LP-based Approximation Algorithms for SetCover

Reminder: (Integer) Linear Programs

Optimize (i.e., minimize or maximize) a linear (objective) function subject to linear inequalities (constraints).

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\textbf{minimize} & c^{\top} x & & \\
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- Important ingredient for approximation: lower bound

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Which lower bounds do we know for the optimum of an ILP?

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LP-based approximation techniques

1. LP Rounding
2. Primal-Dual Method
3. Dual Fitting

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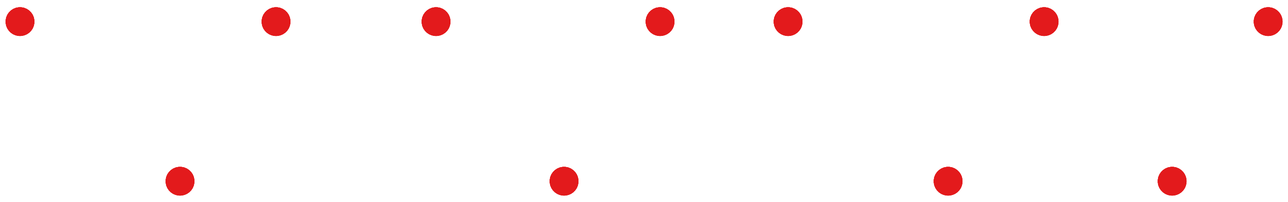
Example: Set Cover

SetCover as an ILP

Ground set U



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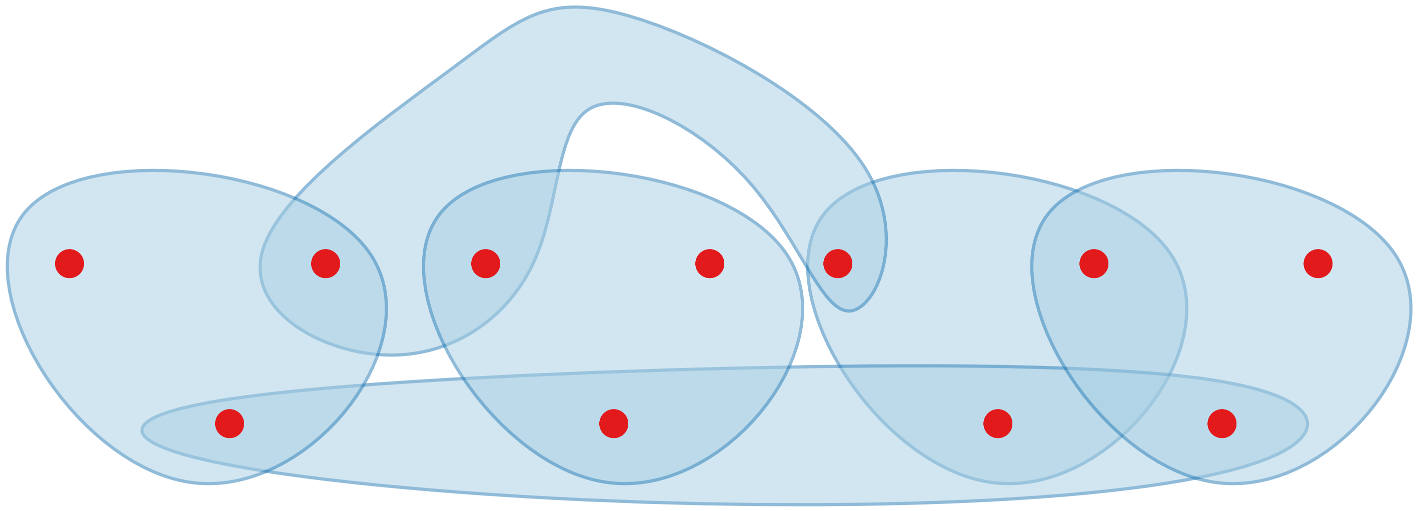
Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$



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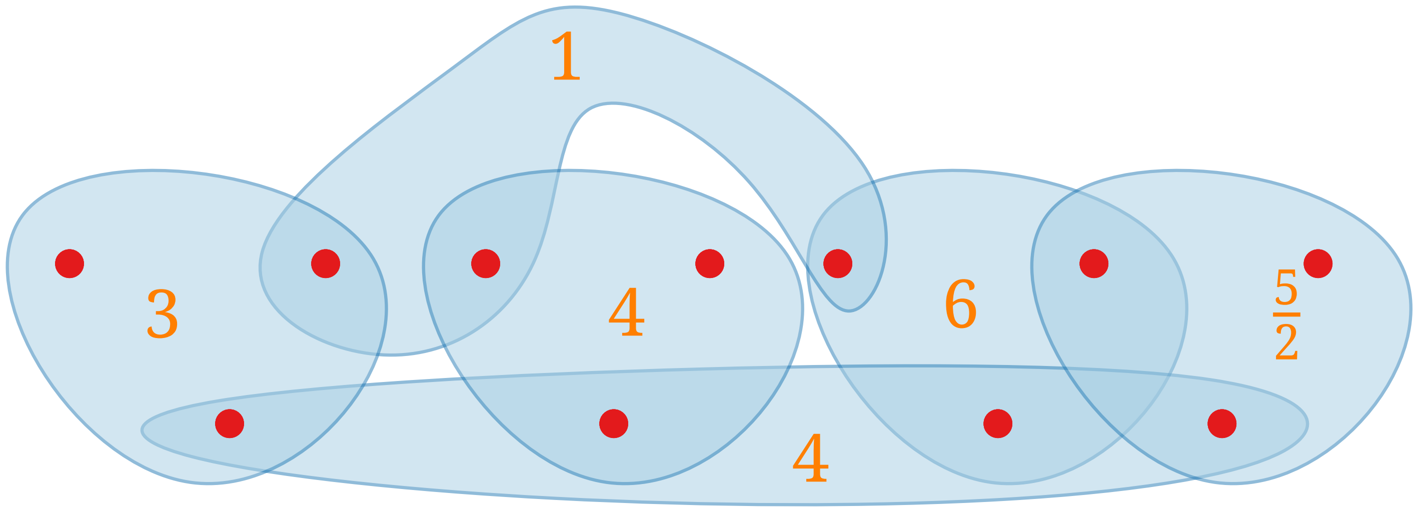

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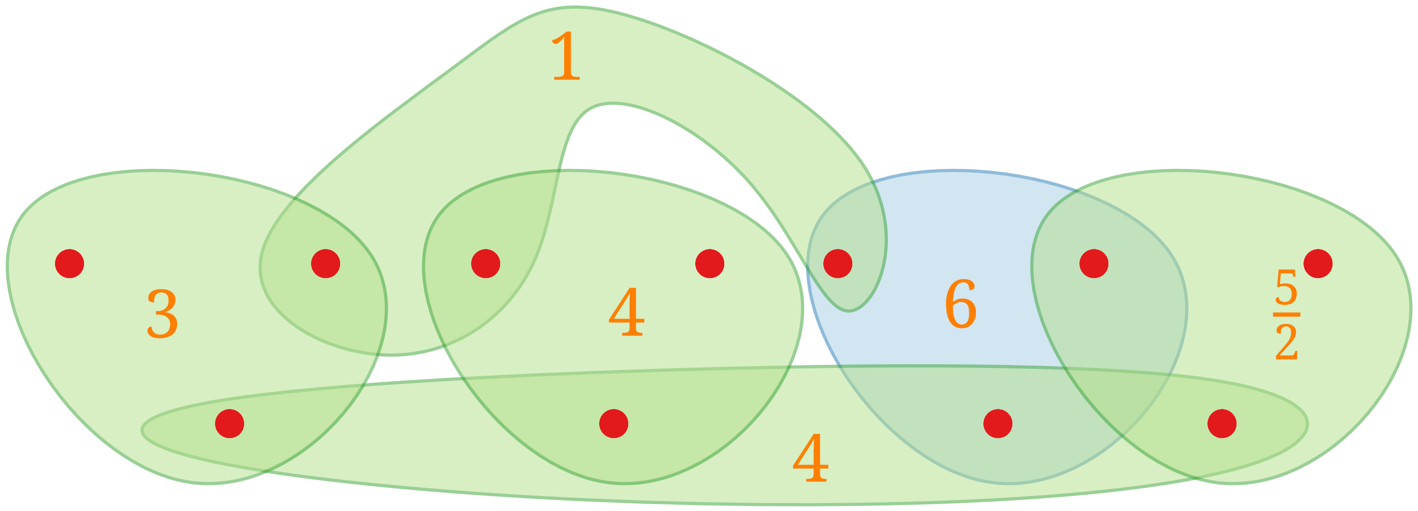

Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

SetCover as an ILP

minimize

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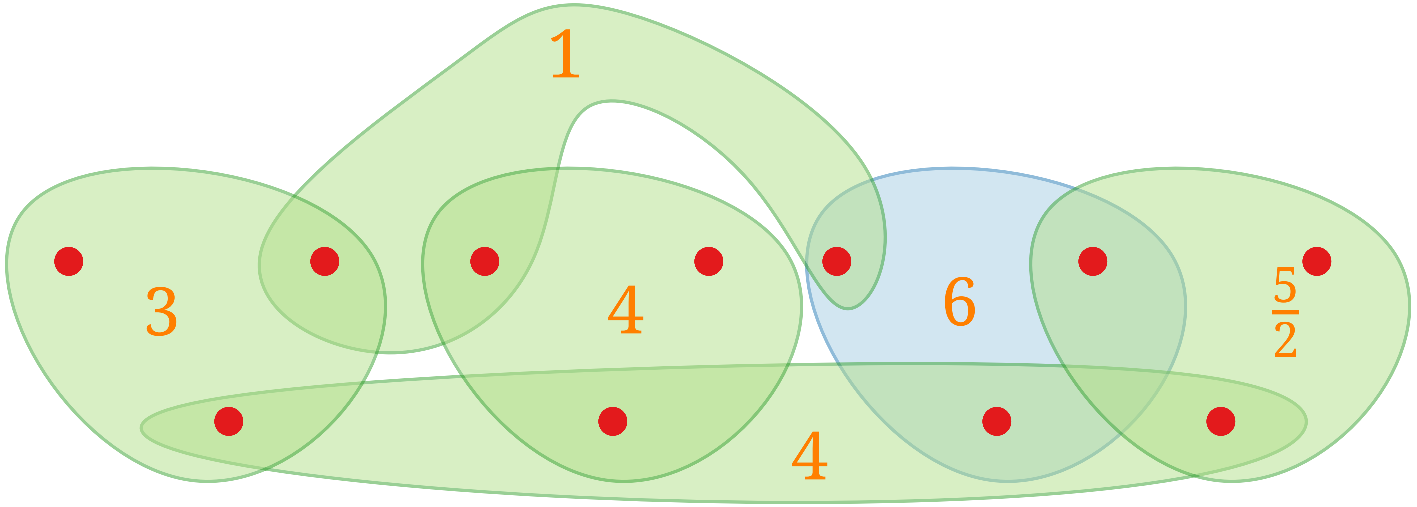

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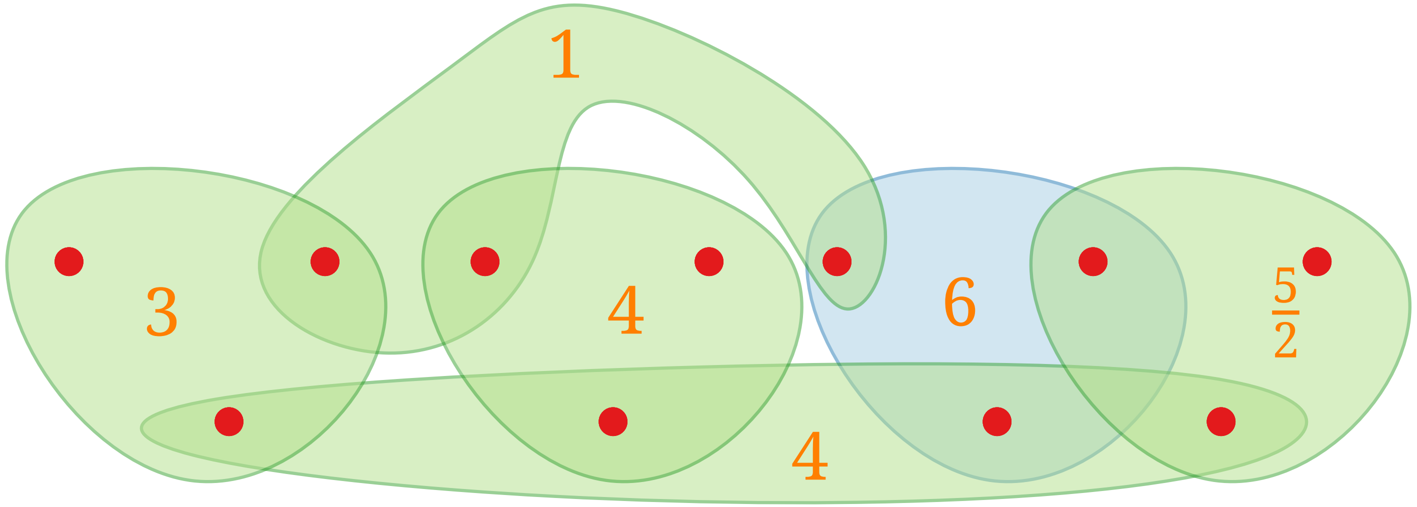

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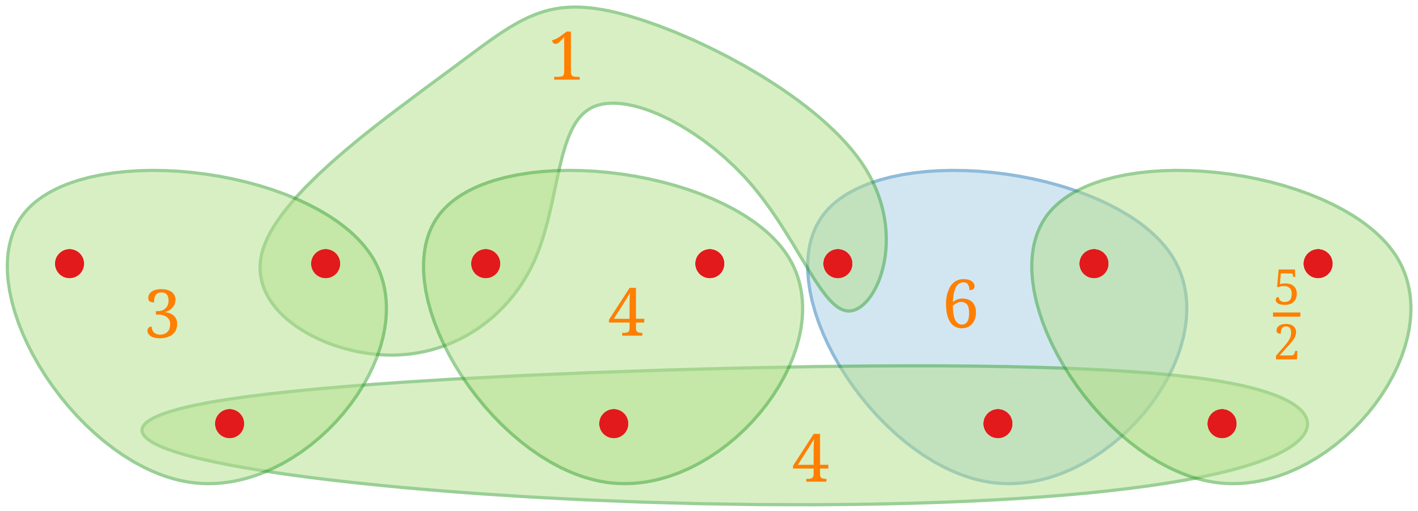

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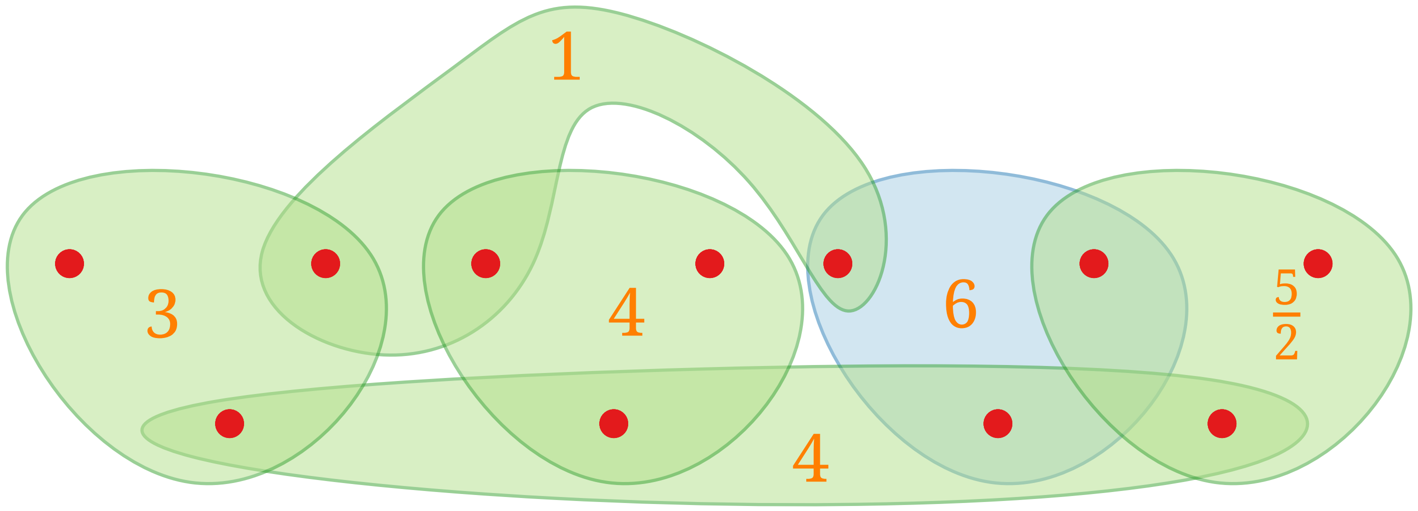

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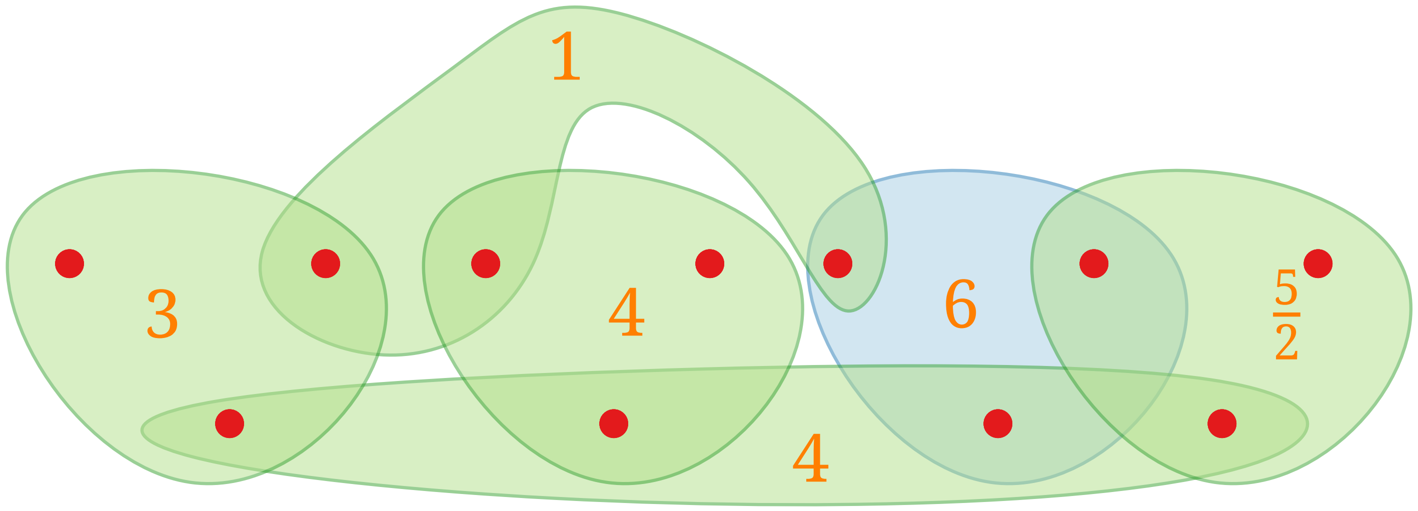

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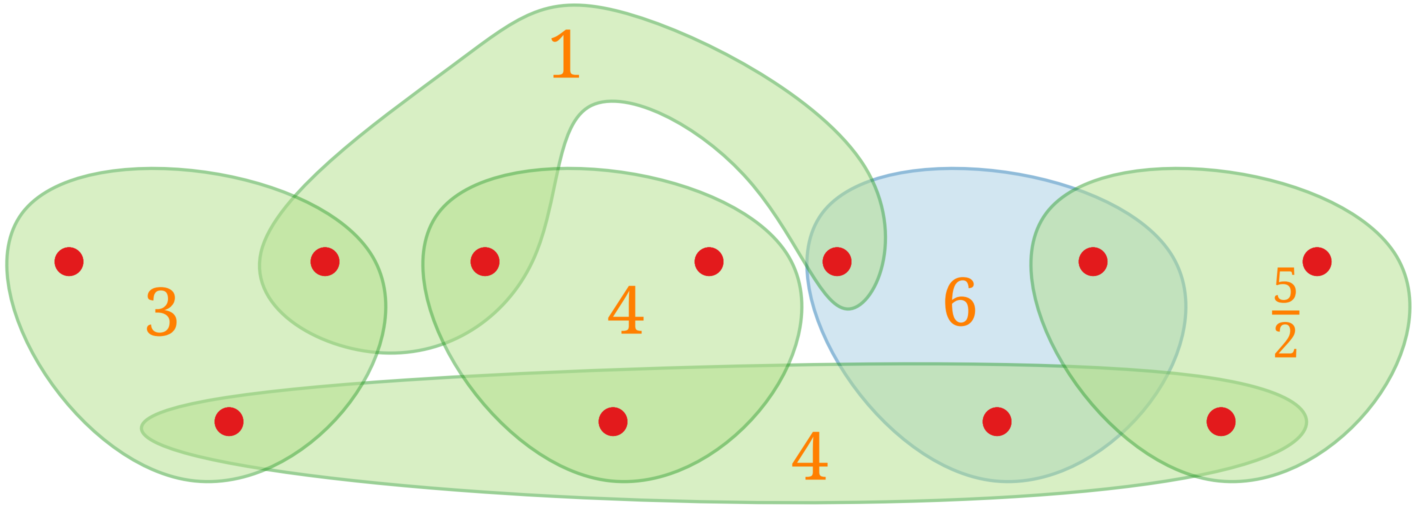

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Relaxed LP: $0 \leq x_S (\leq 1)$ instead
but how does it help?

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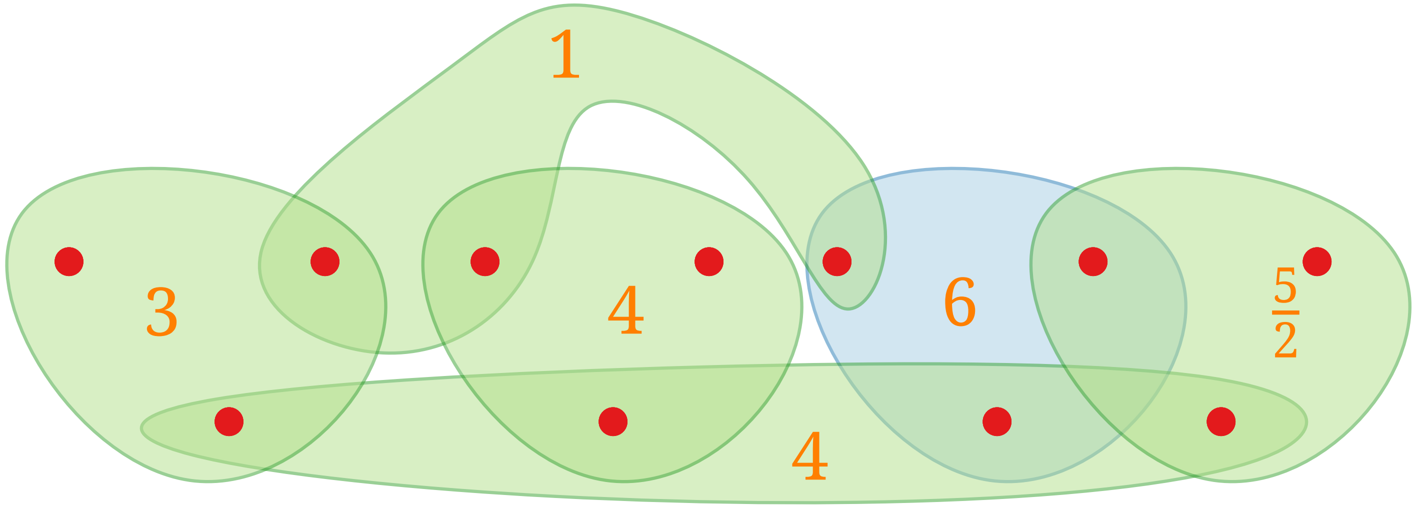

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LP Rounding

Technique I) LP-Rounding

Consider a minimization problem Π in ILP form.

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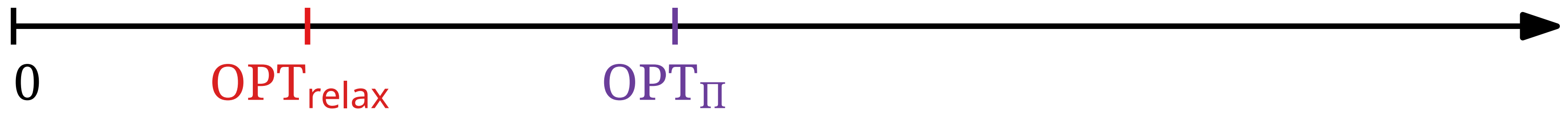

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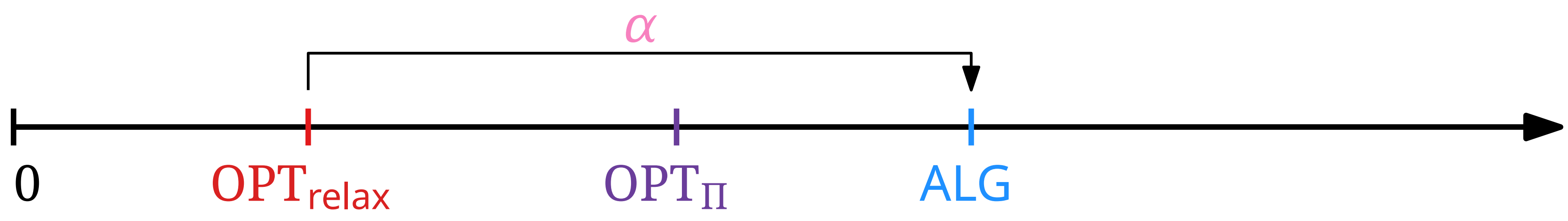

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Approximation factor: $\mathrm{ALG}/\mathrm{OPT}_{\Pi} \leq \mathrm{ALG}/\mathrm{OPT}_{\text{relax}}$.

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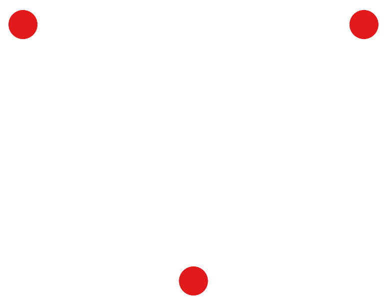

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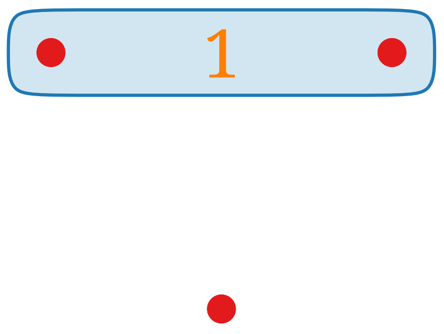

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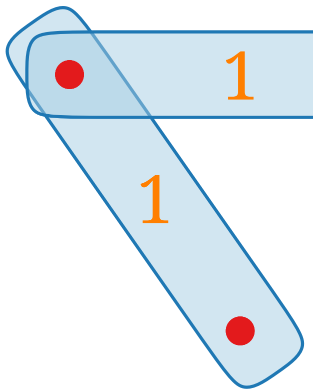

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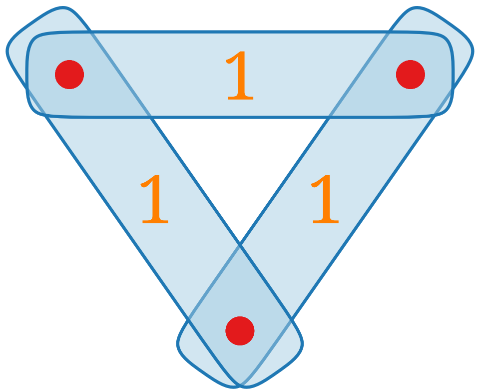

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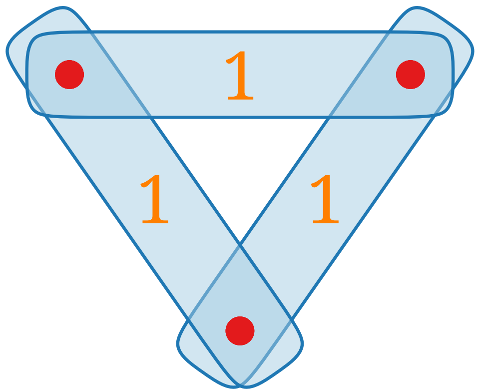

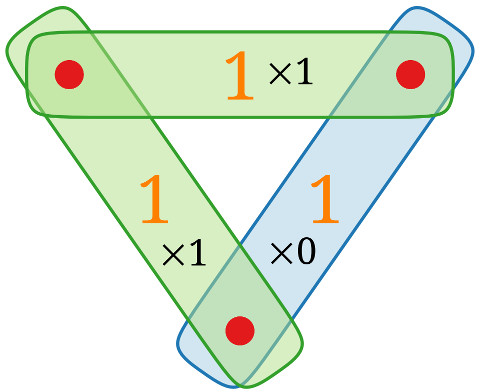

integer: 2

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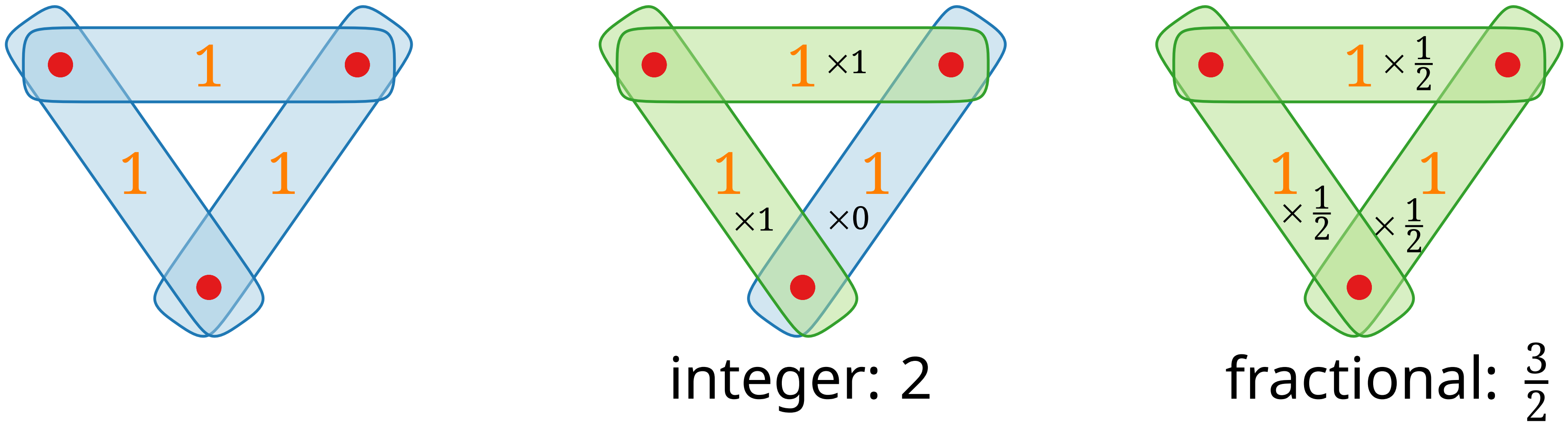

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LP-Rounding-Attempt$(U, \mathcal{S}, c)$

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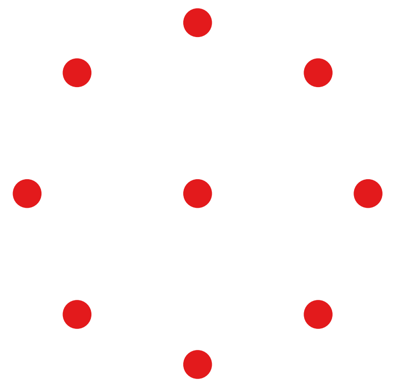

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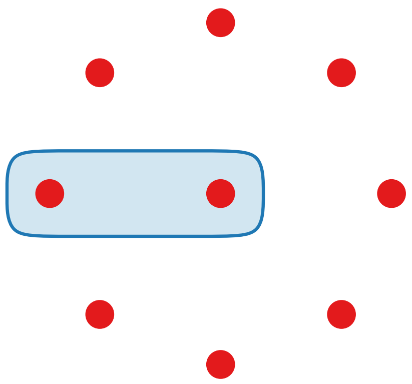

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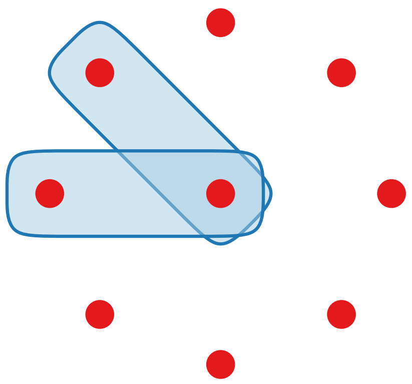

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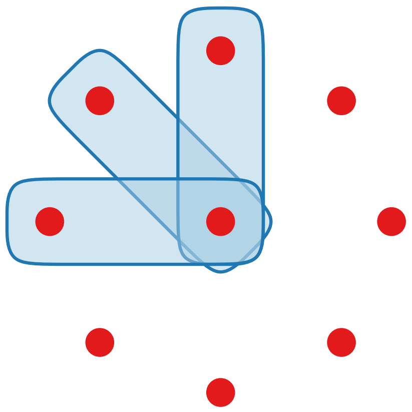

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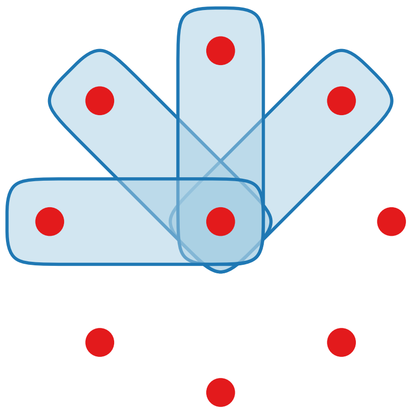

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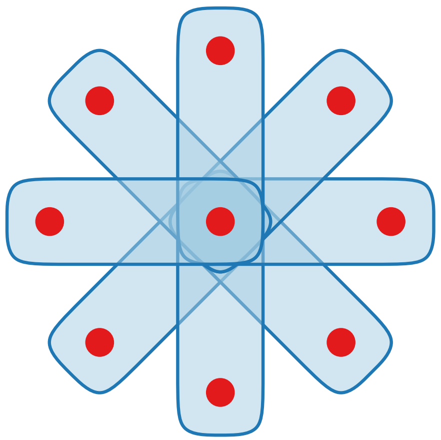


LP-Rounding: First Attempt

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\
\textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\
& x_S \geq 0 & S \in \mathcal{S}
\end{array}
$$

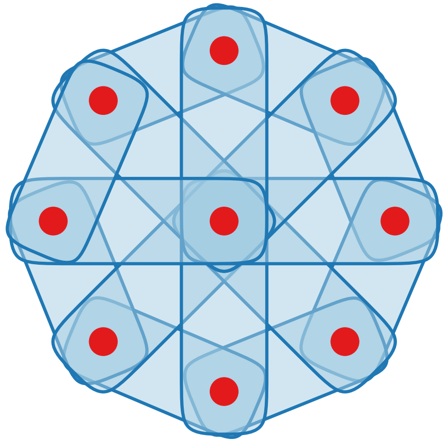
LP-Rounding-Attempt$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

– Generates a valid solution.

– Scaling factor arbitrarily large.



LP-Rounding: First Attempt

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S & \\
\textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 & u \in U \\
& x_S \geq 0 & S \in \mathcal{S}
\end{aligned}$$

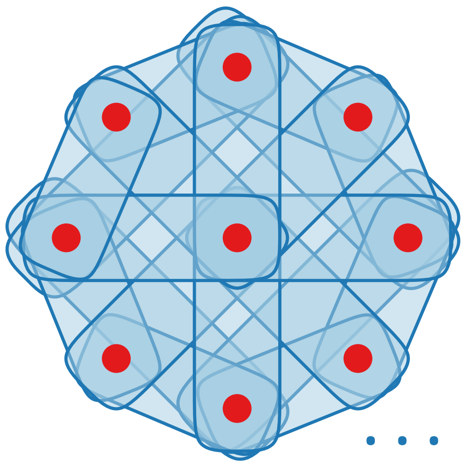
LP-Rounding-Attempt$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

– Generates a valid solution.

– Scaling factor arbitrarily large.



LP-Rounding: First Attempt

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding-Attempt$(U, \mathcal{S}, c)$

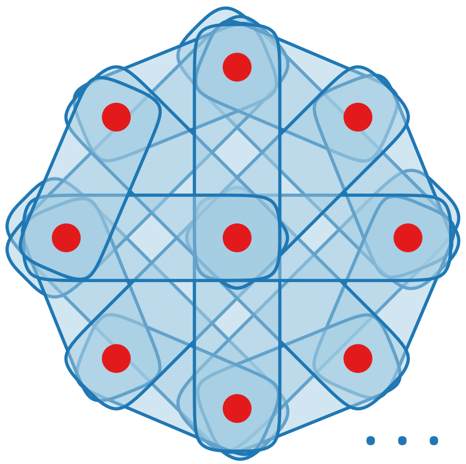
Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

– Generates a valid solution.

– Scaling factor arbitrarily large.

Use frequency f



LP-Rounding

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{aligned}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Why is this feasible?

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.
Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

Why is this feasible?

sum of $\leq f$ elements
$\rightarrow$ at least one $\geq 1/f$
by pigeonhole principle

LP-Rounding

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 && u \in U \\ & x_S \geq 0 && S \in \mathcal{S} \end{aligned}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.
Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

What is the approximation factor?

Why is this feasible?
sum of $\leq f$ elements
$\rightarrow$ at least one $\geq 1/f$
by pigeonhole principle

LP-Rounding

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad\quad\; S \in \mathcal{S} \end{aligned}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

What is the approximation factor?

the rounded solution is bounded by f times the fractional solution

Why is this feasible?

sum of $\leq f$ elements

$\rightarrow$ at least one $\geq 1/f$

by pigeonhole principle

LP-Rounding

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.
Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

Theorem. LP-Rounding gives a factor-f approximation algorithm for SetCover.

LP-Rounding

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

Theorem. LP-Rounding gives a factor-f approximation algorithm for SetCover.

Note: Factor-$O(\log n)$ can be achieved with randomized rounding (next lecture)

Reminder: LP Duality

LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

LP-Duality

primal

$$\begin{array}{llcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual

???

LP-Duality

primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

LP-Duality

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

minimize	$c^\intercal x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\intercal y$			*dual*
subject to	$A^\intercal y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

LP-Duality

minimize	$c^\intercal x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\intercal y$			*dual*
subject to	$A^\intercal y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad\qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

				primal
minimize	$c^\top x$			
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

				dual
maximize	$b^\top y$			
subject to	$\boxed{A^\top y \leq c}$			
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad\qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad\qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

minimize	$c^\top x$		*primal*
subject to	Ax	$\geq$	b
	$\boxed{x \geq 0}$		

maximize	$b^\top y$		*dual*
subject to	$\boxed{A^\top y \leq c}$		
	y	$\geq$	0

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i x_j \right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

primal

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	$\boxed{x \geq 0}$		

dual

maximize	$b^\top y$		
subject to	$\boxed{A^\top y \leq c}$		
	y	$\geq$	0

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i\right)} x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_i x_j\right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x \geq 0} & & \end{array}$$

dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & \boxed{A^\top y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i .$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i .$$

LP-Duality

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	$\boxed{x \geq 0}$			

maximize	$b^\top y$			*dual*
subject to	$\boxed{A^\top y \leq c}$			
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i\right)} x_j = \sum_{i=1}^{m} \left(\boxed{\sum_{j=1}^{n} a_{ij} x_j}\right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

		primal
minimize	$c^\top x$	
subject to	$Ax \geq b$	
	$x \geq 0$	

		dual
maximize	$b^\top y$	
subject to	$A^\top y \leq c$	
	$y \geq 0$	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i\right)} x_j = \sum_{i=1}^{m} \left(\boxed{\sum_{j=1}^{n} a_{ij} x_j}\right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

minimize	$c^\top x$	*primal*
subject to	$\boxed{Ax \geq b}$	
	$\boxed{x \geq 0}$	

maximize	$b^\top y$	*dual*
subject to	$\boxed{A^\top y \leq c}$	
	$\boxed{y \geq 0}$	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{i=1}^{m} \left(\boxed{\sum_{j=1}^{n} a_{ij} x_j} \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

LP-Duality

		primal
minimize	$c^\top x$	
subject to	$\boxed{Ax \geq b}$	
	$\boxed{x \geq 0}$	

		dual
maximize	$b^\top y$	
subject to	$\boxed{A^\top y \leq c}$	
	$\boxed{y \geq 0}$	

Theorem. (weak duality) If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i\right)} x_j = \sum_{i=1}^{m} \left(\boxed{\sum_{j=1}^{n} a_{ij} x_j}\right) y_i \overset{\geq}{\geq} \sum_{i=1}^{m} \boxed{b_i} y_i \,.$$

LP-Duality

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. (weak duality) If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Theorem. (strong duality) primal has a finite optimum $\Leftrightarrow$ dual has a finite optimum. If $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are optimal solutions for the primal and dual program, respectively, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \,.$$

Complementary Slackness

primal

$$\begin{array}{llcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual

$$\begin{array}{llcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

Complementary Slackness

primal

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

dual

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively. Then

Complementary Slackness

primal

$$\begin{array}{llll} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual

$$\begin{array}{llll} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Complementary Slackness

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Complementary Slackness

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Complementary Slackness

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Proof.

Complementary Slackness

minimize	$c^\top x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\top y$			*dual*
subject to	$A^\top y$	$\leq$	c	
	y	$\geq$	0	

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \dots, n$: $\quad x_j = 0 \quad$ or $\quad \sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \dots, m$: $\quad y_i = 0 \quad$ or $\quad \sum_{j=1}^{n} a_{ij} x_j = b_i$

Proof. Follows from LP-duality:

Complementary Slackness

minimize	$c^\intercal x$			*primal*
subject to	Ax	$\geq$	b	
	x	$\geq$	0	

maximize	$b^\intercal y$			*dual*
subject to	$A^\intercal y$	$\leq$	c	
	y	$\geq$	0	

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Proof. Follows from LP-duality:

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

Complementary Slackness

primal

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	$\boxed{x}$	$\boxed{\geq}$	$\boxed{0}$

dual

maximize	$b^\top y$		
subject to	$\boxed{A^\top y}$	$\boxed{\leq}$	$\boxed{c}$
	y	$\geq$	0

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Complementary Slackness

		primal
minimize	$c^\top x$	
subject to	$\boxed{Ax \geq b}$	
	$\boxed{x \geq 0}$	

		dual
maximize	$b^\top y$	
subject to	$\boxed{A^\top y \leq c}$	
	$\boxed{y \geq 0}$	

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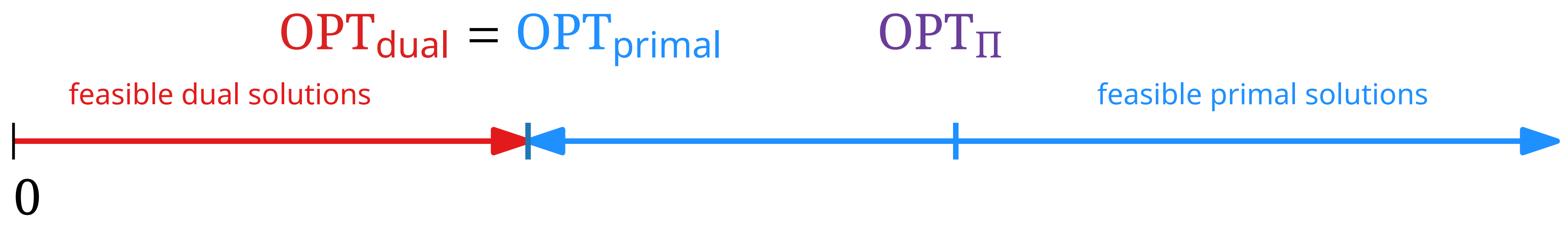
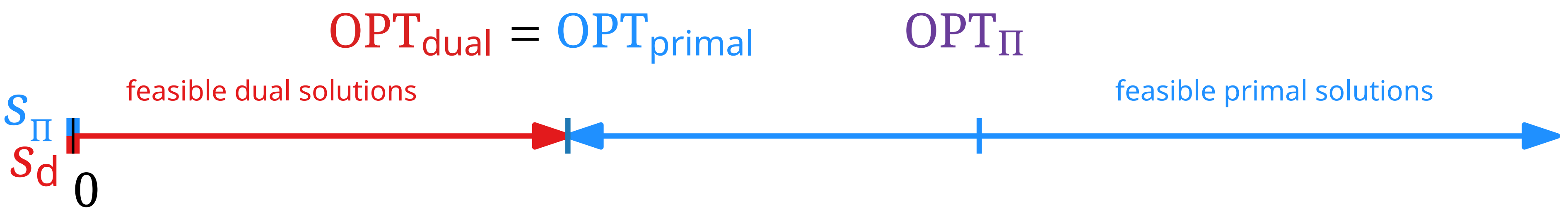
primal-dual method

Technique II) Primal–Dual Method



Consider a minimization problem Π in ILP form.

Technique II) Primal–Dual Method



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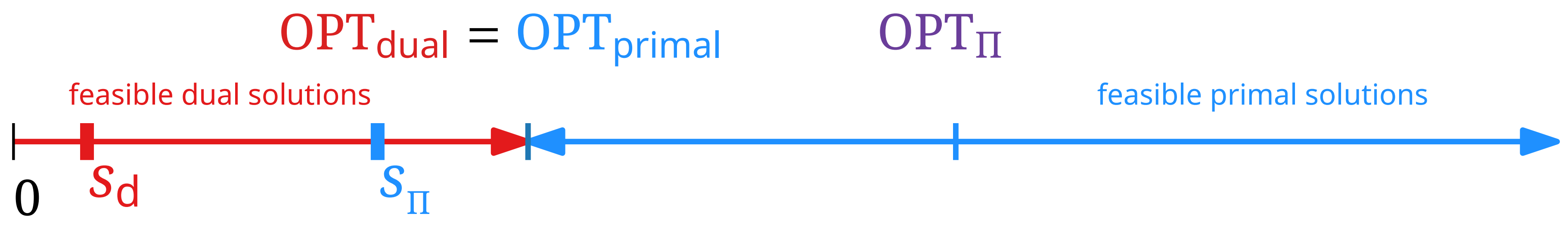

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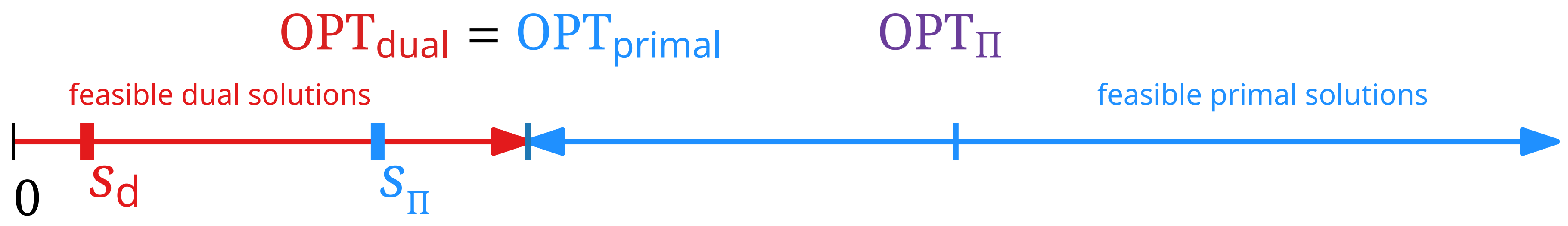

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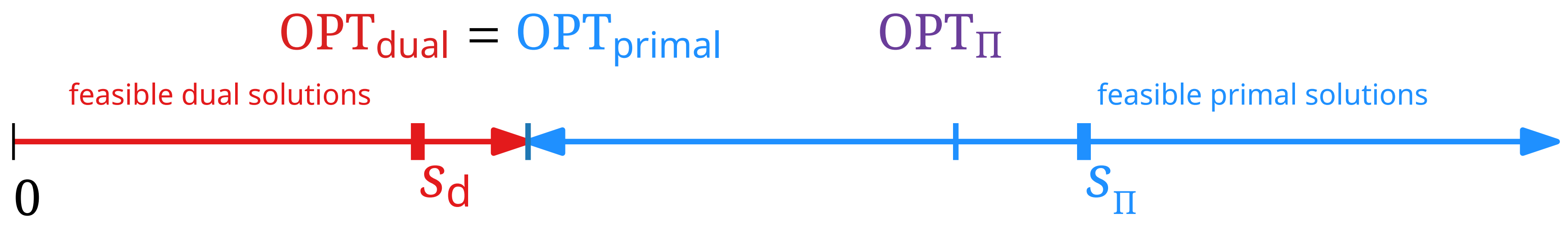

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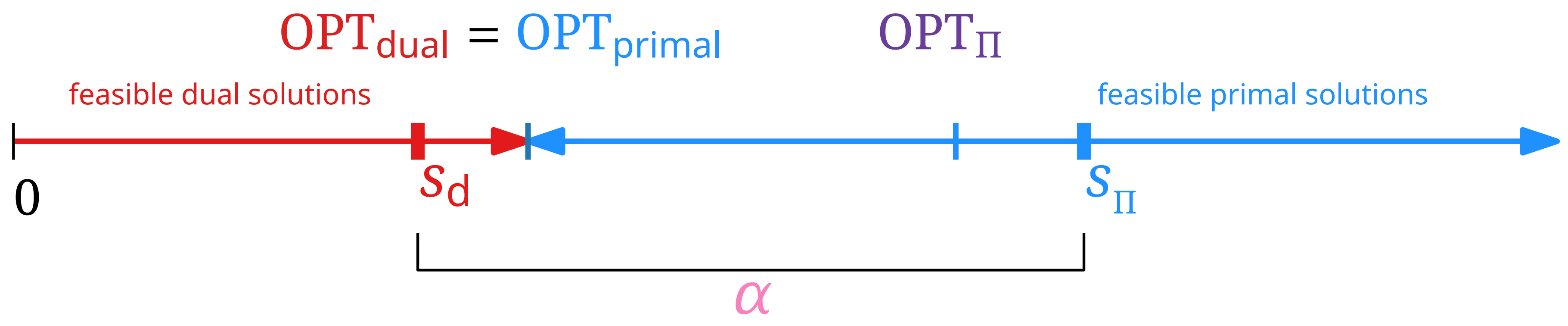

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Approximation factor $\leq \mathrm{obj}(s_{\Pi}) / \mathrm{obj}(s_{\text{d}})$

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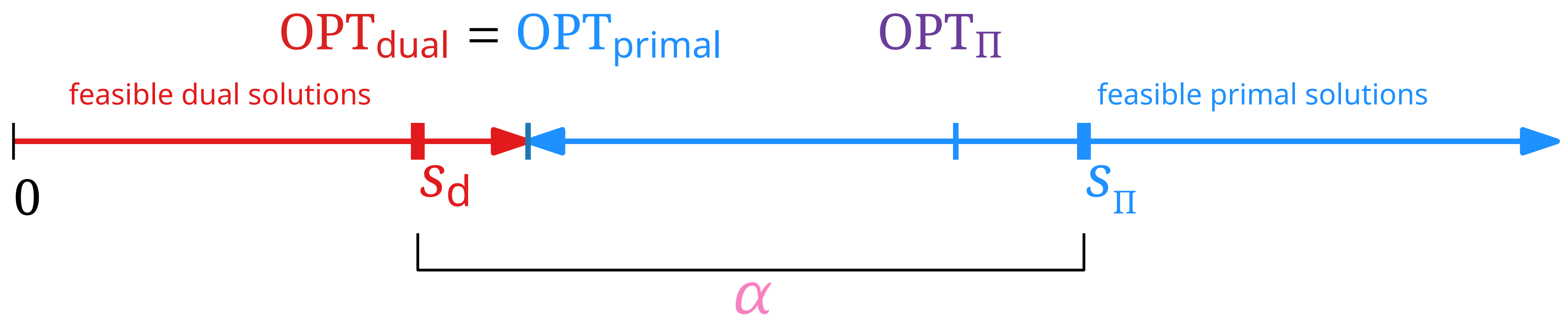

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Approximation factor $\leq \mathrm{obj}(s_{\Pi})/\mathrm{obj}(s_{\mathrm{d}})$

Advantage: Don't need LP-"machinery"; possibly faster, more flexible.

SetCover – Dual LP

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

?

SetCover – Dual LP

minimize $\sum_{S \in \mathcal{S}} c_S x_S$

subject to $\sum_{S \ni u} x_S \geq 1 \qquad u \in U$

$x_S \geq 0 \qquad S \in \mathcal{S}$

maximize

subject to

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maximize

subject to

$y_u \geq 0 \qquad u \in U$

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$$\text{maximize} \quad \sum_{u \in U} y_u$$

subject to

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SetCover – Dual LP

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SetCover – Dual LP

Covering LP

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Packing LP

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{u \in U} y_u & \\ \text{subject to} & \displaystyle\sum_{u \in S} y_u \leq c_S & S \in \mathcal{S} \\ & y_u \geq 0 & u \in U \end{array}$$

SetCover – Dual LP

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$$\text{minimize} \quad \sum_{S \in \mathcal{S}} c_S x_S$$

$$\text{subject to} \quad \sum_{S \ni u} x_S \geq 1 \qquad u \in U$$

$$x_S \geq 0 \qquad S \in \mathcal{S}$$

Packing LP

$$\text{maximize} \quad \sum_{u \in U} y_u$$

$$\text{subject to} \quad \sum_{u \in S} y_u \leq c_S \qquad S \in \mathcal{S}$$ “no overpacking”

$$y_u \geq 0 \qquad u \in U$$

Relaxing Complementary Slackness

$$\begin{array}{llll} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

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~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: $x_j = 0$ or $\cancel{\sum_{i=1}^m a_{ij} y_i = c_j}$

$$c_j/\alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

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For each $i = 1, \ldots, m$: $y_i = 0$ or $\cancel{\sum_{j=1}^{n} a_{ij} x_j = b_i}$

$$b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

Relaxing Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

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$$b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$$

$$\cancel{\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i} \quad \Rightarrow \sum_{j=1}^n c_j x_j \leq \alpha\beta \sum_{i=1}^m b_i y_i \leq \alpha\beta \cdot \mathrm{OPT}_{\mathrm{LP}}$$

Primal–Dual Method

Start with a feasible dual and infeasible primal solution (often trivial).

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"Improve" the feasibility of the primal solution...

. . . and simultaneously the objective value of the dual solution.

Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer-valued.

The feasibility of the primal solution and the relaxed CS conditions provide an approximation ratio.

Relaxed CS for SetCover

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{u \in U} y_u & \\ \textbf{subject to} & \displaystyle\sum_{u \in S} y_u \leq c_S & S \in \mathcal{S} \\ & y_u \geq 0 & u \in U \end{array}$$

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(Unrelaxed) primal CS:

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

Relaxed CS for SETCOVER

minimize $\sum_{S \in \mathcal{S}} c_S x_S$	maximize $\sum_{u \in U} y_u$
subject to $\sum_{S \ni u} x_S \geq 1 \quad u \in U$	subject to $\sum_{u \in S} y_u \leq c_S \quad S \in \mathcal{S}$
$x_S \geq 0 \quad S \in \mathcal{S}$	$y_u \geq 0 \quad u \in U$

critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

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Relaxed dual CS:

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
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maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq \overset{?}{\beta} \cdot 1$

Relaxed CS for SetCover

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Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

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trivial for binary x

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \displaystyle\sum_{S \ni u} x_S \leq f \cdot 1$

Primal–Dual Schema for SETCOVER

PrimalDualSetCover$(U, \mathcal{S}, c)$

$x \leftarrow 0,\ y \leftarrow 0$

repeat

until all elements are covered.

return x

Primal–Dual Schema for SetCover

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Select an uncovered element u.

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  Mark all elements in these sets as covered.

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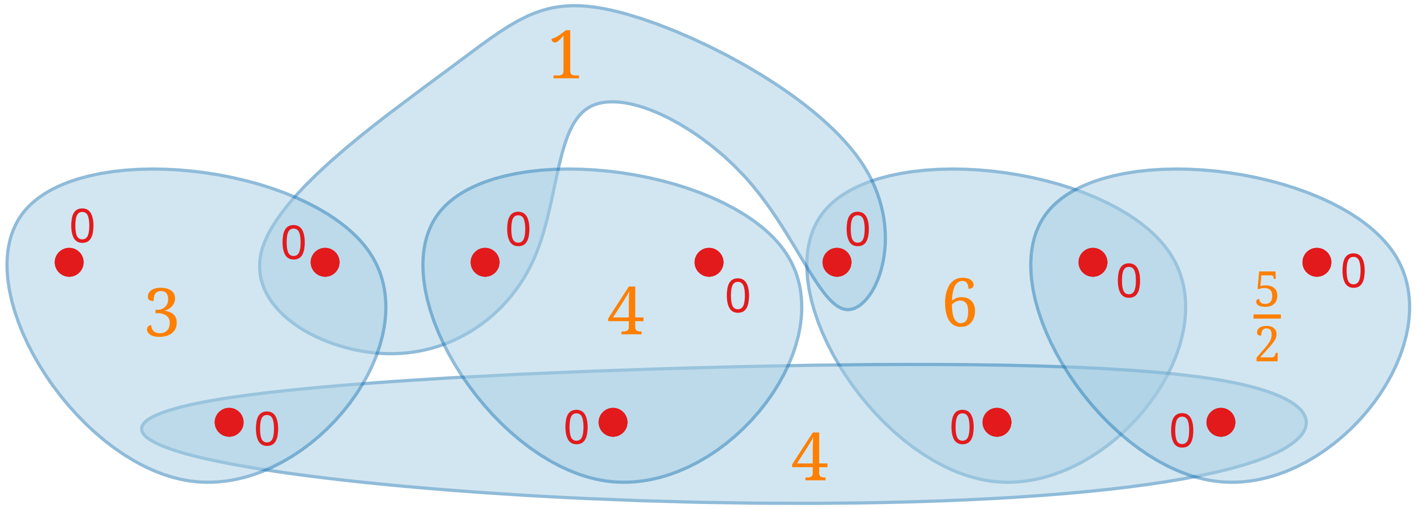


Primal–Dual Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0$, $y \leftarrow 0$

repeat

Select an uncovered element u.

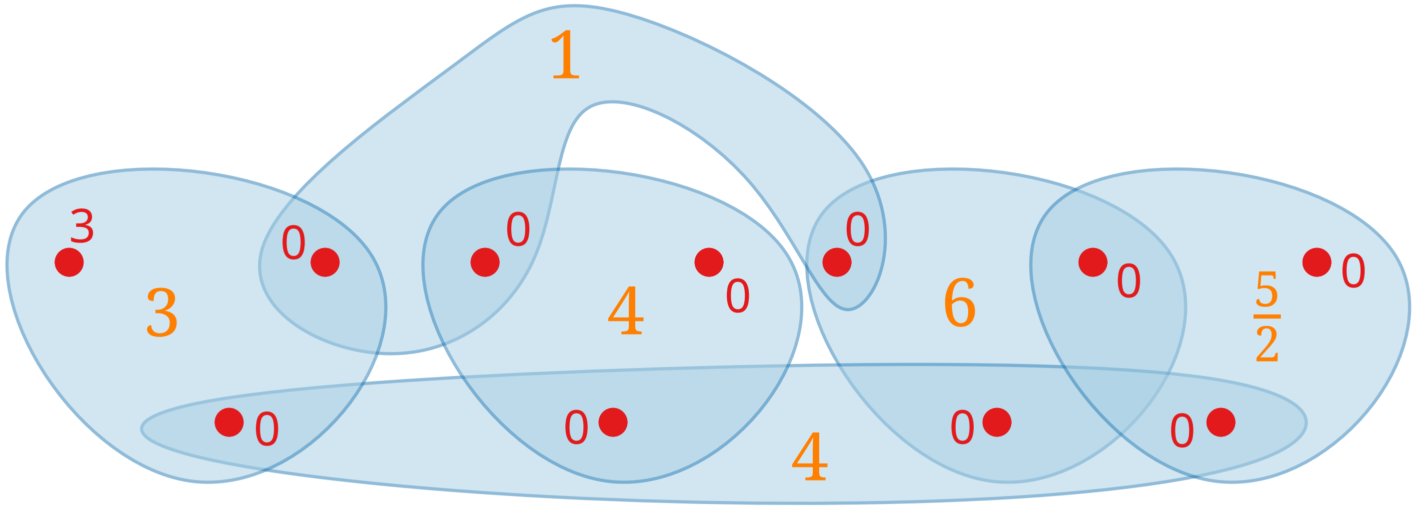
Increase y_u until a set S is critical ($\sum_{u' \in S} y_{u'} = c_S$).

Select all critical sets and update x.

Mark all elements in these sets as covered.

until all elements are covered.

return x



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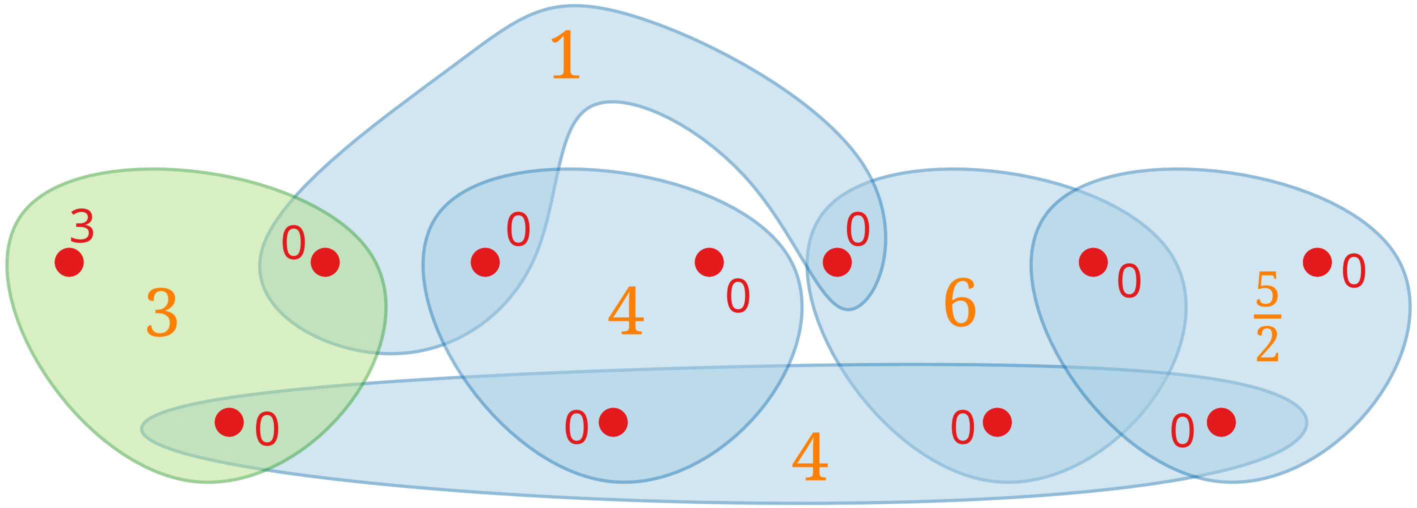

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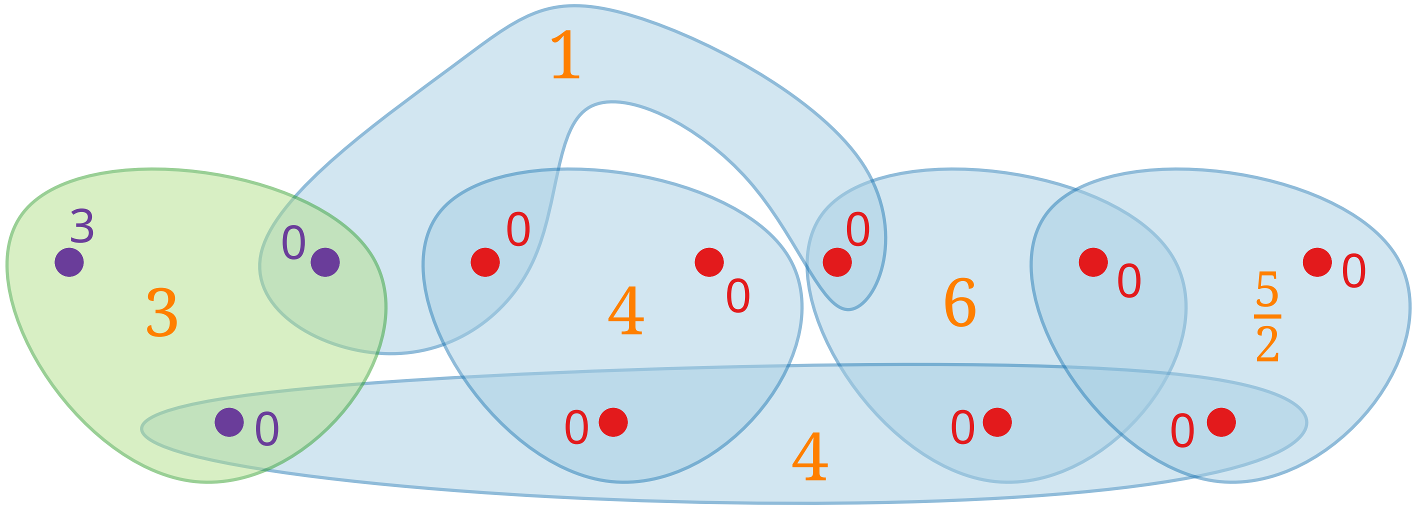

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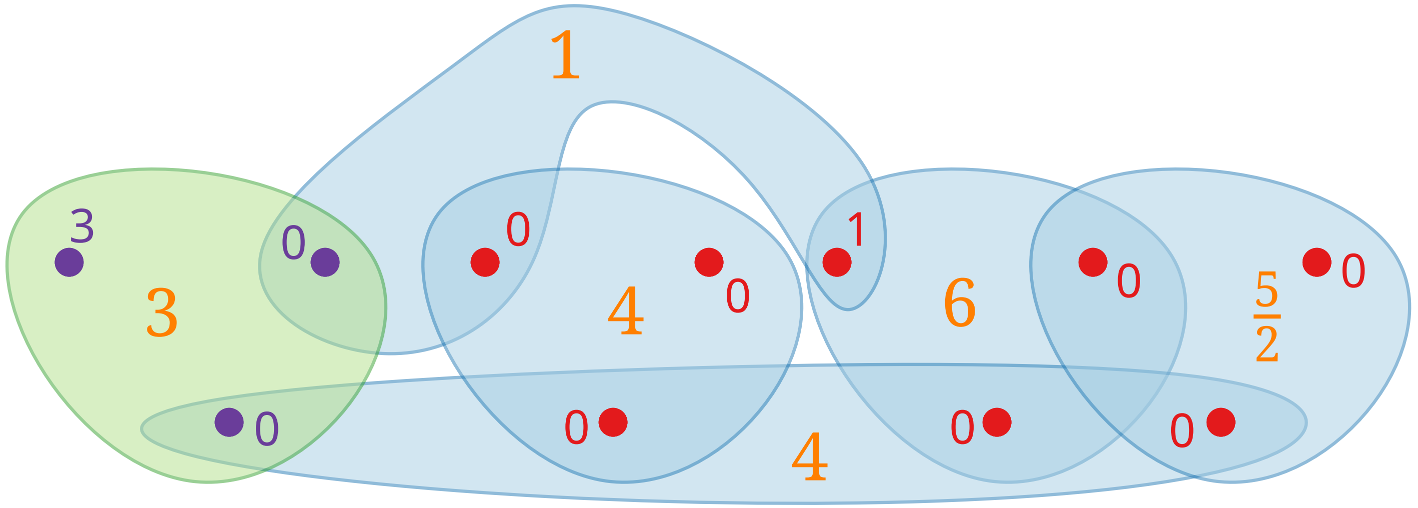

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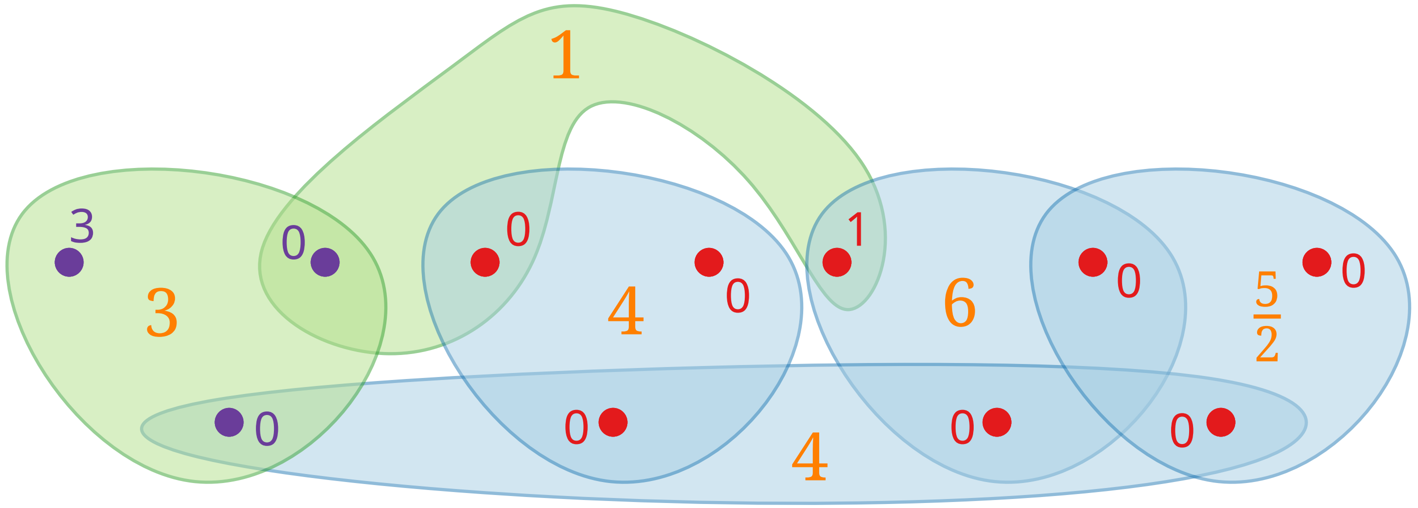

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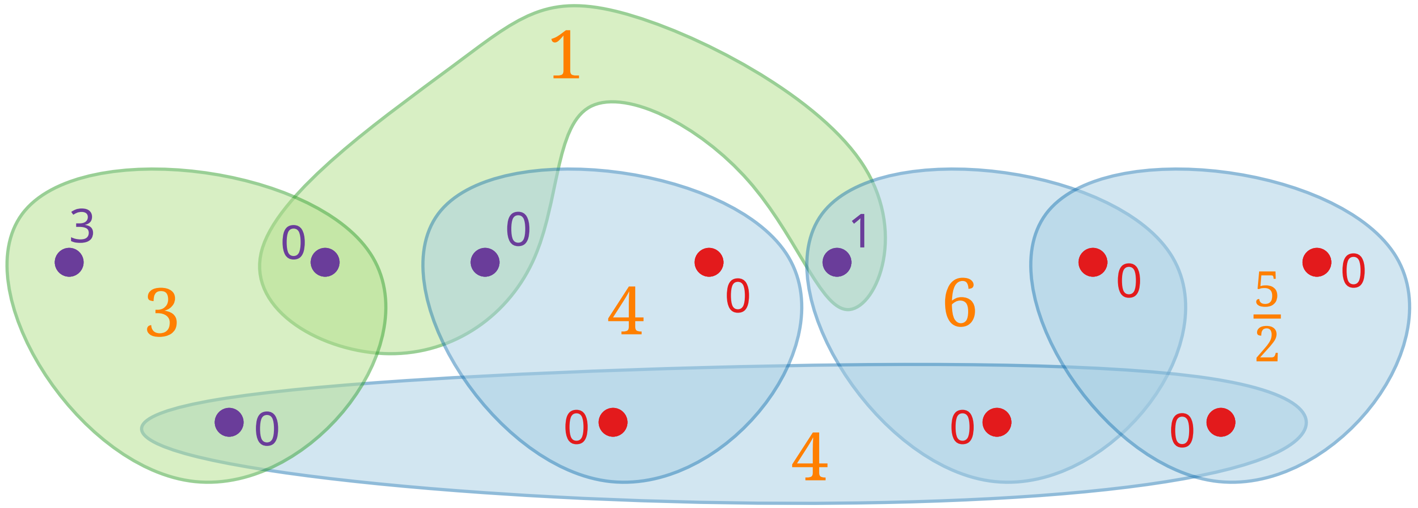

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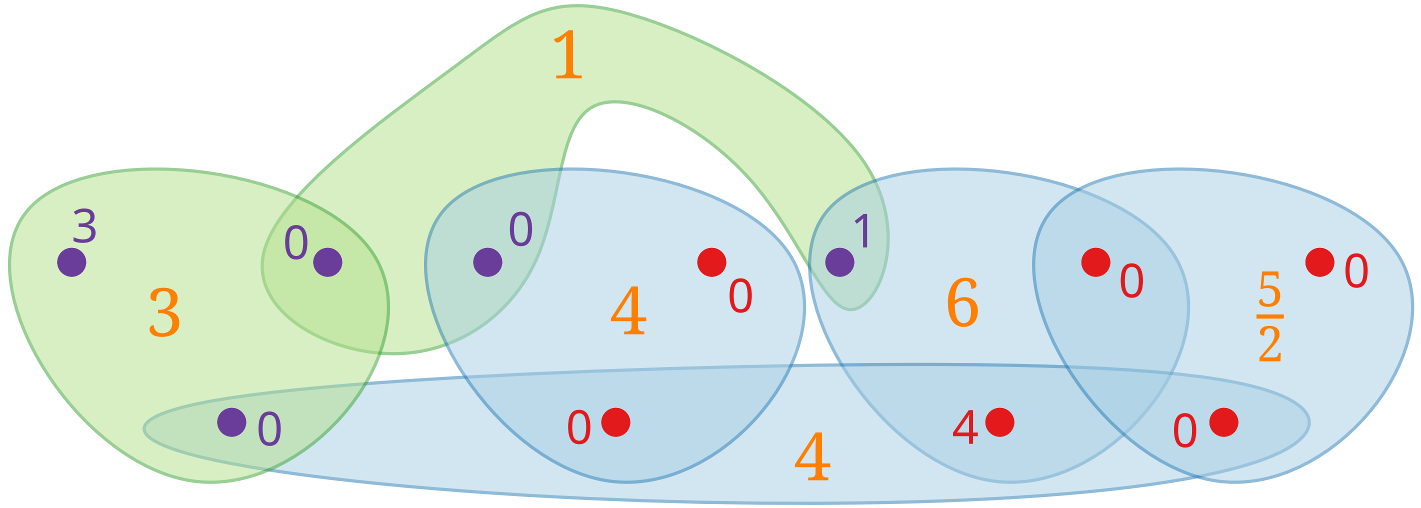

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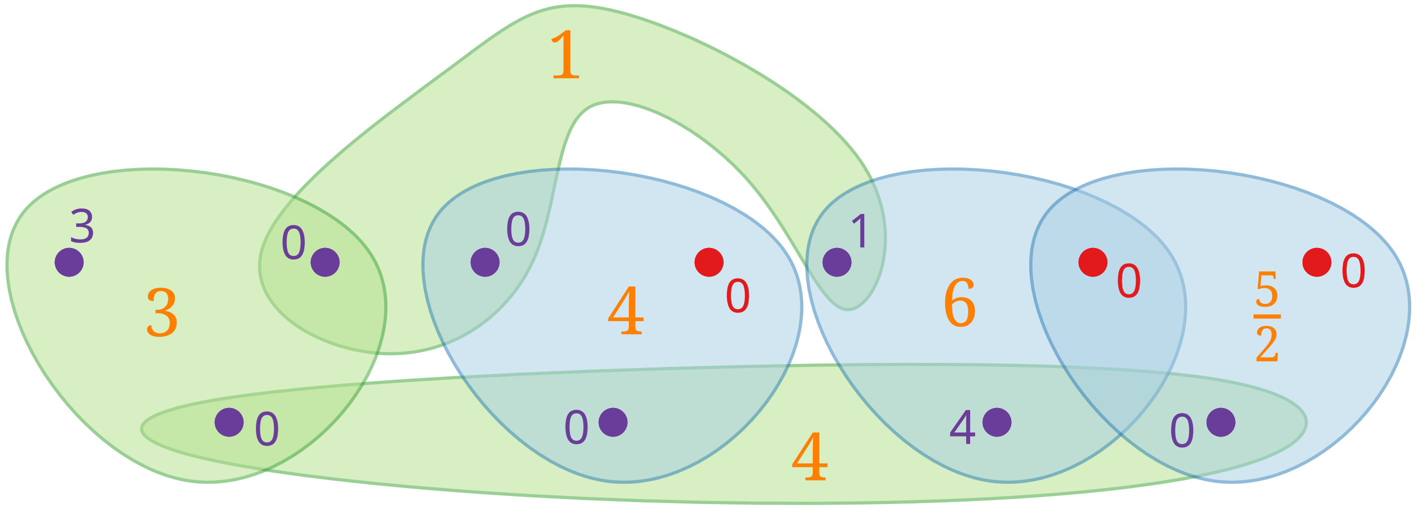

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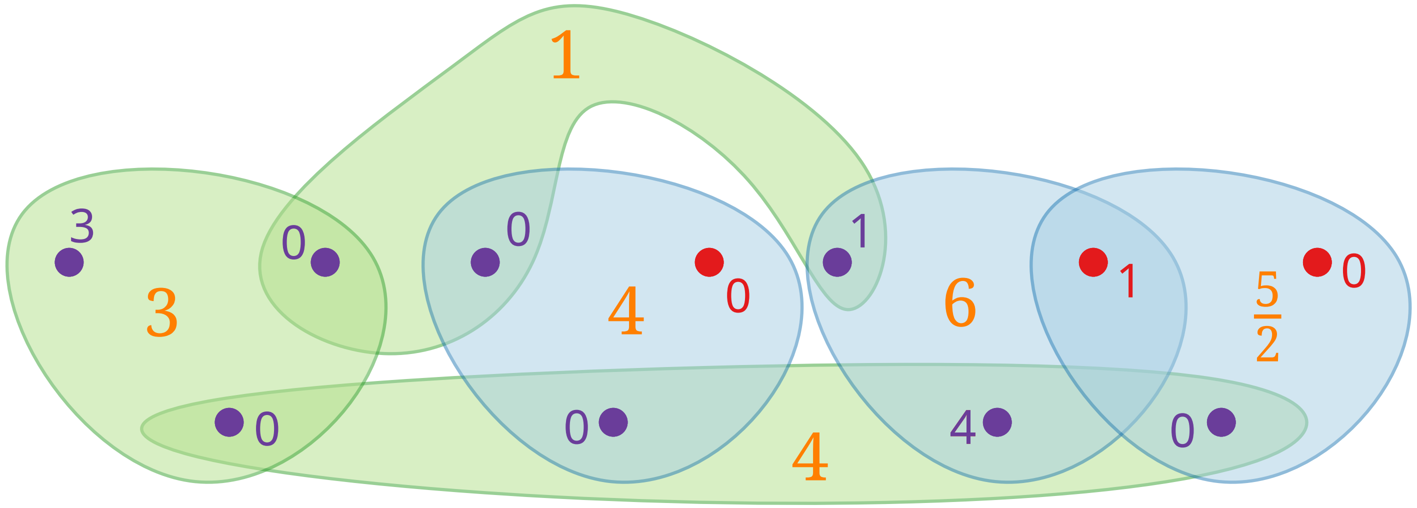

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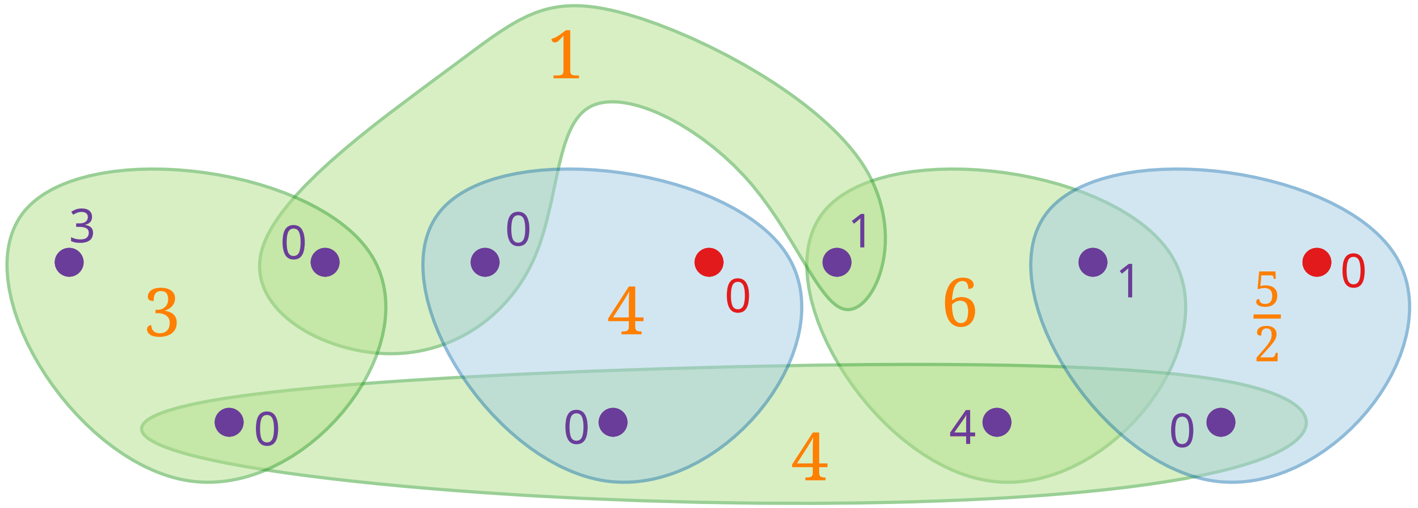

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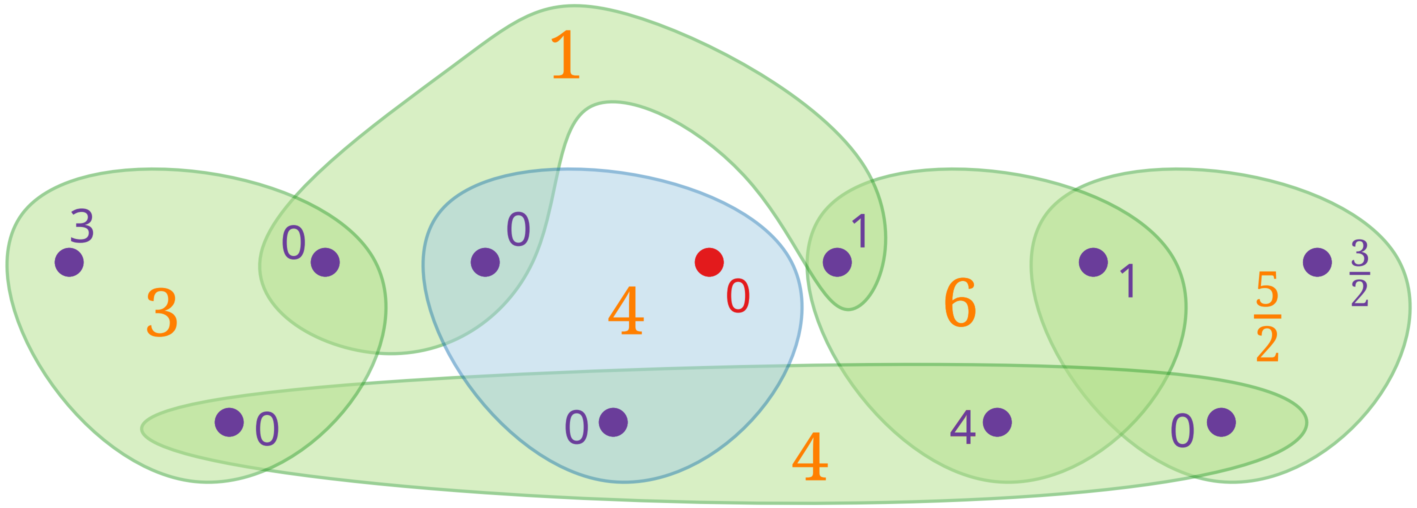

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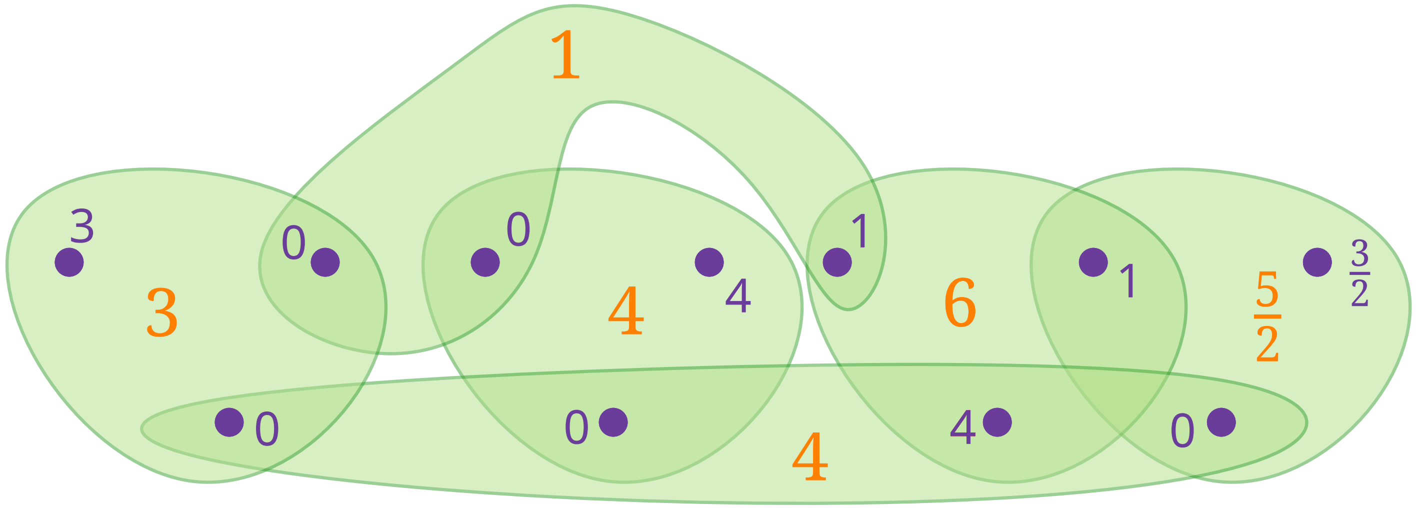

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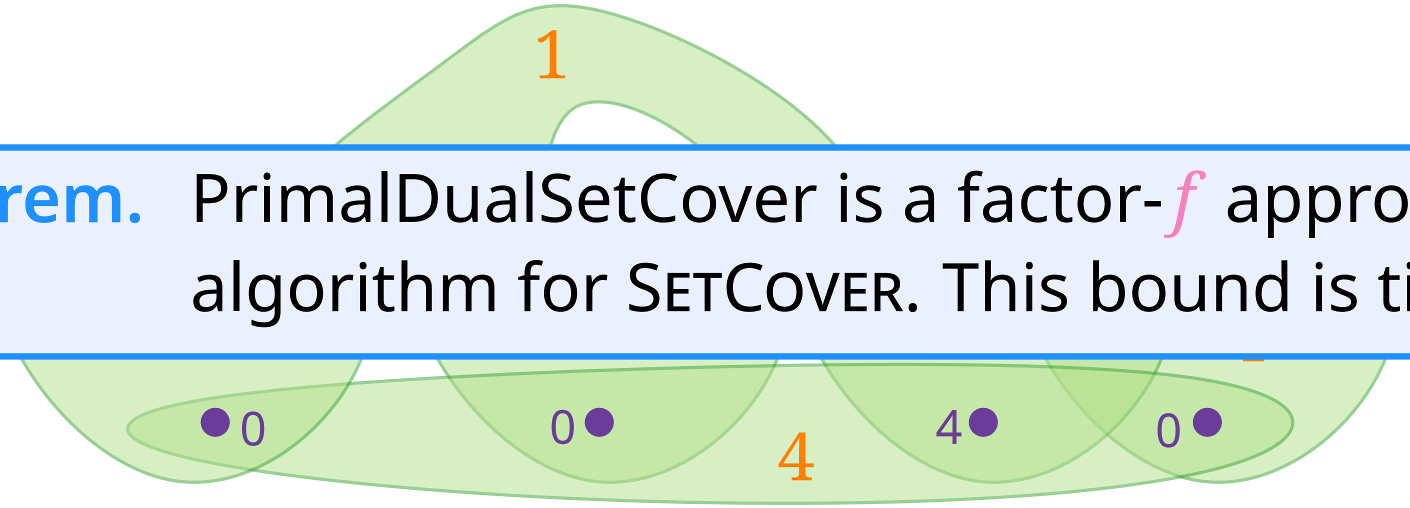

Theorem. PrimalDualSetCover is a factor-f approximation algorithm for SetCover. This bound is tight.

Tight Example

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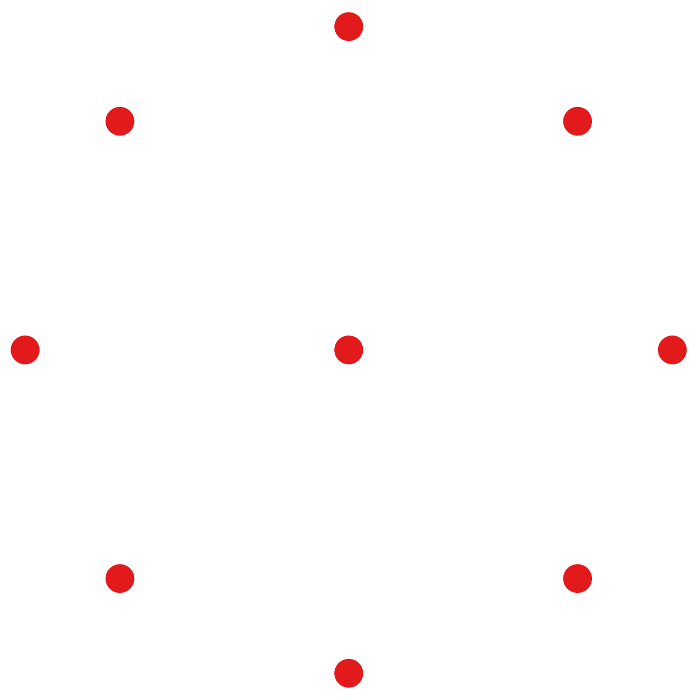

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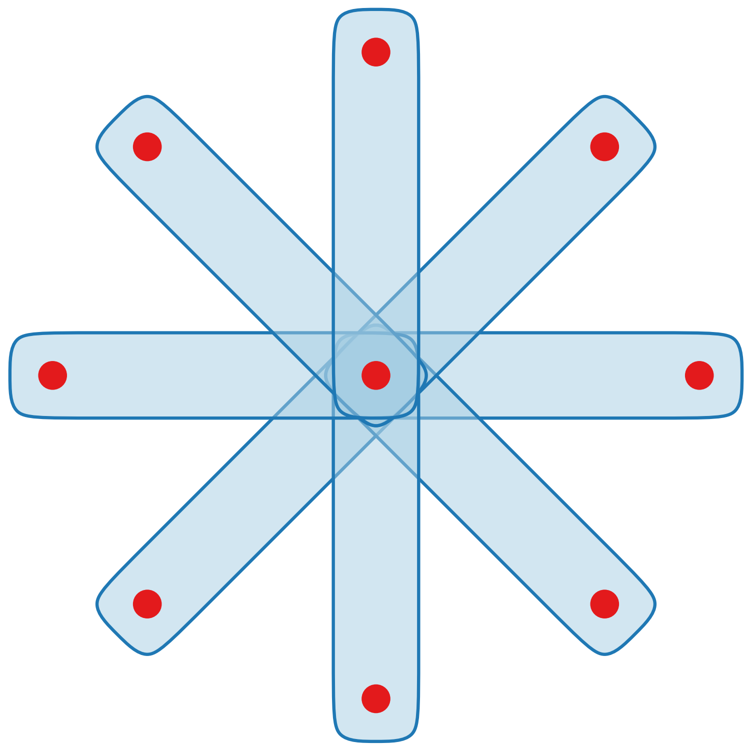

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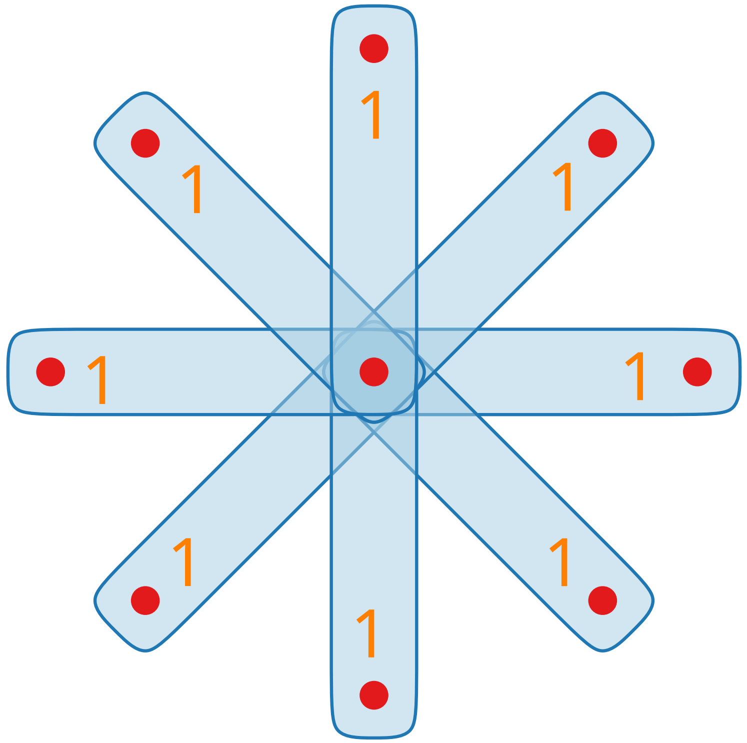

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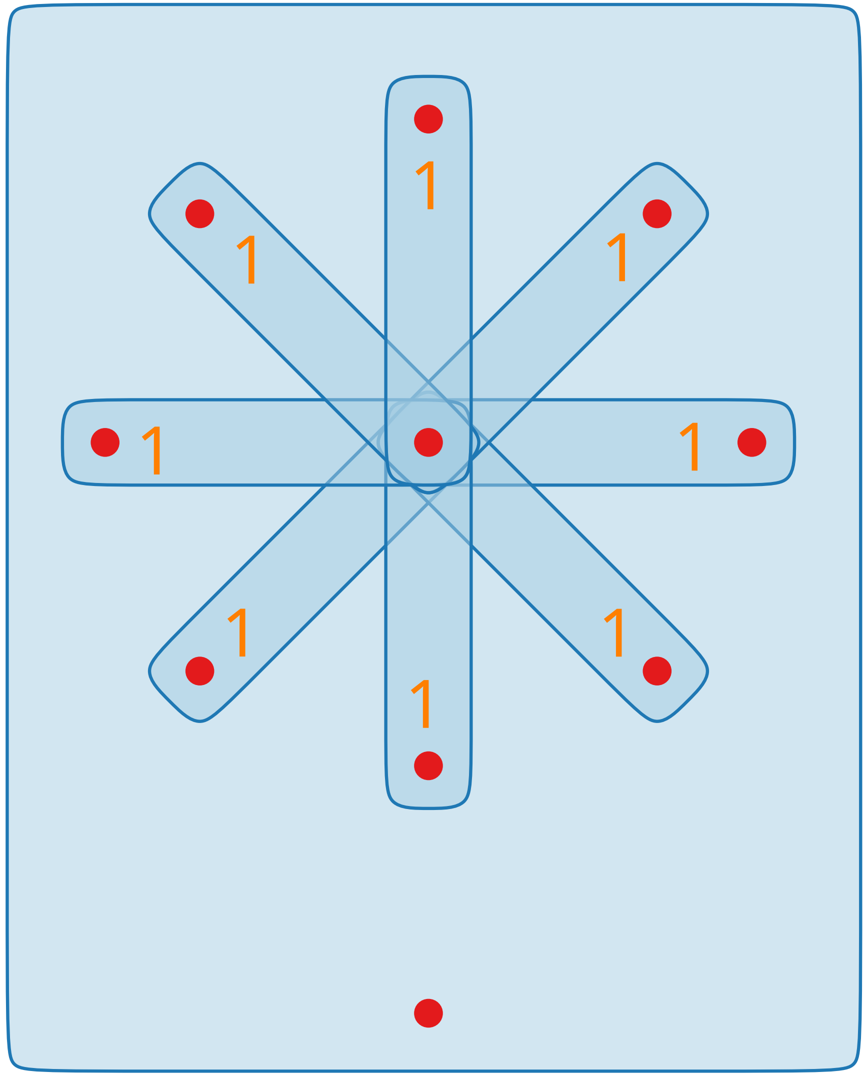

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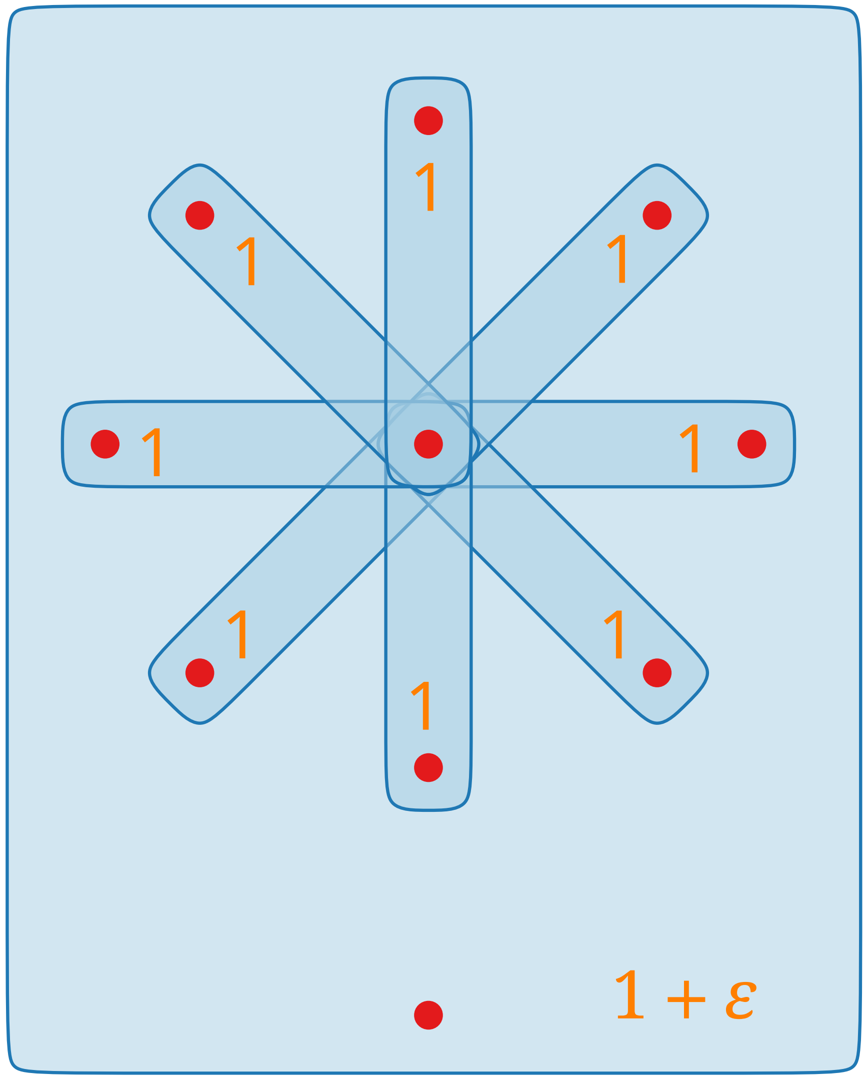

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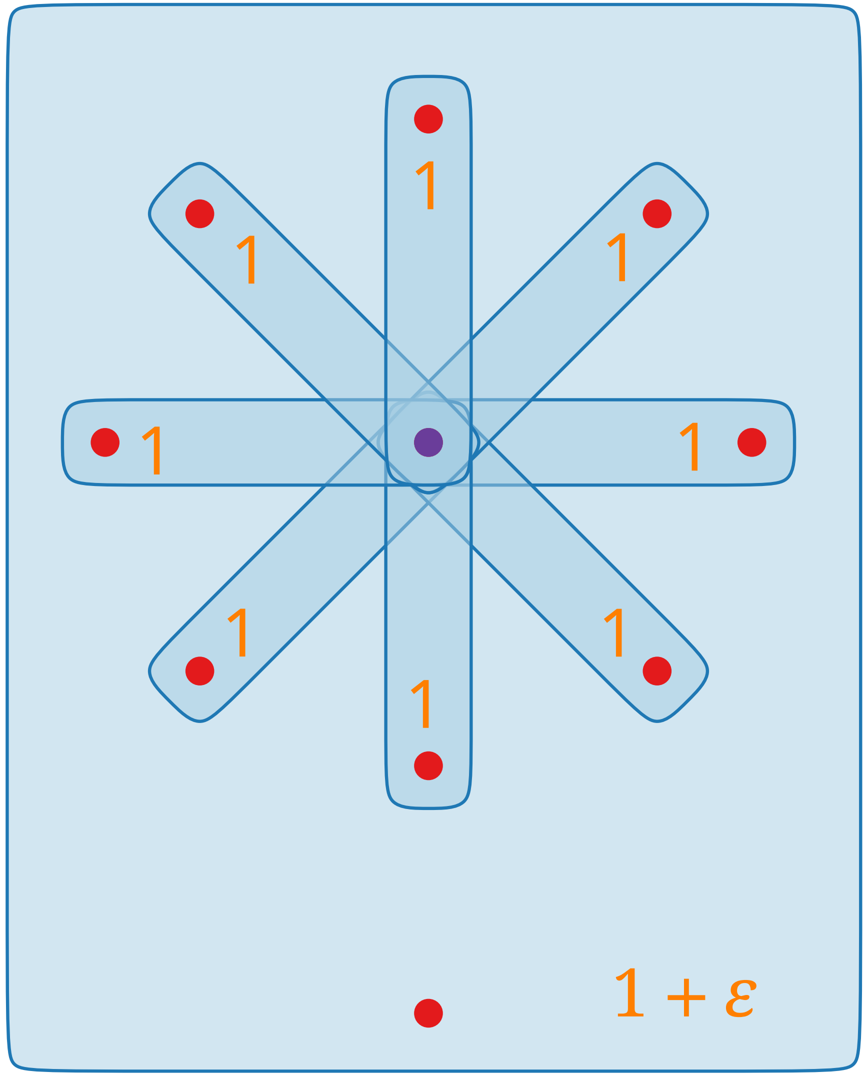

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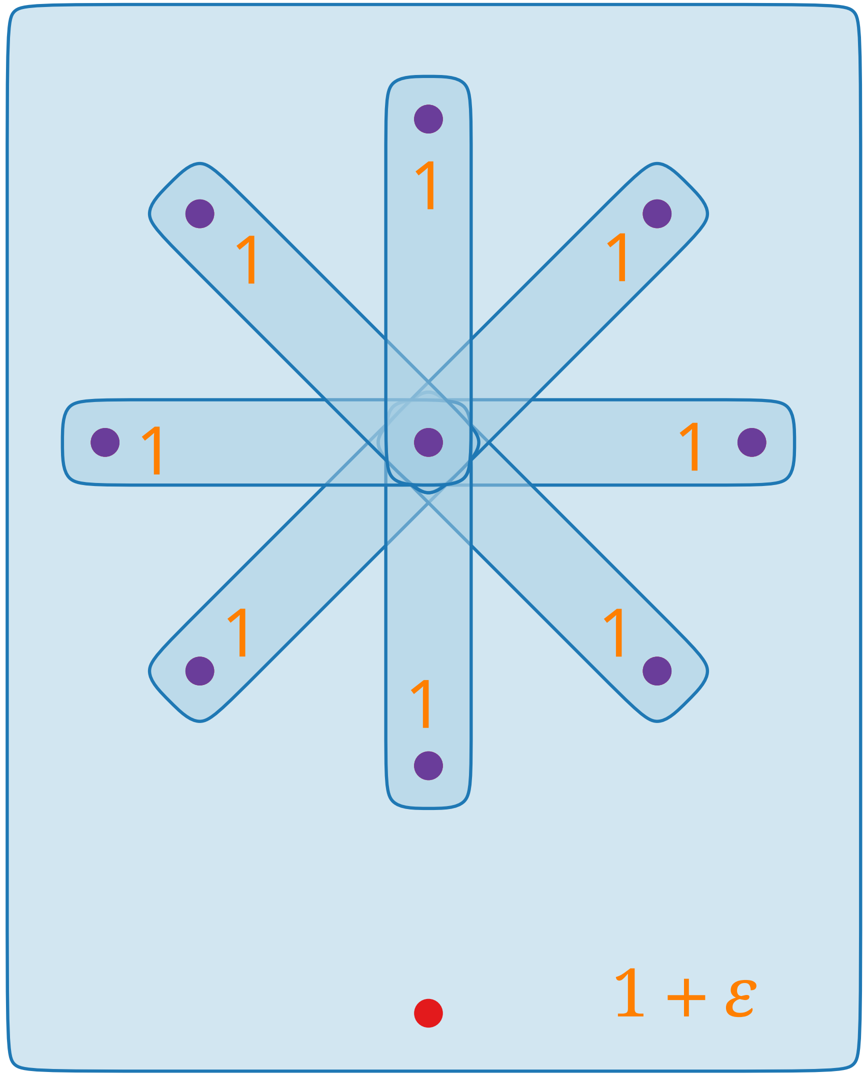

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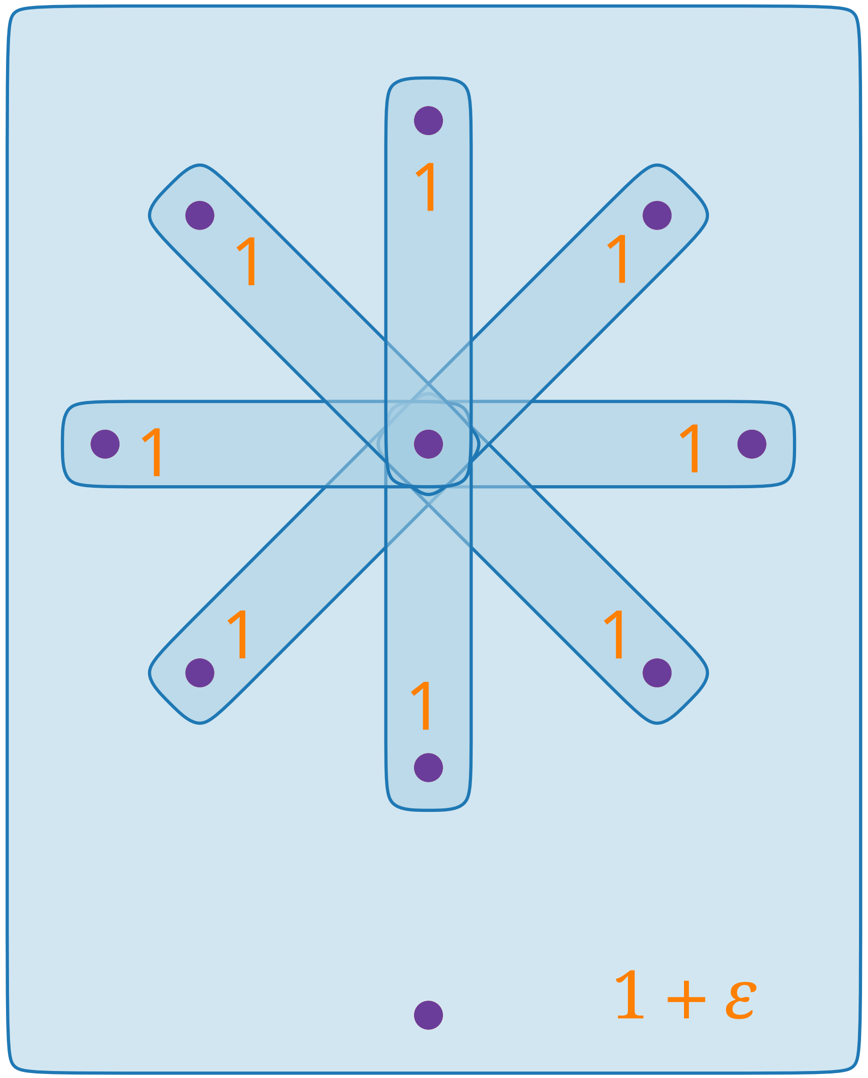

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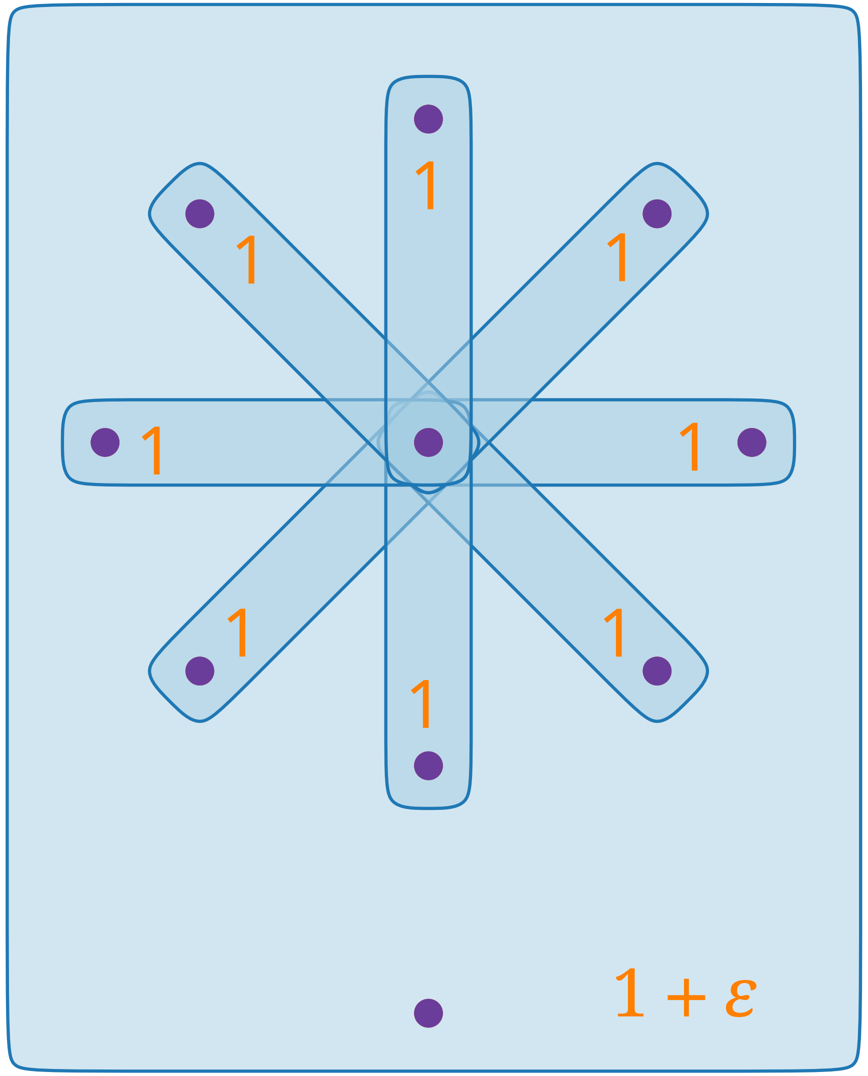
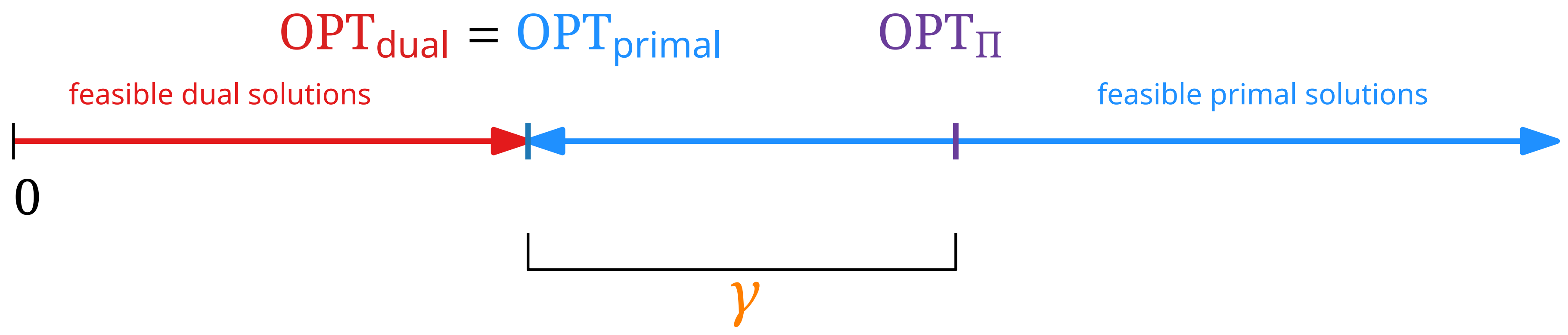


Algorithm may choose all sets at cost $n + \varepsilon$ whereas the optimum has cost $1 + \varepsilon$

Integrality Gap



Consider a minimization problem Π in ILP form.

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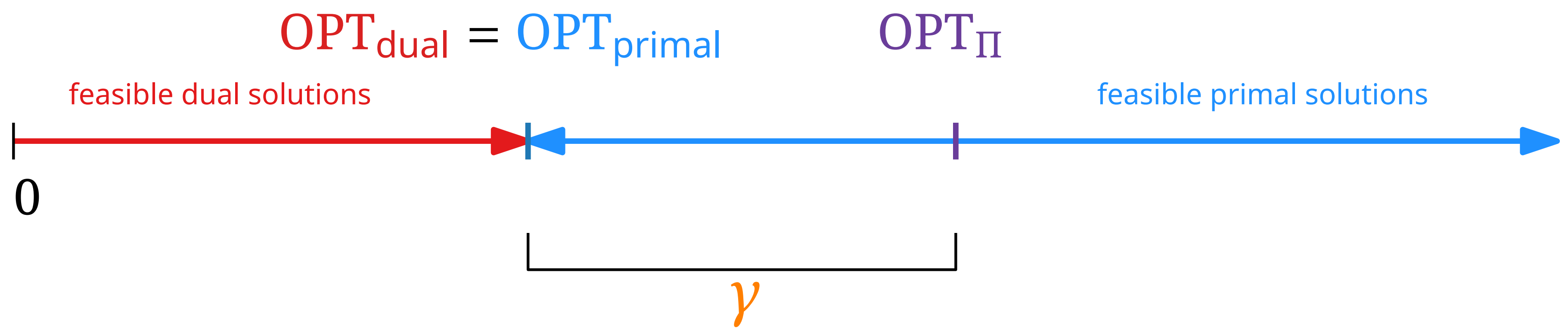

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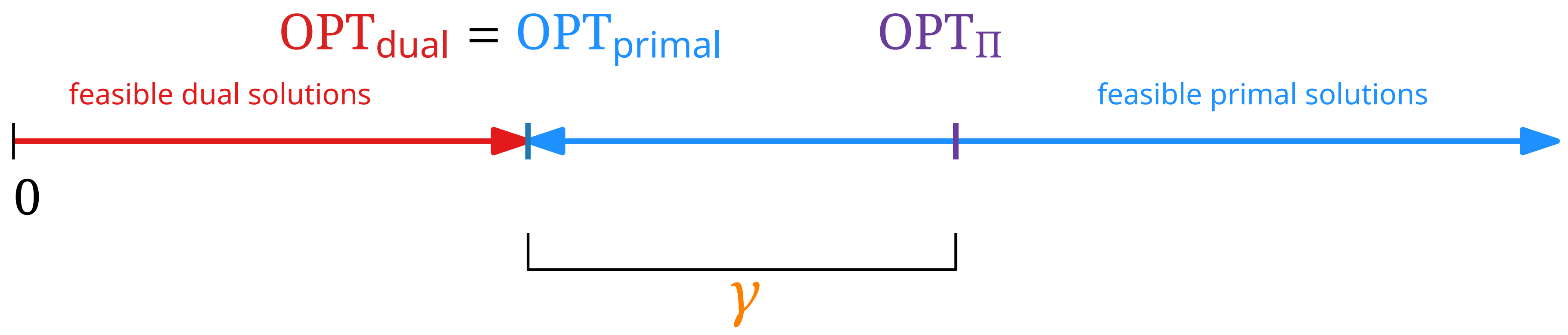

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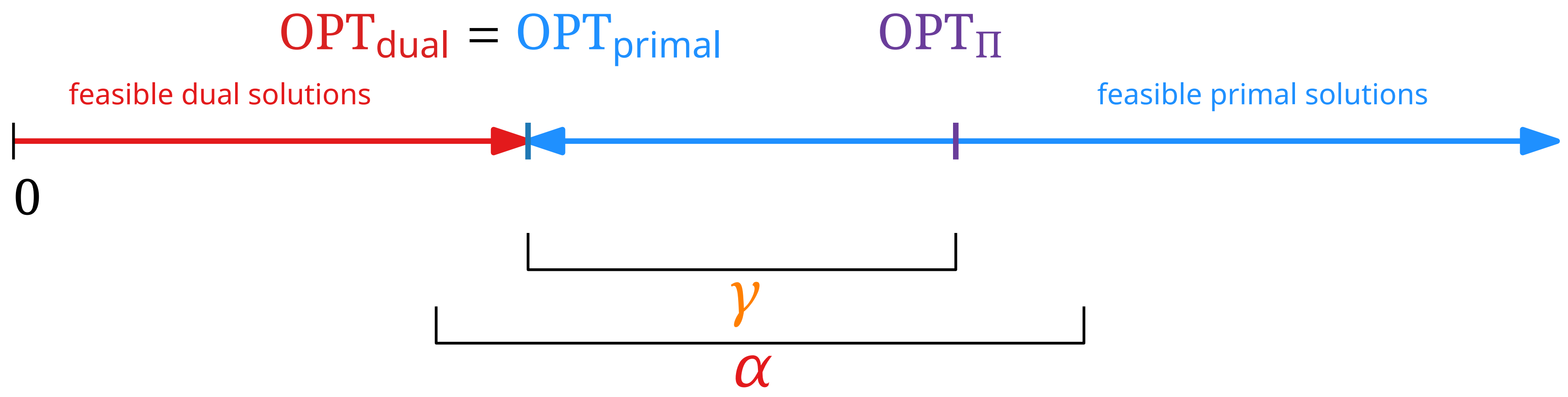

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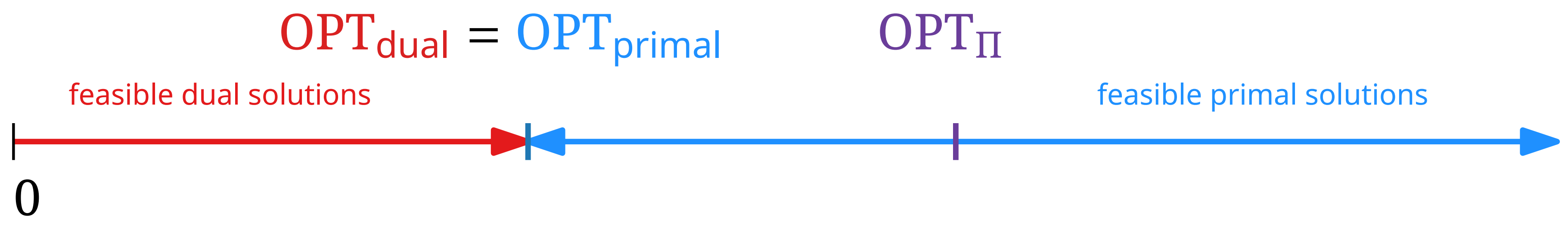
$$\alpha \geq \gamma = \sup_I \frac{\mathrm{OPT}_{\Pi}(I)}{\mathrm{OPT}_{\text{primal}}(I)}$$

dual fitting

Technique III) Dual Fitting



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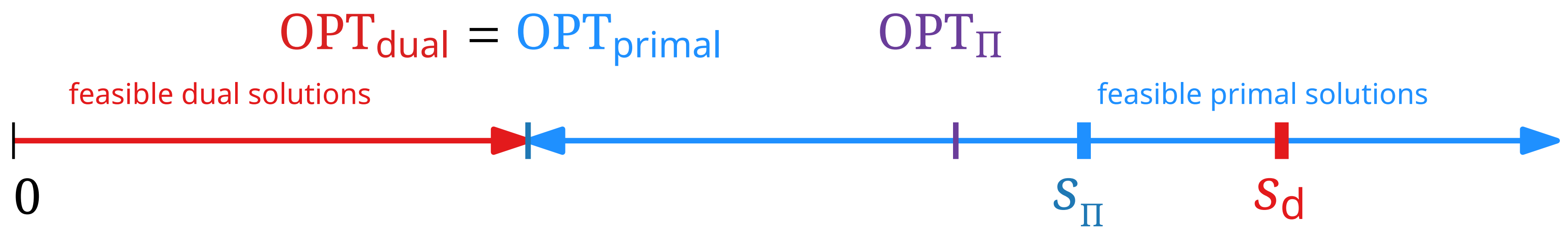

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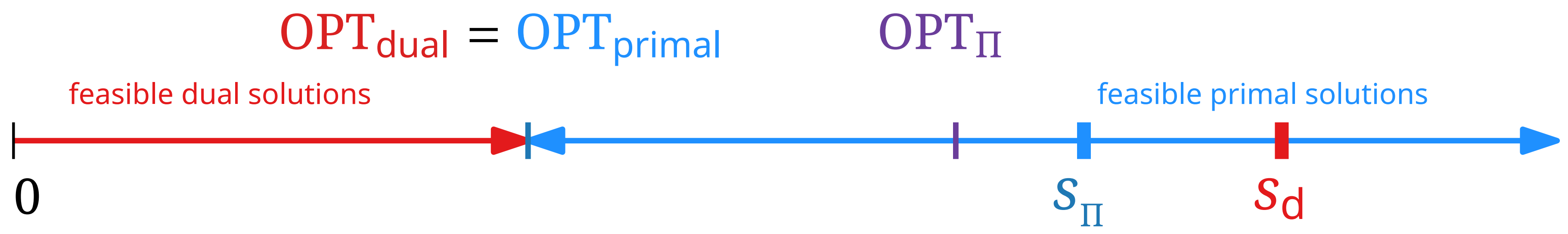

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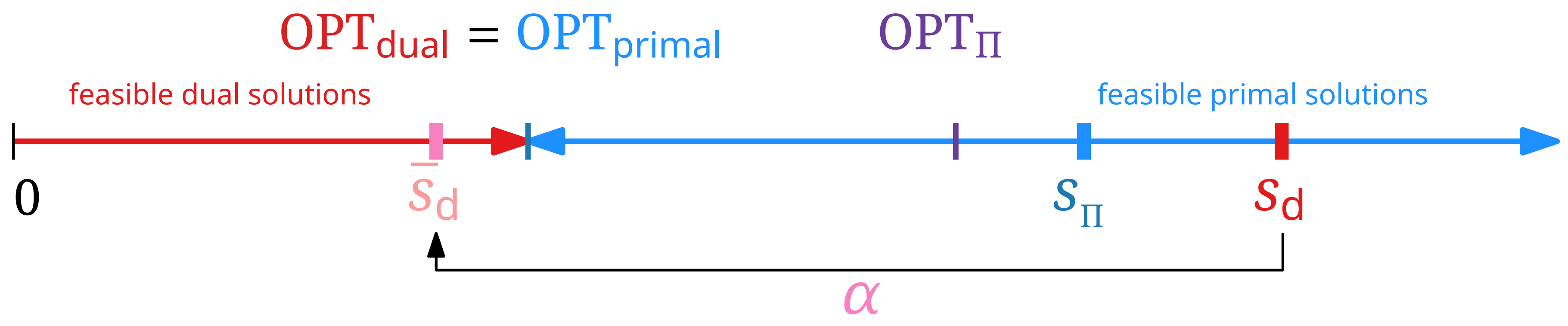

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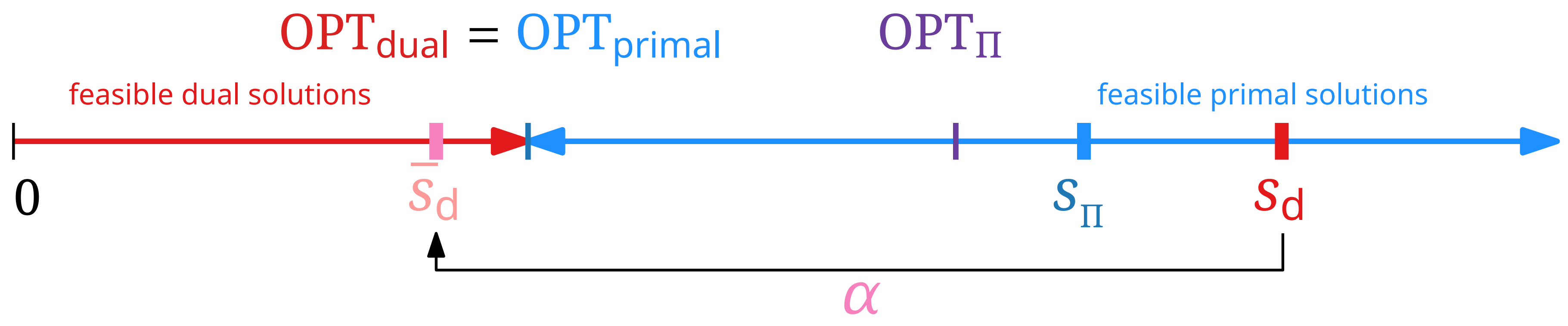

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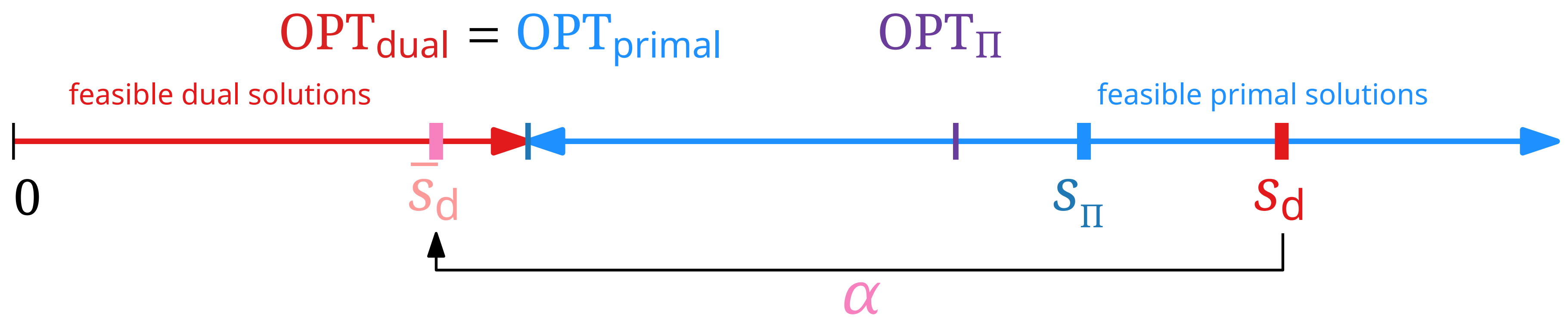

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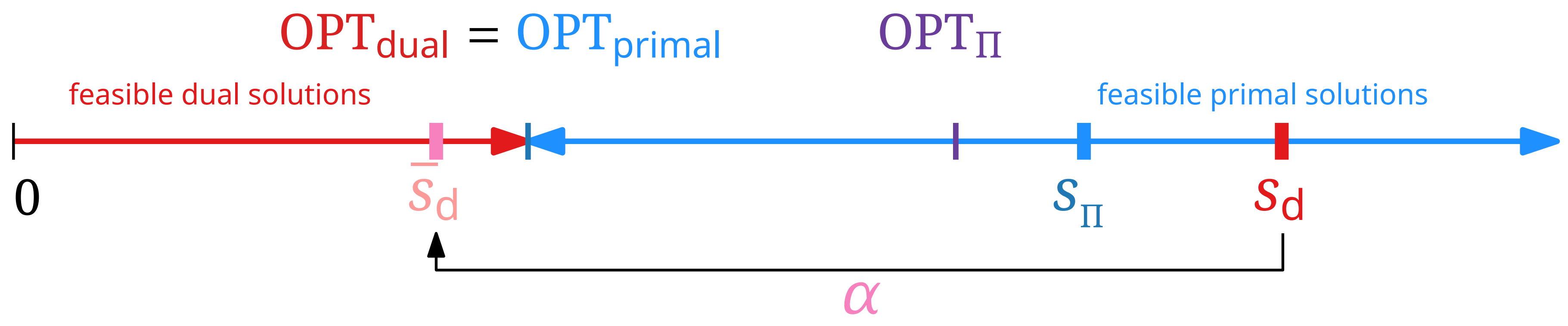

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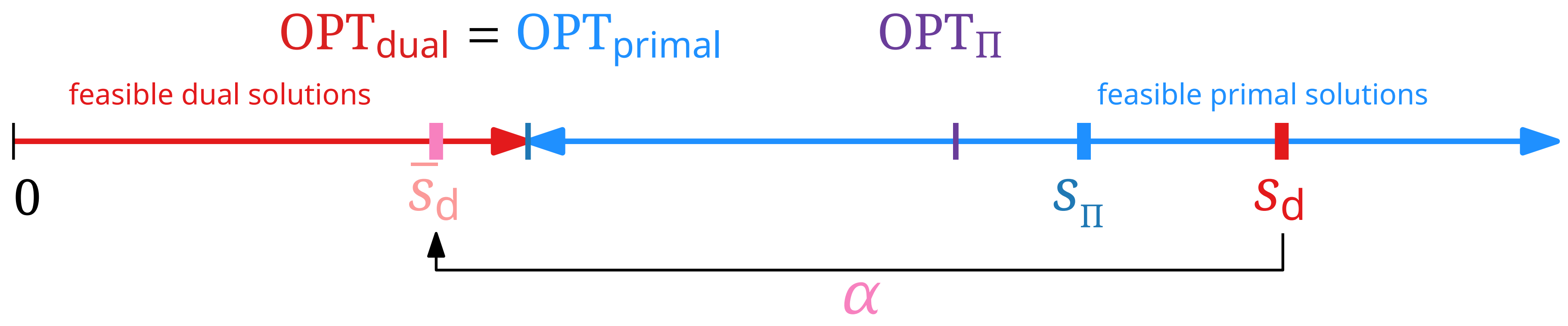

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$\Rightarrow$ Scaling factor α is approximation factor

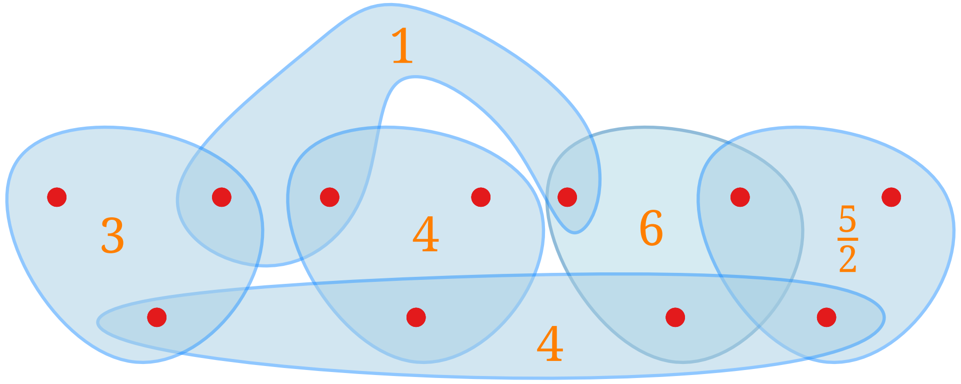
Dual Fitting for SetCover

Combinatorial (greedy) algorithm:

GreedySetCover$(U, \mathcal{S}, c)$

- $C \leftarrow \emptyset$
- $\mathcal{S}' \leftarrow \emptyset$
- **while** $C \neq U$ **do**
  - $S \leftarrow$ Set from $\mathcal{S}$ that minimizes $\frac{c(S)}{|S \setminus C|}$
  - **foreach** $u \in S \setminus C$ **do**
    - price$(u) \leftarrow \frac{c(S)}{|S \setminus C|}$
  - $C \leftarrow C \cup S$
  - $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{S\}$
- **return** $\mathcal{S}'$ // Cover of U

What does the algorithm do on this example?



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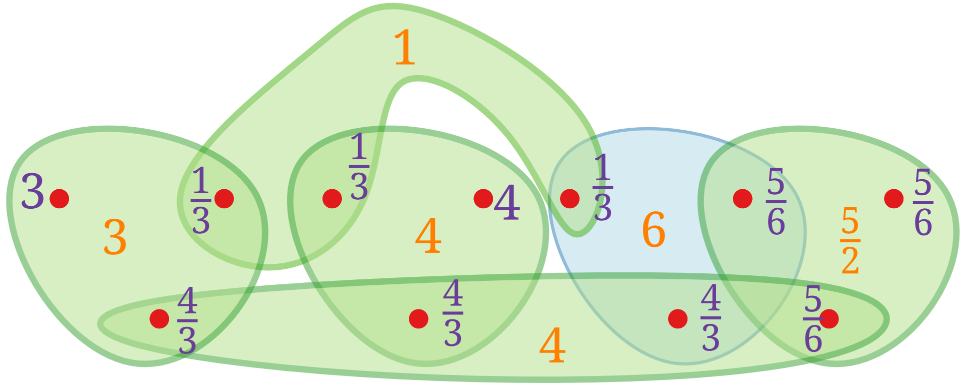

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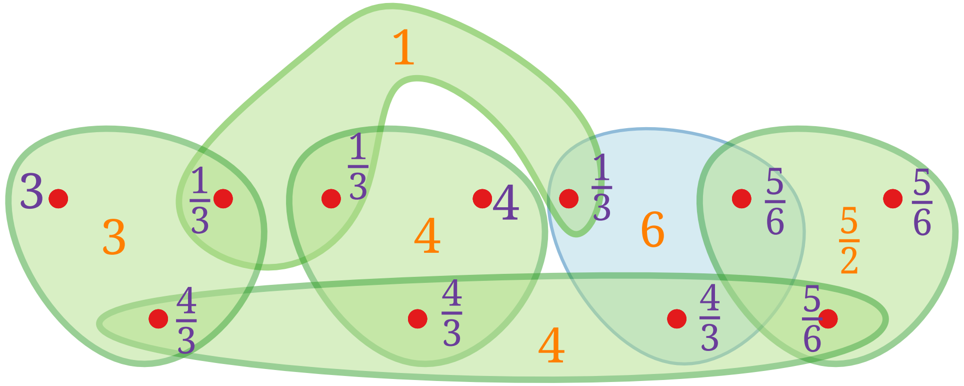

Reminder: $\sum_{u \in U}$ price(u) …

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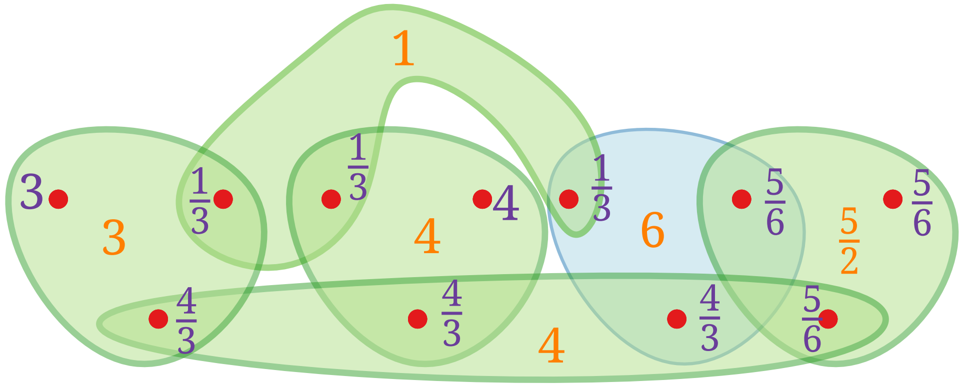

Reminder: $\sum_{u \in U} \text{price}(u)$ completely pays for $\mathcal{S}'$.

New: LP-based Analysis

Observation. For each $u \in U$, $\mathrm{price}(u)$ is a dual variable

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maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

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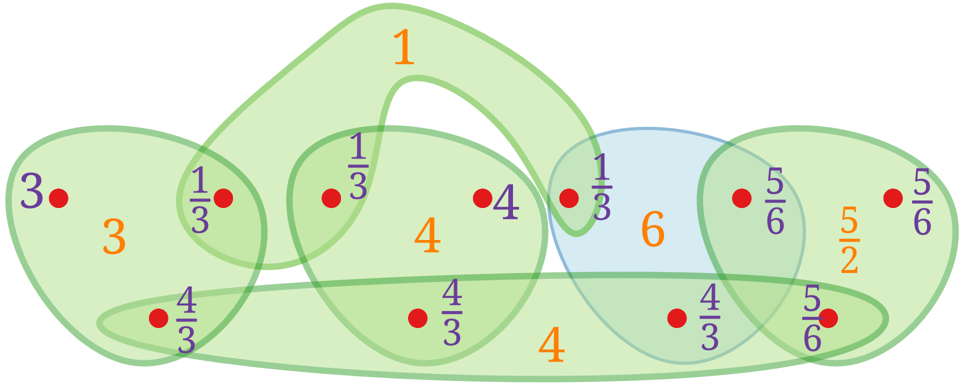
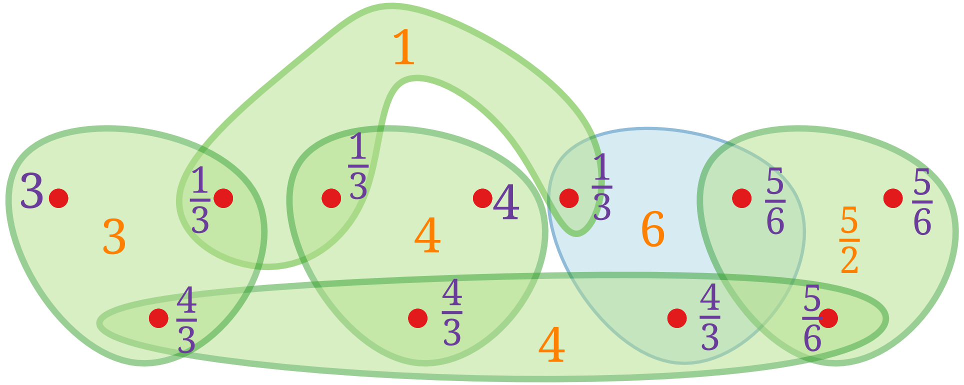


$$
\begin{array}{lll}
\textbf{maximize} & \displaystyle\sum_{u \in U} y_u & \\
\textbf{subject to} & \displaystyle\sum_{u \in S} y_u \le c_S & S \in \mathcal{S} \\
 & y_u \ge 0 & u \in U
\end{array}
$$

New: LP-based Analysis

Observation. For each $u \in U$, $\text{price}(u)$ is a dual variable
But this dual solution is in general not feasible.



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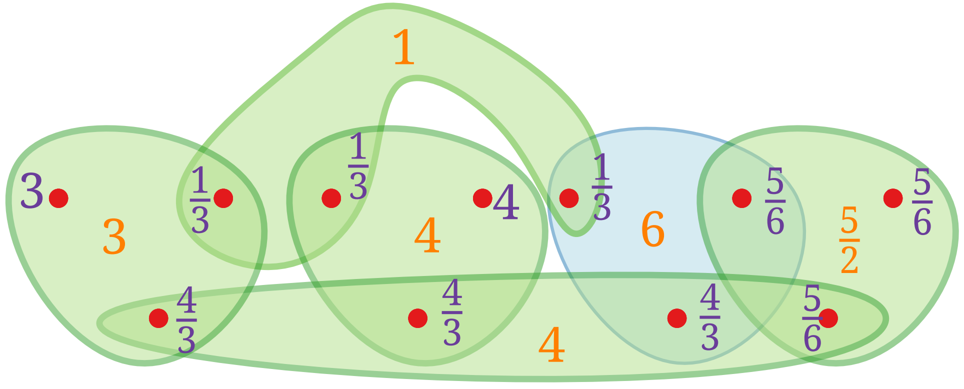

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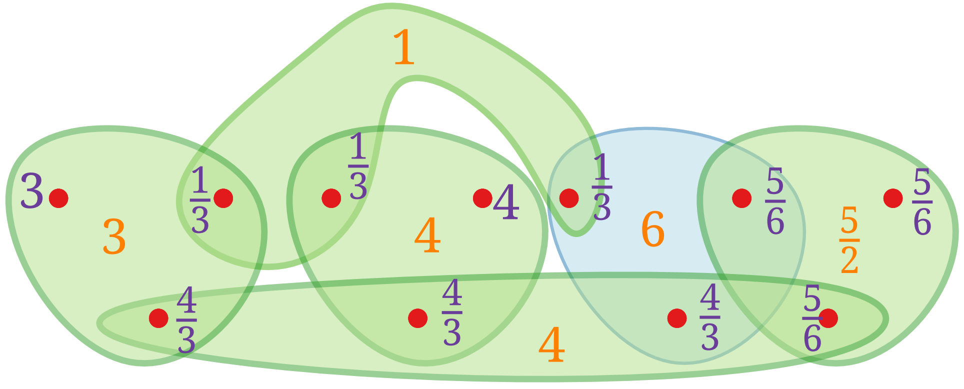

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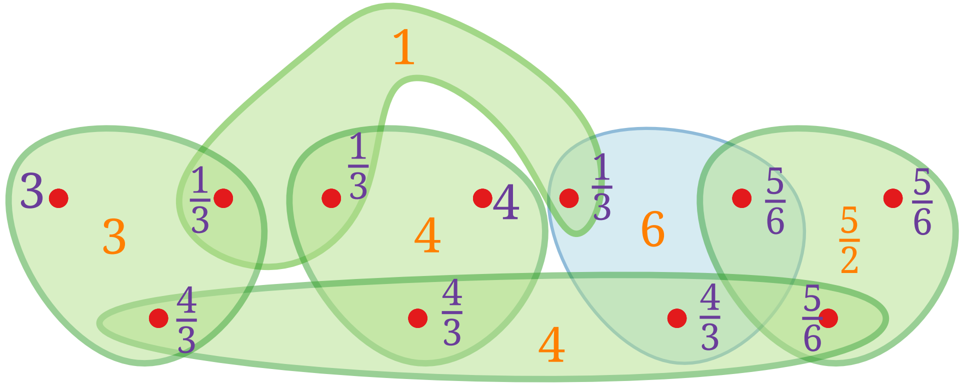

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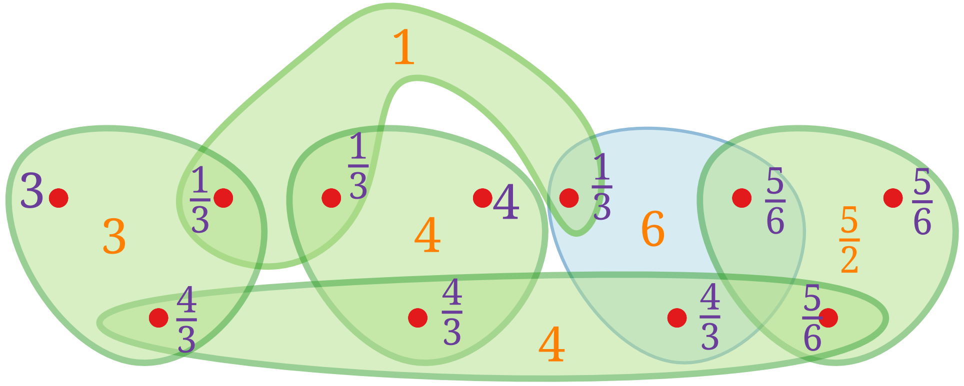

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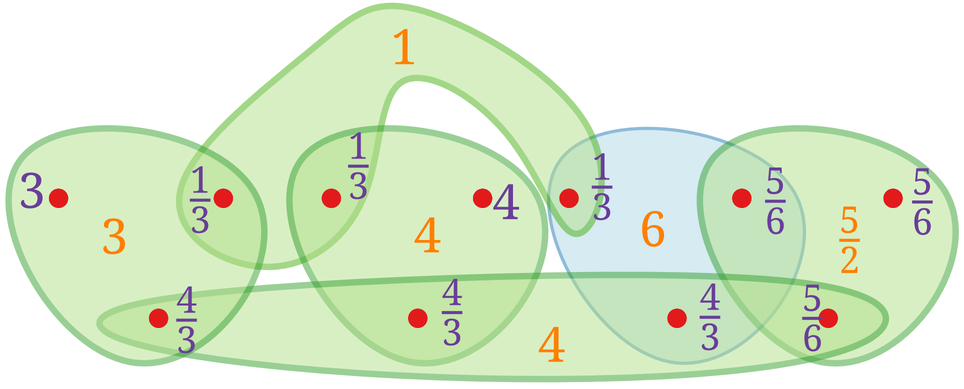

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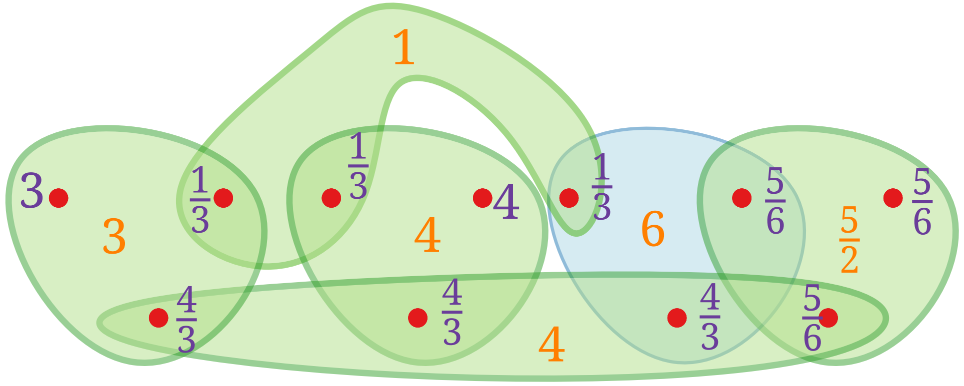

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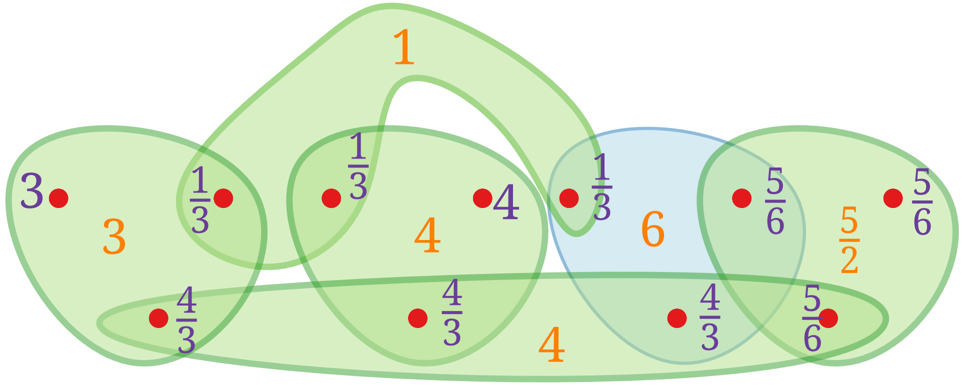

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\mathrm{ALG} = c(\mathcal{S}') \leq \sum_{u \in U} \mathrm{price}(u) &= \mathcal{H}_k \cdot \sum_{u \in U} \bar{y}_u \\
&\leq \mathcal{H}_k \cdot \mathrm{OPT}_{\mathrm{relax}} \\
&\leq \mathcal{H}_k \cdot \mathrm{OPT} \qquad \square
\end{aligned}
$$

Also shows: Integrality gap of relaxation $\leq \mathcal{H}_k$

Dual solution allows a per-instance estimation $c(\mathcal{S}')/\mathrm{OPT}_{\mathrm{relax}}$

of the quality of the greedy solution

…which may be stronger than the worst-case bound $\mathcal{H}_k$.

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next: Randomized Rounding