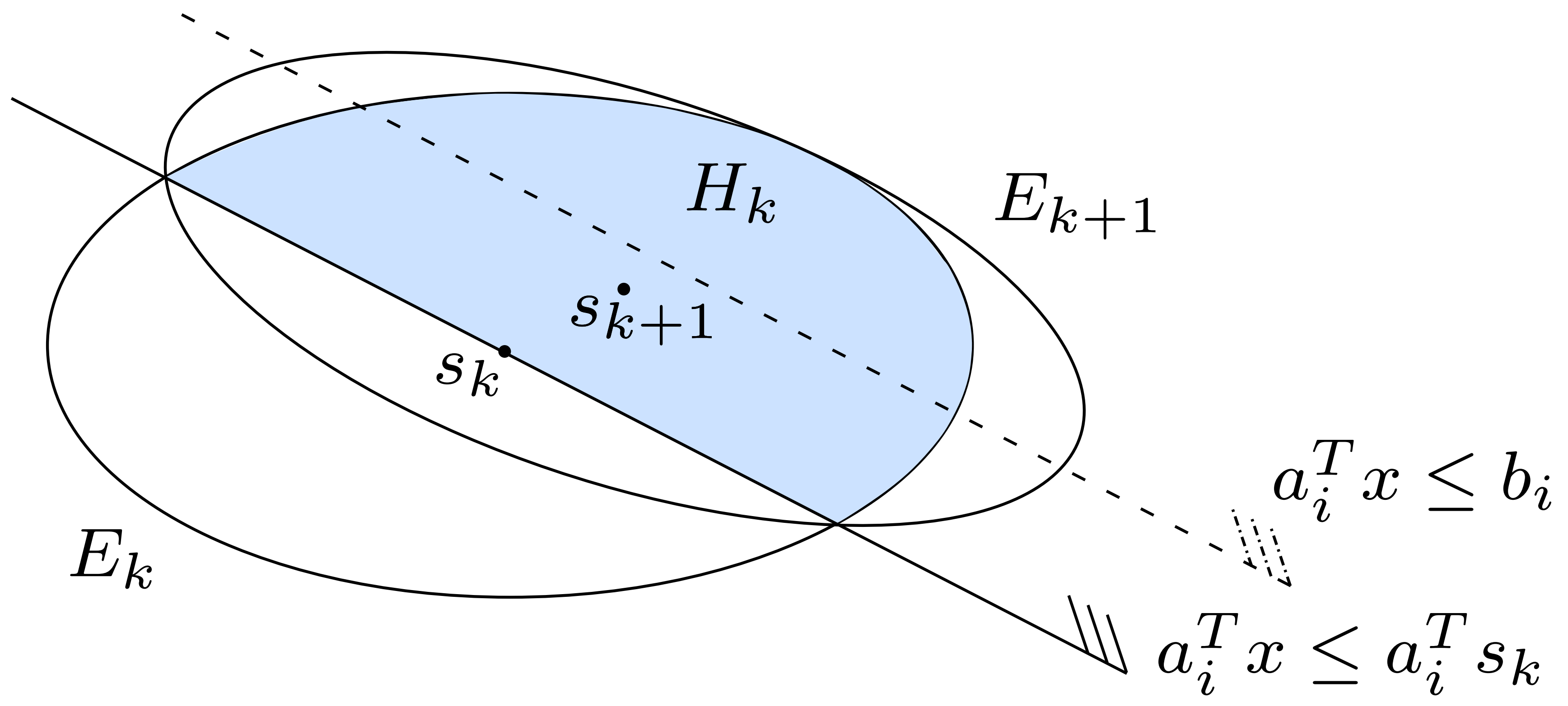

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Ellipsoid Algorithm

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n here because for $n - dim.$ disk of rad. r: volume grows proportional to r^n

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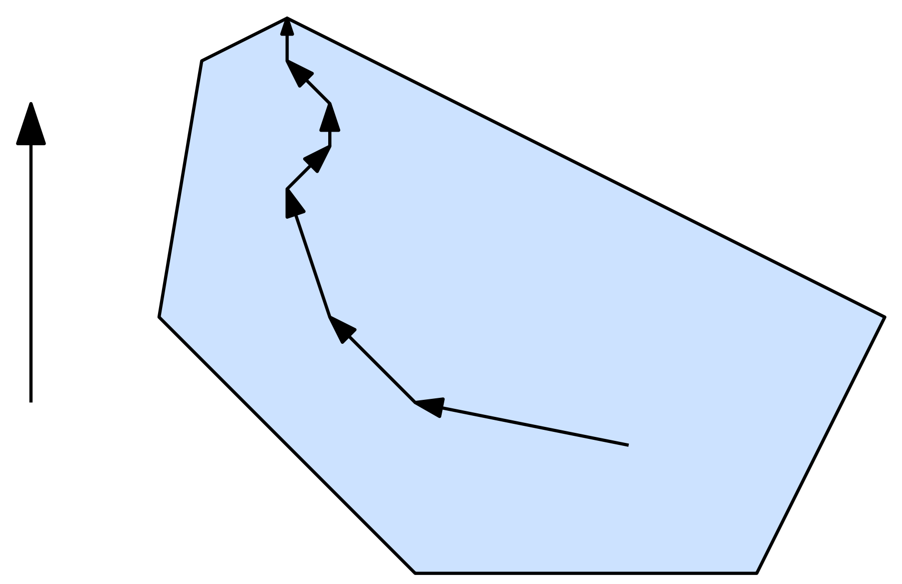
- Ellipsoids could be replaced with other "rich-enough" families of convex sets,
  like simplices.
- All we need is a "separation oracle", we don't need to know $Ax \leq b$.
  Could even have infinitely many constraints ($\rightarrow$ semidefinite programming)

Interior Point Methods

for Linear Programming



Interior point methods

- Used for nonlinear problems since 1950’s

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The New York Times

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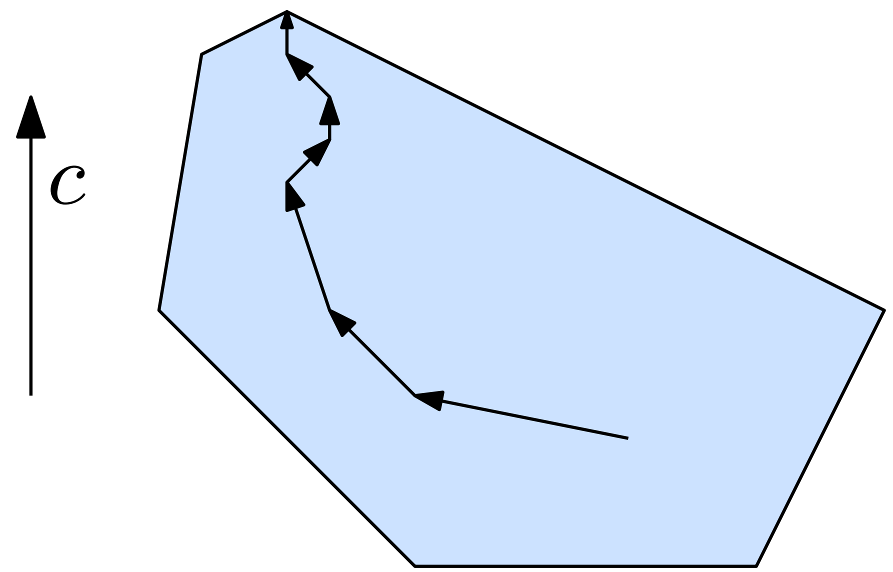
- Now competitive with the simplex method, especially on large problems (too large for the 1970’s): Rely on powerful routines for sparse system of equations.

Interior point methods

- Interior methods walk along the interior, until they hop to an exact optimum via a last rounding step.



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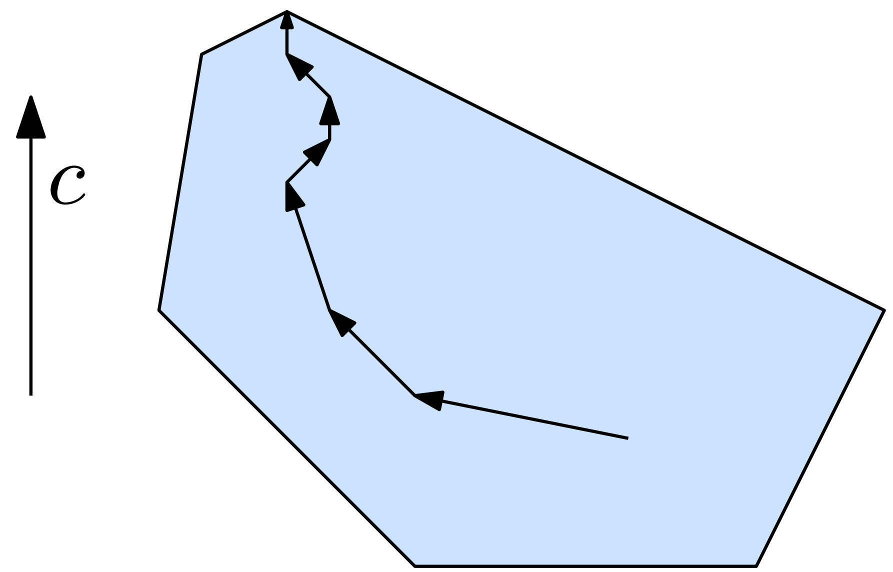

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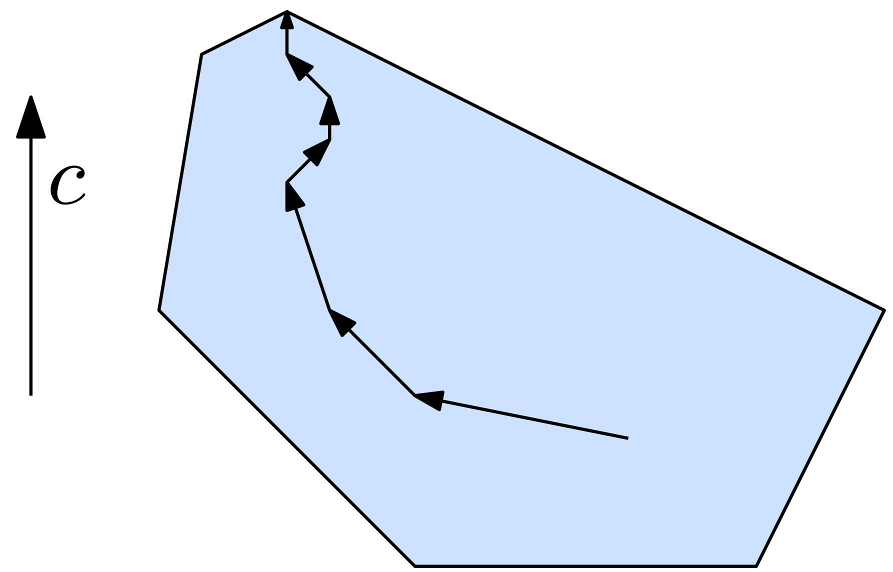

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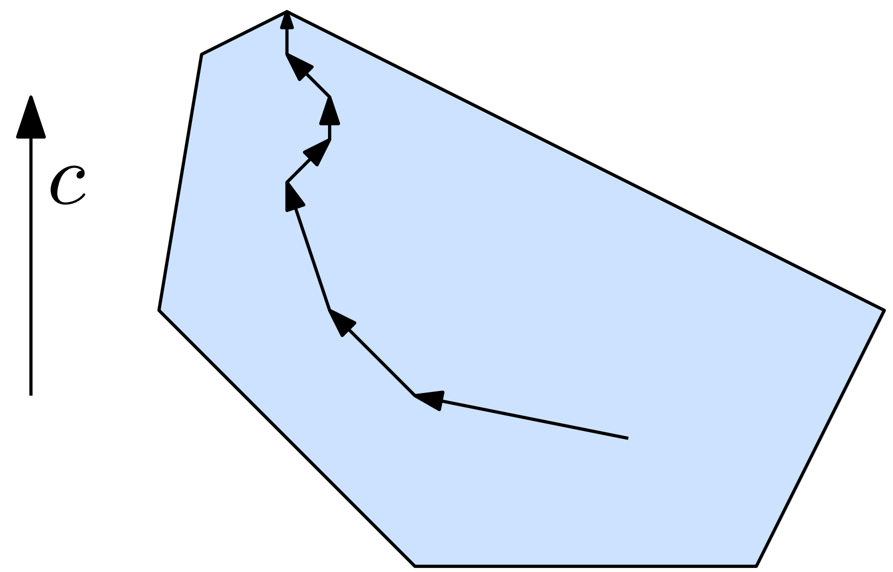


Types of interior point methods:

Central path, potential reduction, affine scaling.

For each of them:

primal, dual, primal-dual, or self-dual.

The Central Path

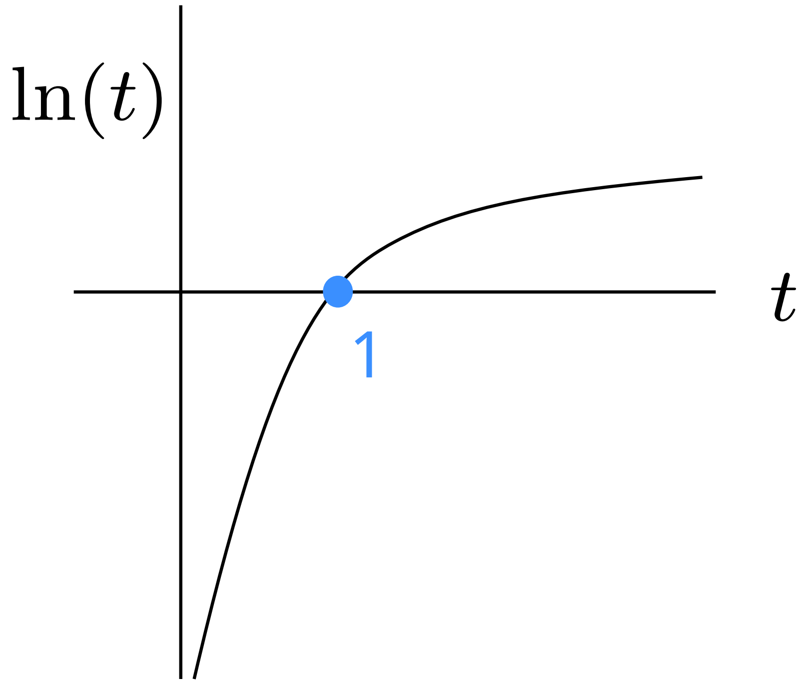
Maximize $c^T x$ subject to $Ax \leq b$

For $\mu > 0$, define

$$f_\mu(x) = c^T x + \mu \sum_{i=1}^{m} \ln(b_i - a_i x)$$

where a_i is the i-th row of A.

Logarithmic barrier



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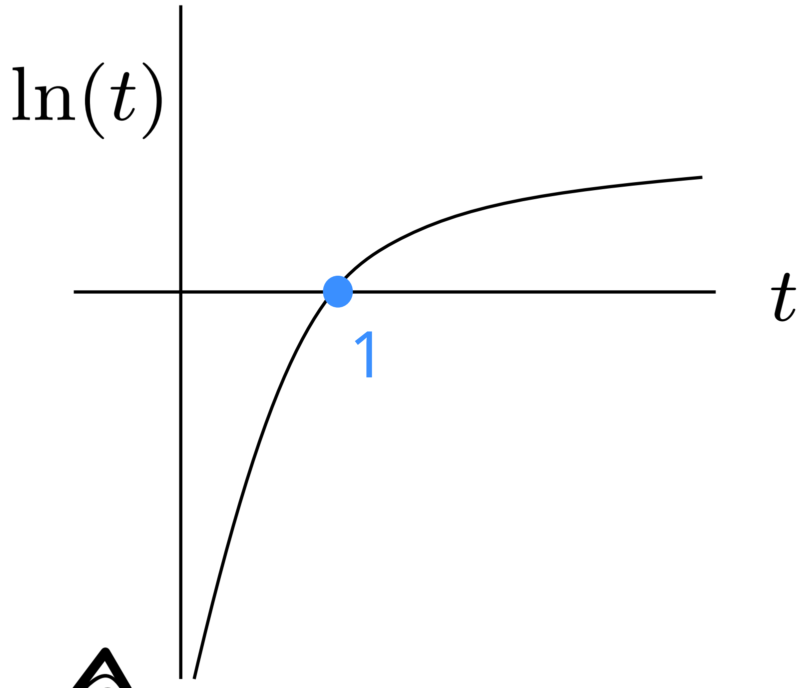
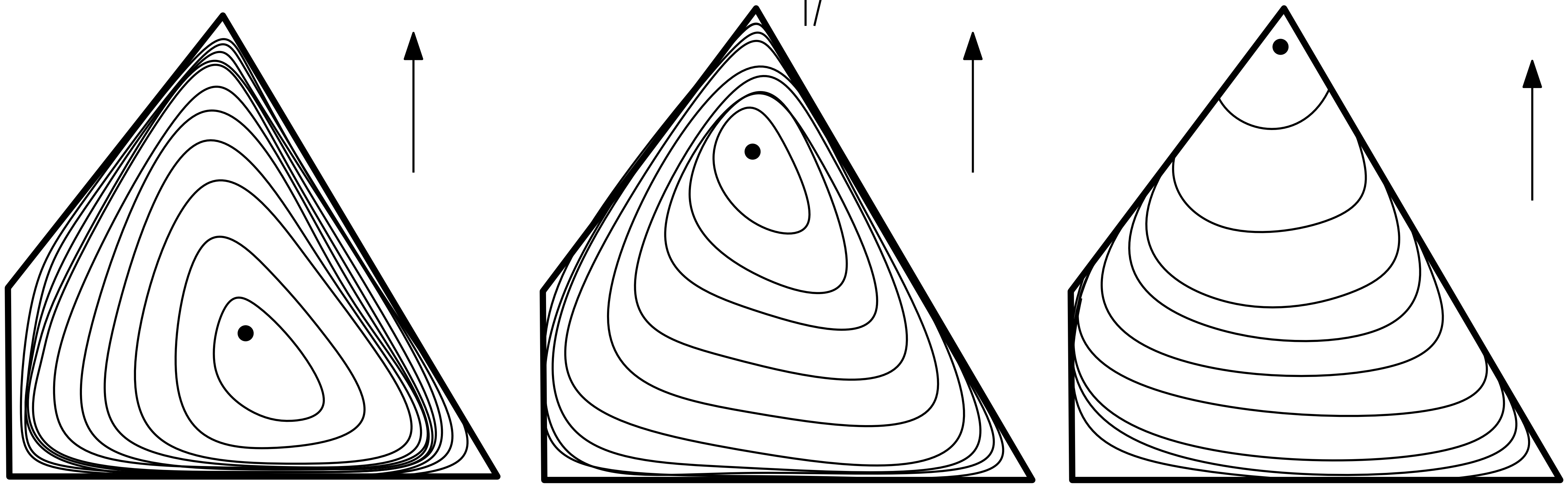
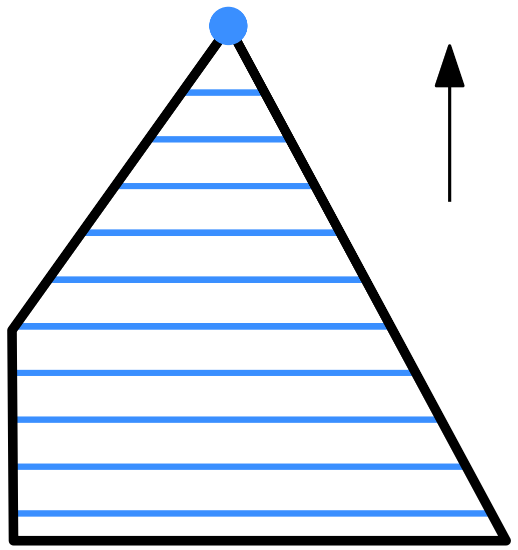


"furthest point
from boundary"



$\mu = 100$ $\mu = 1$ $\mu = 1/100$

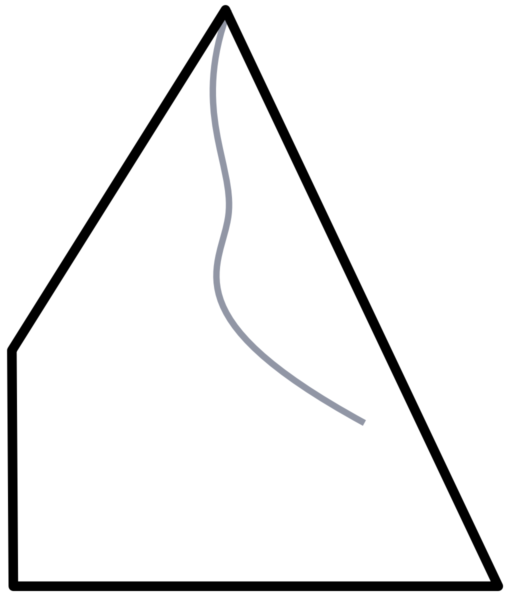
The Central Path



$\mu = 0$ (the limit)

Let $x^*(\mu)$ be the unique solution to
Maximize $f_\mu(x)$subject to $Ax \leq b$.

redundant



Def: The central path is the curve $x^*(\mu)$ for $\mu > 0$.

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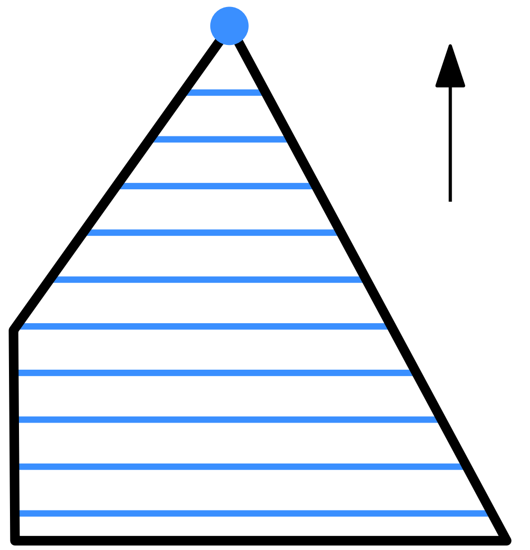

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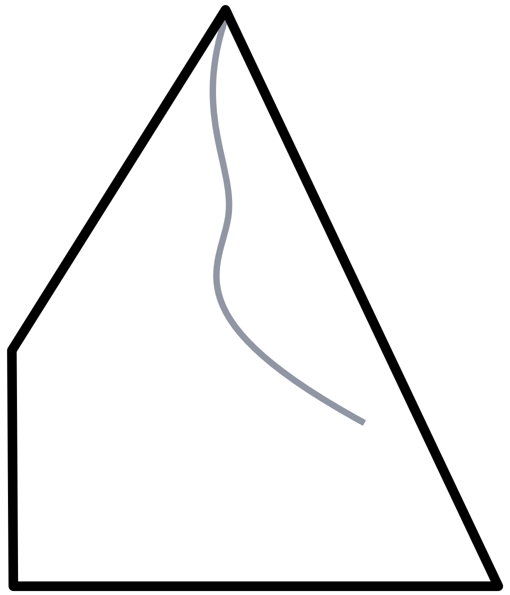

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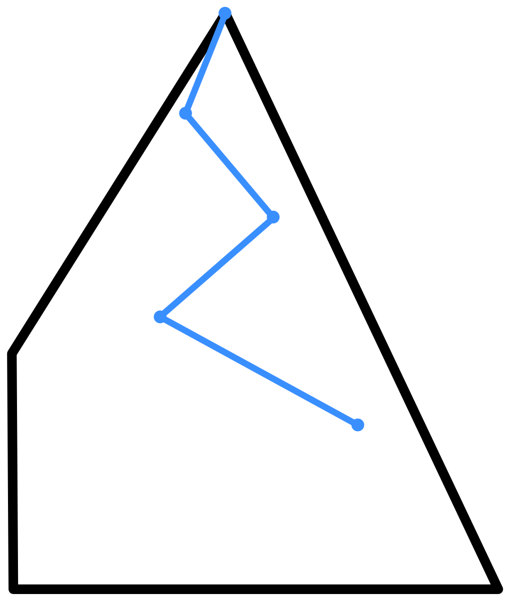
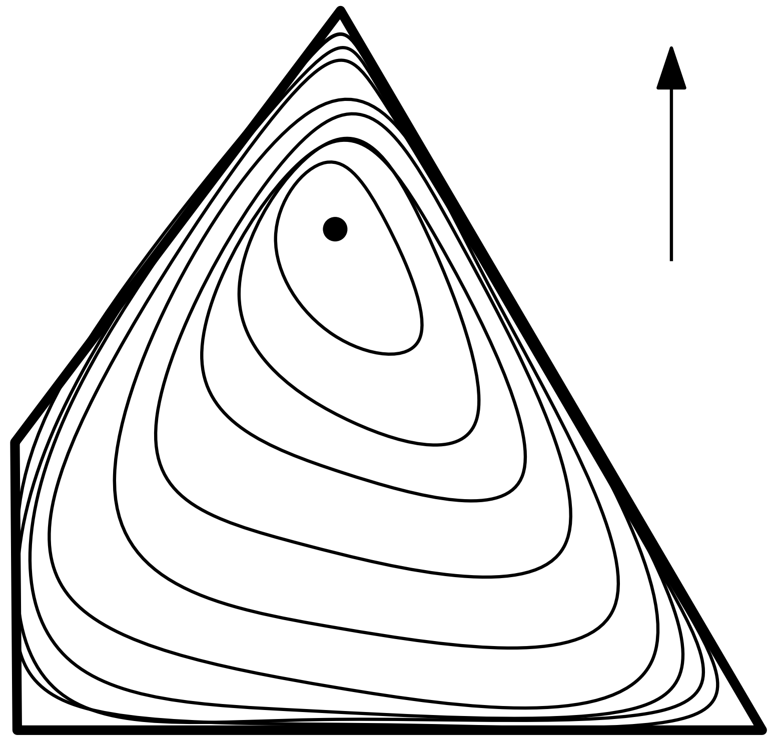


Actual algorithms will only approximately follow this path.

The Central Path

For $\mu > 0$ fixed, why is there a unique optimum $x^*(\mu)$?

- An optimum exists since we have a continous function
$$f_\mu(x) = c^T x + \mu \sum_i \ln(b_i - a_i x)$$
on a compact set
$\{x : Ax \leq b$ and $f_\mu(x) \geq f_\mu(y)\}$.



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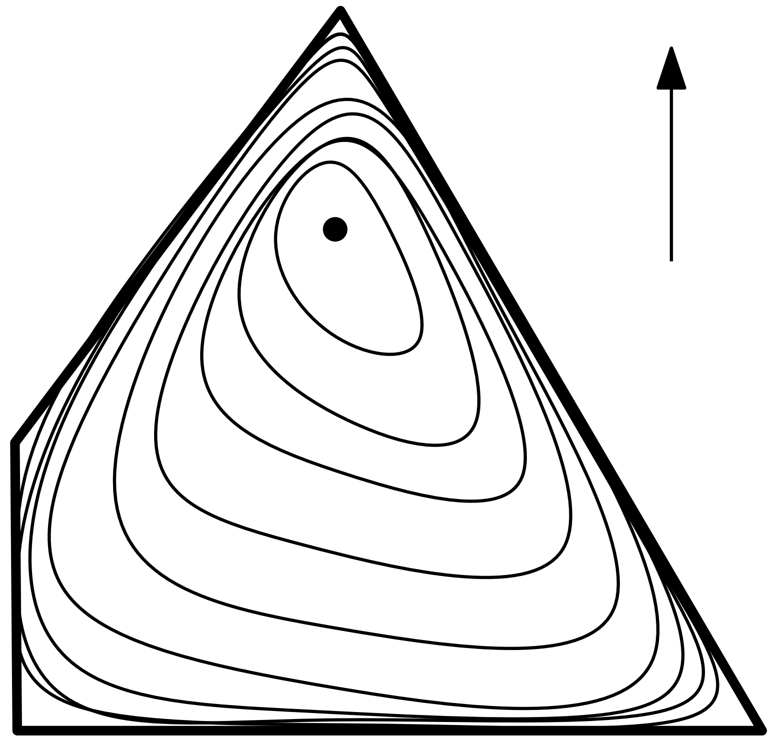
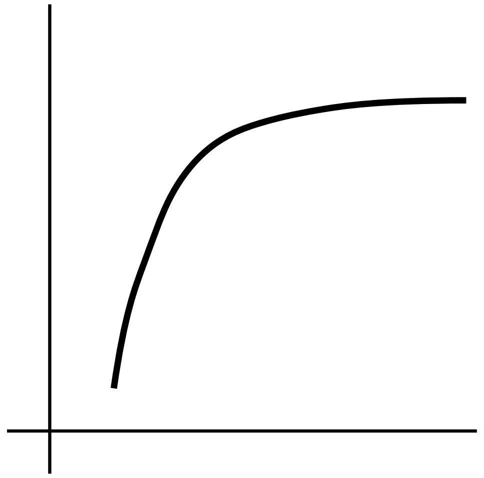
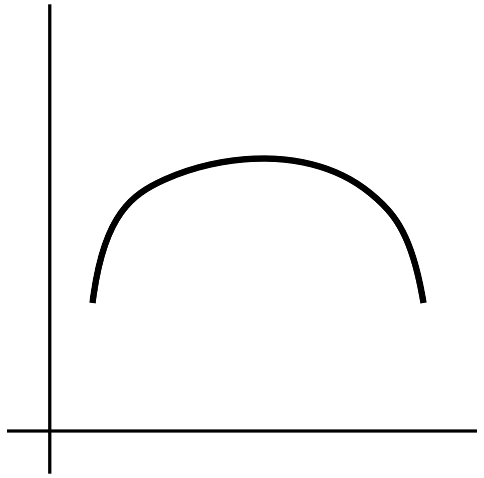


- The optimum is unique since f_μ is strictly concave for $\mu > 0$.





The primal-dual Central Path (equational form)

Maximize $c^T x$ subject to $Ax = b,\ x \geq 0$

size $m \times n$

$$f_\mu(x) = c^T x + \mu \sum_{j=1}^{n} \ln(x_j)$$

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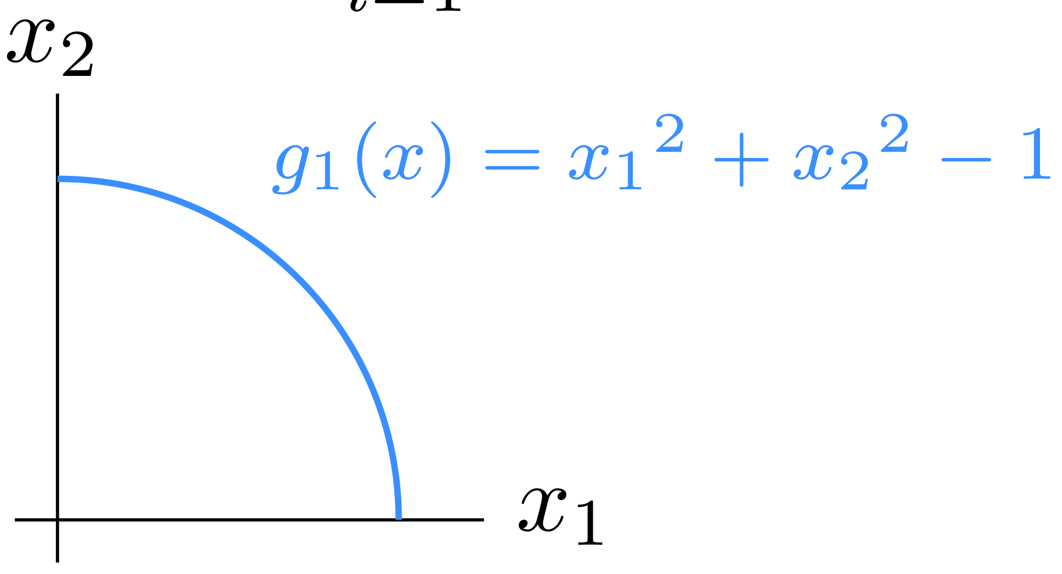
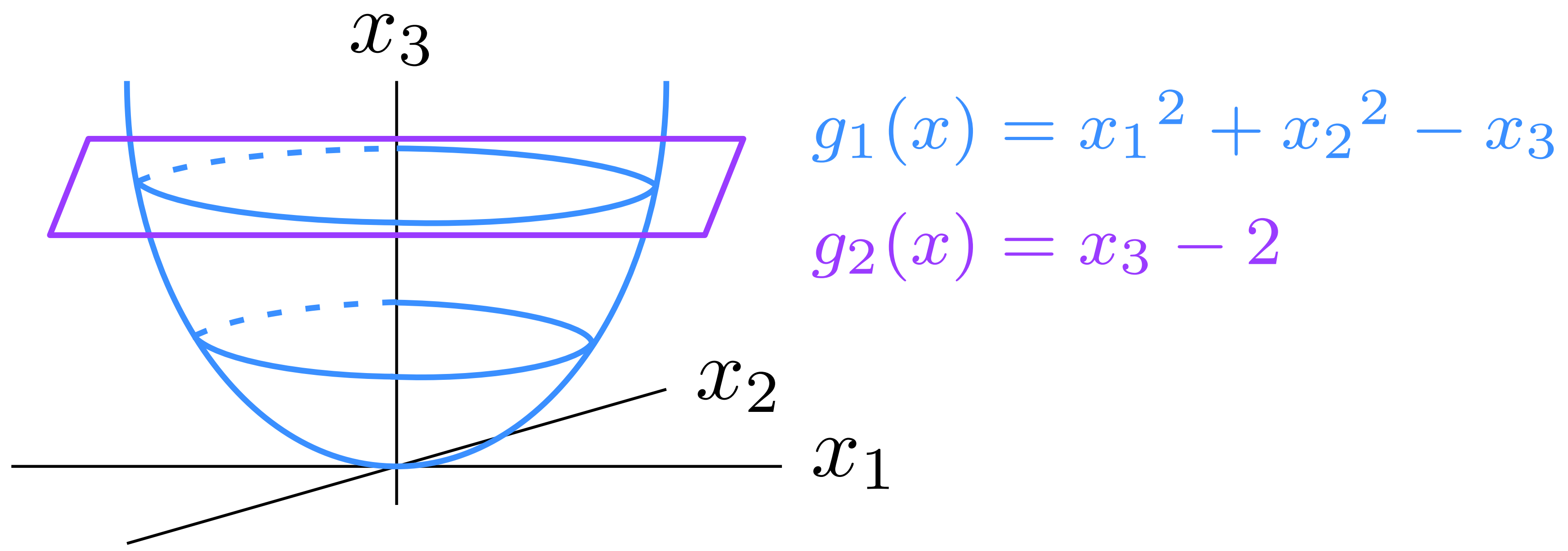
Lagrange Multipliers:

A maximum of $f(x)$ subject to $g_1(x) = g_2(x) = \ldots = g_m(x) = 0$ satisfies

$$\nabla f(x) = \sum_{i=1}^{m} y_i \nabla g_i(x)$$

(Functions are $\mathbb{R}^n \to \mathbb{R}$, gradients are row vectors.)





The primal-dual Central Path (equational form)

Apply Lagrange multipliers to

$$g_i(x) = a_i x - b_i$$

Maximize $\quad f_\mu(x) = c^T x + \mu \sum_{j=1}^{m} \ln(x_j) \quad$ subject to $\; Ax = b, \; x \geq 0$

to get $\quad c + \mu \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) = \nabla f(x) = \sum_{i=1}^{m} y_i \nabla g_i(x) = \sum_{i=1}^{m} y_i a_i = A^T y$

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Introduce the nonnegative vector $s = \mu \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n} \right)$ to get

$$\begin{gathered} Ax = b \\ A^T y - s = c \\ (s_1 x_1, s_2 x_2, \ldots, s_n x_n) = (\mu, \mu, \ldots, \mu) \qquad \text{(Not linear)} \\ x, s \geq 0 \qquad y \in \mathbb{R}^m \end{gathered}$$

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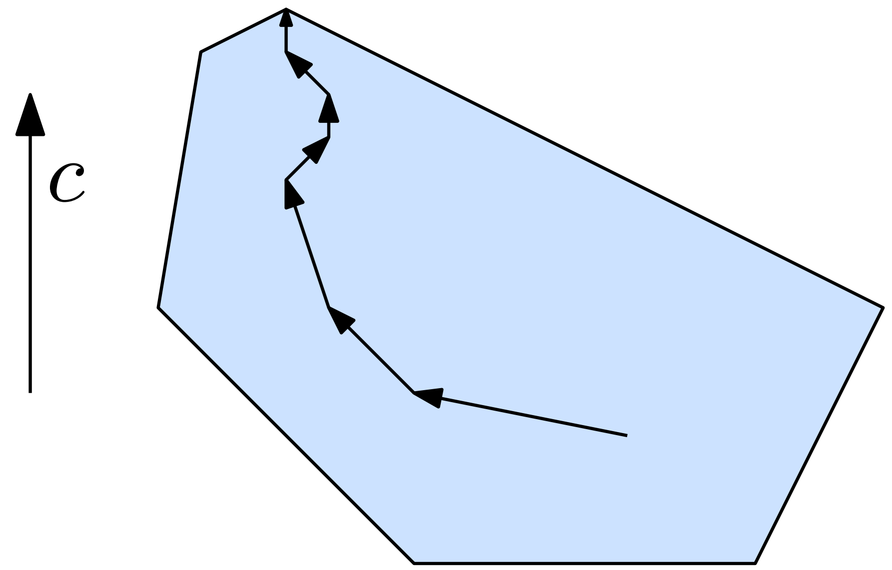
The primal-dual central path is

$$\left\{ (x^*(\mu), y^*(\mu), s^*(\mu)) \in \mathbb{R}^{2n+m} : \mu > 0 \right\}$$



The primal-dual Central Path (equational form)

In some sense, Lagrange multipliers recover the duality of linear programming!

We started with

Maximize $c^T x$ subject to $Ax = b,\ x \geq 0$

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We derived

$$\boxed{\begin{array}{rcll} Ax &=& b & \\ A^T y - s &=& c & \\ (s_1 x_1, s_2 x_2, \ldots, s_n x_n) &=& (\mu, \mu, \ldots, \mu) & \text{(Not linear)} \\ x, s &\geq& 0 & y \in \mathbb{R}^m \end{array}}$$

for $\mu > 0$, but setting $\mu = 0$ gives $(s_1 x_1, \ldots, s_n x_n) = (0, \ldots, 0)$, i.e.,

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So
$$\begin{aligned} 0 &= s^T x \\ &= \left(A^T y - c\right)^T x \\ &= y^T A x - c^T x \\ &= y^T b - c^T x \qquad \text{Since } Ax = b \end{aligned}$$

So x is a feasible solution of the primal,

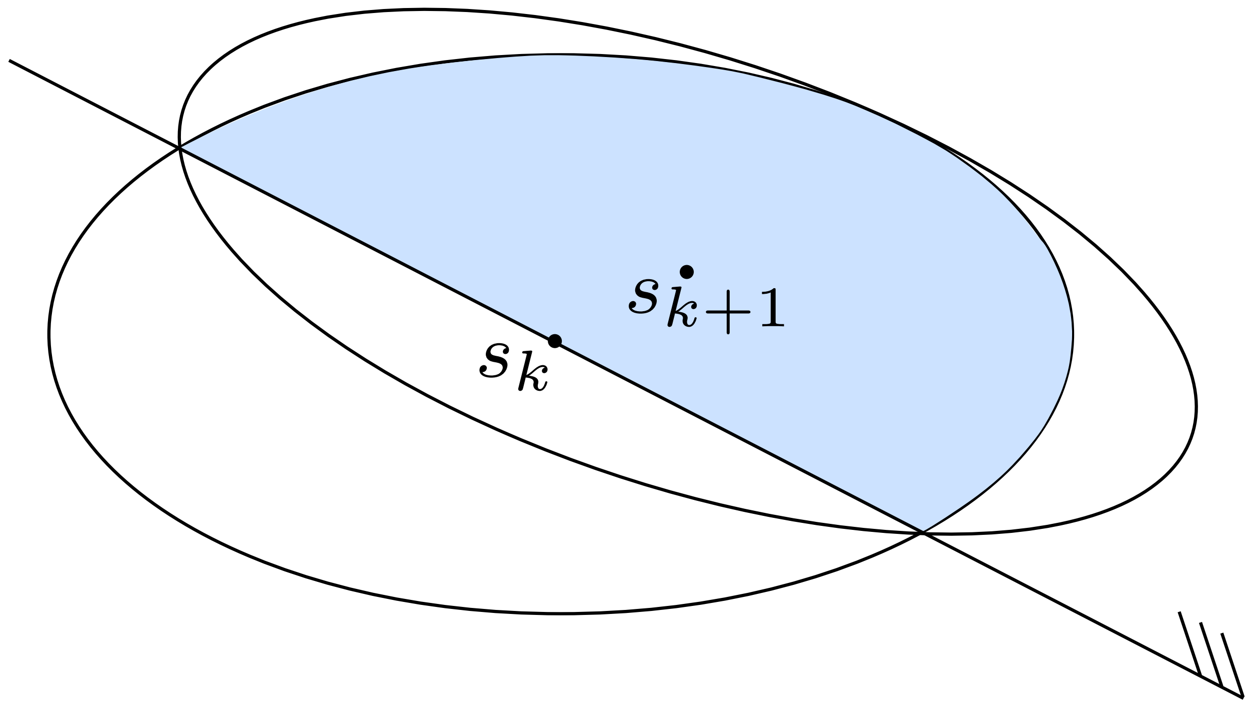
y is a feasible solution of the dual (with slack variables s_j)

and $y^T b = c^T x$ implies these feasible solutions are optimal, by strong duality.

Summary

Ellipsoid method:

find smaller and smaller ellipsoids containing set of feasible solution inside; cutting by halfspaces



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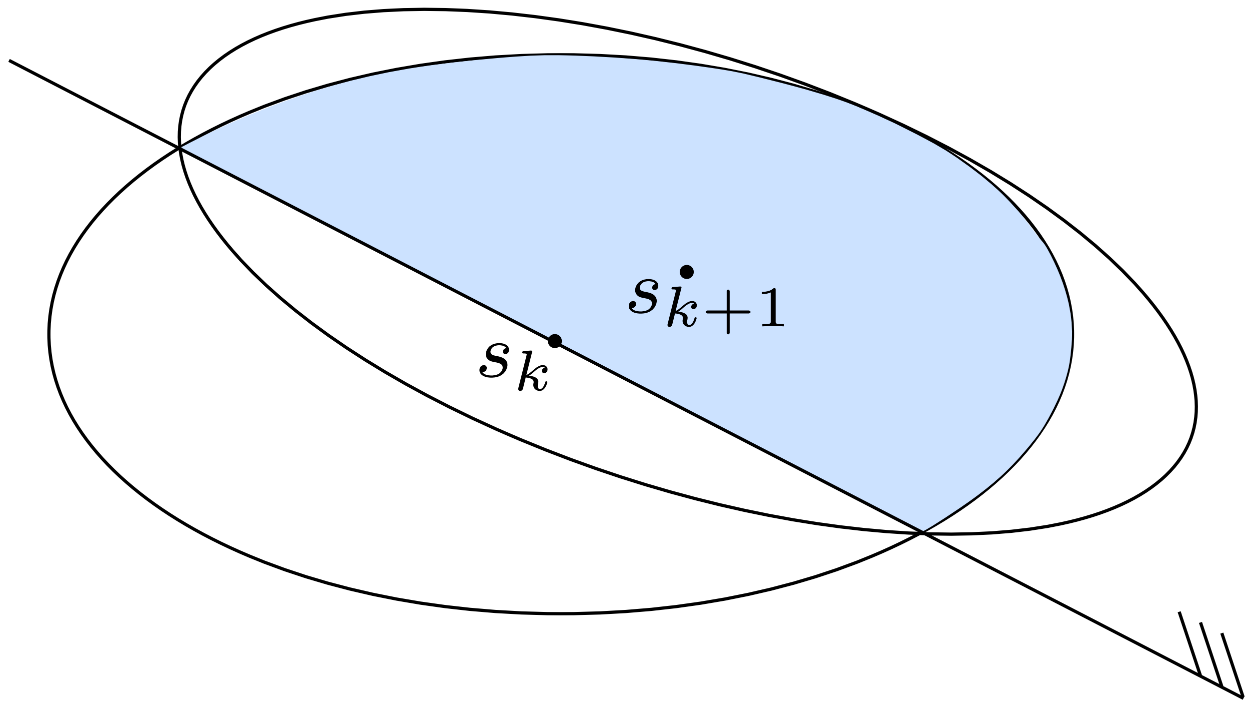

Interior point method – central paths:

force solution inside by adding a “barrier” to the objective function. Central path: by lowering effect μ of barrier moves towards LP solution. Uses Lagrange Multipliers.