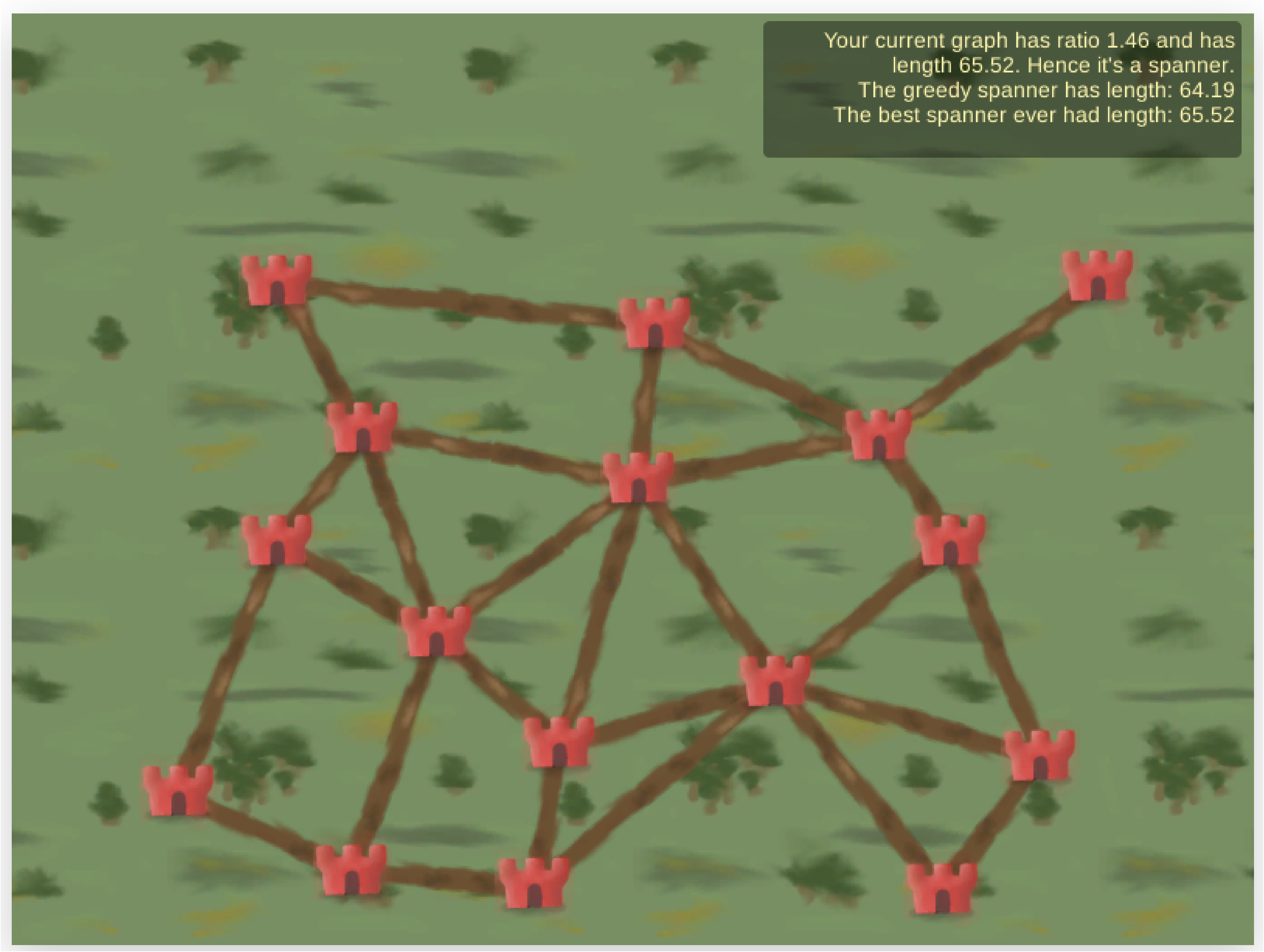
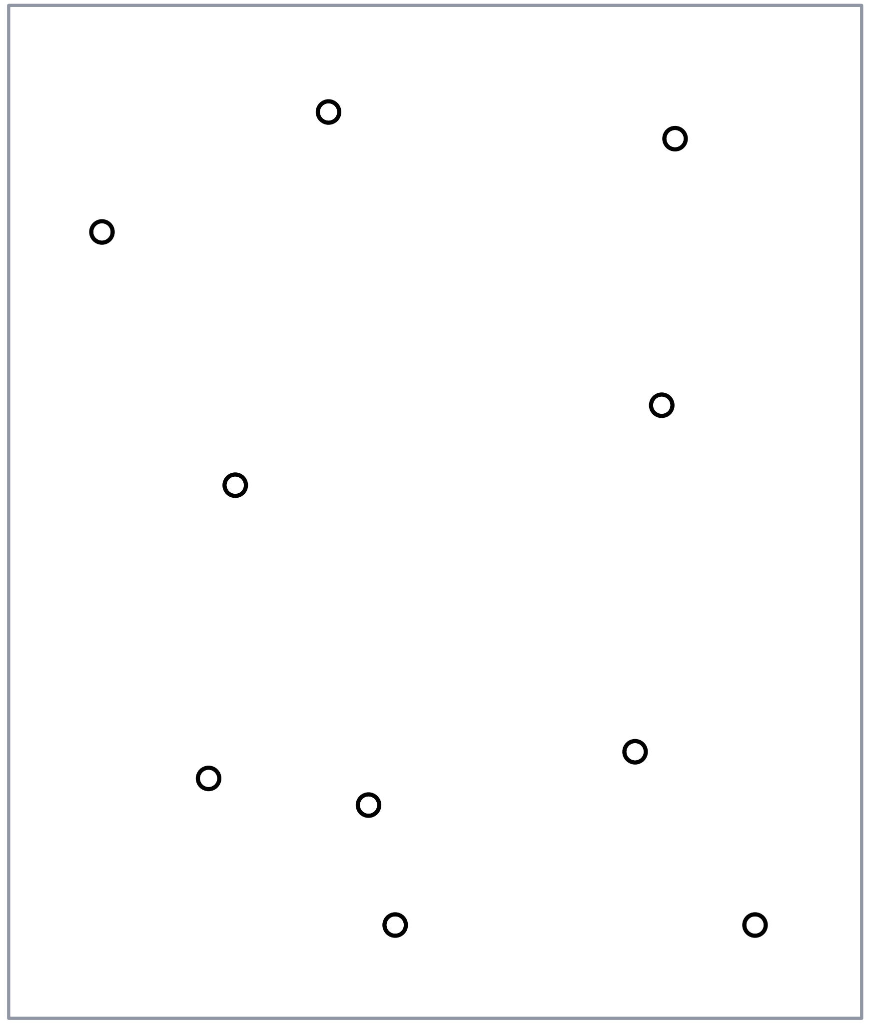
Well-Separated Pair Decomposition

Application: geometric spanners
Construction and size



https://kbuchin.github.io/ruler/

Motivation



Problem: Connect a set of cities by a new street network.

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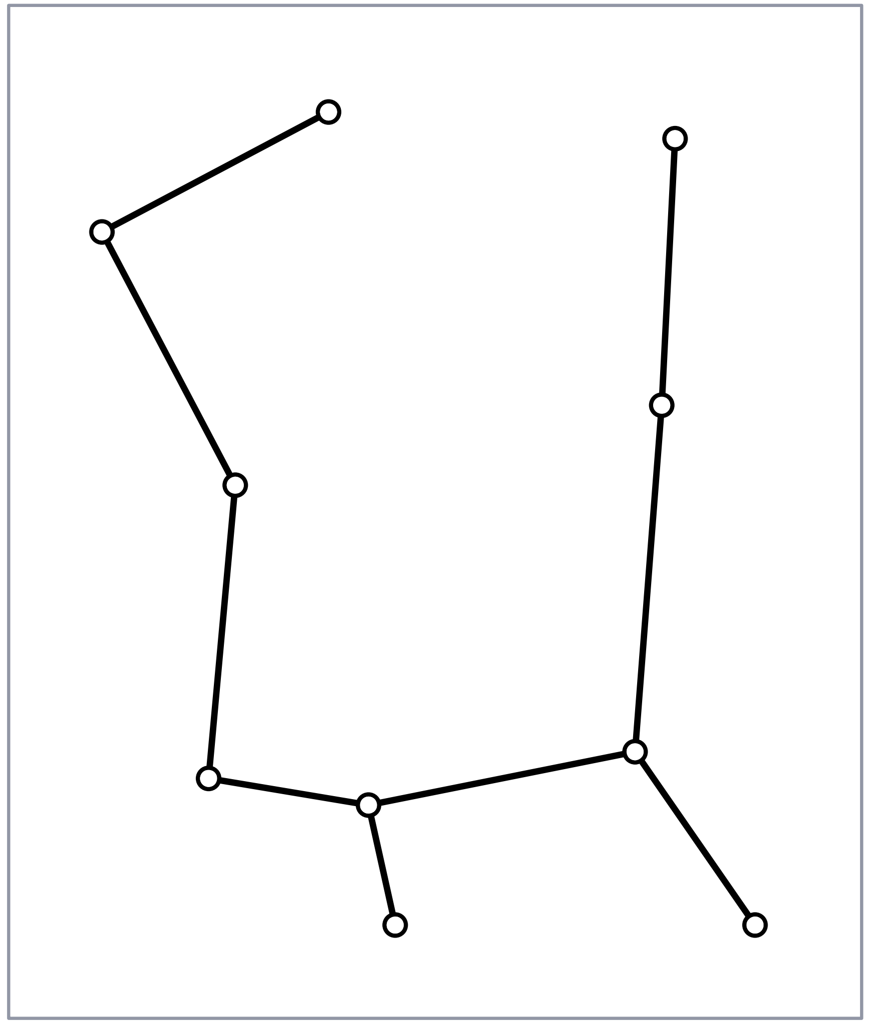

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1. Idea: Euclidean MST

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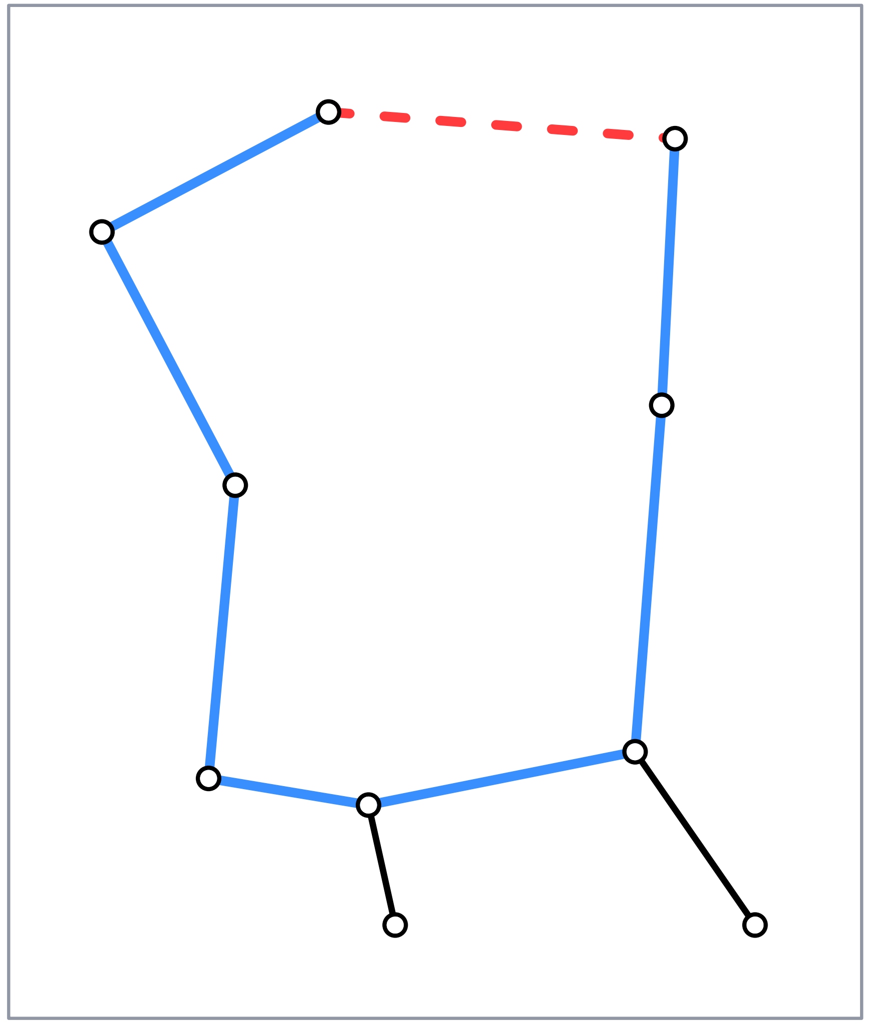

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But for any pair (x, y) the graph distance shouldn't be much longer than $\|xy\|$

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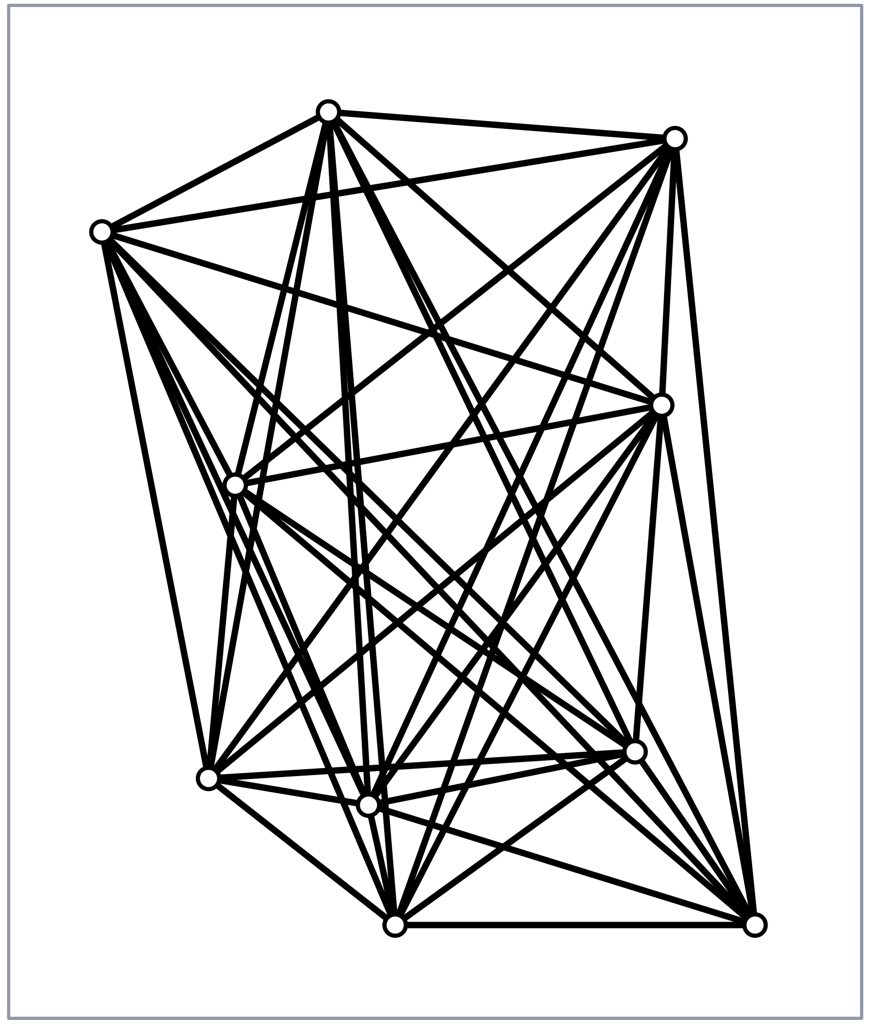

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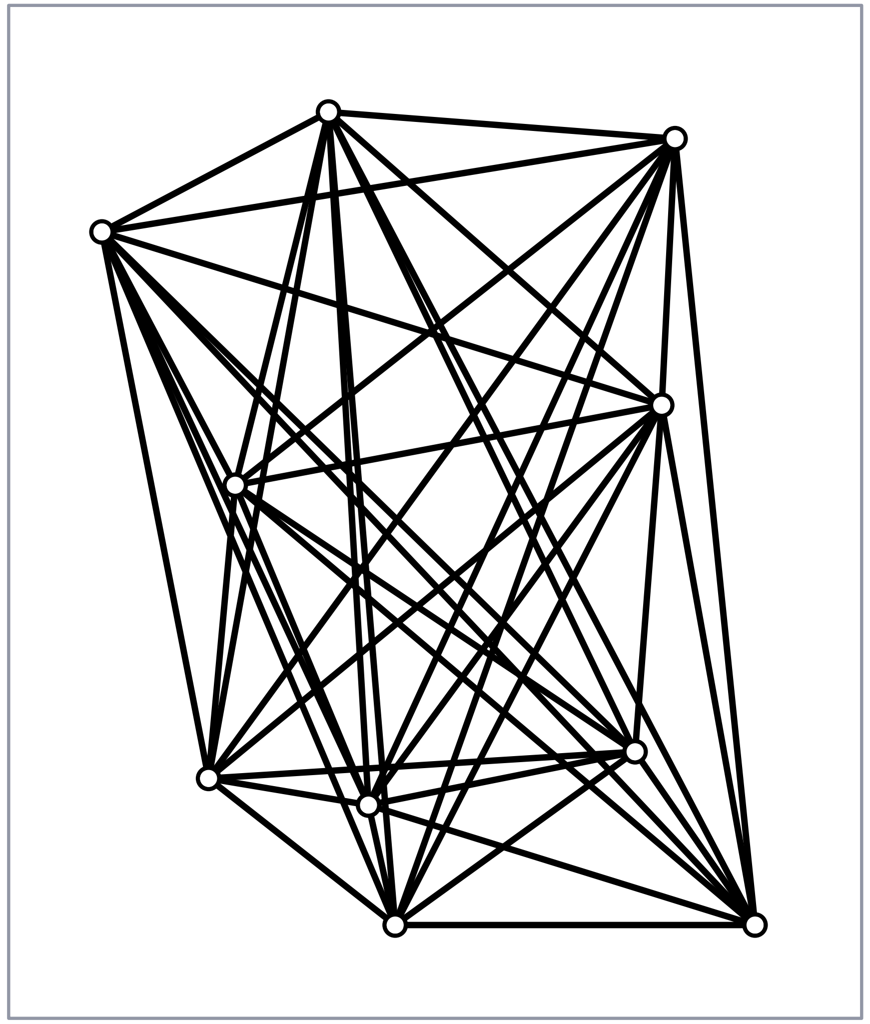

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   The budget for roads only pays for $O(n)$ roads.

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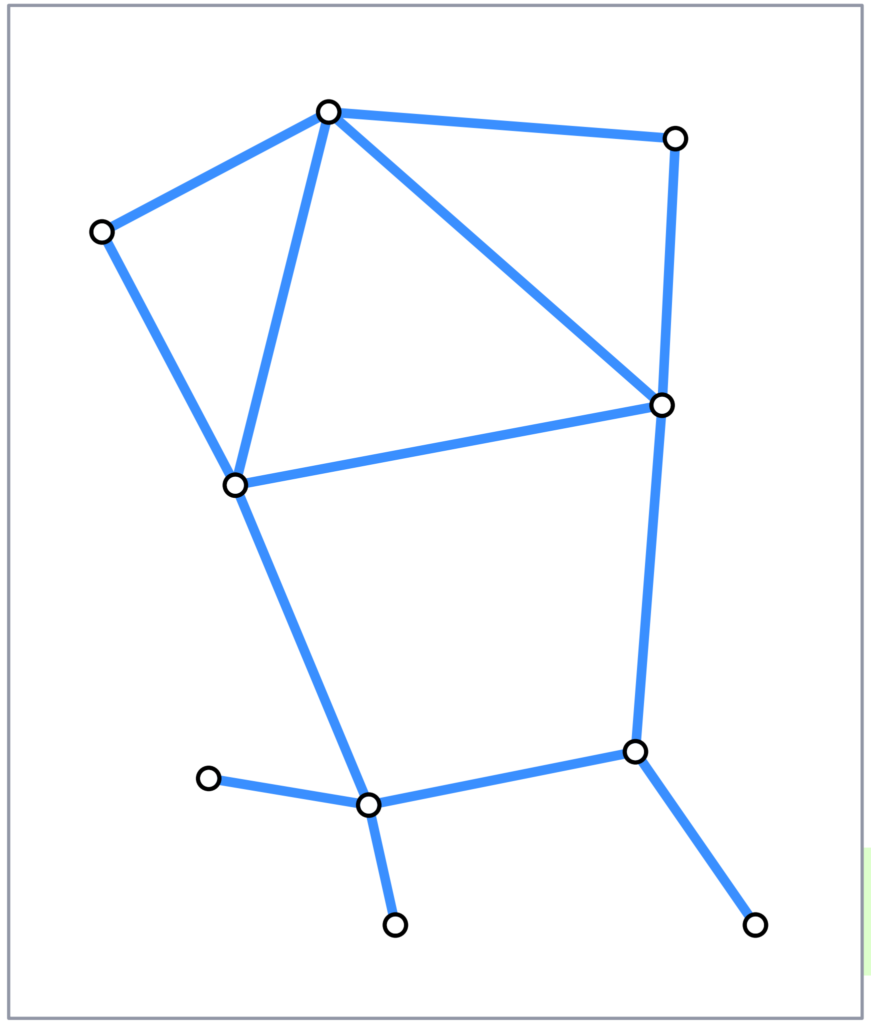

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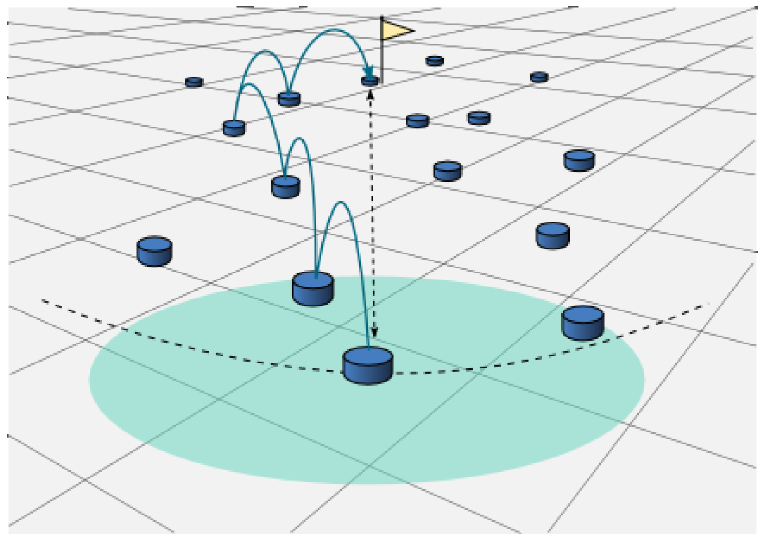
3. Idea: sparse t-spanner

$O(n)$ edges

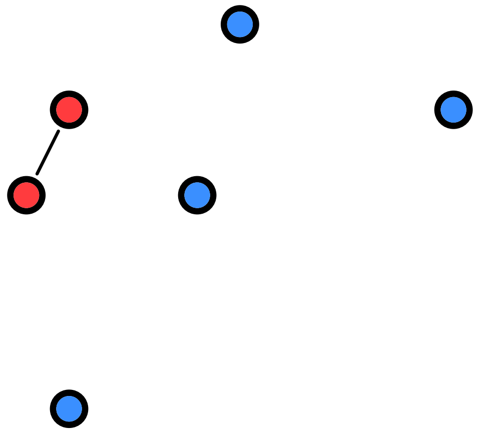
detour $\leq t \cdot$shortest path

Applications



communication and connectivity in networks

- topology control in wireless networks
- routing in networks
- network analysis



fast, approximate distance computation

- geometric approximation algorithms for diameter etc.
- exact algorithms: closest pair, . . . , Voronoi diagrams

t-spanner

For a set P of n points in $\mathbb{R}^d$ the Euclidean graph $\mathcal{EG}(P) = (P, \binom{P}{2})$ is the complete, weighted graph with Euclidean distances as edge weights.

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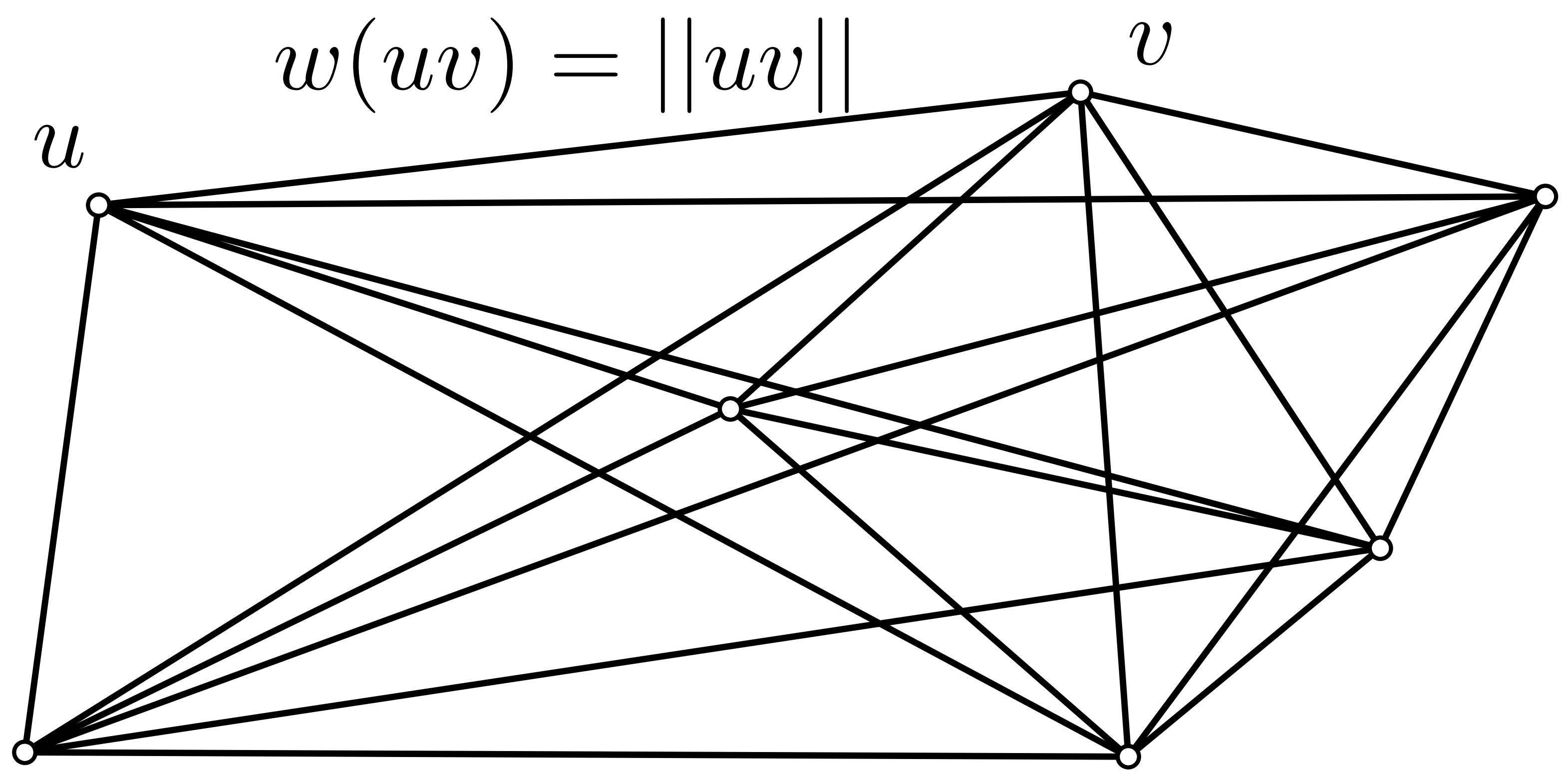

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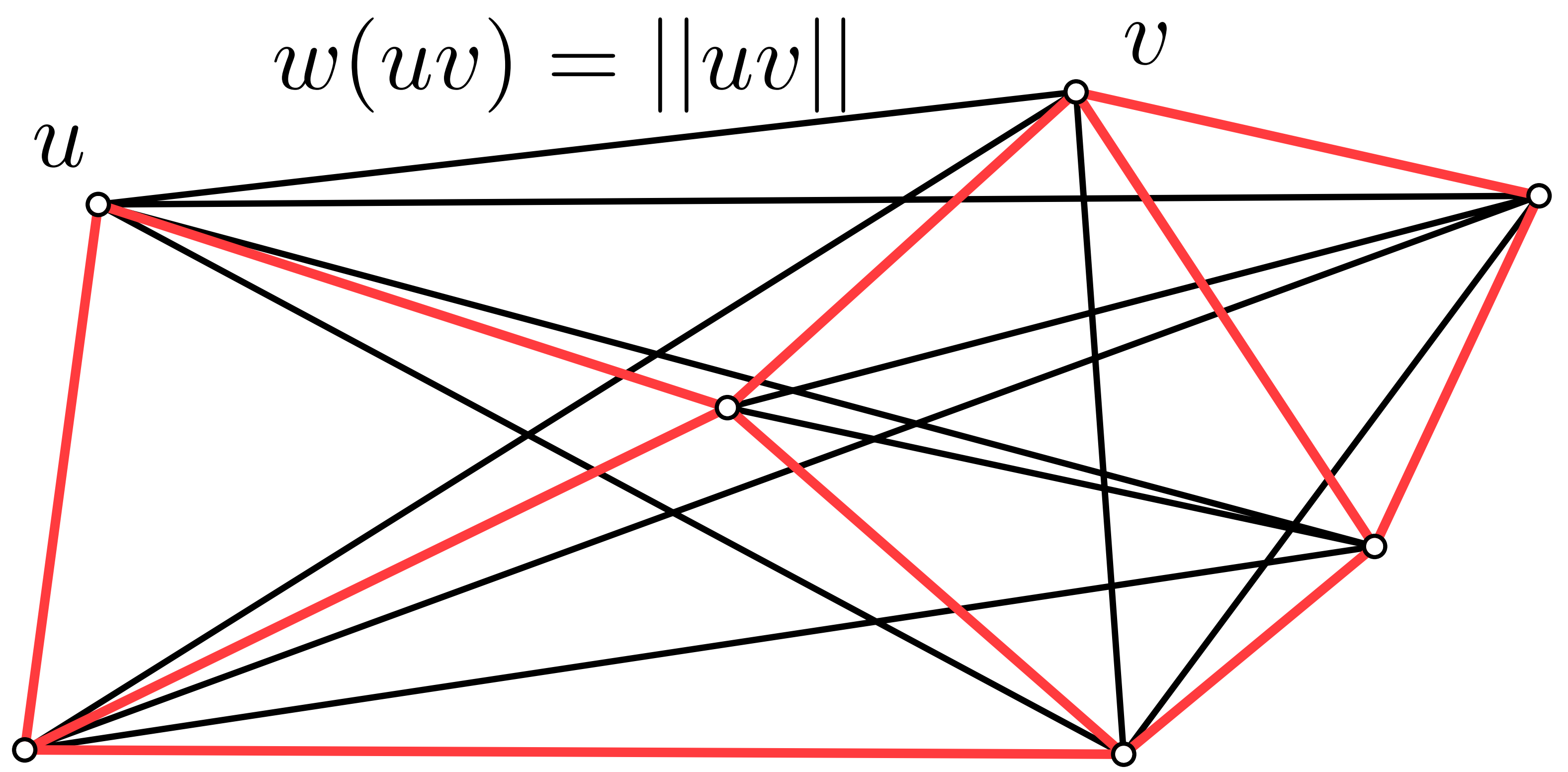
Since $\mathcal{EG}(P)$ has $\Theta(n^2)$ edges, we want a sparse graph with $O(n)$ edges such that the shortest paths in the graph approximate the edge weights of $\mathcal{EG}(P)$.



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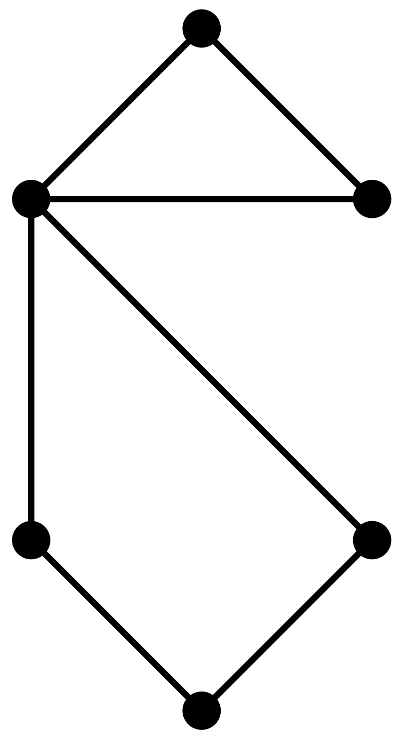
Definition: A weighted graph G with vertex set P is called t-spanner for P and a stretch factor $t \geq 1$, if for all pairs $x, y \in P$:

$$||xy|| \leq \delta_G(x, y) \leq t \cdot ||xy|| \, ,$$

where $\delta_G(x, y) =$ length of the shortest x-to-y path in G.

Quiz

What is the smallest t for which the following graph is a t-spanner?



A: $\sqrt{2}$

B: 2

C: $\sqrt{2} + 1$

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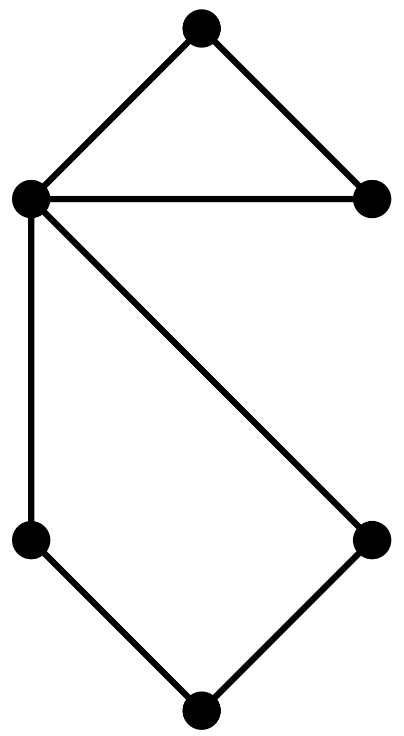

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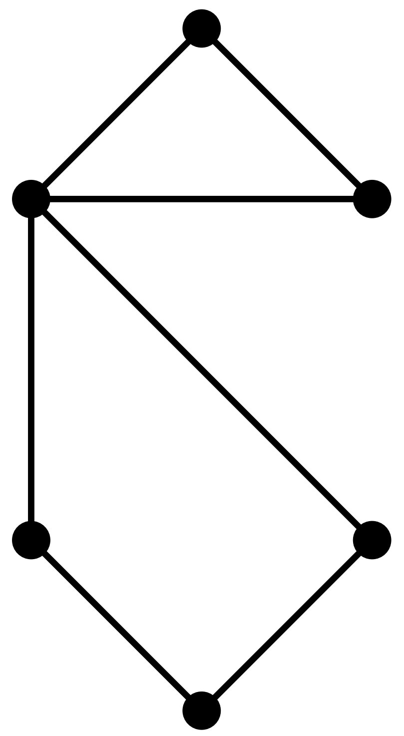

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How can we compute a t-spanner?

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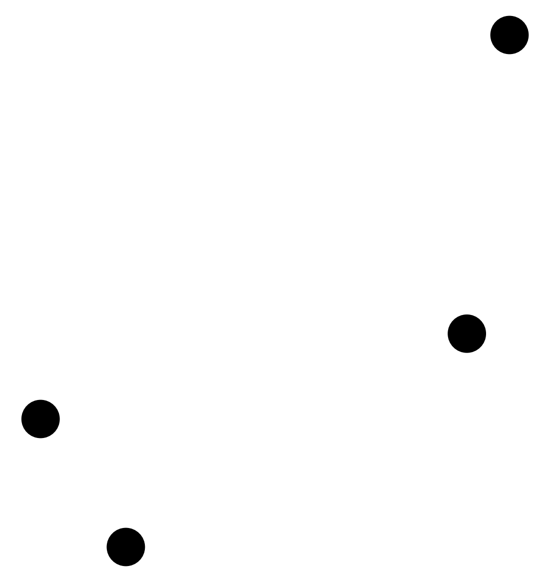
Spanner construction paradigms

greedy

- sort point pairs by distance, start with no edges
- if for the next point pair the dilation is $> t$ then add corresponding edge



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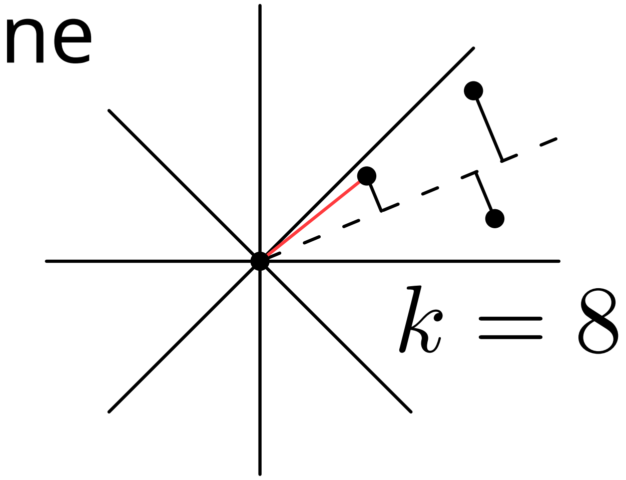
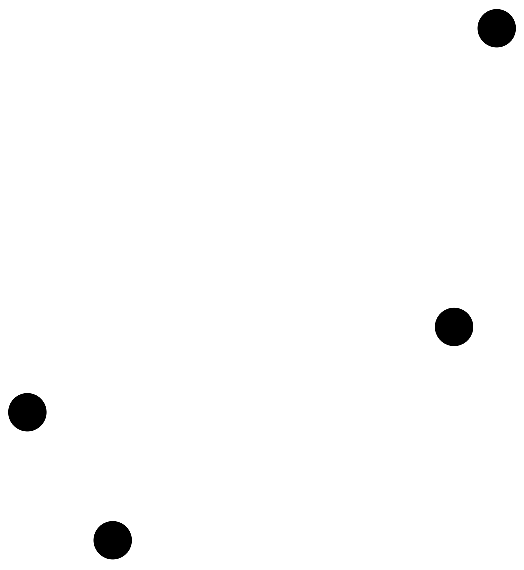
- subdivide space around each point into $k > 6$ non-overlapping cones with angle $\phi = 2\pi/k$
- connect to “closest” point in each cone





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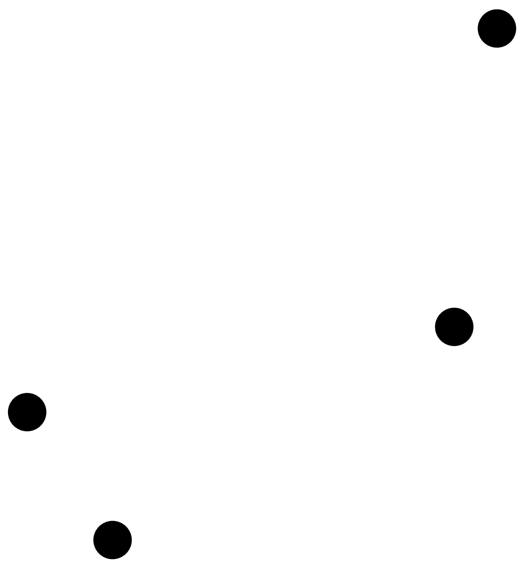

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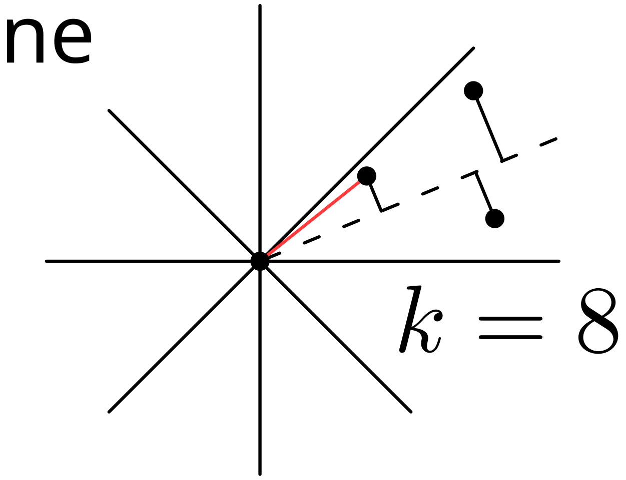


distance approximation

- well-separated pair decomposition (next!)

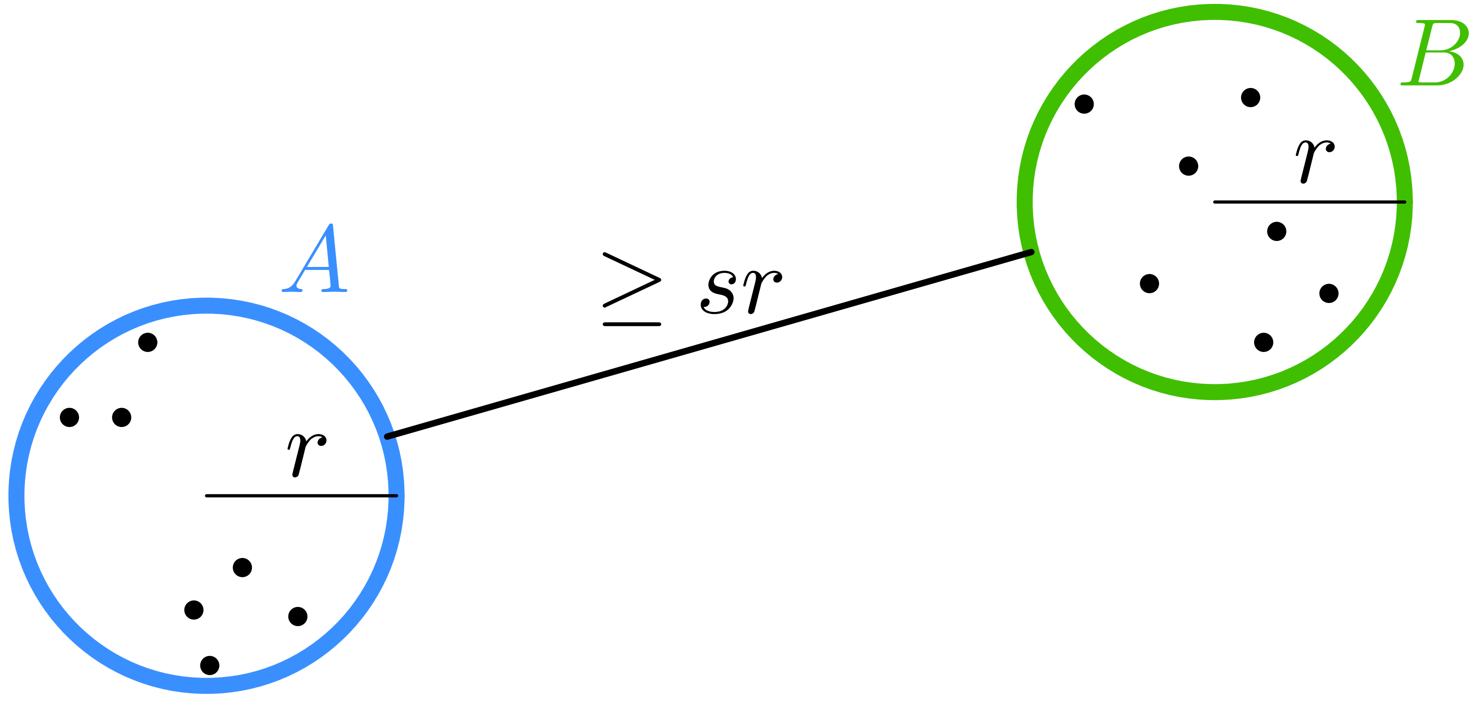
Well-Separated Pair Decomposition

Definition

Compressed Quadtrees

Well-Separated Pairs

Definition: A pair of disjoint point sets A and B in $\mathbb{R}^d$ is called s-well separated for an $s > 0$, if A and B both can be covered by a ball of radius r and the distance between the balls is at least sr.



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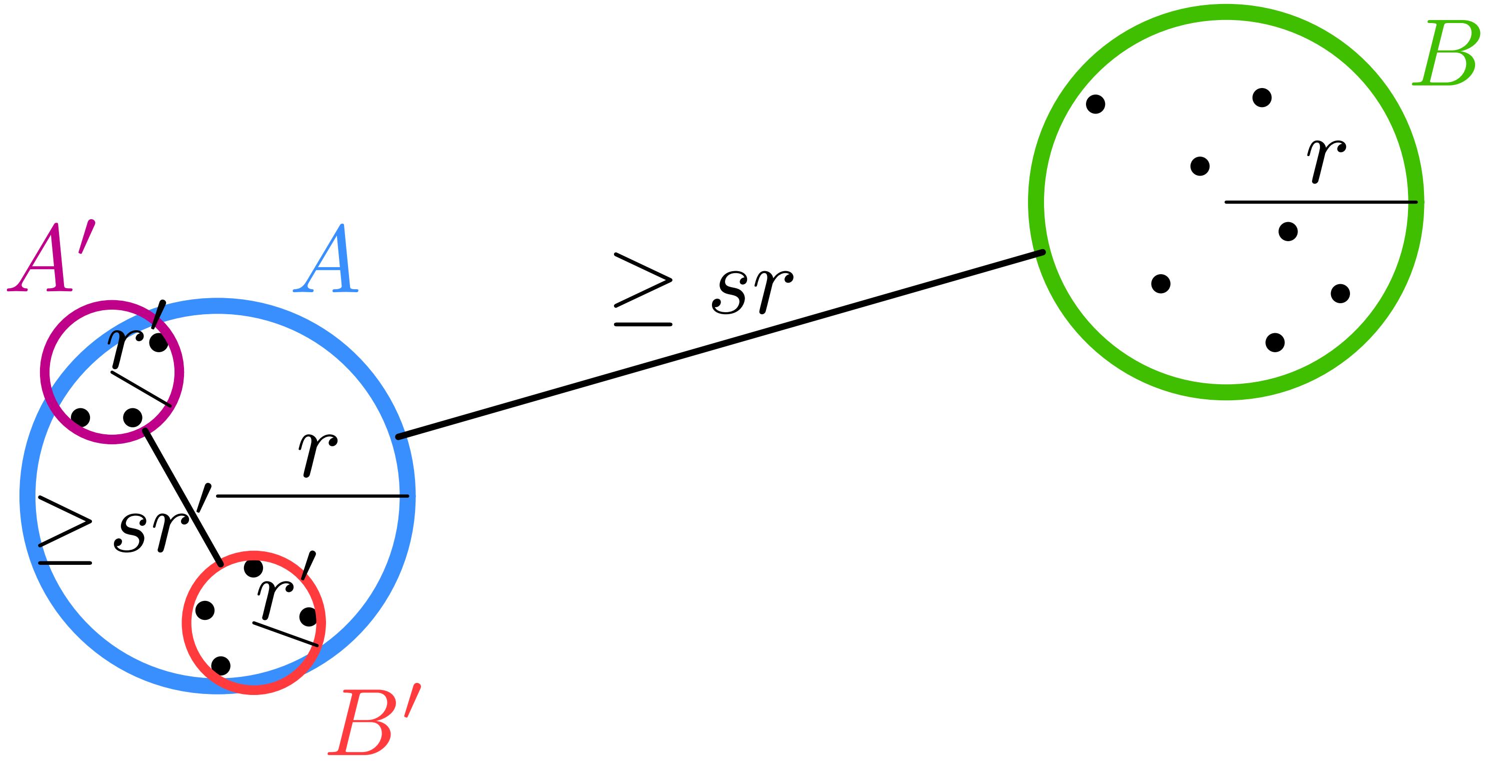

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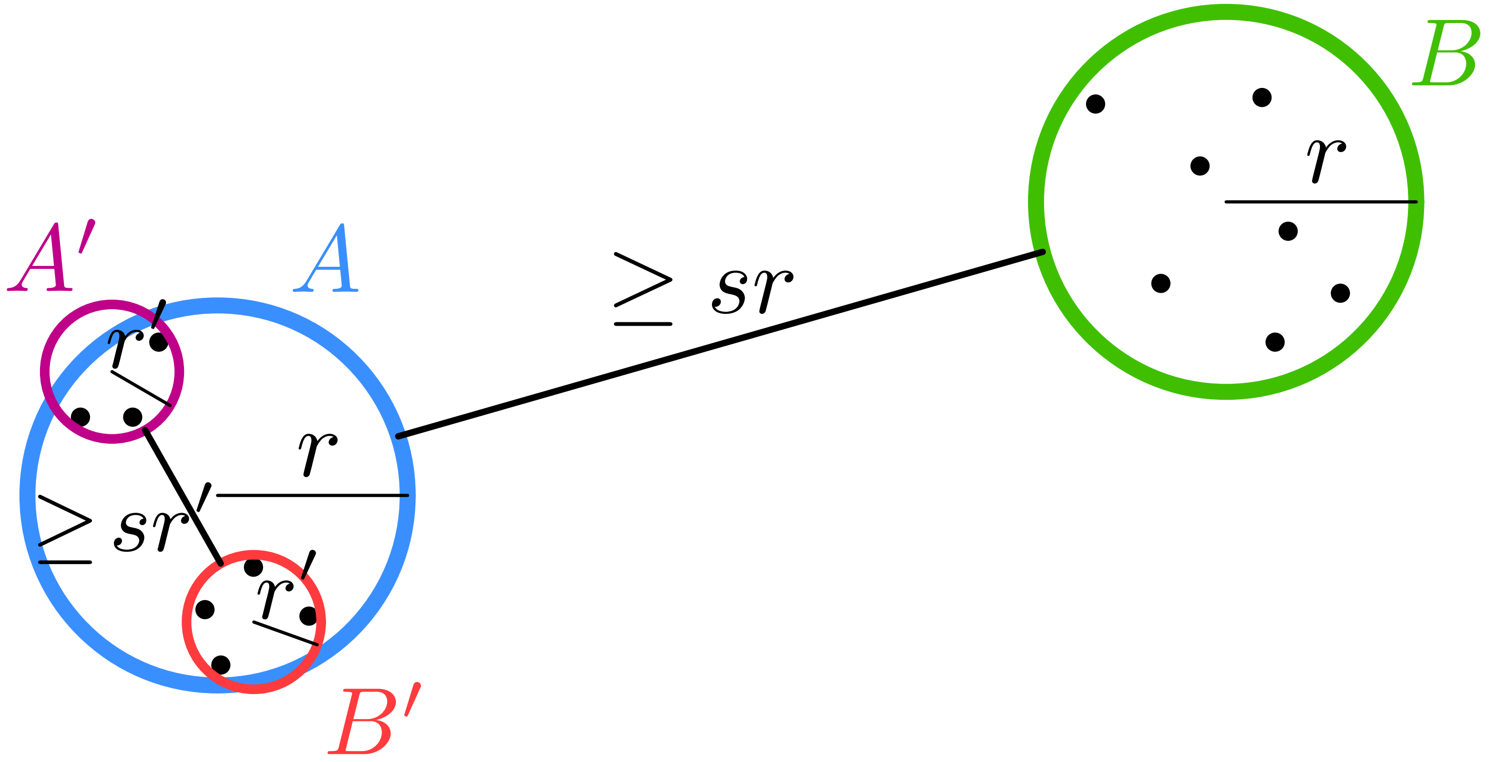

Observation:

- s-well separated $\Rightarrow$ s'-well separated for all $s' \leq s$
- singletons $\{a\}$ and $\{b\}$ are s-well separated for all $s > 0$

Well-Separated Pair Decomposition

For a well-separated pair $\{A, B\}$ the distance between all point pairs in $A \otimes B := \{\{a, b\} \mid a \in A, b \in B, a \neq b\}$ is similar.

Goal: $o(n^2)$-data structure that approximates all $\binom{n}{2}$ pairwise distances of a point set $P = \{p_1, \ldots, p_n\}$.

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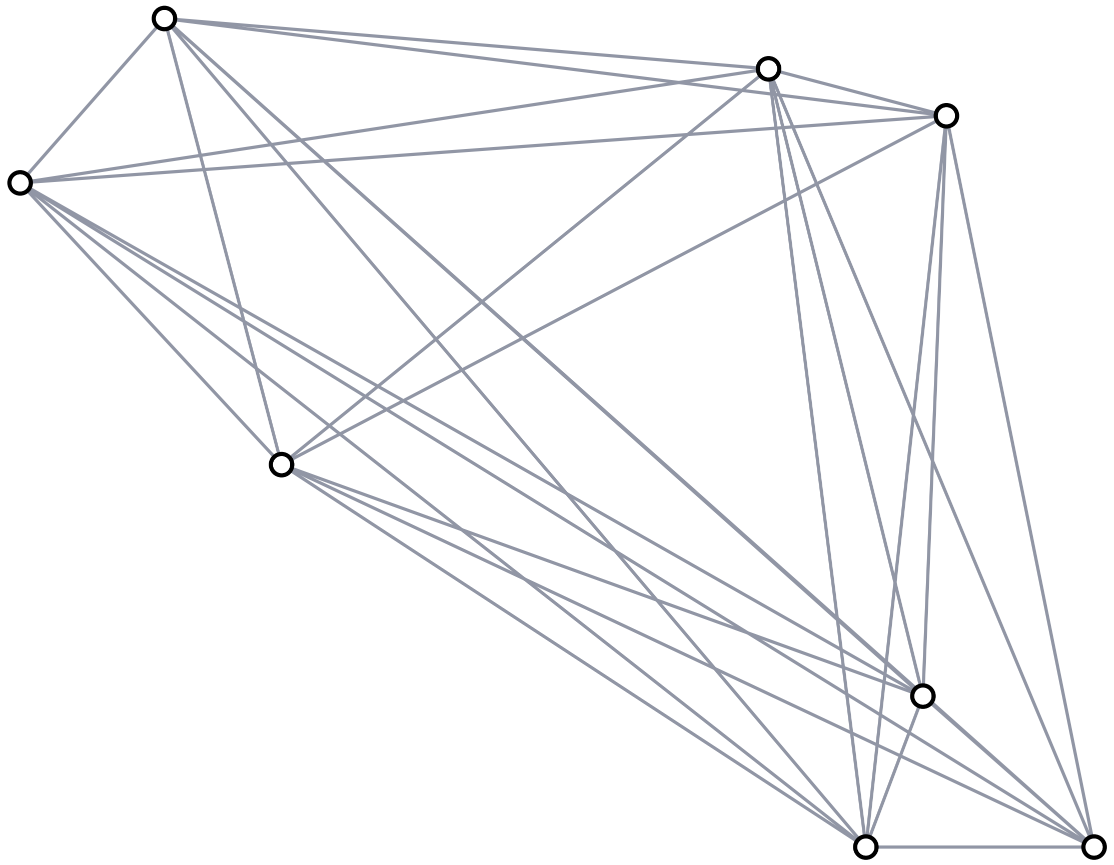
Definition: For a set of points P and $s > 0$ an s-well separated pair decomposition (s-WSPD) is a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ with

- $A_i, B_i \subset P$ for all i
- $A_i \cap B_i = \varnothing$ for all i
- $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$
- $\{A_i, B_i\}$ s-well separated for all i

Example



28 pairs of points

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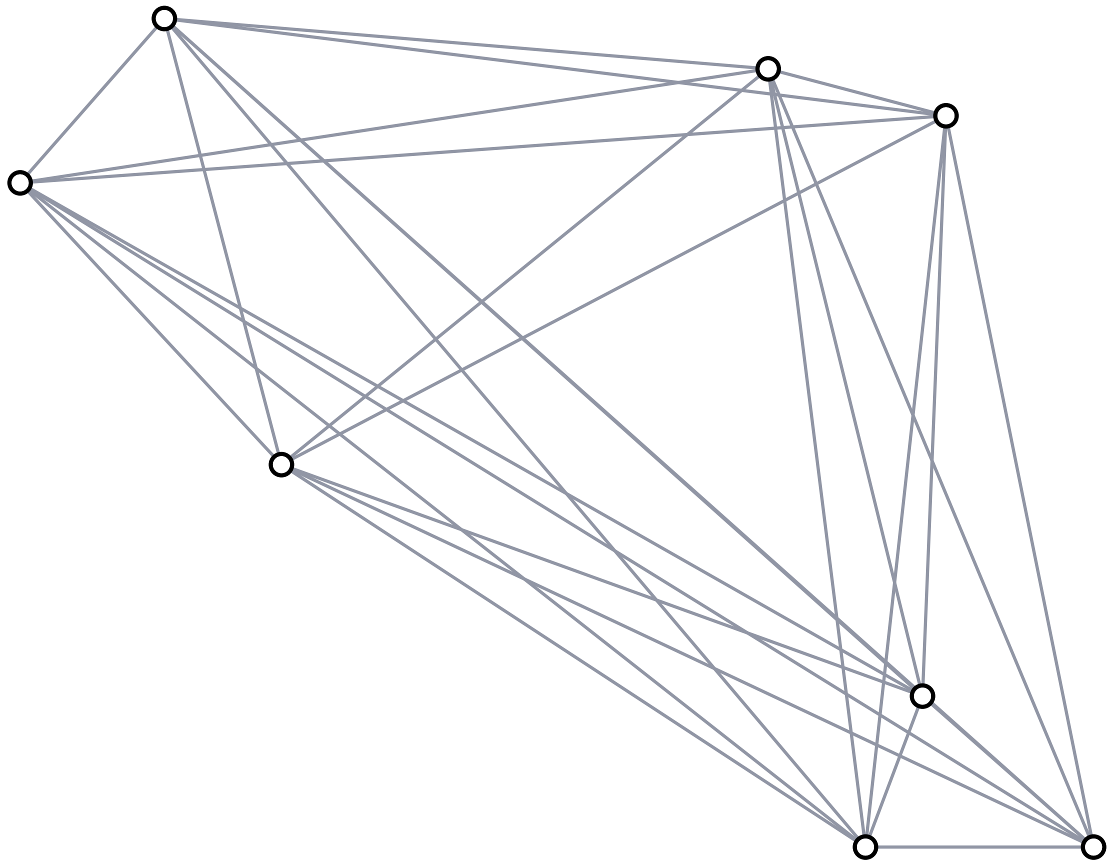

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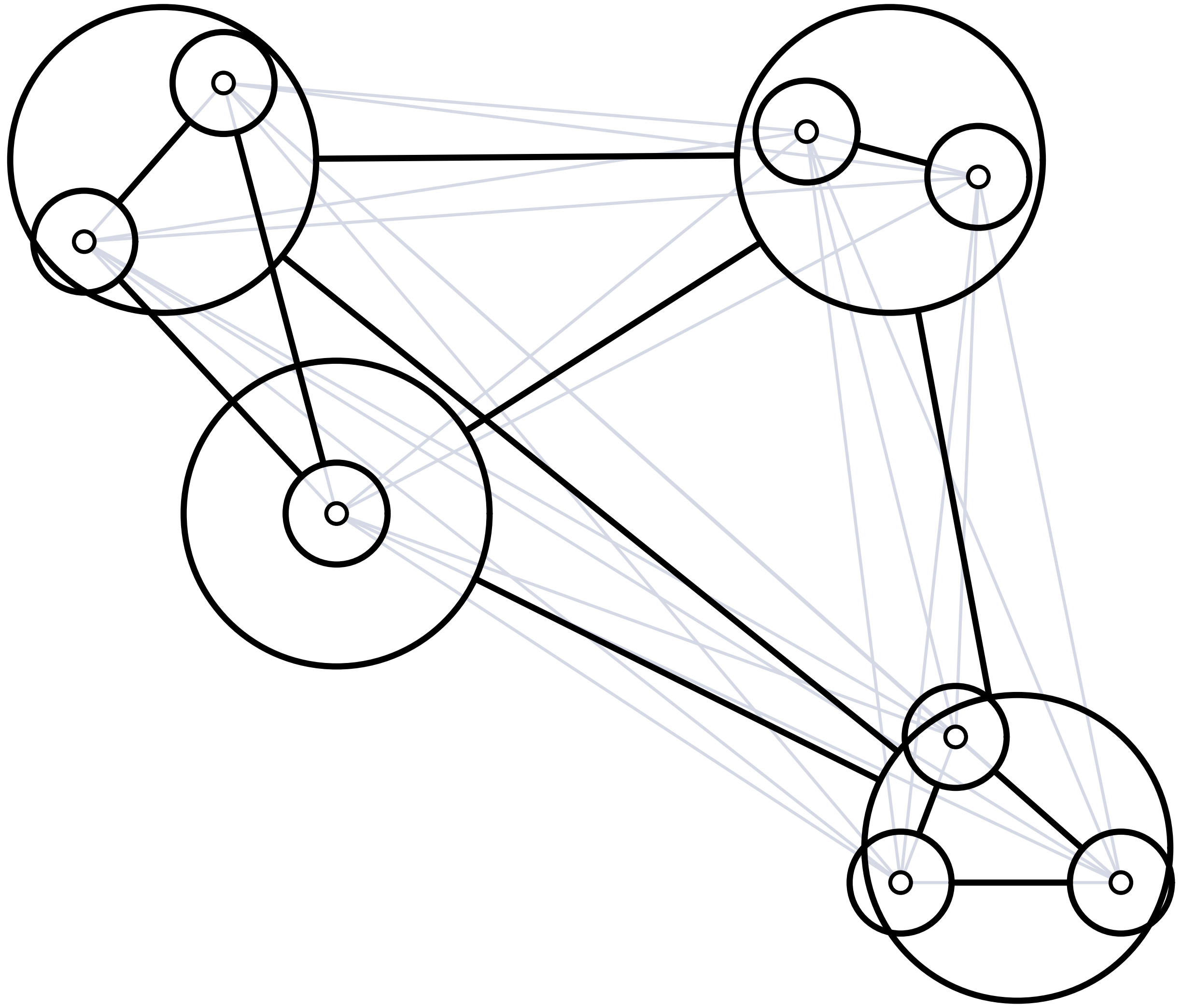

12 s-well separated pairs

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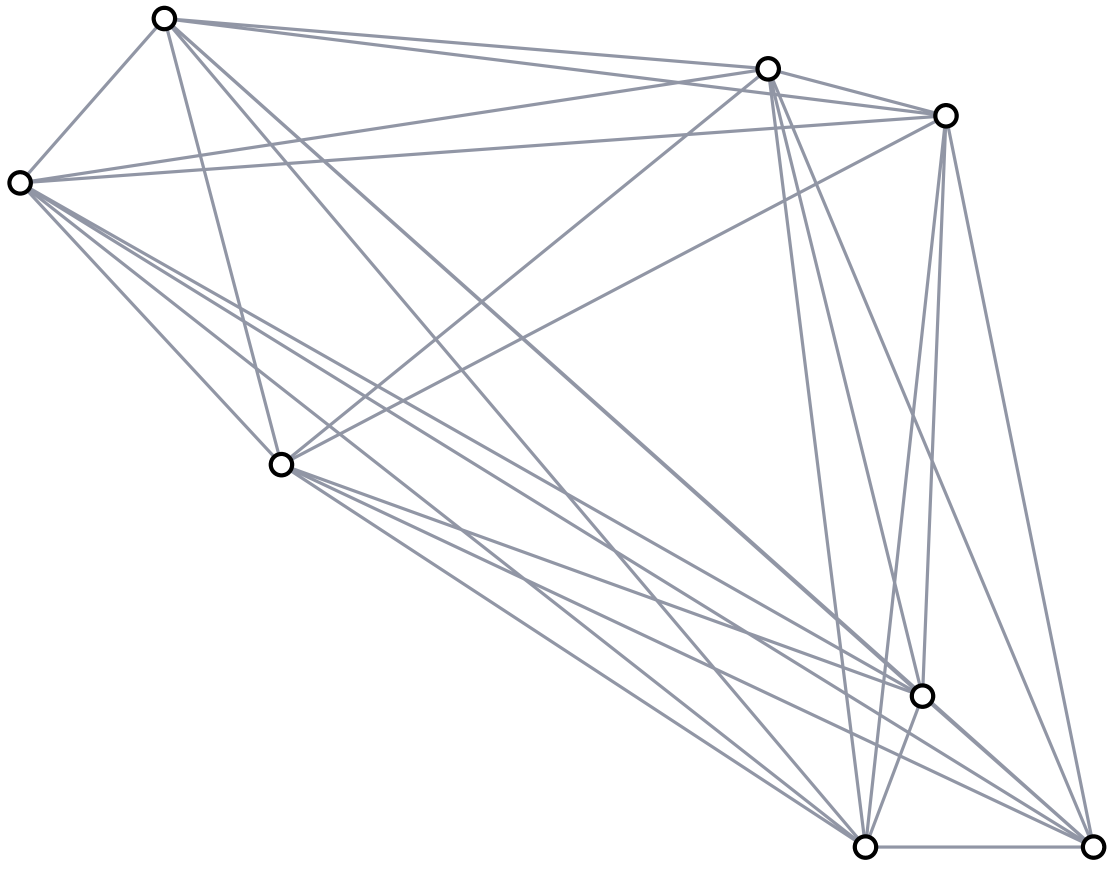

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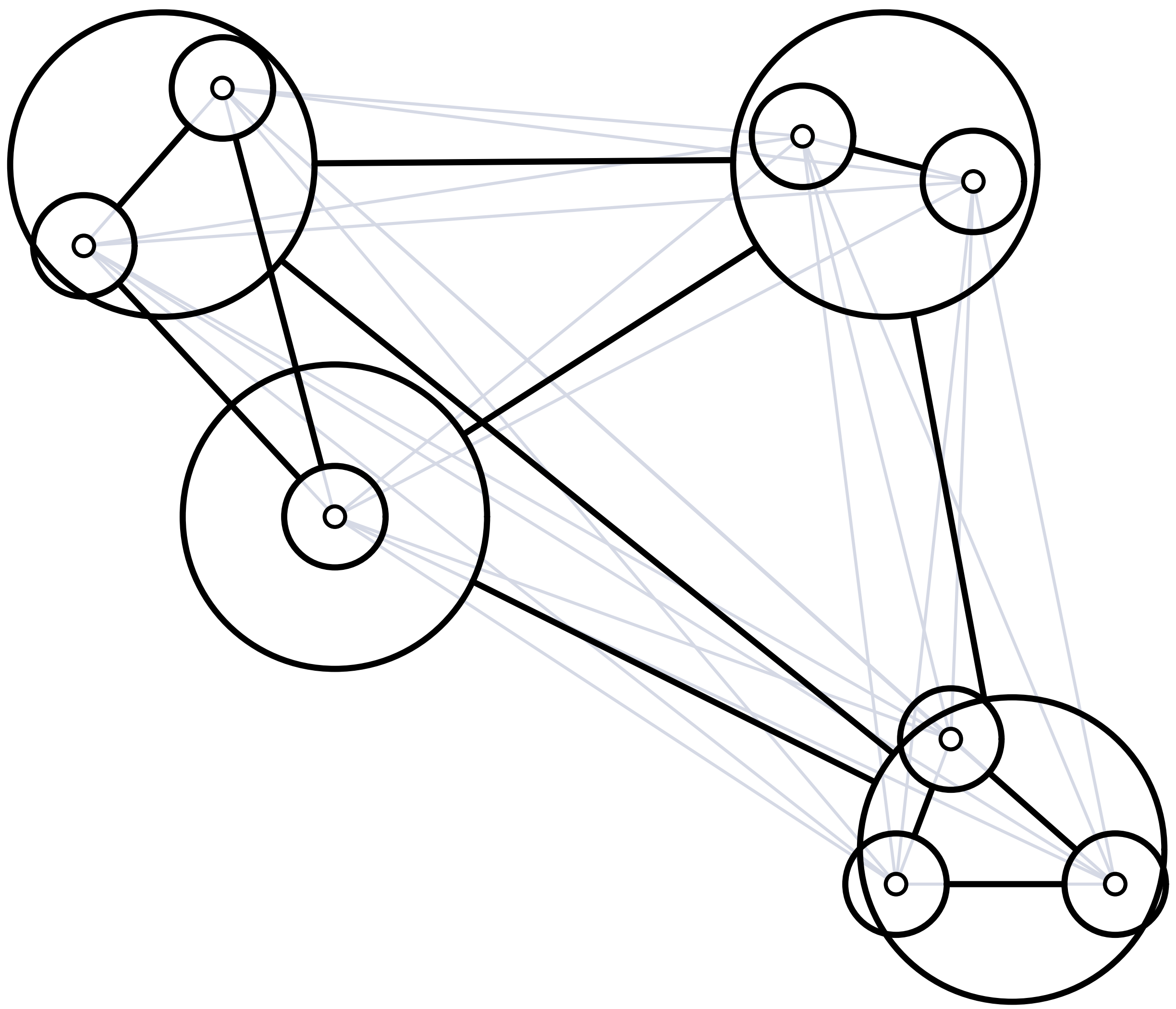

12 s-well separated pairs

WSPD of size $O(n^2)$ is trivial.

What is the ‘size’? Can we get size $O(n)$?

Quiz

What size does a 2-WSPD on the following point set have at least?

A: 3

B: 4

C: 5

D: 6

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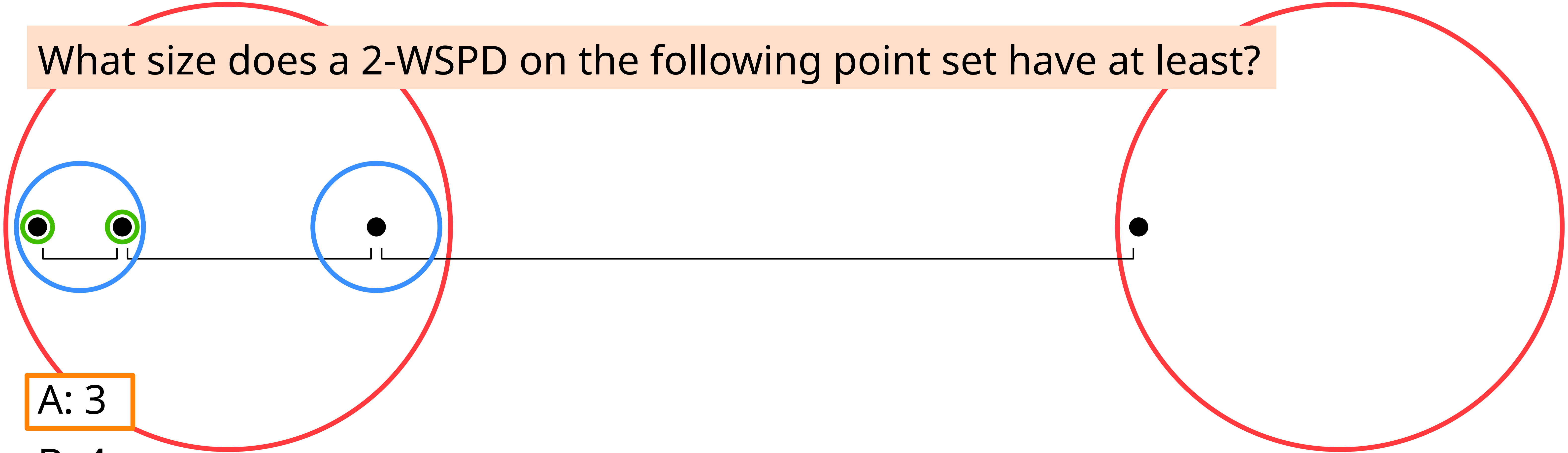

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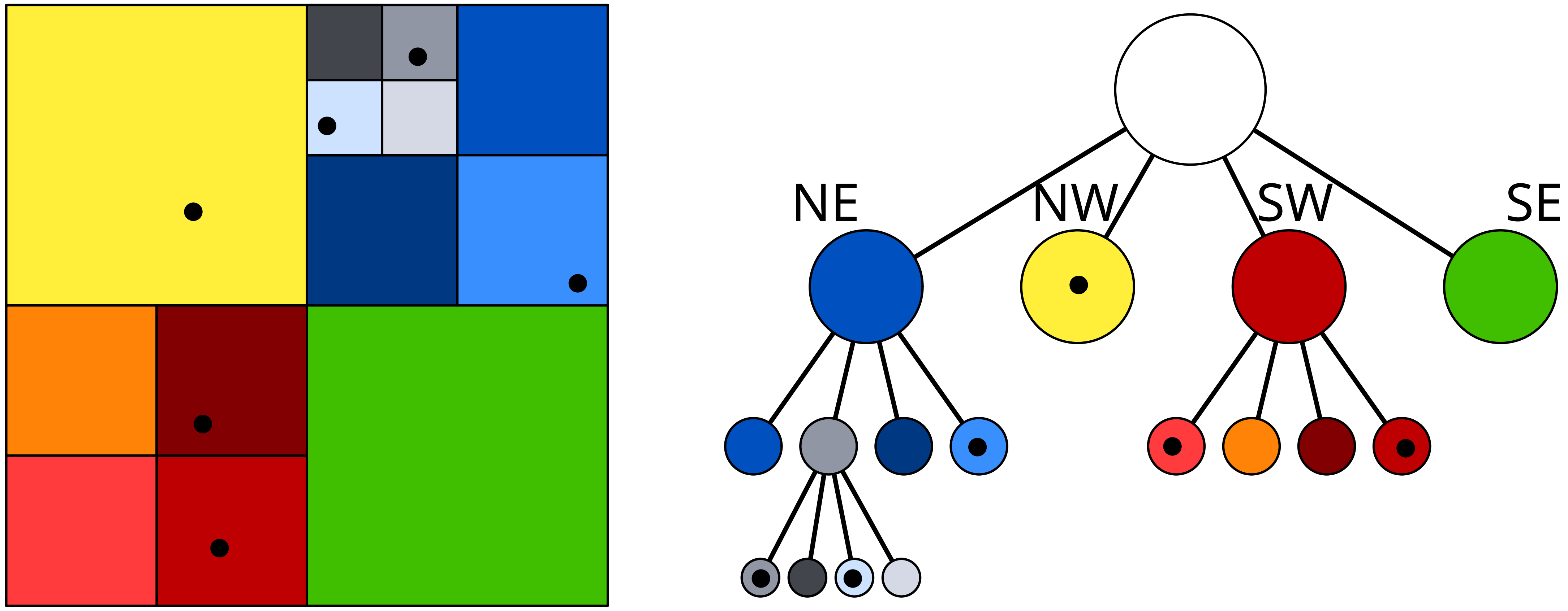
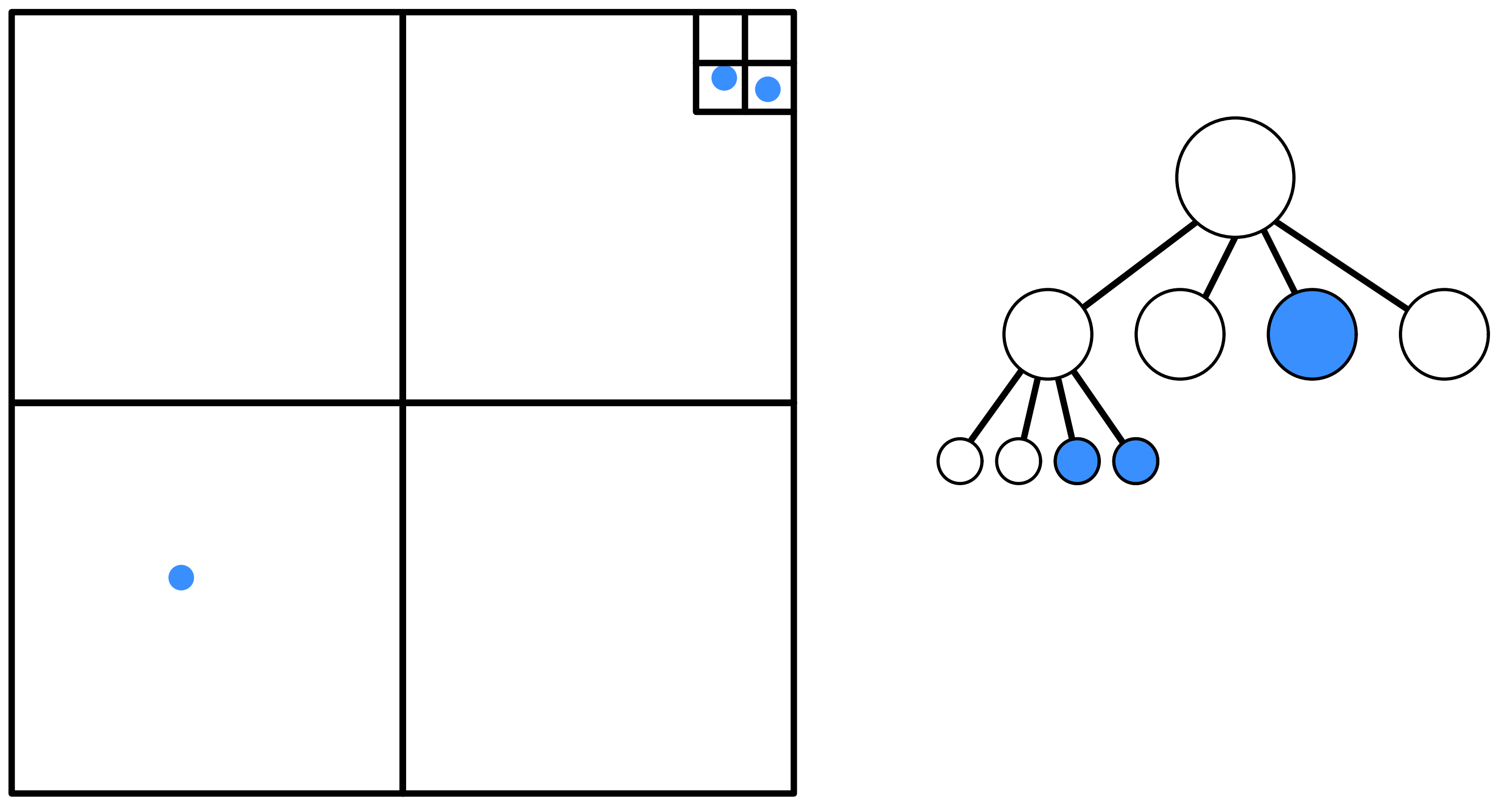
Reminder: quadtrees

Definition: A quadtree is a rooted tree, in which every interior node has 4 children. Every node corresponds to a square, and the squares of children are the quadrants of the parent's square.



Compressed quadtrees

Definition: A compressed quadtree is a quadtree in which paths of non-separating inner nodes are compressed to an edge.

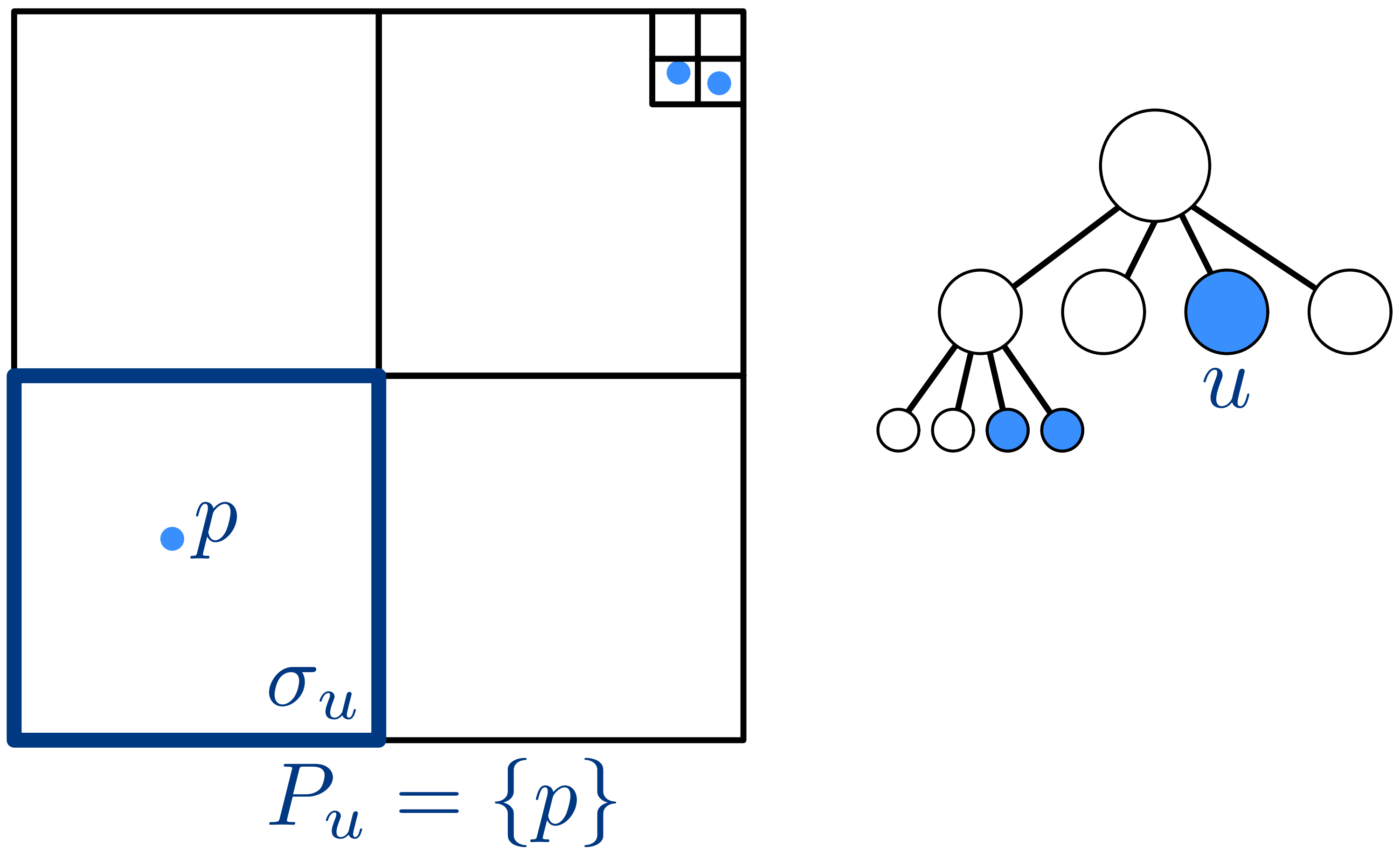


Theorem 2: A compressed quadtree for n points in $\mathbb{R}^d$ for fixed d has size $O(n)$ and can be computed in $O(n \log n)$ time.

(without proof)

Representative and Level

Definition: For every node u of a quadtree $\mathcal{T}(P)$ let $P_u = \sigma_u \cap P$, where σ_u is the square corresponding to u.



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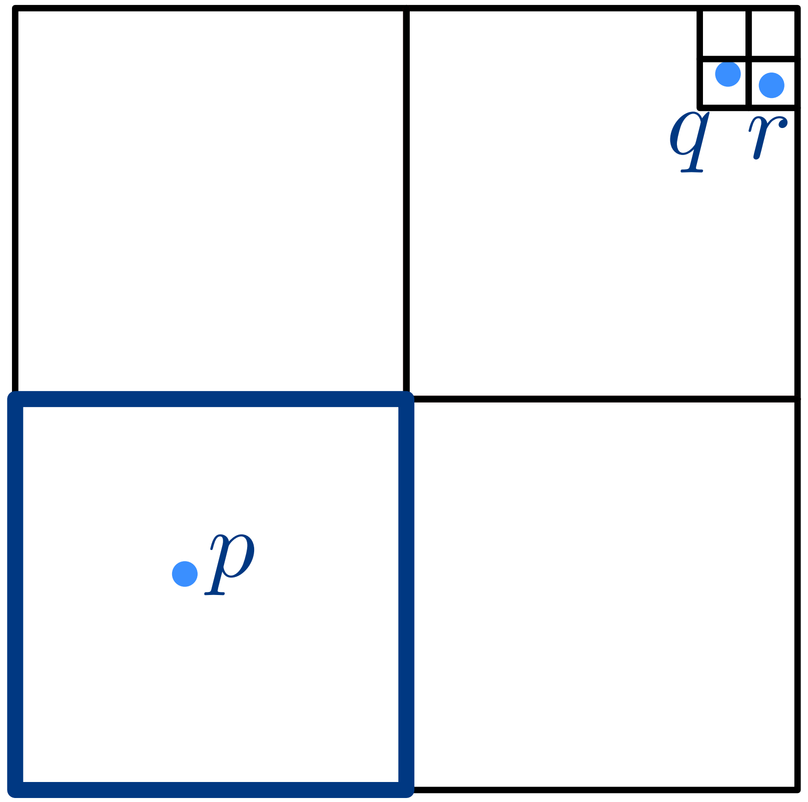

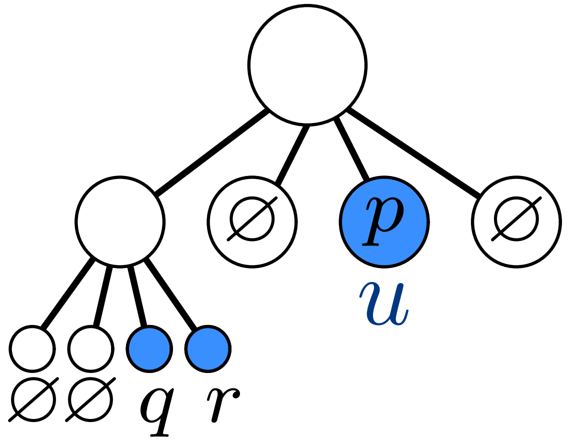

For every leaf u define the representative

$$\mathrm{rep}(u) = \begin{cases} p & \text{if } P_u = \{p\} \text{ (} u \text{ is leaf)} \\ \varnothing & \text{otherwise.} \end{cases}$$

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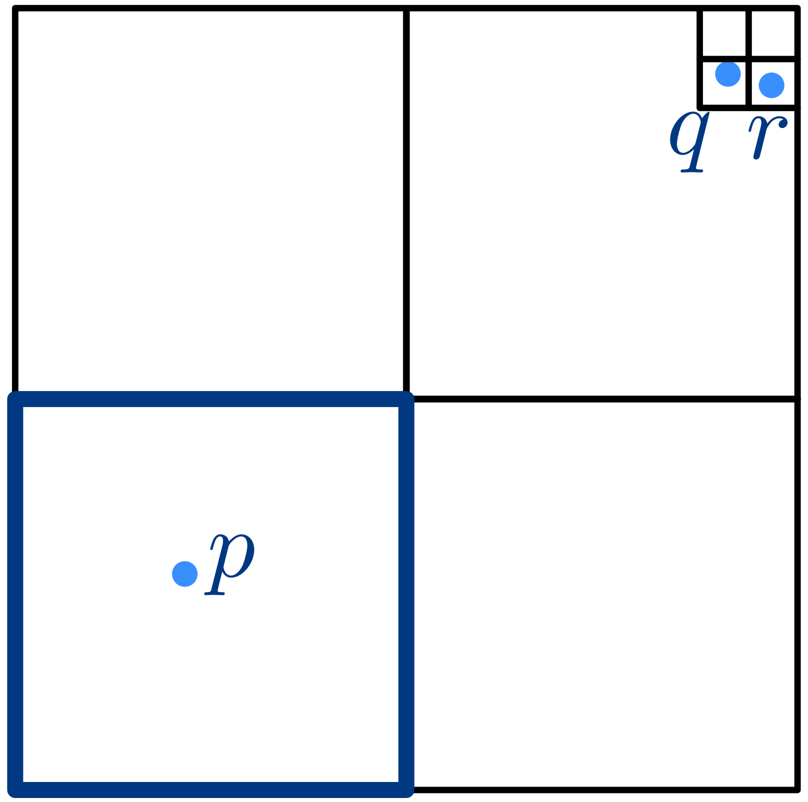

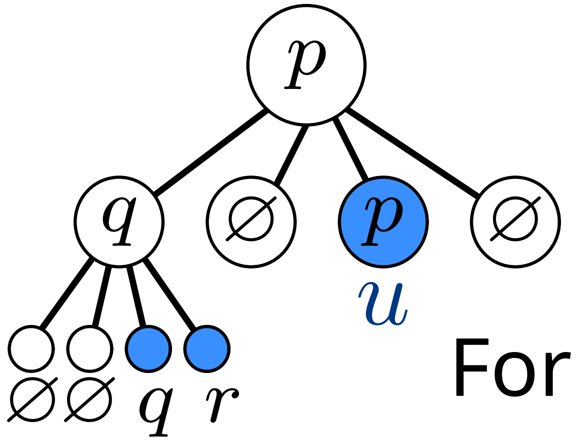

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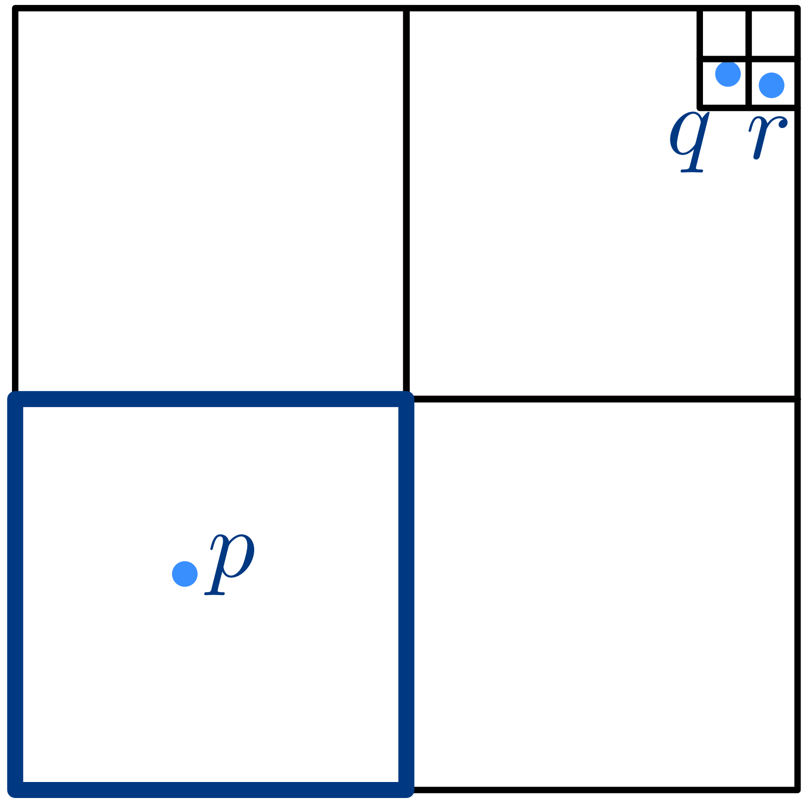

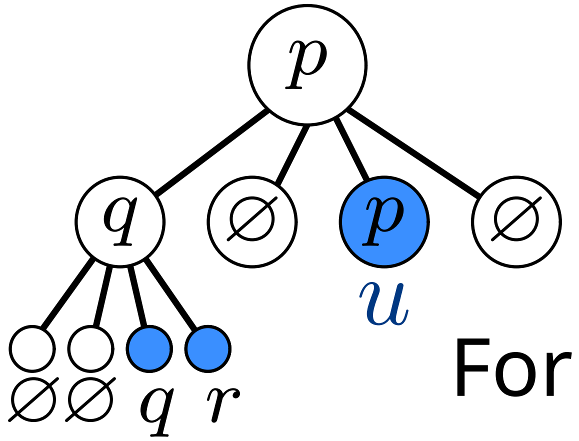

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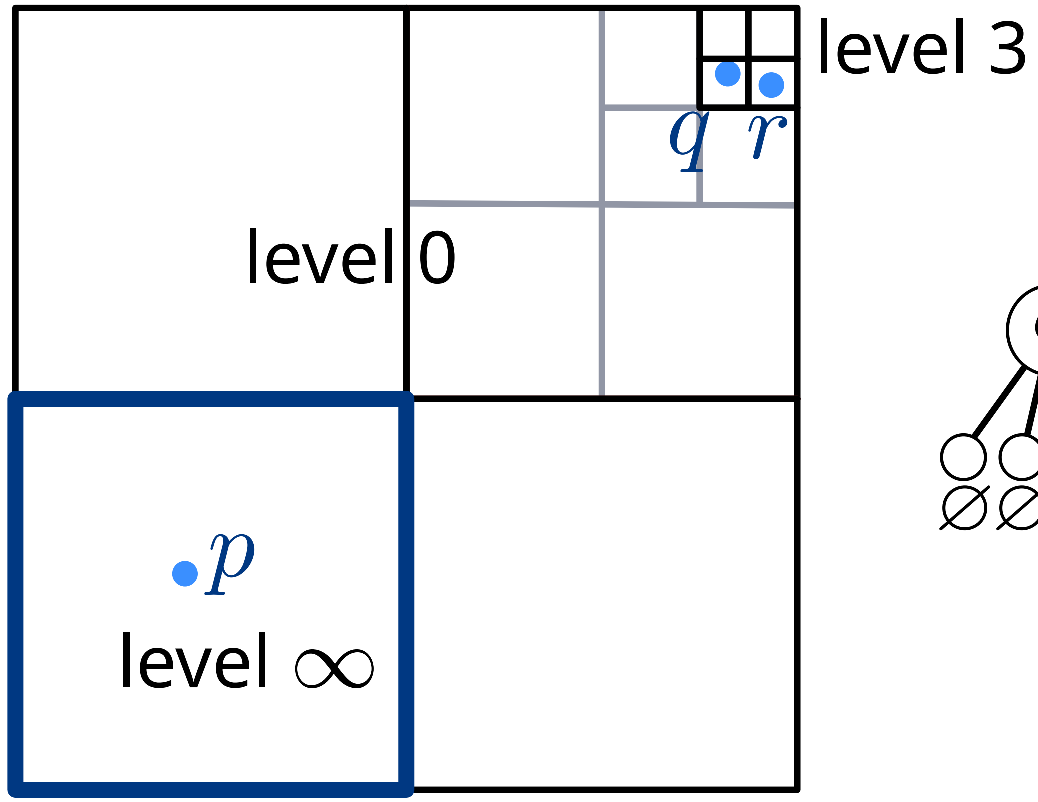

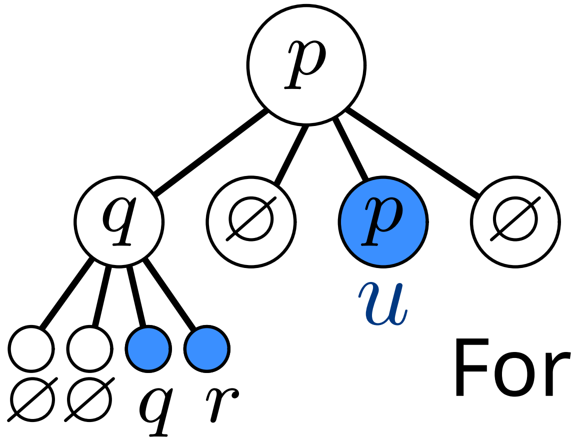

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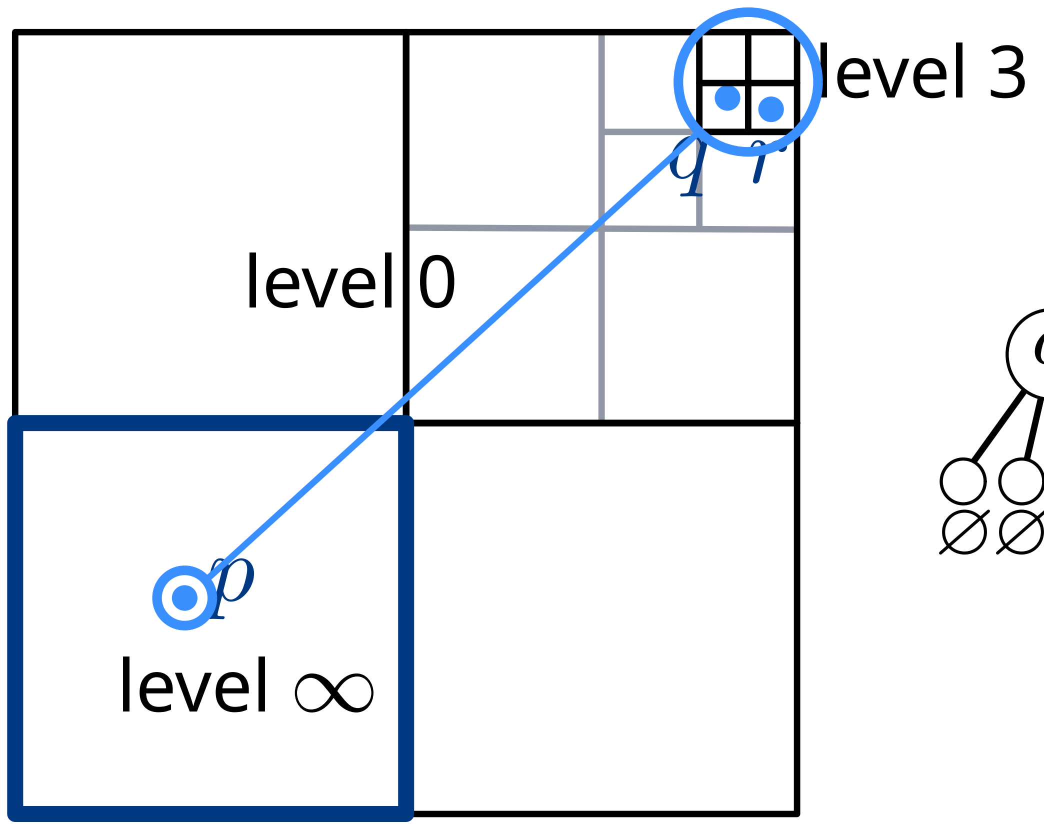

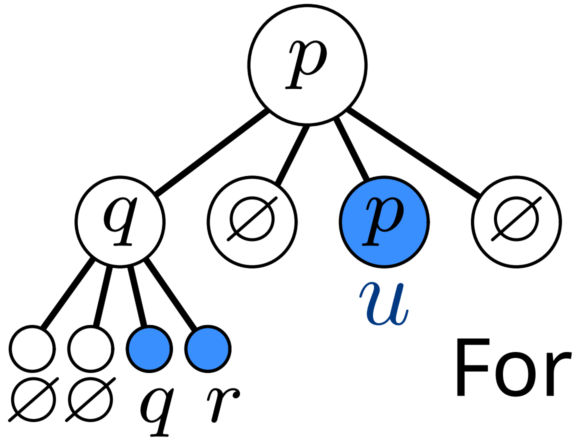

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next: using quadtree to compute WSPD

Well-Separated Pair Decomposition

Construction

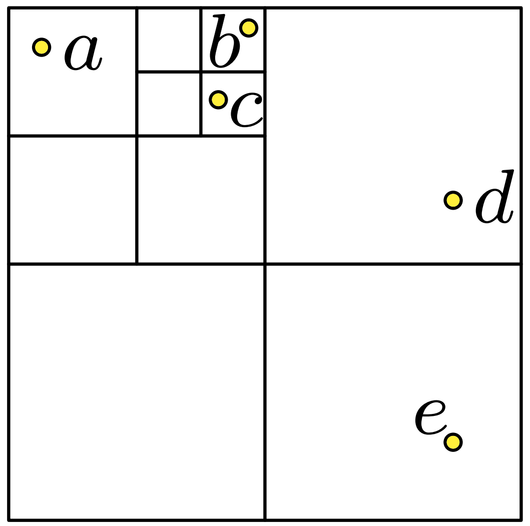
Construction of a WSPD

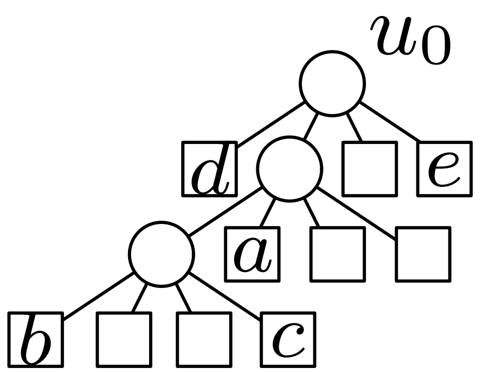
wsPairs$(u, v, \mathcal{T}, s)$

Input: quadtree nodes u, v, quadtree $\mathcal{T}$, $s > 0$

Output: WSPD for $P_u \otimes P_v$

if rep$(u) = \varnothing$ or rep$(v) = \varnothing$ or leaves $u = v$ **then return** $\varnothing$
else if P_u and P_v s-well separated **then return** $\{\{u, v\}\}$
else
    if level$(u) >$ level(v) **then** exchange u and v
    $(u_1, \ldots, u_m) \leftarrow$ children of u in $\mathcal{T}$
    return $\bigcup_{i=1}^{m}$ wsPairs$(u_i, v, \mathcal{T}, s)$





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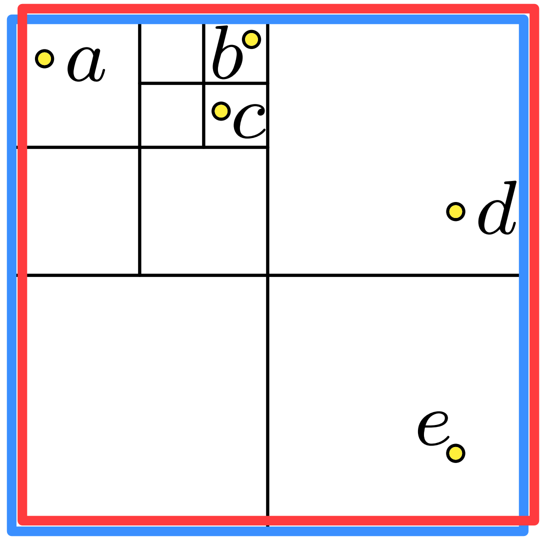

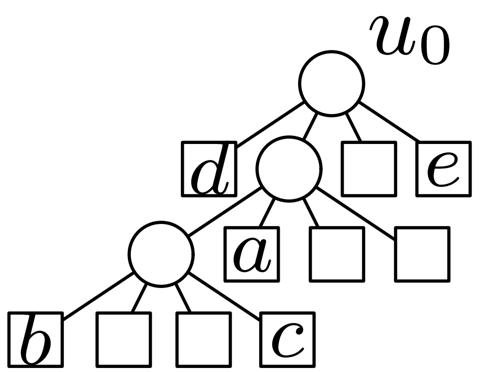

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Output: WSPD for $P_u \otimes P_v$

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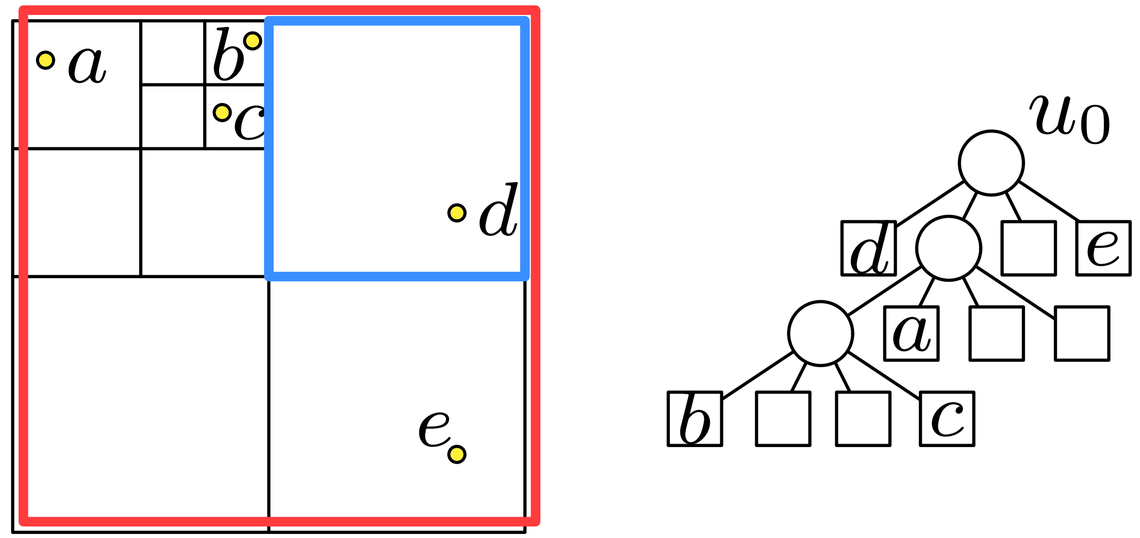

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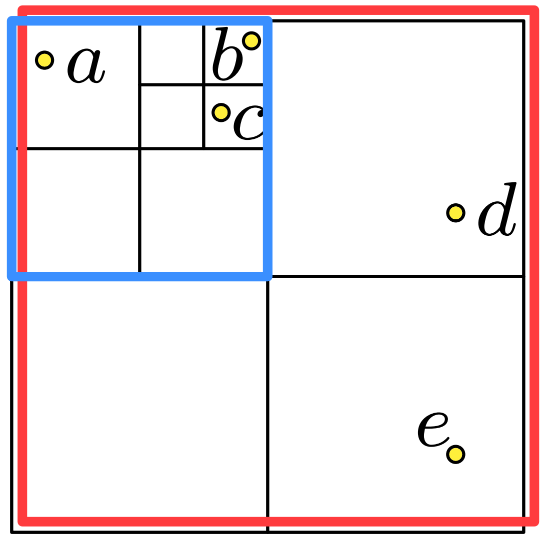

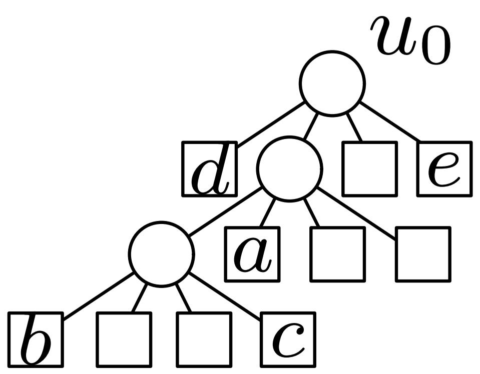

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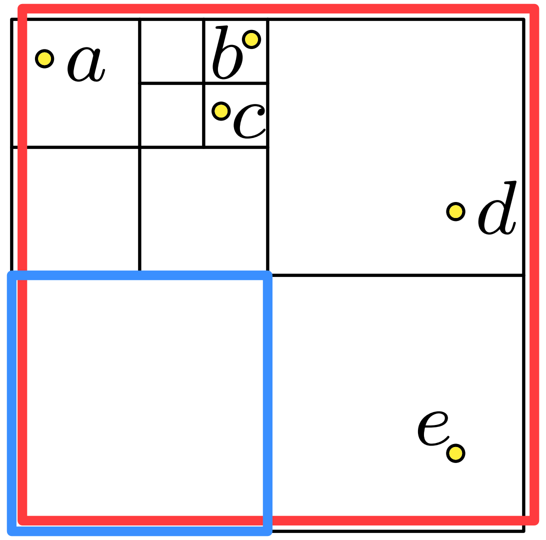

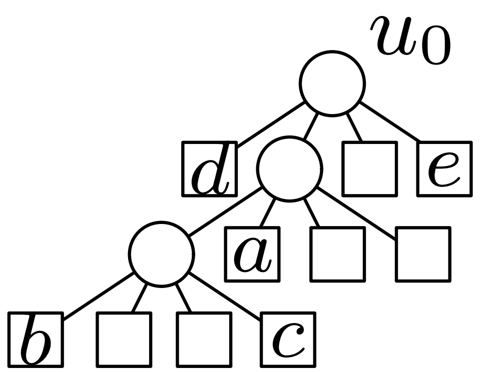

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```
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else
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```

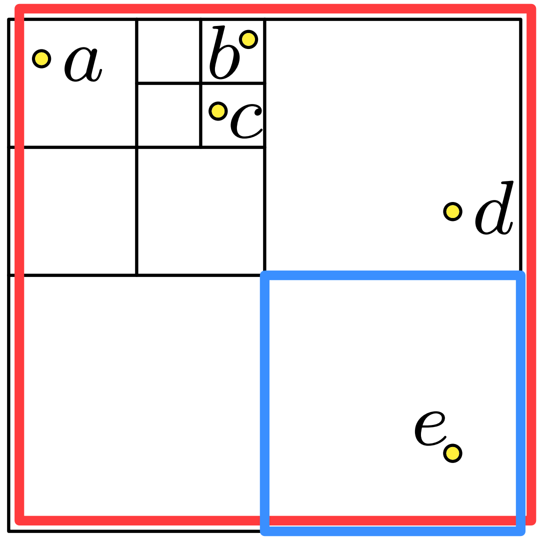

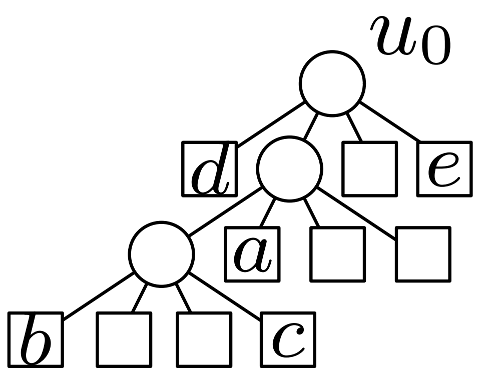

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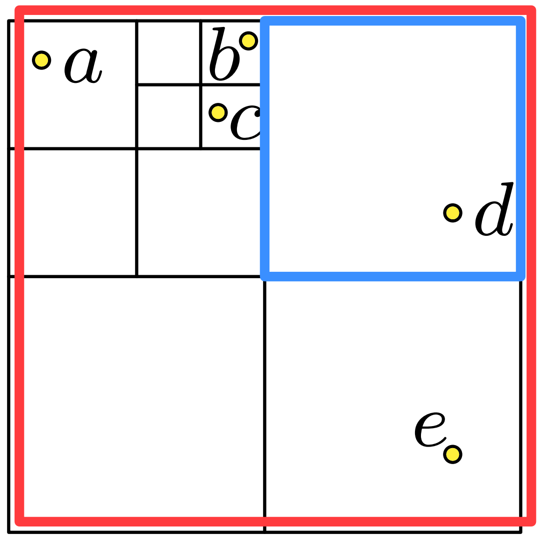

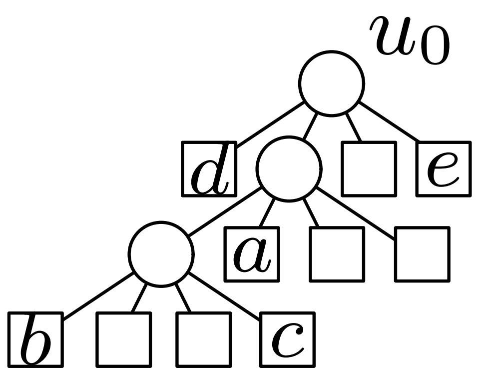

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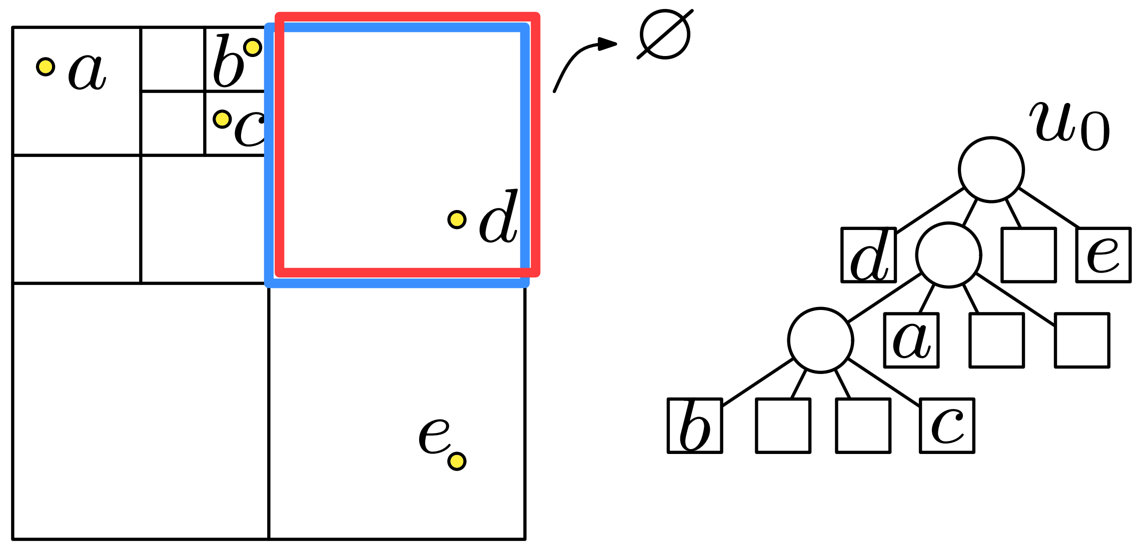

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circles around σ_u and σ_v (or radius 0 for point in a leaf),
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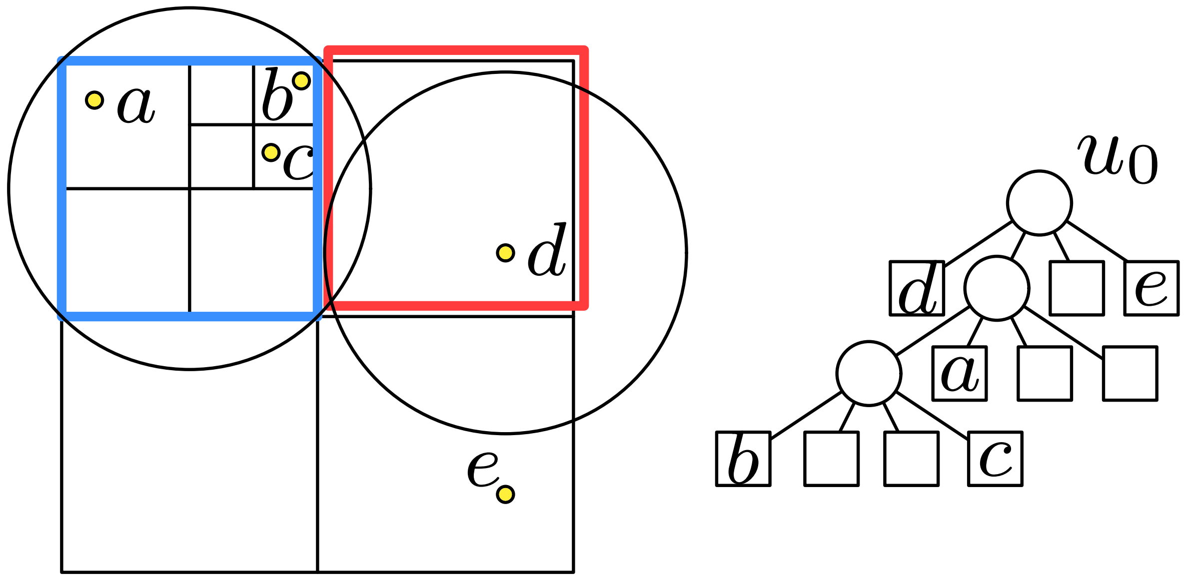

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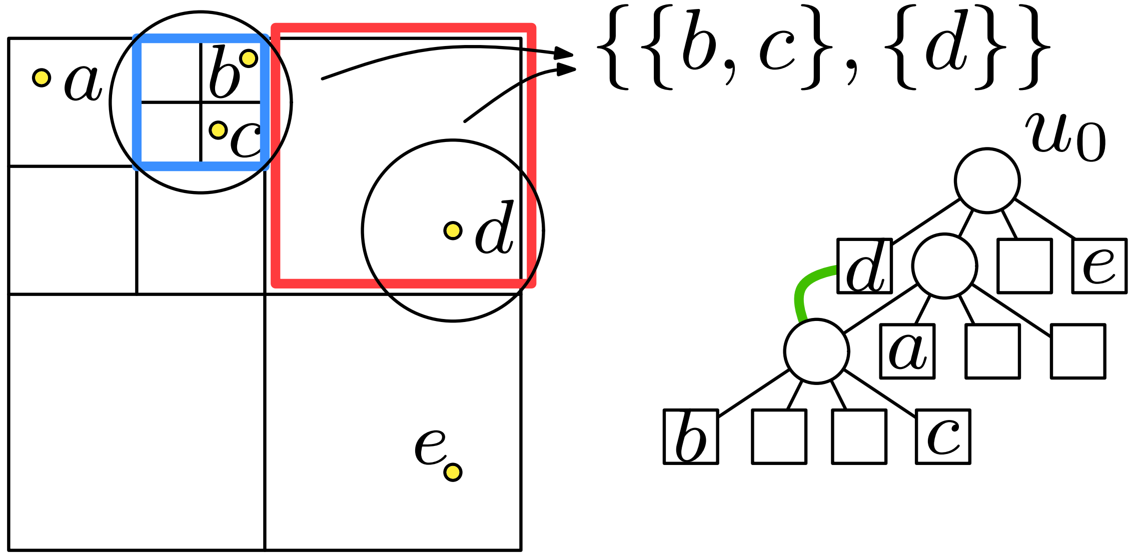

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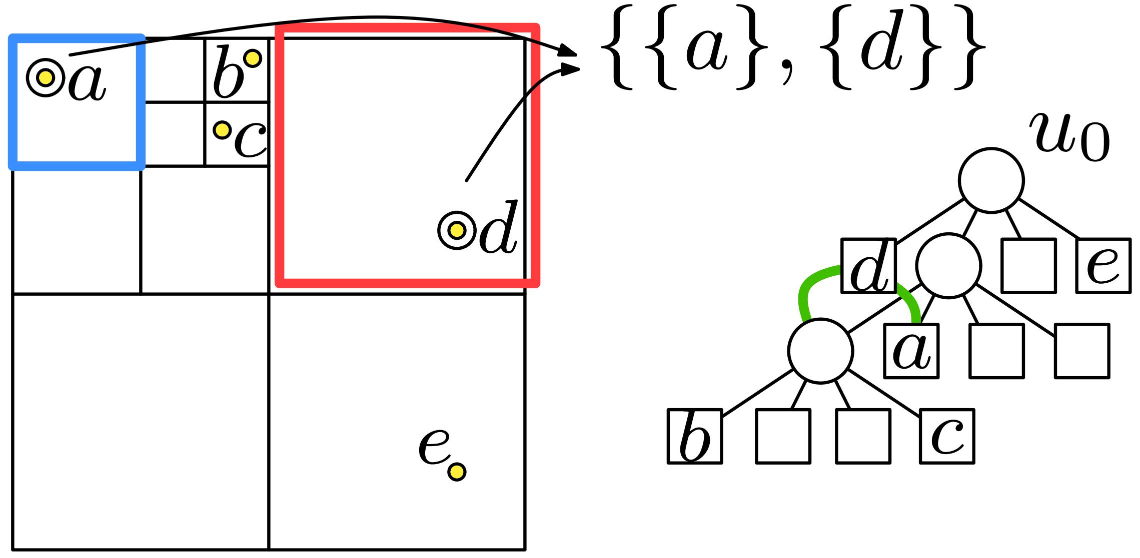

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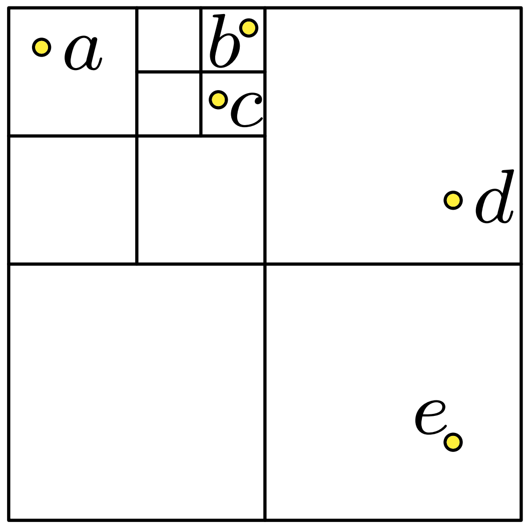

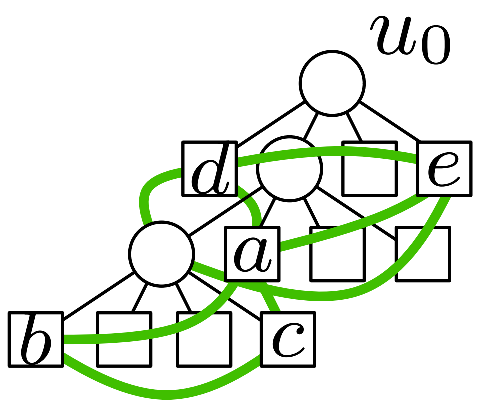

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- initial call $\textsc{wsPairs}(u_0, u_0, \mathcal{T}, s)$
- avoid duplicate $\textsc{wsPairs}(u_i, u_j, \mathcal{T}, s)$ and $\textsc{wsPairs}(u_j, u_i, \mathcal{T}, s)$
- pairs of leaves are s-well separated $\rightarrow$ algorithm terminates
- output are pairs of quadtree nodes

Quiz

Is the size of the s-WSPD constructed minimal?

A: Yes, because the s-WSPD is unique.

B: Yes, because all s-WSPDs have the same size.

C: No, not necessarily.

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Question: How many pairs are generated by the algorithm?

Well-Separated Pair Decomposition

Complexity

Analysis of WSPD-Construction

Theorem: For a point set P in $\mathbb{R}^d$ and $s \geq 1$ we can construct an s-WSPD with $O(s^d n)$ pairs in time $O(n \log n + s^d n)$.

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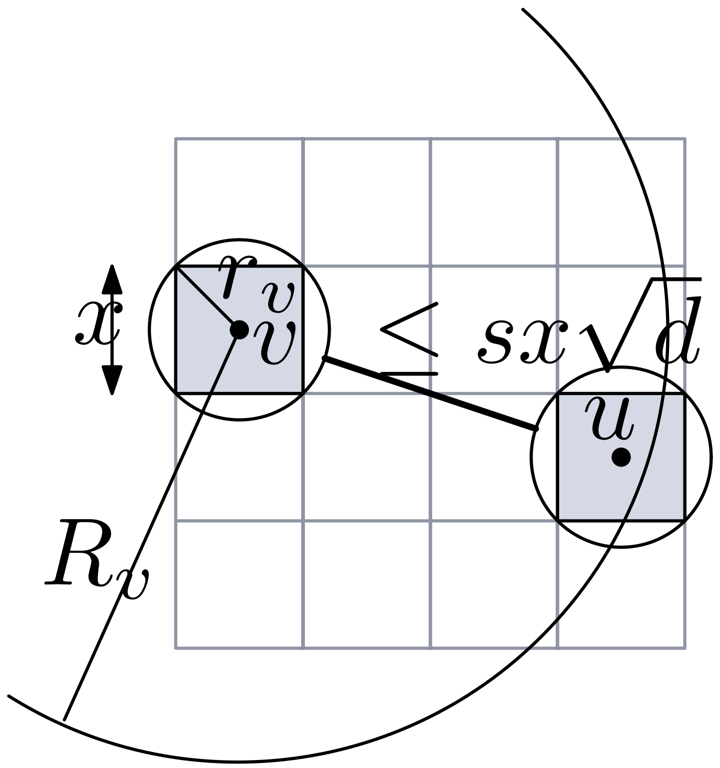
Proof sketch:

Assumptions: $s \geq 1$, QT uncompressed.

Count the non-terminal calls.



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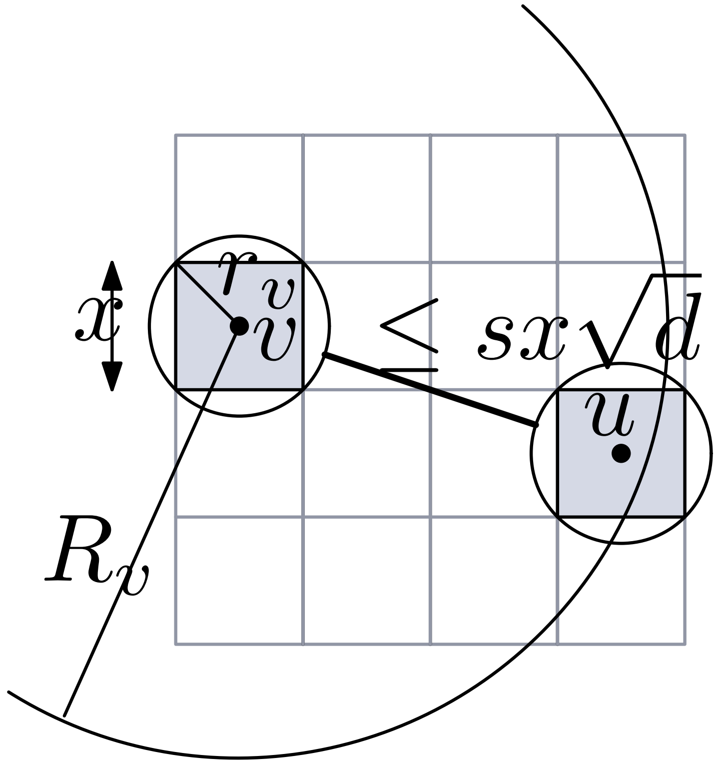

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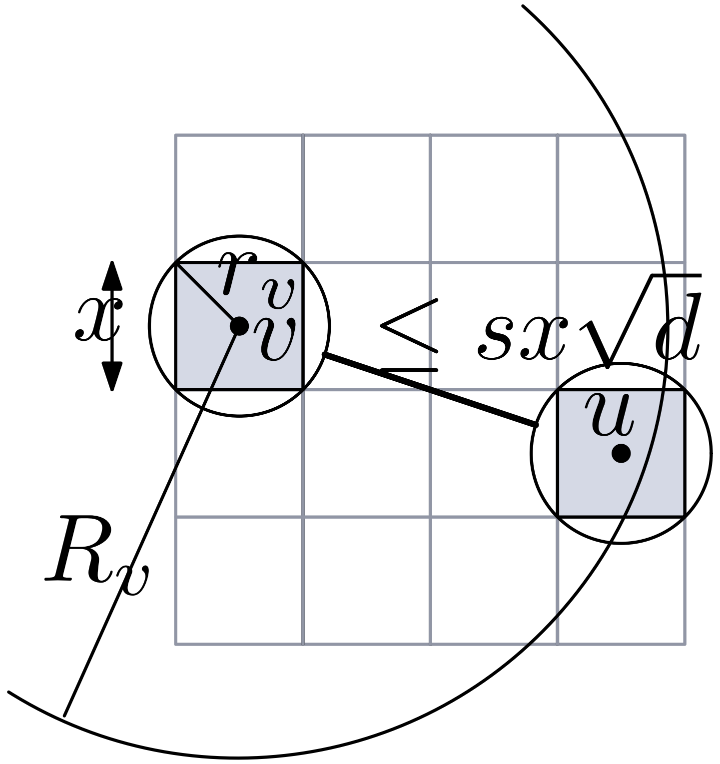
claim: $O(s^d)$ charges to each square



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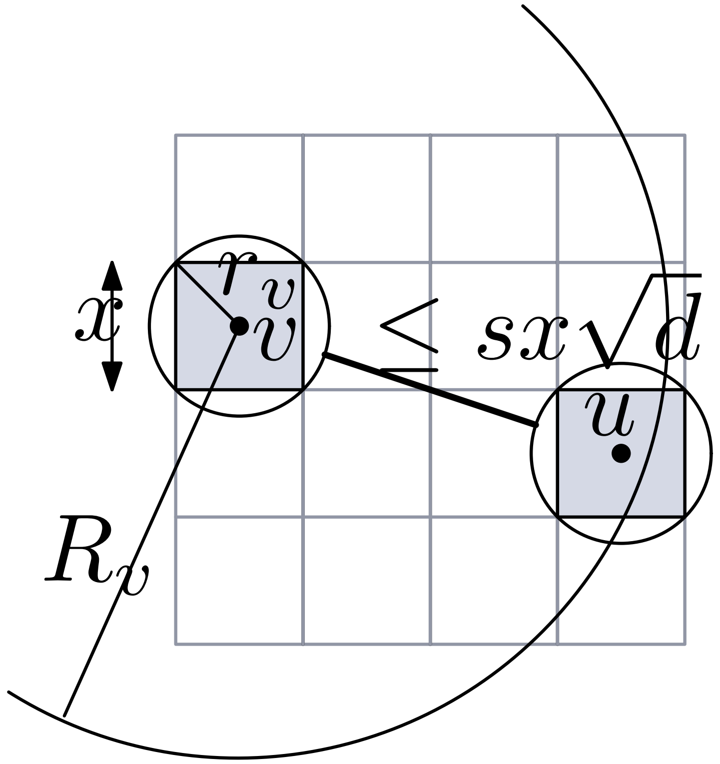

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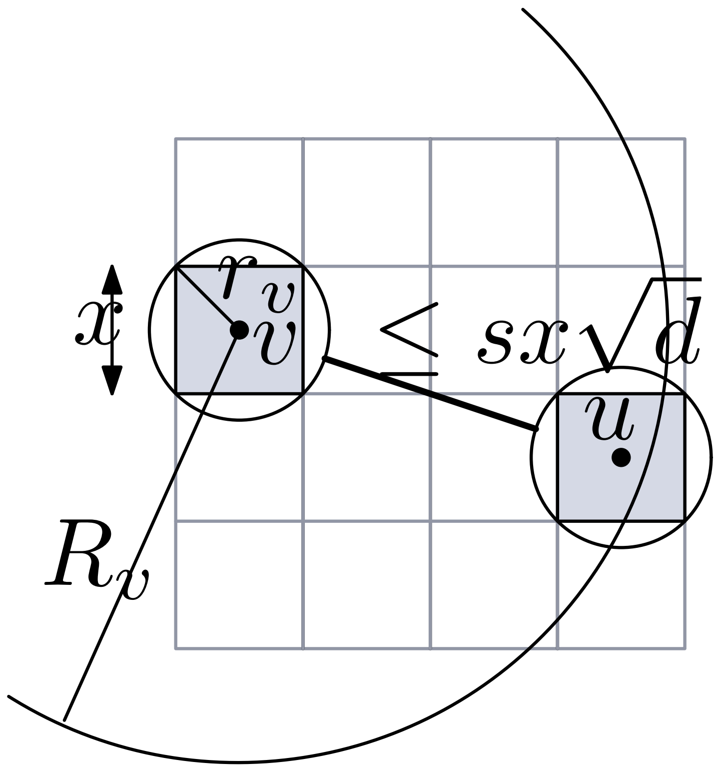

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u, v are not separated,

u is at most factor 2 larger than v

$\Rightarrow$ distance between the balls

$\leq s \max(r_u, r_v) \leq 2sr_v = sx\sqrt{d}$

$\Rightarrow$ distance between their centers

$\leq (1/2 + 1 + s)x\sqrt{d} \leq 3sx\sqrt{d} =: R_v$

packing lemma: only $O(s^d)$ such squares.

Packing Lemma

Lemma: Let B be a ball of radius r in $\mathbb{R}^d$ and X a set of pairwise disjoint quadtree cells with side length $\geq x$, that intersect B. Then

$$|X| \leq (1 + \lceil 2r/x \rceil)^d.$$

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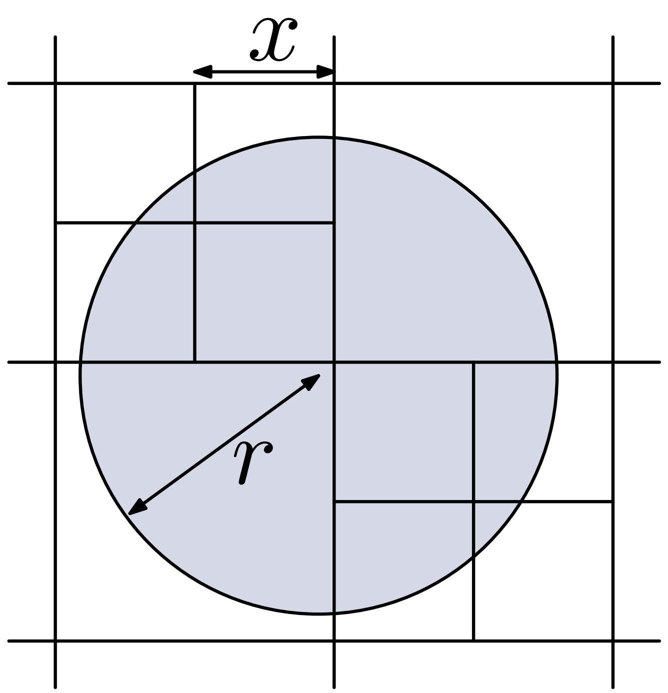

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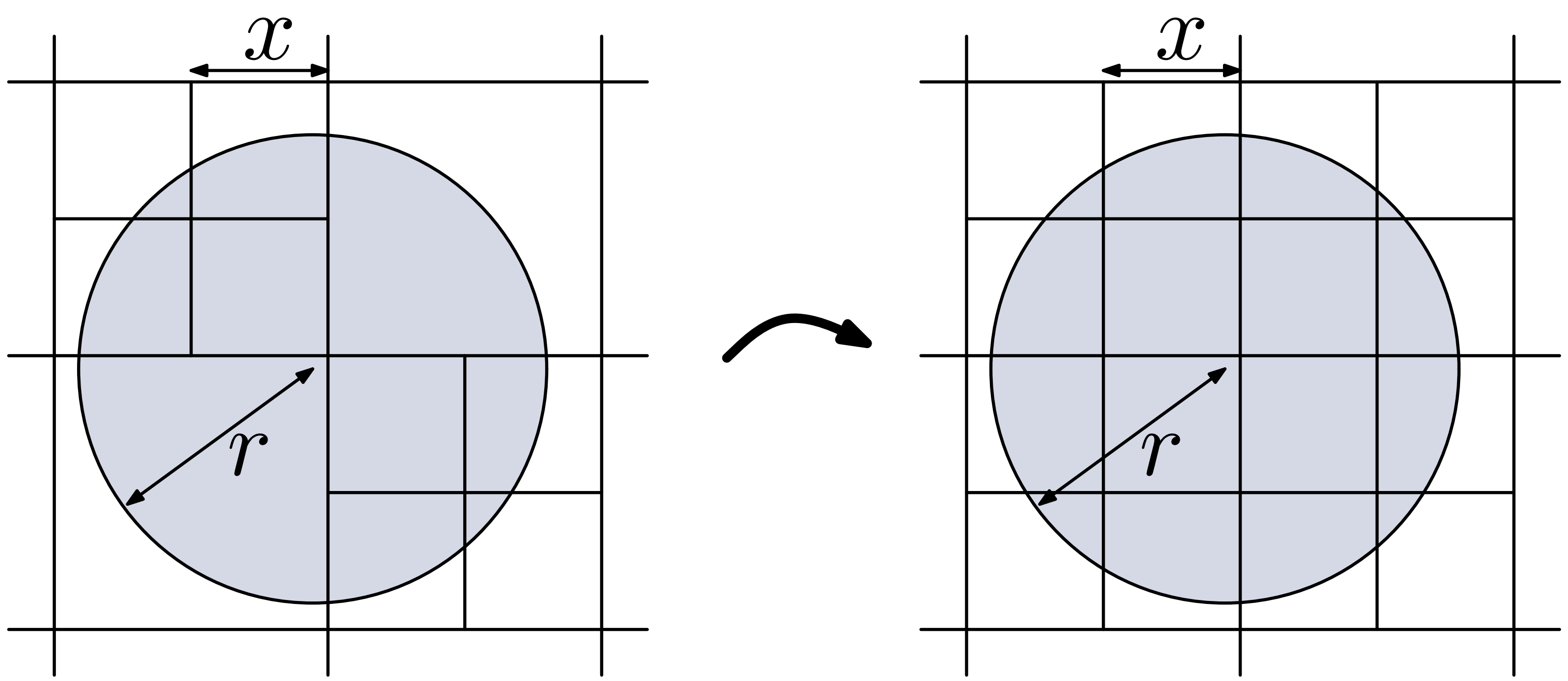

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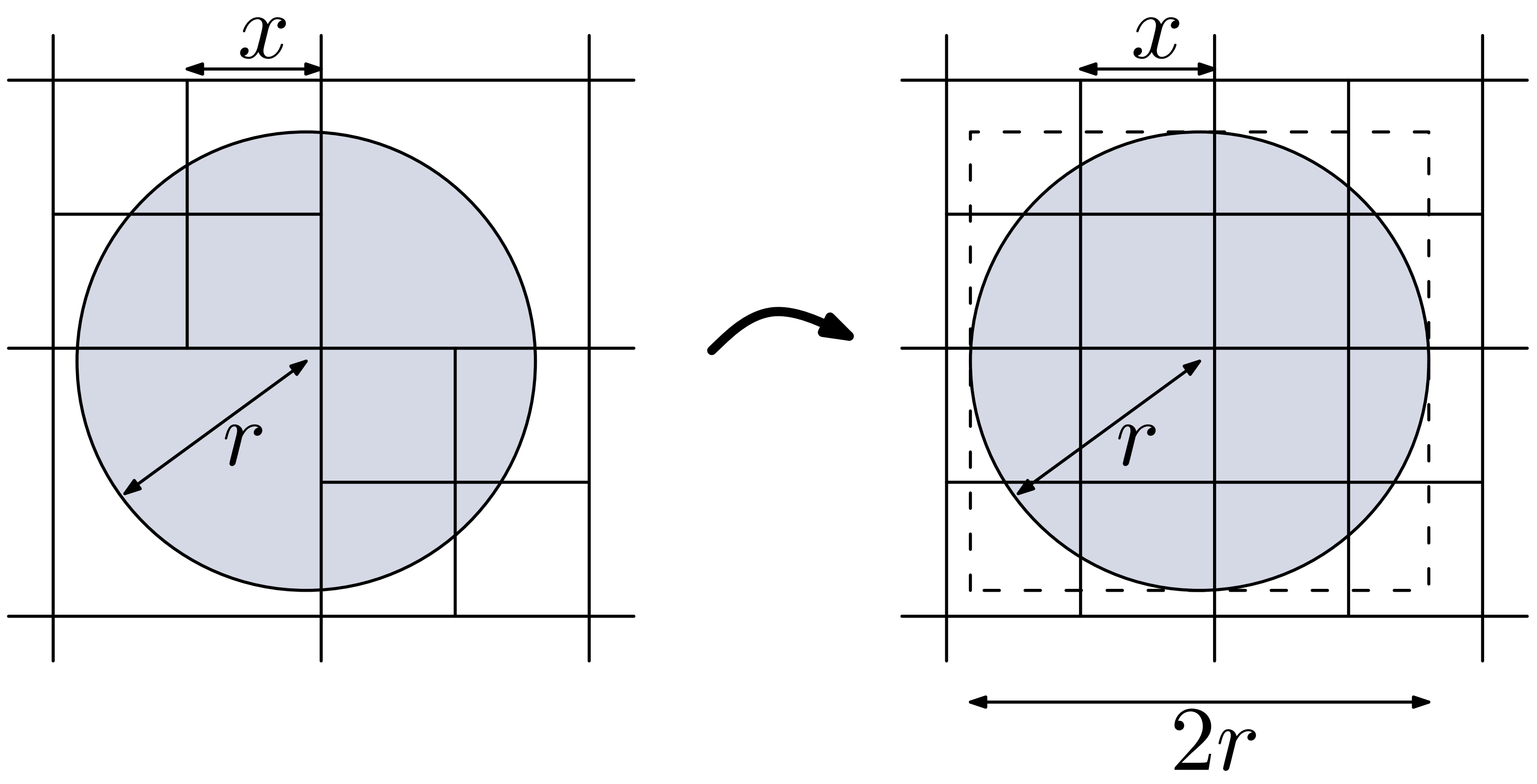


in every dimension at most $1 + \lceil 2r/x \rceil$ squares can intersect the ball

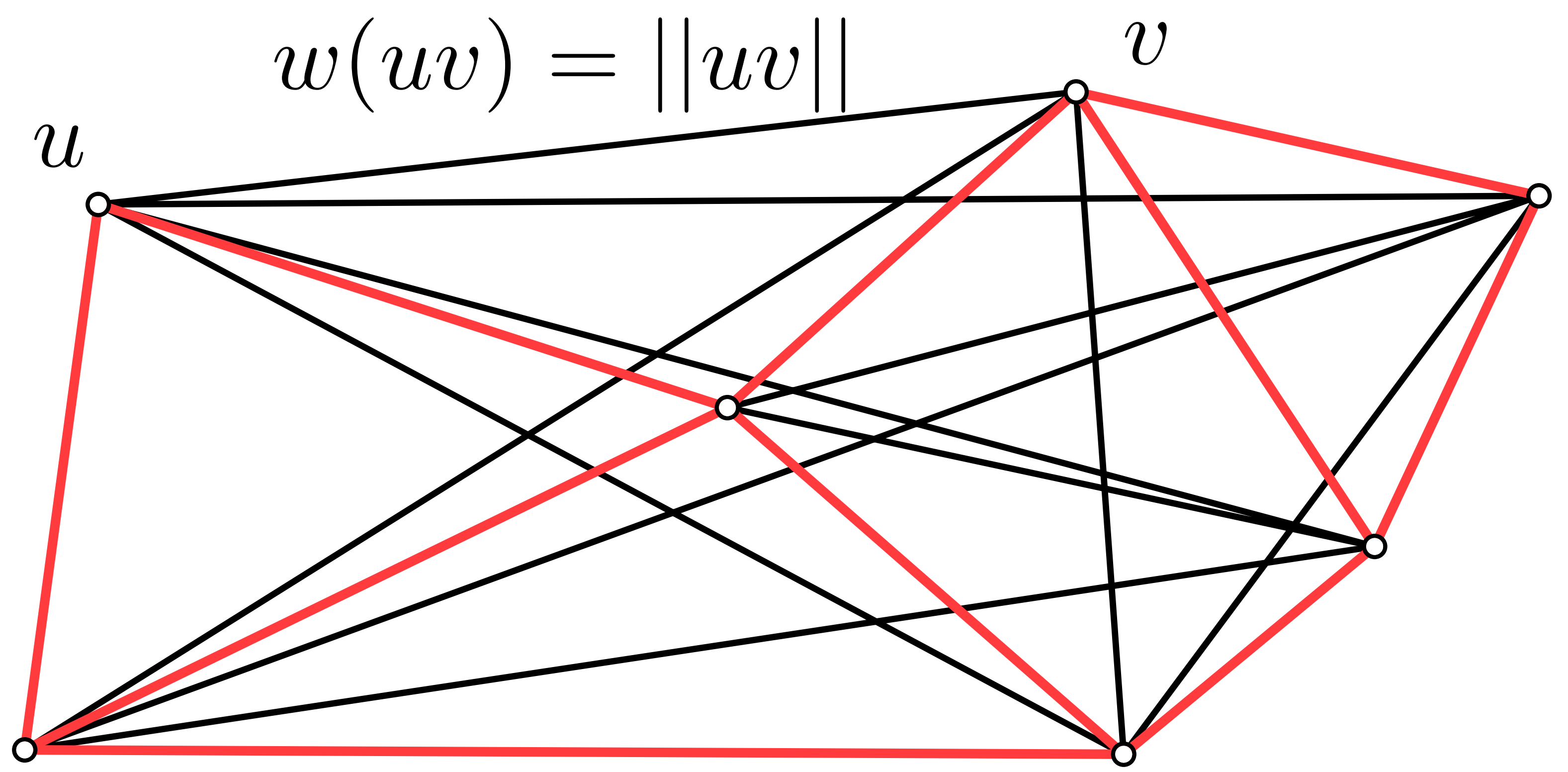
Well-Separated Pair Decomposition

Application: t-spanner

t-spanner

For a set P of n points in $\mathbb{R}^d$ the Euclidean graph $\mathcal{EG}(P) = (P, \binom{P}{2})$ is the complete, weighted graph with Euclidean distances as edge weights.

Since $\mathcal{EG}(P)$ has $\Theta(n^2)$ edges, we want a sparse graph with $O(n)$ edges such that the shortest paths in the graph approximate the edge weights of $\mathcal{EG}(P)$.



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Definition: A weighted graph G with vertex set P is called t-spanner for P and a stretch factor $t \geq 1$, if for all pairs $x, y \in P$:

$$||xy|| \leq \delta_G(x, y) \leq t \cdot ||xy||\,,$$

where $\delta_G(x, y) =$ length of the shortest x-to-y path in G.

WSPD and t-Spanner

Definition: For n points P in $\mathbb{R}^d$ and a WSPD W of P define the graph $G = (P, E)$ with $E = \{\{x, y\} \mid \{u, v\} \in W$ and $\text{rep}(u) = x, \text{rep}(v) = y\}$.

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Reminder: every pair $\{u, v\} \in W$ corresponds to two quadtree nodes u and v. From each quadtree node a representative is selected in the following way. For leaf u define as representative

$$\text{rep}(u) = \begin{cases} p & \text{if } P_u = \{p\} \text{ (}u\text{ is leaf)} \\ \varnothing & \text{otherwise.} \end{cases}$$

For an inner node v set $\text{rep}(v) = \text{rep}(u)$ for a non-empty child u of v.

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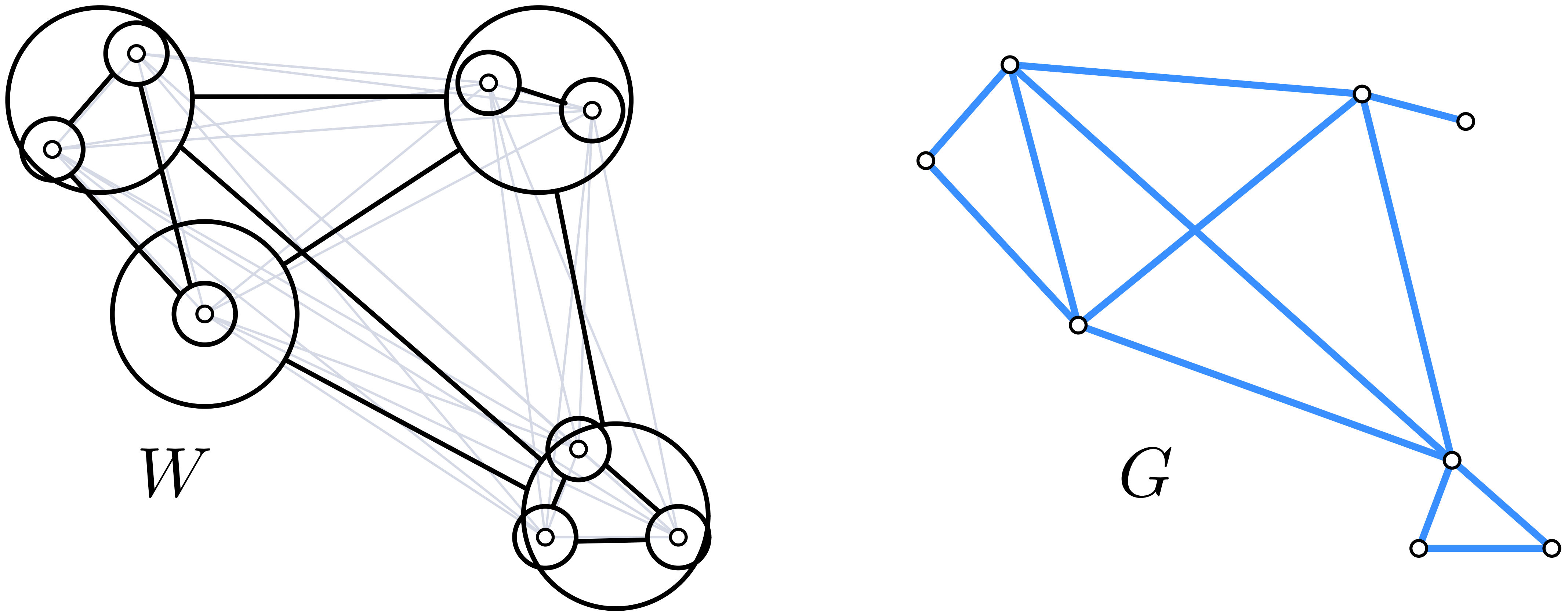

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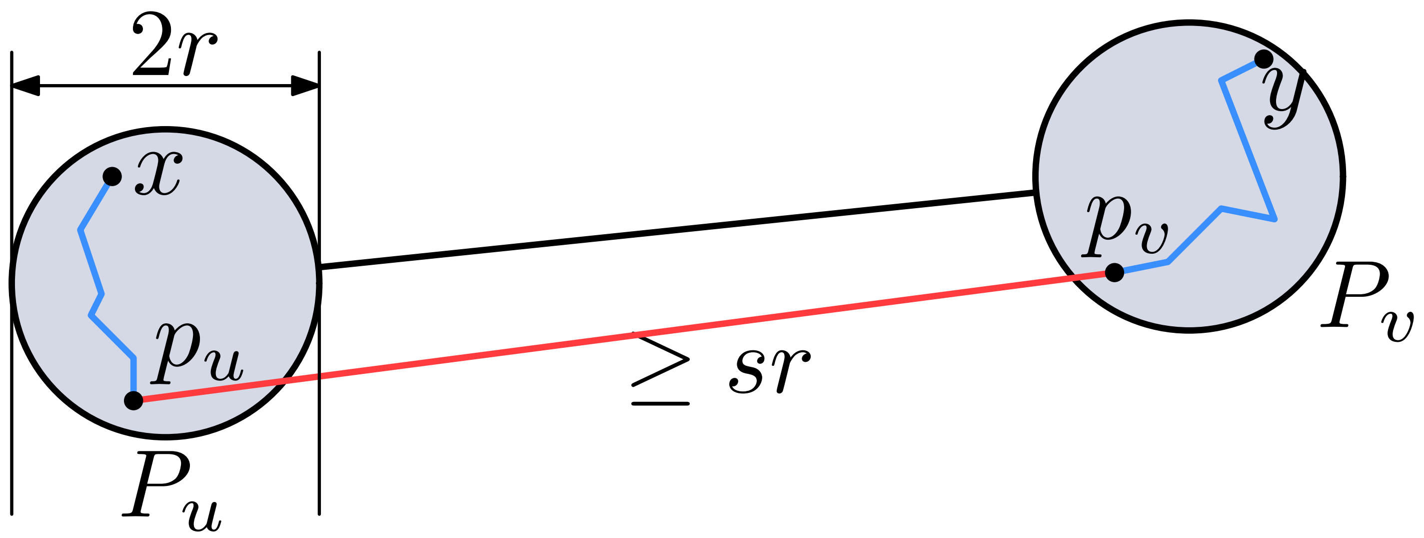

WSPD and t-Spanner

Definition: For n points P in $\mathbb{R}^d$ and a WSPD W of P define the graph $G = (P, E)$ with $E = \{\{x, y\} \mid \{u, v\} \in W$ and $\text{rep}(u) = x, \text{rep}(v) = y\}$.

Lemma: If W is an s-WSPD for $s = 4 \cdot \frac{t+1}{t-1}$, then G is a t-spanner for P with $O(s^d n)$ edges.

Question: How large does s need to be if $t = 1 + \varepsilon$

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B: $O(1/\varepsilon)$

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P

$\downarrow$ $O(n \log n)$

compressed quadtree

$\downarrow$ $O(n/\varepsilon^d)$

WSPD

$\downarrow$ $O(n/\varepsilon^d)$

$(1+\varepsilon)$-spanner

$\square$

Discussion

Applications of the WSPD?

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Fun fact

Constructing the greedy spanner naturally uses $\Omega(n^2)$ space, but resulting from a previous project of this course, we know how to compute the greedy spanner in near-quadratic time using linear space (partially using WSPD) [Alewijnse et al. 2014]