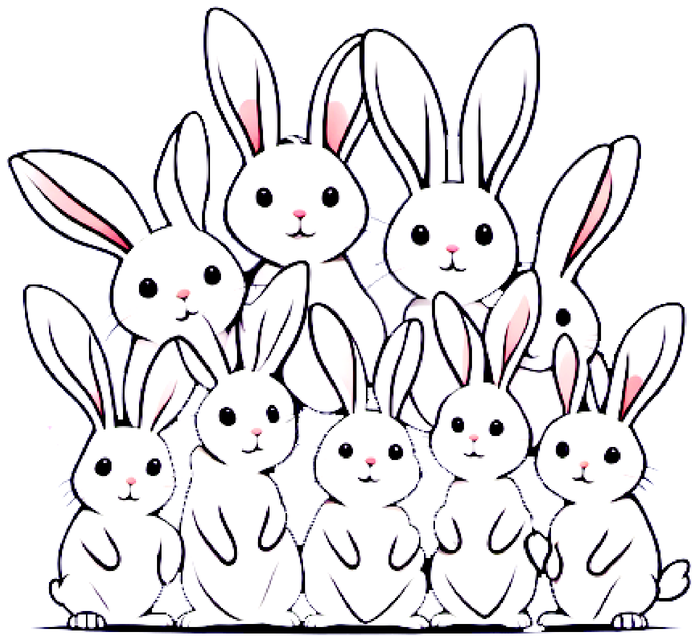
Fibonacci Heaps



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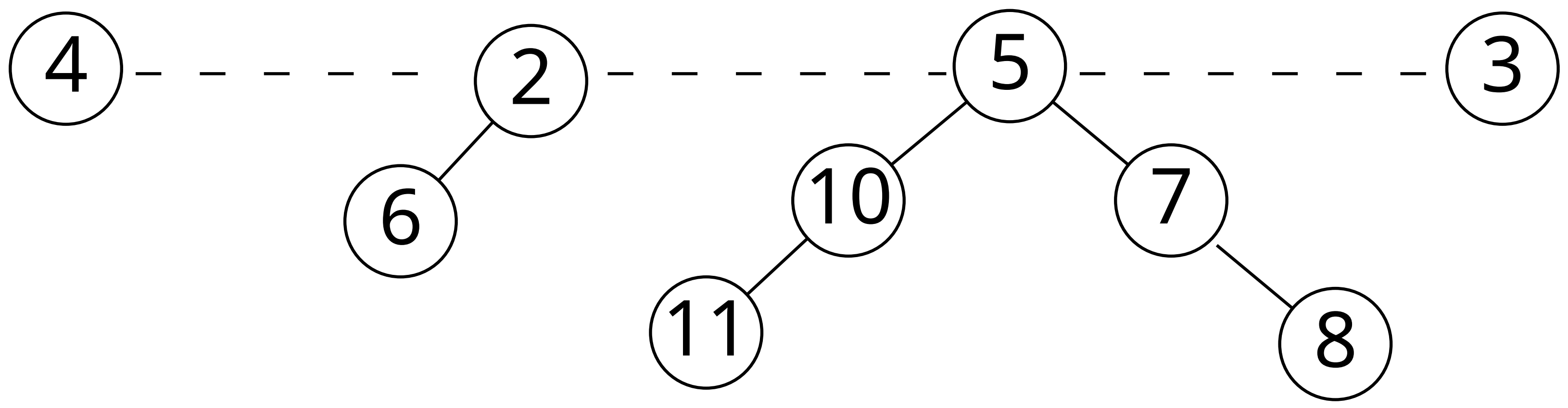
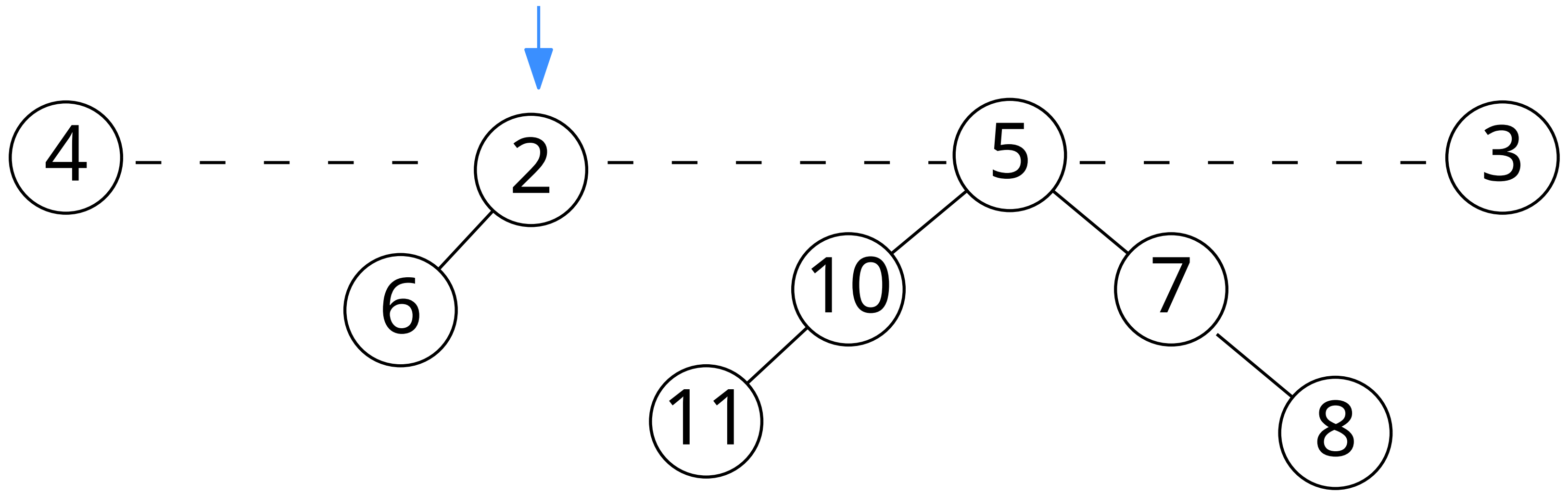


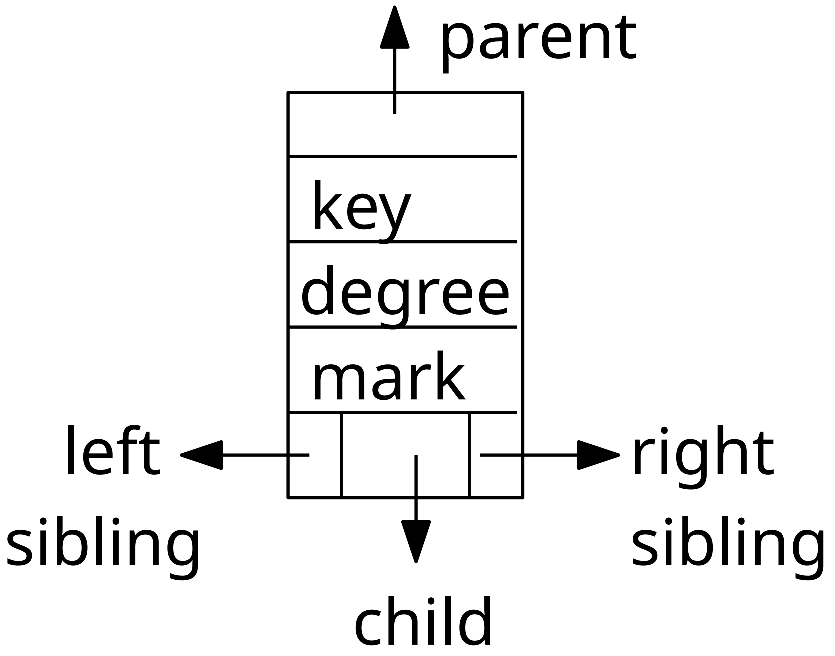
Fibonacci Heaps

Goal: Efficiently mergeable heaps with efficient decreaseKey

Fibonacci Heap is a set of trees with min-heap property, in it

- each node has four pointers, one each to parent, child, left sibling, right sibling as well as two attributes: degree and mark
- roots of all trees are stored in a doubly linked list
- pointer to minRoot in list





Idea: like binomial heaps, but more flexible structure; link trees of equal degree

Fibonacci Heaps

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Fibonacci Heap is a set of trees with min-heap property,

We will use the potential method for the amortized analysis with potential function

$$\Phi(H) = \underbrace{t(H)}_{\#\text{ trees}} + 2\underbrace{m(H)}_{\#\text{ marks}}$$

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We will analyze the amortized cost with respect to $D(n)$, which is the maximum degree of a node in a Fibonacci heap on n nodes. Afterwards we will bound

$$D(n) = O(\log n).$$

Operations

Idea: Operations (except for decreaseKey) like Binomial Heap with lazy union

Amortised costs: $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ $\quad\quad \Phi(H) = \underbrace{t(H)}_{\text{\# trees}} + 2\underbrace{m(H)}_{\text{\# marks}}$

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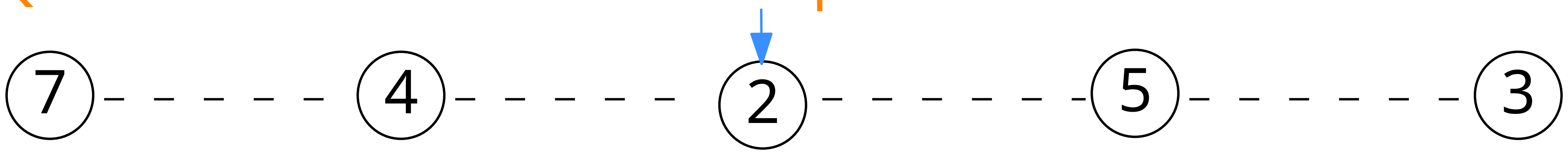

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$$\leq \underbrace{(D(n) + 1 + 2m(H))}_{\text{potential after}} - \underbrace{(t(H) + 2m(H))}_{\text{potential before}} = D(n) + 1 - t(H)$$

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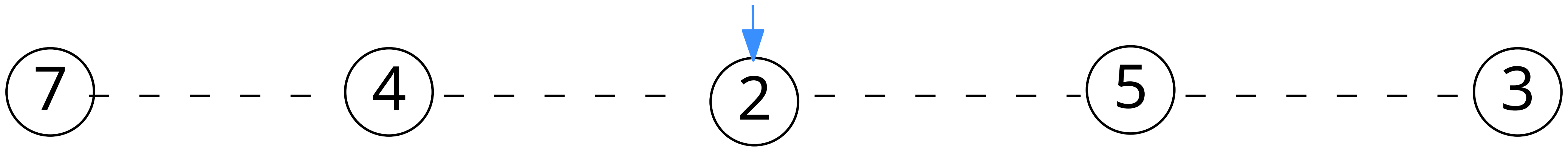

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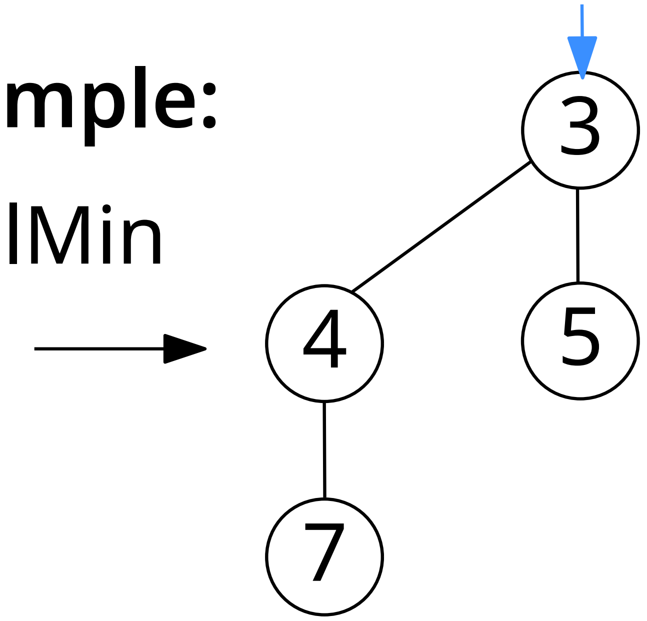

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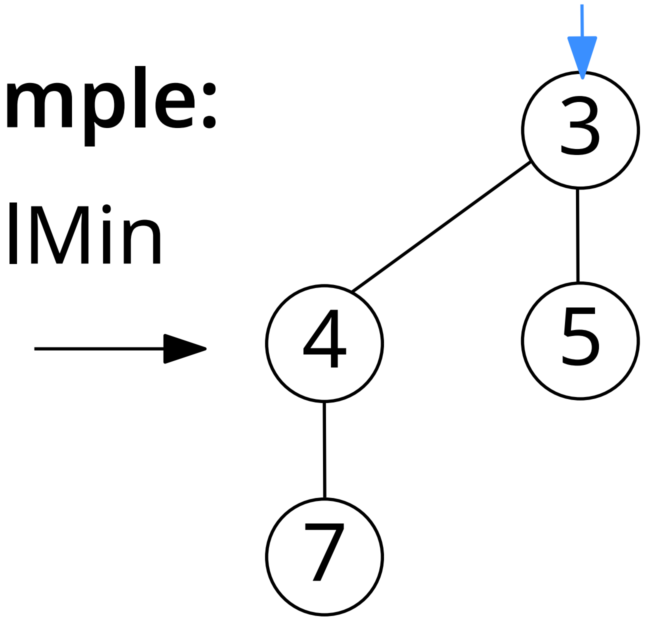
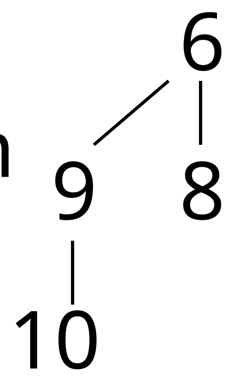


union



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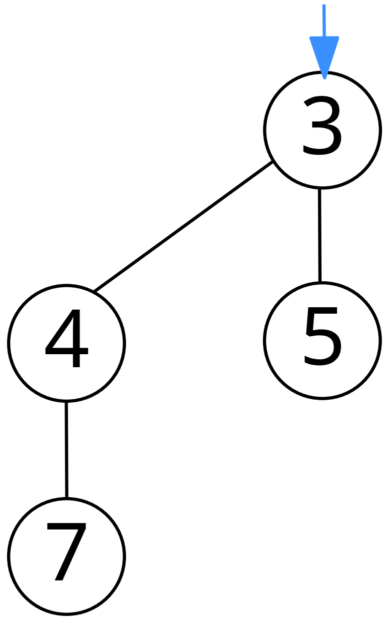
Example:

delMin $\longrightarrow$



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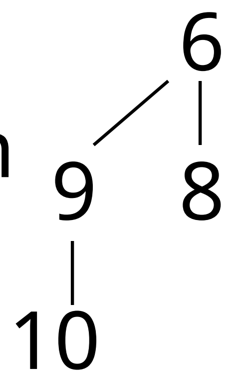

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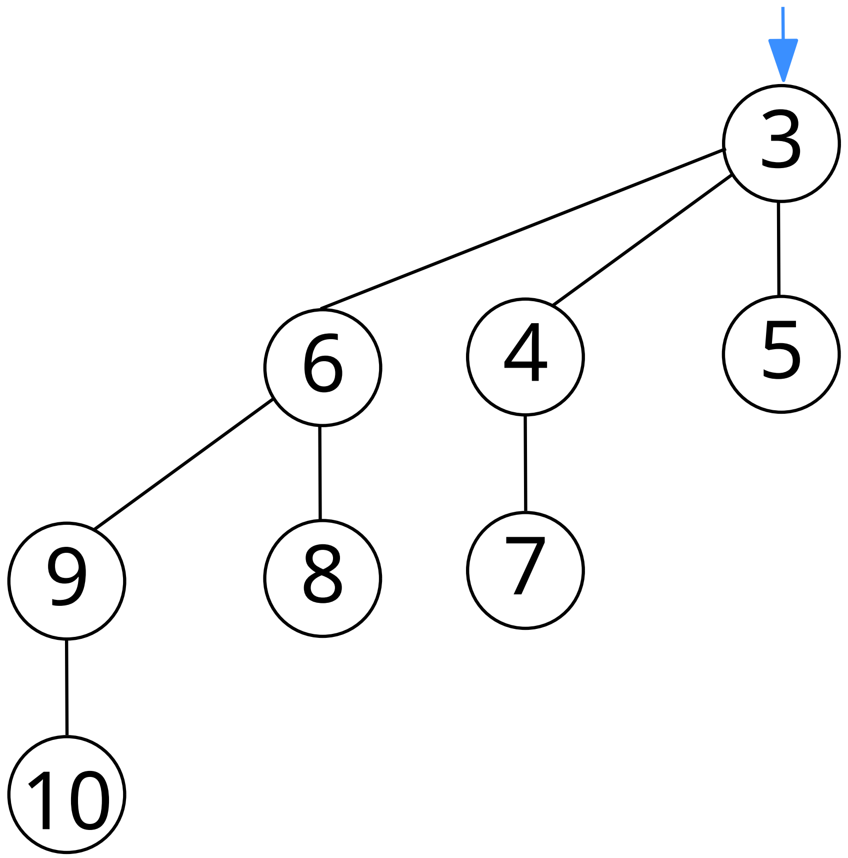

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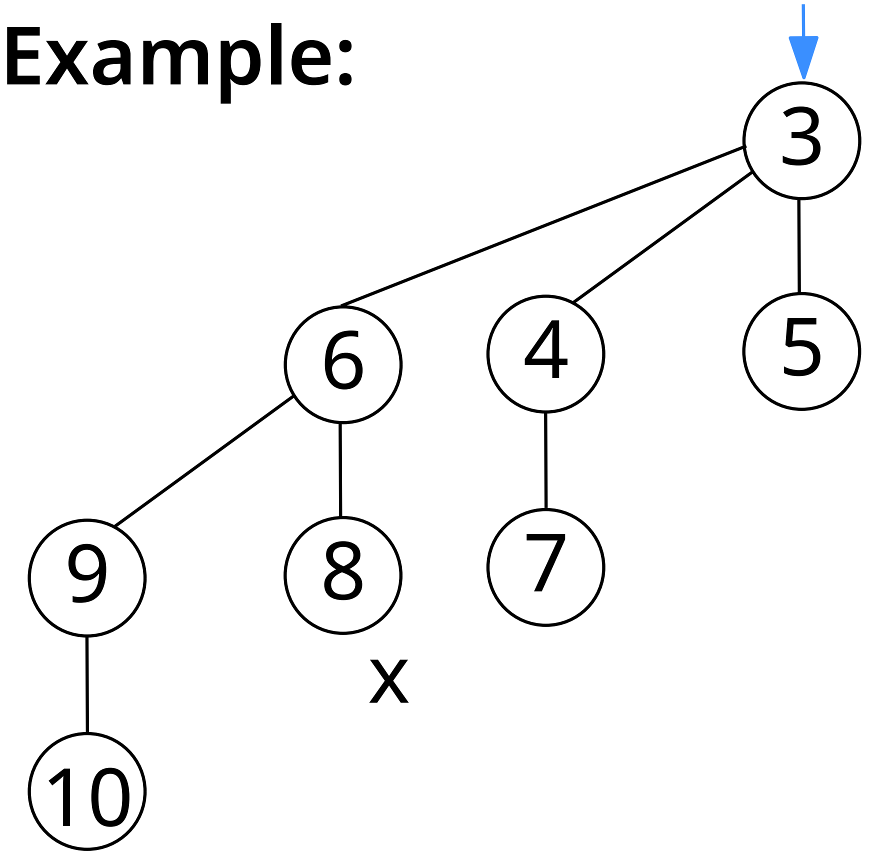

decKey(x,2)

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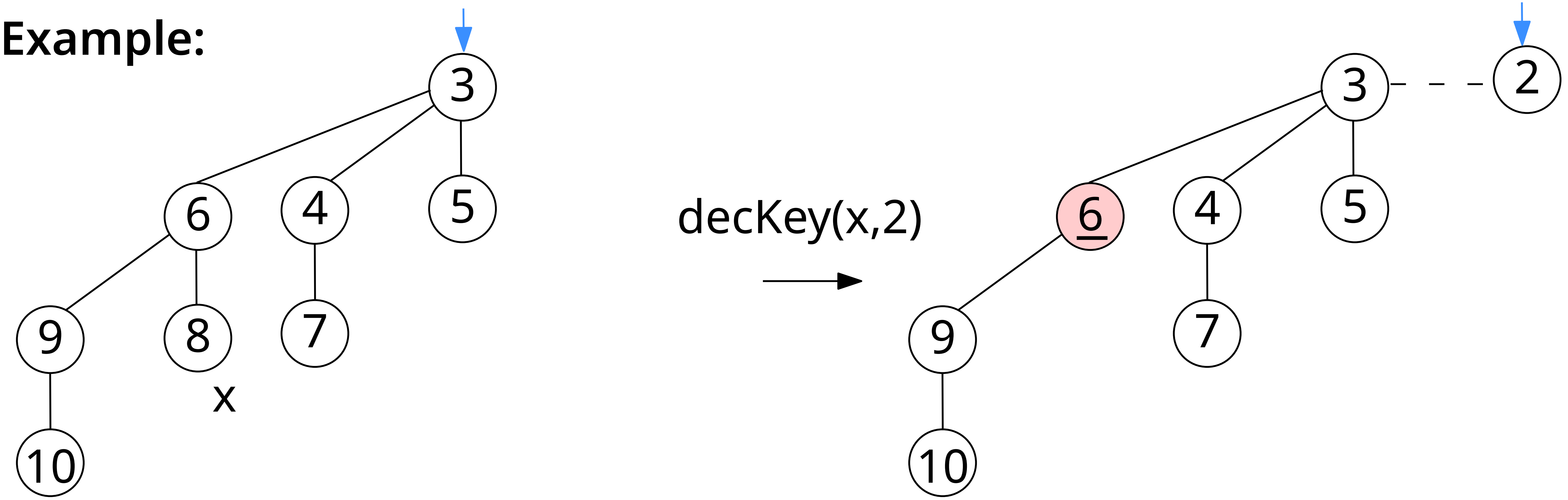

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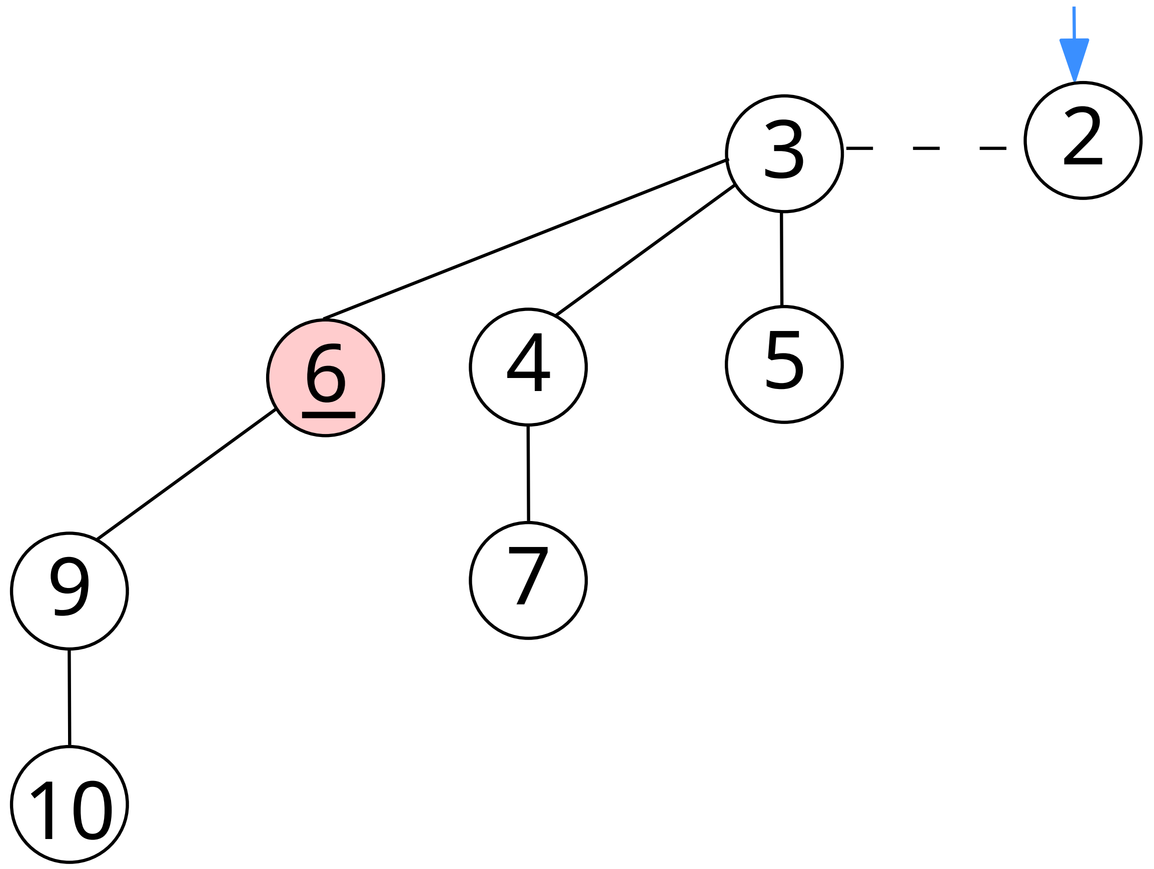

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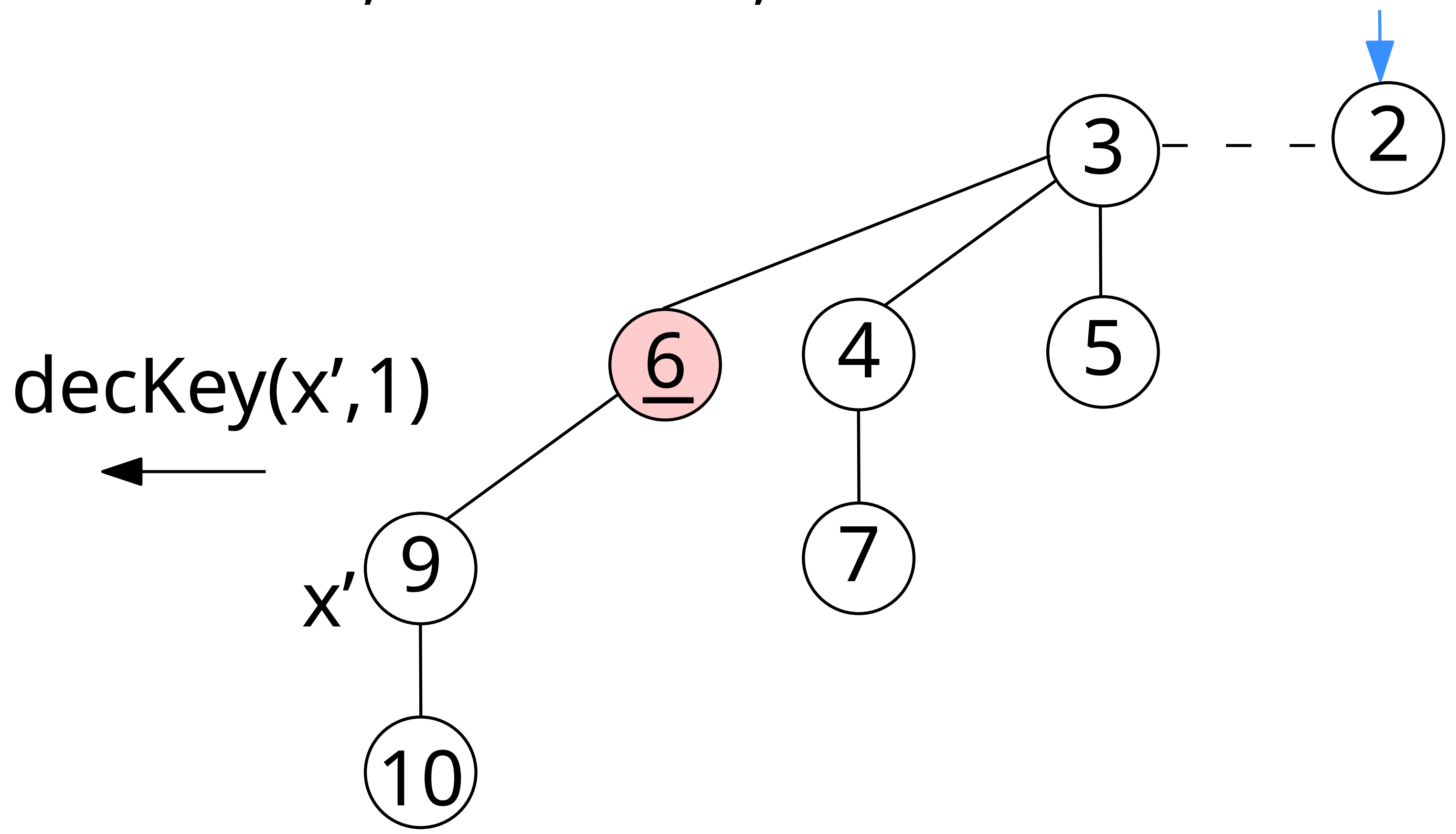

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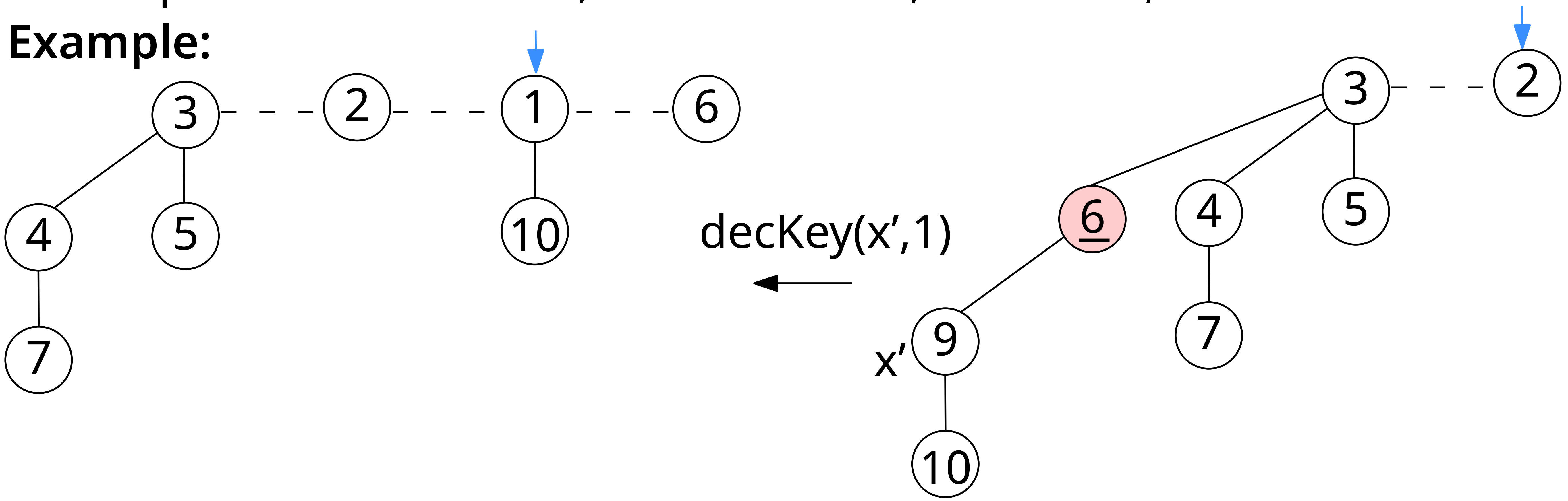

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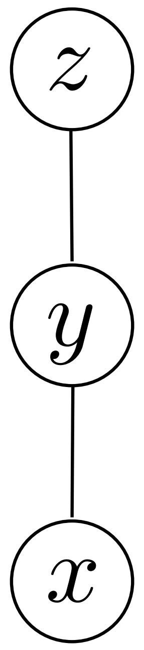
Amortised costs: $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ $\quad\quad \Phi(H) = \underbrace{t(H)}_{\text{\# trees}} + 2\underbrace{m(H)}_{\text{\# marks}}$

- decKey:

decKey(x, k)
  x.key $= k$
  $y = x$.parent
  if $y \neq$ nil and x.key $< y$.key **then**
    cut(x)
    cascade-cut(y)
  if $k <$ min **then**
    min $= k$

cascade-cut(y)
  $z = y$.parent
  if $z \neq$ nil **then**
    if y.mark = false **then**
      y.mark = true
    else
      cut(y)
      cascade-cut(z)



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change in potential:

$$\leq \underbrace{t(H) + \underset{\substack{\uparrow\\ \text{new trees}}}{k} + 2(m(H) \underline{- k + 2})}_{\text{potential after}} - \underbrace{(t(H) + 2m(H))}_{\text{potential before}} = 4 - k$$

$-(k-1)$ for cascading cuts, $+1$ new mark

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hence amortised cost $\hat{c}_i = O(k) + 4 - k = O(1)$

again by appropriate scaling of the potential

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one for paying for a cut, one for paying for the new tree

Runtimes

	amortised
make	$O(1)$
min	$O(1)$
insert	$O(1)$
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Can we bound $D(n)$?

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if we would never cut $\rightarrow$ Binomial heap

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size(x) = 2^k (by induction).

size of subtree with x as root

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if we cut, we can cut at most one child per node

then: degree of any node decreases by at most 1,

inductive argument still works for size(x) $\geq \phi^k$.

(golden ratio)

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Q: Could we still bound $D(n)$ if we were allowed to cut more than 1 child per node?

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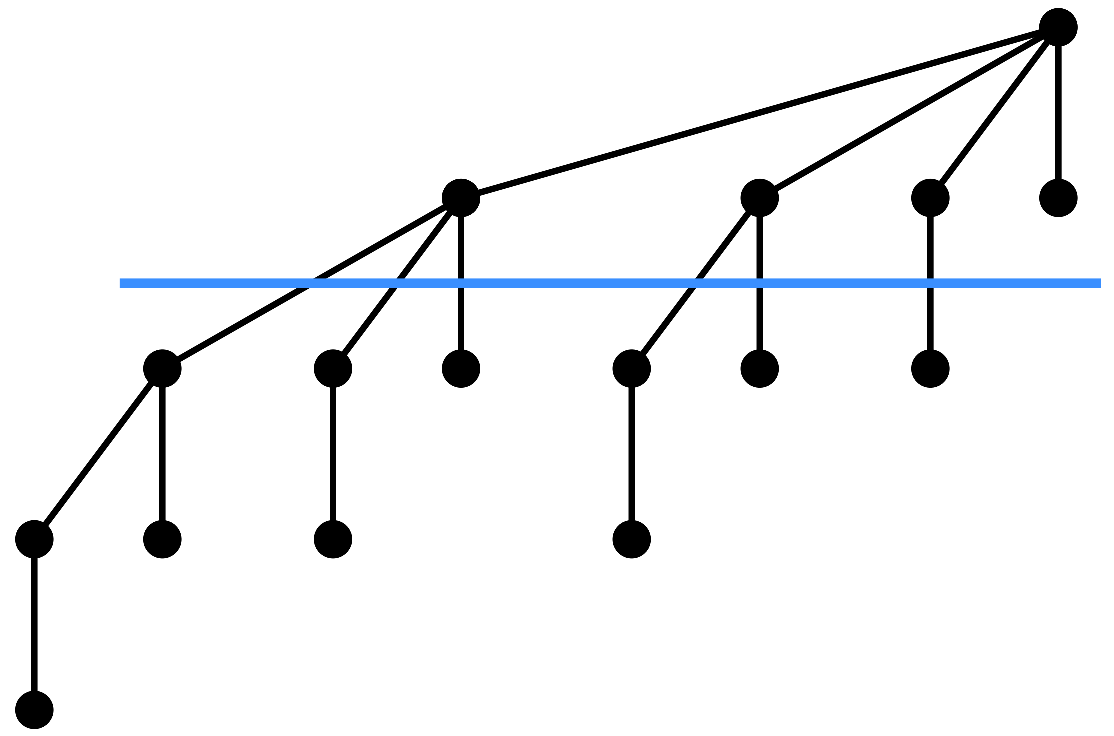

Bounding the maximal degree

Goal: show $D(n) \leq \lfloor \log_\phi n \rfloor$ where $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio.

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($k+2$)-nd Fibonacci number

Recall: $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots F_{k+2} = F_k + F_{k+1}$

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This implies $n \geq$ size$(x) \geq \phi^{D(n)}$ and thus $D(n) \leq \lfloor \log_\phi n \rfloor$.

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Lemma 1: Let x be an arbitrary node of degree k in a Fibonacci Heap H.
Let $y_1, \ldots, y_k$ be the children of x in order of age.
Then degree$[y_1] \geq 0$ and degree$[y_i] \geq i-2$ for $i = 2, \ldots, k$.

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Proof: The first statement is trivial.
For the second, observe that when y_i was linked to x, it was degree$[y_i]$ = degree$[x] \geq i - 1$. Then at most one child was deleted.

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Proof: by Induction(k)

$k = 0 : F_2 = 1 + 0 = 1 + F_0$

$k > 0 : F_{k+2} = F_k + F_{k+1} = F_k + 1 + \sum_{i=0}^{k-1} F_i = 1 + \sum_{i=0}^{k} F_i.$

ind. hyp.

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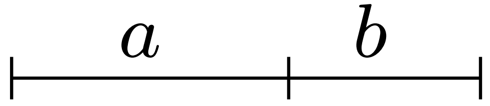
$k = 0 : F_2 = 1 = \phi^0$, $k = 1 : F_3 = 2 > \phi^1$
$k > 2 : F_{k+2} = F_{k+1} + F_k \underset{\text{ind. hyp.}}{\geq} \phi^{k-1} + \phi^{k-2} = \phi^{k-2}\underbrace{(\phi + 1)}_{= \phi^2} = \phi^k$

Recall:


$\phi = \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{1}{\phi}$
$\Rightarrow \phi^2 = 1 + \phi$

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Lemma 4: Let x be an arbitrary node of degree k in a Fibonacci Heap H.
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Q: What is s_0 and s_1?

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Now let z be a node of degree k and size s_k, and let $y_1, \ldots, y_k$ be the ordered children of z. Then size$(x) \geq s_k \underset{z,\, y_1}{\geq} 2 + \sum_{i=2}^{k} s_{deg(y_i)} \underset{\text{Lm 1}}{\geq} 2 + \sum_{i=2}^{k} s_{i-2}$.

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For $k \geq 2$ we have

$$s_k = 2 + \sum_{i=2}^{k} s_{i-2} \underset{\text{ind. hyp.}}{\geq} 2 + \sum_{i=2}^{k} F_i = 1 + \sum_{i=0}^{k} F_i \underset{\text{Lm 2}}{=} F_{k+2}$$

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Corollary 5: $D(n) = O(\log n)$

Comparison of Runtimes

	Binary Heap	Binomial Heap lazy union	Fibonacci Heap
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min	$O(1)$	$O(1)$	$O(1)$
insert	$O(\log n)$	$O(1)$	$O(1)$
union	$O(n)$	$O(1)$	$O(1)$
deleteMin	$O(\log n)$	$O(\log n)^*$	$O(\log n)^*$
decKey	$O(\log n)$	$O(\log n)$	$O(1)^*$

* amortised