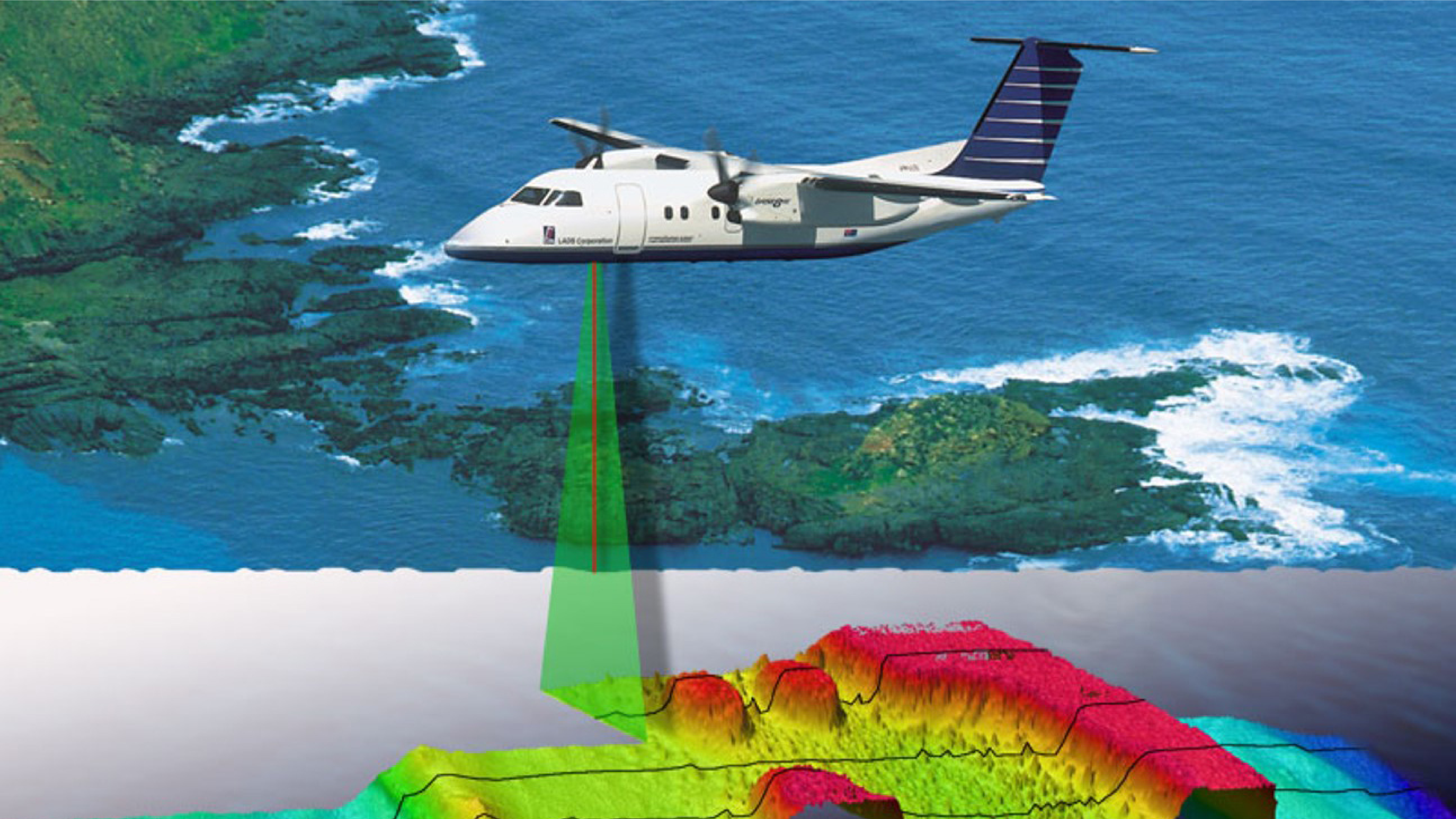
Delaunay Triangulations and Voronoi Diagrams

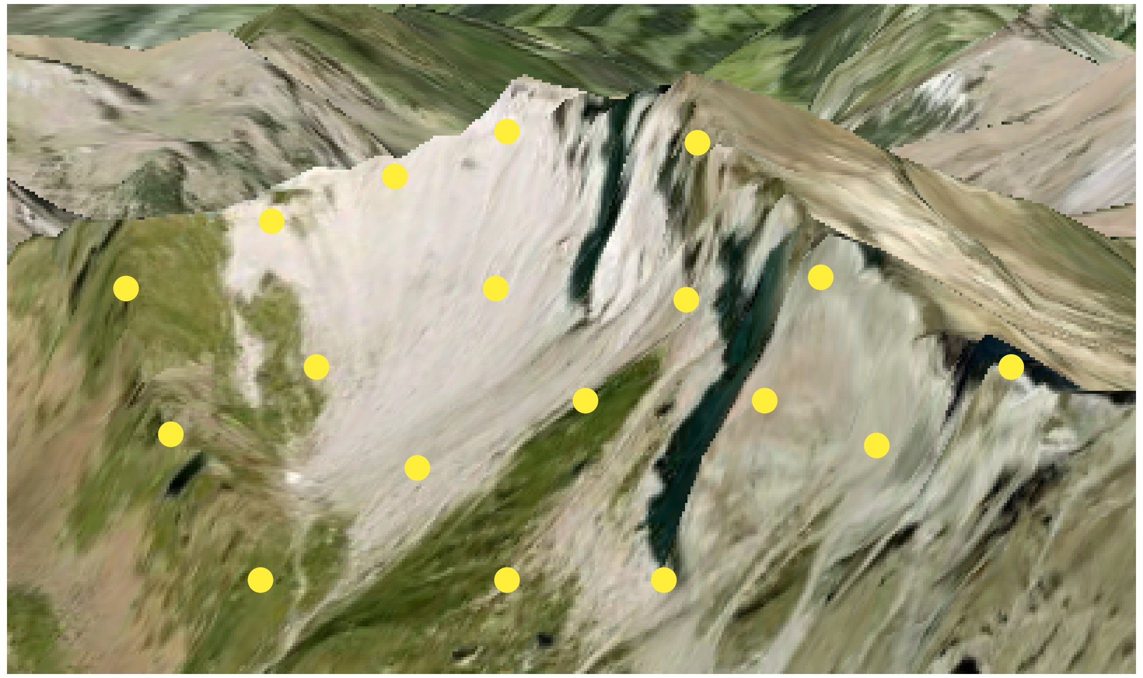
Motivation: Spatial Interpolation, Nearest Neighbor Queries

Algorithmic Technique: Randomized Incremental Construction

Data Structures: Voronoi diagrams, Delaunay triangulations



Spatial Interpolation



height measurements

$p = (p_x, p_y, p_z)$

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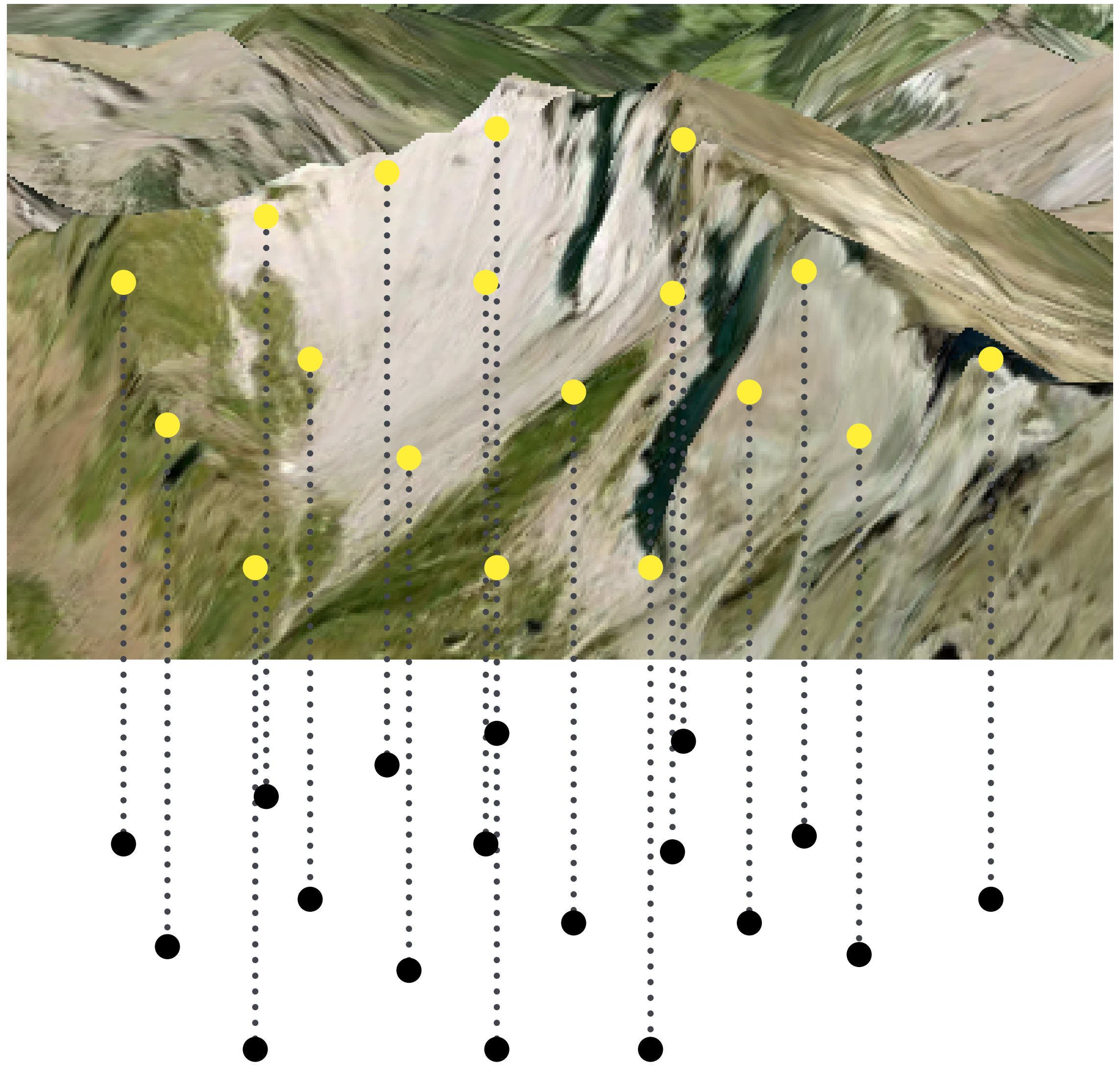

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projection

$$\pi(p) = (p_x, p_y, 0)$$

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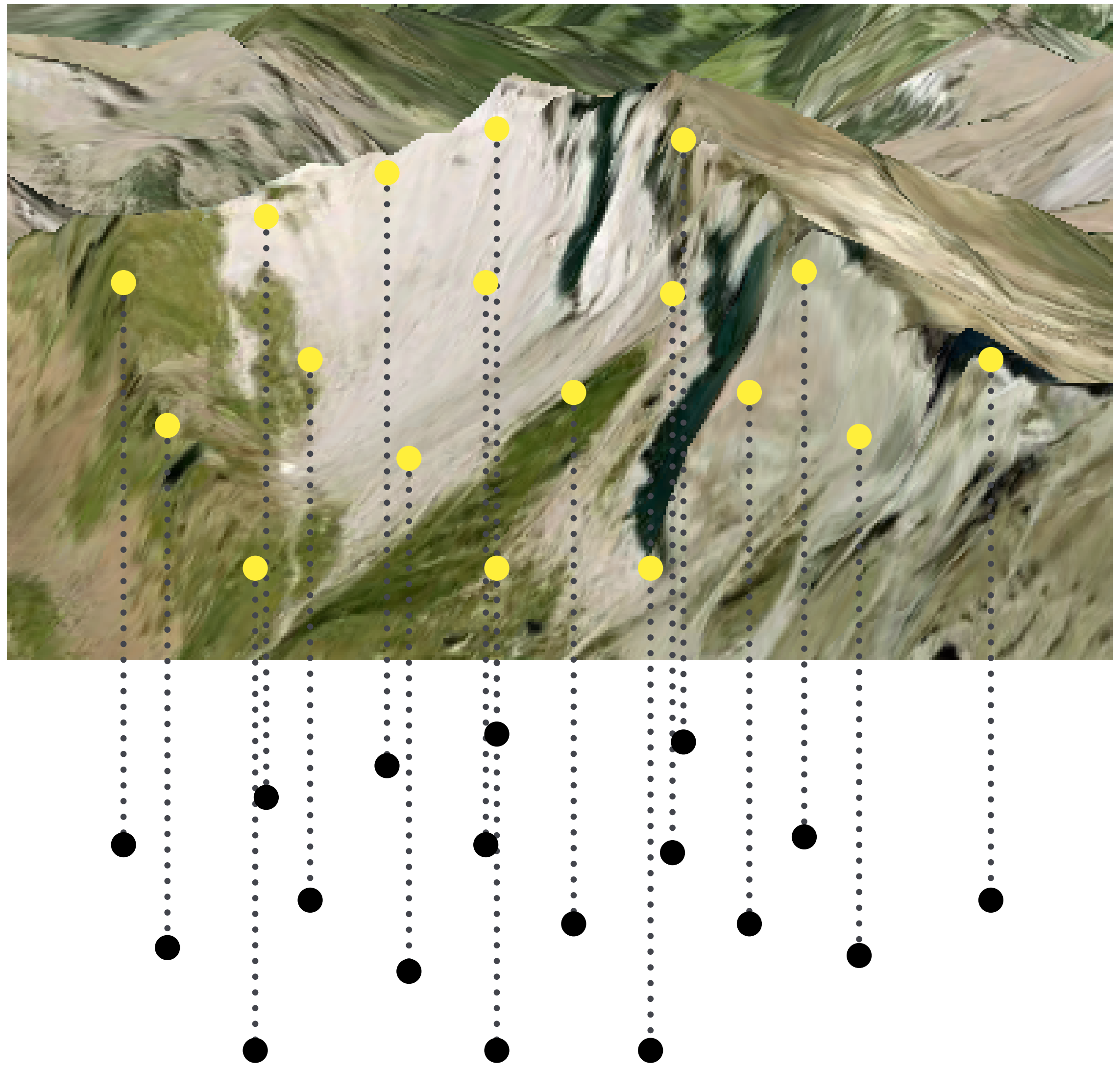

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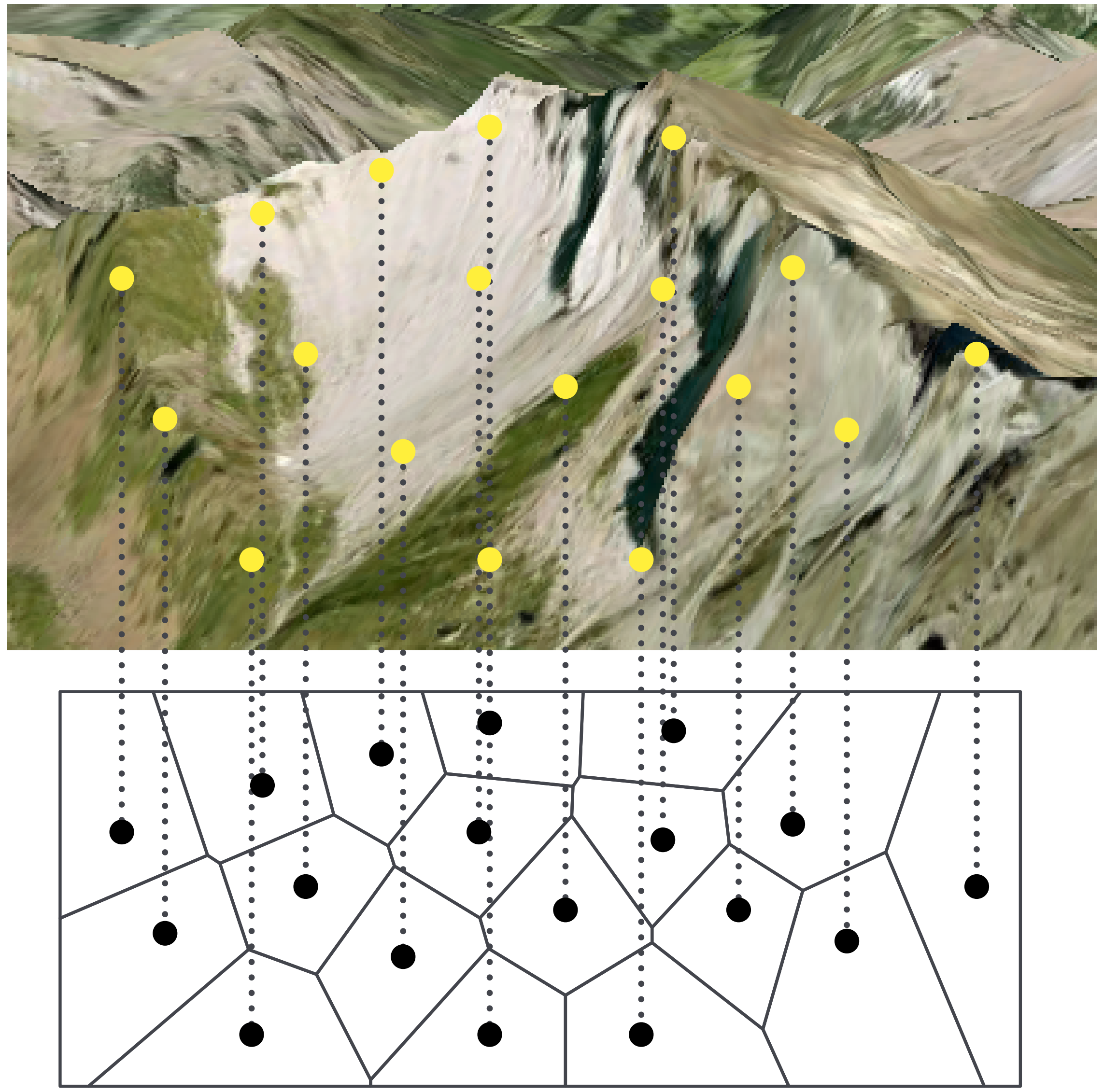
Question: How do we estimate the height at (x, y)?

Spatial Interpolation



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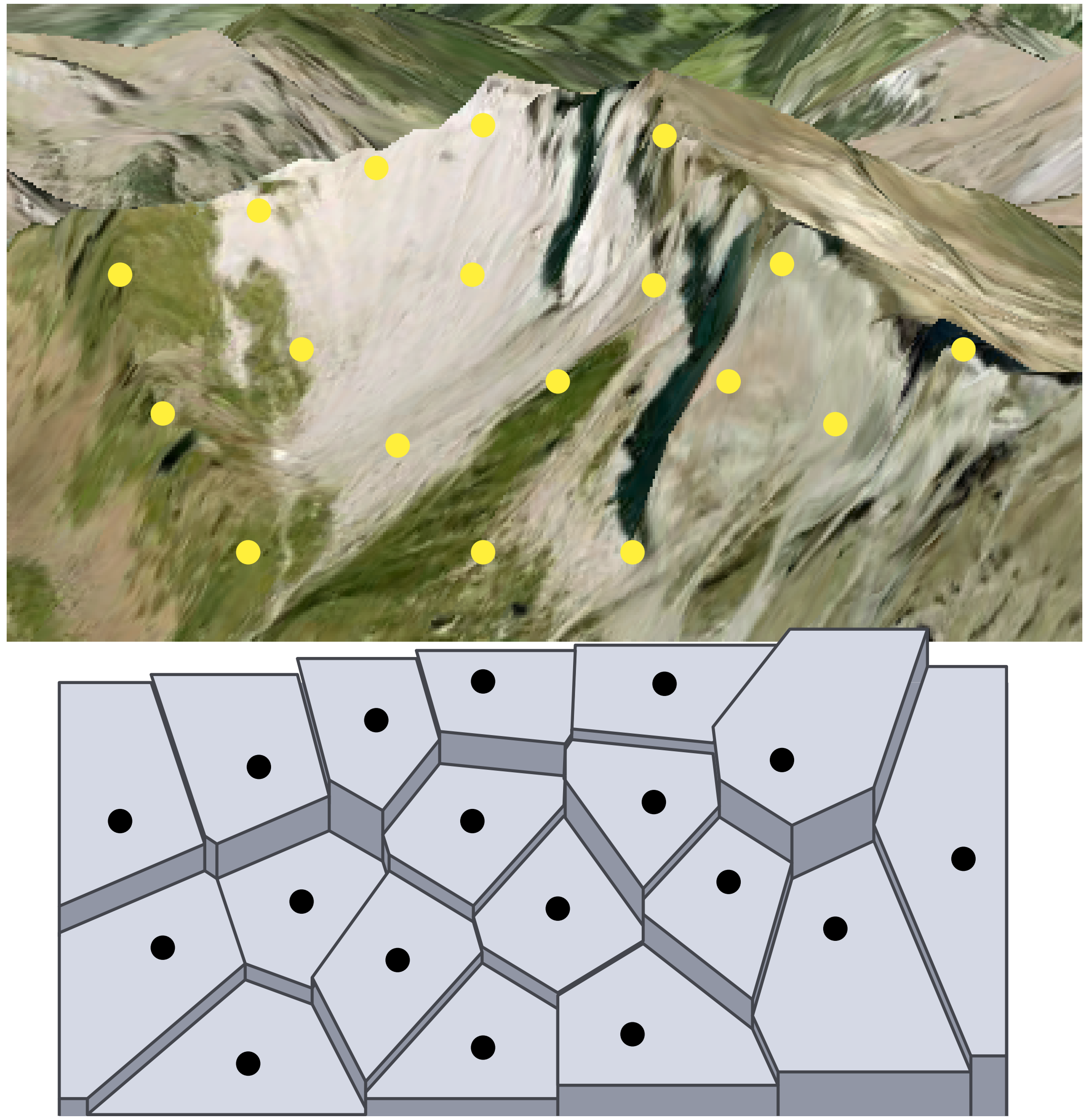
Interpolation 1: assign height of nearest neighbor

Spatial Interpolation



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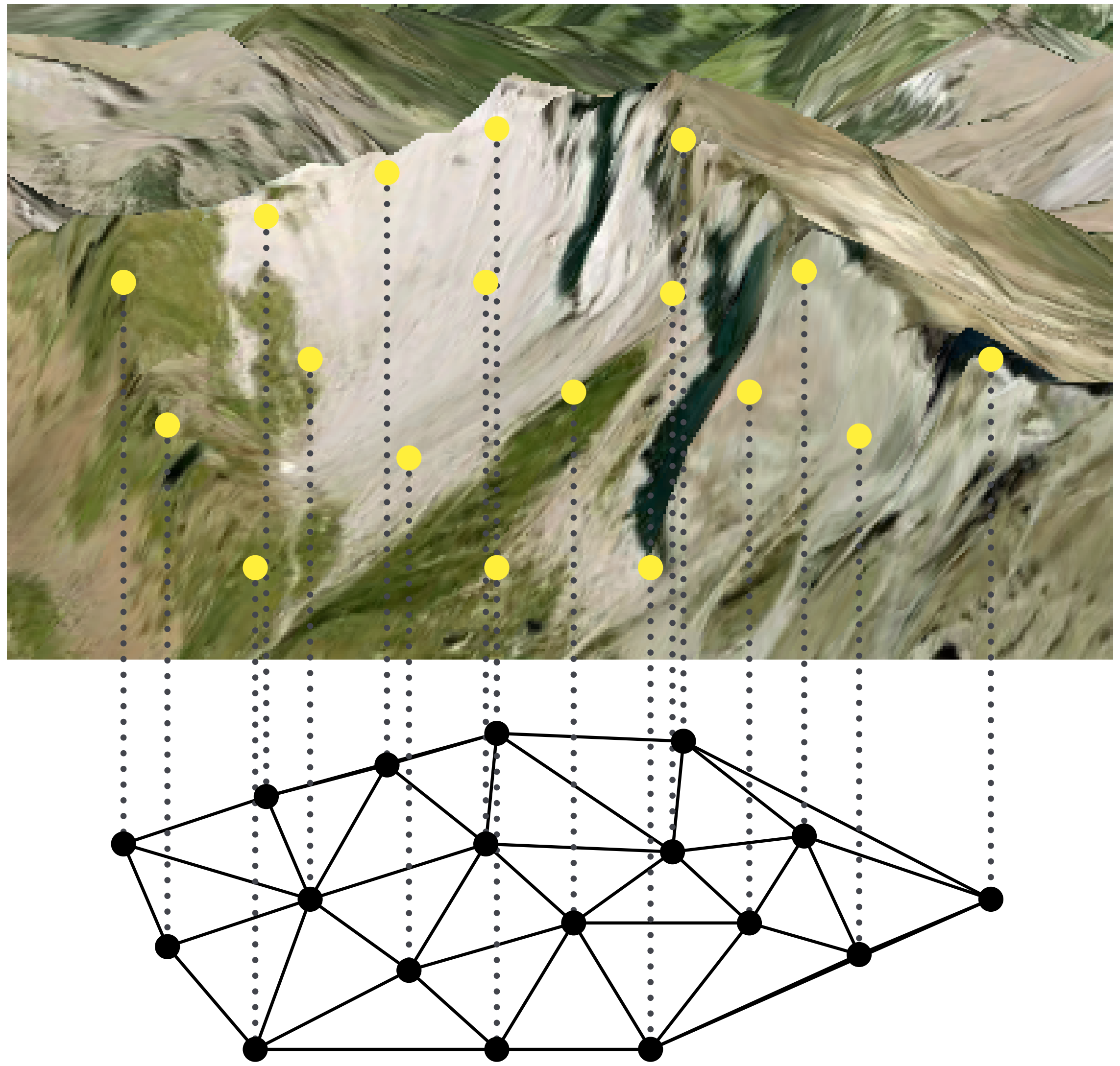

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Interpolation 2: triangulate & interpolate within triangles

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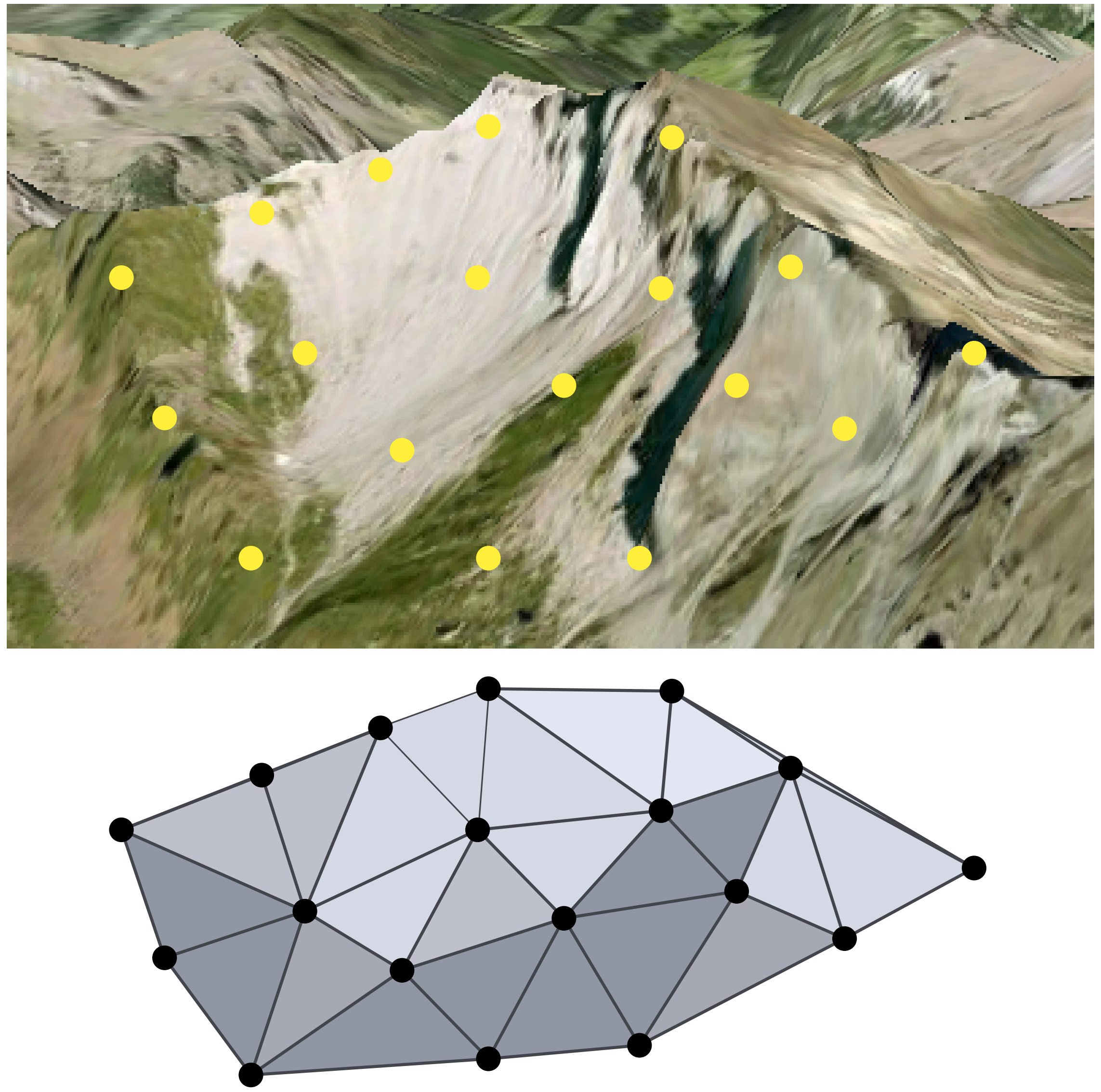

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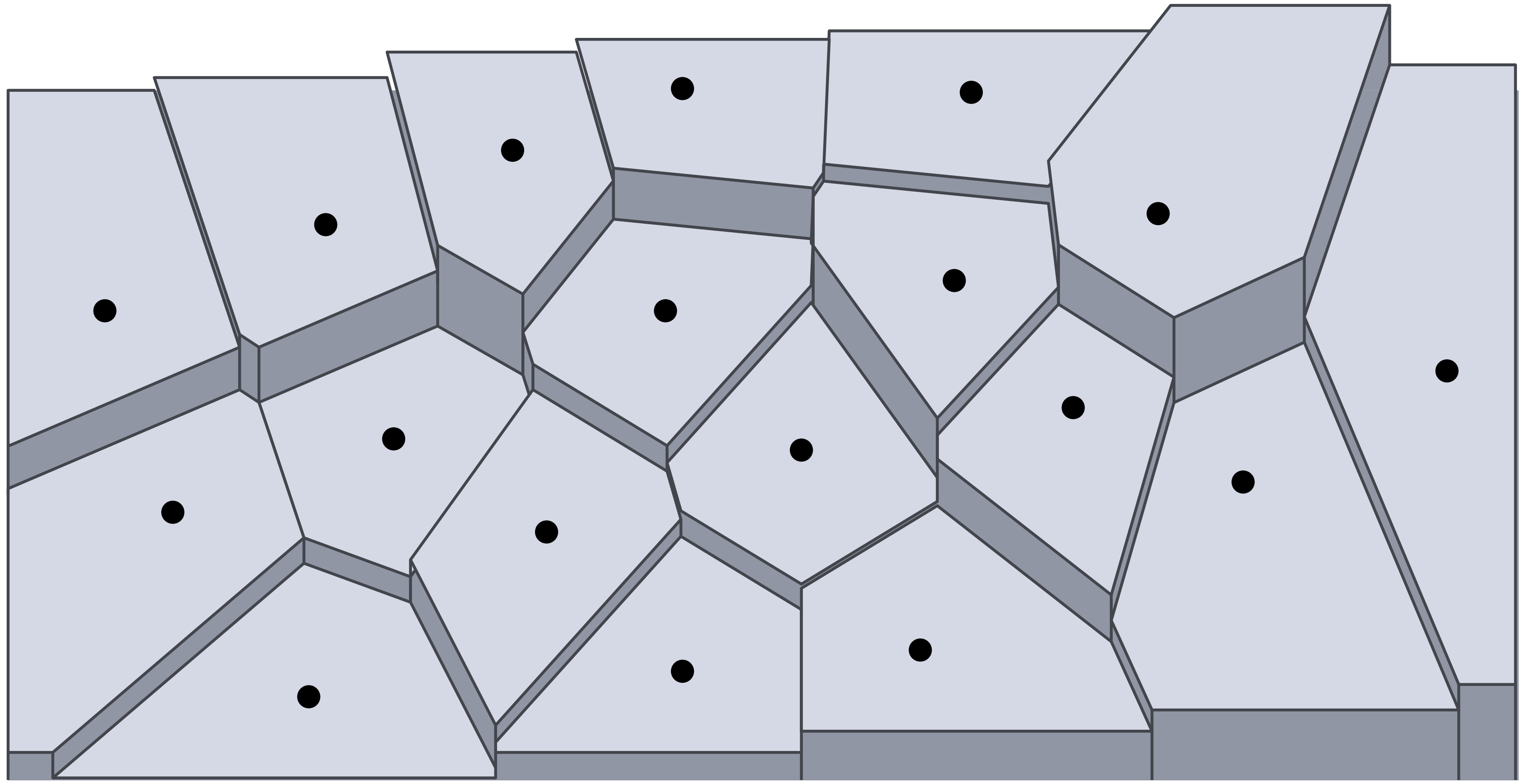
Interpolation 2: triangulate & interpolate within triangles

Voronoi Diagrams



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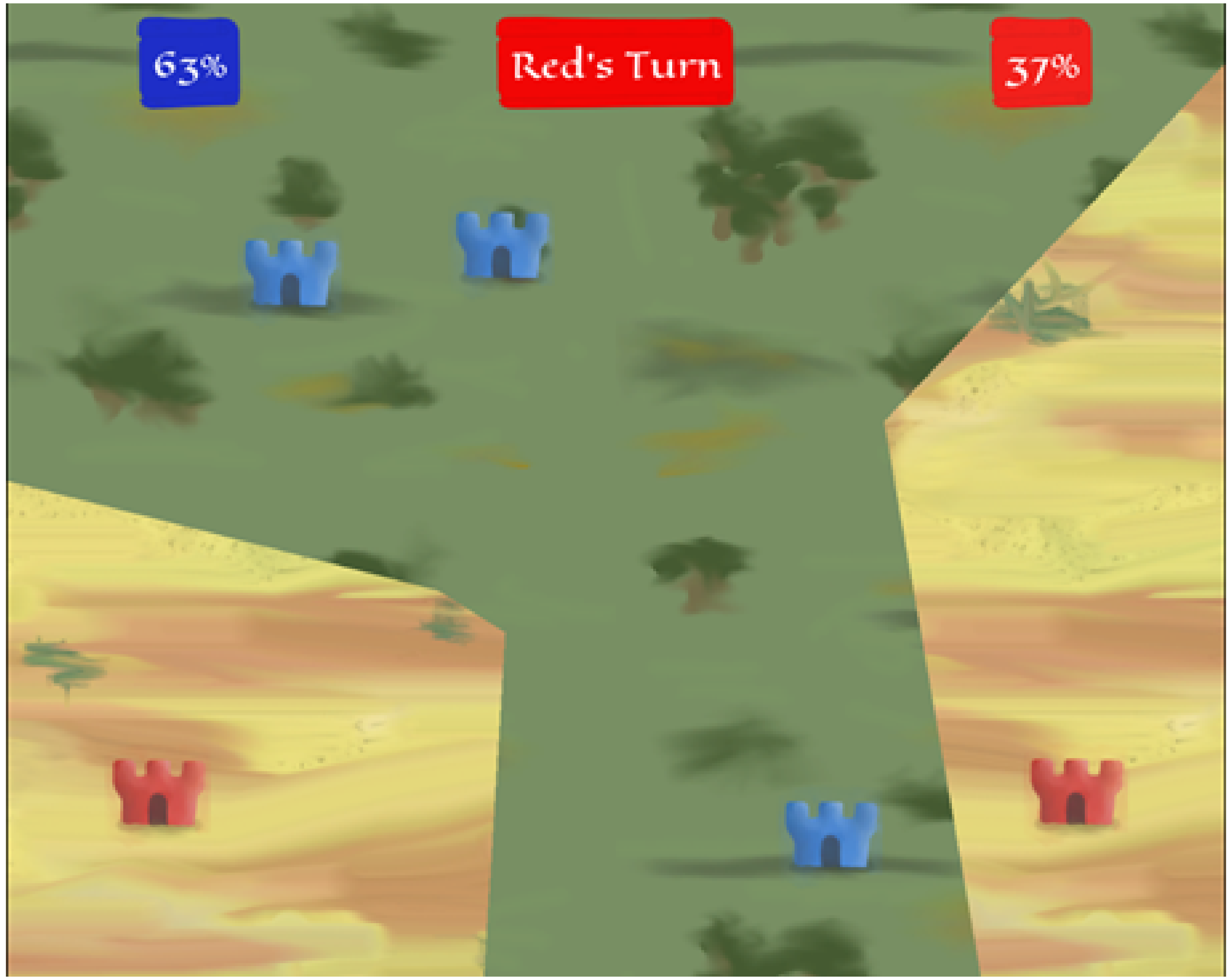
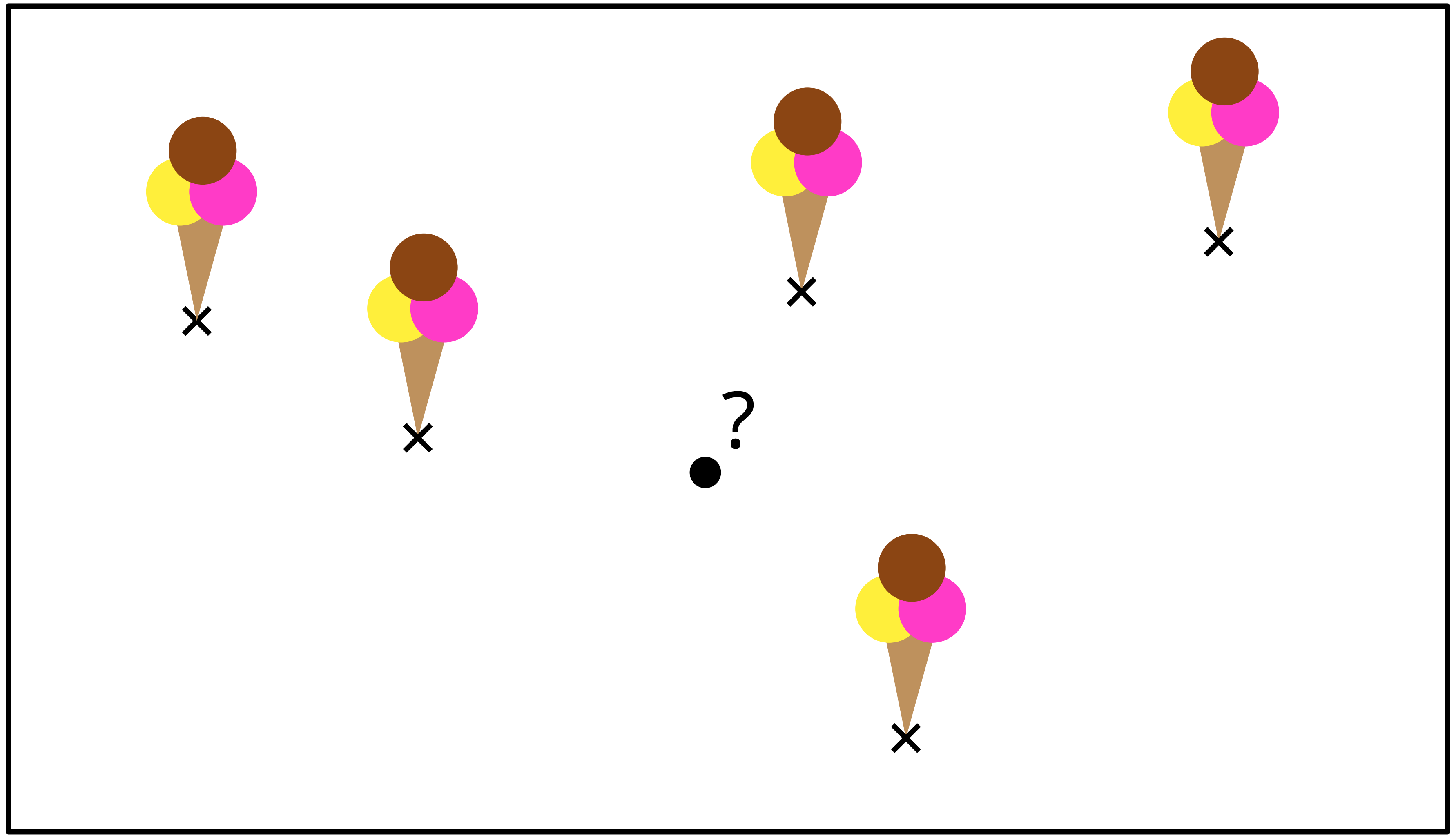


Motivation

Voronoi Diagrams



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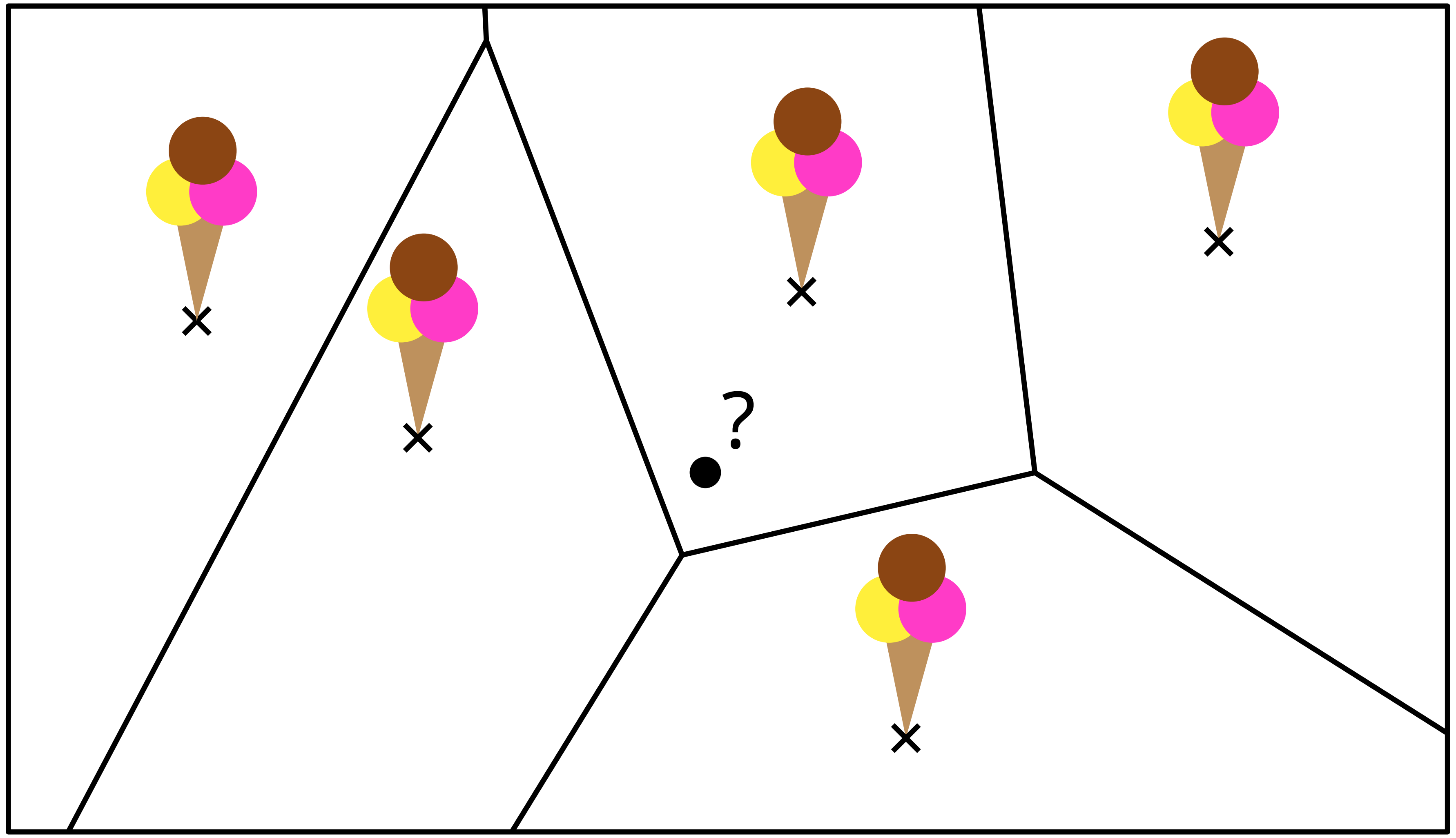

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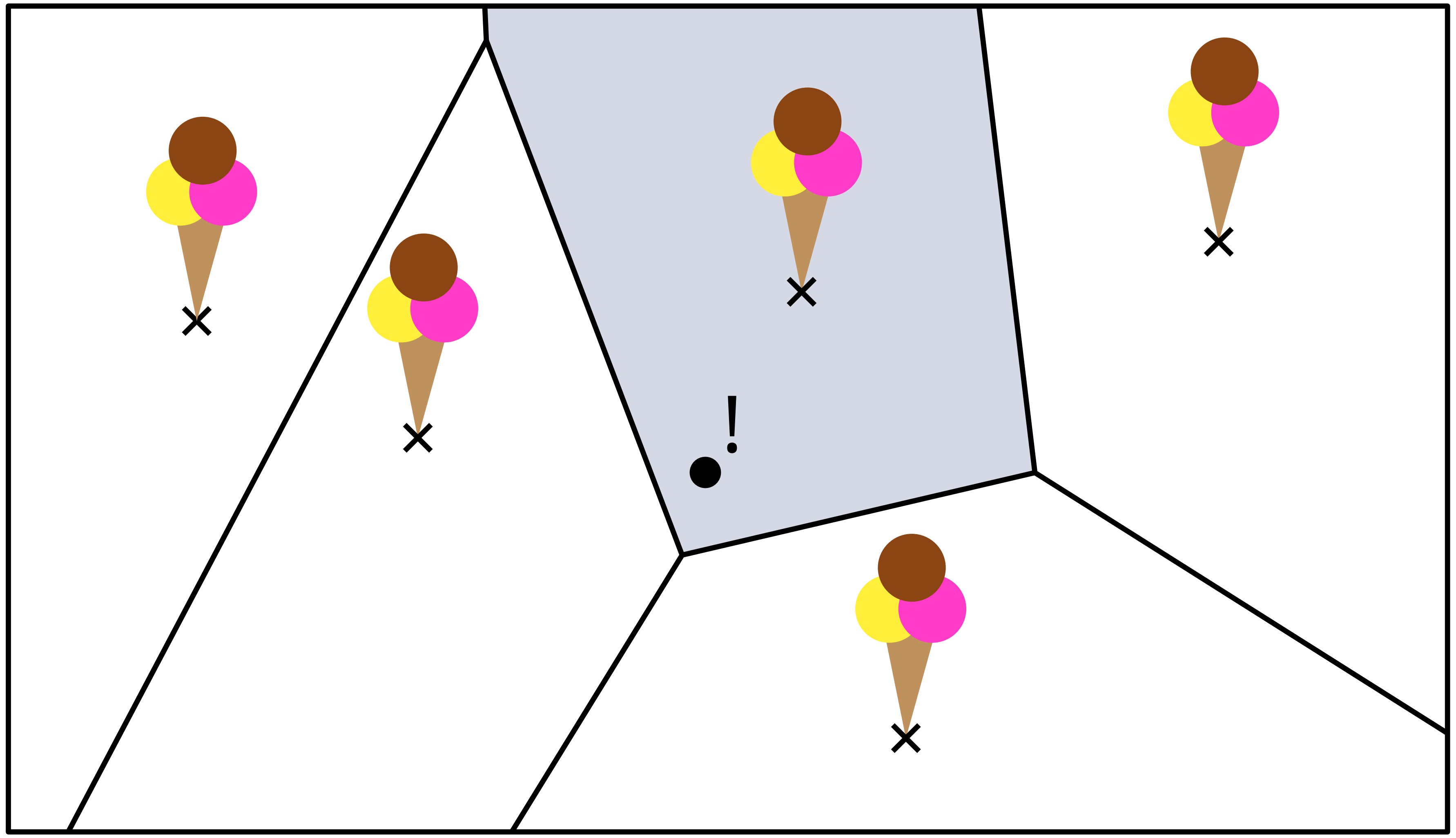

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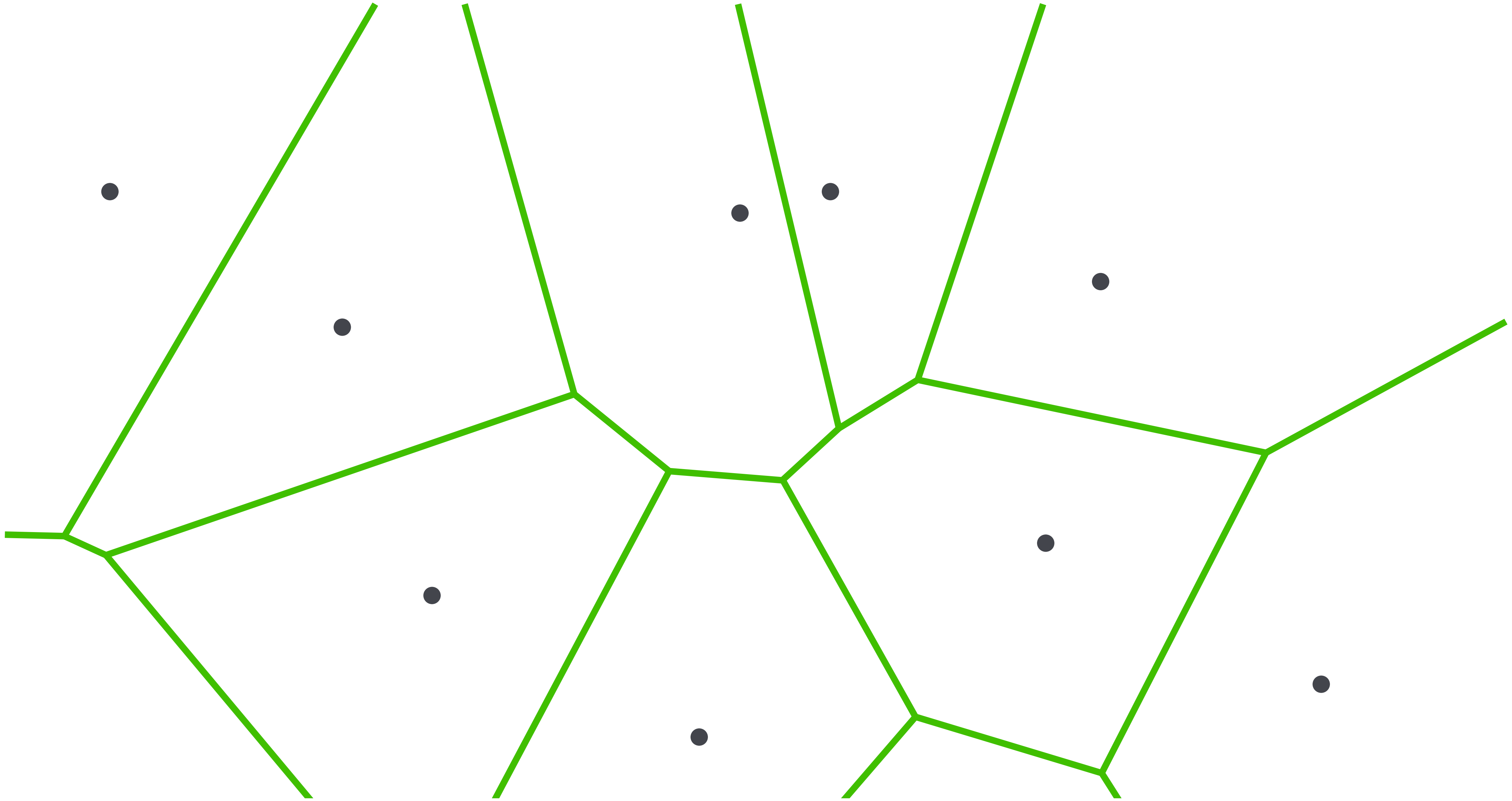
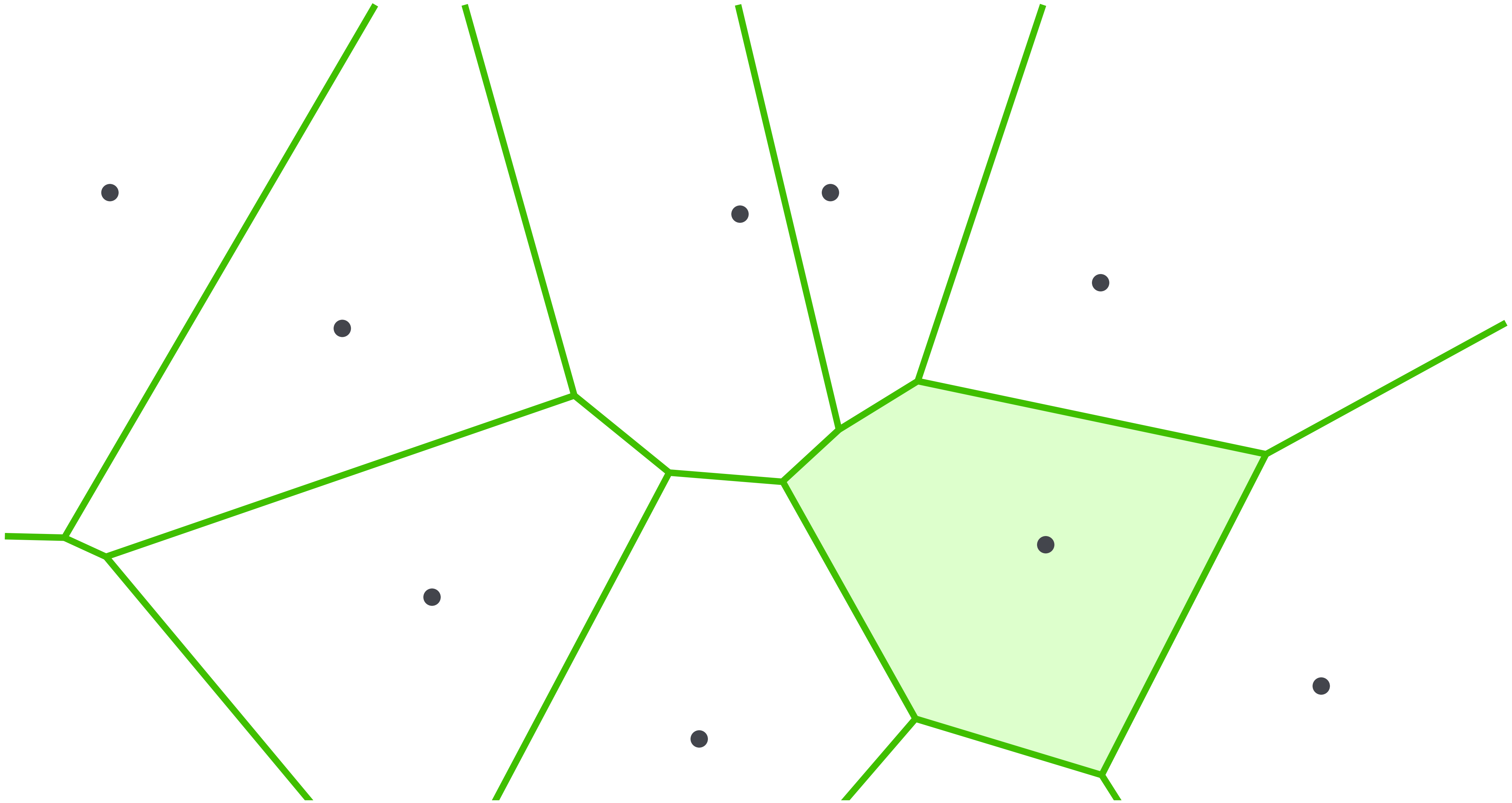


Voronoi diagram of $\{p_1, \ldots, p_n\}$: subdivision of the plane into cells, such that a point q is in the cell of p_i if and only if $dist(q, p_i) < dist(q, p_j)$ for all $i \neq j$.

Voronoi diagrams



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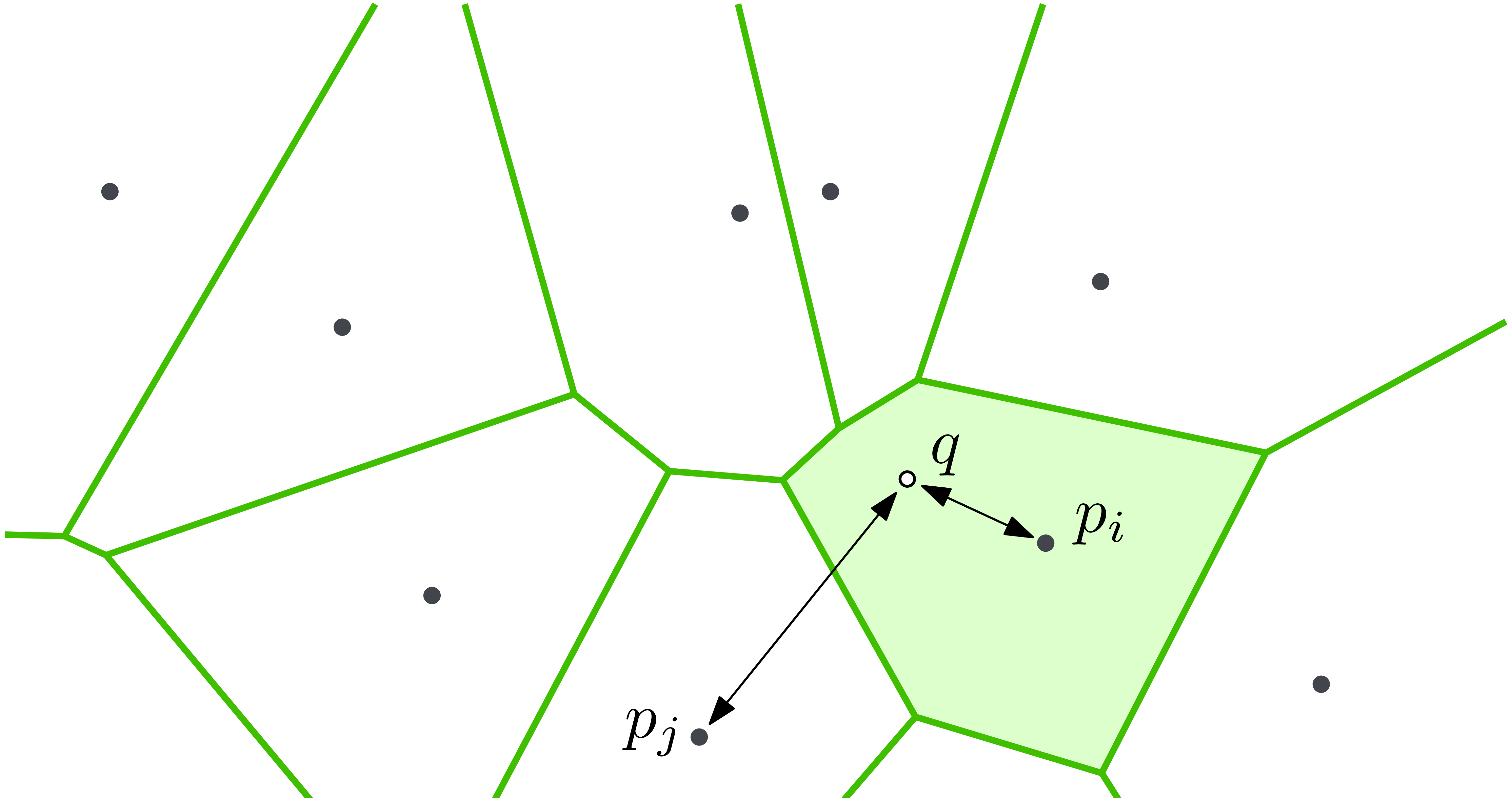

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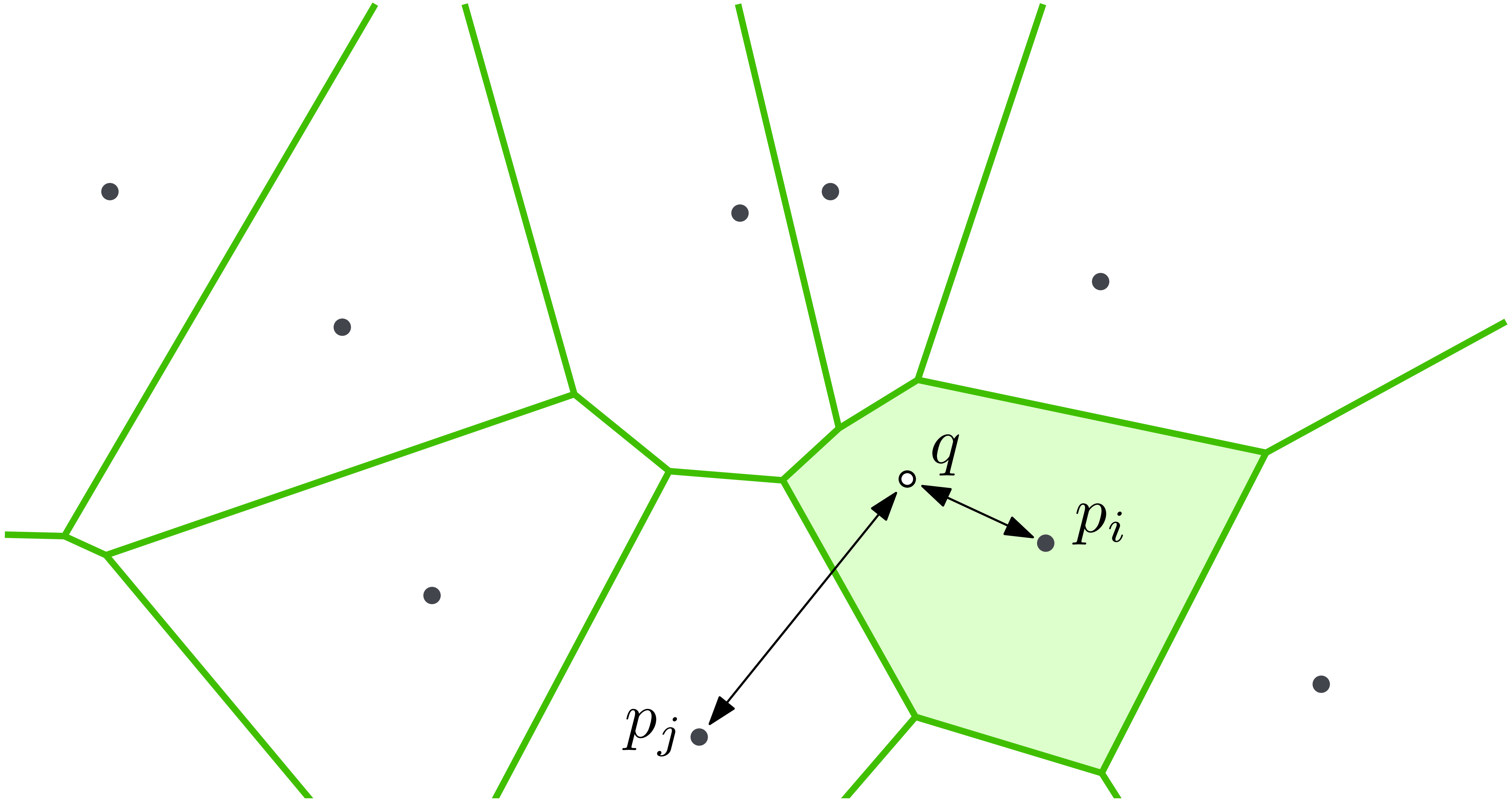

long history

- Descartes 1644
- Dirichlet 1850
- Voronoi 1907

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Voronoi diagrams

for 2 points

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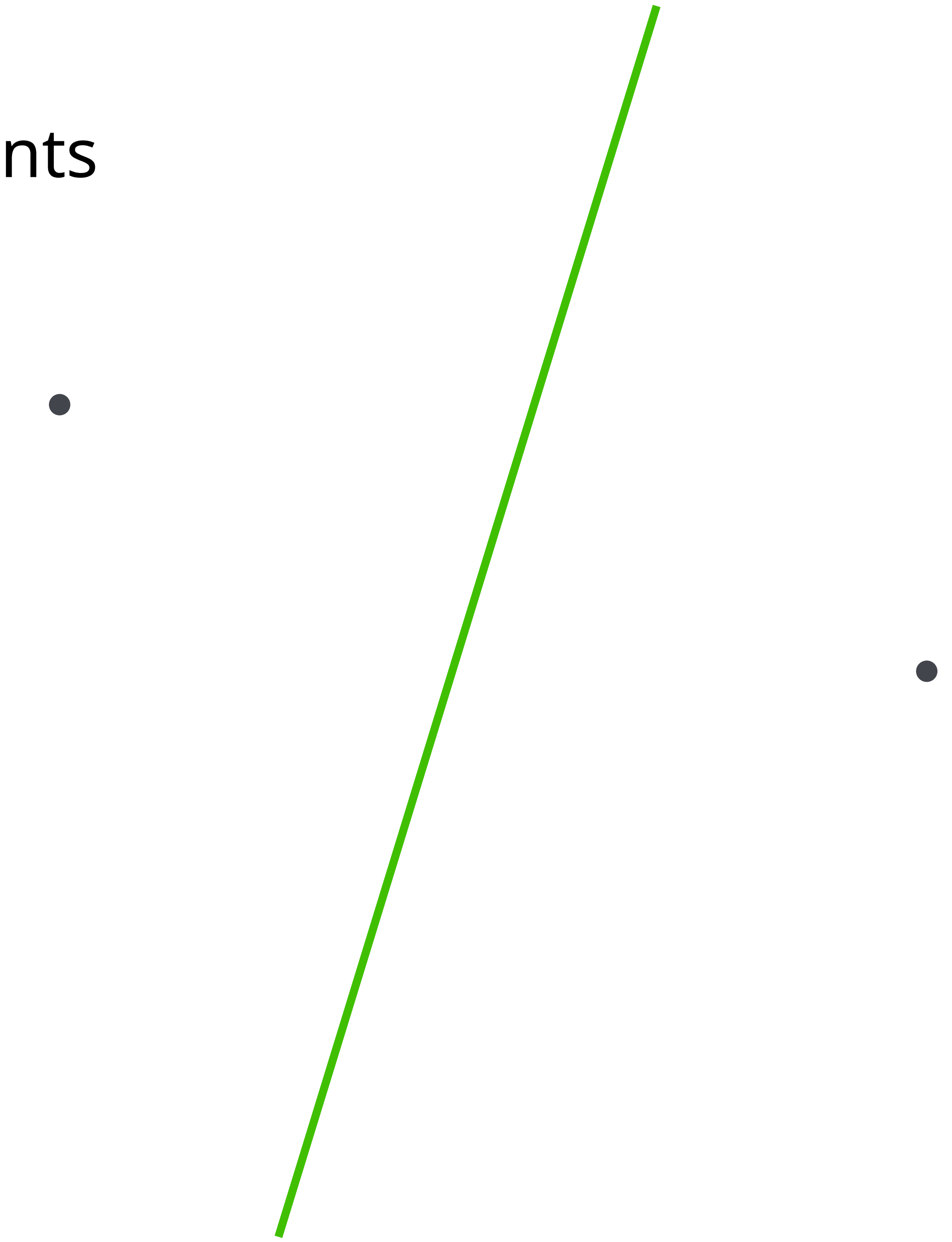

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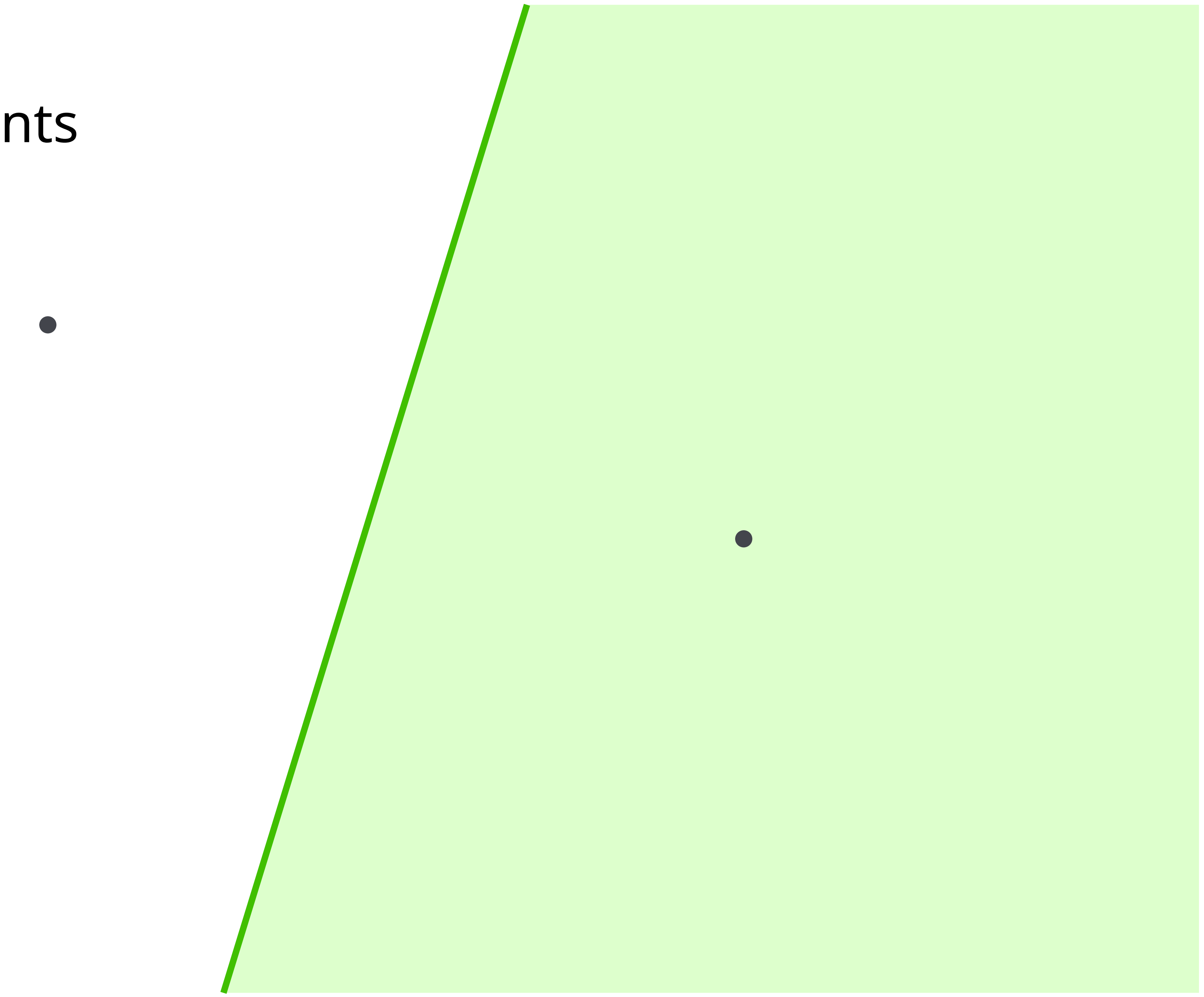

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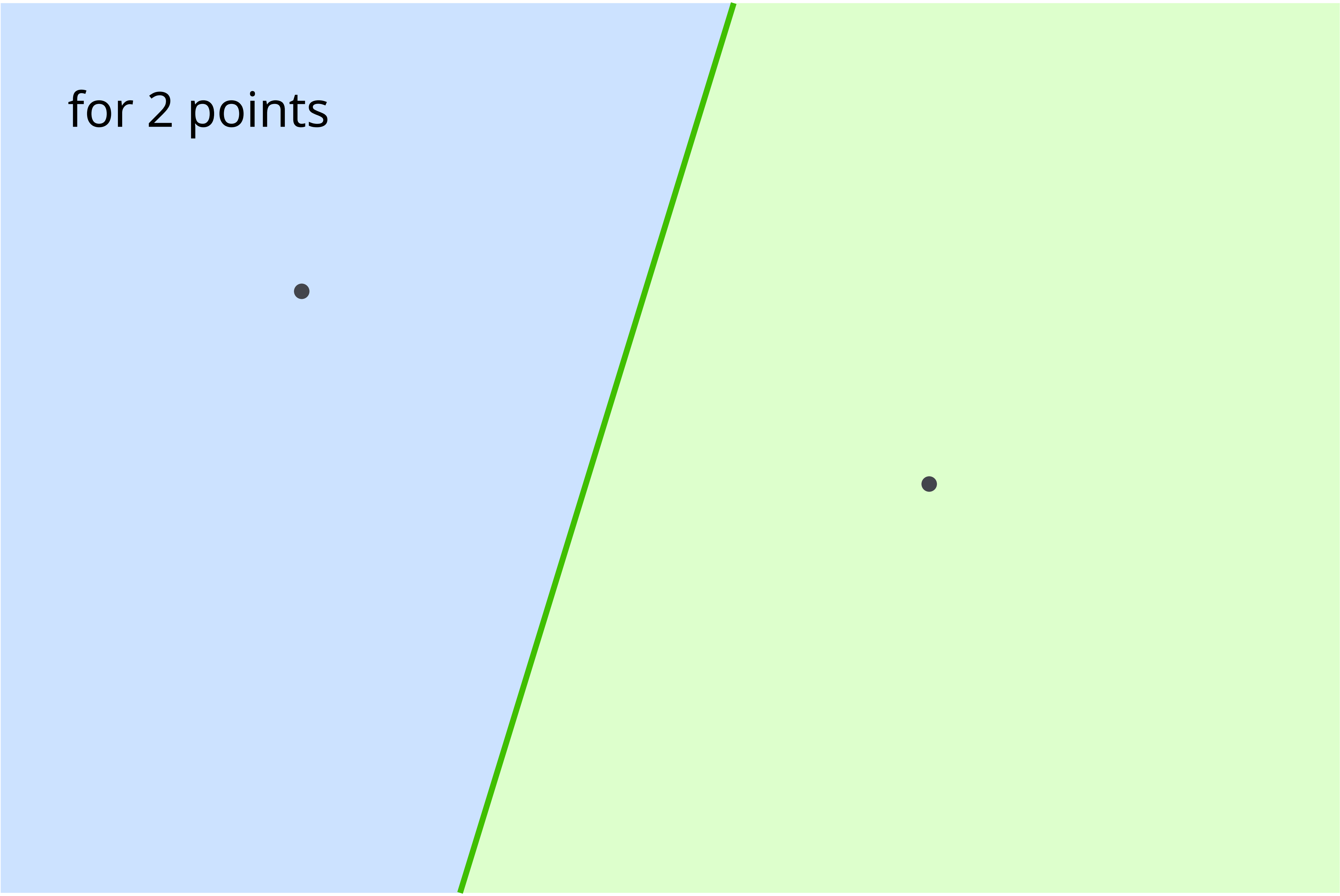

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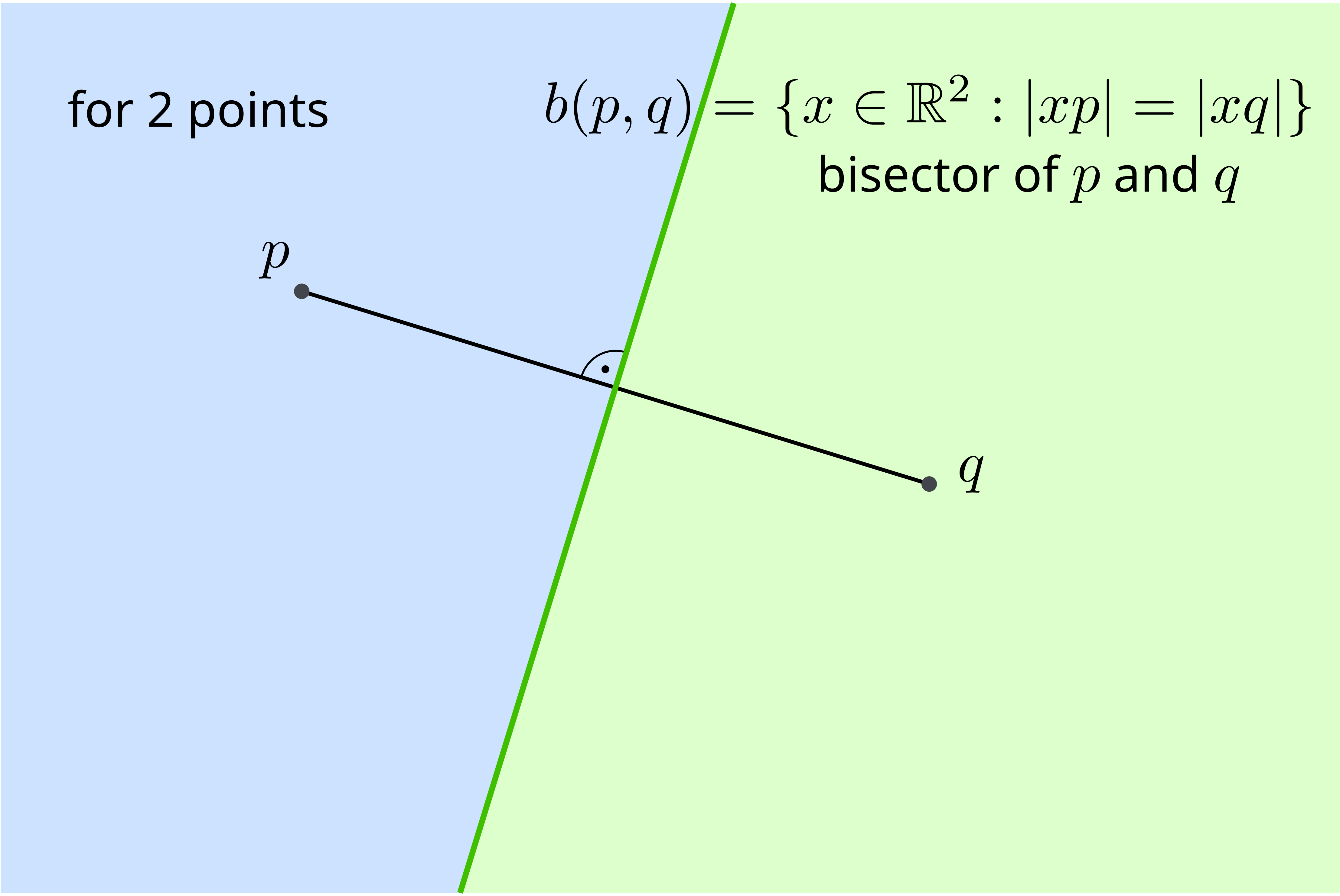

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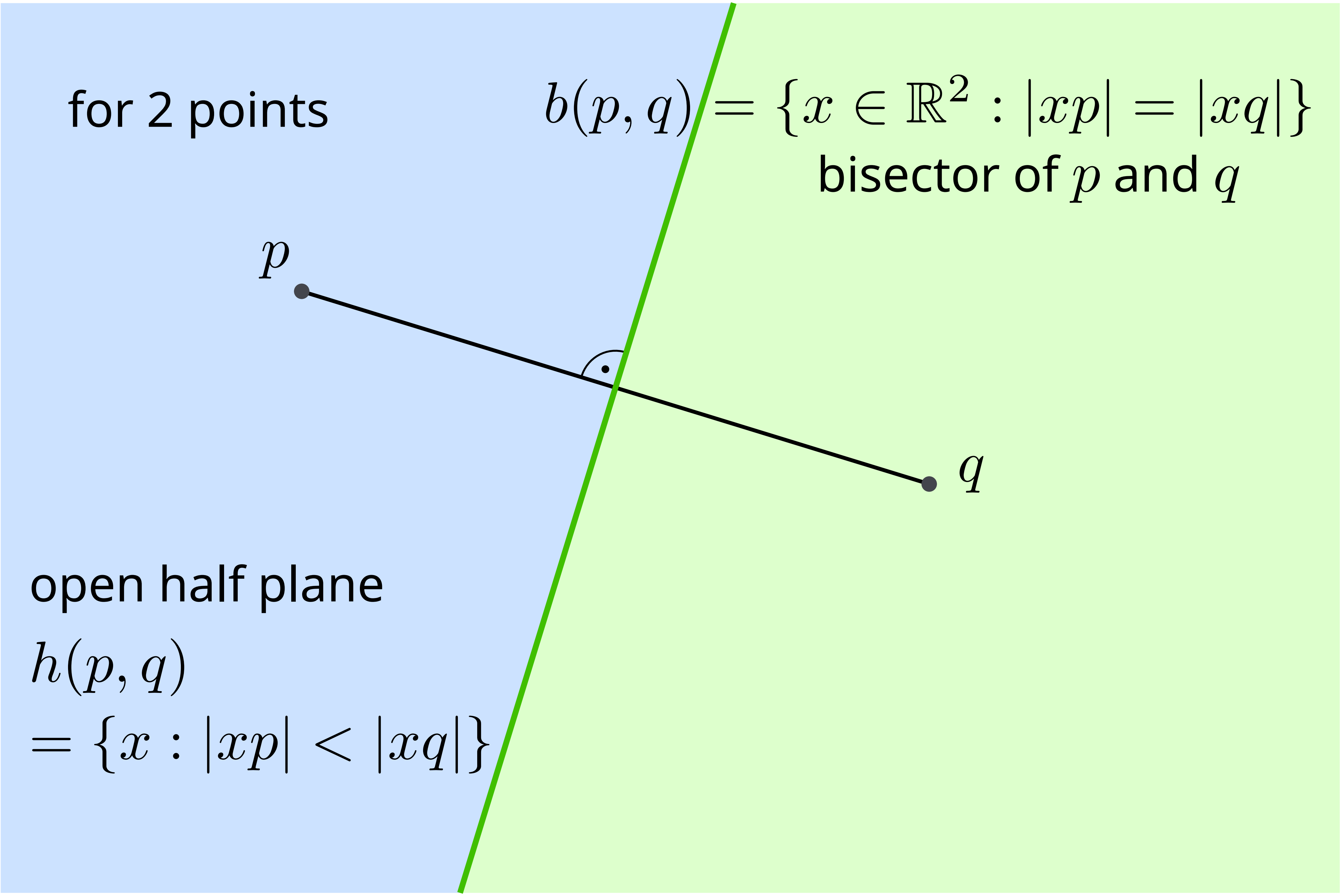

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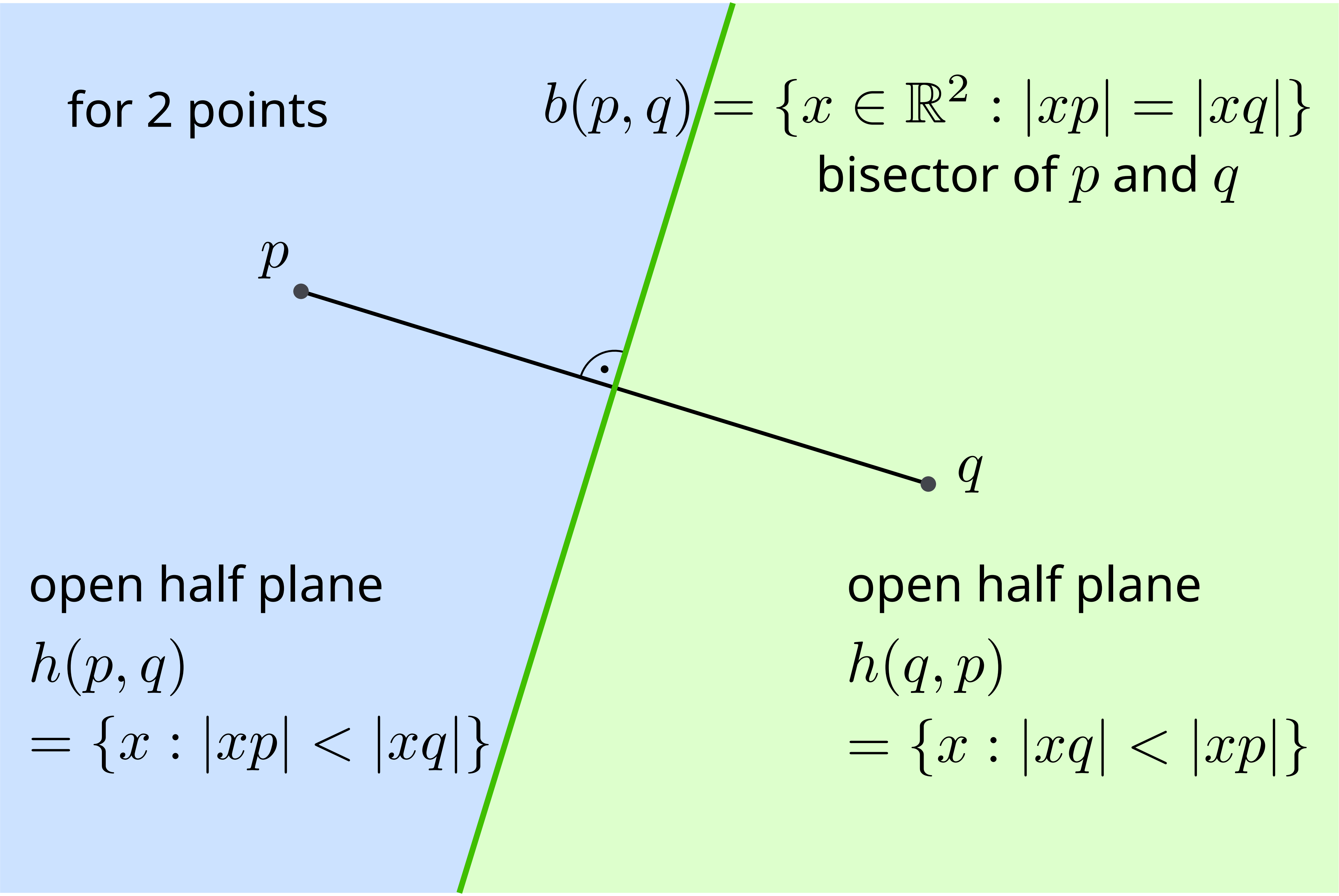

Voronoi diagrams

for n points $P = \{p_1, p_2, \ldots, p_n\}$

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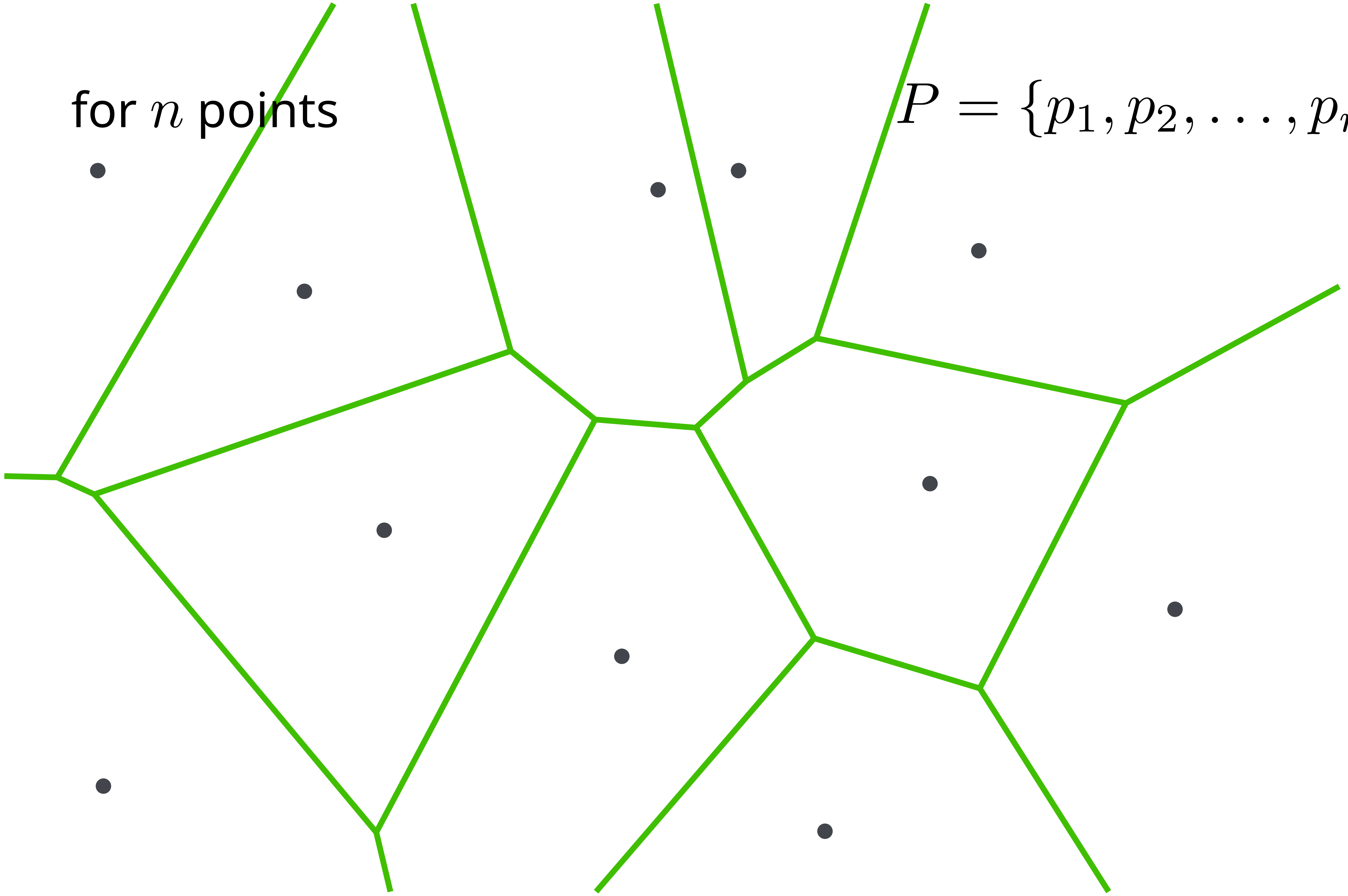

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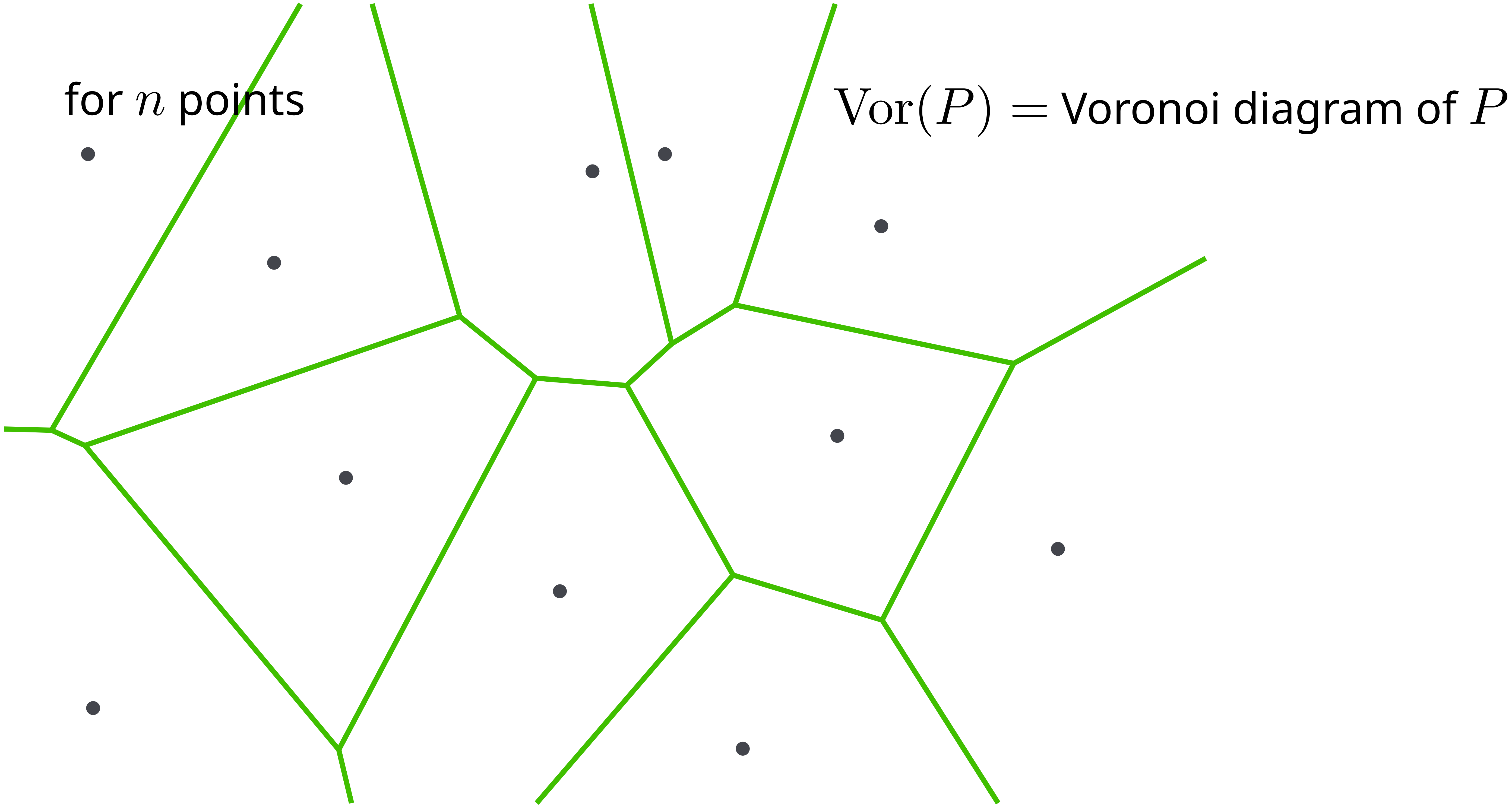

Quiz

Are Voronoi cells convex?

A: yes

B: no

C: only if they are bounded

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Quiz

Are Voronoi cells convex?

$\boxed{\text{A: yes}}$ for this lets have a closer look at cells

B: no

C: only if they are bounded

Voronoi diagrams

Let P be a set of points in the plane and $p, p', p'' \in P$.

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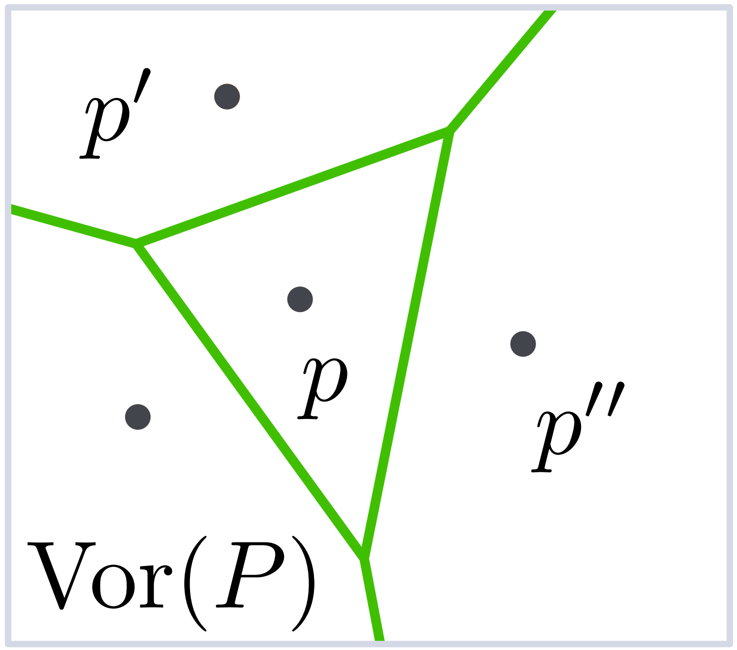

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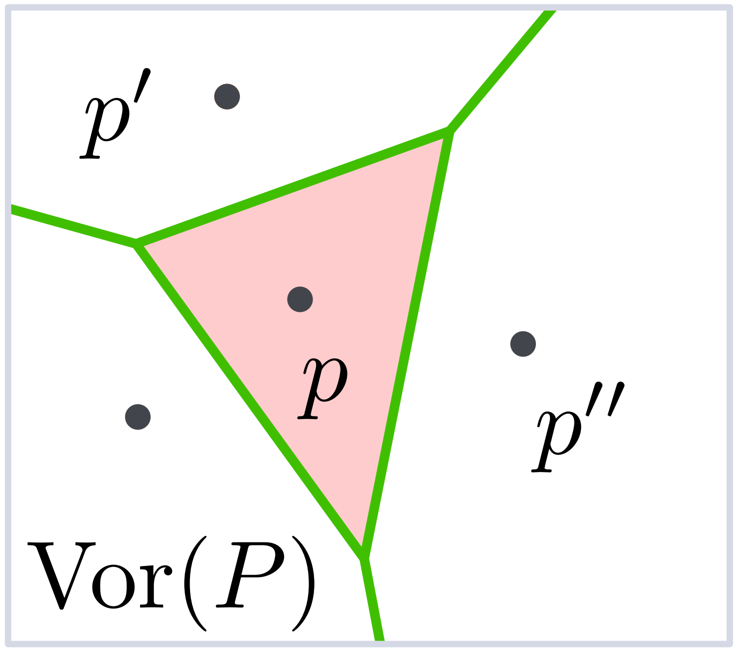

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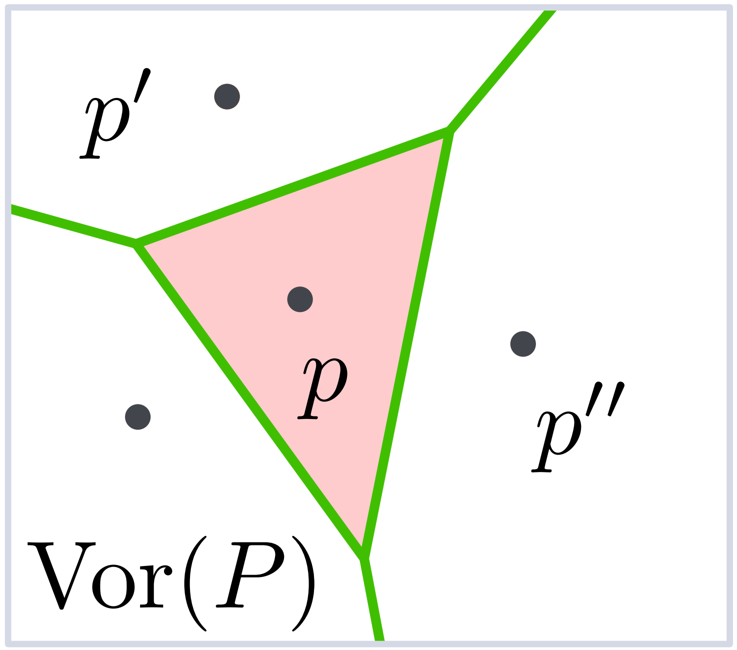
Voronoi diagram:

Voronoi cell

$\mathcal{V}(\{p\}) =$



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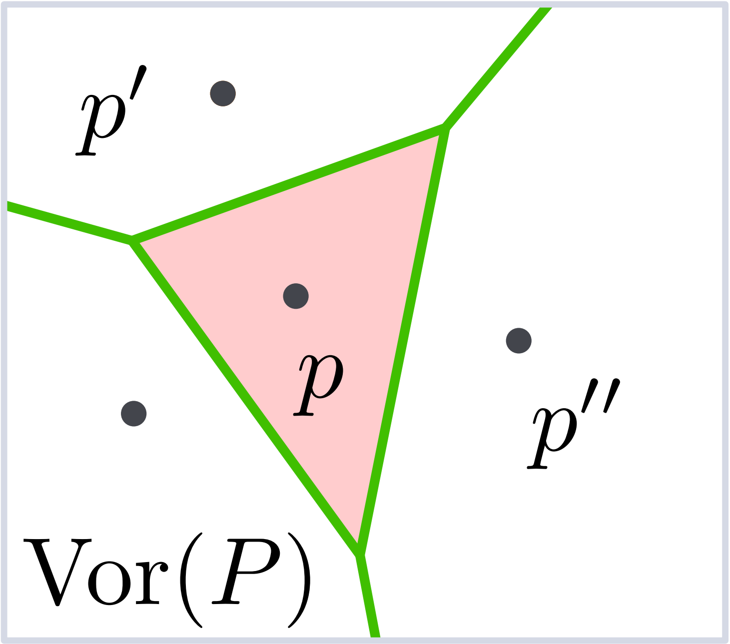

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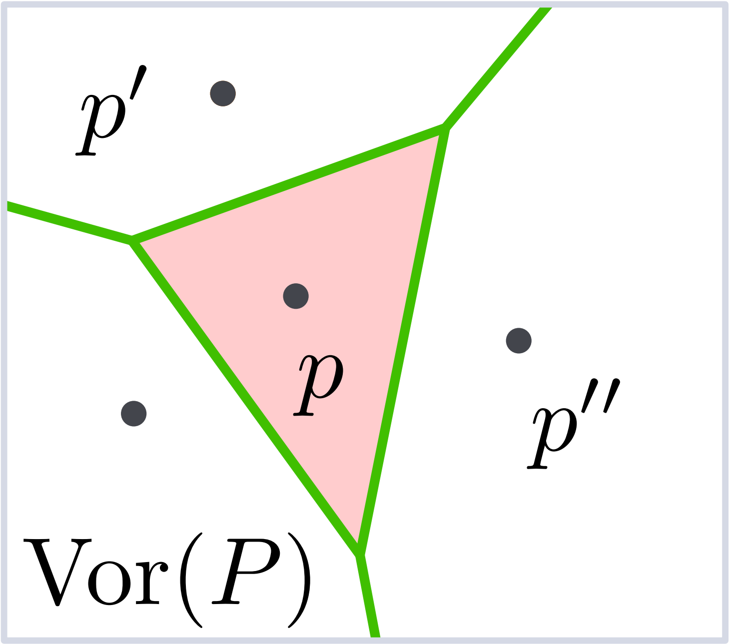

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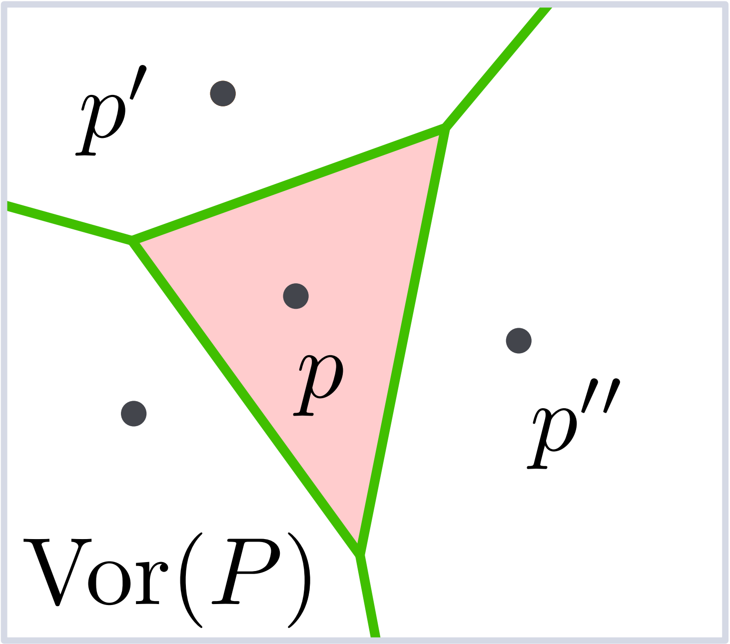

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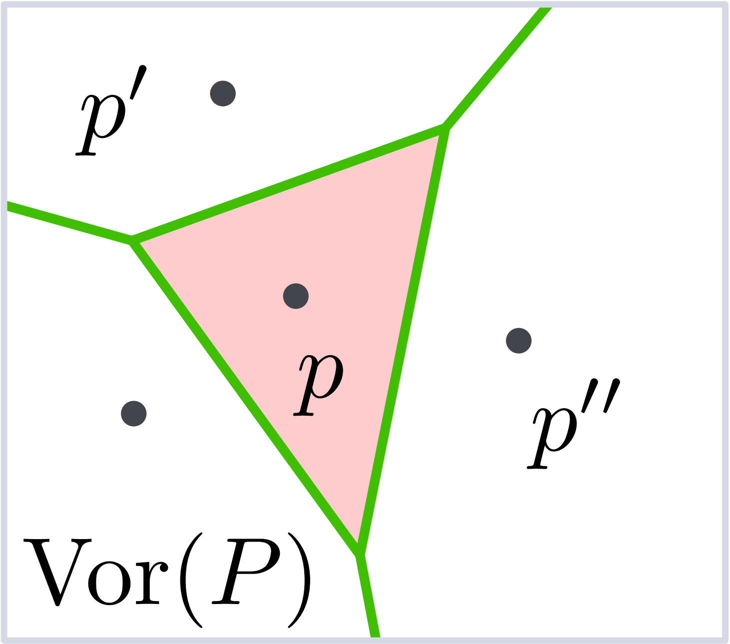
intersection of convex sets $\rightarrow$ cells are convex



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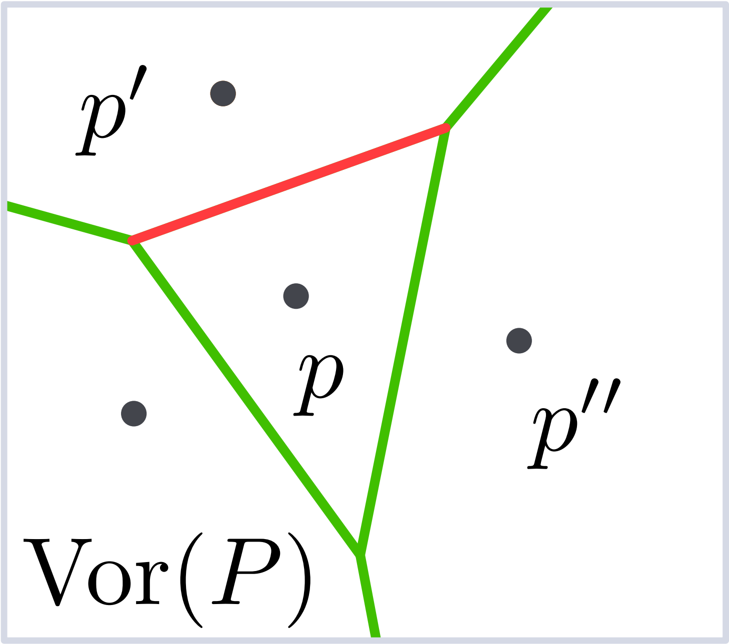

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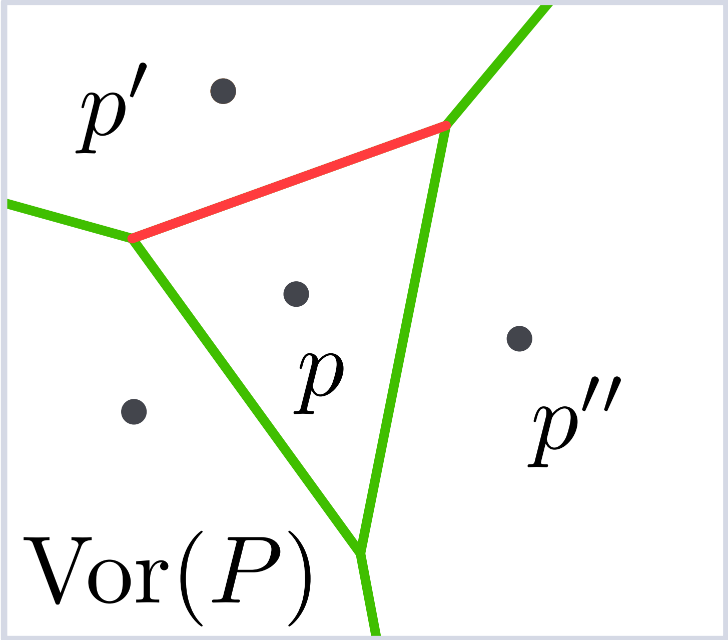

Voronoi edge

$$\mathcal{V}(\{p, p'\}) =$$

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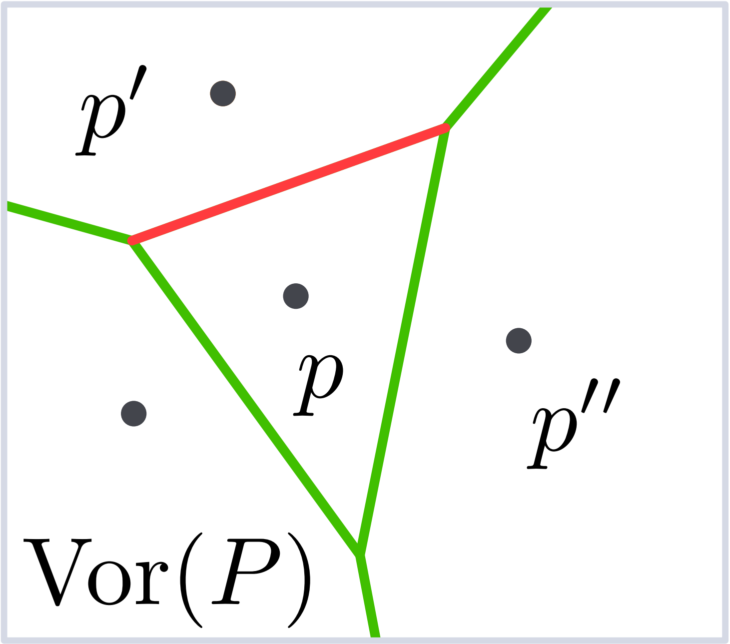

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$$\mathcal{V}(\{p, p'\}) = \{x : |xp| = |xp'| \text{ and } |xp| < |xq| \;\; \forall q \neq p, p'\}$$

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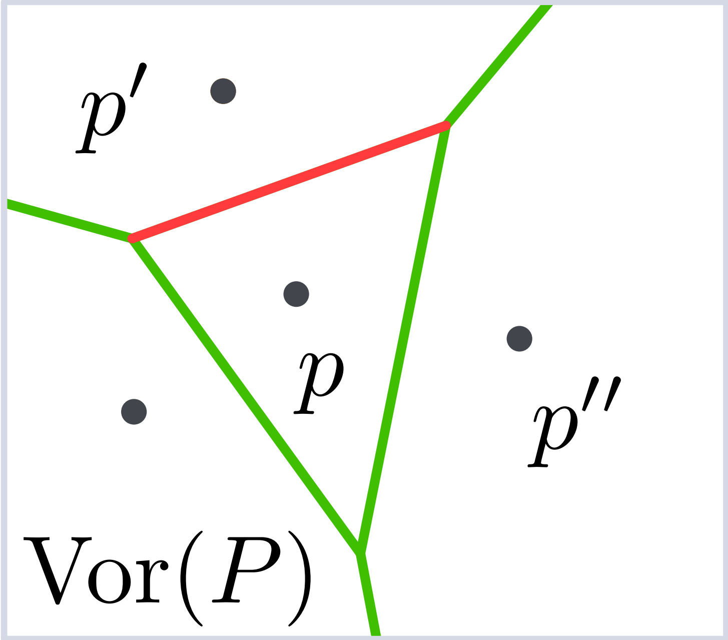

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$$= \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')$$

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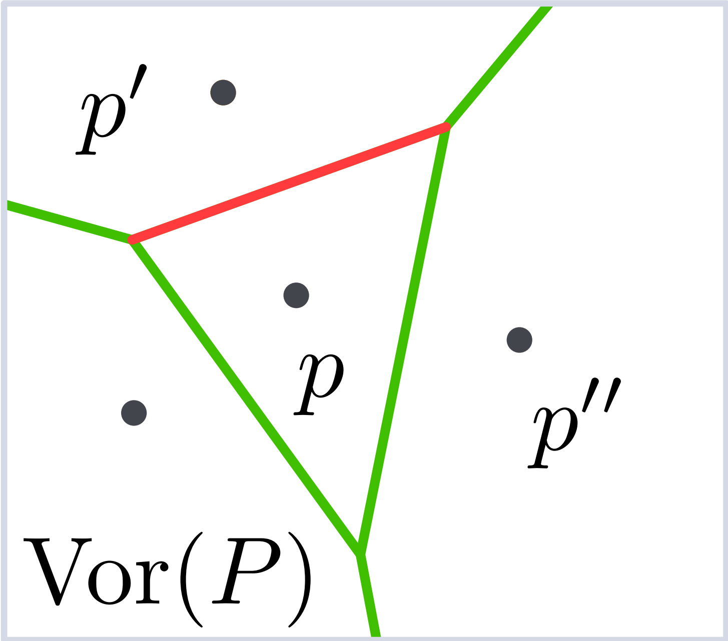

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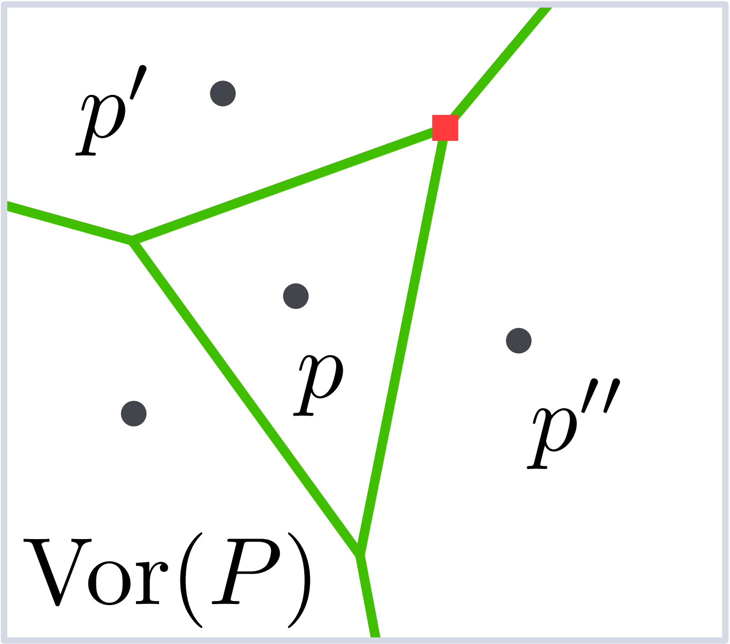

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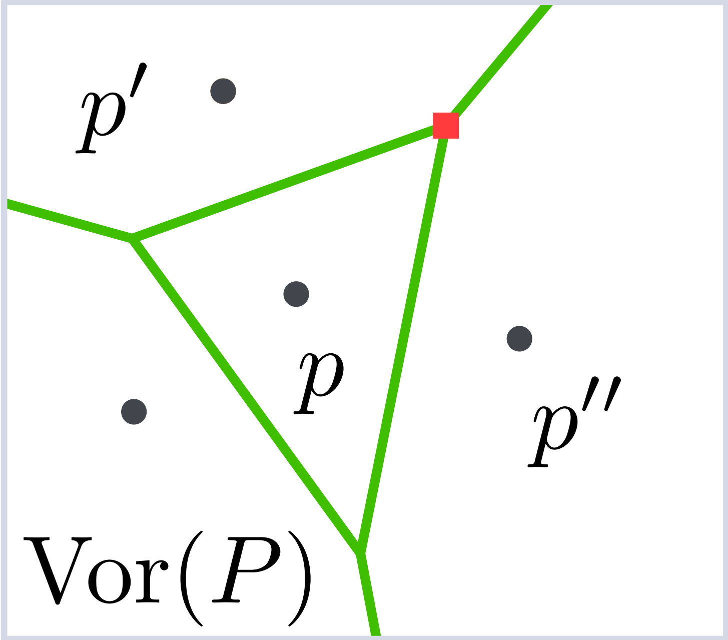

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Voronoi vertex

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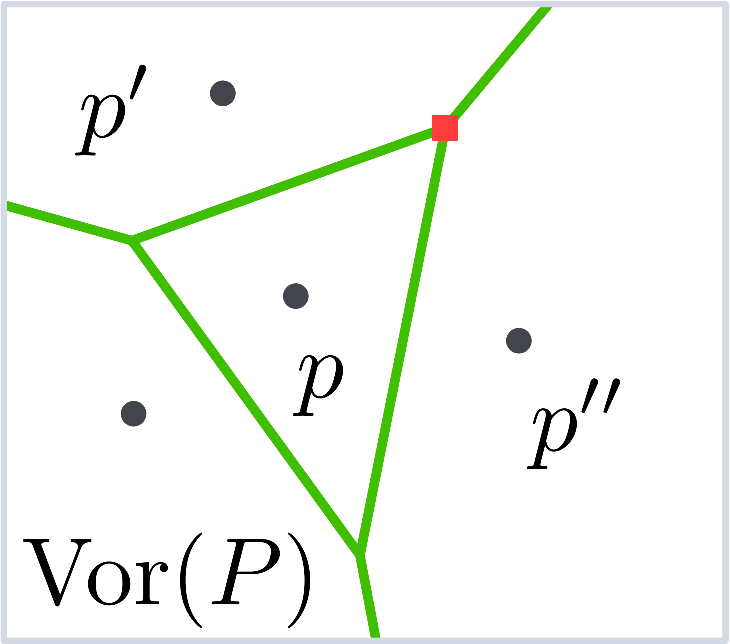

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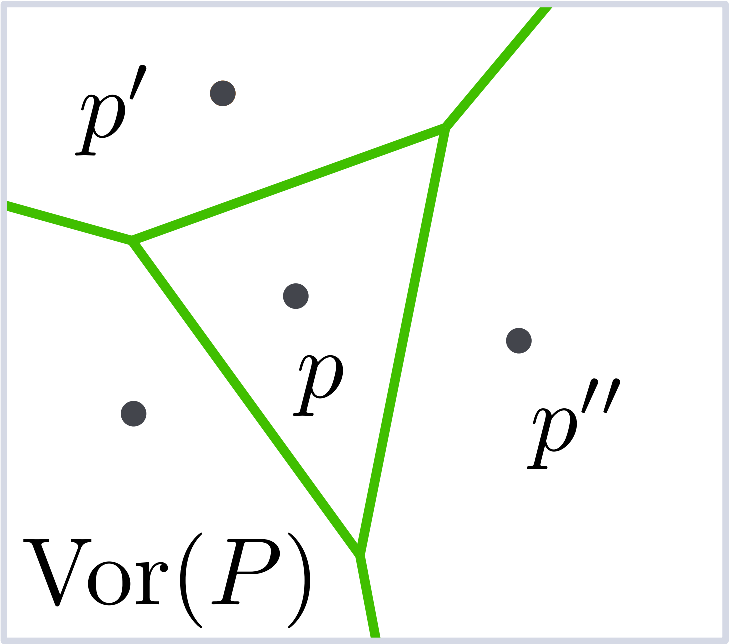
Voronoi diagrams

Let P be a set of points in the plane and $p, p', p'' \in P$.

Voronoi diagram:

Voronoi cell

a subdivision



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Voronoi vertex

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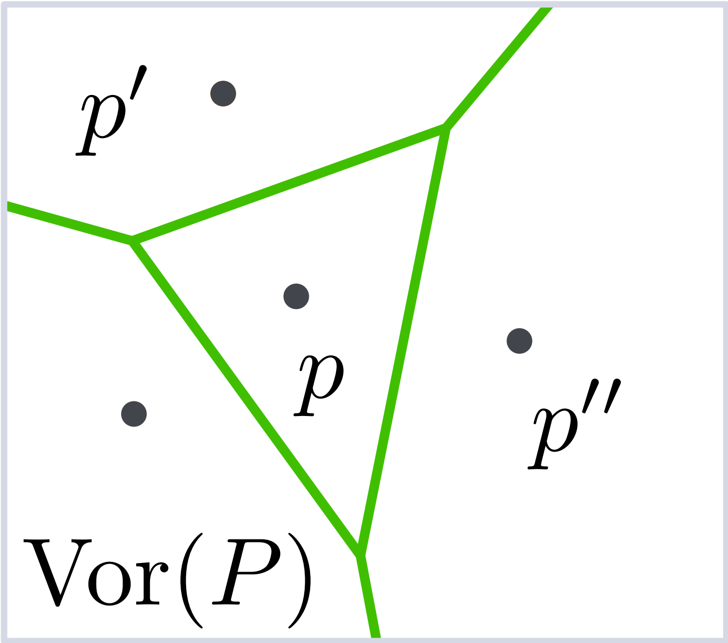


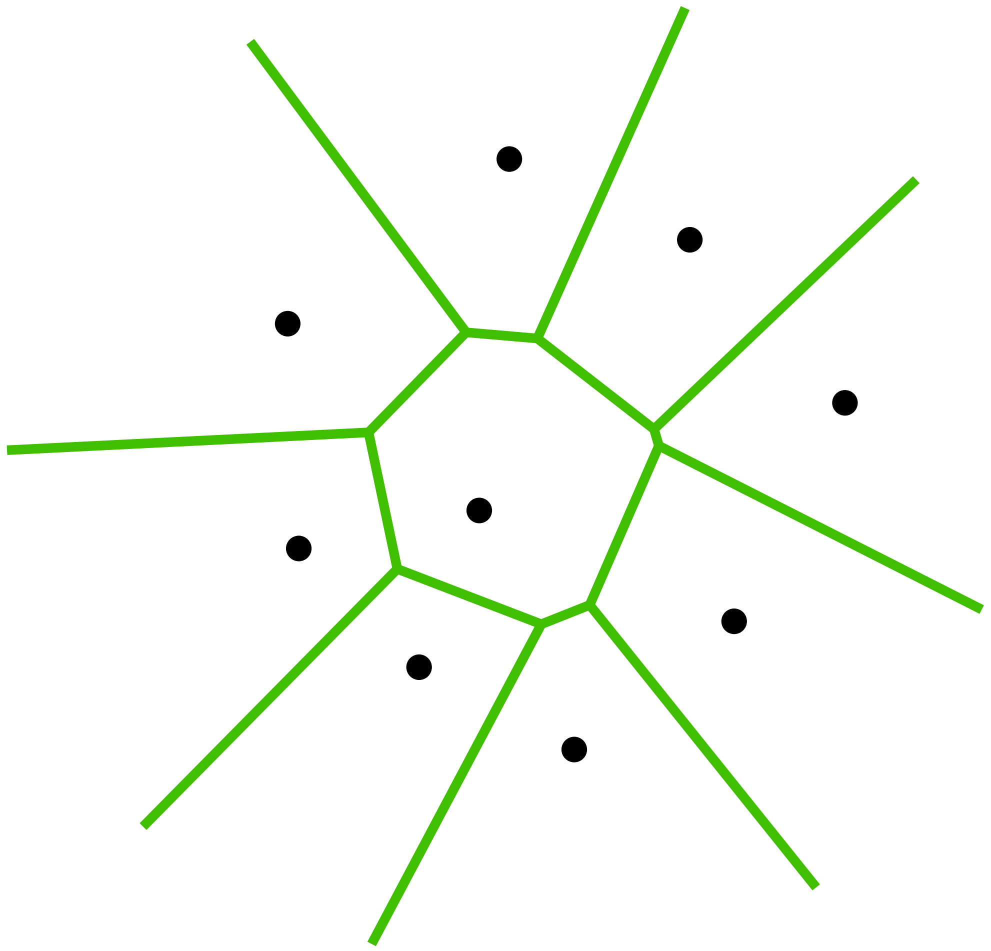
Voronoi edge

a plane graph

Voronoi vertex

Structure

If points in P are not all collinear, then Vor(P) is connected



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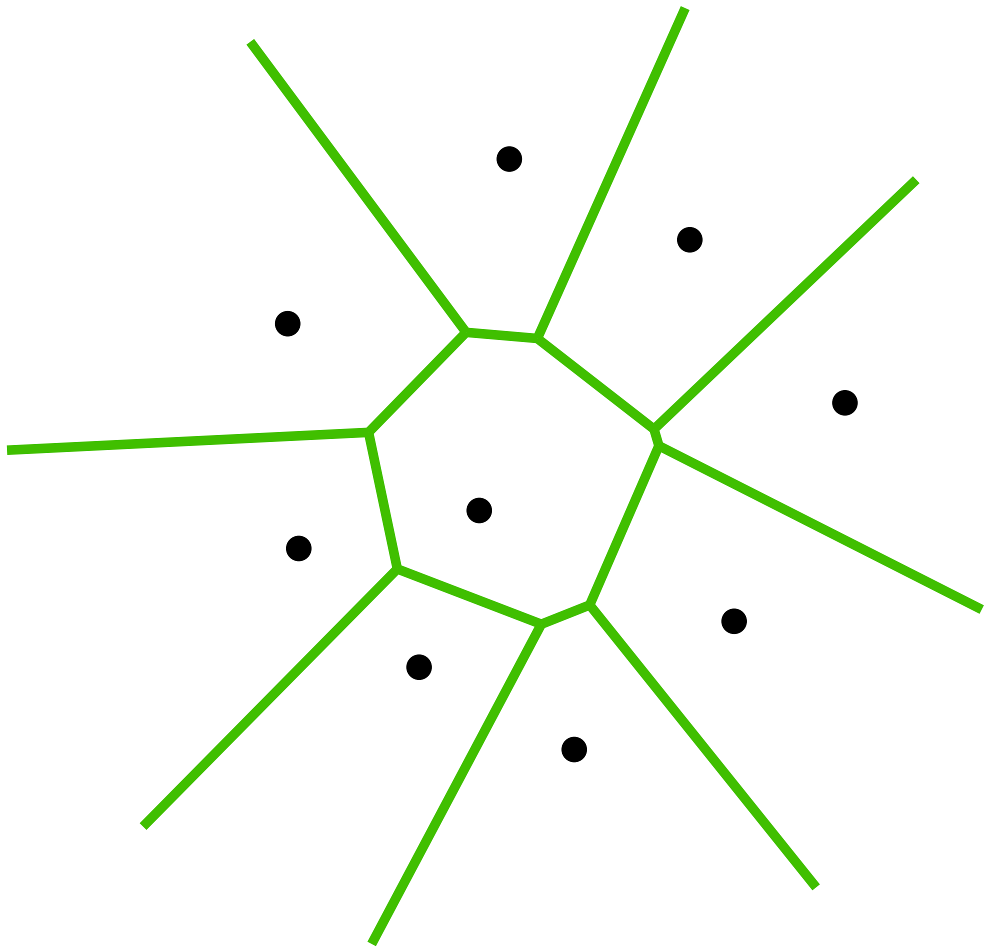
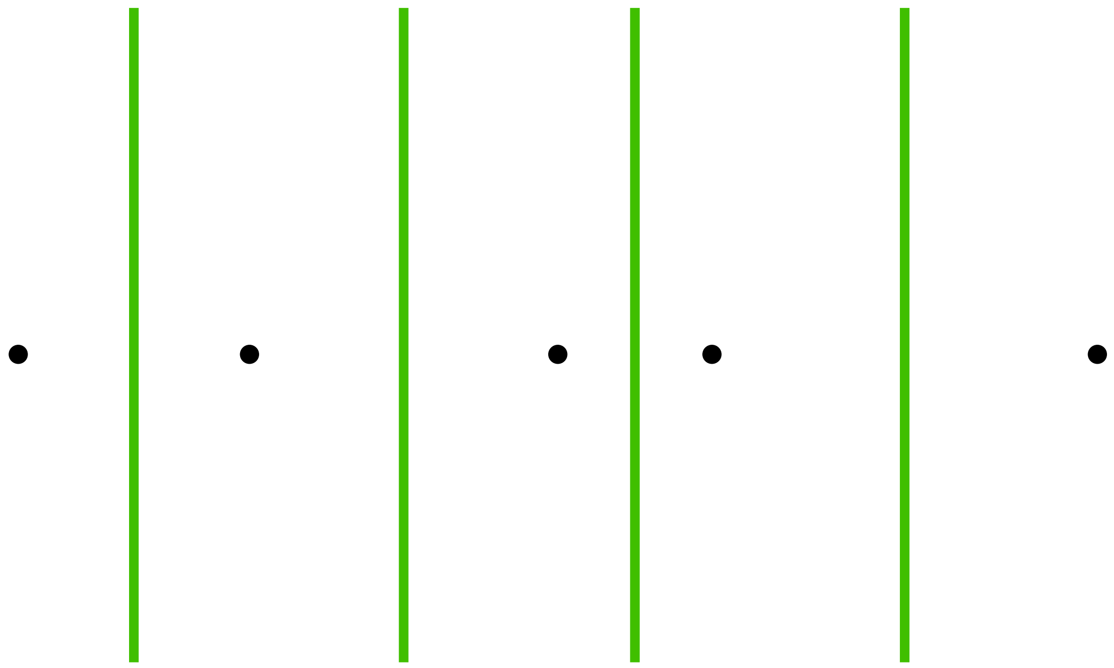
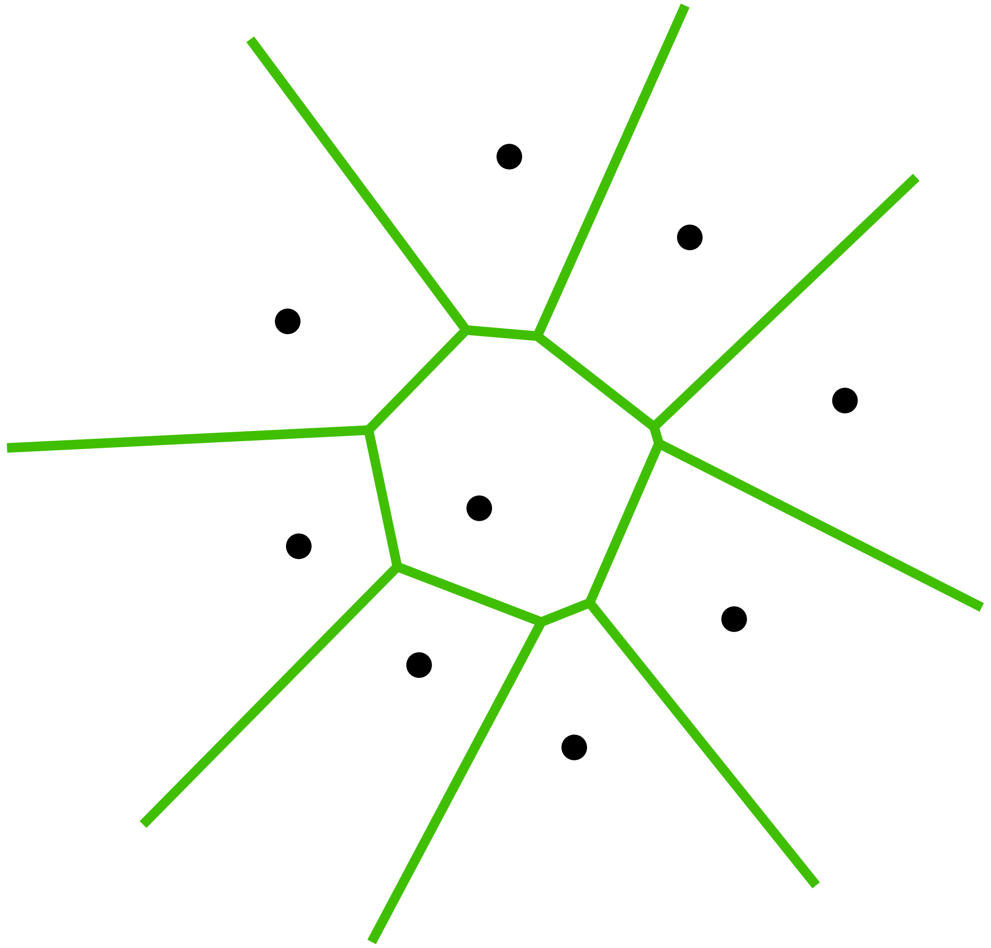
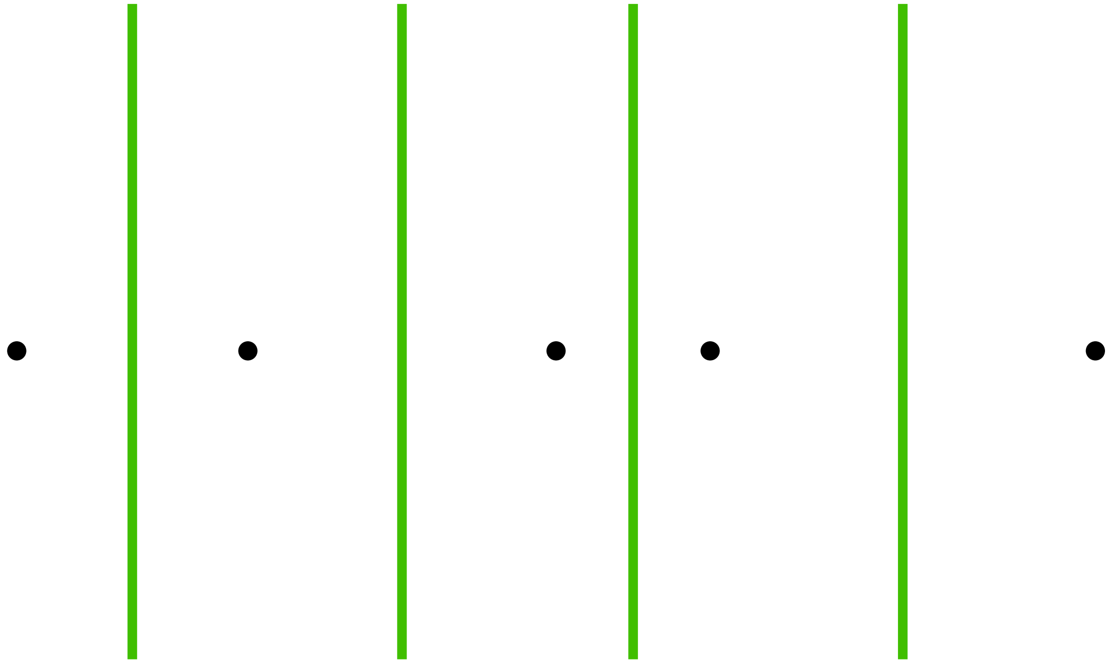




Structure

If points in P are not all collinear, then Vor(P) is connected

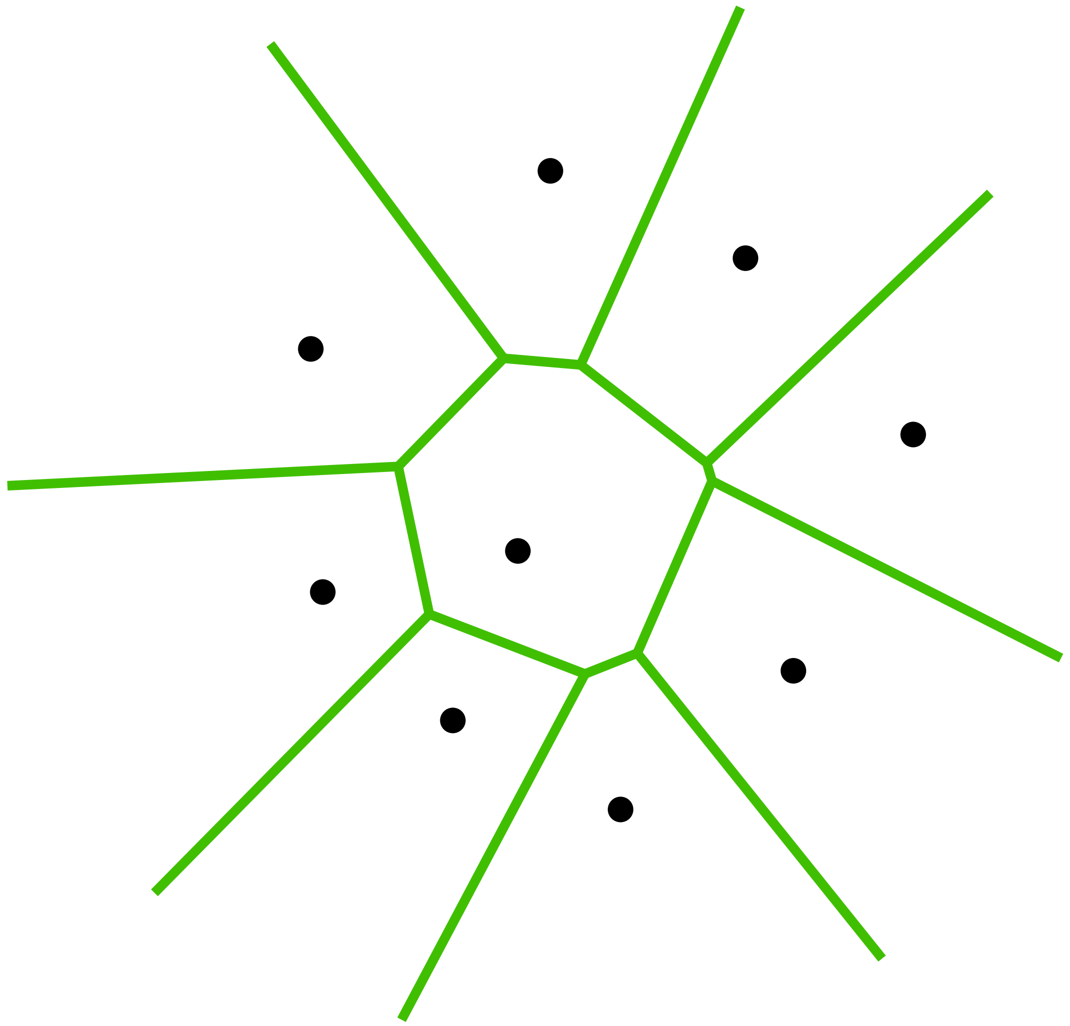




next: Complexity of Vor(P)

Voronoi Diagrams

Complexity



Quiz

How many sides can a single cell have?

A: 6

B: $n/2$

C: $n - 1$

Quiz

How many sides can a single cell have?

A: 6

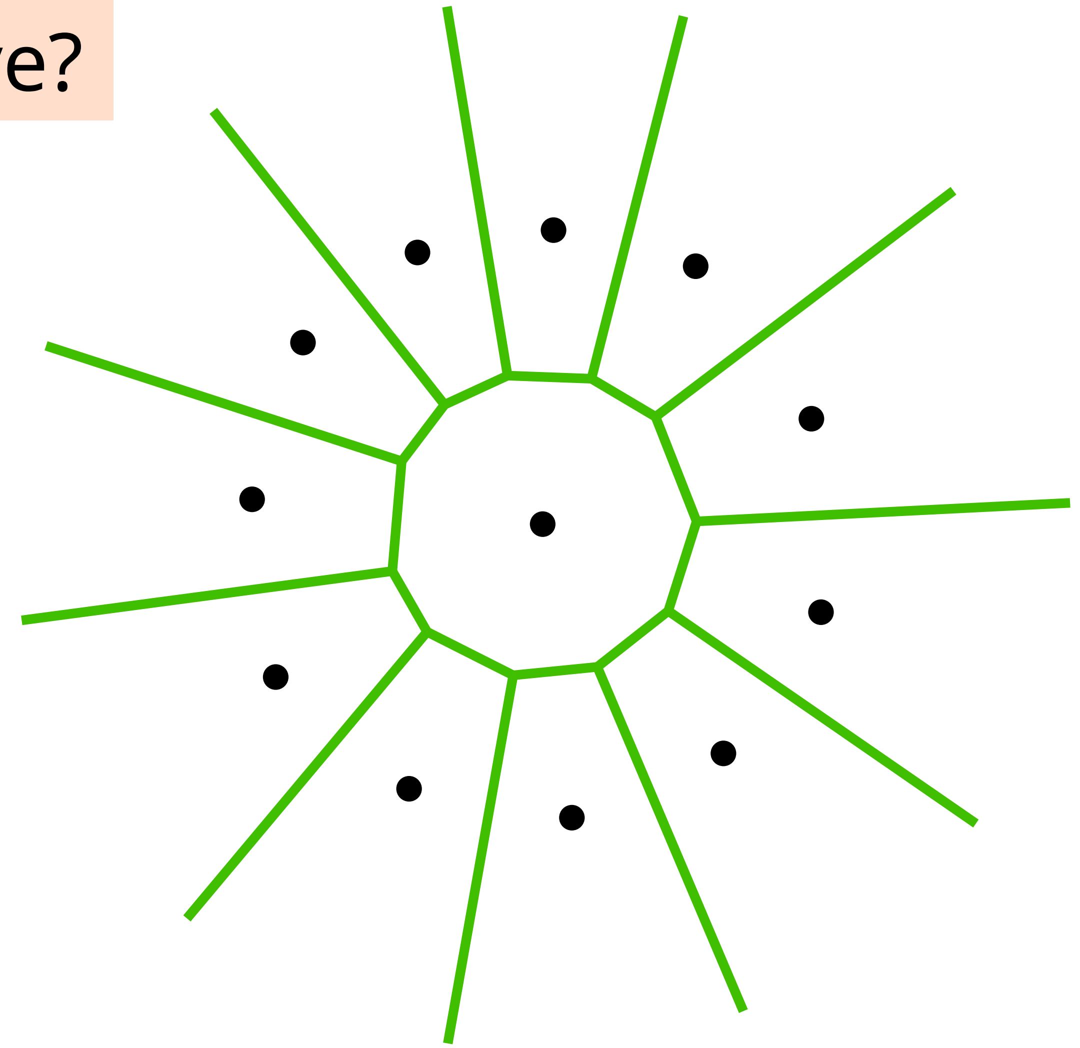
B: $n/2$

$\boxed{\text{C: } n - 1}$

Complexity of a Cell

How many sides can a single cell have?

A cell may have up to $n-1$ sides



Complexity of a Cell

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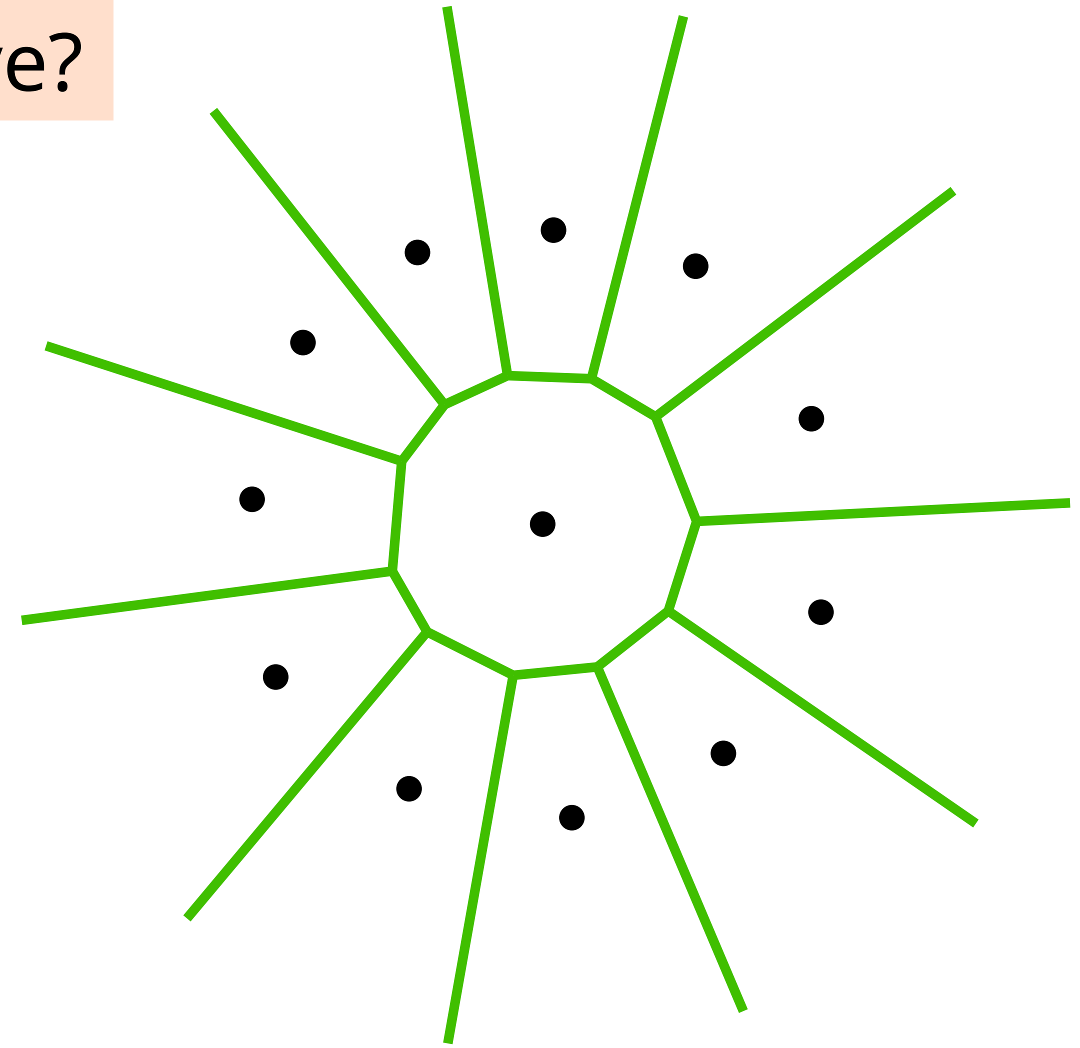

A cell may have up to $n-1$ sides

Can all cells have $n-1$ sides?

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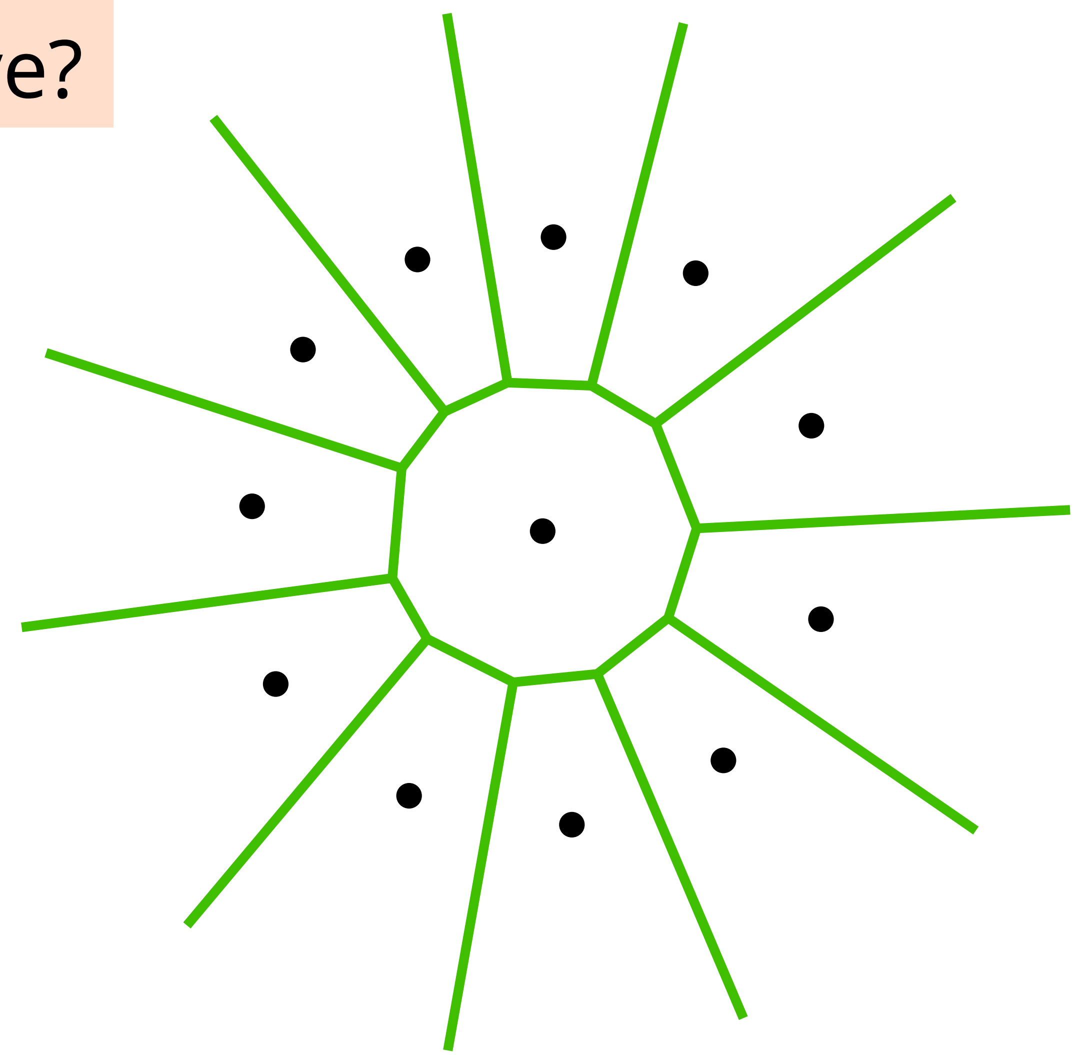


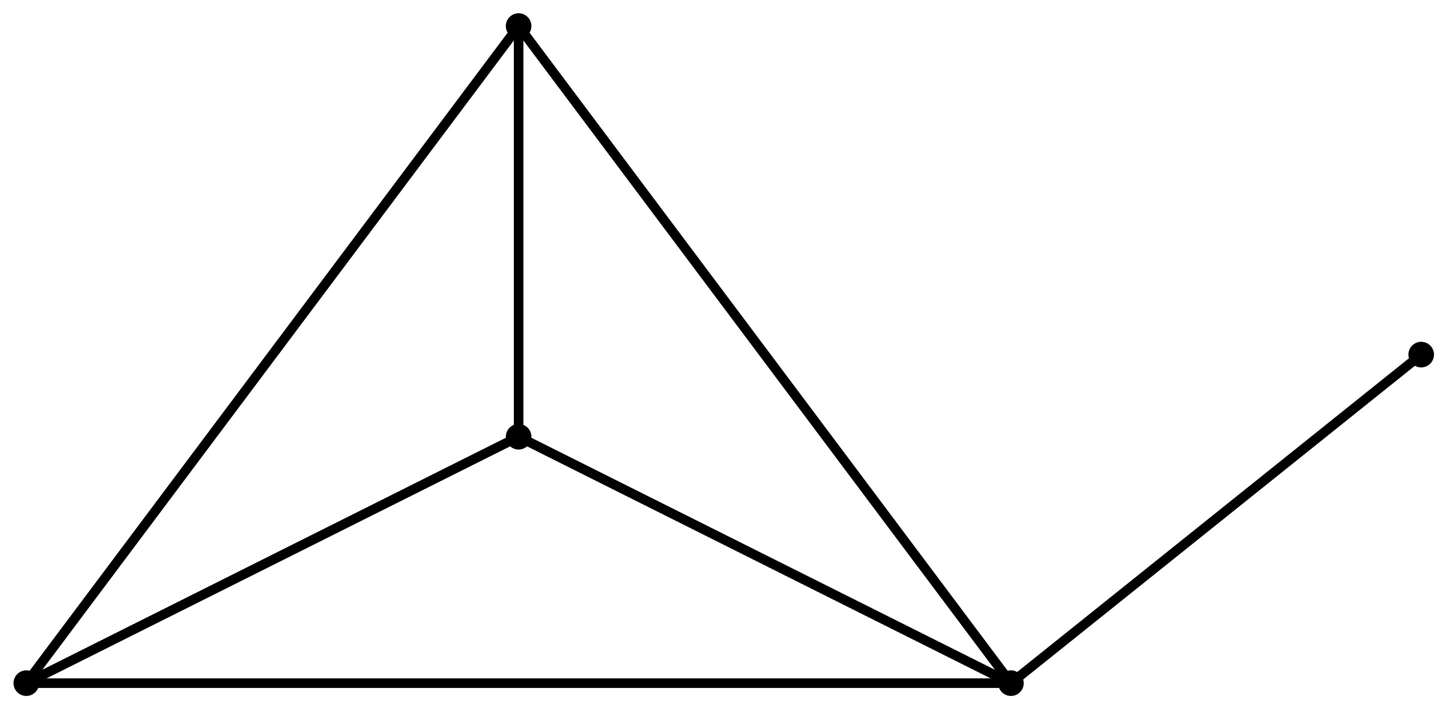
A cell may have up to $n-1$ sides

Can all cells have $n-1$ sides?

How many cells/edges/vertices may $\mathrm{Vor}(P)$ have?

Complexity

Euler's formula for plane connected graphs: $V - E + F = 2$



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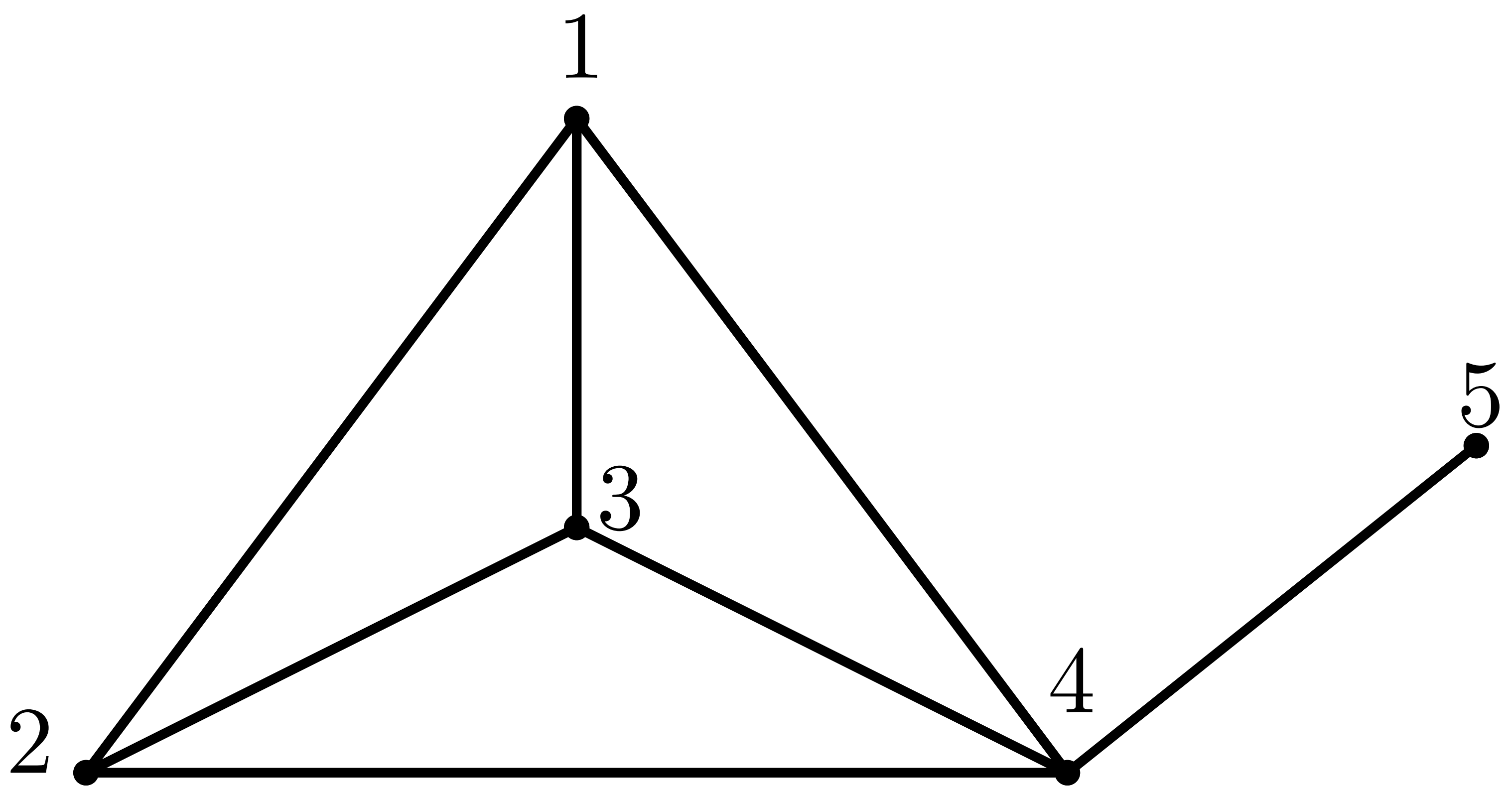
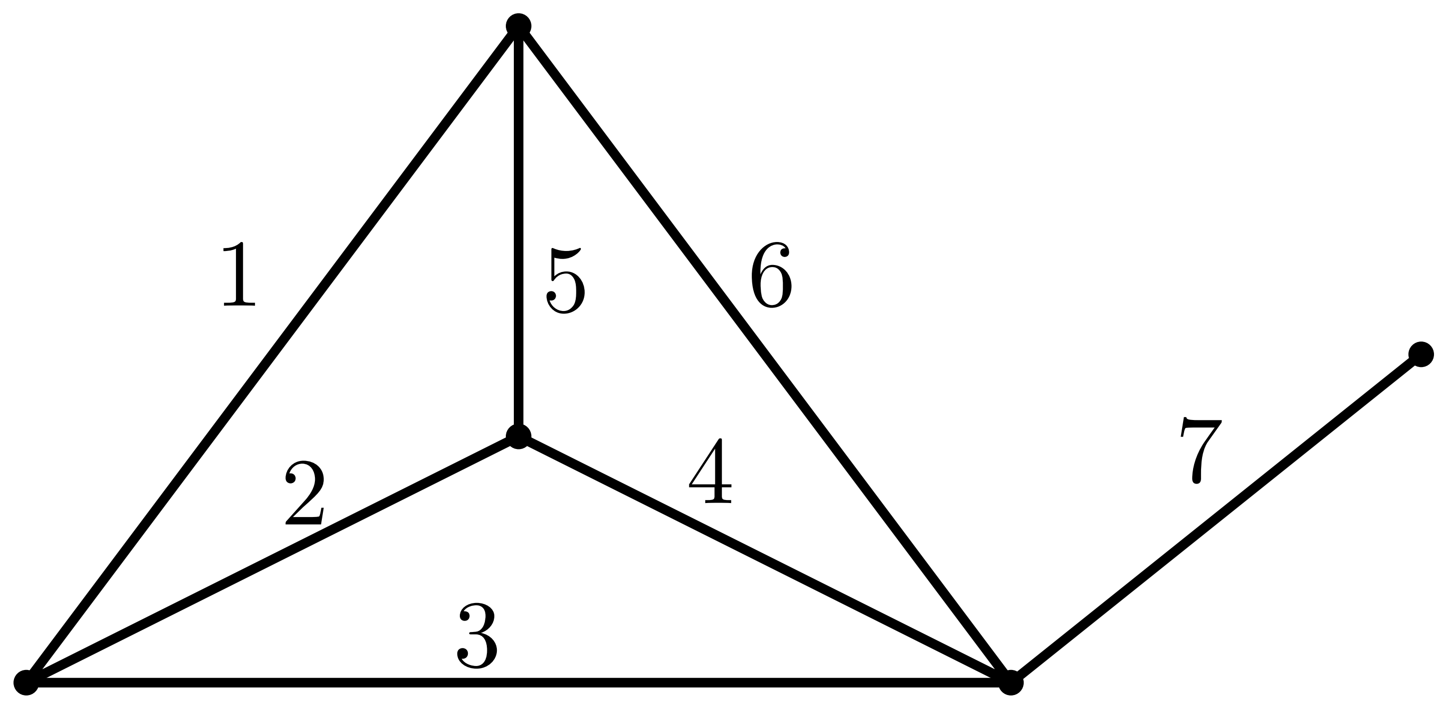


5

Complexity

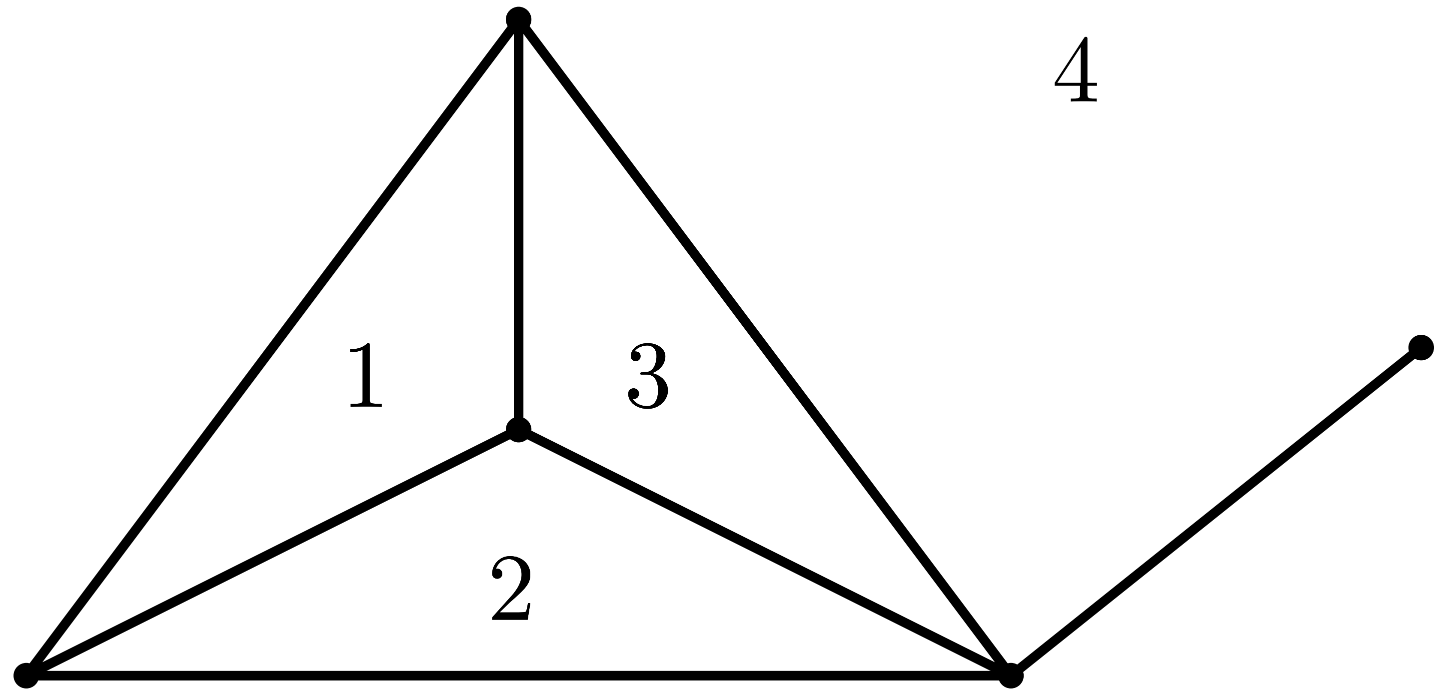
Euler's formula for plane connected graphs: $V - E + F = 2$



$5 - 7$

Complexity

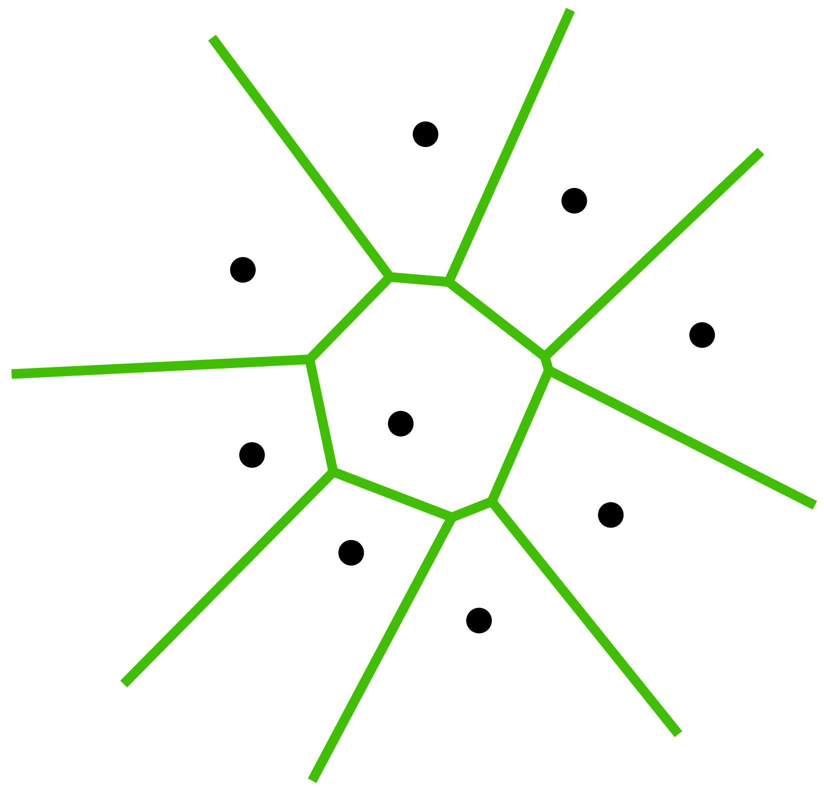
Euler's formula for plane connected graphs: $V - E + F = 2$



$$5 - 7 + 4 = 2$$

Complexity

Euler's formula for plane connected graphs: $V - E + F = 2$



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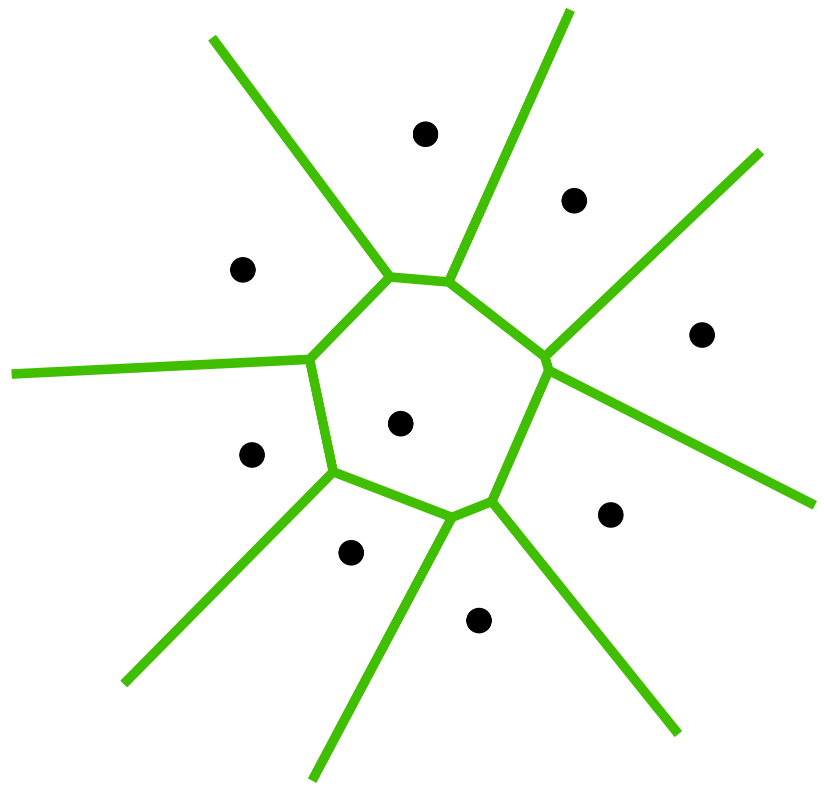
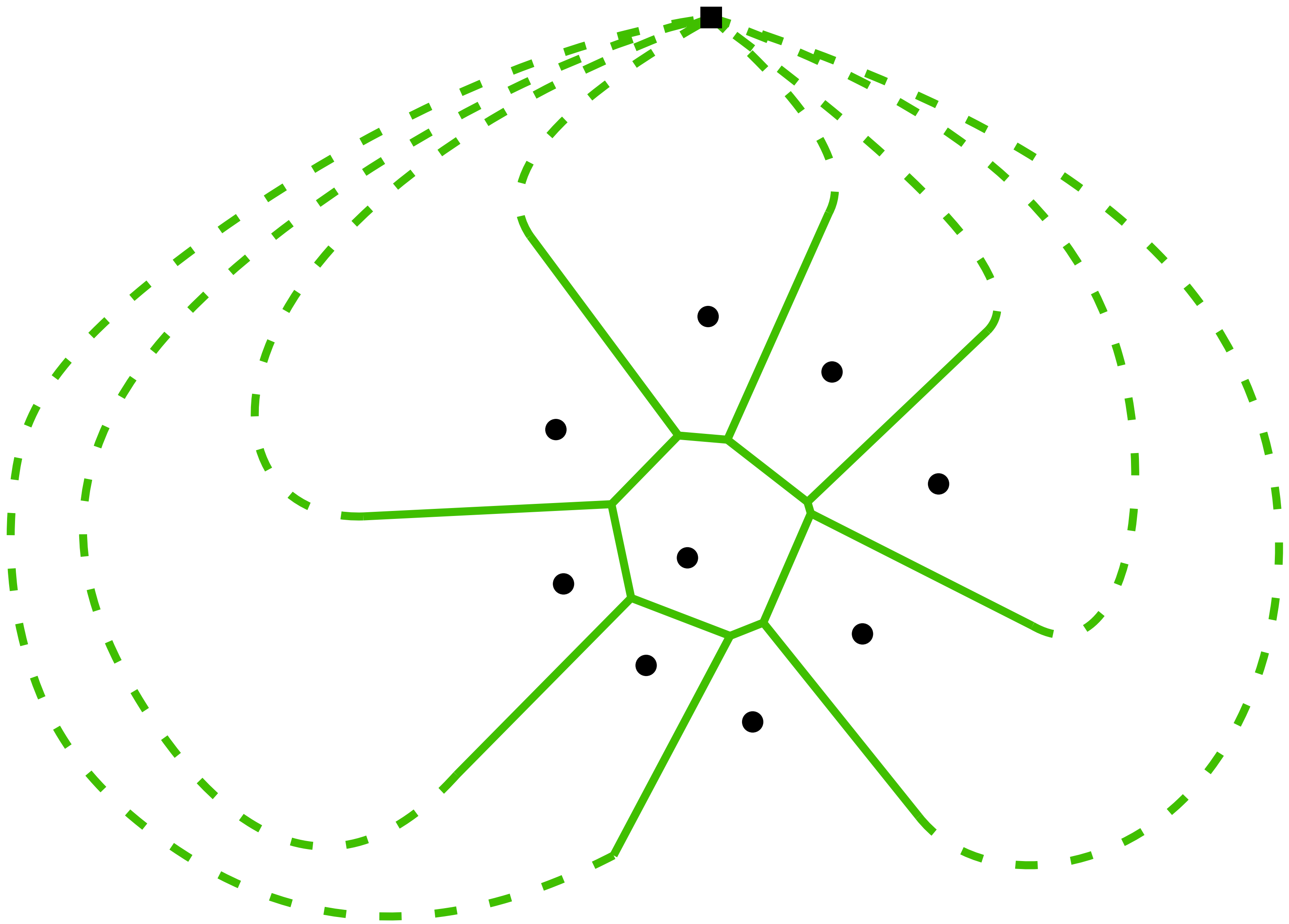


Complexity

Euler's formula for plane connected graphs: $V - E + F = 2$

$$(n_v + 1) - n_e + n = 2$$



Complexity

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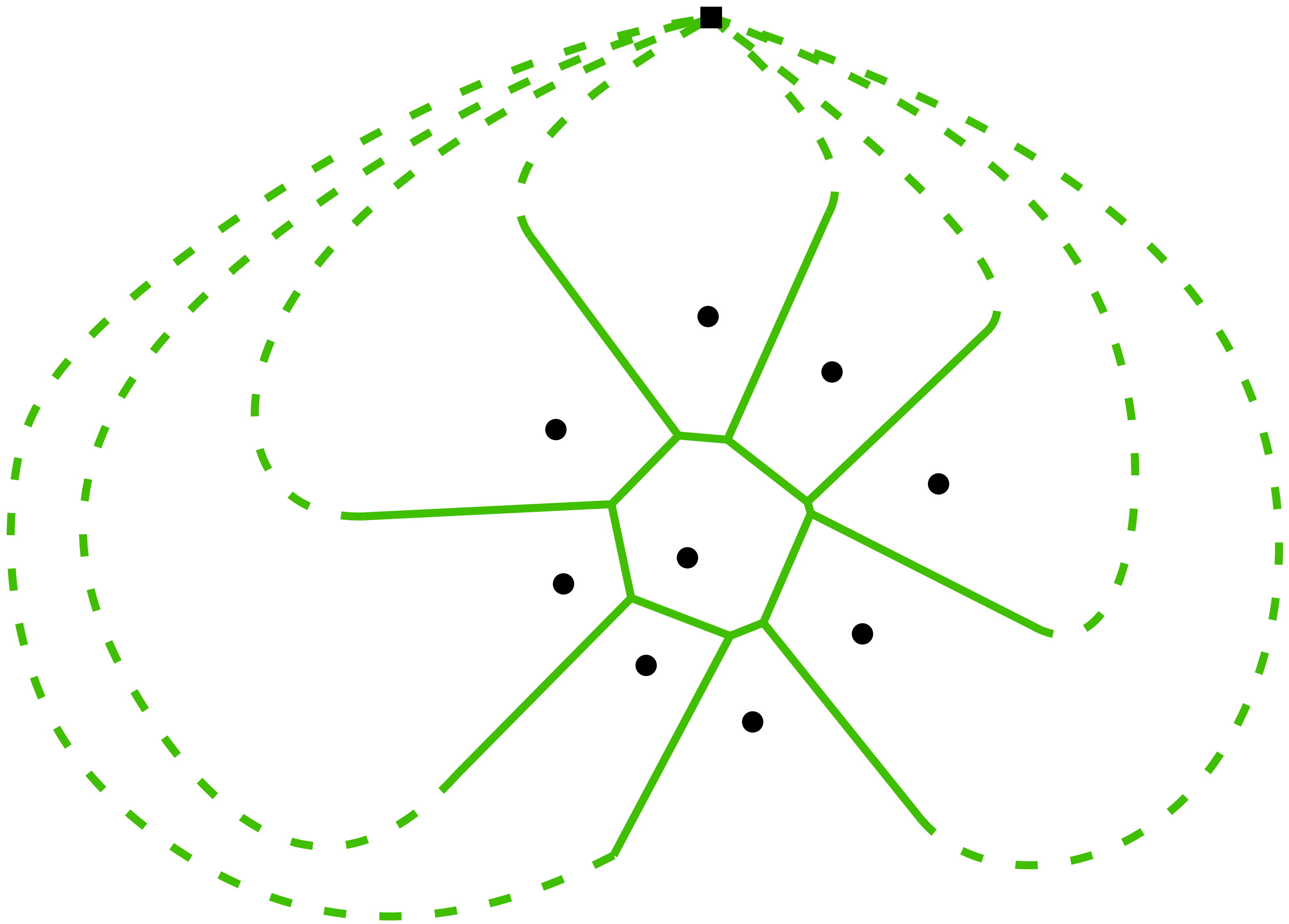
$(n_v + 1) - n_e + n = 2$

Handshaking lemma:

$2n_e = \sum_v deg(v)$



Complexity

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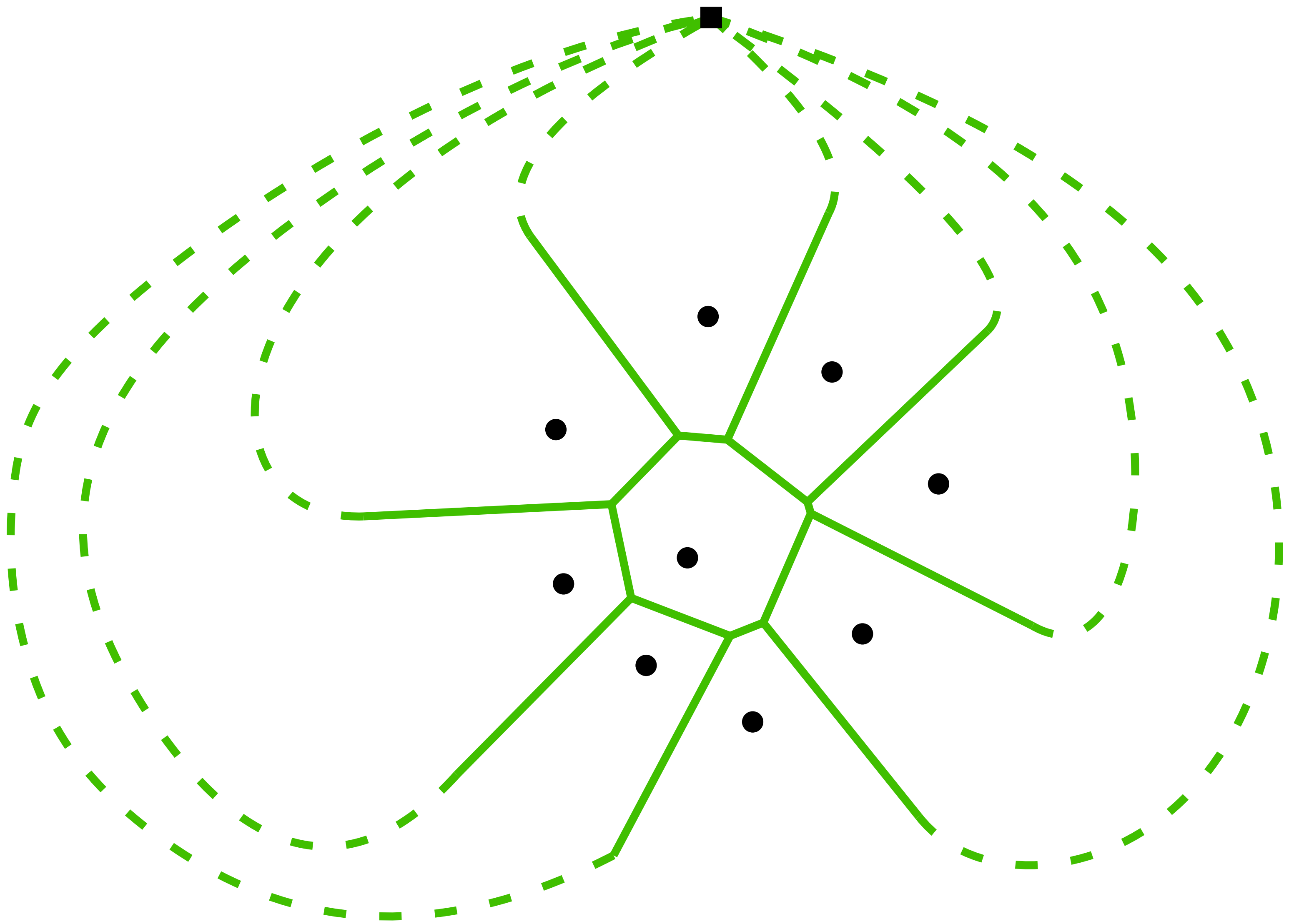
$$(n_v + 1) - n_e + n = 2$$

Handshaking lemma:

$$2n_e = \sum_v deg(v)$$
$$\geq 3(n_v + 1)$$

(degree at least 3)



Quiz

1. $(n_v + 1) - n_e + n = 2$
2. $2n_e \geq 3(n_v + 1)$

Which bounds does this imply on n_v and n_e (as tight as possible)?

A: $n_e \leq n(n+1)/2, n_v \leq n(n+1)/3$

B: $n_e \leq 3n - 6, n_v \leq 2n - 5$

C: $n_e = n_v = n$

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Plug in 1. into 2.:

$$2n_e \geq 3(2 + n_e - n)$$

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Plug in 3. into 1.:

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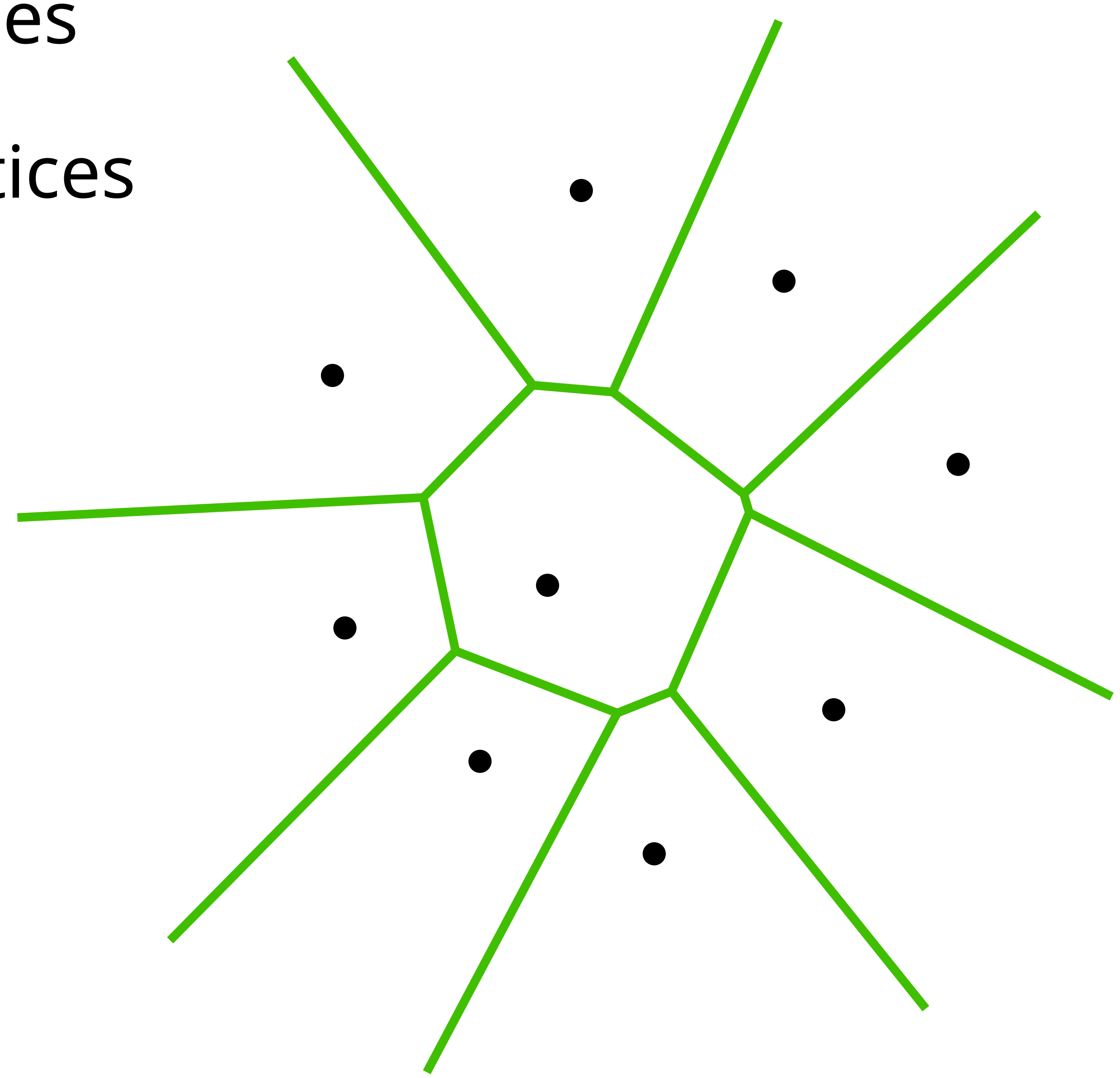
Voronoi Diagram Complexity

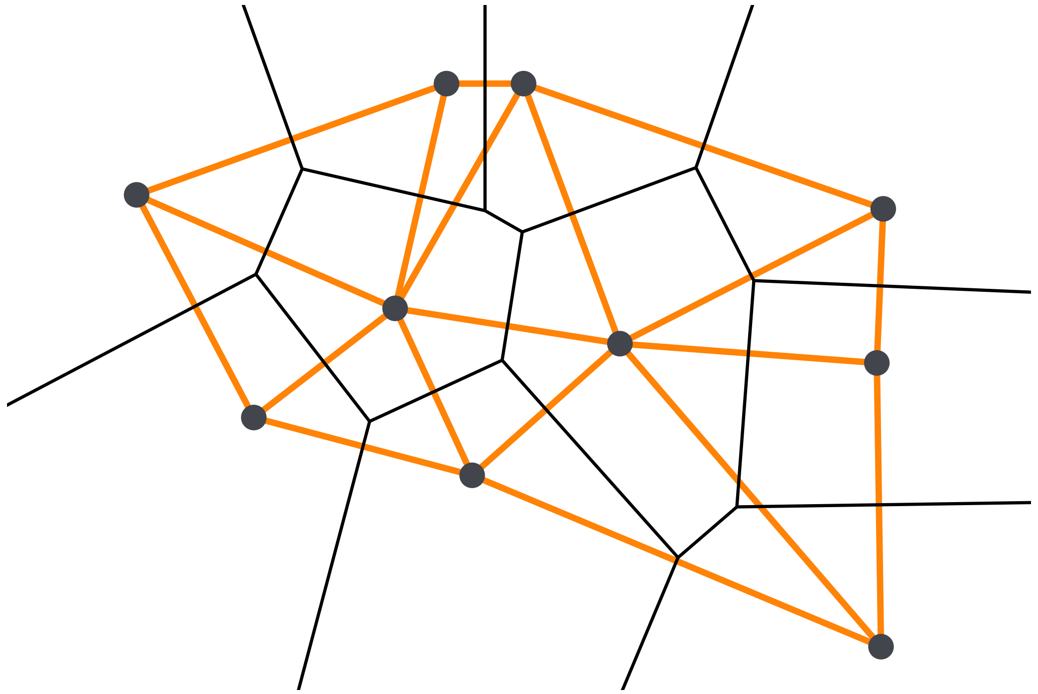
The Voronoi diagram of n points in the plane has at most

$3n - 6$ Voronoi edges

$2n - 5$ Voronoi vertices

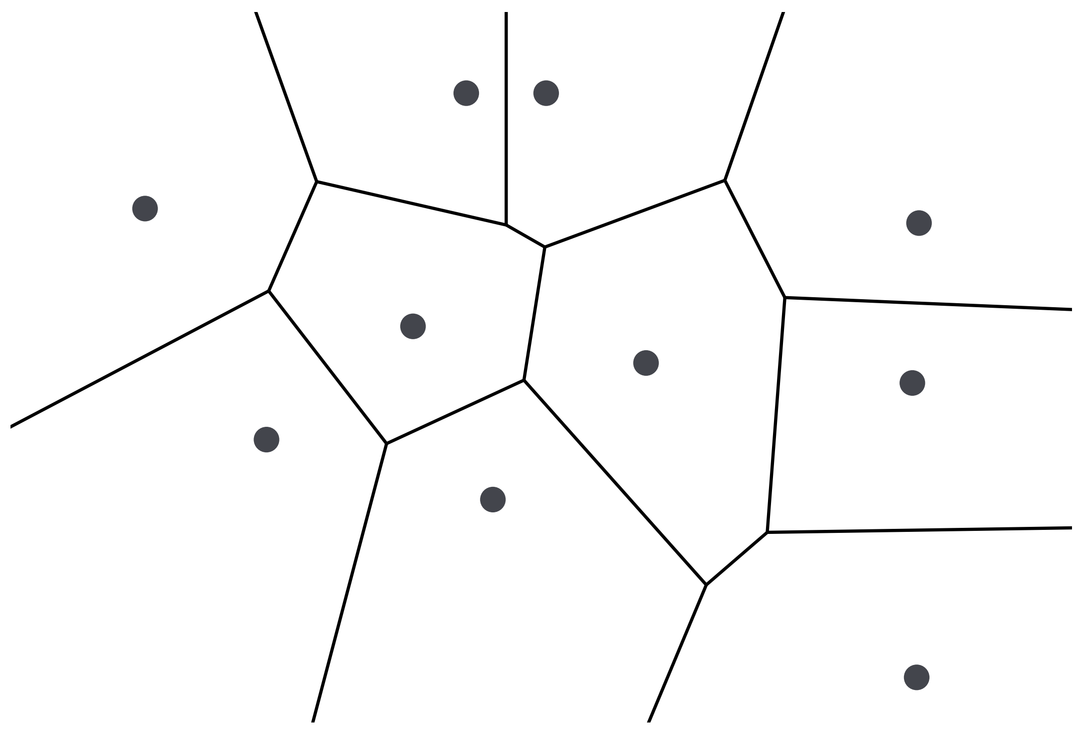


Delaunay Triangulations



Delaunay triangulation

Let $\mathrm{Vor}(P)$ be the Voronoi diagram of P.



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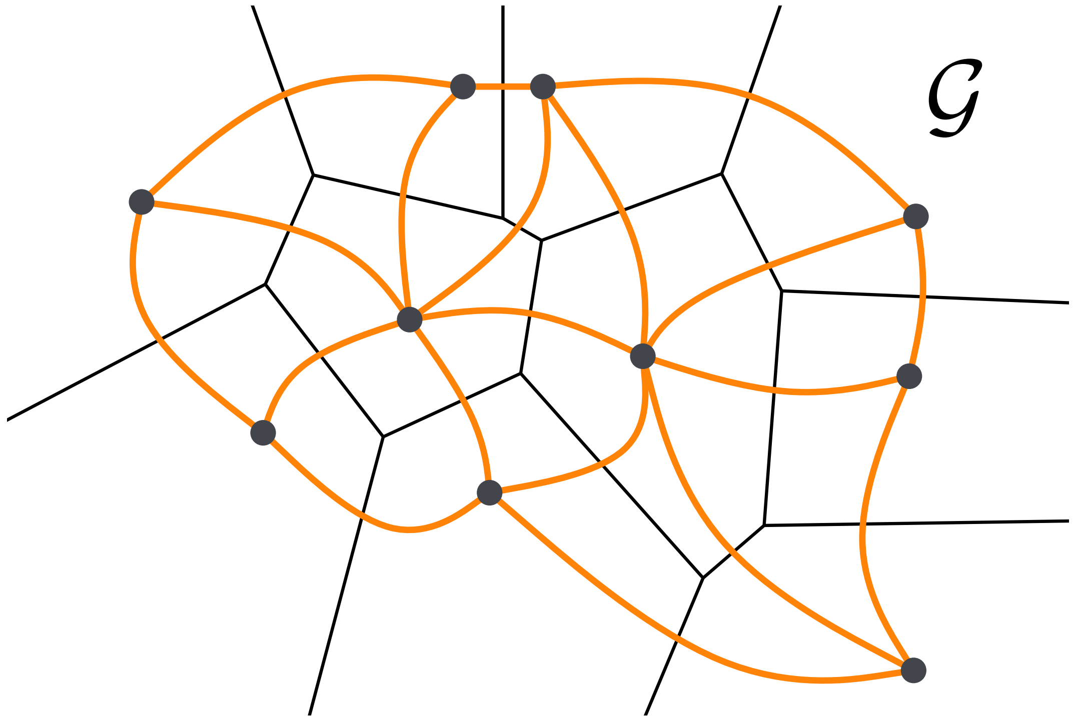
Let $\mathrm{Vor}(P)$ be the Voronoi diagram of P.

Definition: The graph $\mathcal{G} = (P, E)$ with

$$E = \{(p,q) \mid \mathcal{V}(p) \text{ and } \mathcal{V}(q) \text{ are adjacent}\}$$

is called the dual graph of $\mathrm{Vor}(P)$.



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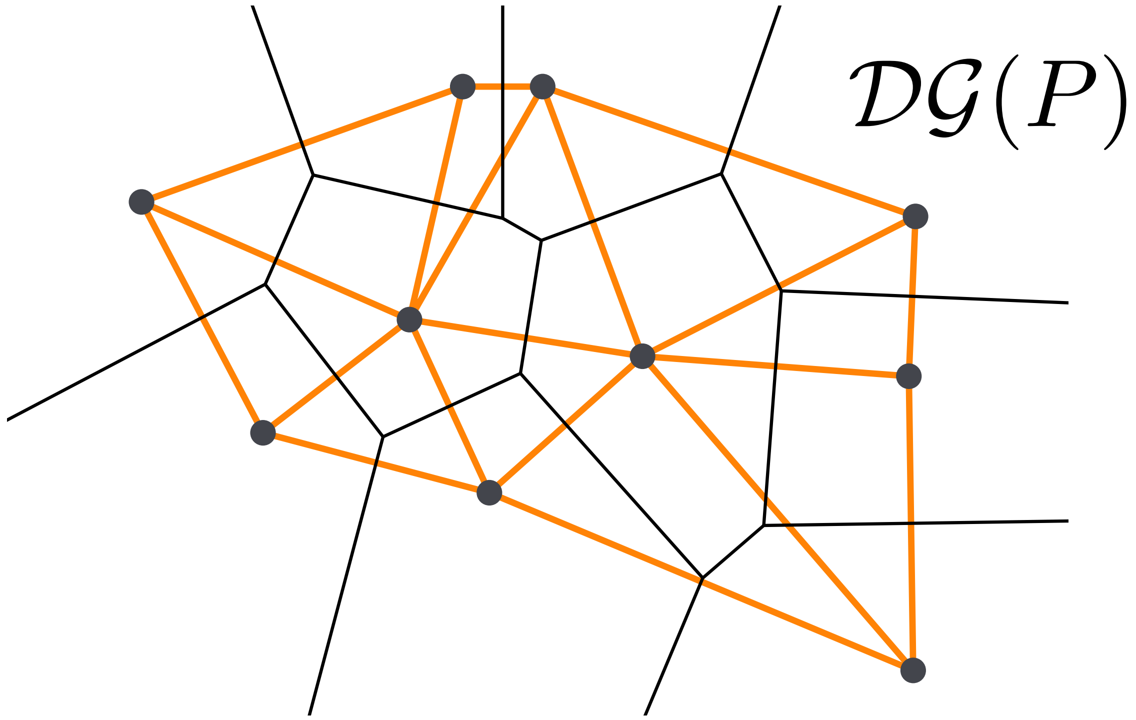
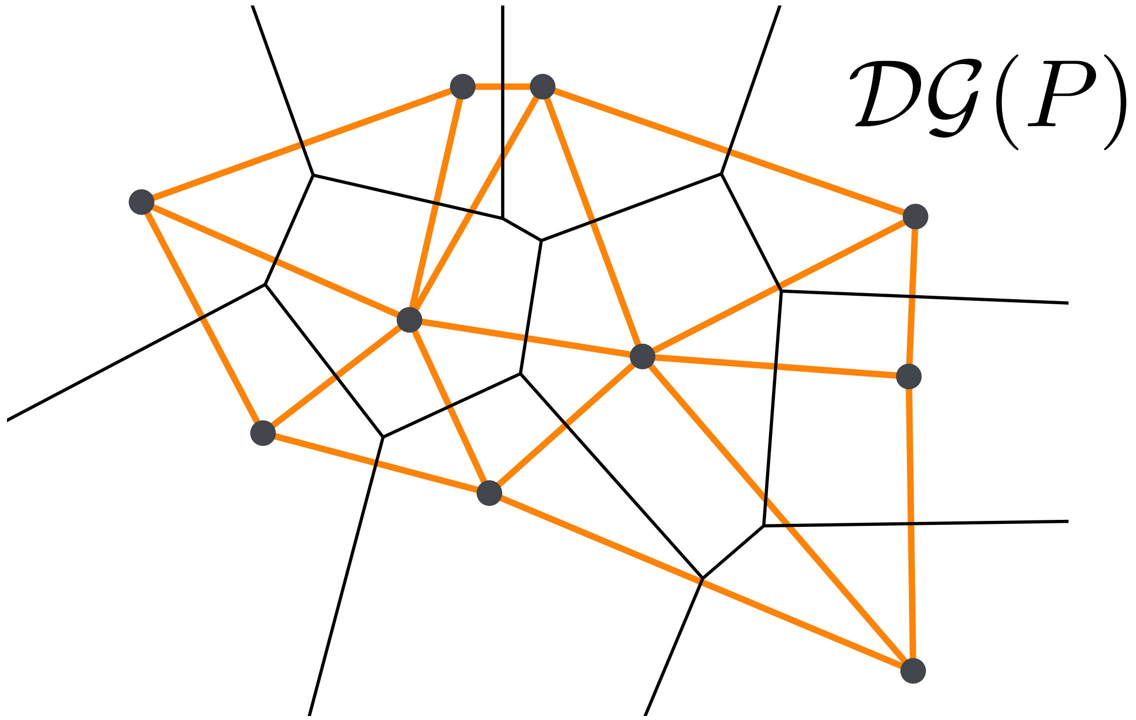
Definition: The straight-line drawing of $\mathcal{G}$ is called Delaunay graph $\mathcal{DG}(P)$.



Properties

$\mathcal{DG}(P)$ has no crossing edges.



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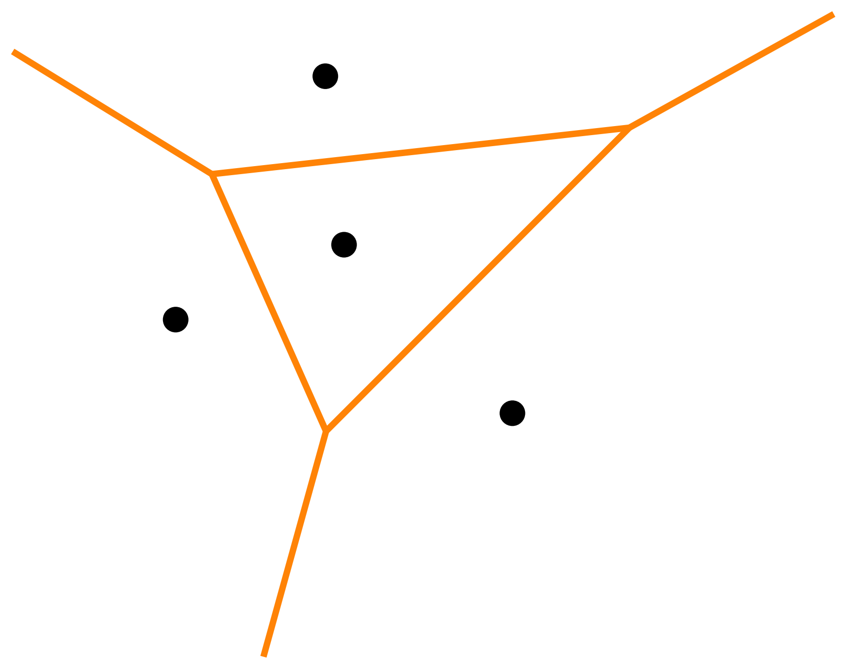

Proof by contradiction. Suppose there is an intersection. Then it is determined by 4 points.

Properties

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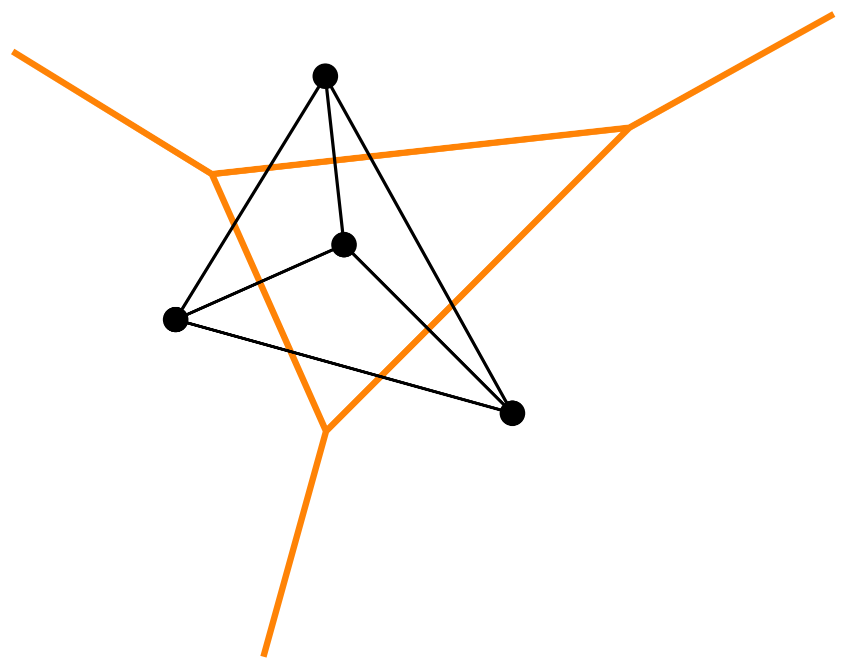

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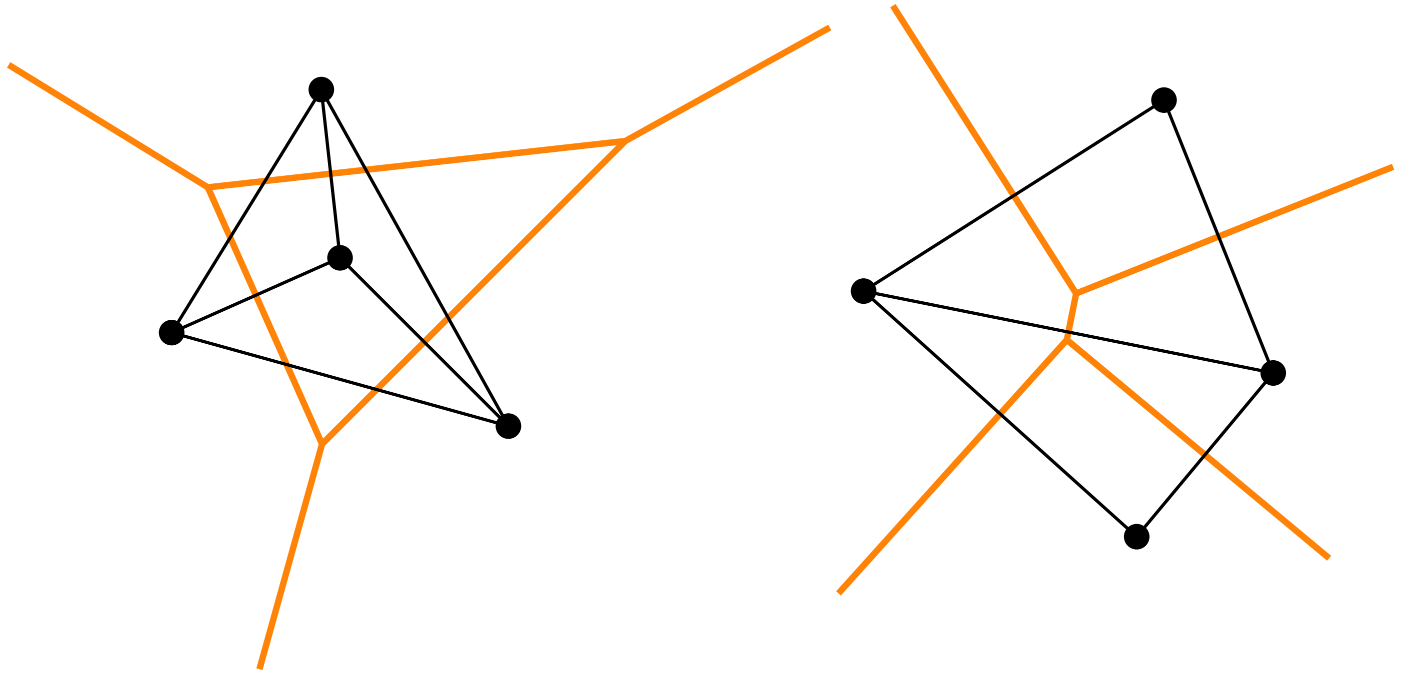

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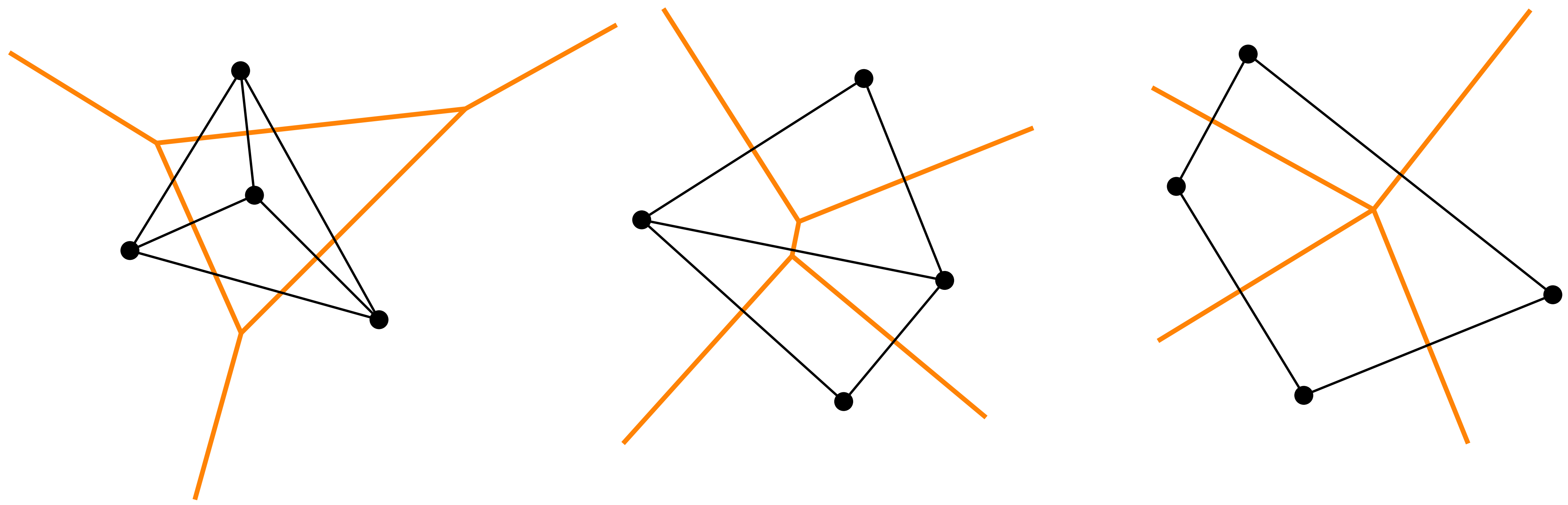

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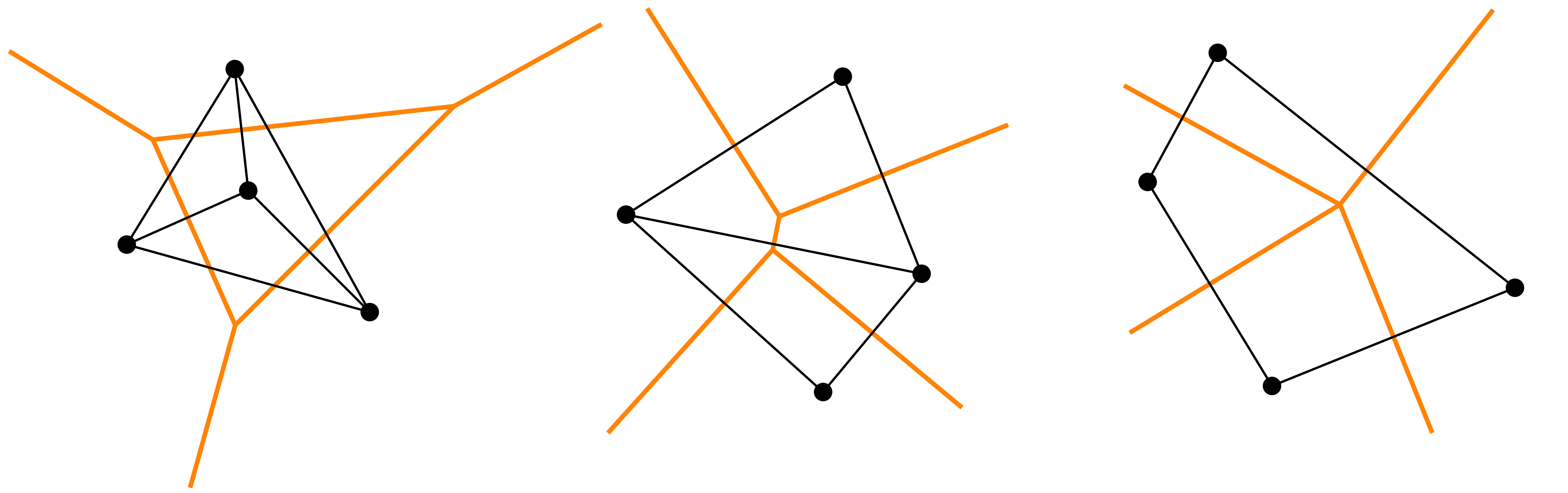
Delaunay triangulation: add edges until all faces are triangles



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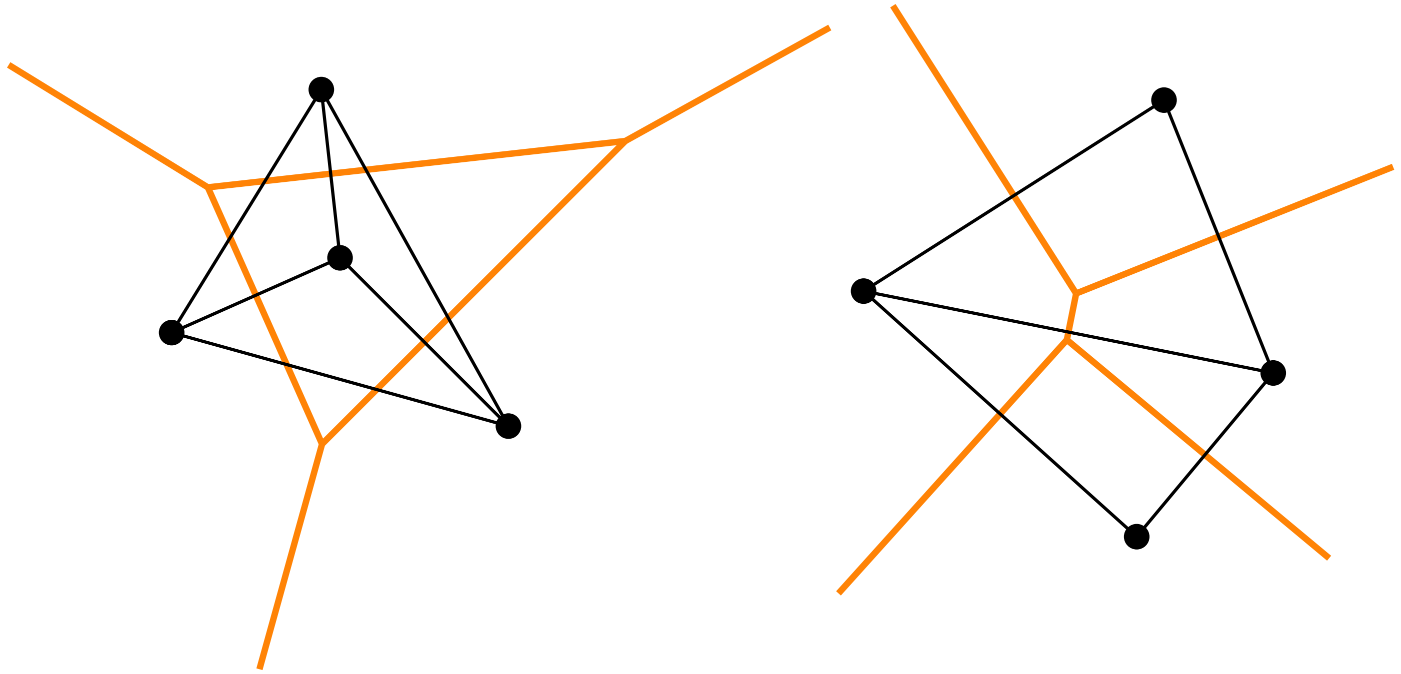
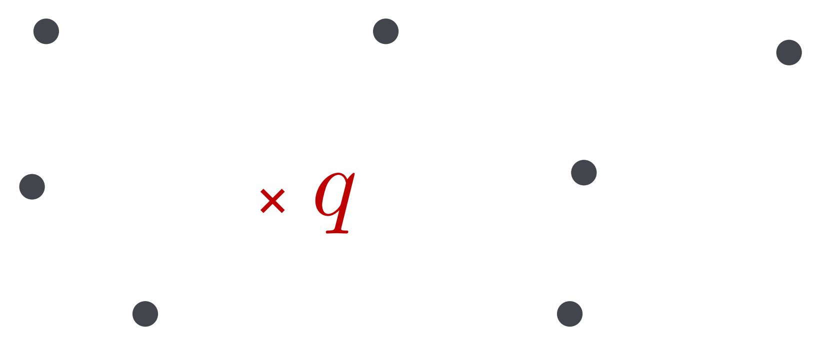


Characterization of Voronoi vertices/edges

Definition: Let q be a point. Define $C_P(q)$ as the largest disk with center q containing no points of P in its interior.



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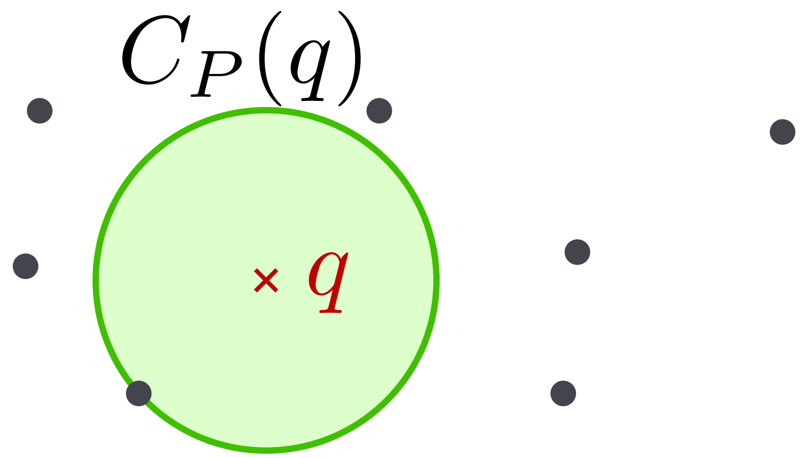

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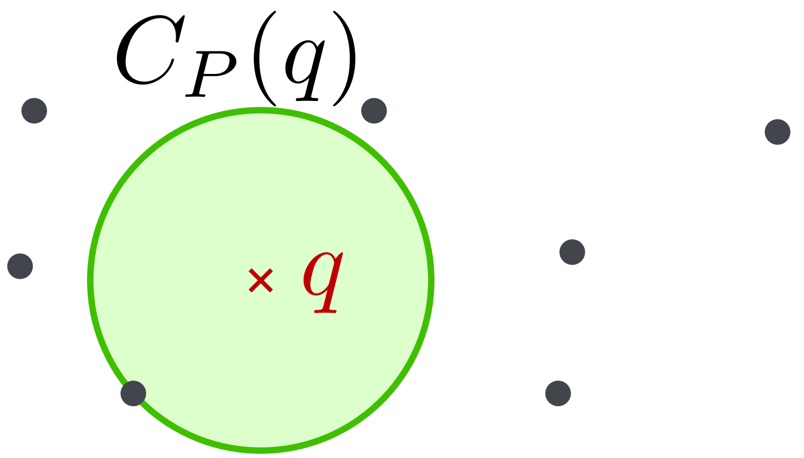
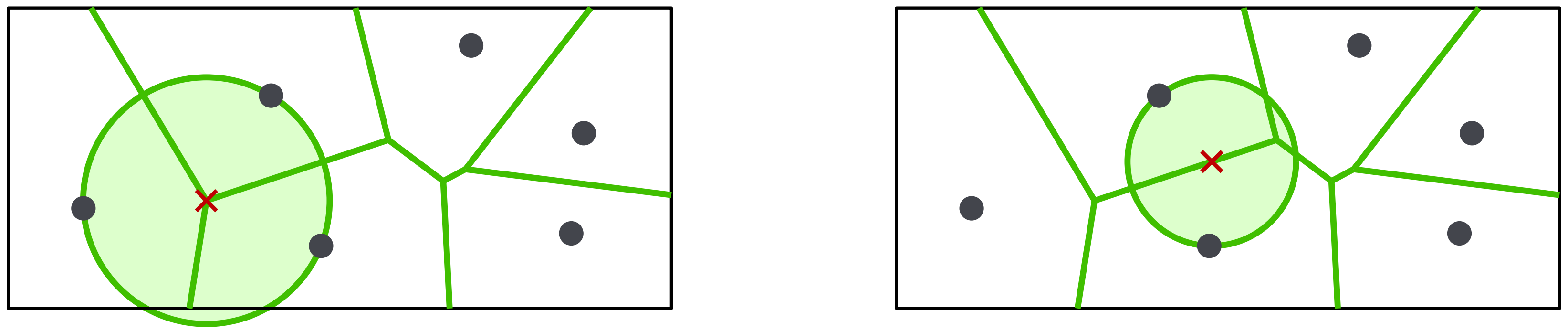


Observation:

- A point q is a Voronoi vertex $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
- The bisector $b(p_i, p_j)$ defines a Voronoi edge
  $\Leftrightarrow \exists q \in b(p_i, p_j)$ with $C_P(q) \cap P = \{p_i, p_j\}$.



Empty-circle property

Theorem about Voronoi diagrams:

- a point q is a Voronoi vertex
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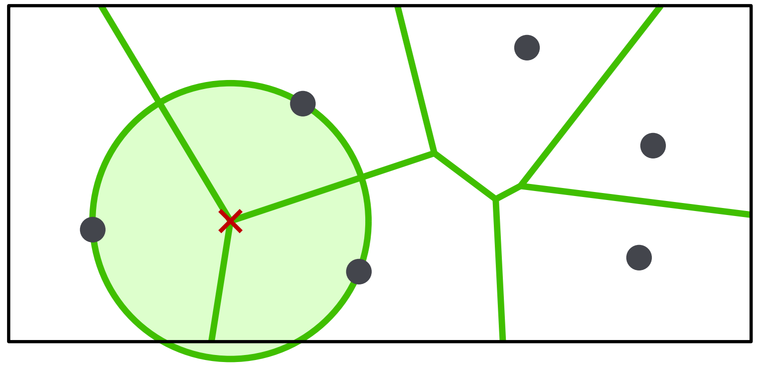

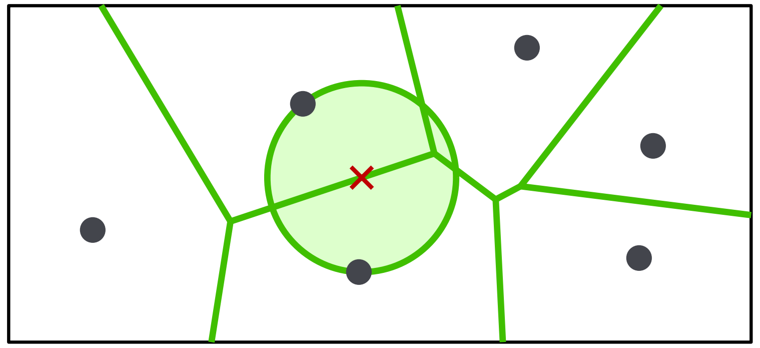

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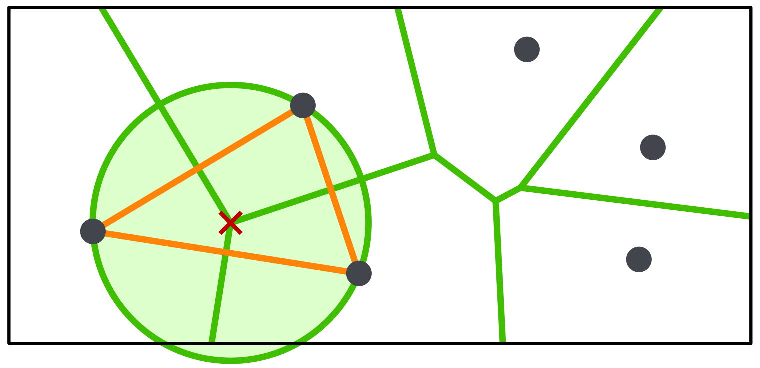

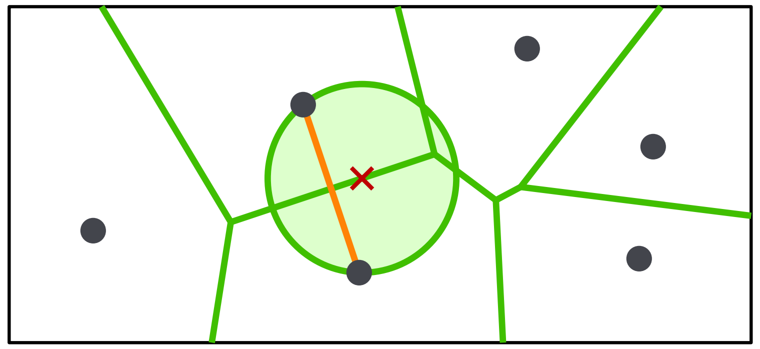

Theorem: Let P be a set of points.

- points p, q, r are vertices of the same face
  in $\mathcal{DG}(P) \Leftrightarrow$ circle through p, q, r is empty,
- edge pq is in $\mathcal{DG}(P)$
  $\Leftrightarrow$ there is an empty circle $C_{p,q}$ through p and q.

Empty-circle property

Theorem about Voronoi diagrams:

- a point q is a Voronoi vertex
  $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
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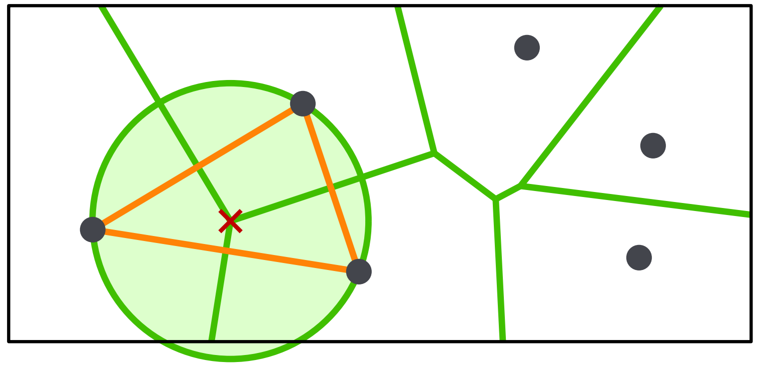

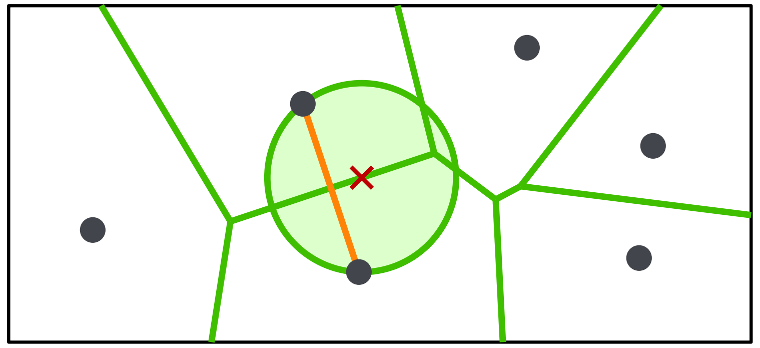

Theorem: Let P be a set of points.

- points p, q, r are vertices of the same face in $\mathcal{DG}(P) \Leftrightarrow$ circle through p, q, r is empty,
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Corollary: Let P be a set of points and $\mathcal{T}$ a triangulation of P. $\mathcal{T}$ is a Delaunay triangulation $\Leftrightarrow$ circumcircle of every triangle is empty.

Quiz

How many triangles does a (Delaunay) triangulation of n points contain at most?

A: $2n/3 - 1$

B: $2n - 5$

C: $3n - 6$

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How many triangles does a (Delaunay) triangulation of n points contain at most?

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Quiz

How many triangles does a (Delaunay) triangulation of n points contain at most?

A: $2n/3 - 1$

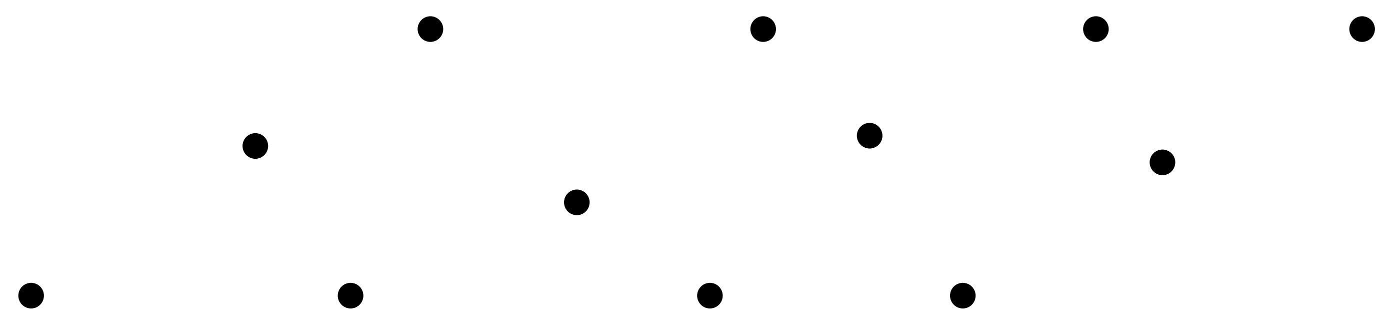
$\boxed{\text{B:} \quad 2n - 5}$ = no. Voronoi vertices (or use Euler's formula)

C: $3n - 6$

Angle-optimal Triangulations

Triangulation of a point set

Definition: A triangulation of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with vertex set P.



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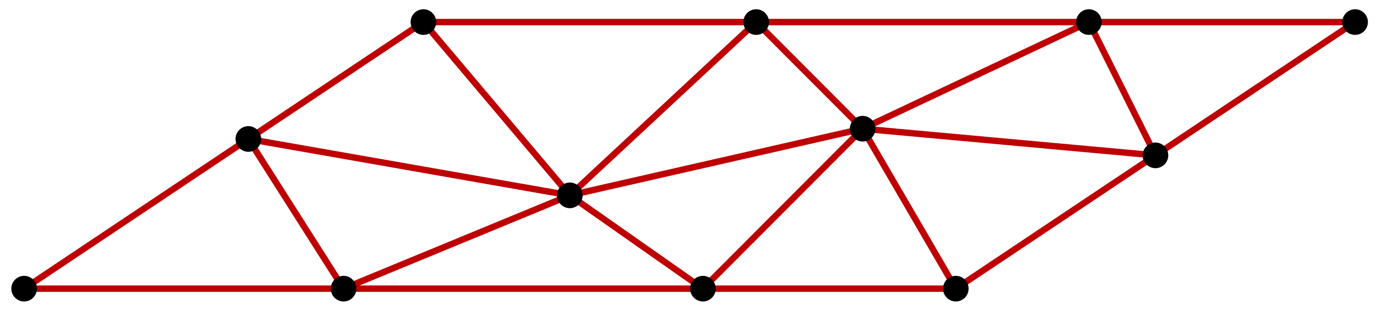

Observations:

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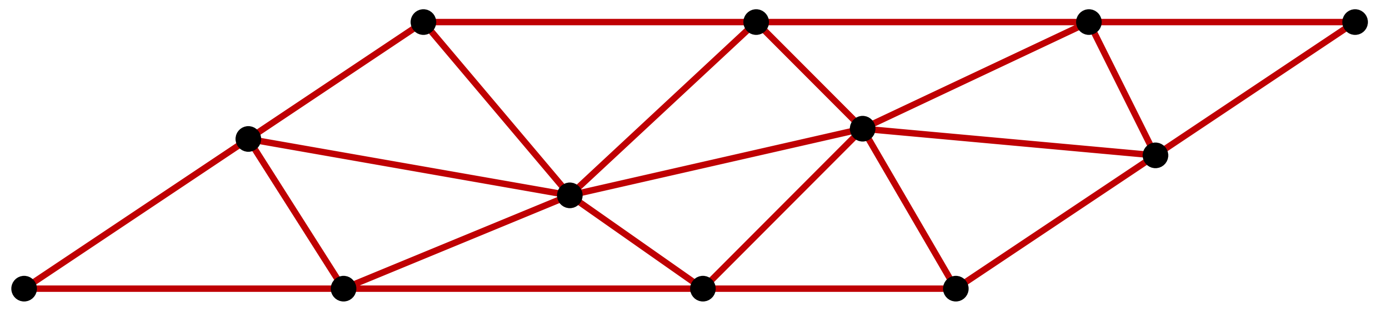

Observations:

- all inner faces are triangles

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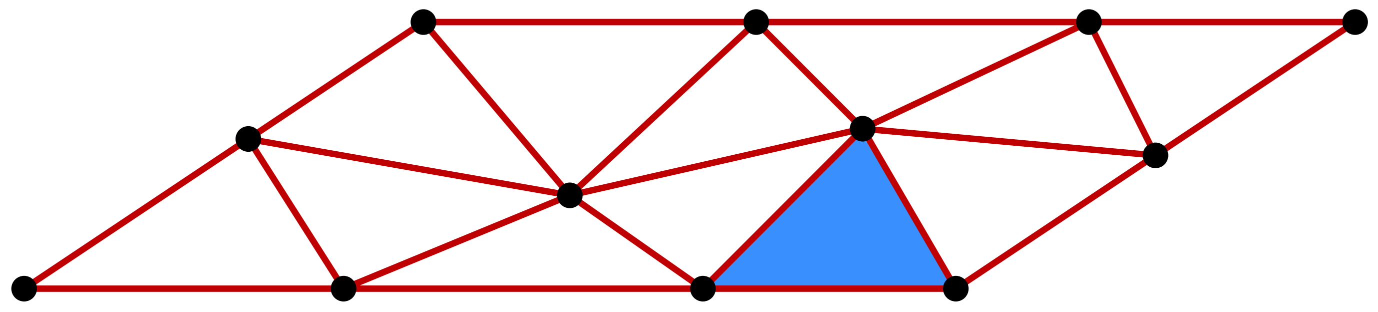

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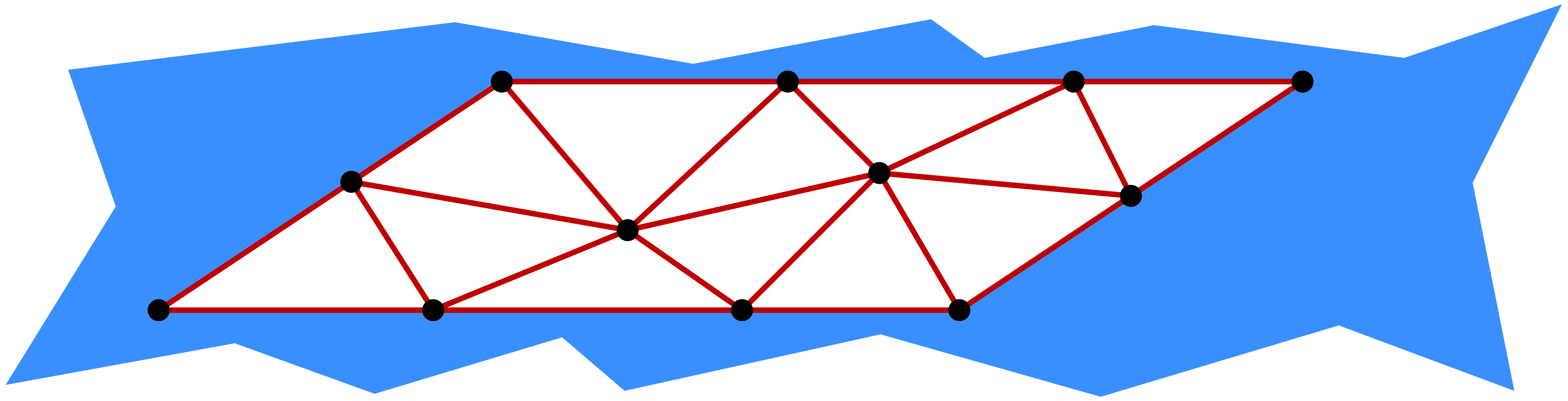

Observations:

- all inner faces are triangles
- outer face is complement of convex hull

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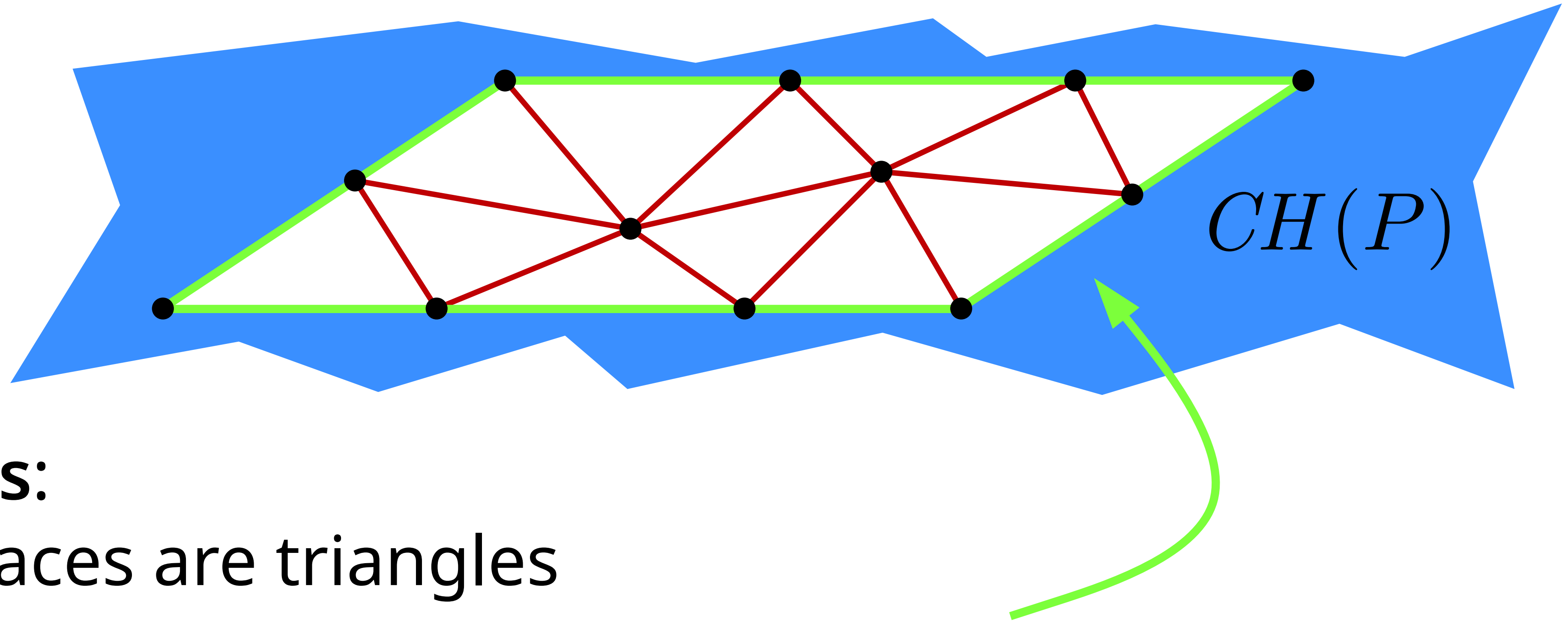

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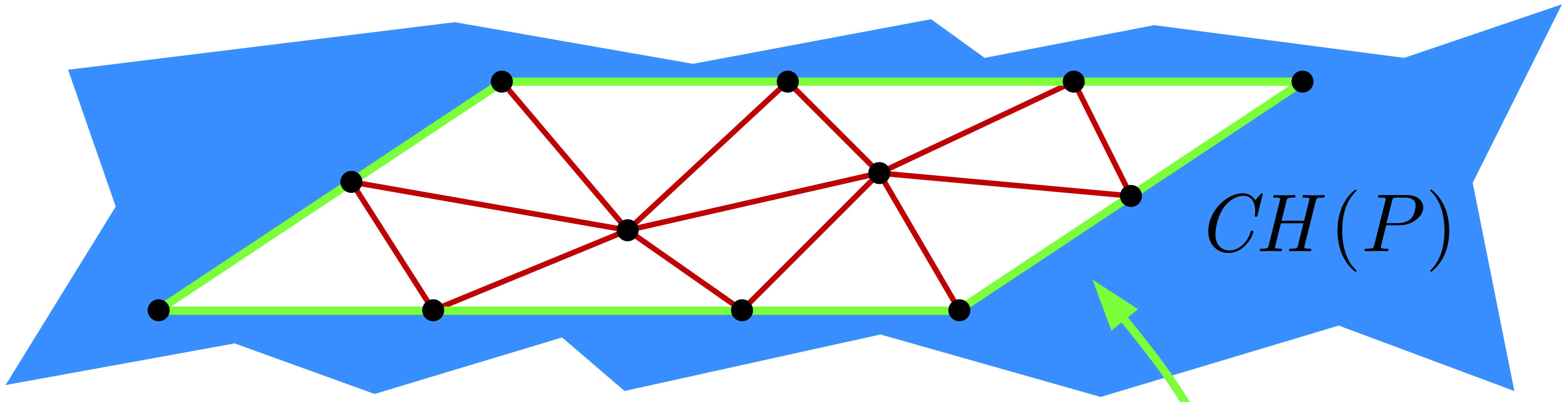

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Theorem 1: Let P be a set of n non-collinear points and let h be the number of vertices of $CH(P)$. Then every triangulation of P has $t(n,h)$ triangles and $e(n,h)$ edges.

Triangulation of a point set

Definition: A triangulation of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with vertex set P.

Compute $t(n, h)$ and $e(n, h)$!

Observations:
- all inner faces are triangles
- outer face is complement of convex hull

Theorem 1: Let P be a set of n non-collinear points and let h be the number of vertices of $CH(P)$. Then every triangulation of P has $t(n, h)$ triangles and $e(n, h)$ edges.

Triangulation of a point set

Definition: A triangulation of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with vertex set P.

Compute $t(n, h)$ and $e(n, h)$!

Euler's formula for connected plane graphs:

$\#$ faces $-$ $\#$ edges + $\#$ vertices $=$ 2,

also counting the outer face.

Observations:

- all inner faces are triangles
- outer face is complement of convex hull

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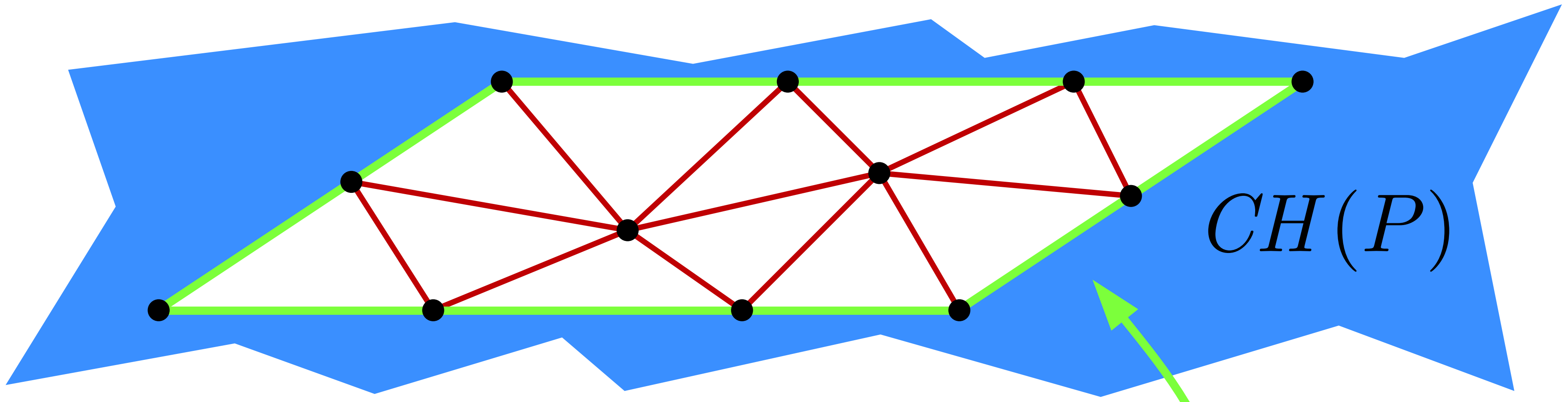

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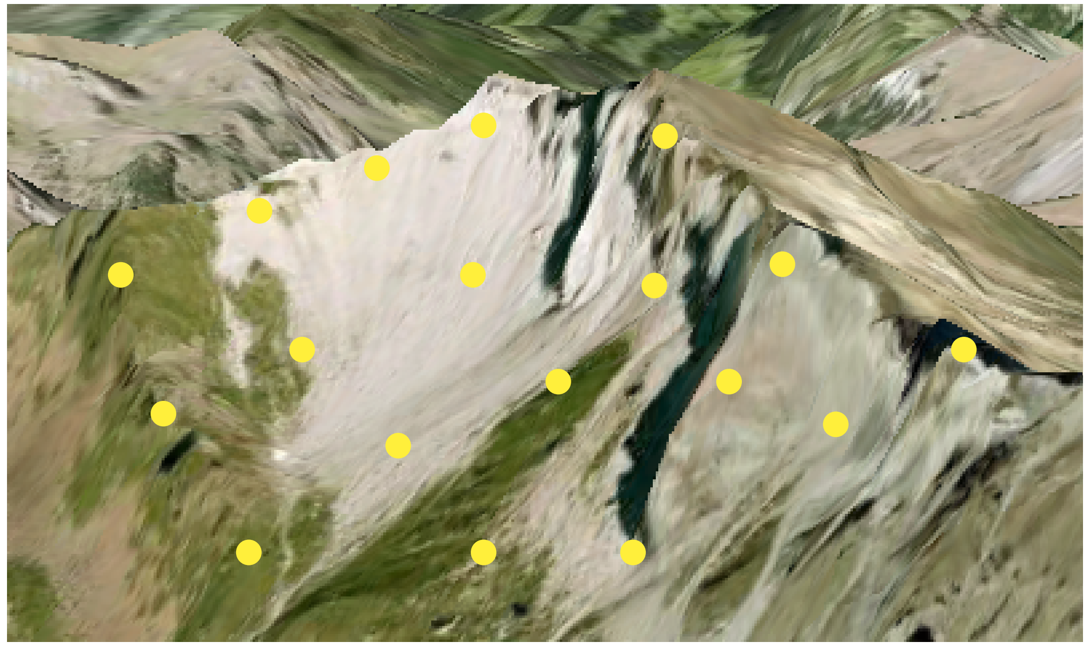
Theorem 1: Let P be a set of n non-collinear points and let h be the number of vertices of $CH(P)$. Then every triangulation of P has $(2n - 2 - h)$ triangles and $(3n - 3 - h)$ edges.

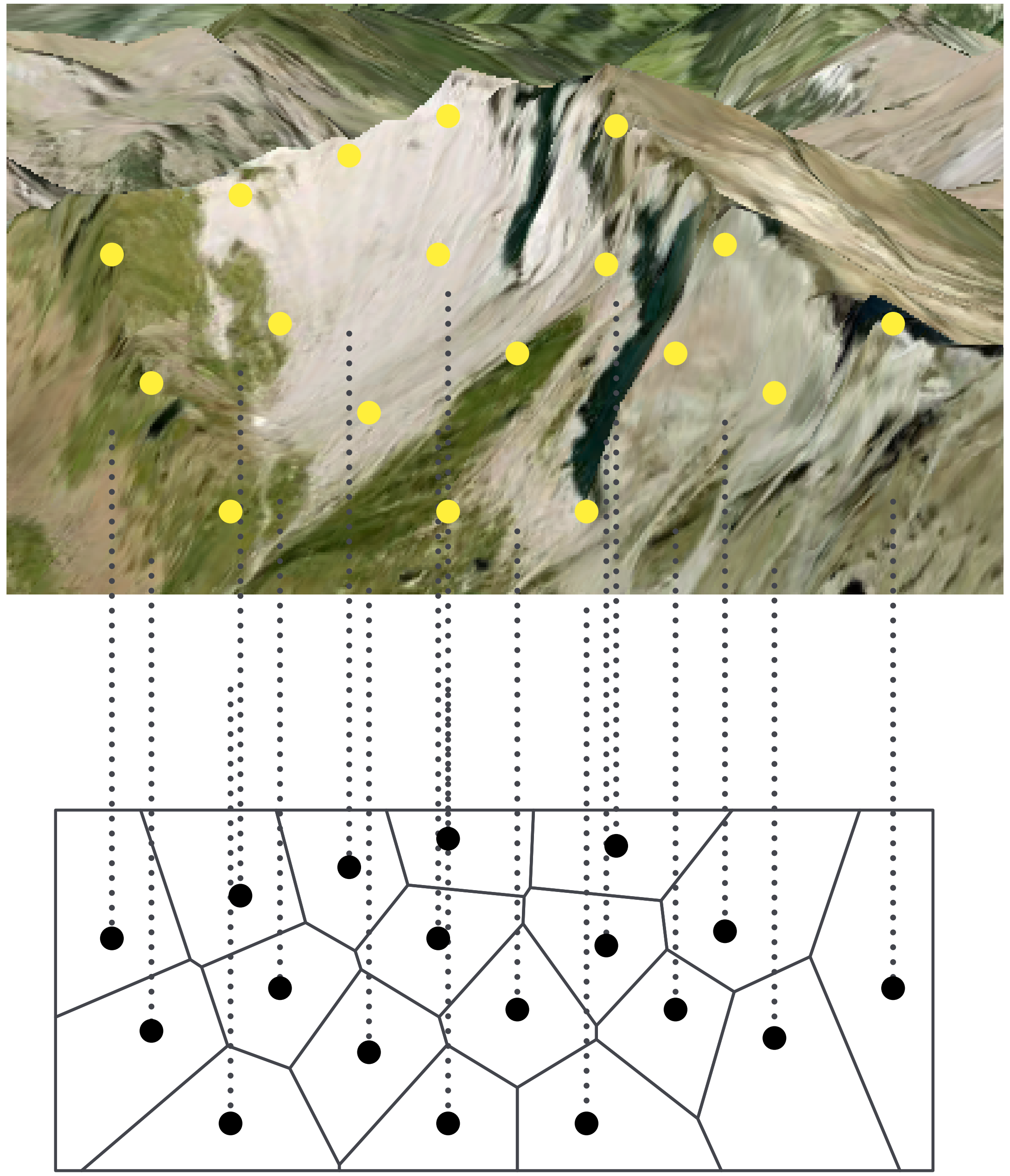
Motivation revisited



height measurements

$p = (p_x, p_y, p_z)$

Motivation revisited



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$p = (p_x, p_y, p_z)$

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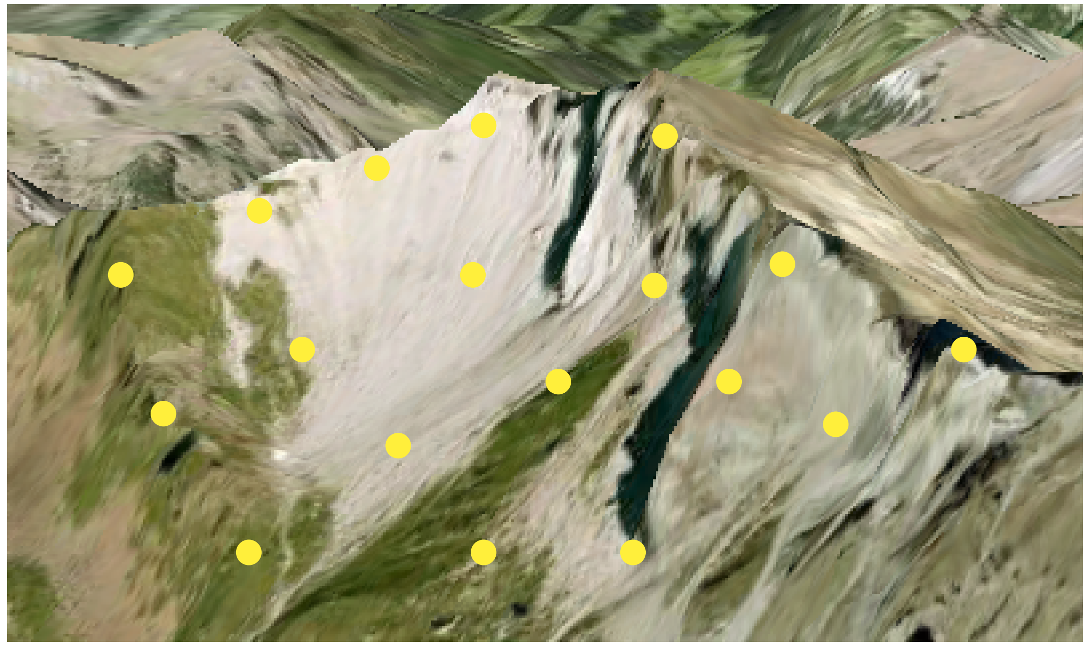
projection $\pi(p) = (p_x, p_y, 0)$

Interpolation 1: assign height of nearest neighbor

$\rightarrow$ Voronoi diagrams

Motivation revisited



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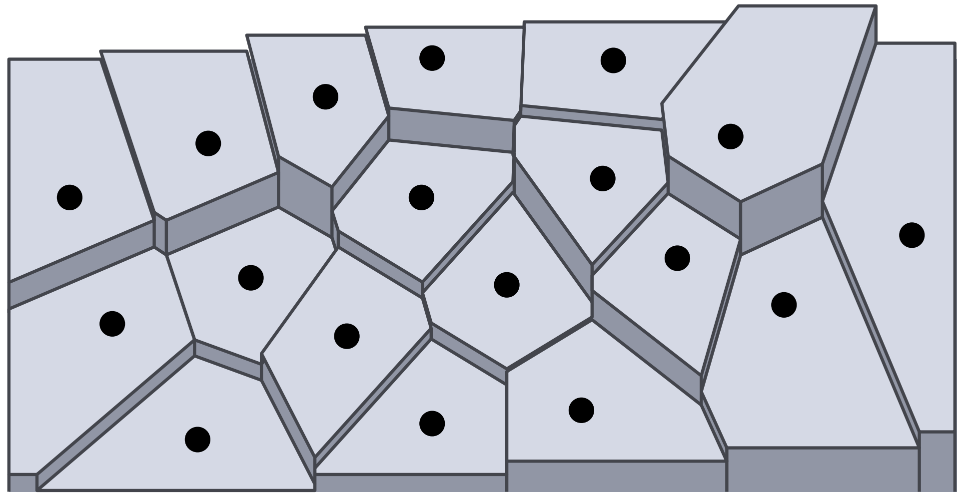

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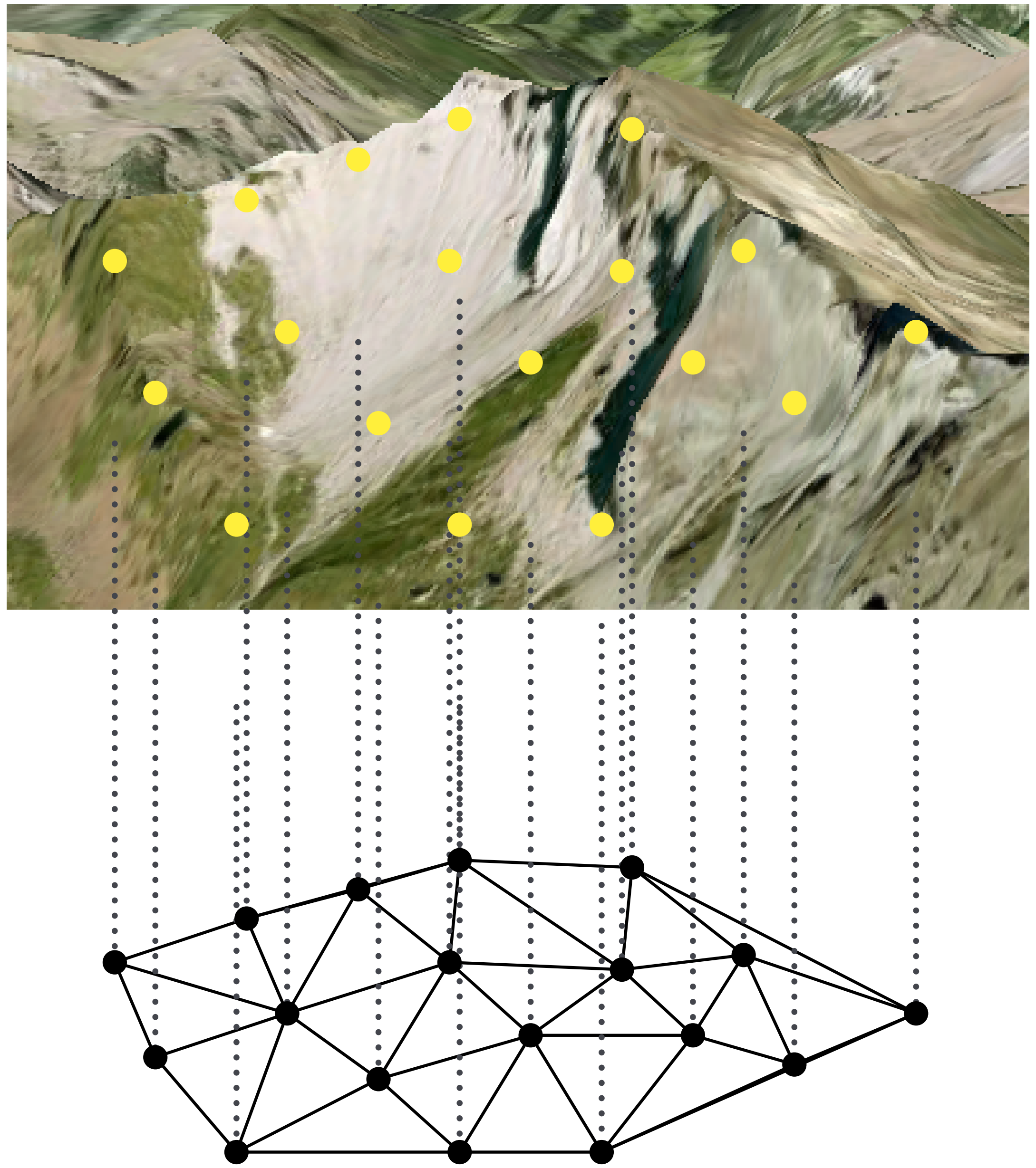

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Interpolation 2: triangulate & interpolate within triangles

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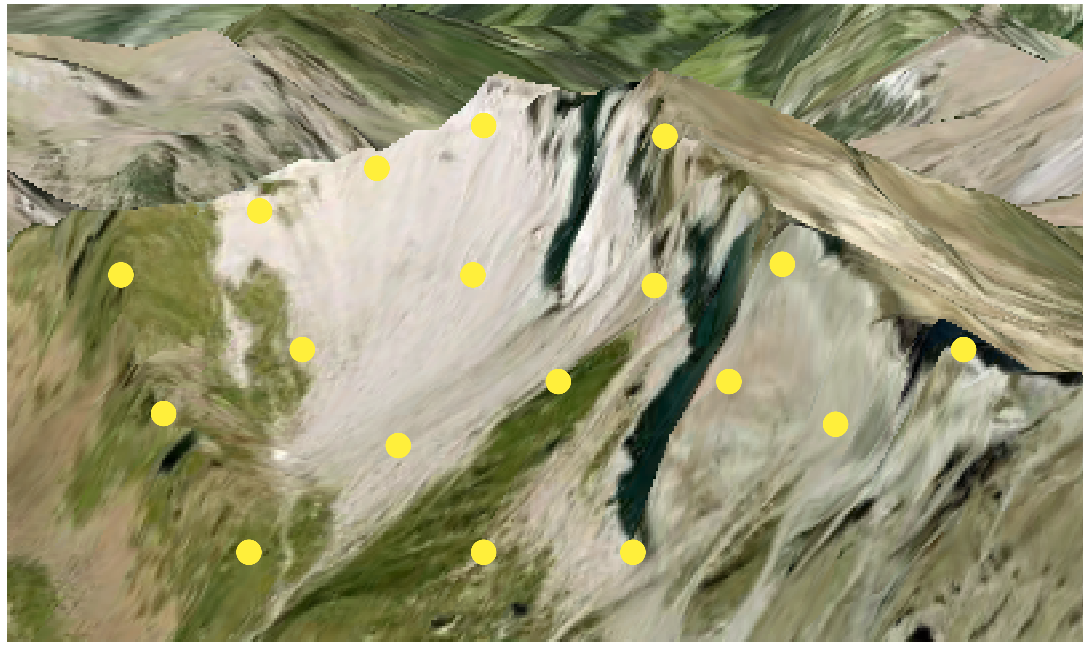

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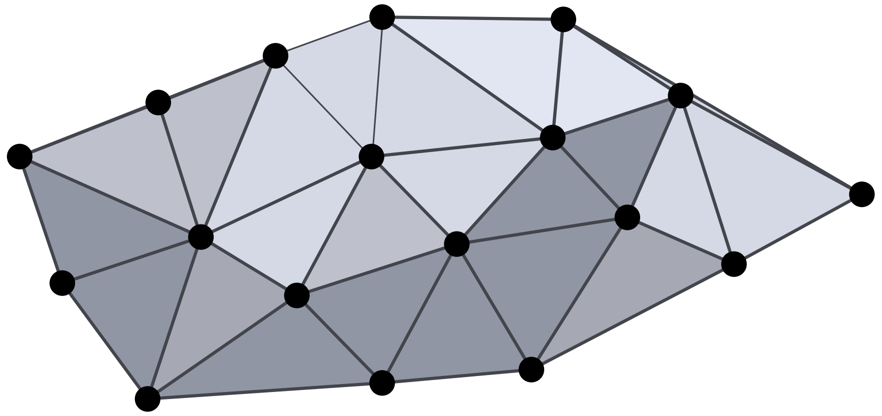

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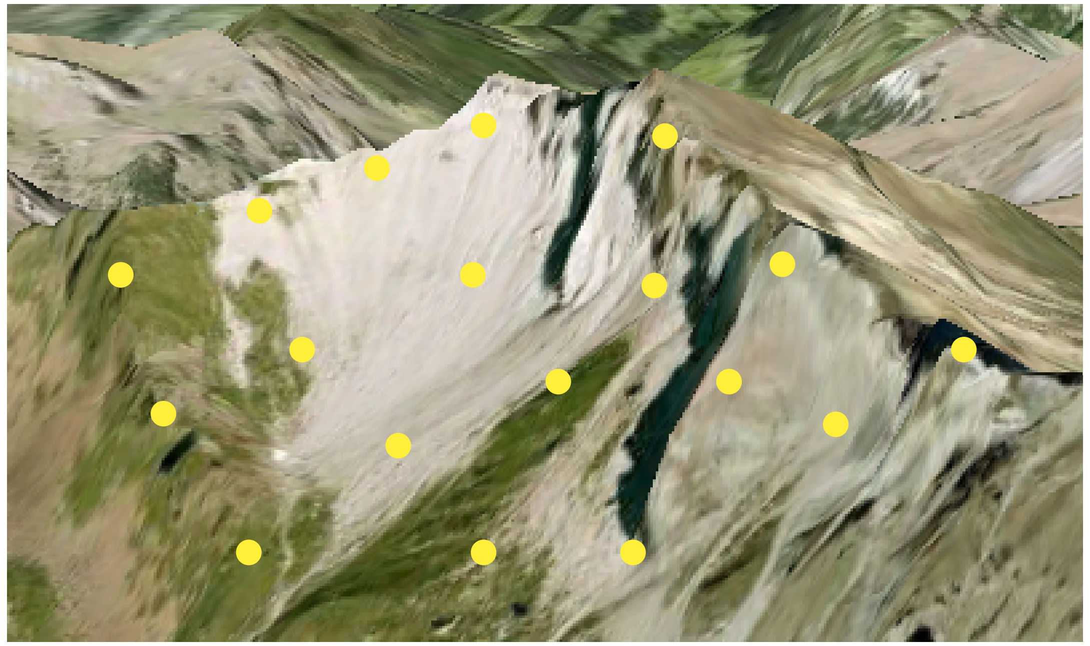

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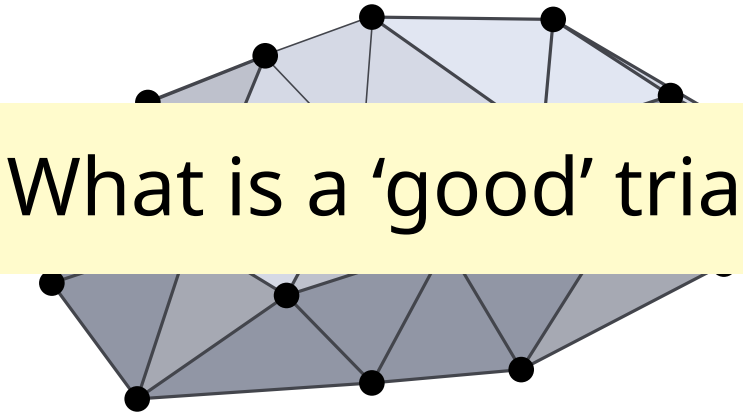
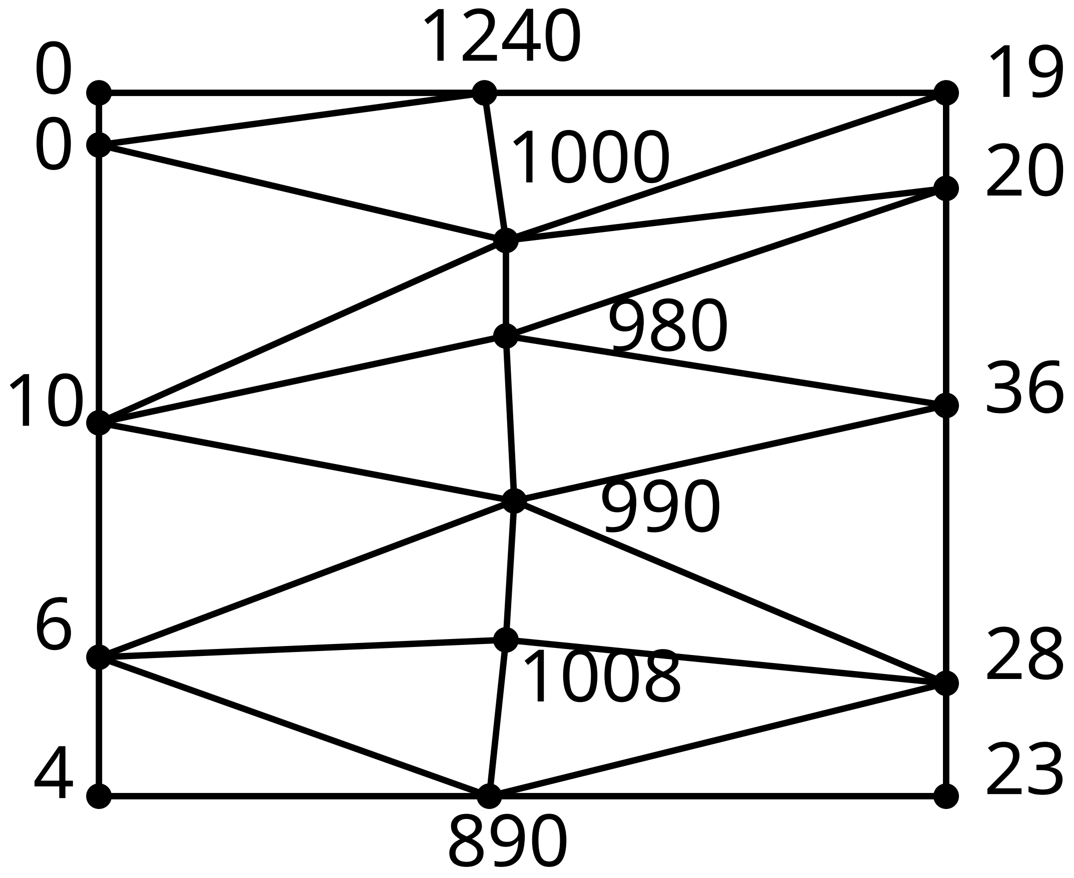


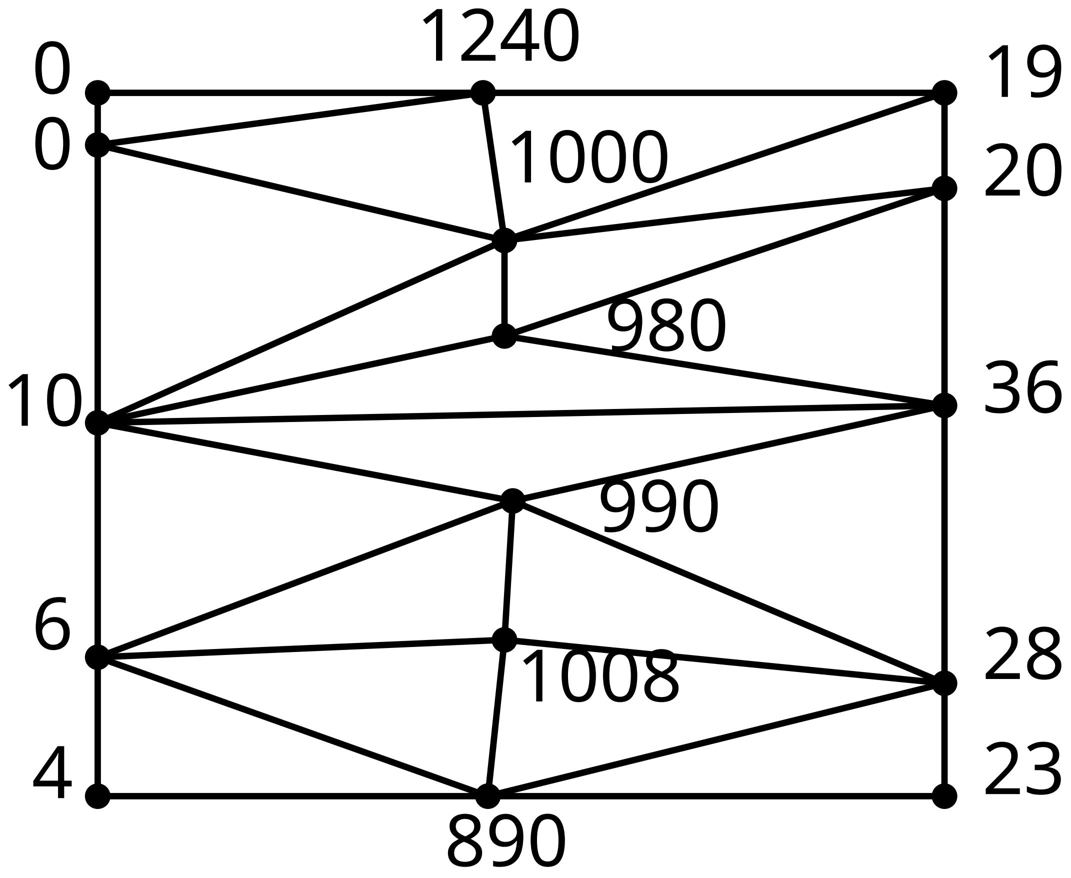
What is a ‘good’ triangulation?

Interpolation 2: triangulate & interpolate within triangles

Back to height interpolation

Lets look at the interpolation along an edge:





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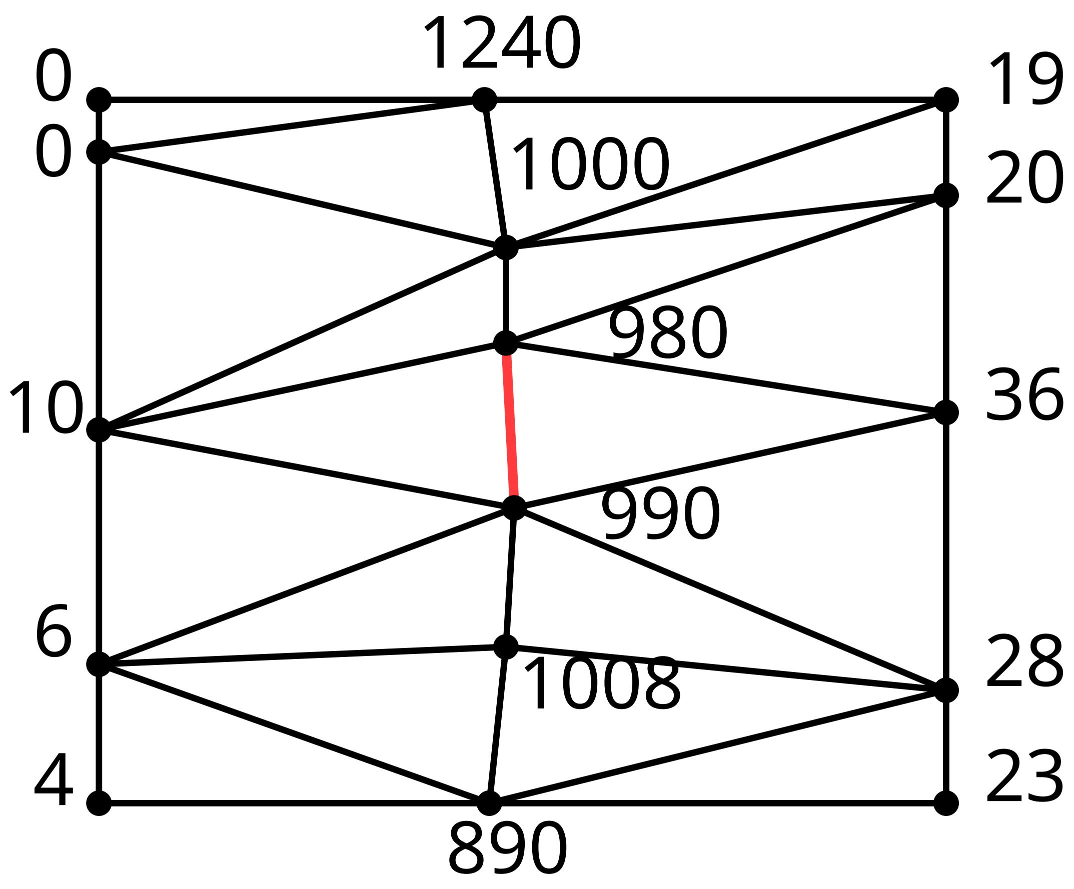

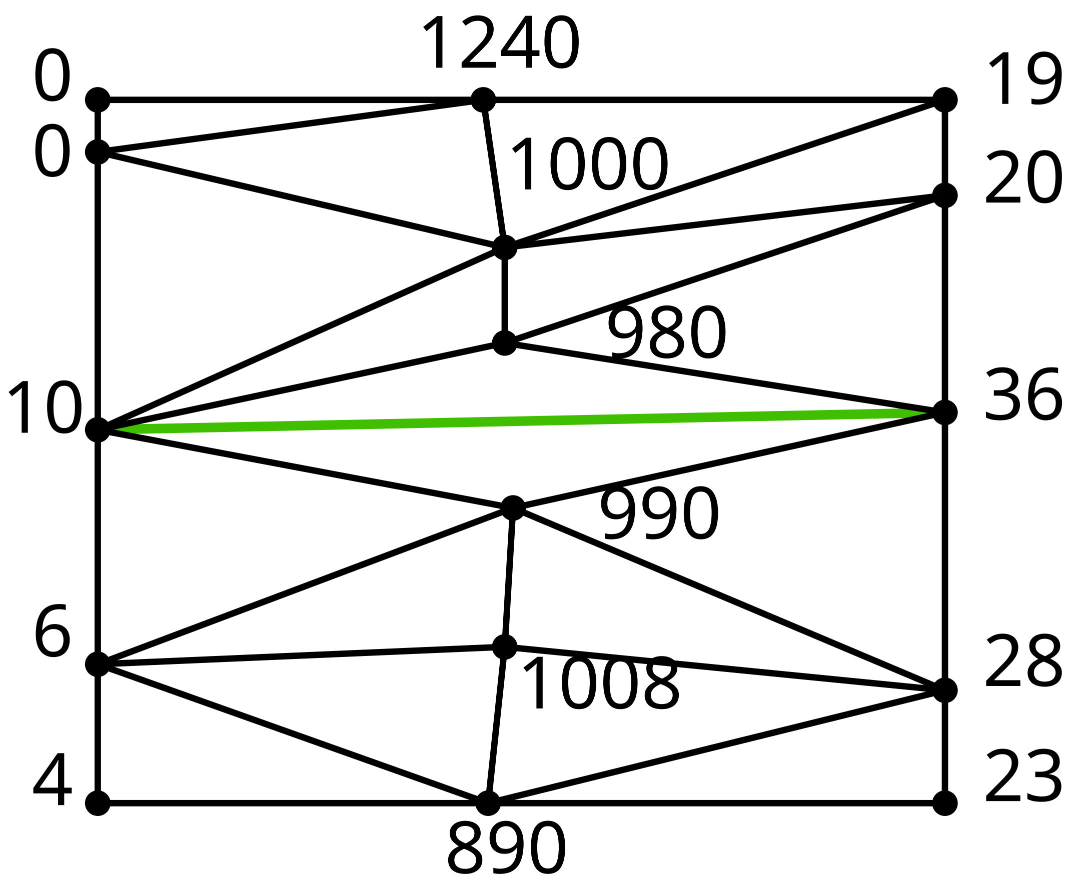

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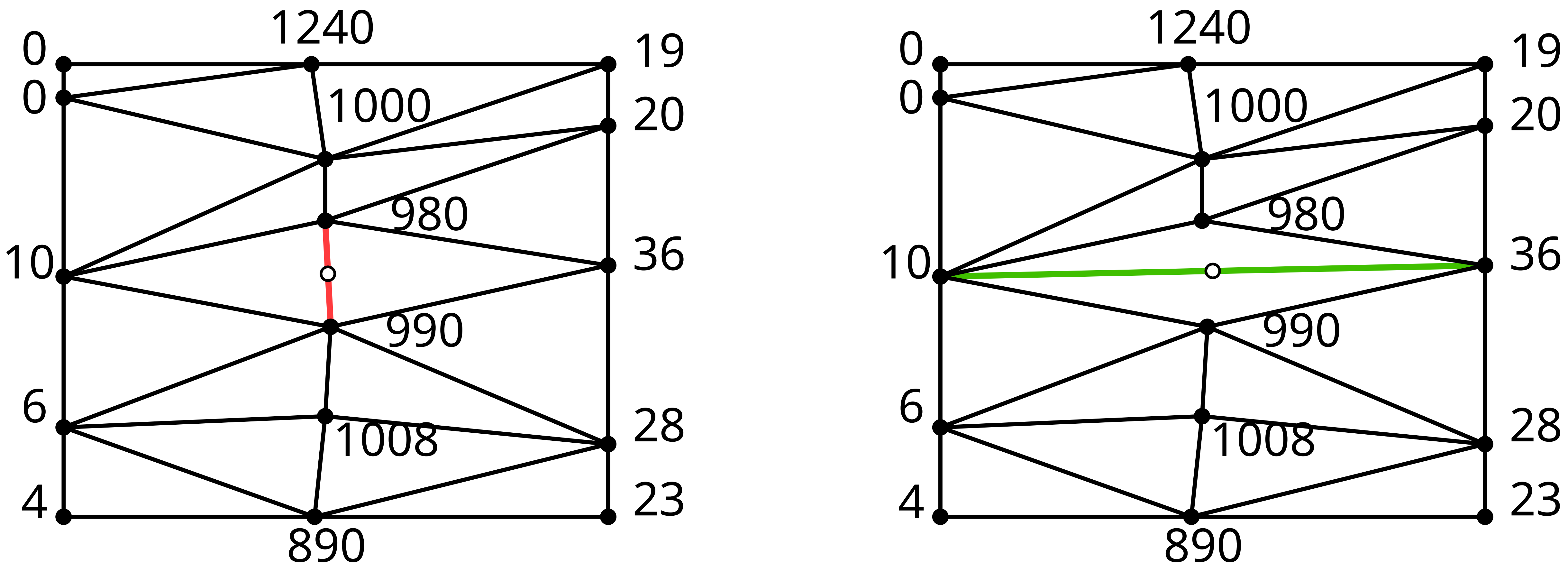

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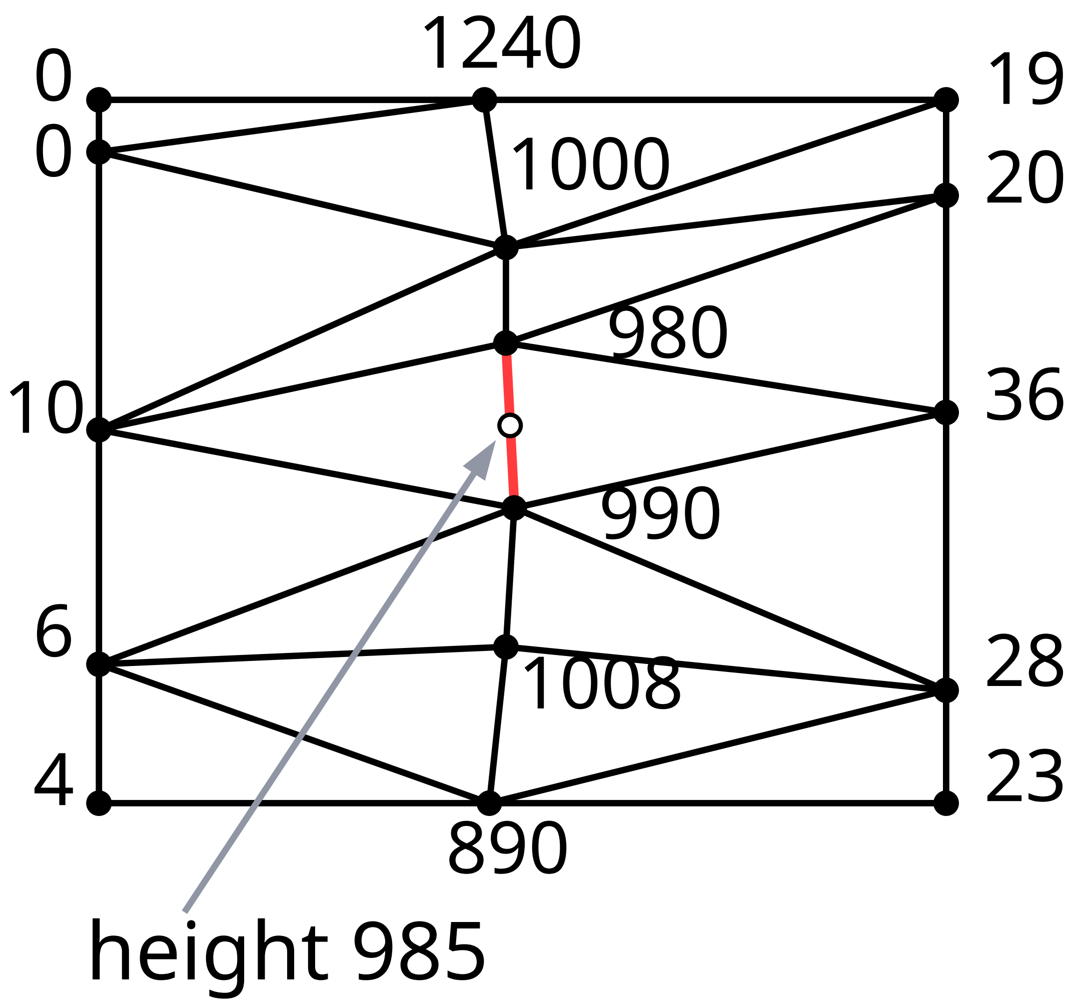

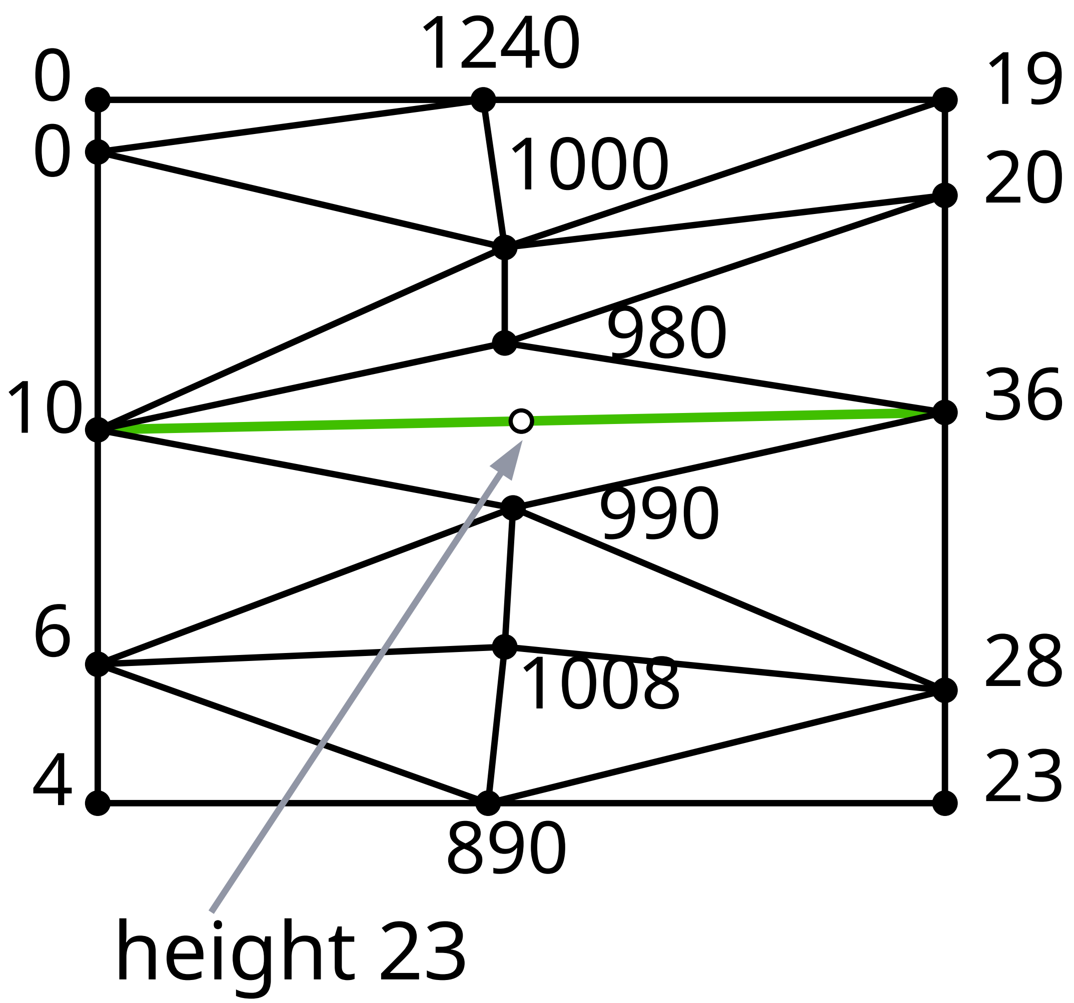

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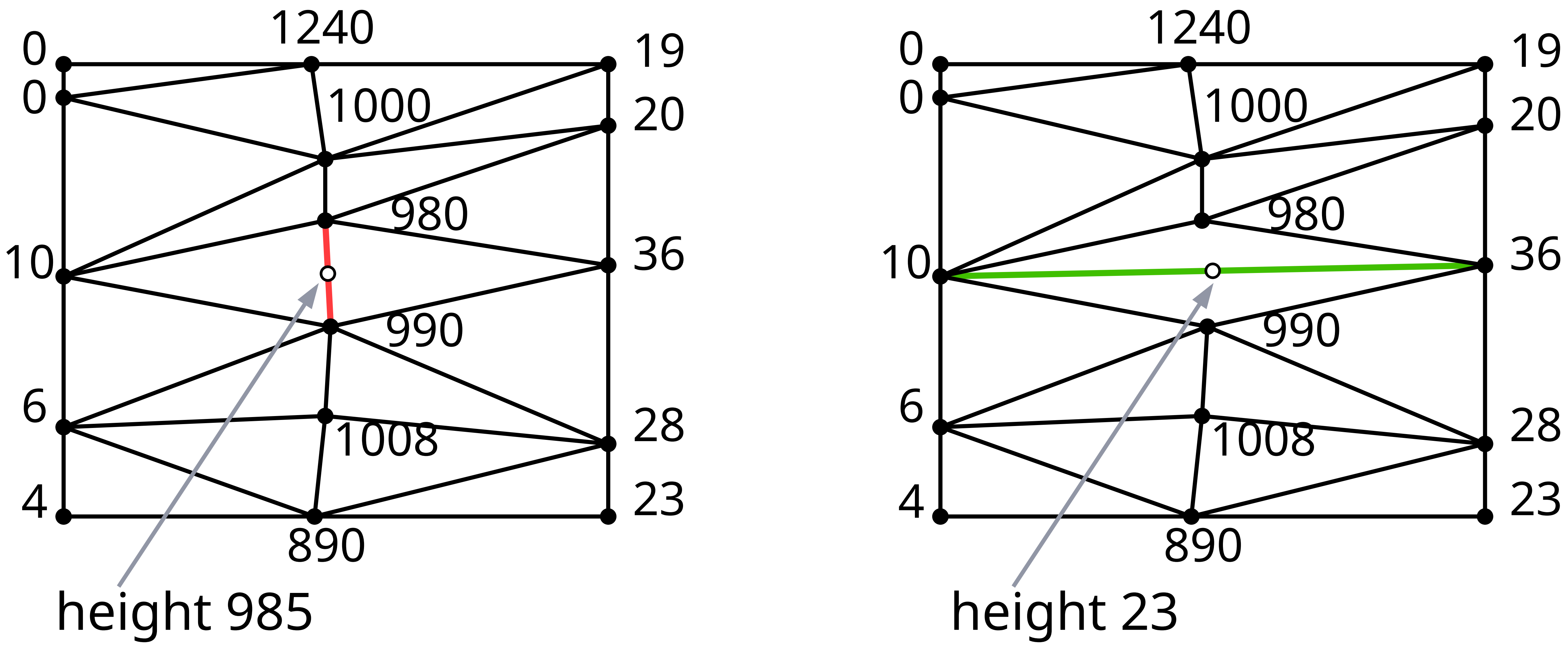

Intuition: avoid ‘thin’ triangles!

Back to height interpolation

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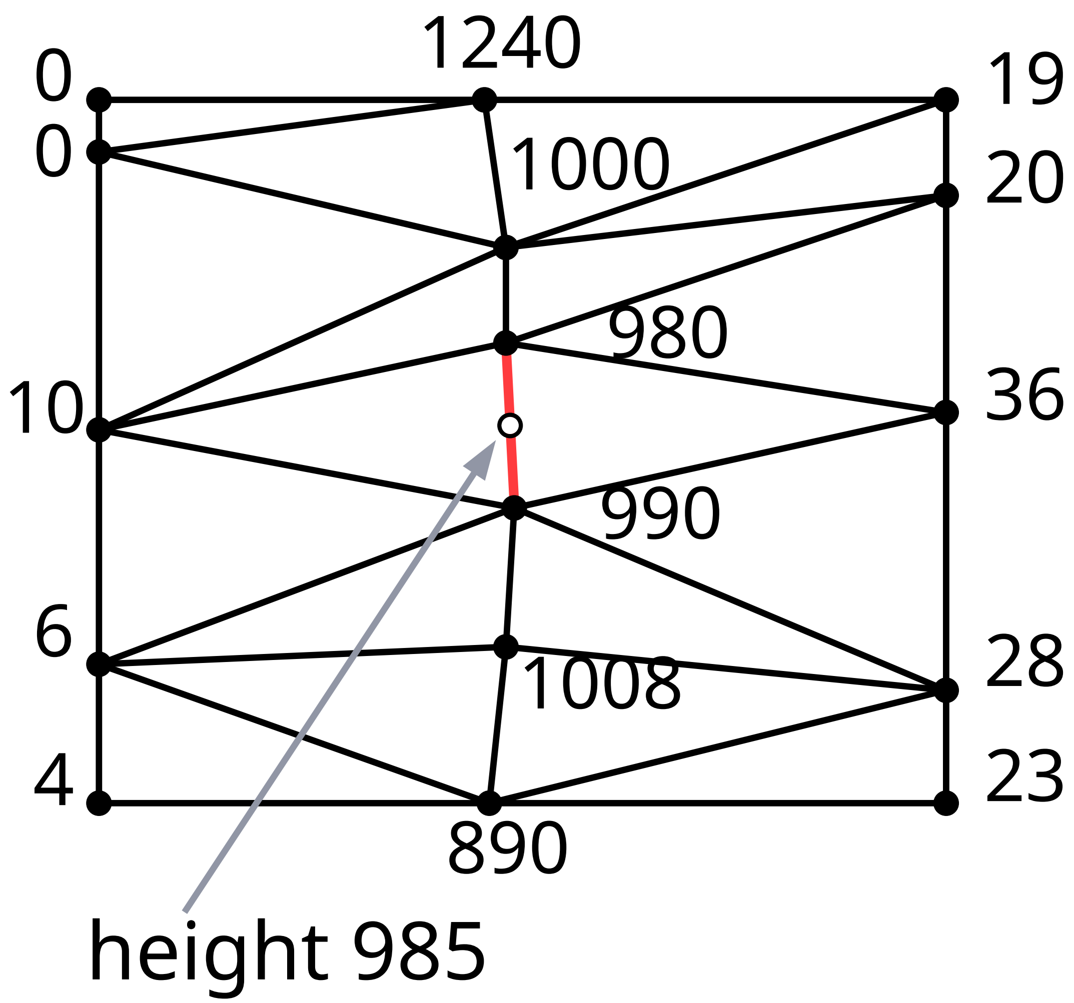
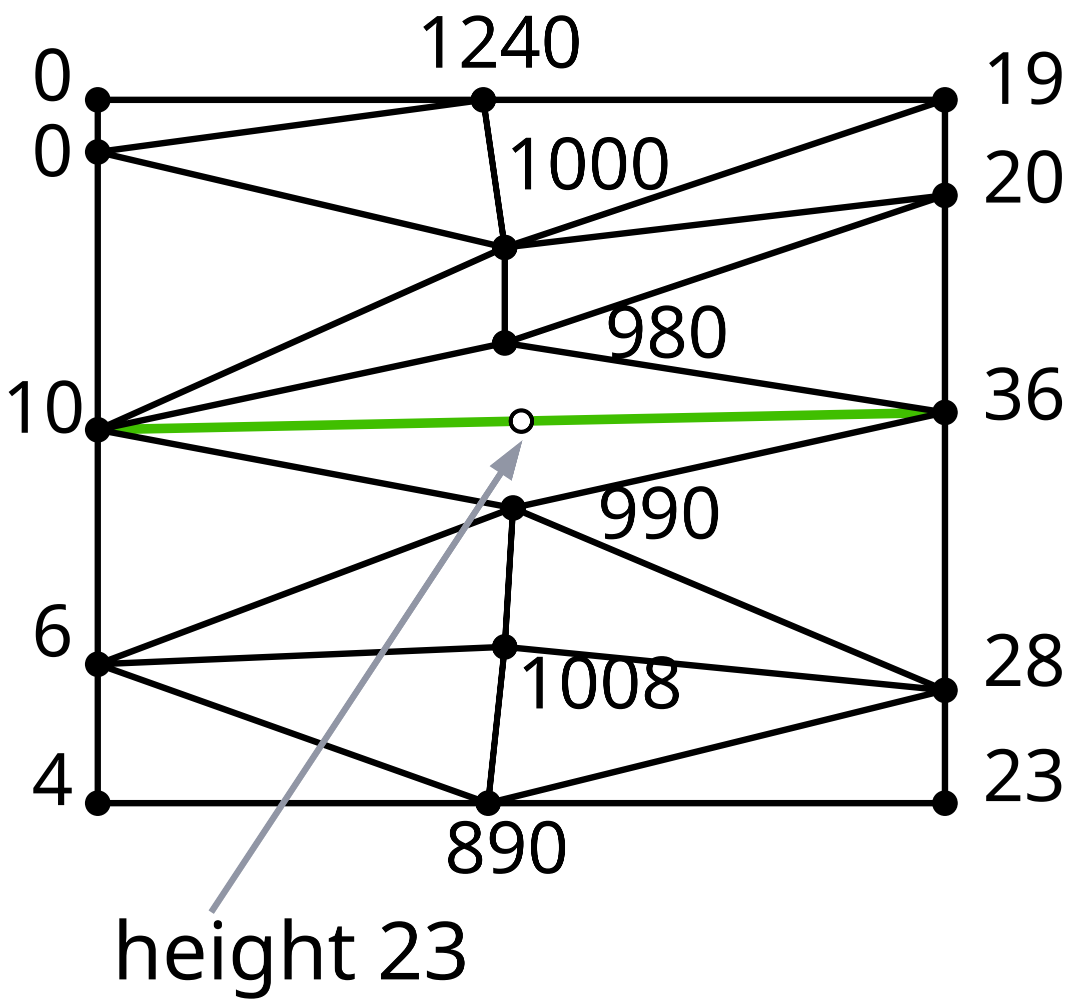




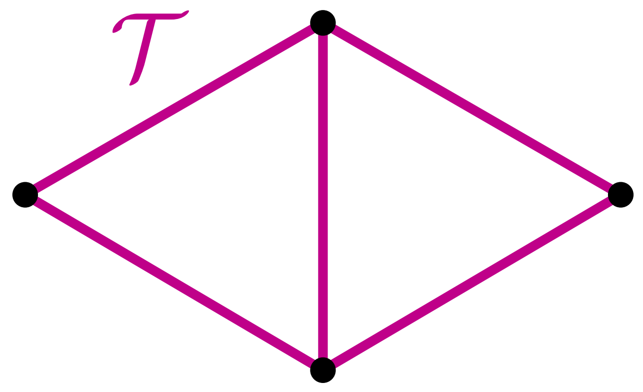
Intuition: avoid ‘thin’ triangles!

or: maximize the smallest angle within triangles!

Angle-optimal triangulations

Definition: Let $\mathcal{T}$ be a triangulation of P with m triangles and $3m$ vertices. Its angle vector is $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$, where $\alpha_1, \ldots, \alpha_{3m}$ are the angles of $\mathcal{T}$ sorted by increasing value.

$$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$$



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- For two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of P define order $A(\mathcal{T}) > A(\mathcal{T}')$ according to the lexicographical order.

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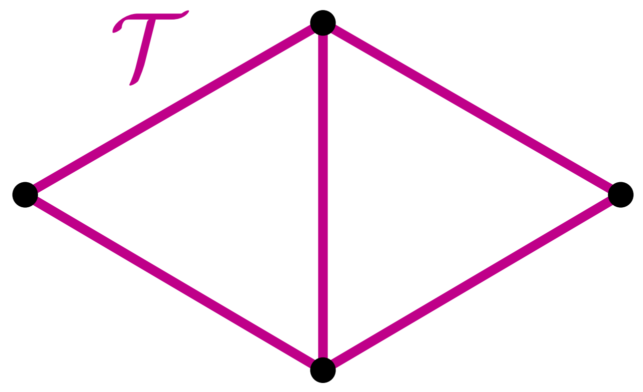

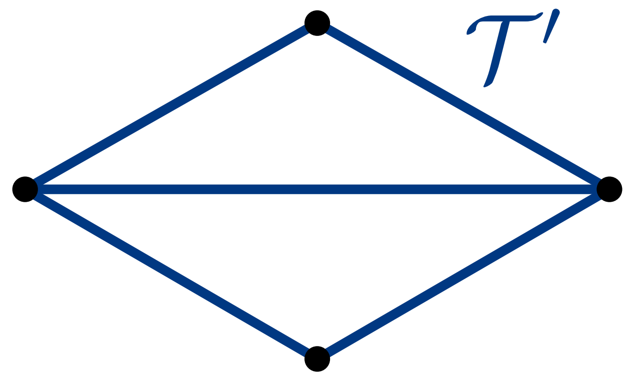

$$A(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$$

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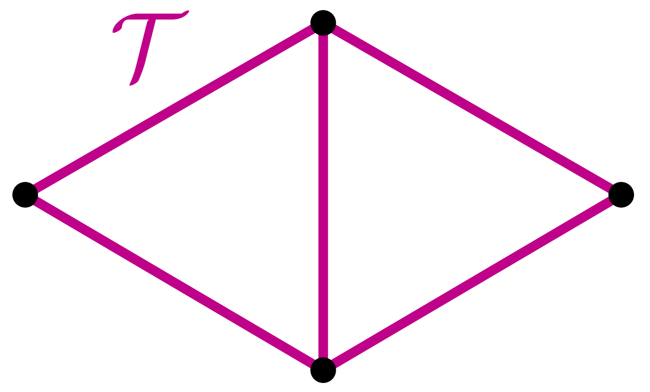
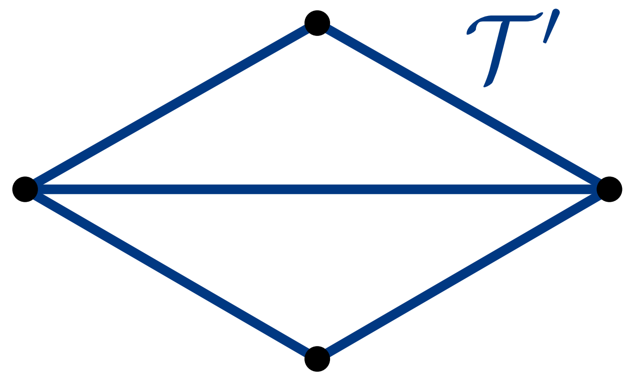
- For two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of P define order $A(\mathcal{T}) > A(\mathcal{T}')$ according to the lexicographical order.
- $\mathcal{T}$ is angle optimal, if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.

$$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$$

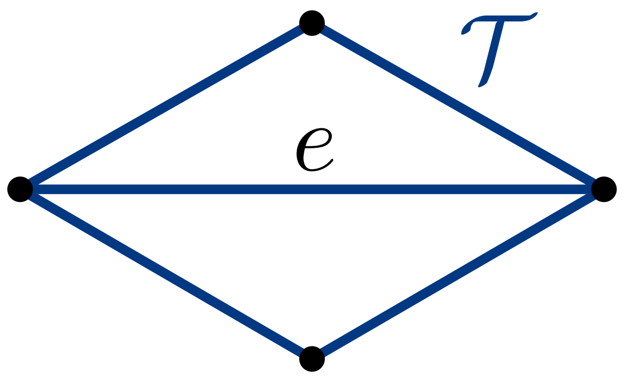




$$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$$

Edge flips

Definition: Let $\mathcal{T}$ be a triangulation. An edge e of $\mathcal{T}$ is illegal, if the smallest angle of the triangles incident to e can be increased by flipping e.



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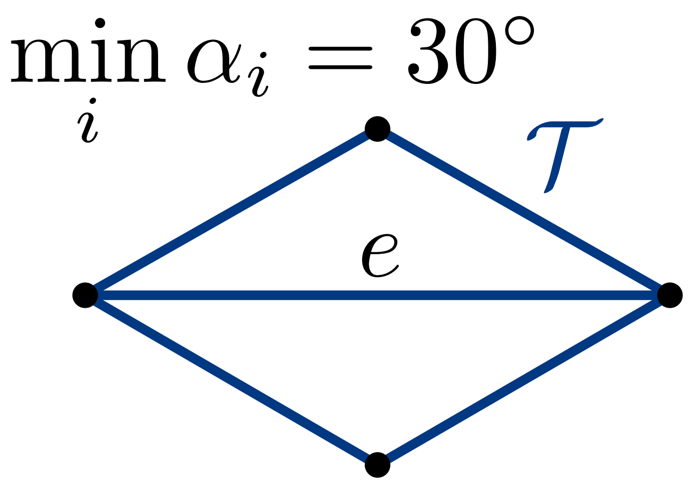

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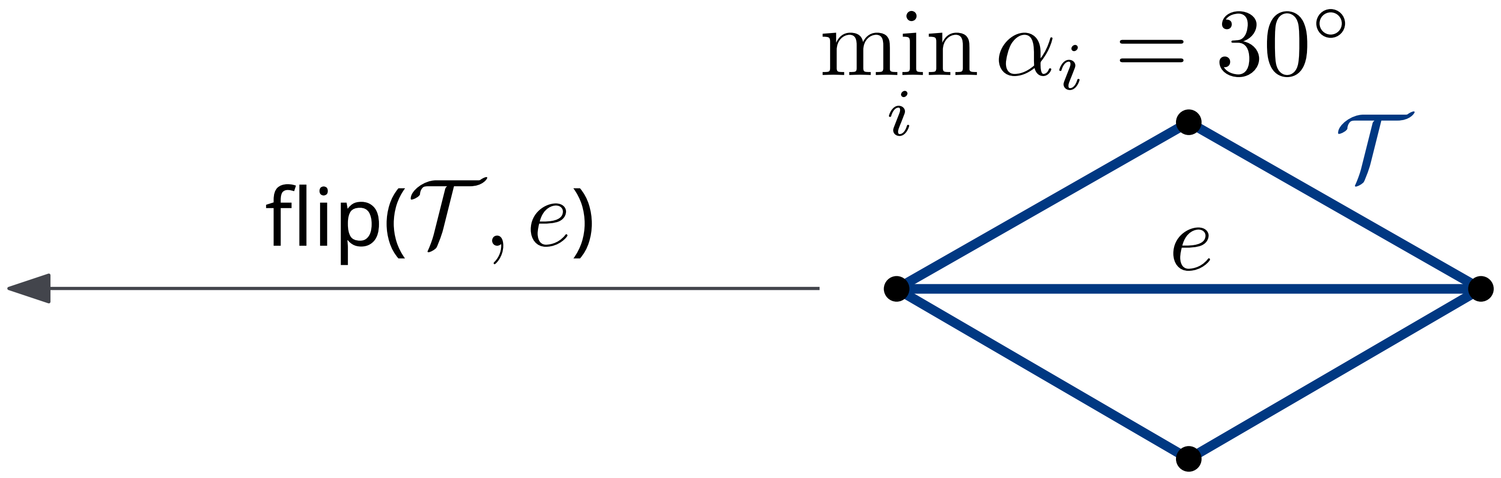

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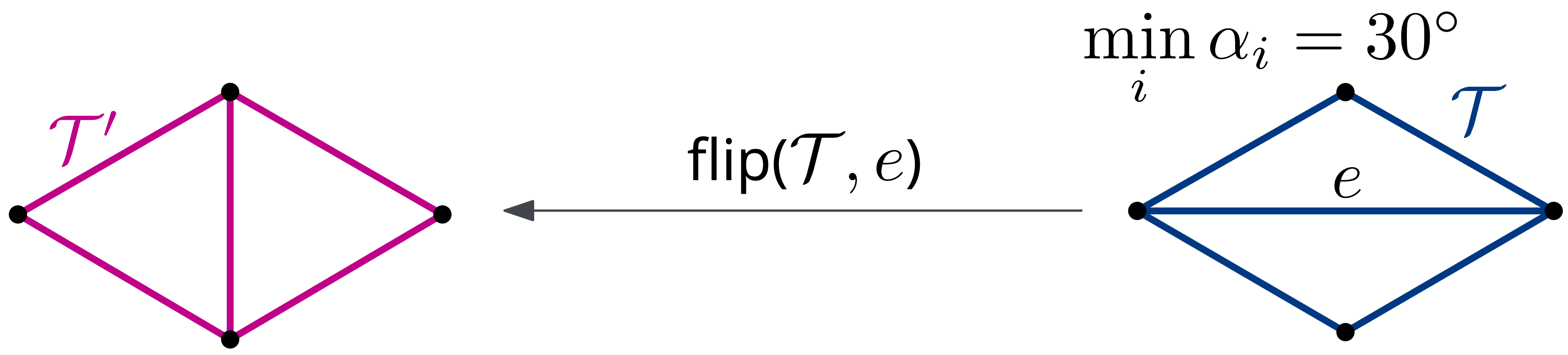

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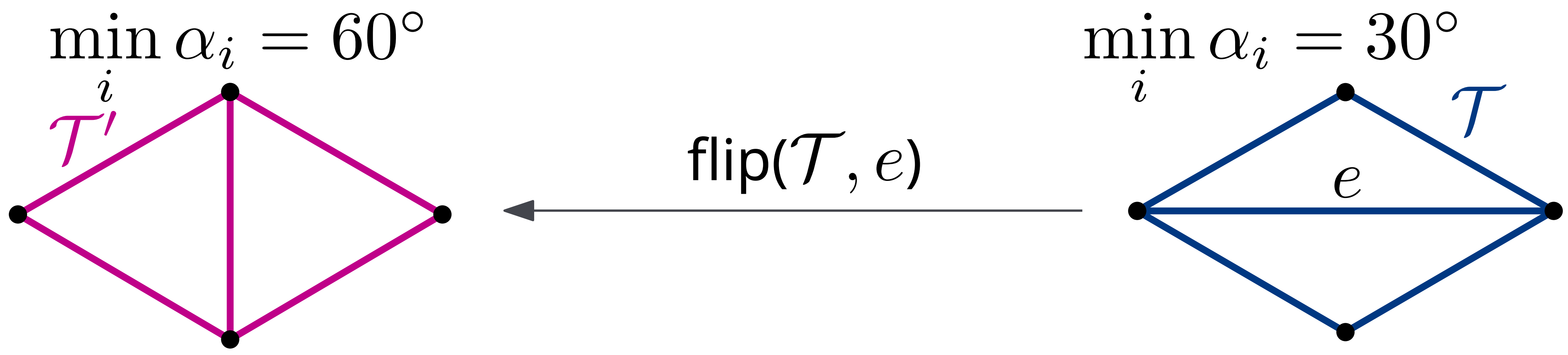

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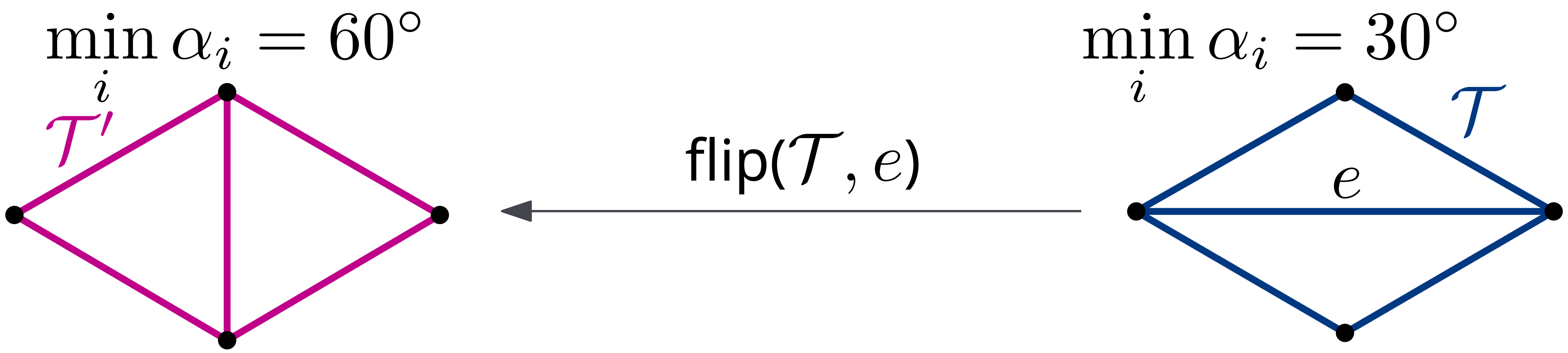
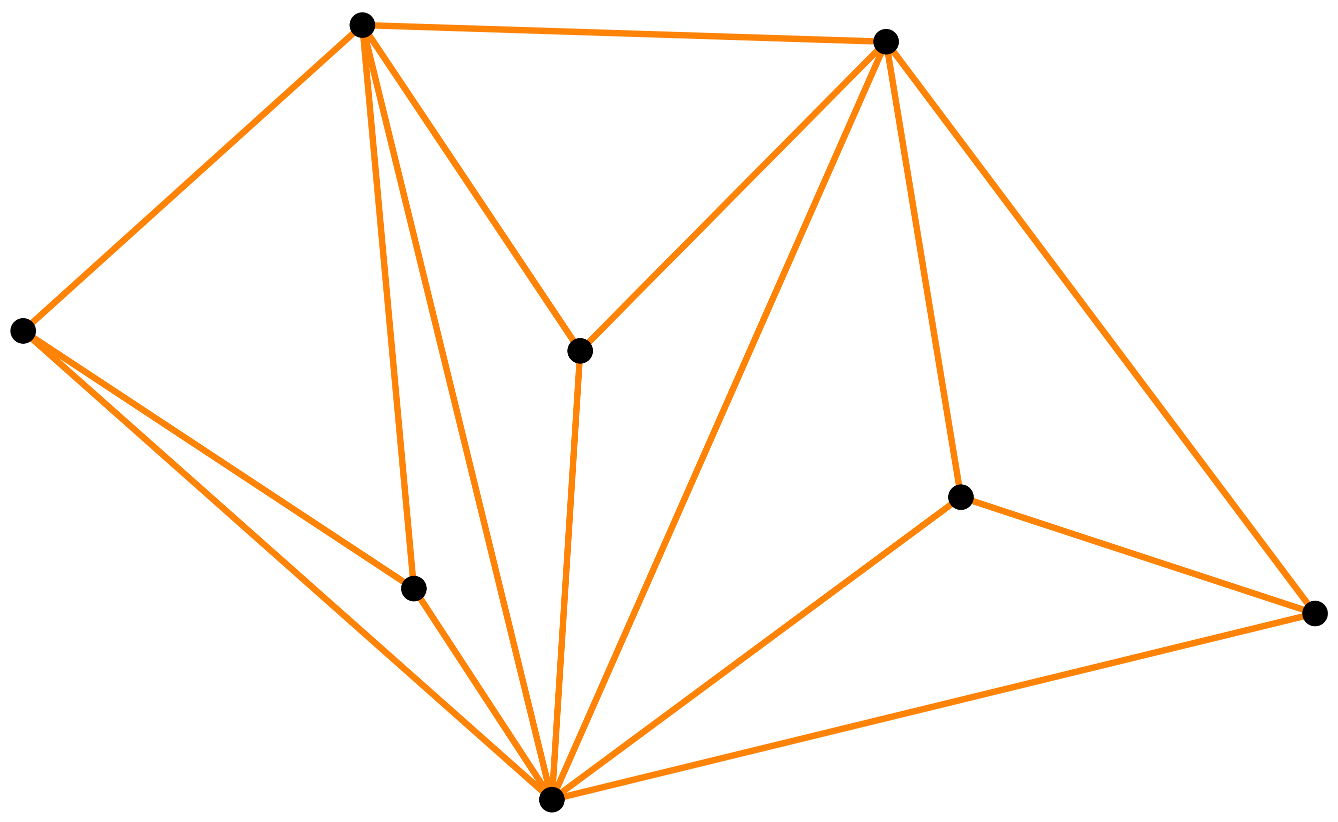
Observation: Let e be an illegal edge in $\mathcal{T}$ and let $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.



Quiz

How many edge flips are needed to remove all illegal edges?



A: 0

B: 2

C: 3

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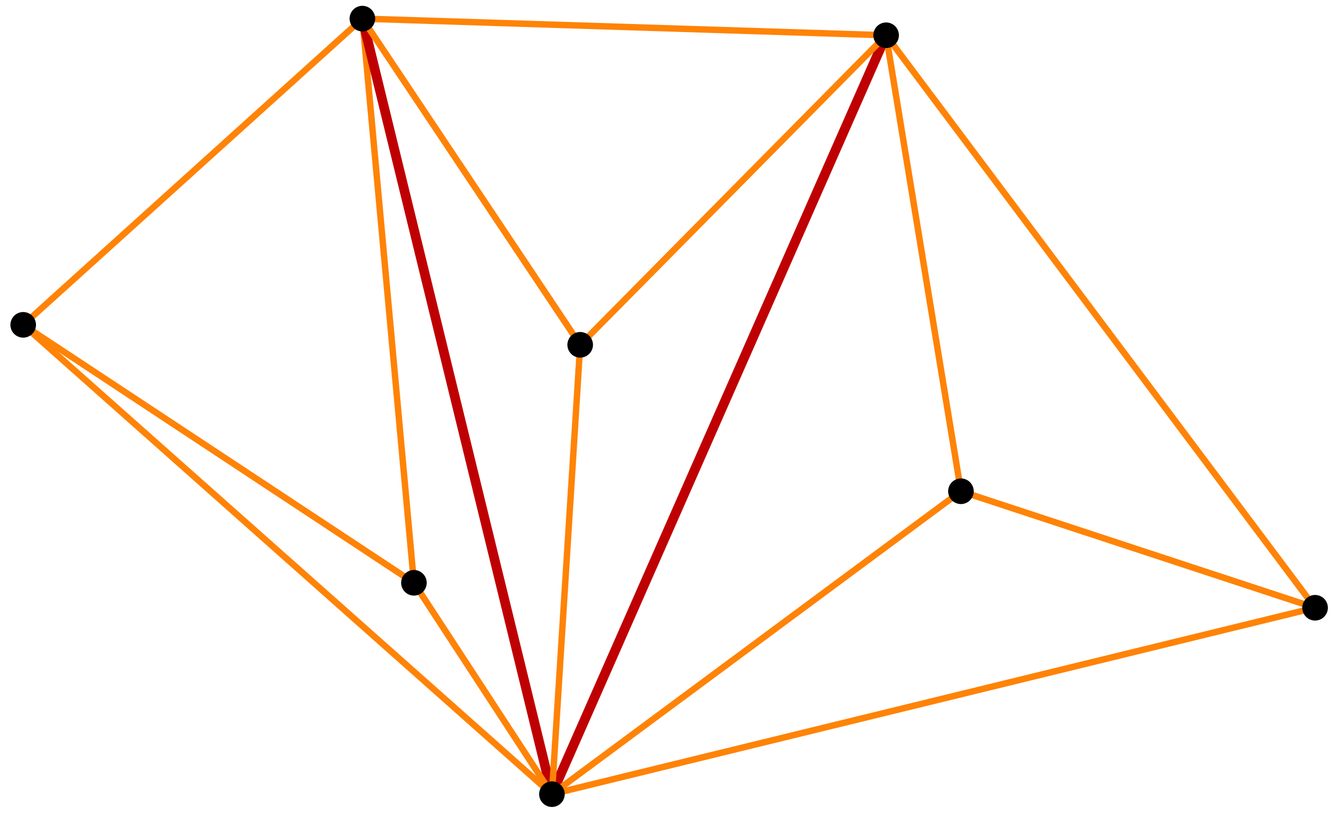

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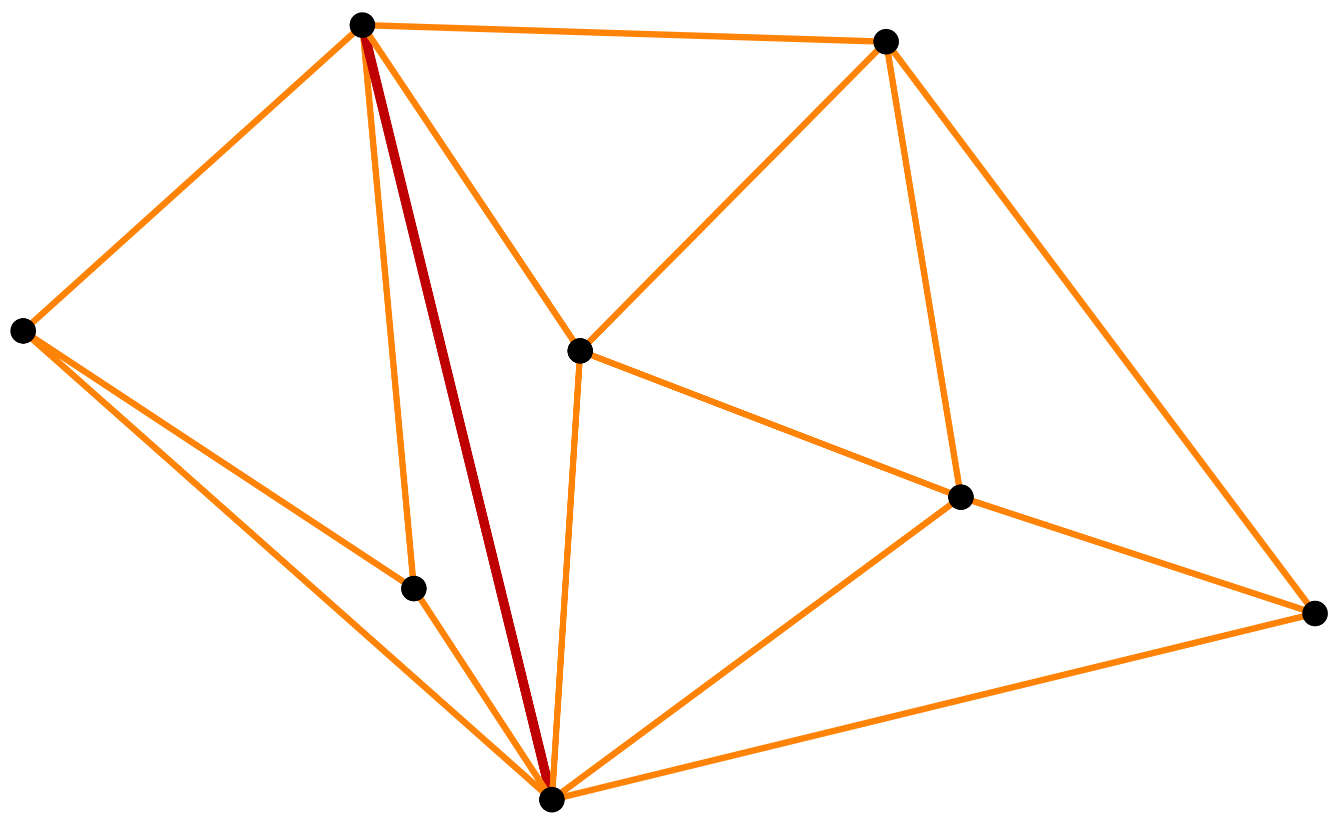

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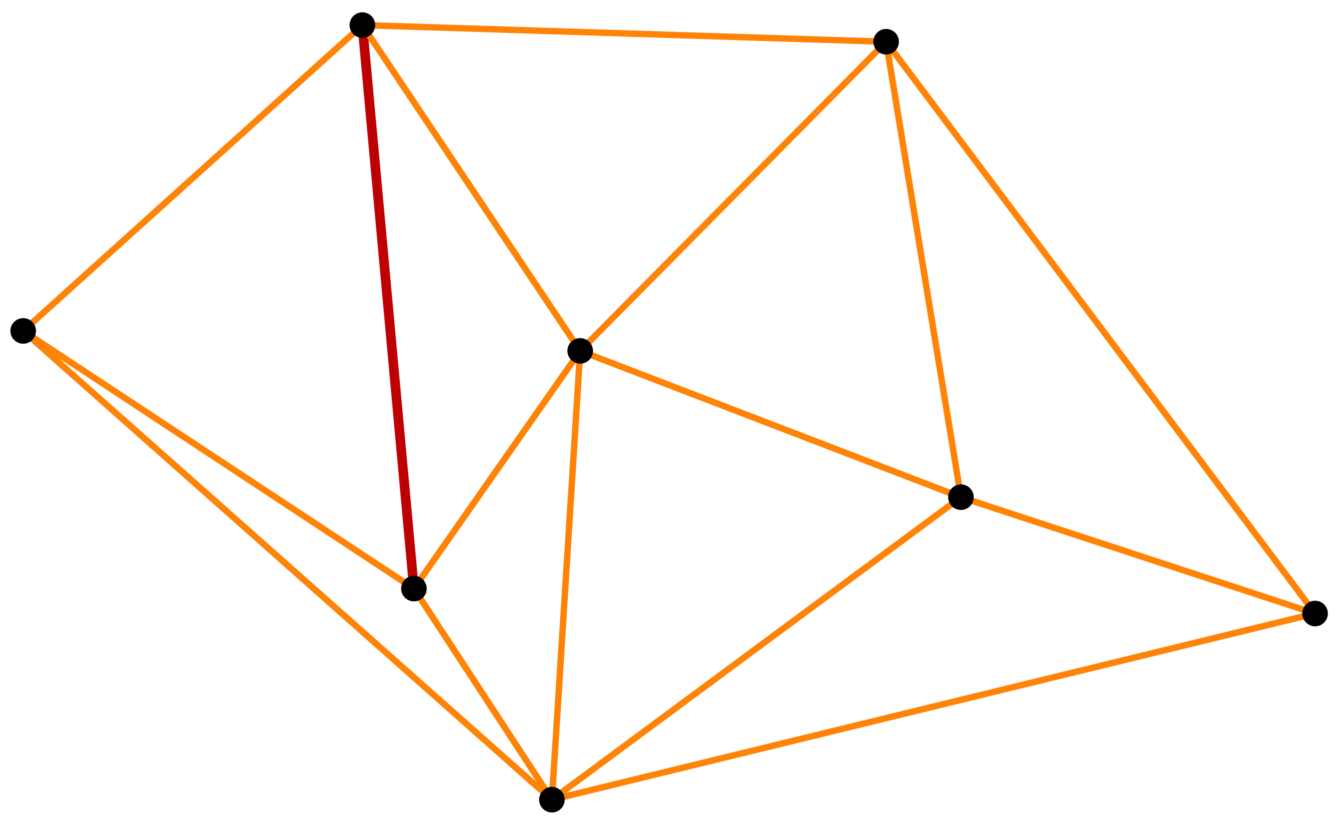

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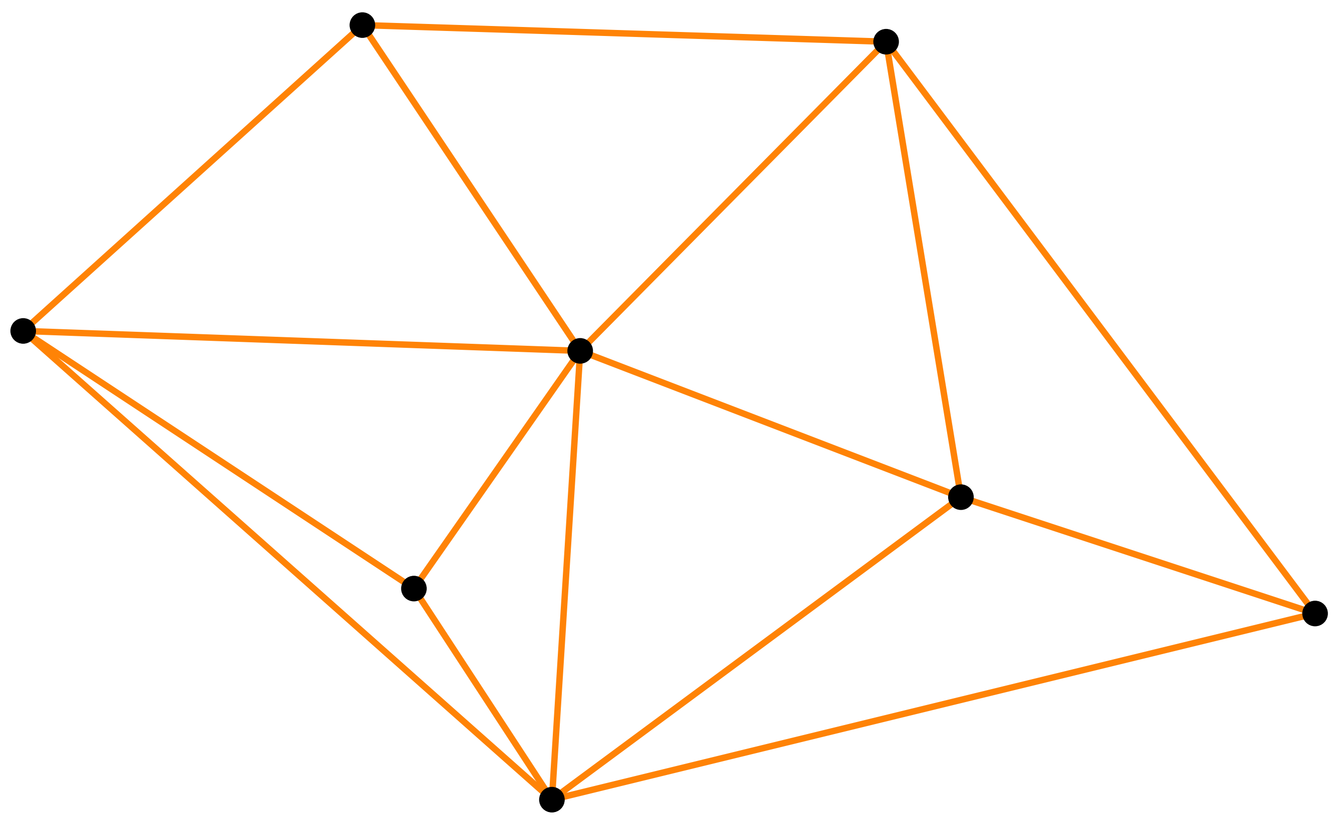

A: 0

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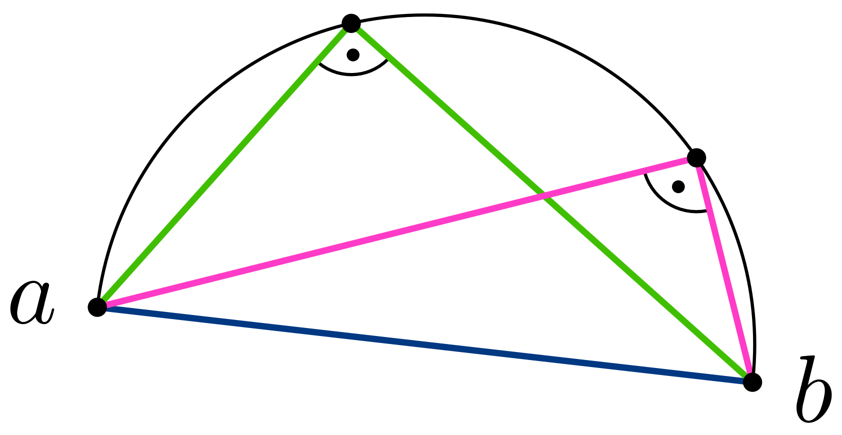
Next:

How can we check whether an edge is illegal without comparing angles?

Legal Triangulations

Thales theorem

Theorem: If ab is a diameter, then the angle at any third point on the circle c is $90°$.



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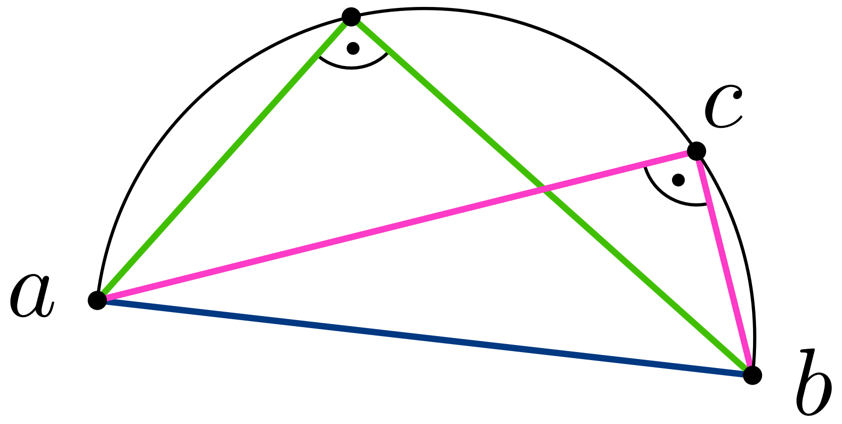
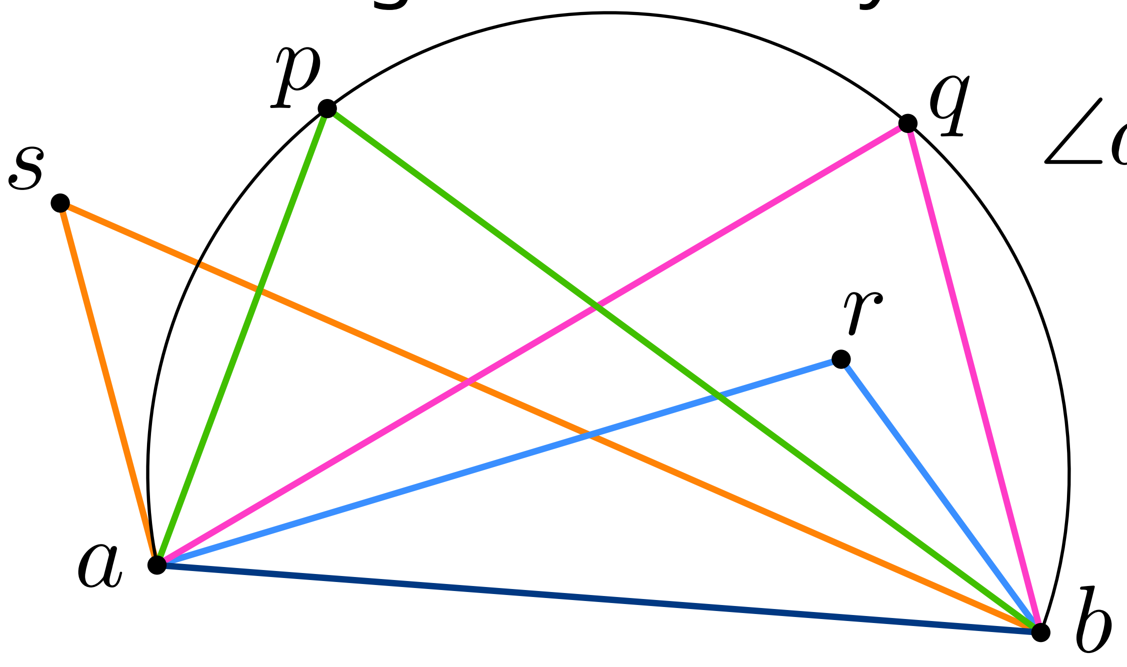


Theorem: Let C be a circle, ℓ a line intersecting C in points a and b, and p, q, r, s points lying on the same side of ℓ. Suppose that p, q lie on C, r lies inside C, and s lies outside C. Then $\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb$, where $\measuredangle abc$ denotes the smaller angle defined by three points a, b, c.



$$\angle asb < \angle apb = \angle aqb < \angle arb$$

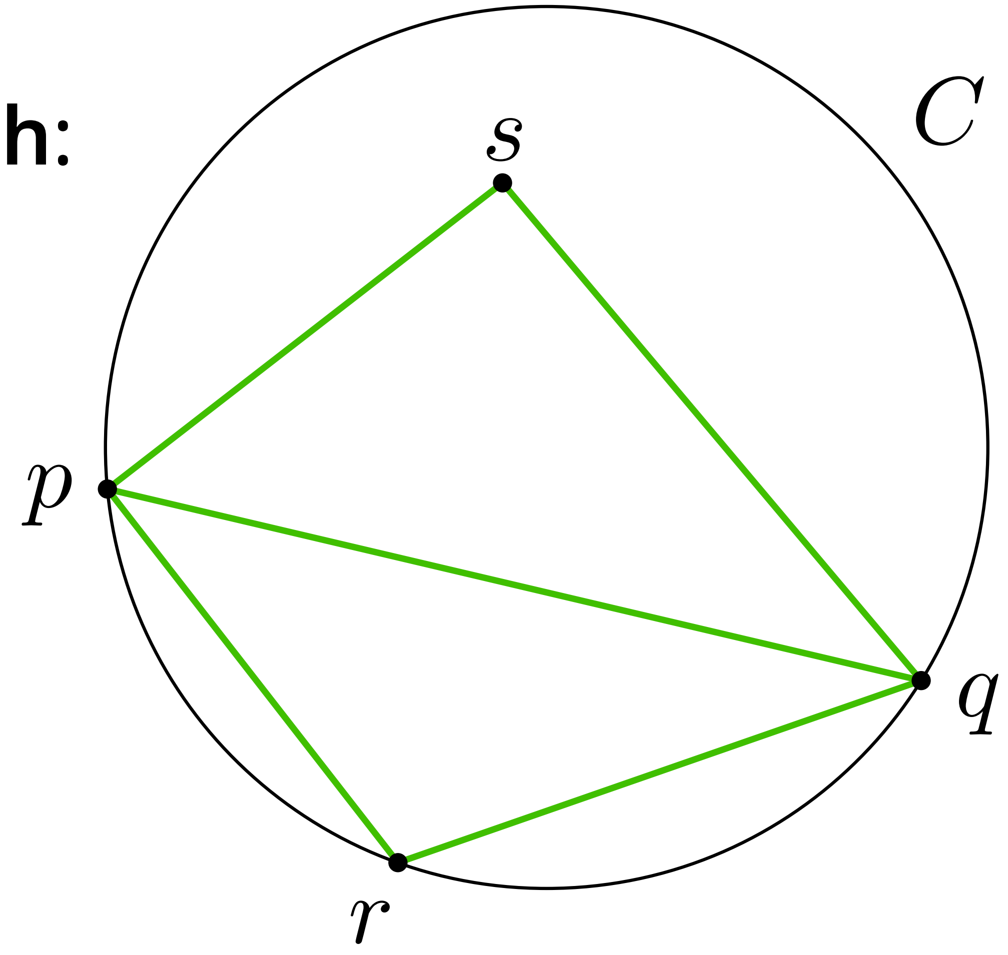
Legal triangulations

Lemma 1: Let Δprq and Δpqs be two adjacent triangles in $\mathcal{T}$ and C the circumcircle of Δprq. Then:

$$\overline{pq} \text{ is illegal} \quad \Leftrightarrow \quad s \in \text{int}(C).$$

If p, q, r, s form a convex quadrilateral and $s \notin \partial C$, then either $\overline{pq}$ or $\overline{rs}$ is illegal.

Proof sketch:



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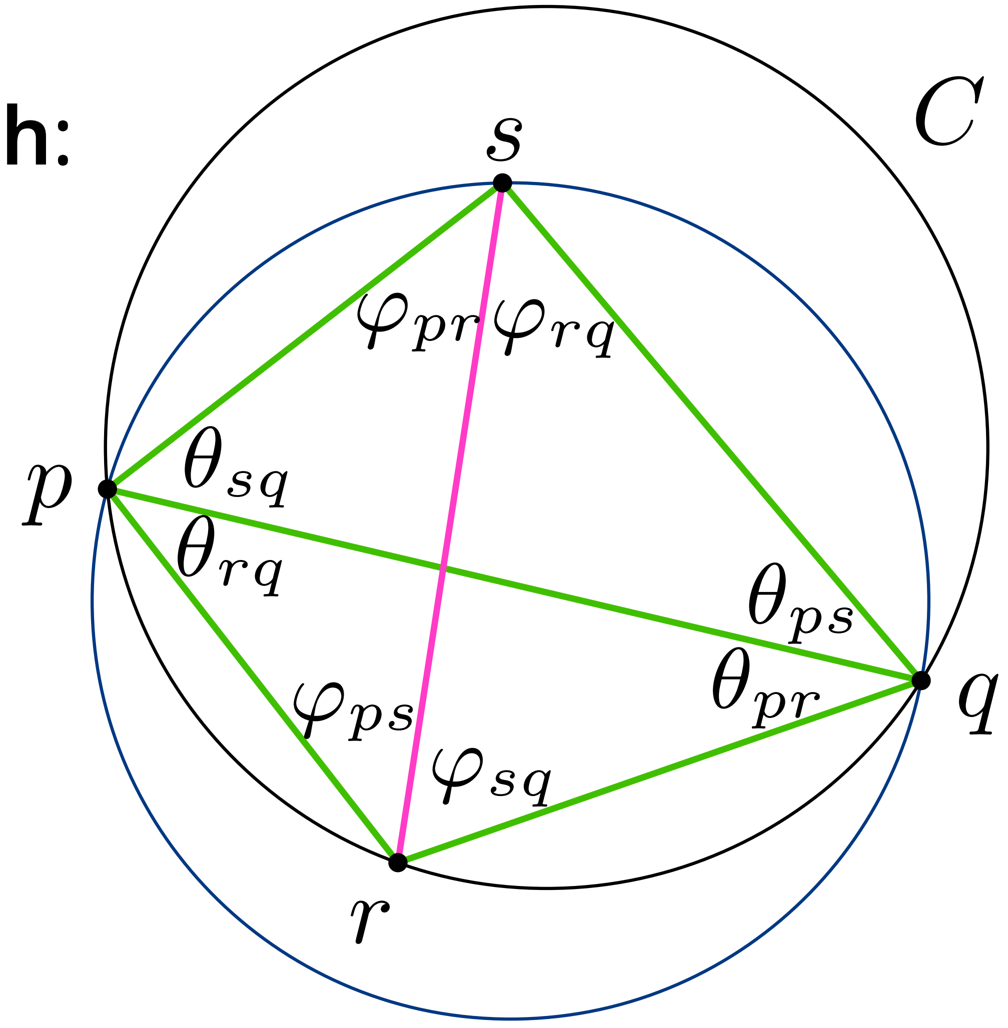

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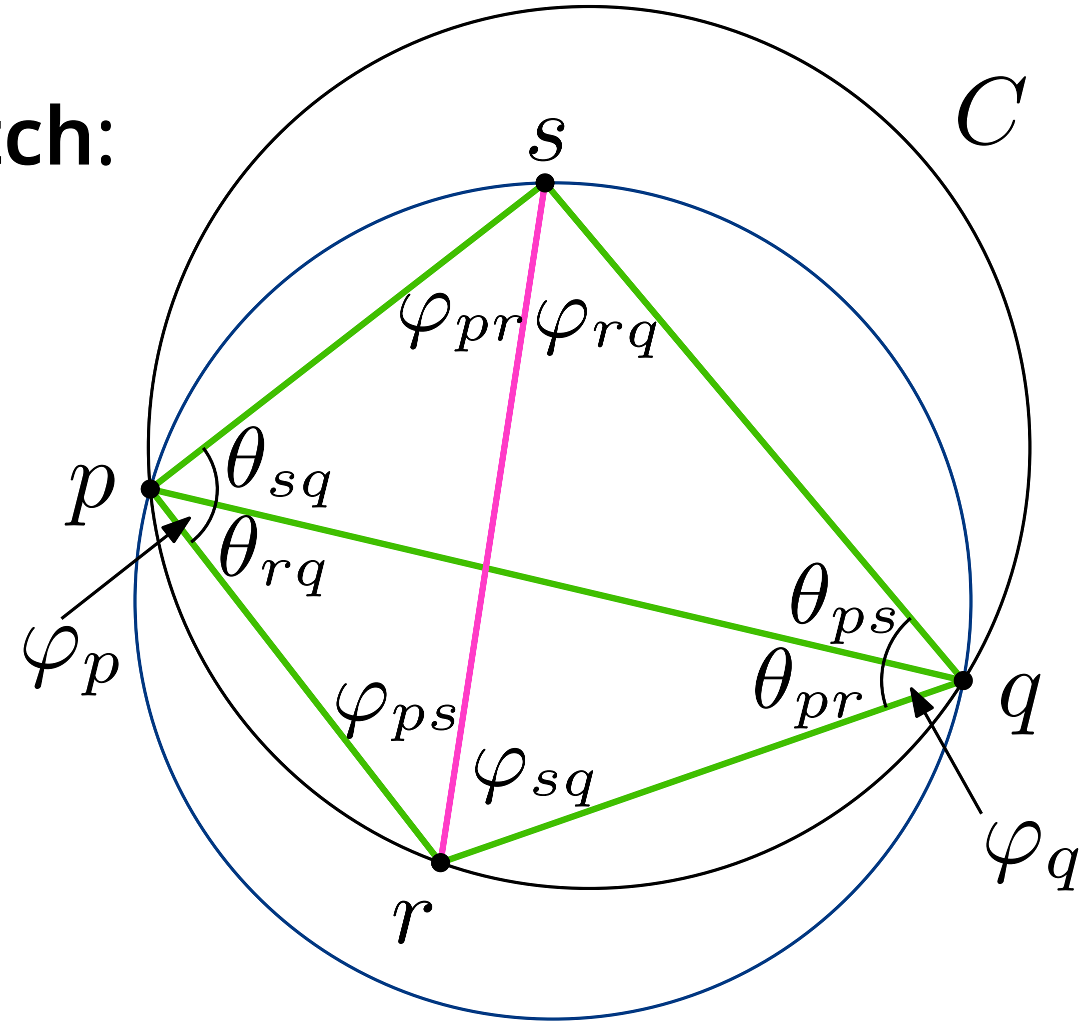

$$\varphi_{pr} > \theta_{pr}$$
$$\varphi_{ps} > \theta_{ps}$$
$$\varphi_{rq} > \theta_{rq}$$
$$\varphi_{sq} > \theta_{sq}$$
$$\varphi_p = \theta_{rq} + \theta_{sq}$$
$$\varphi_q = \theta_{pr} + \theta_{ps}$$

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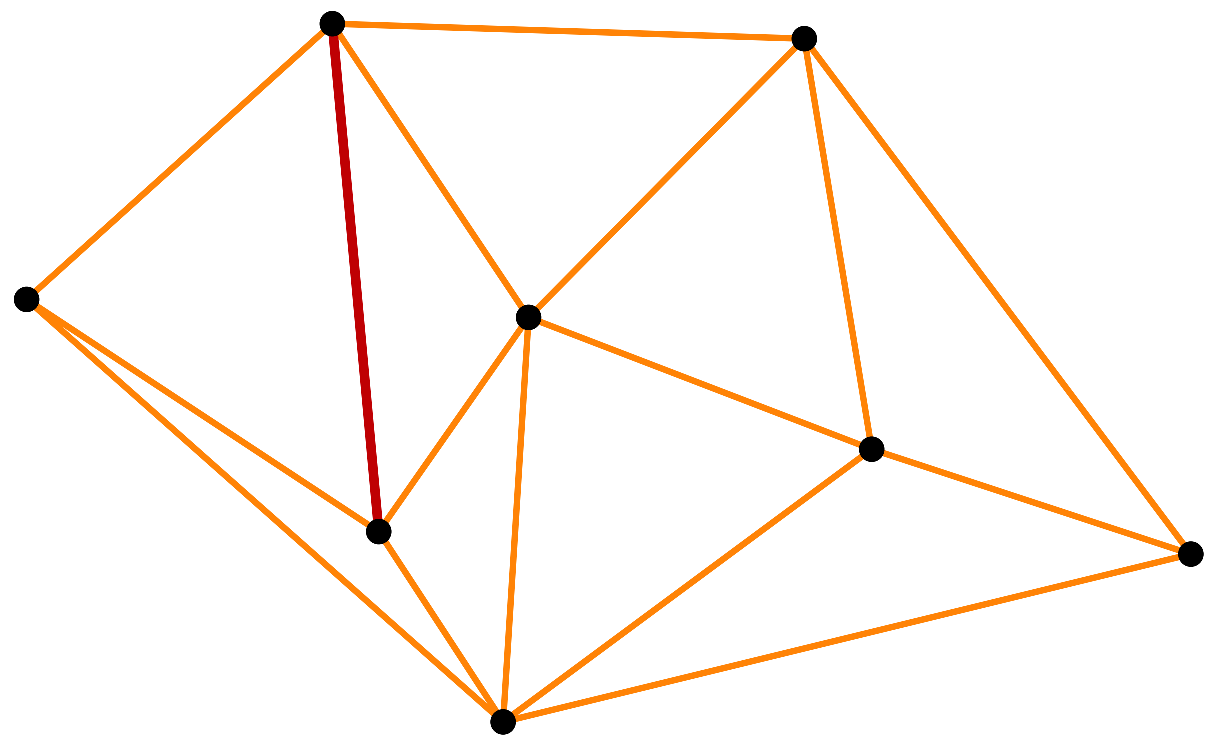

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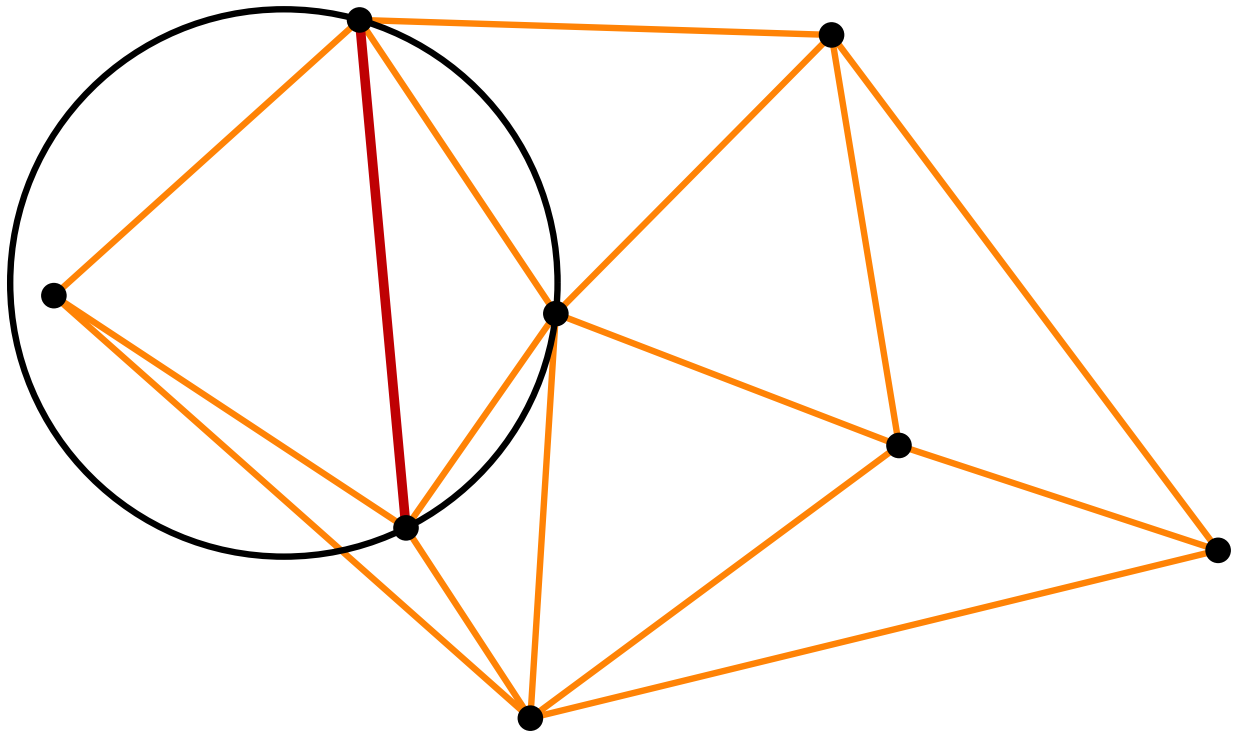

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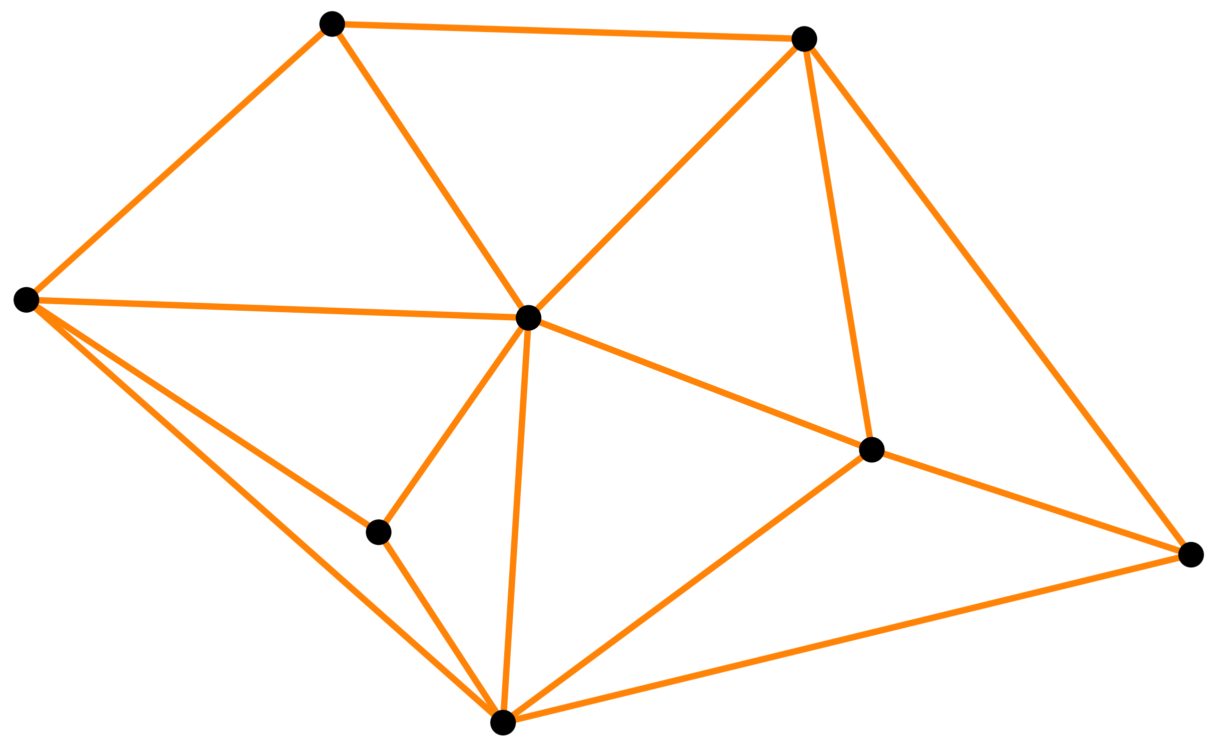

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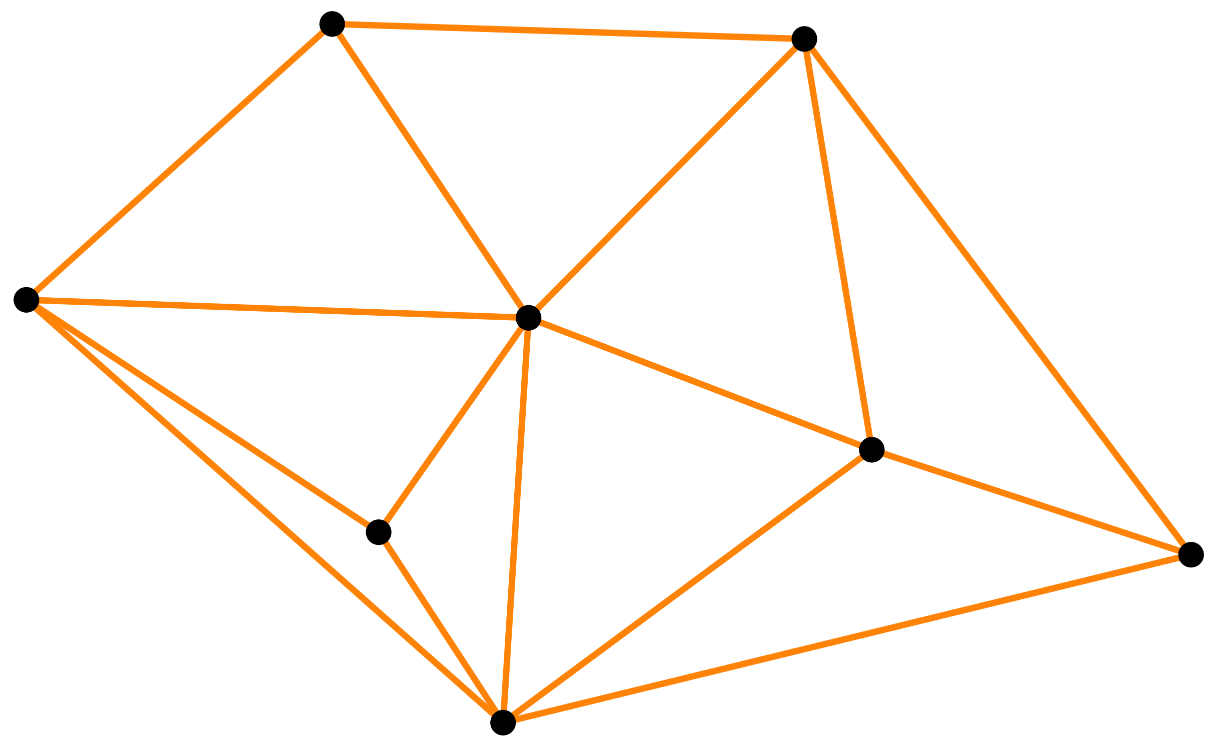
Definition: A triangulation without illegal edges is called legal triangulation.



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Are there legal triangulations?

Legal triangulations

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Definition: A triangulation without illegal edges is called legal triangulation.

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Legal triangulations

Lemma 1: Let Δprq and Δpqs be two adjacent triangles in $\mathcal{T}$ and C the circumcircle of Δprq. Then:

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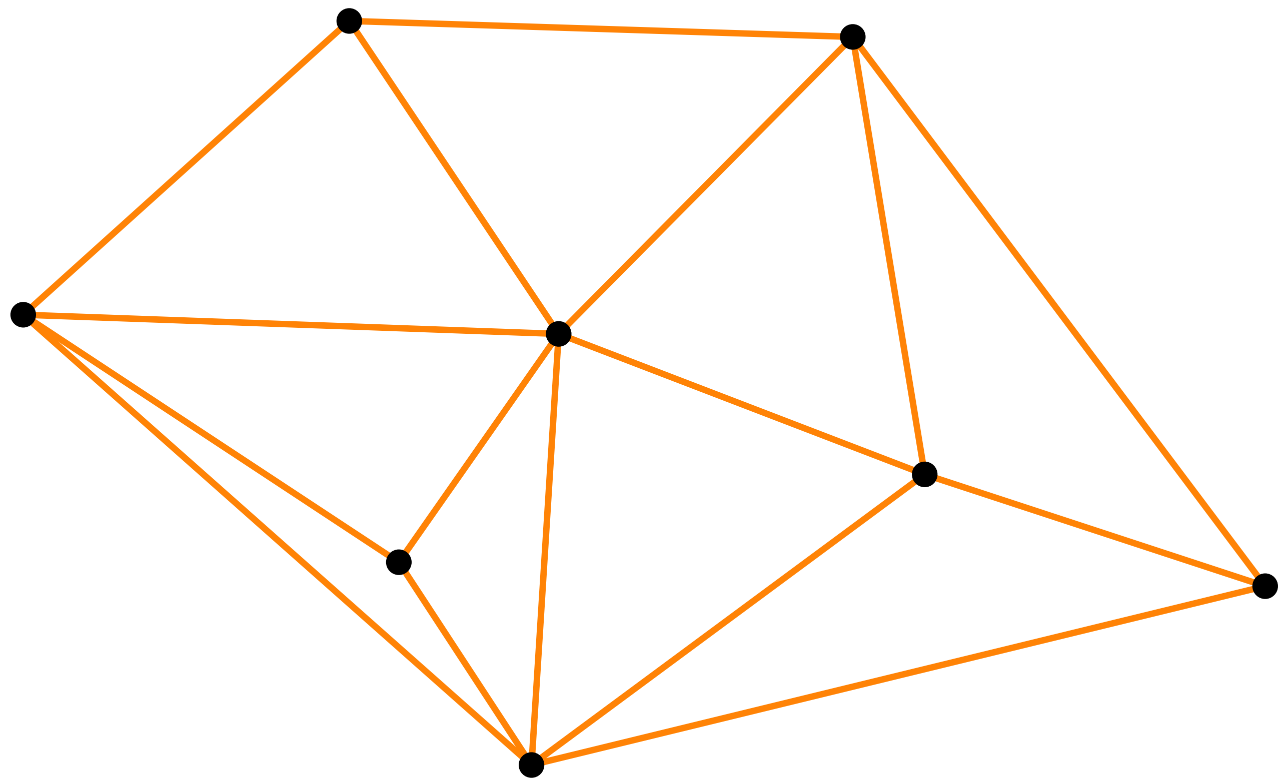
Does the algorithm terminate?

yes, since $A(\mathcal{T})$ increases and #triangulations is finite

Legal vs angle-optimal

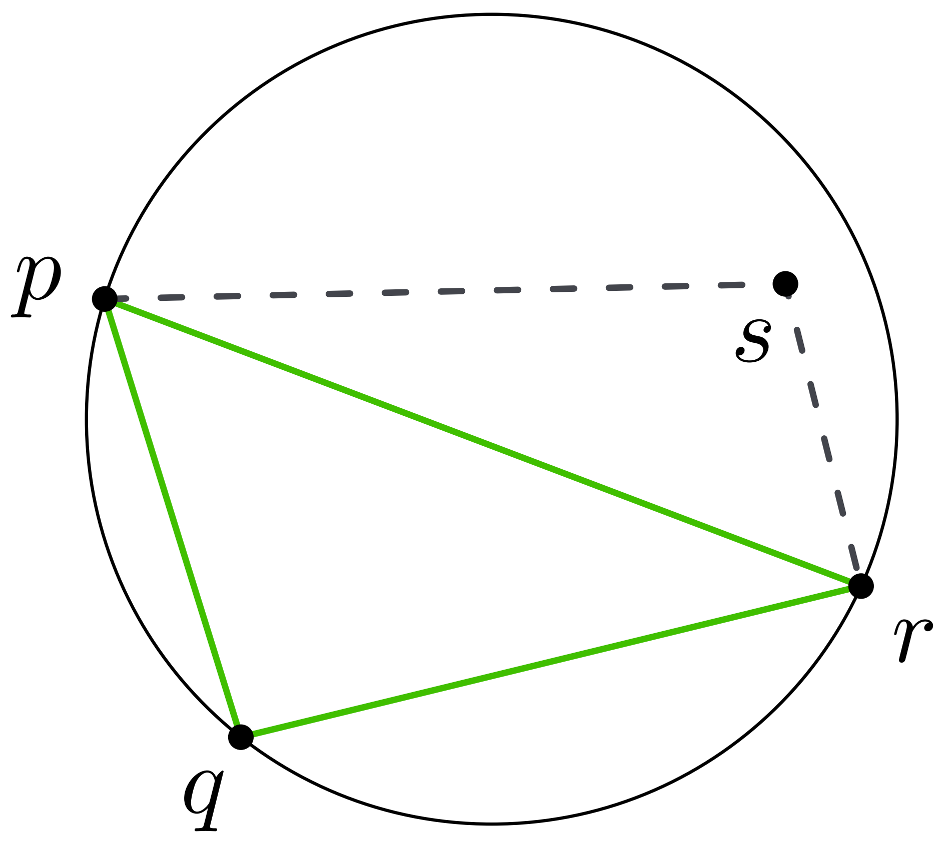
We know: Every angle-optimal triangulation is legal.

But is every legal triangulation angle-optimal?



Legal vs Delaunay

Theorem: Let P be a set of points in $\mathbb{R}^2$. A triangulation $\mathcal{T}$ of P is legal if and only if $\mathcal{T}$ is a Delaunay triangulation.



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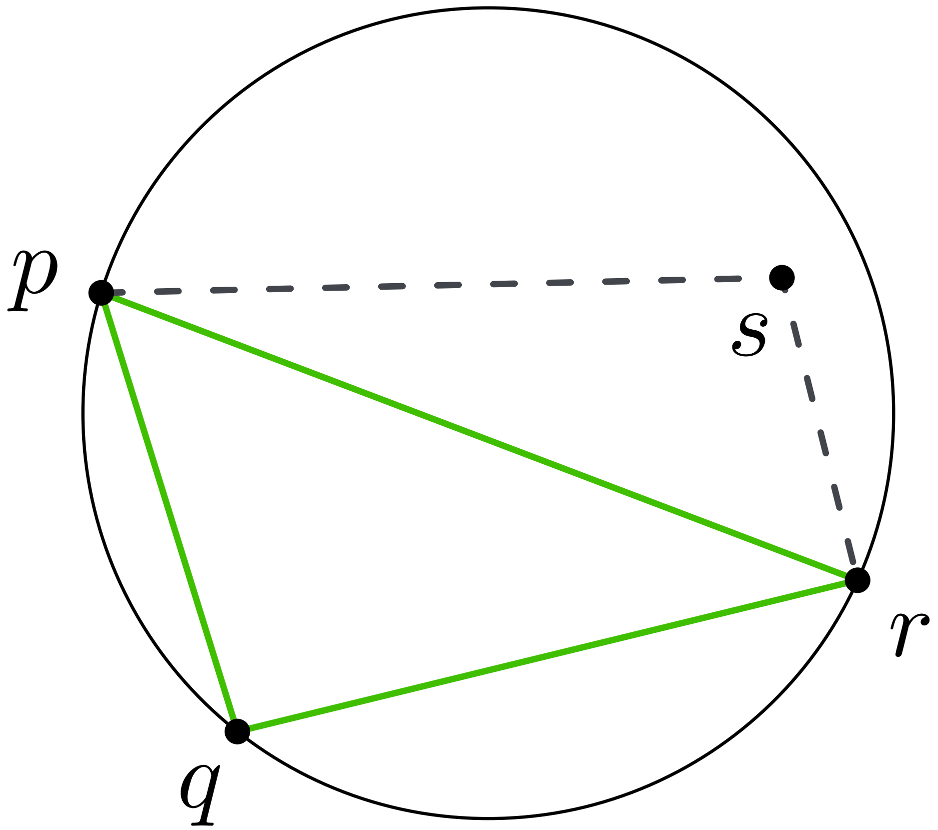
Proof sketch:

"$\Leftarrow$" obvious, use



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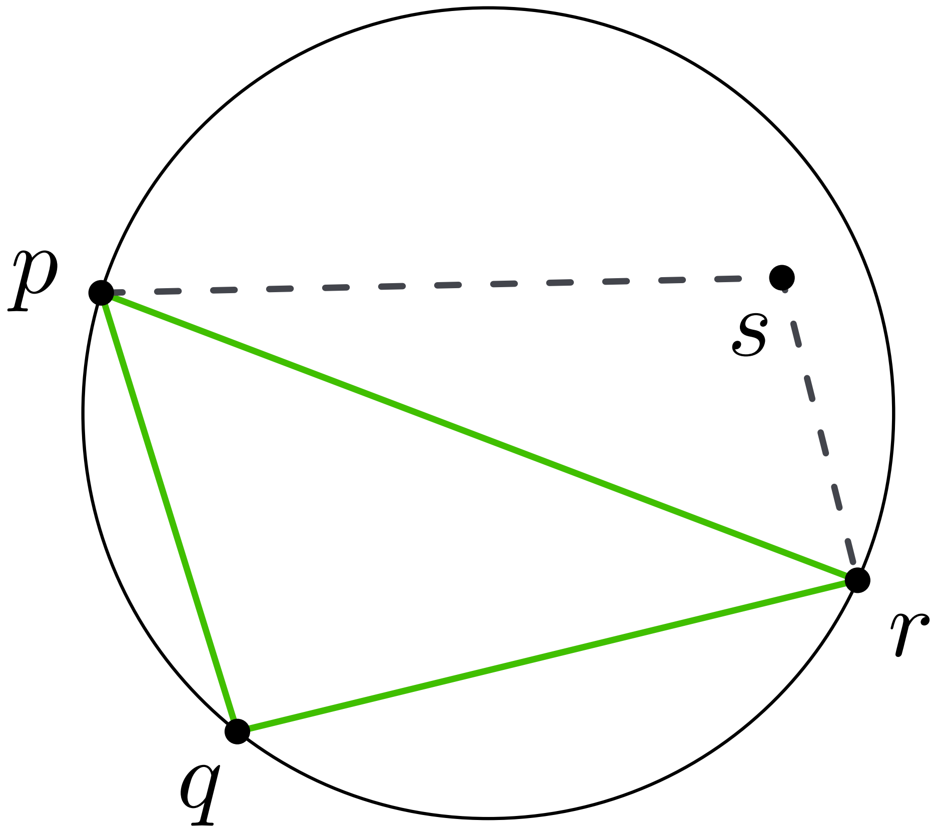

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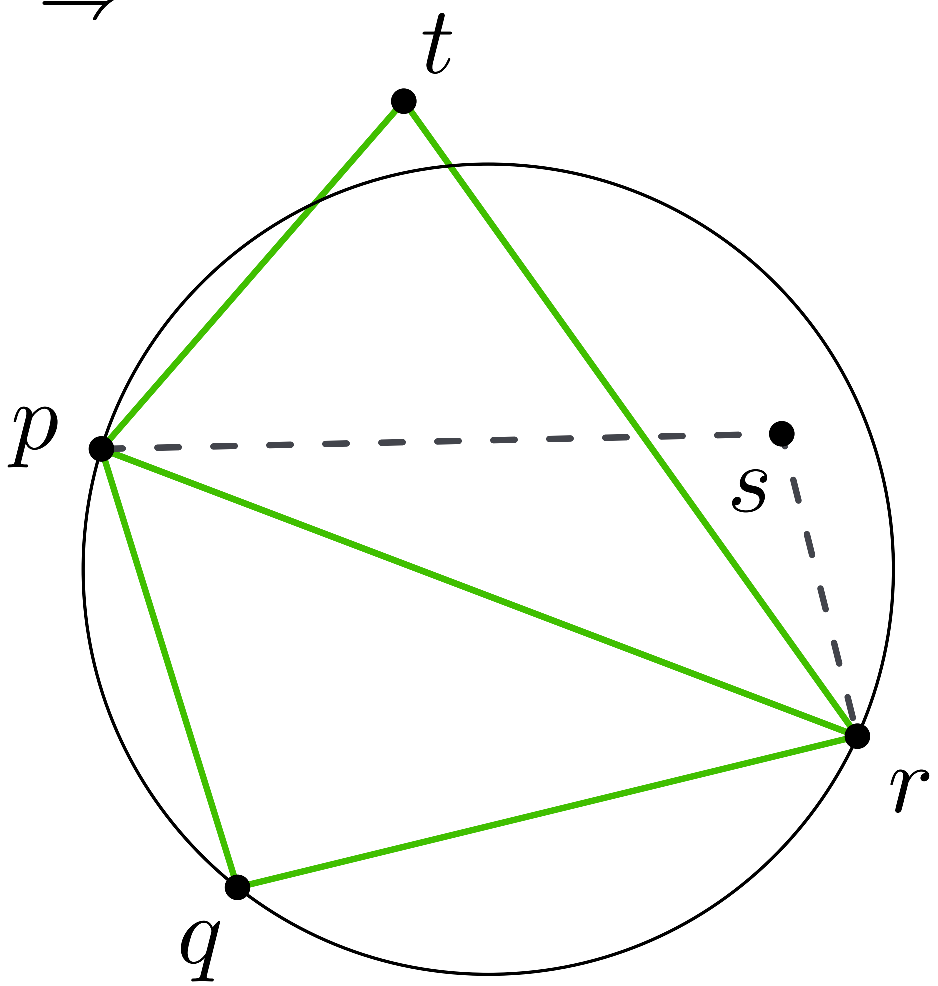

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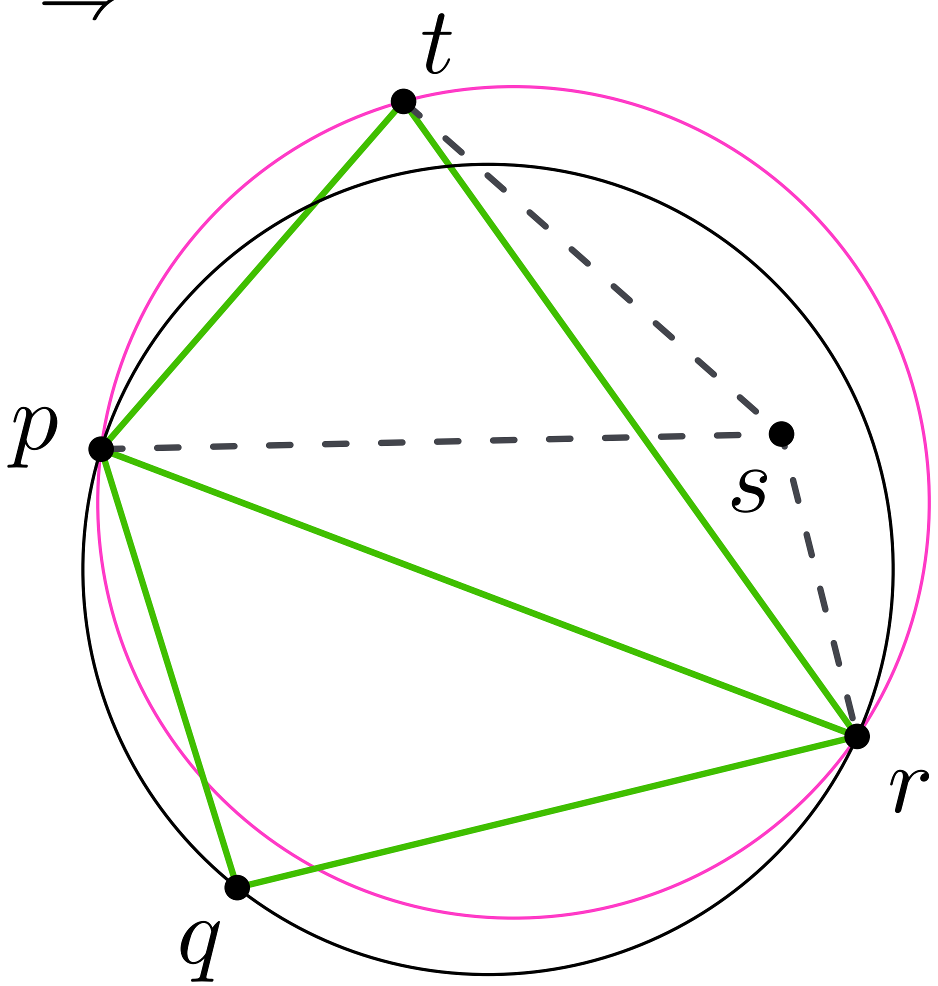

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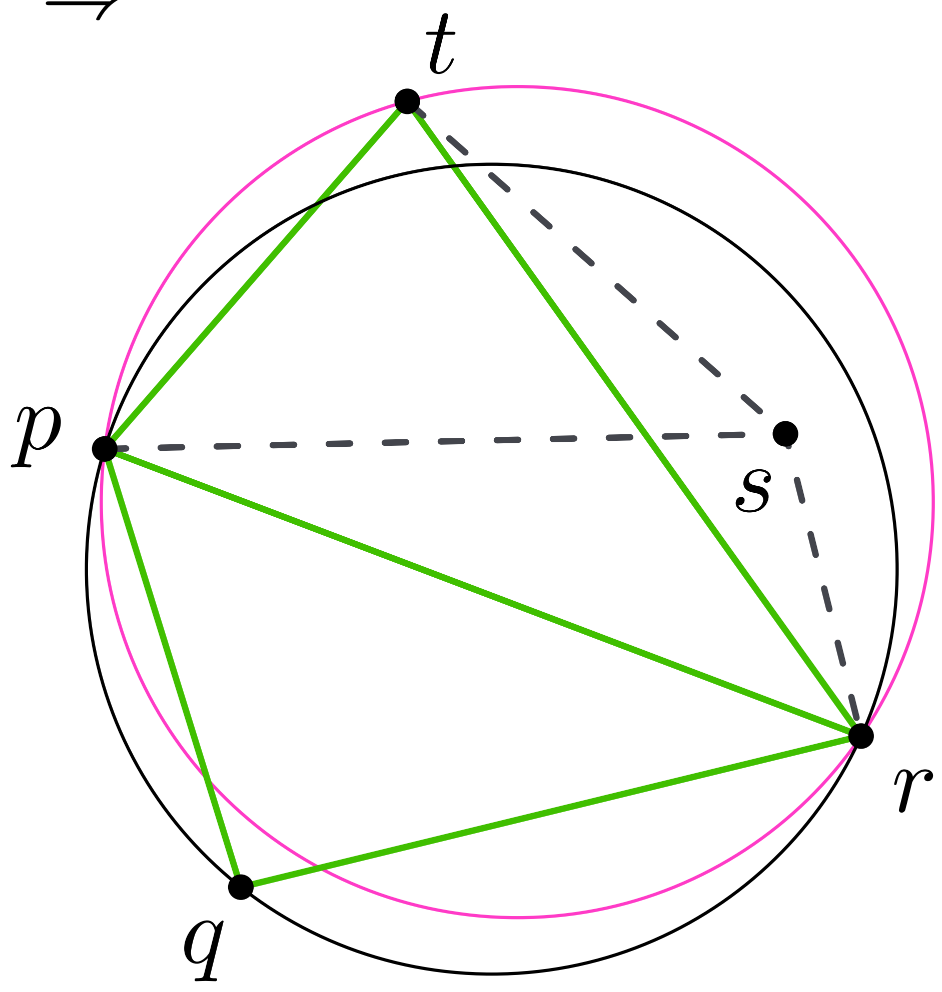

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- Thales theorem: $\angle tsr > \angle psr$
- *Contradiction to choice of* $\overline{pr}$

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Observation: If P is in general position, then the Delaunay triangulation of P is unique.

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$\Rightarrow$ angle-optimal triangulation is $\mathcal{DG}(P)$!

Quiz

In general position: legal $\Leftrightarrow$ Delaunay $\Leftrightarrow$ angle-optimal

Question: If points are not in general position, are Delaunay triangulations still angle optimal?

A: yes, but the proof is more complicated

B: no, but the smallest angle is still maximized

C: no, not even the smallest angle is maximized

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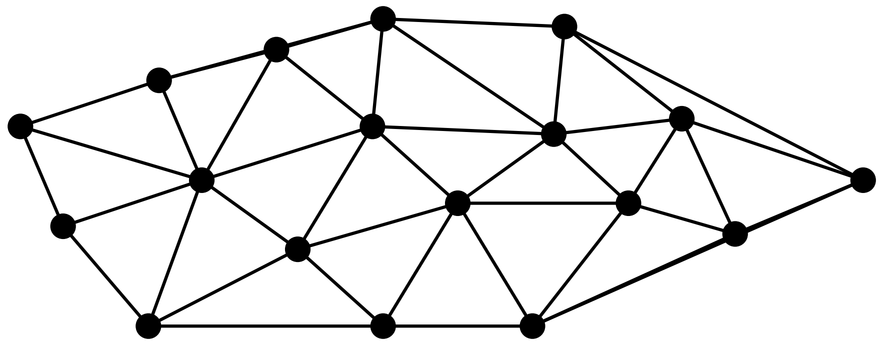
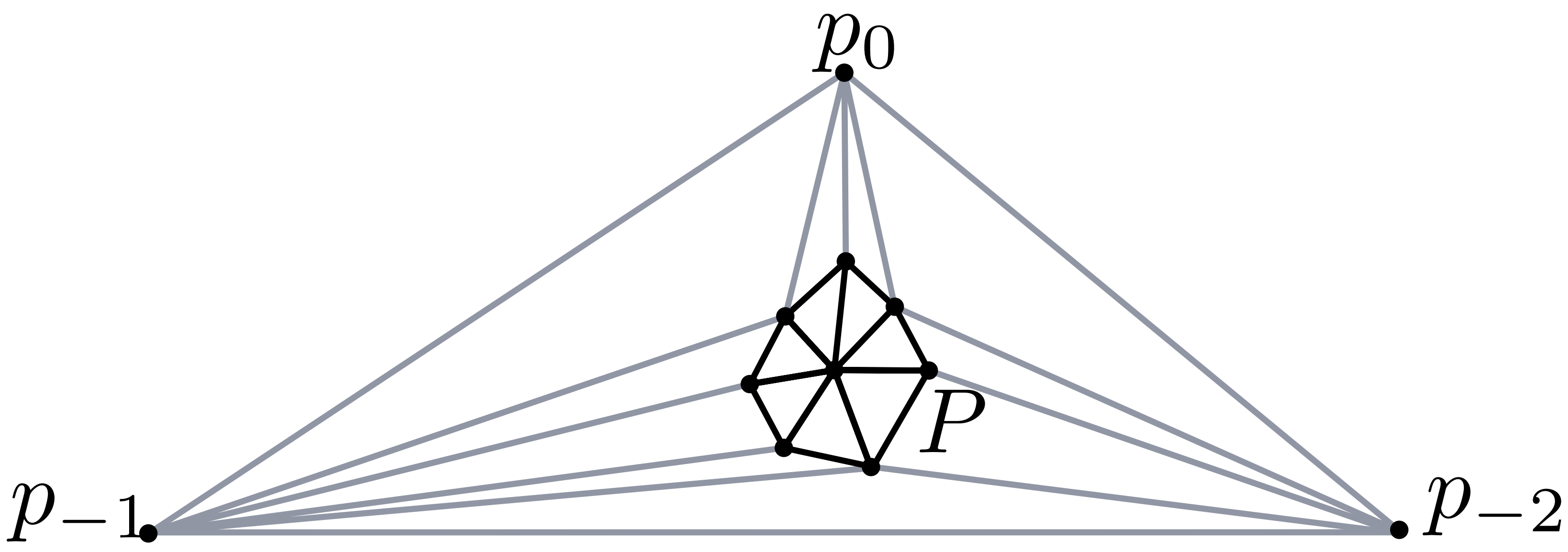
If P not in general position, then the smallest angle in every triangulation of the "large" faces in $\mathcal{DG}(P)$ is the same.
(proof uses Thales theorem)



Computing the Delaunay triangulation

Randomized Incremental Construction using Edge Flips

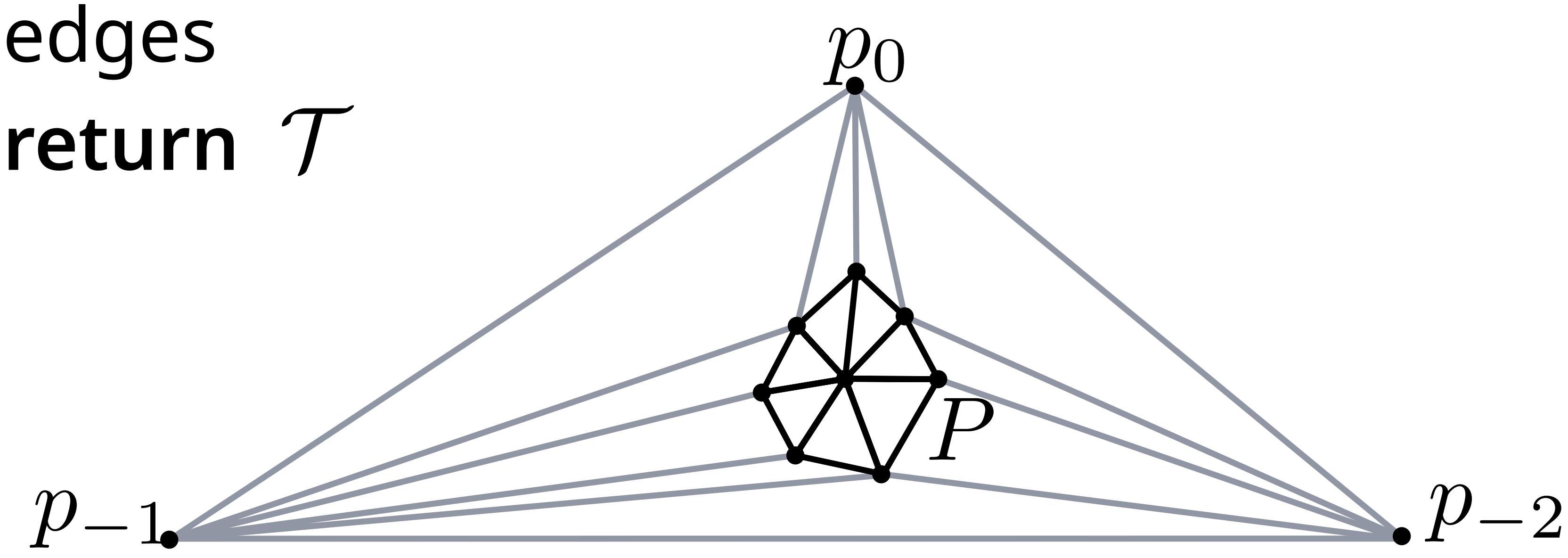
Randomized incremental construction



Randomized incremental construction

Algorithm DELAUNAYTRIANGULATION(P)

1: Initialize $\mathcal{T}$ as a large triangle $\triangle p_0 p_{-1} p_{-2}$ containing all points from P
2: Compute a random permutation of $p_1, \ldots, p_n$
3: **for** $r \leftarrow 1$ **to** n **do**
4: INSERT$(p_r, \mathcal{T})$
5: Discard p_0, p_{-1} and p_{-2} with all their incident edges
6: **return** $\mathcal{T}$



Randomized incremental construction

Algorithm INSERT$(p_r, \mathcal{T})$

1: find triangle $\in \mathcal{T}$ containing p_r

2: **if** p_r lies in $\triangle p_i p_j p_k$ **then**

3: add edges from p_r to p_i, p_j, p_k

4: LEGALIZEEDGE$(p_r, \overline{p_i p_j}, \mathcal{T})$

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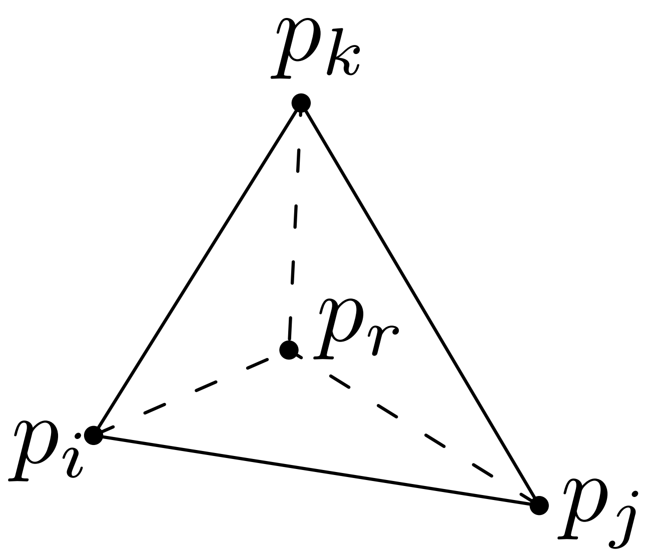

Randomized incremental construction

Algorithm $\textsc{Insert}(p_r, \mathcal{T})$

find triangle $\in \mathcal{T}$ containing p_r
if p_r lies in $\triangle p_i p_j p_k$ **then**
    add edges from p_r to p_i, p_j, p_k
    $\textsc{LegalizeEdge}(p_r, \overline{p_i p_j}, \mathcal{T})$
    $\textsc{LegalizeEdge}(p_r, \overline{p_j p_k}, \mathcal{T})$
    $\textsc{LegalizeEdge}(p_r, \overline{p_k p_i}, \mathcal{T})$
else // p_r lies on edge $\overline{p_i p_j}$
    add edges from p_r to p_l, p_k
    $\textsc{LegalizeEdge}(p_r, \overline{p_i p_l}, \mathcal{T})$
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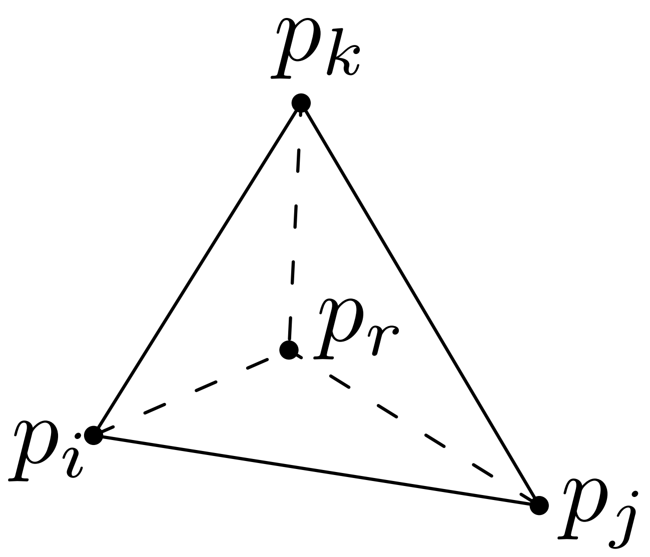
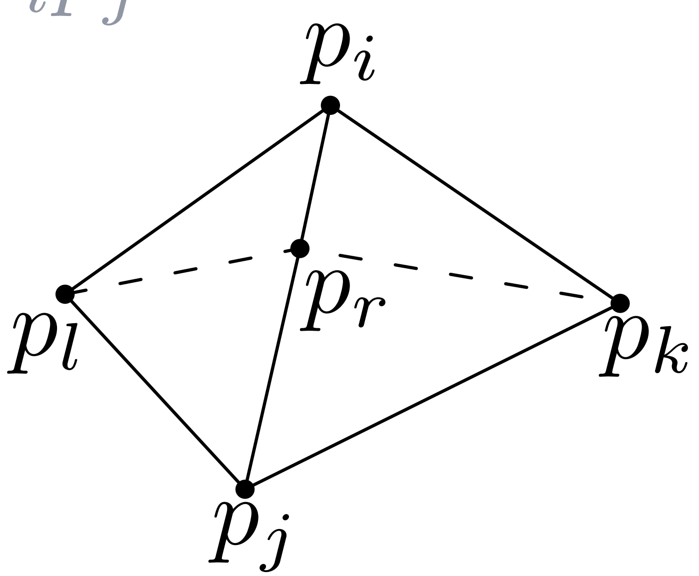
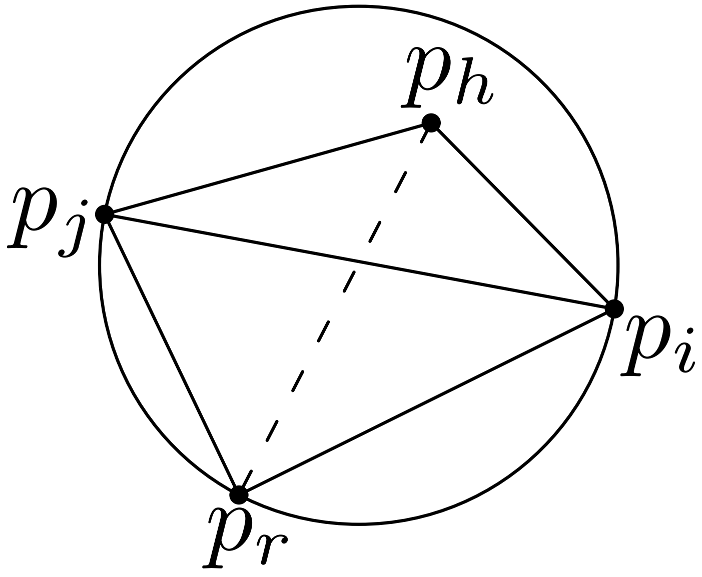




Randomized incremental construction

Algorithm LEGALIZEEDGE$(p_r, \overline{p_i p_j}, \mathcal{T})$

if $\overline{p_i p_j}$ is illegal **then** // Edge flip
    let $\triangle p_i p_j p_h$ be the adjacent triangle
    replace $\overline{p_i p_j}$ by $\overline{p_r p_h}$
    LEGALIZEEDGE$(p_r, \overline{p_j p_h}, \mathcal{T})$
    LEGALIZEEDGE$(p_r, \overline{p_h p_i}, \mathcal{T})$



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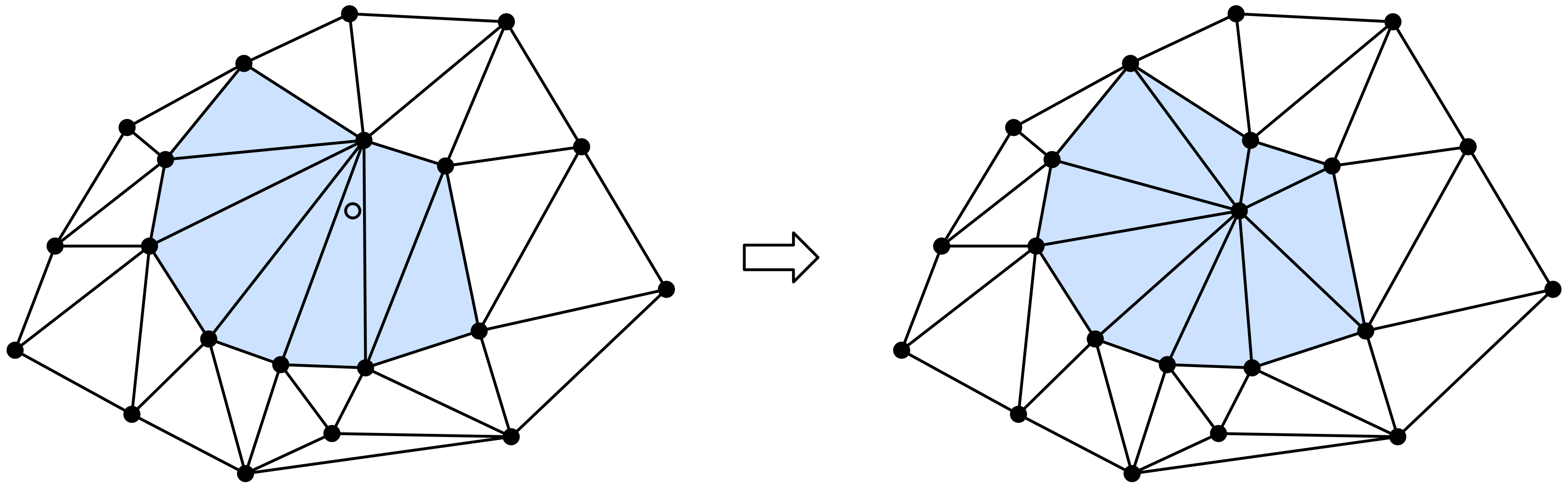

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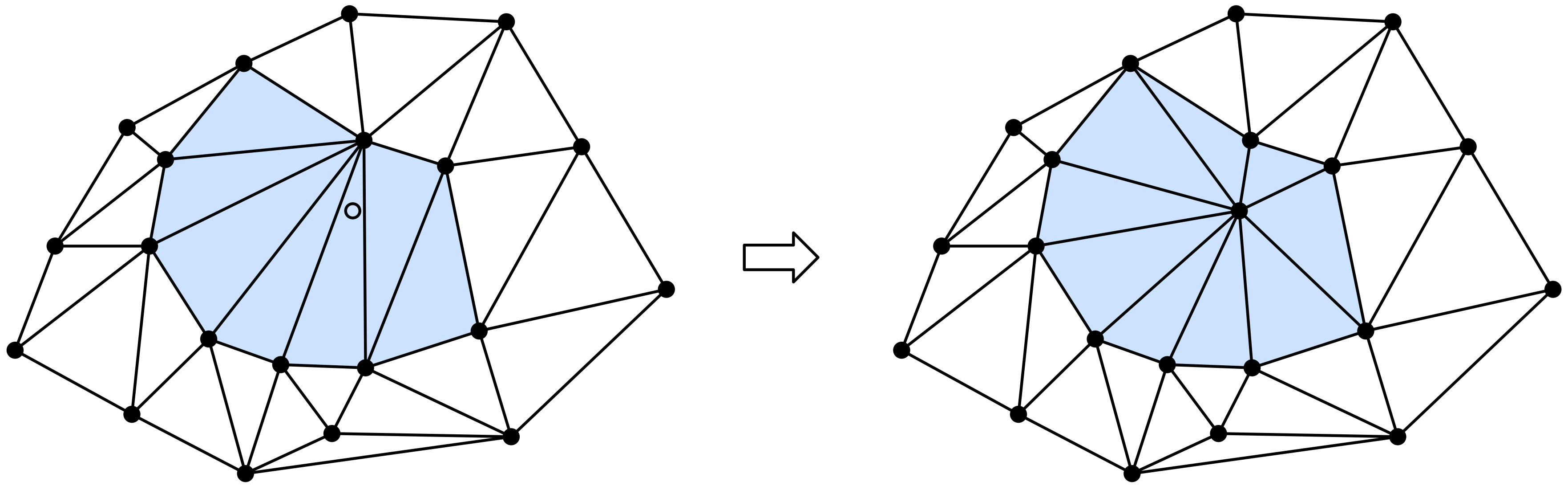
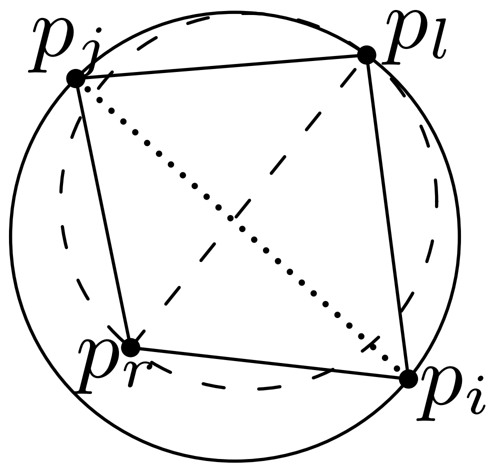


   - All edges inserted are Delaunay edges.



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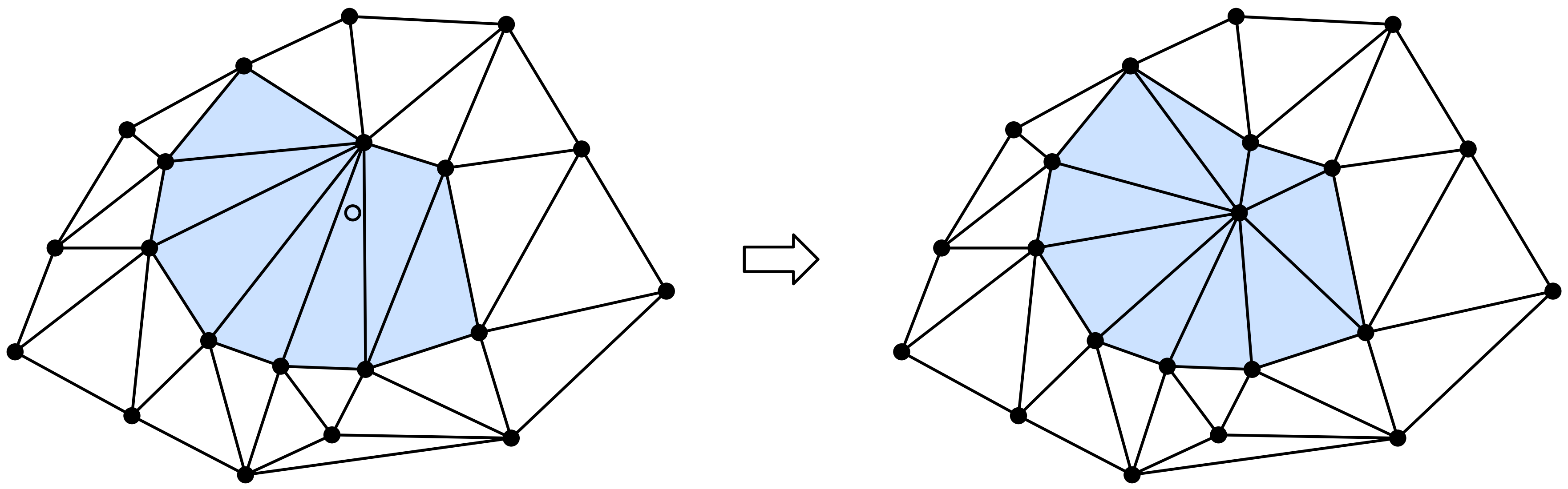

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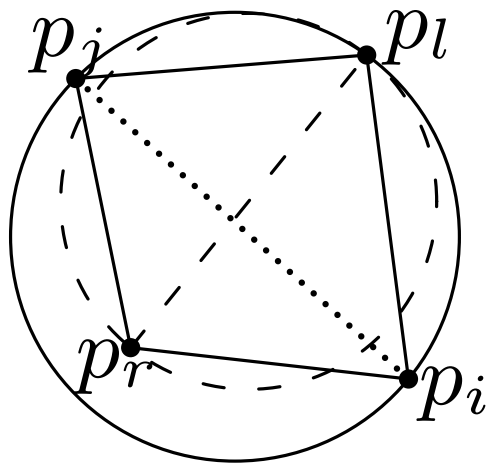
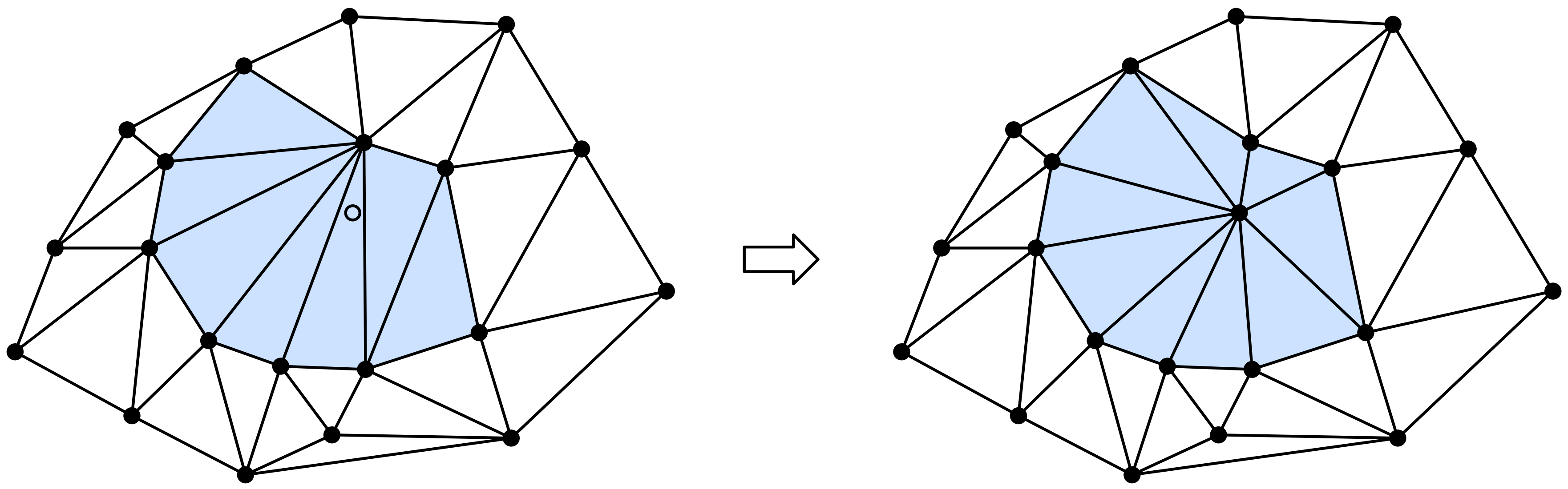


2. All other edges are legal
   - an edge can only be illegal if it is incident to a new triangle.

Quiz

If $deg(p_r) = k$ after inserting p_r (not on an edge), how many triangles were created during the insertion process?



A: 3

B: k

C: $2k - 3$

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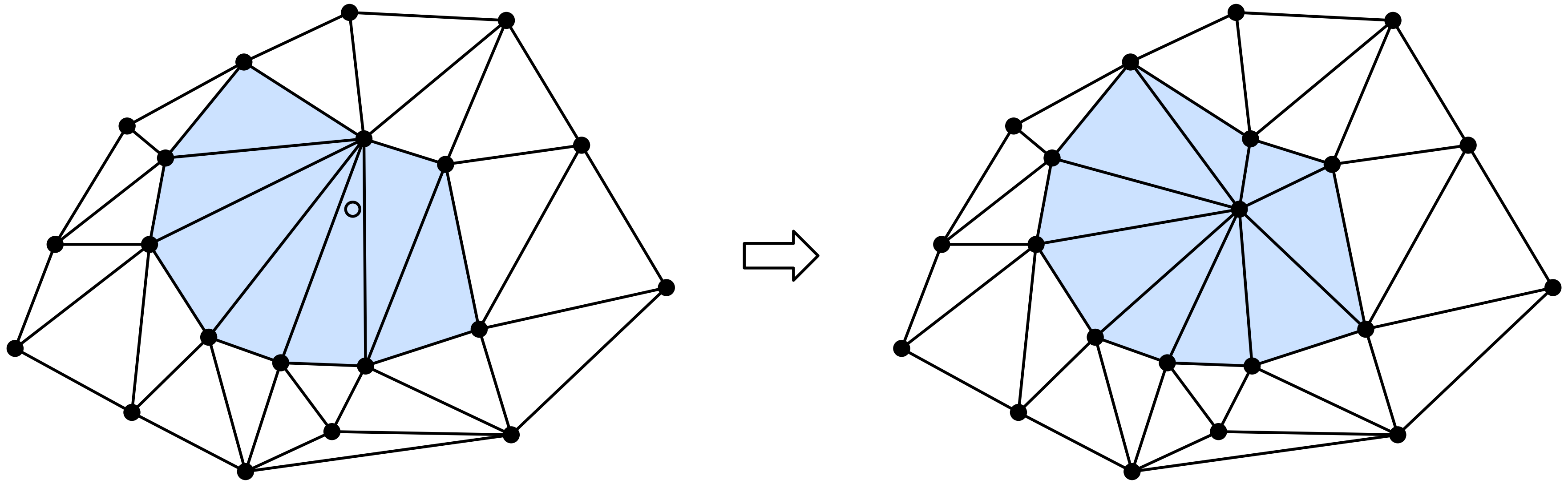

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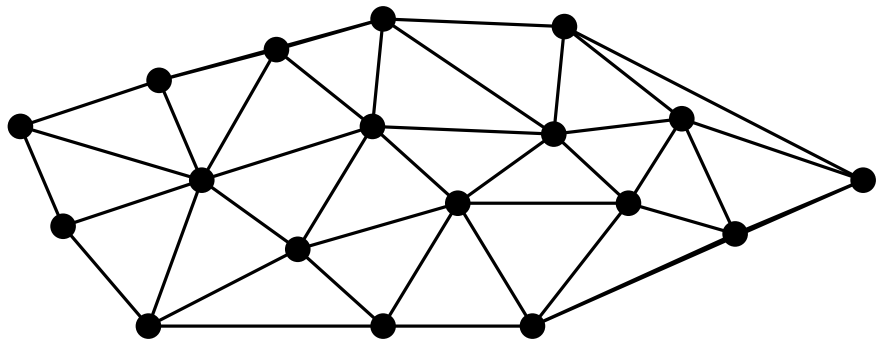
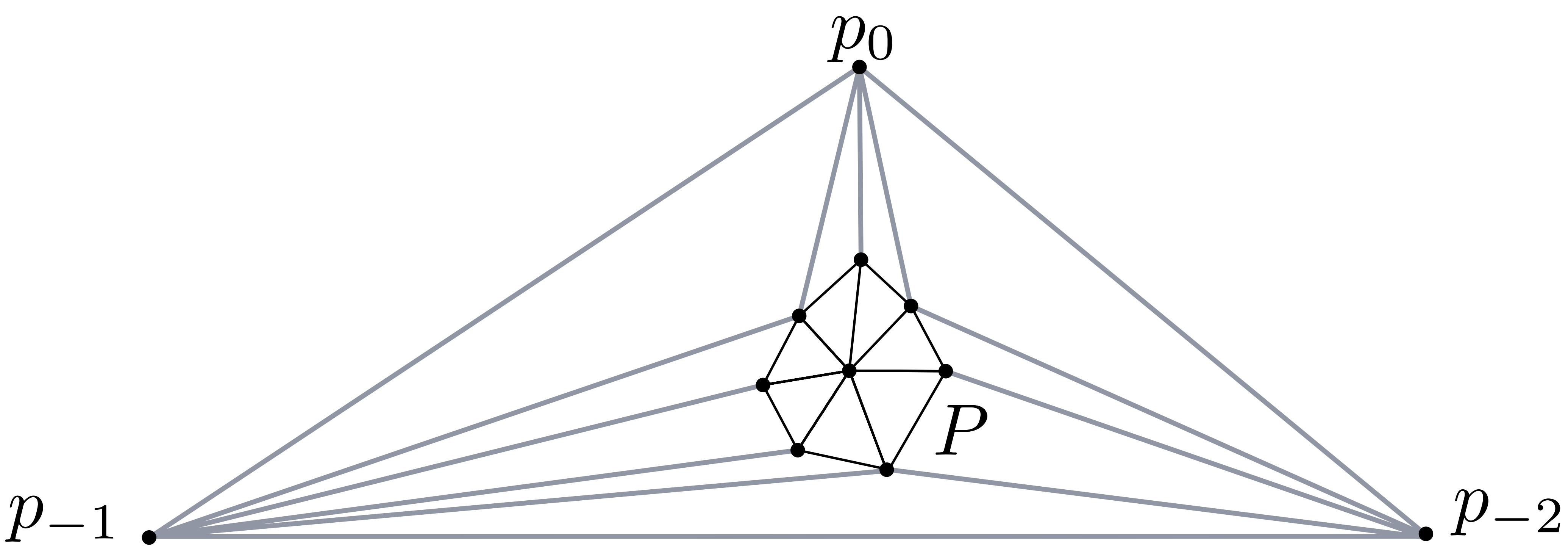


Computing the Delaunay triangulation

Search Structure and Analysis

Initialization

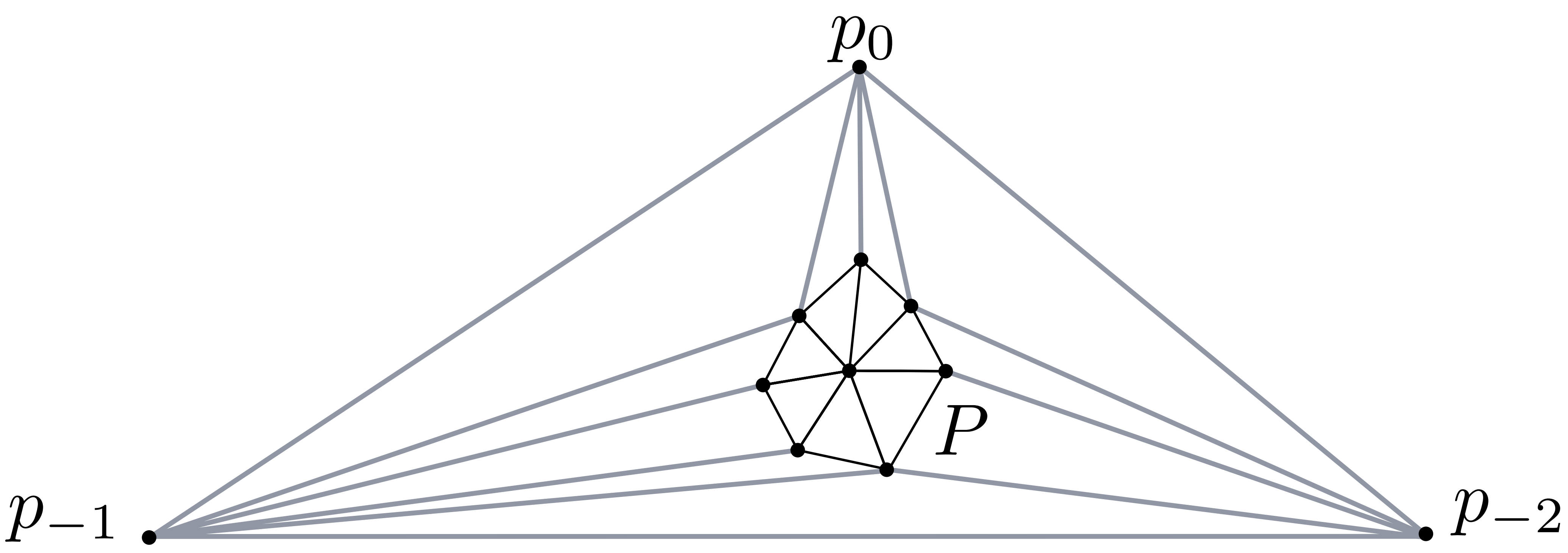
Choose p_0, p_{-1}, p_{-2} far enough away from P, such that they lie in none of the circles of P and such that P lies in their triangle.



Initialization

Better:

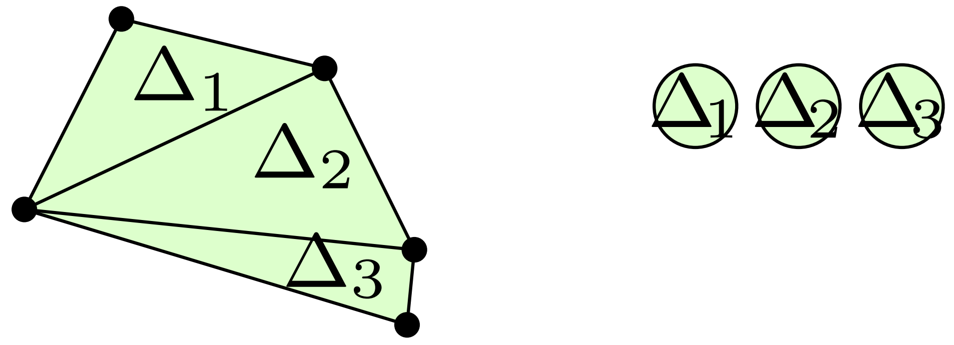
Treat p_0, p_{-1}, p_{-2} *symbolically* by modifying tests/predicates used for point location and testing illegal edges.



Search structure

Build search structure for point location: directed acyclic graph with

- leaves: current triangles
- inner nodes: deleted triangles



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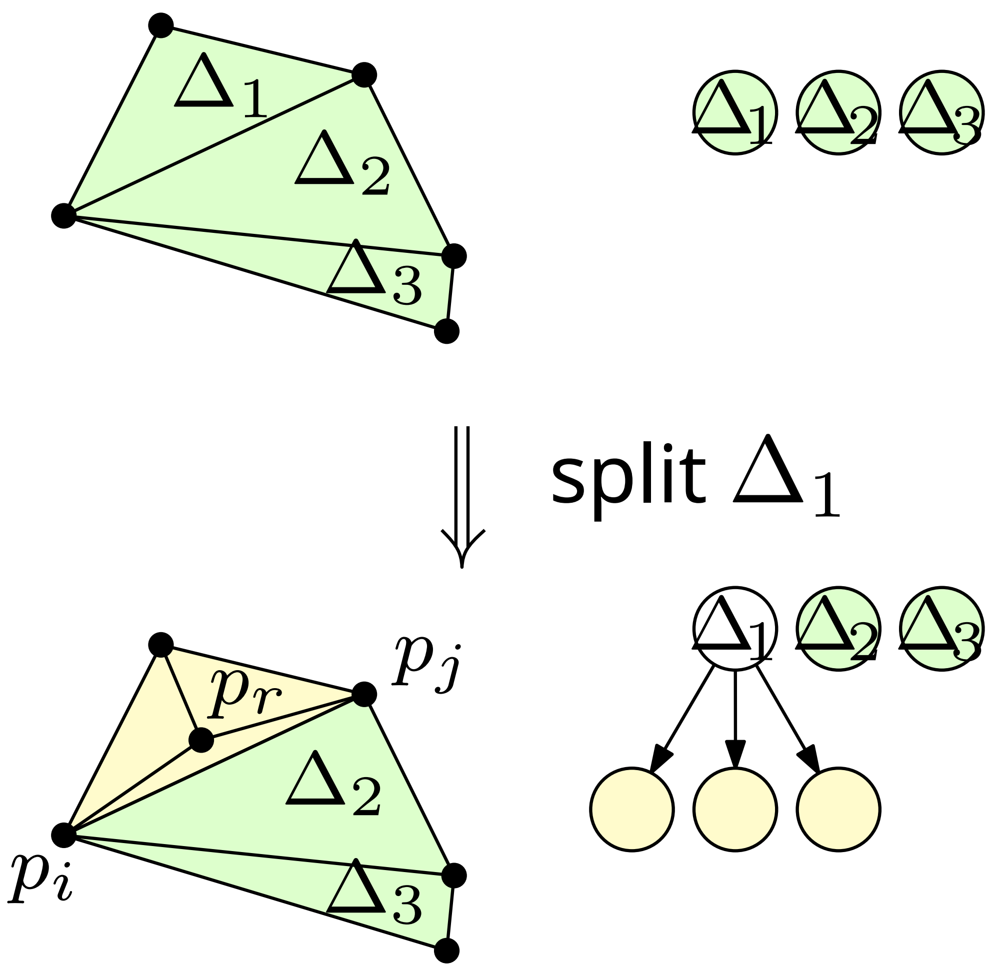

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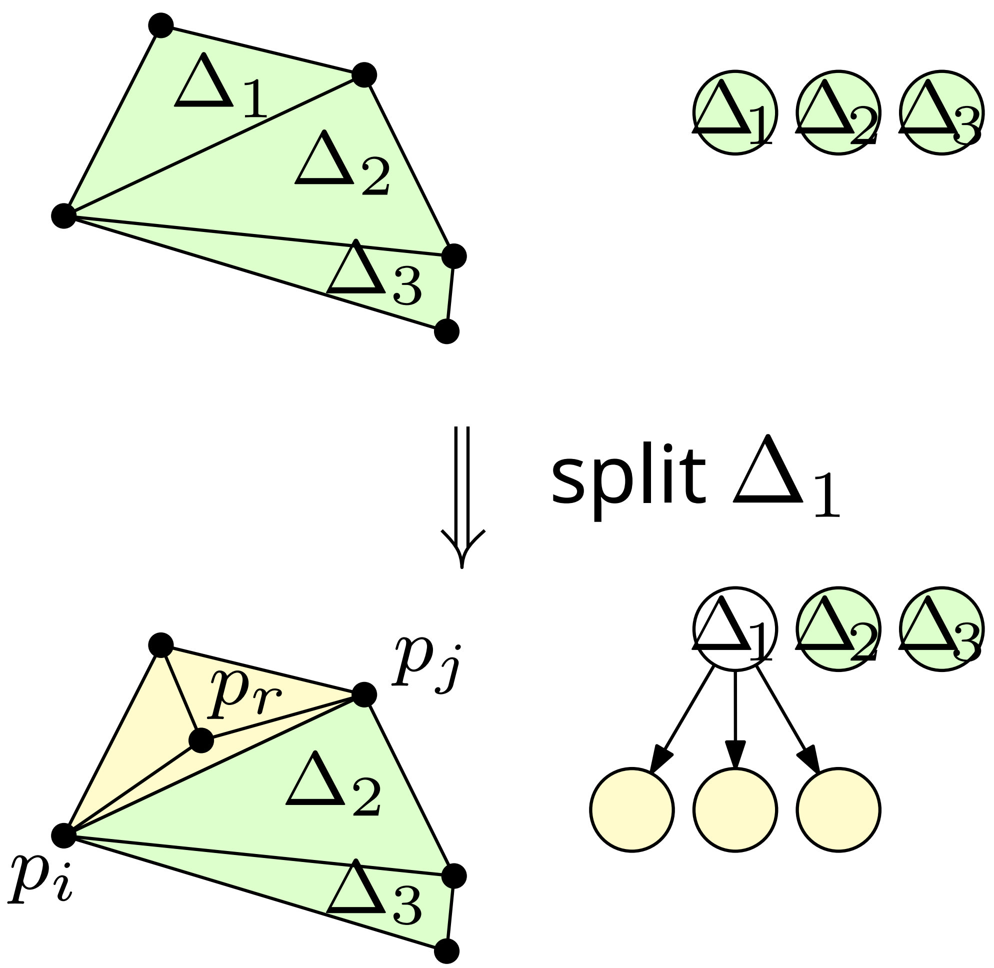

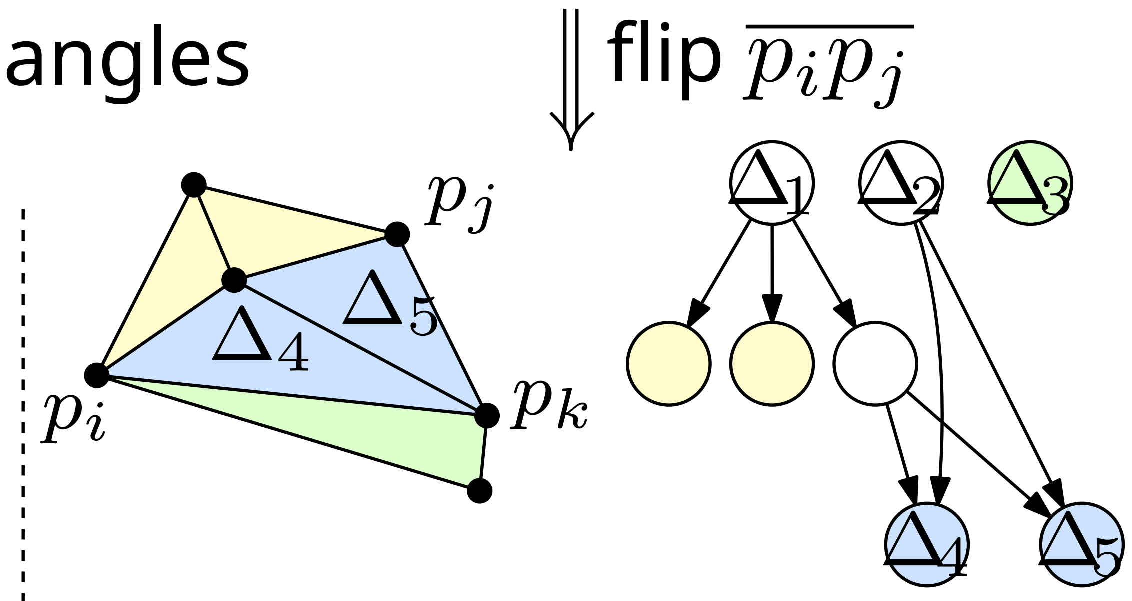

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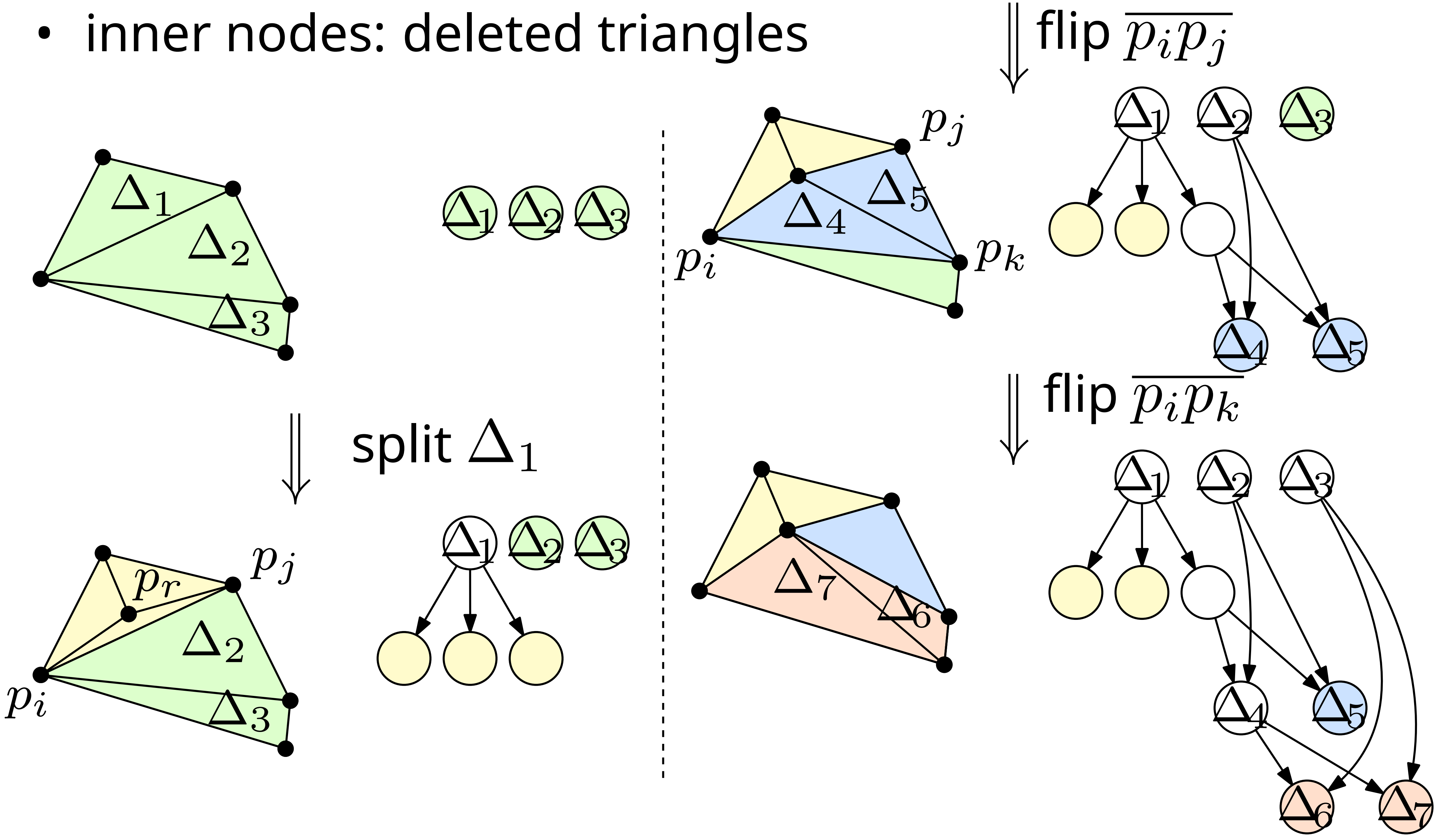

Analysis

Lemma: The expected number of triangles created is at most $9n + 1 = O(n)$.

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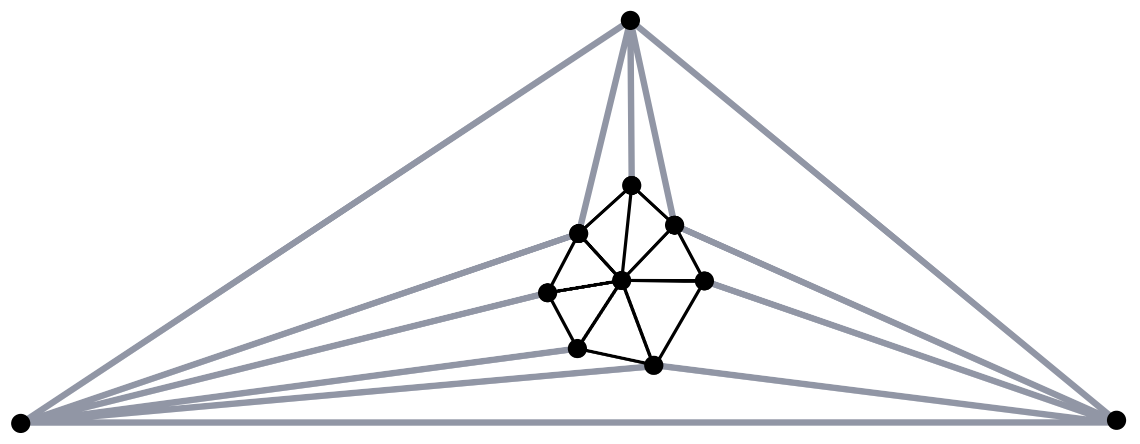
Proof: How many triangles are created when p_r is inserted?



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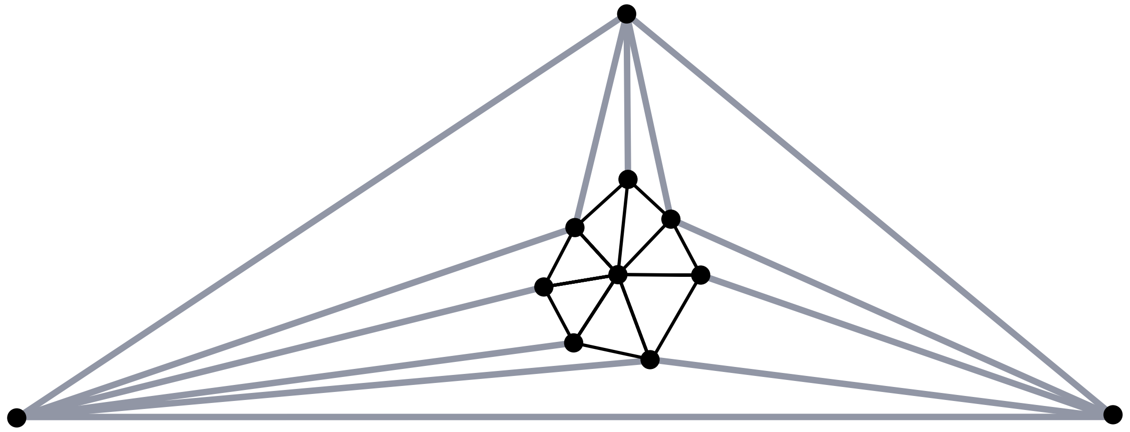

Backwards analysis:

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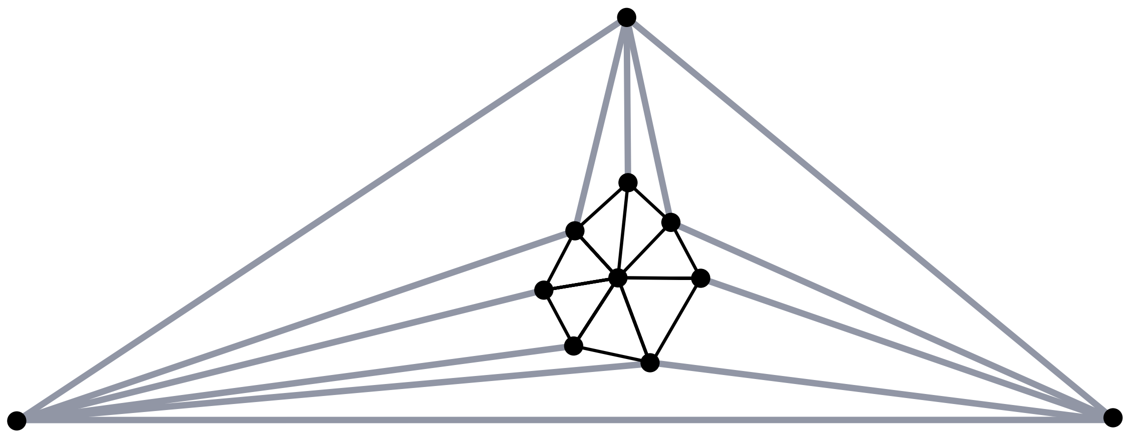

Backwards analysis:

- every point in $p_1, \ldots, p_r$ has the same probability $1/r$ to be the last point
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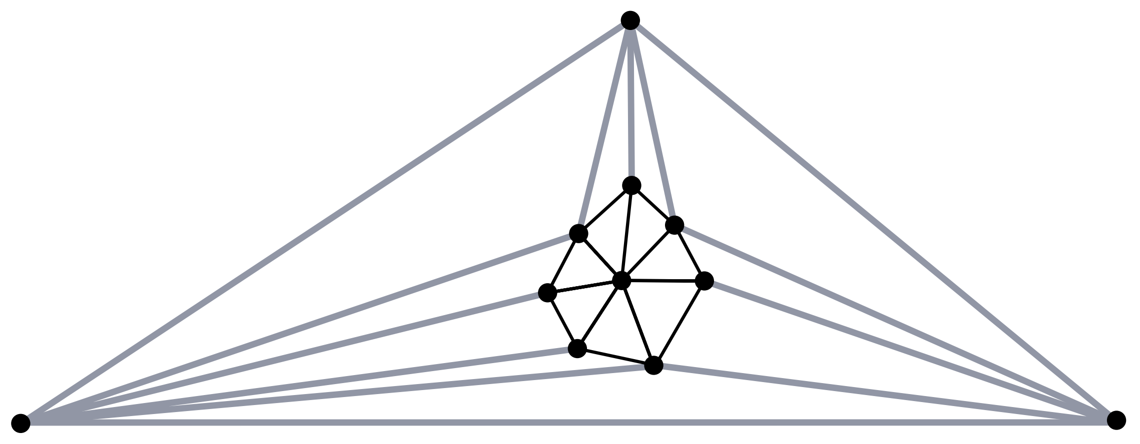

Backwards analysis:

- every point in $p_1, \ldots, p_r$ has the same probability $1/r$ to be the last point
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- number of triangles created at p_r is $\leq 2(degree(p_r)) - 3$

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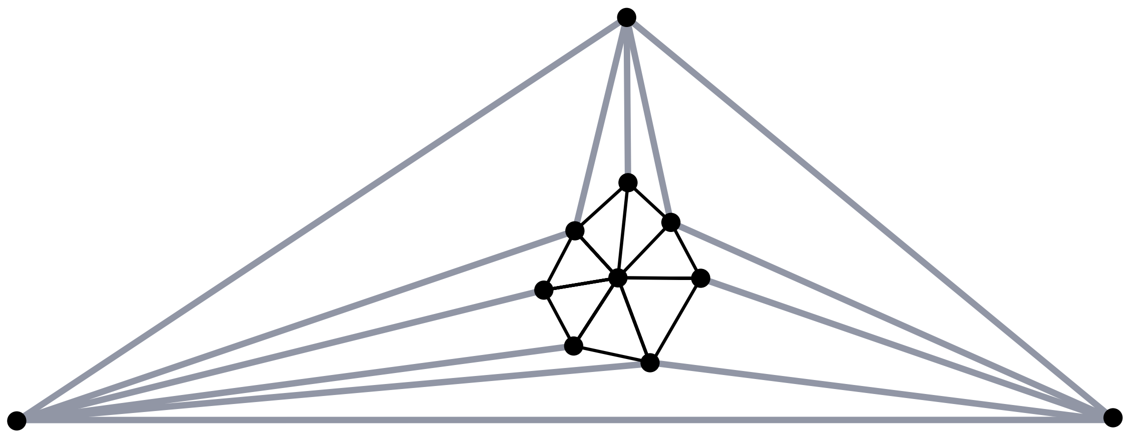
Proof: How many triangles are created when p_r is inserted?



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- overall $\leq 2 \cdot 6 - 3 = 9$; plus 1 for the outer triangle

Analysis

Lemma: The expected number of triangles created is at most $9n + 1 = O(n)$.

Lemma: The expected number of triangles which are visited in the search structure during the construction is $O(n \log n)$.

Proof in the book.

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Theorem: The Delaunay triangulation of n points can be computed in $O(n \log n)$ expected time using randomized incremental construction.

Quiz

How fast can we compute the Voronoi diagram of n points?

A: $\Theta(n)$ expected time

B: $\Theta(n \log n)$ expected time

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We can compute the Voronoi diagram in $O(n)$ time from the Delaunay triangulation

With other algorithmic paradigms (divide&conquer, sweepline) we can compute Delaunay triangulations and Voronoi diagrams also deterministically in $\Theta(n \log n)$ time.

Summary

Theorem: The Delaunay triangulation of n points can be computed in $O(n \log n)$ expected time using randomized incremental construction.

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