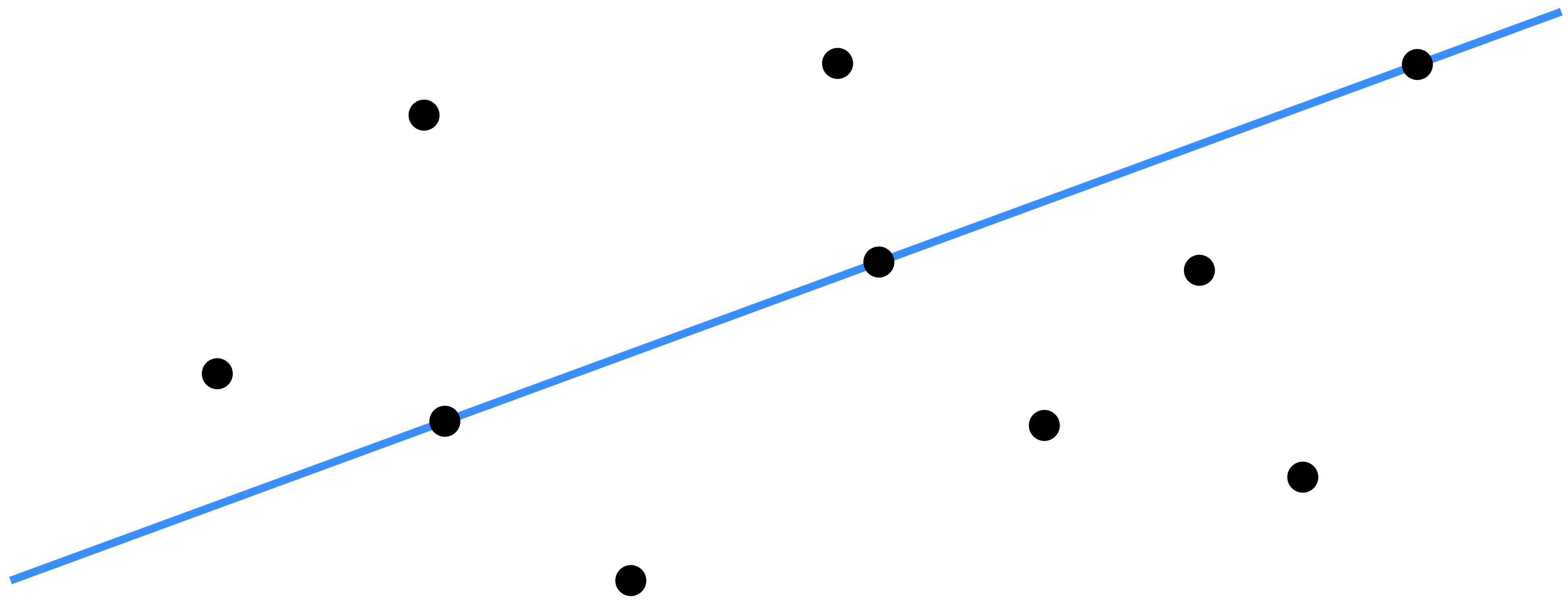
Arrangements and Duality

3 points on a line

Problem: Given a set P of n points in $\mathbb{R}^2$, determine whether there are three points on a line.



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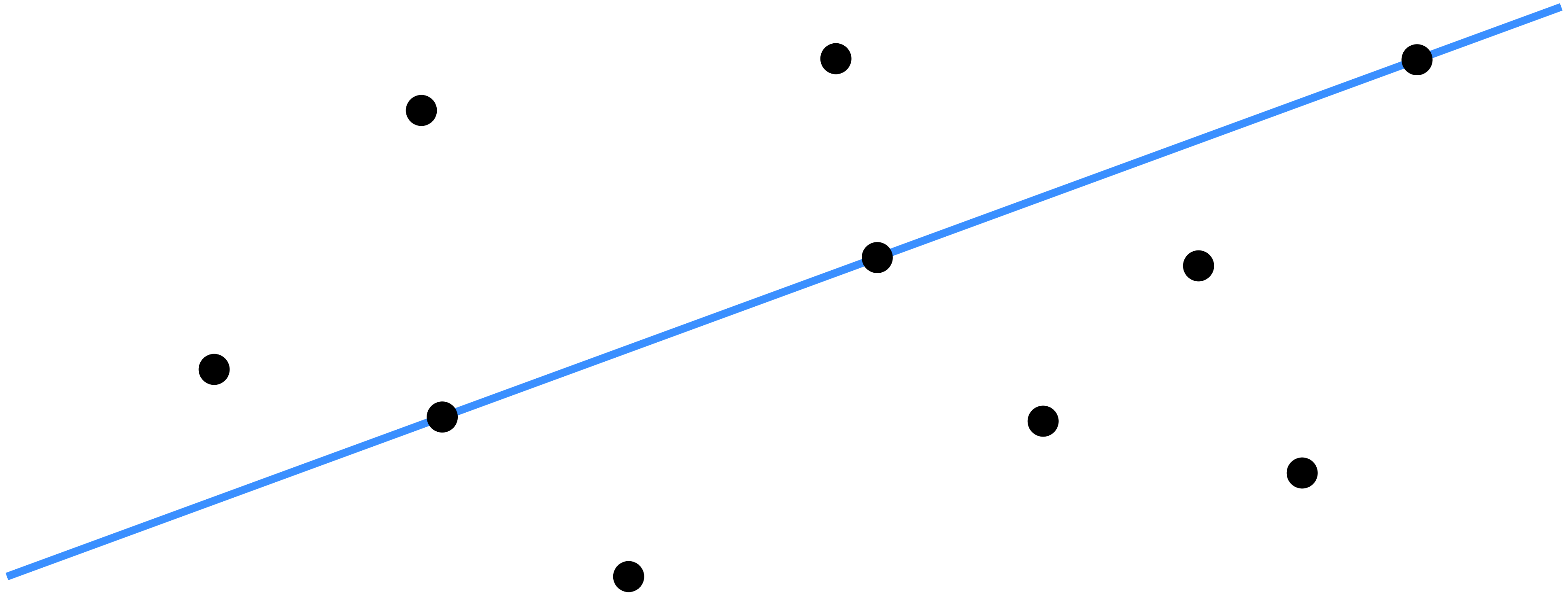
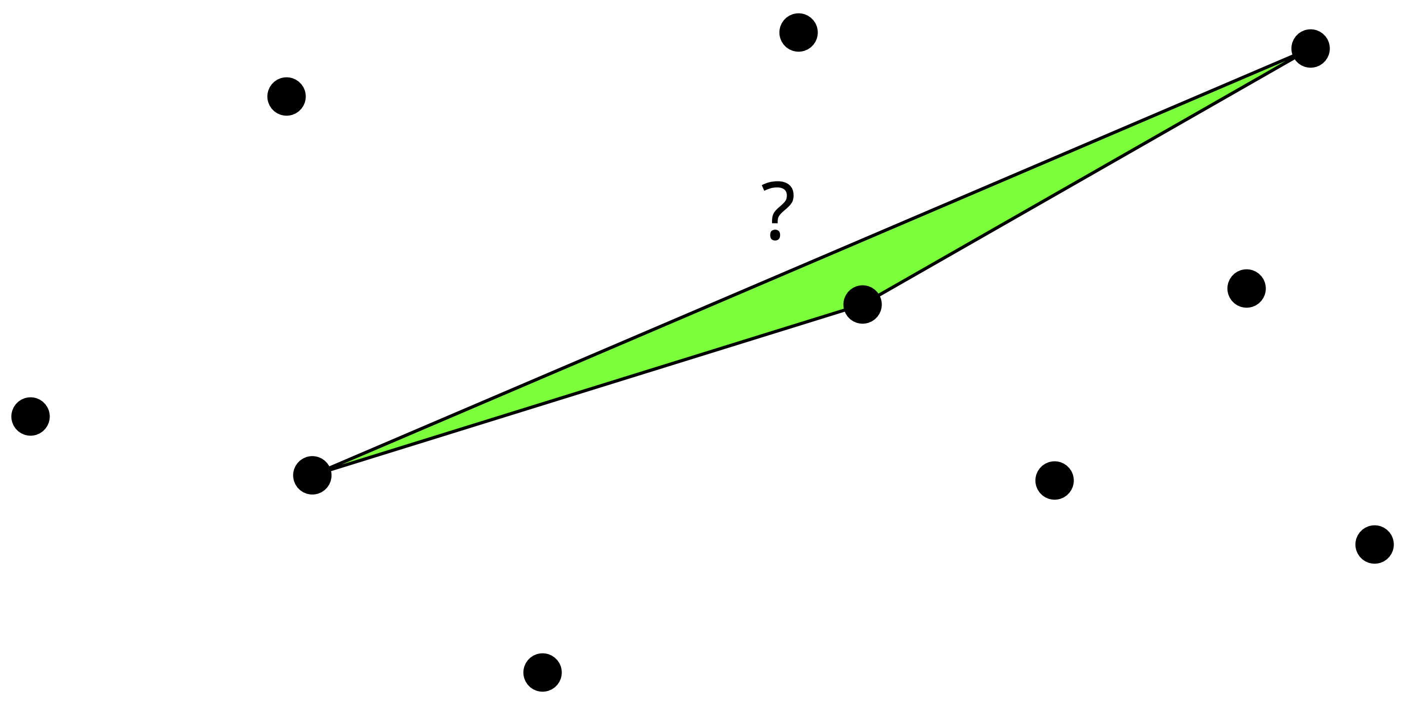
naive $O(n^3)$



Min-area triangle

Problem: Given a set P of n points in $\mathbb{R}^2$, find the smallest-area triangle with corners $p, q, r \in P$.



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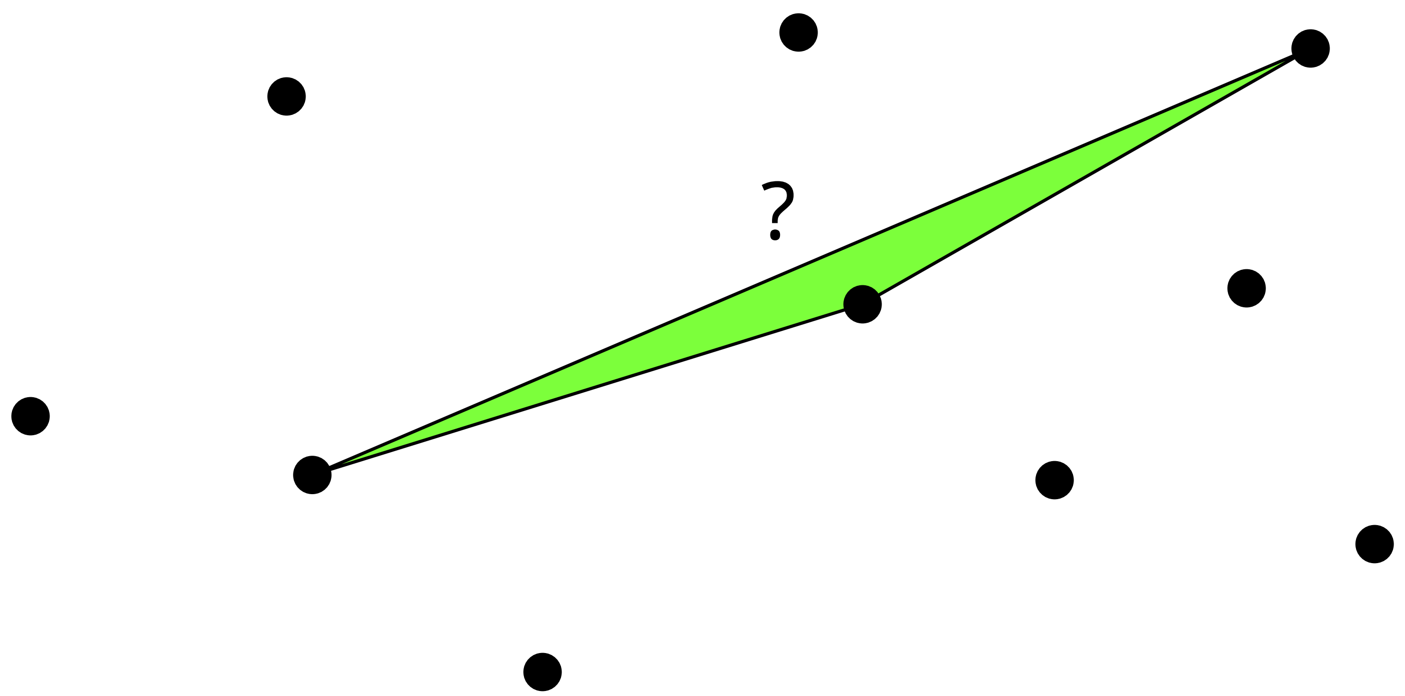
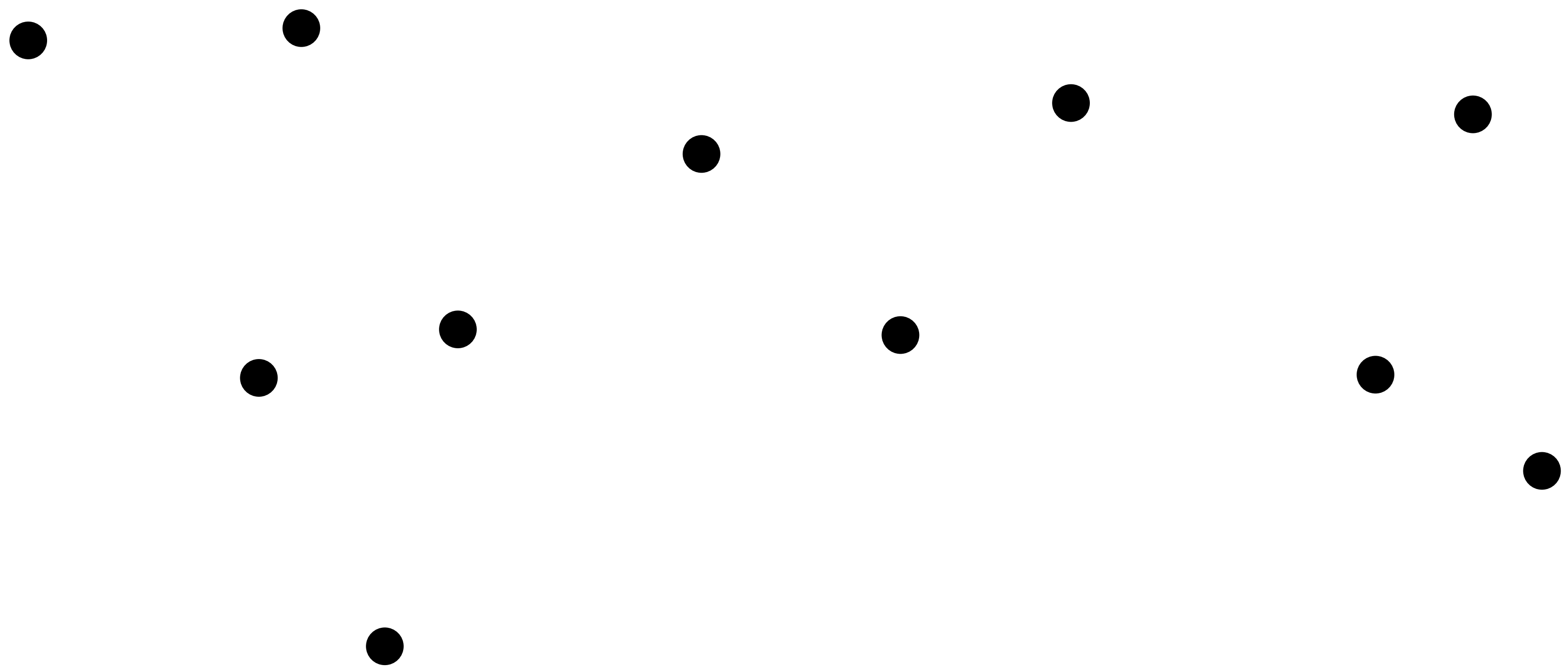


Angular sequences

Problem: Given a set P of n points in $\mathbb{R}^2$, compute for every p the angular order of all $p' \in P \setminus \{p\}$ around it.



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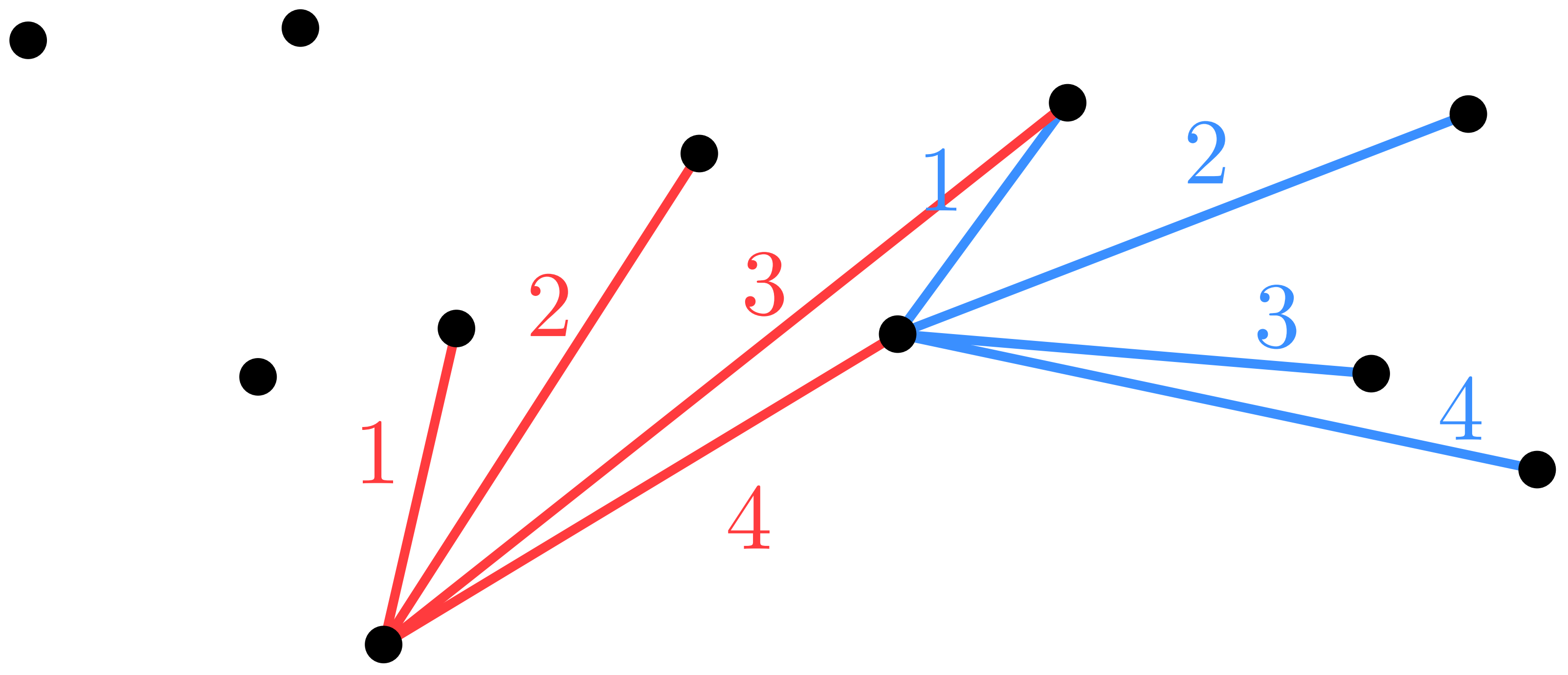
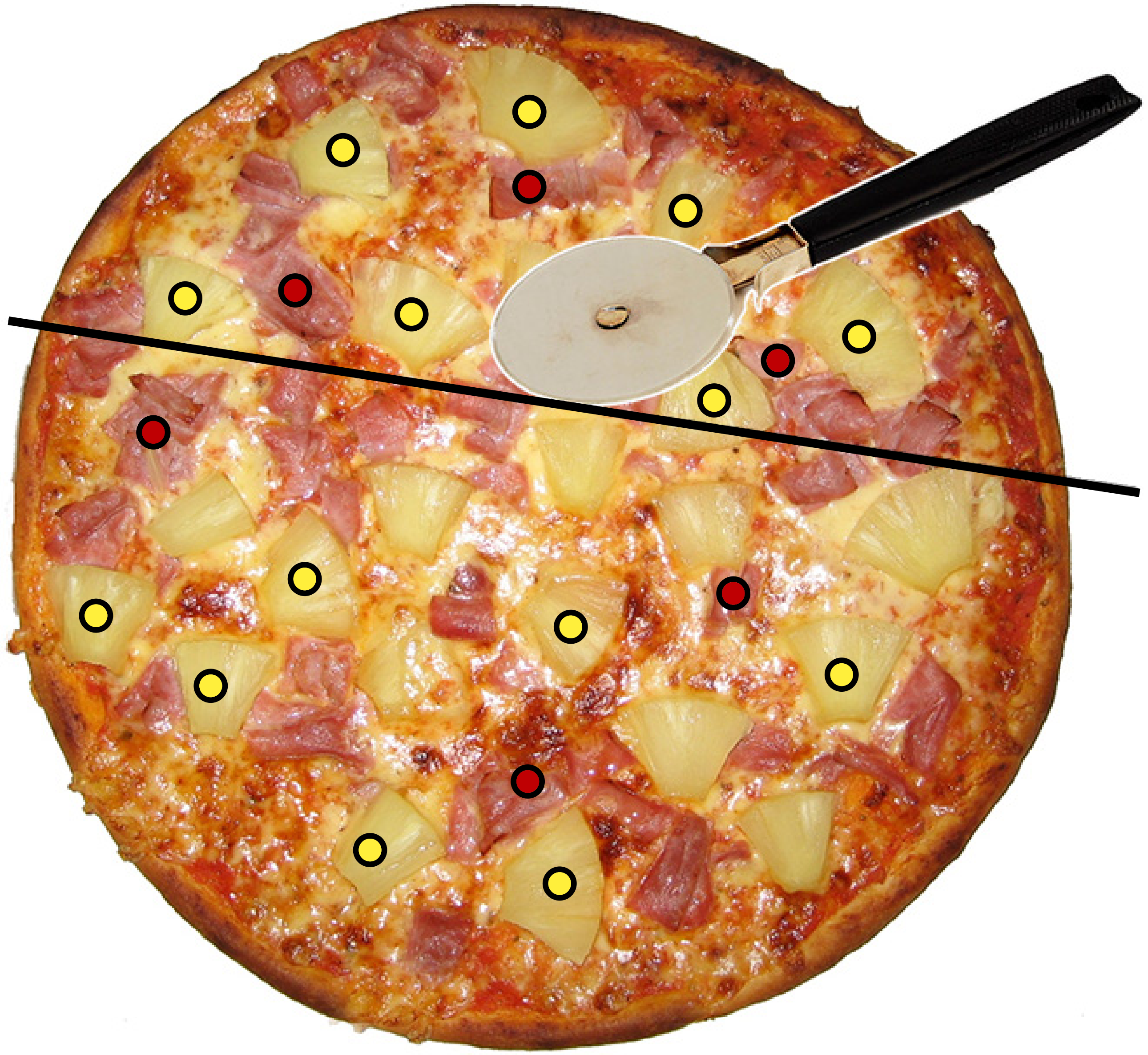


naive $O(n^2 \log n)$

Ham-sandwich cut

Problem: Alice and Bob want to share a Hawaiian pizza. Can they cut it with one straight cut so that each slice has half of the pineapple and half of the ham?



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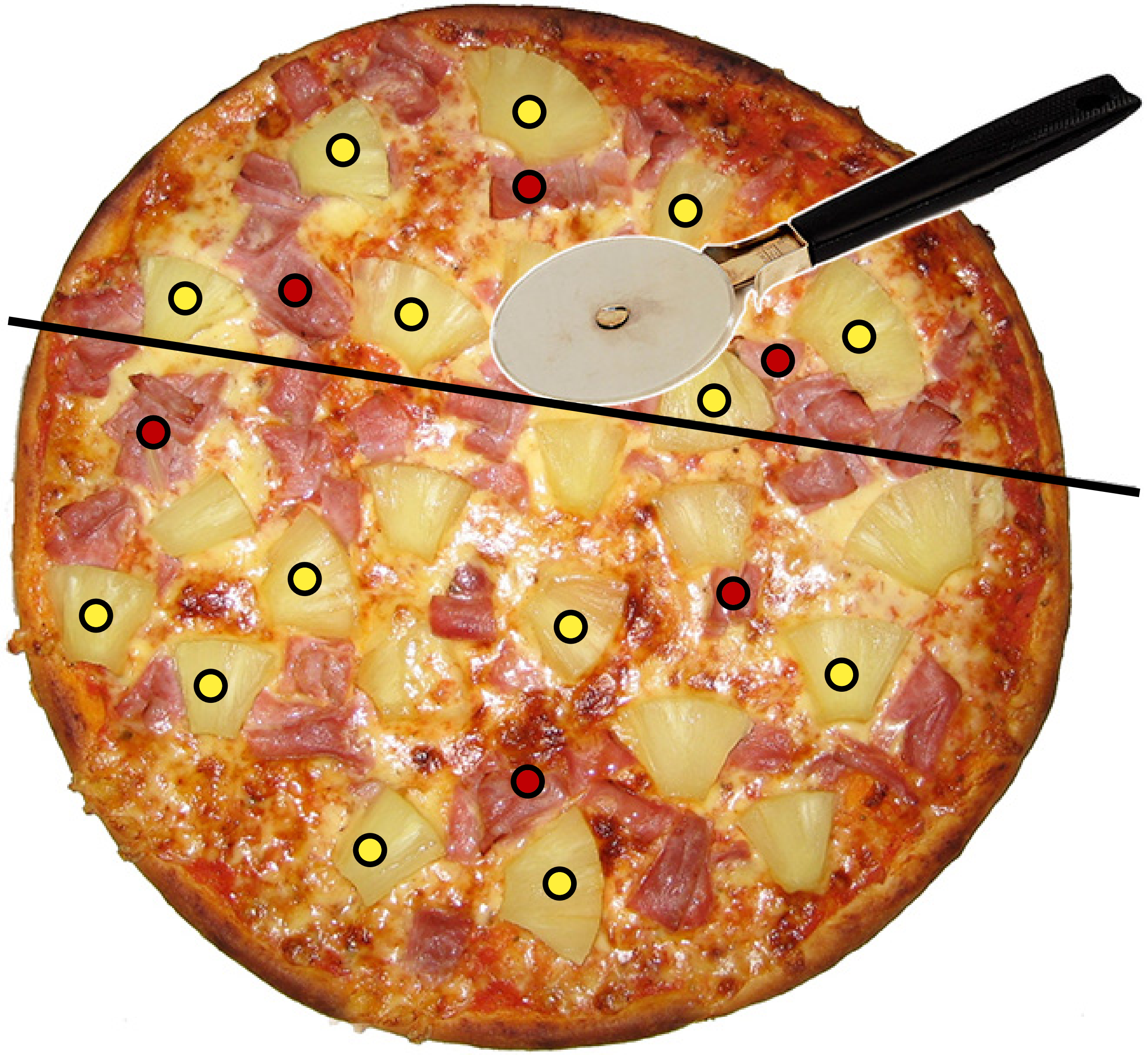

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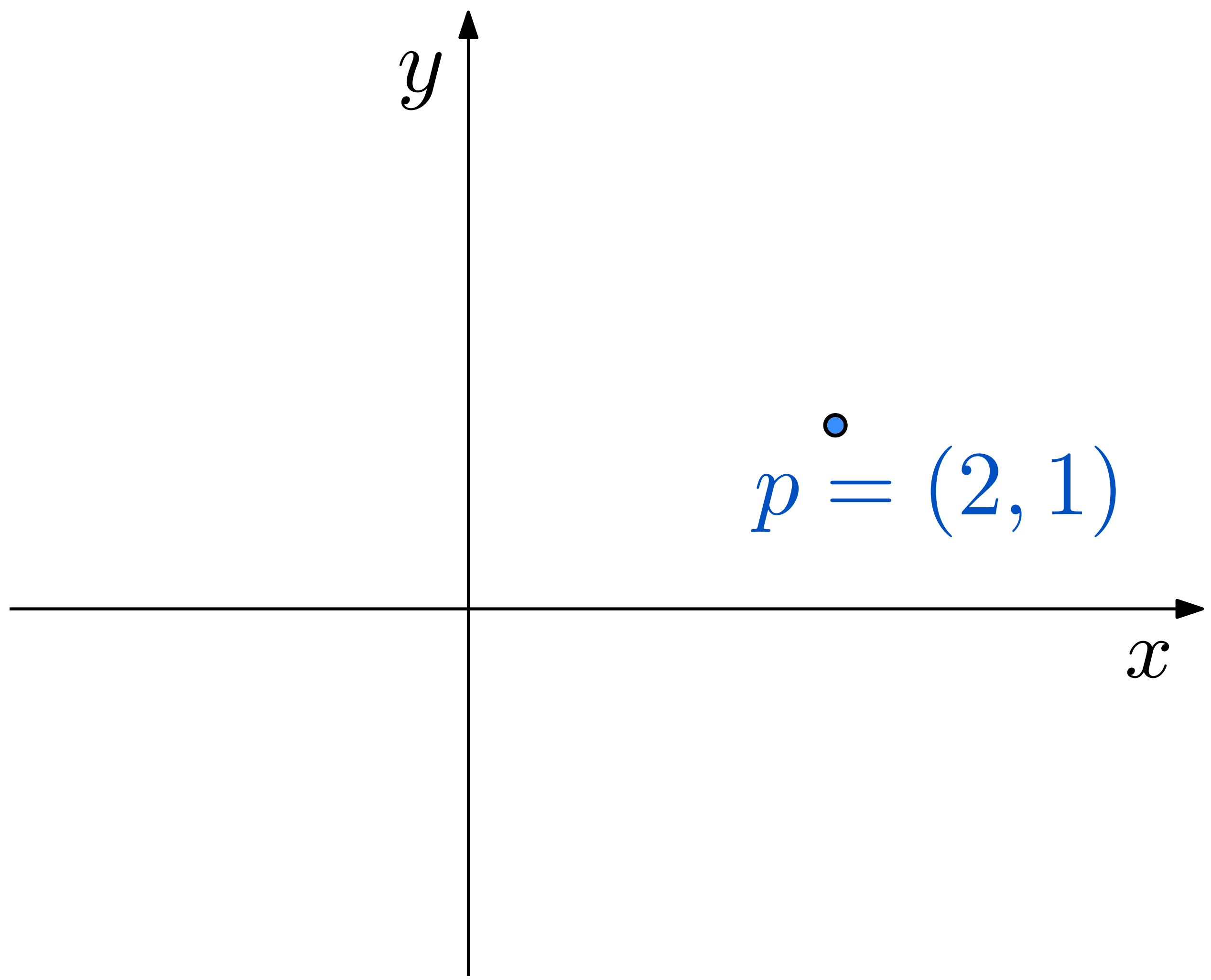
faster algorithms using duality and arrangements

Duality between points and lines



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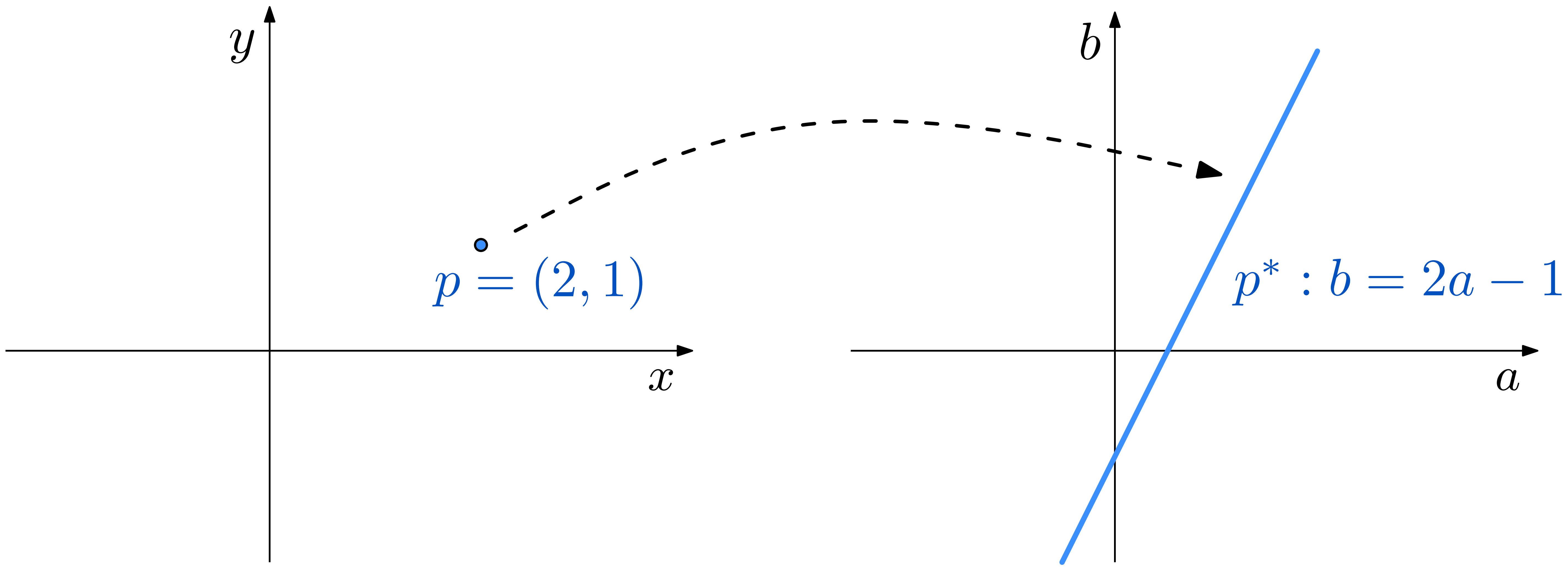

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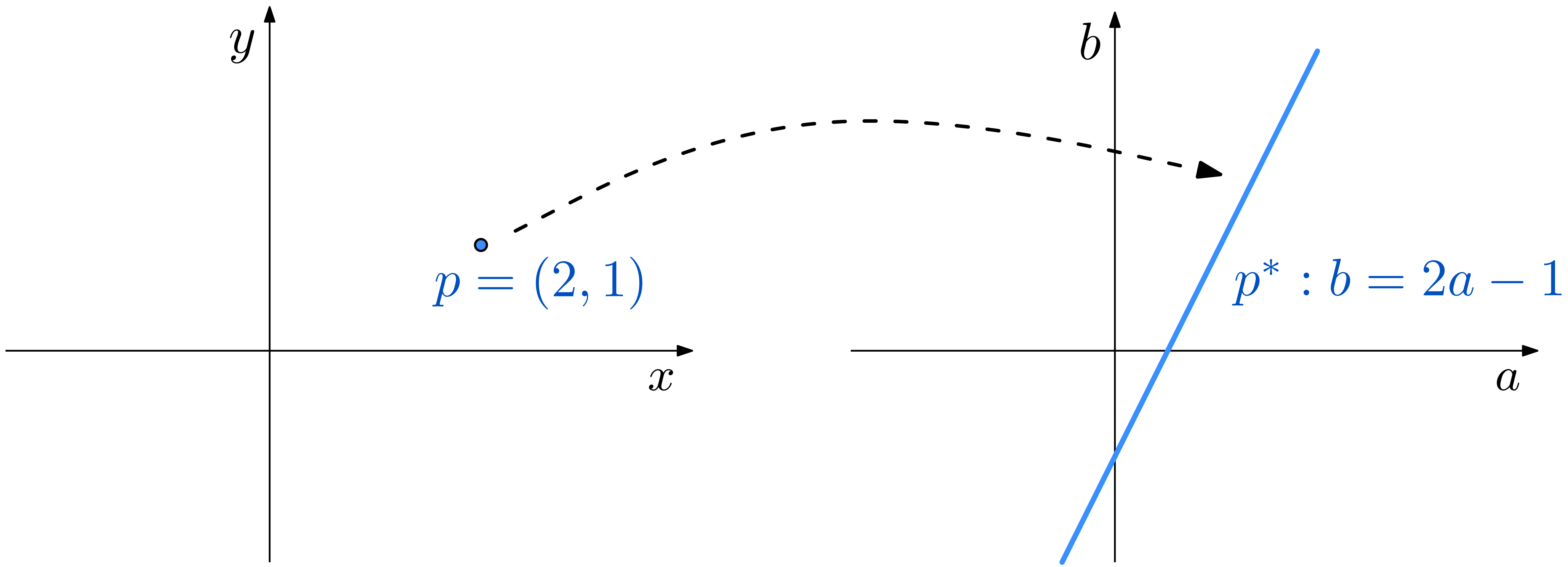

$$p = (p_x, p_y) \quad \mapsto \quad p^* : y = p_x x - p_y$$

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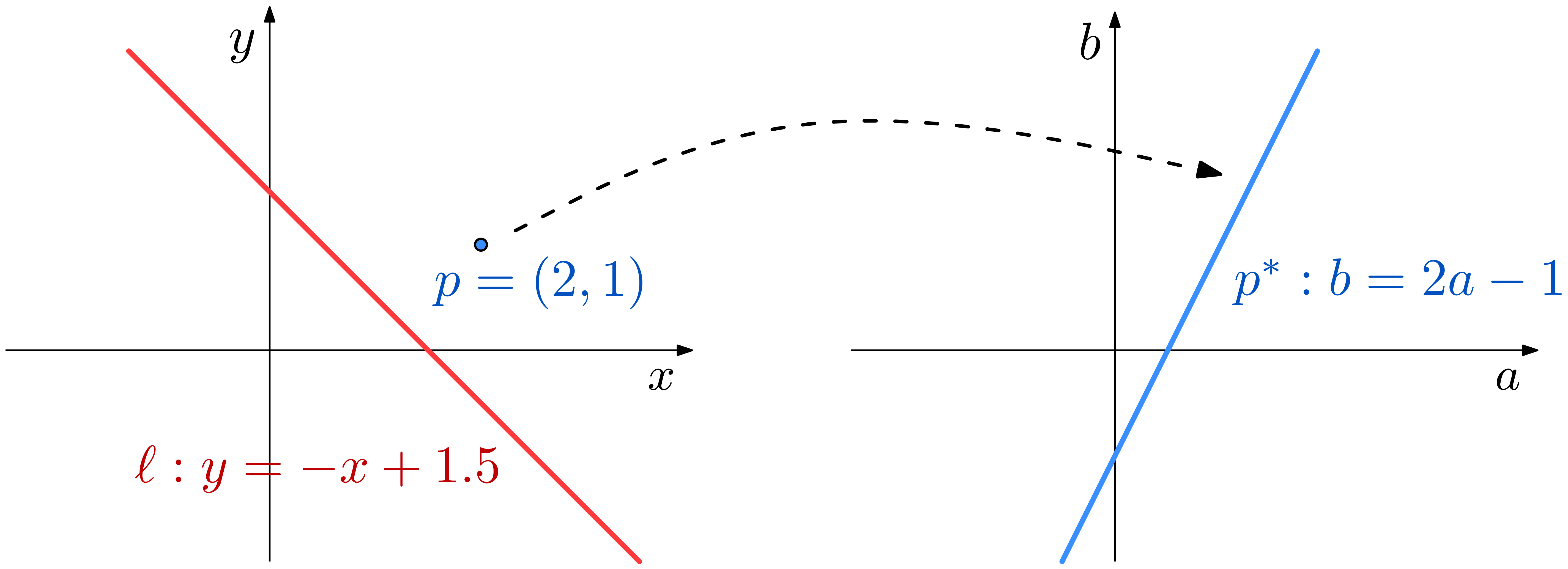

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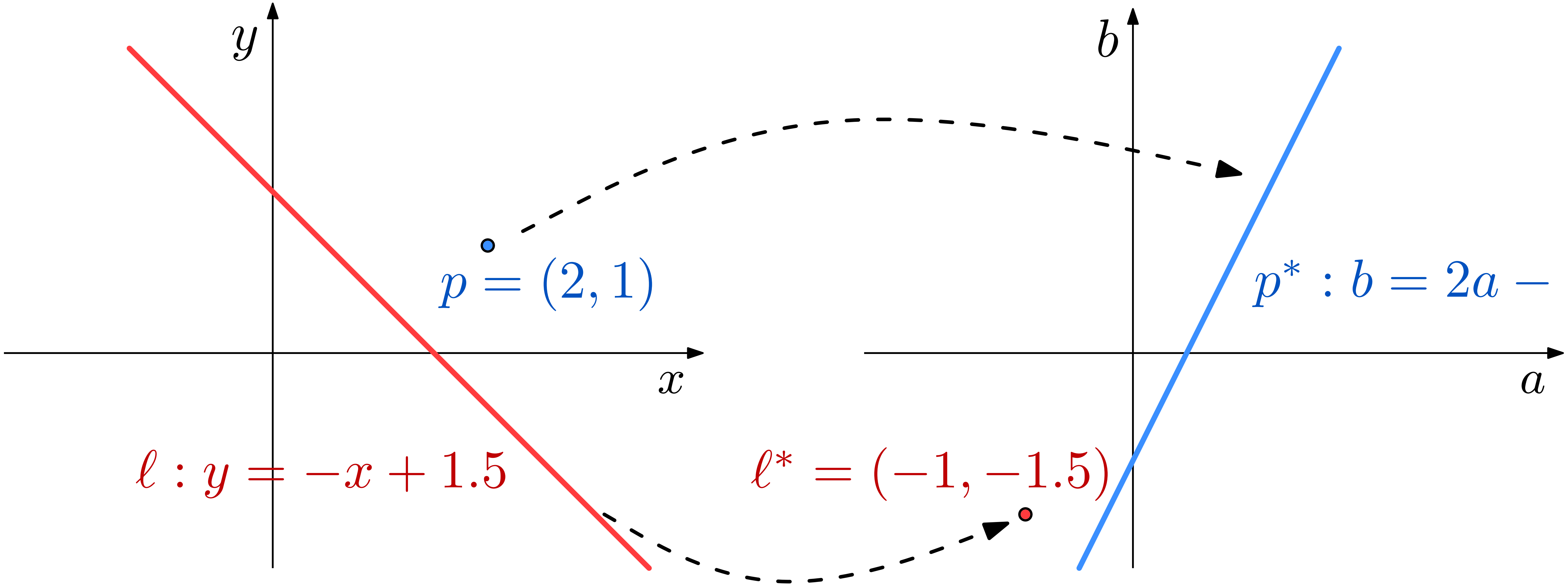

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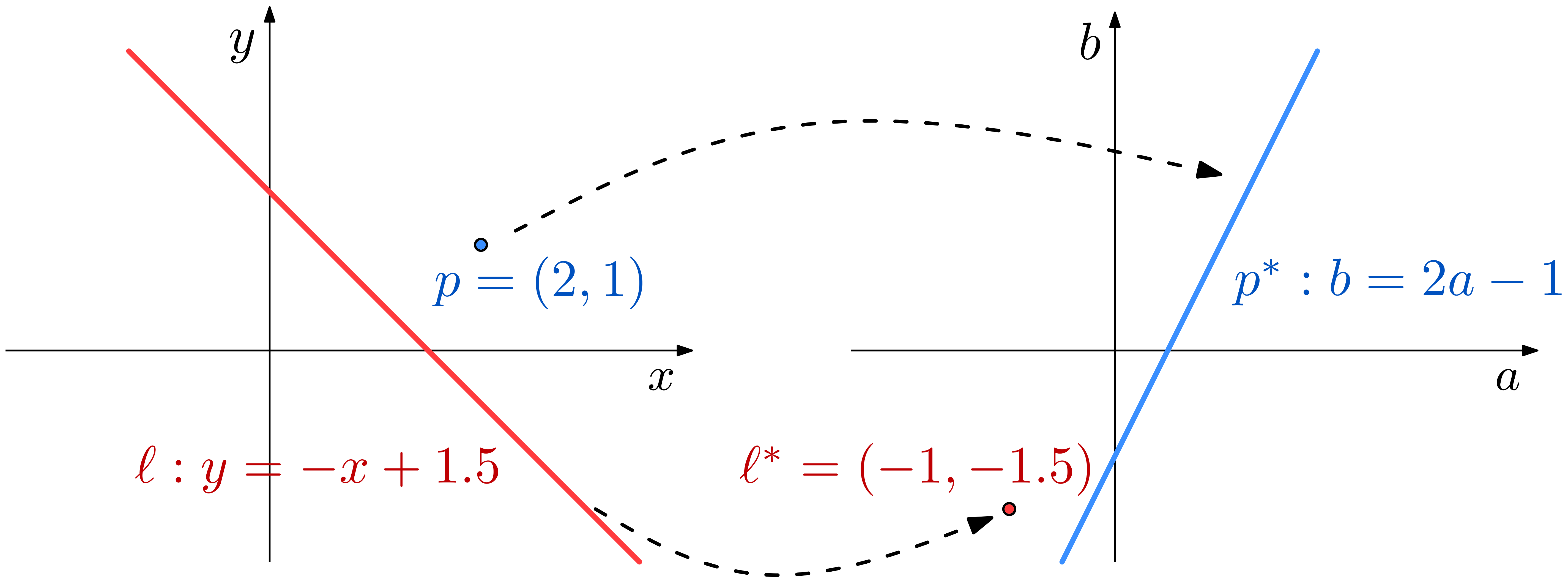

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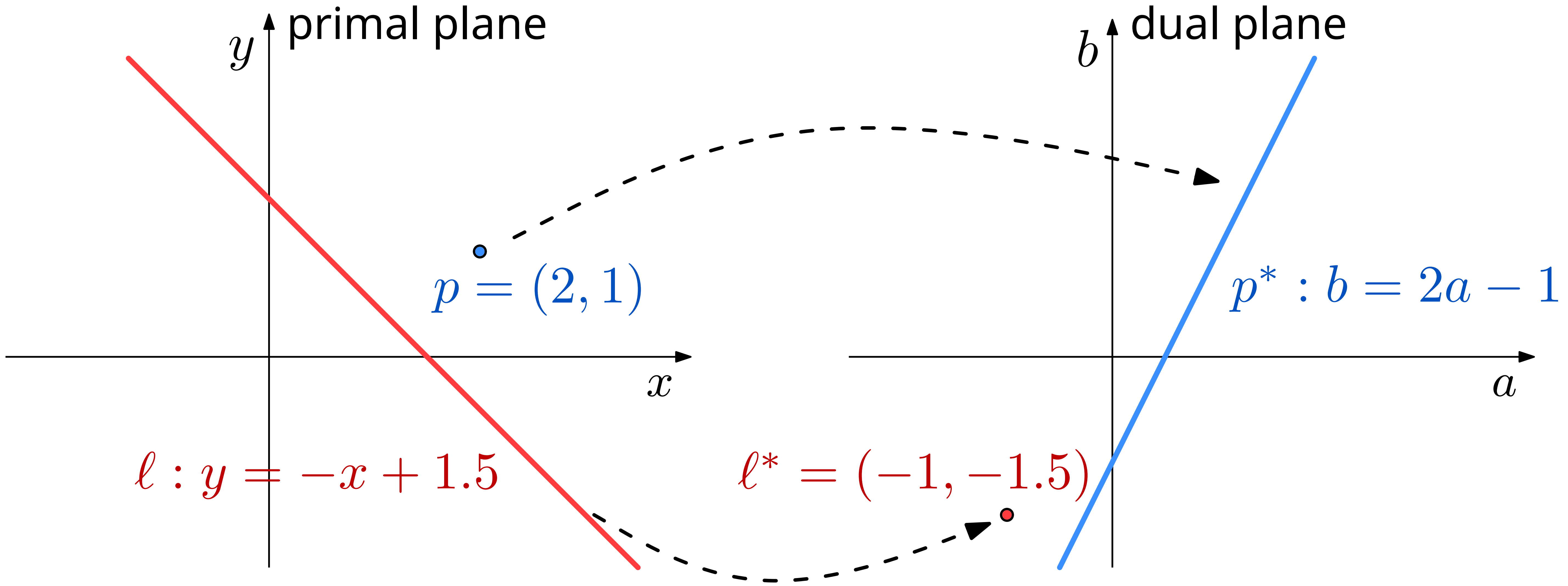

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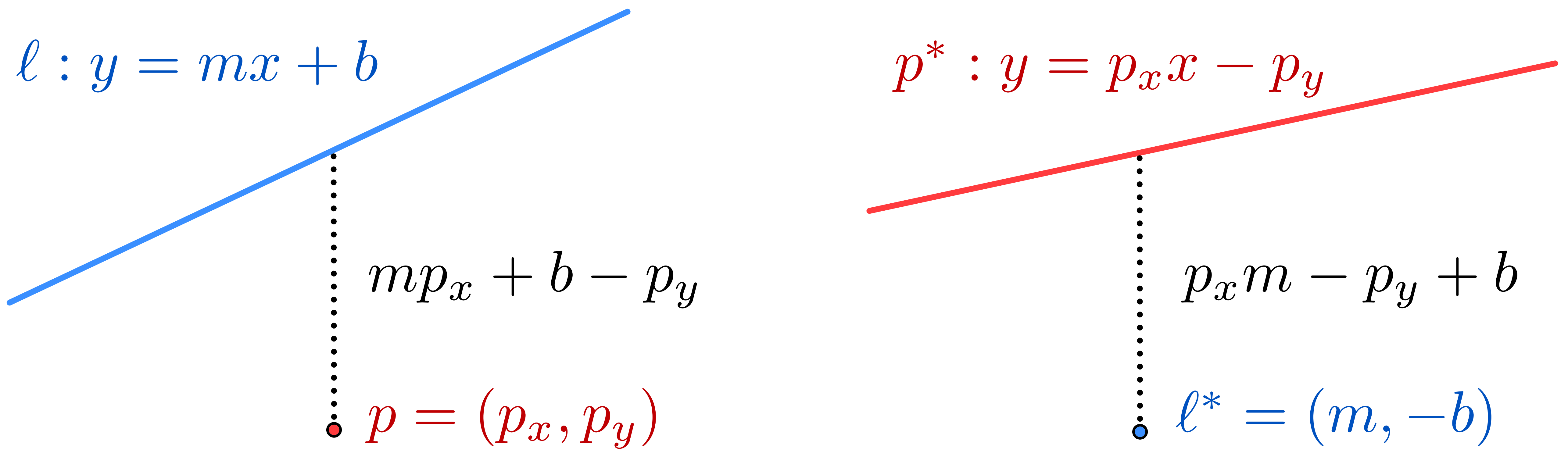
Properties



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- self-inverse: $(p^*)^* = p$ and $(\ell^*)^* = \ell$
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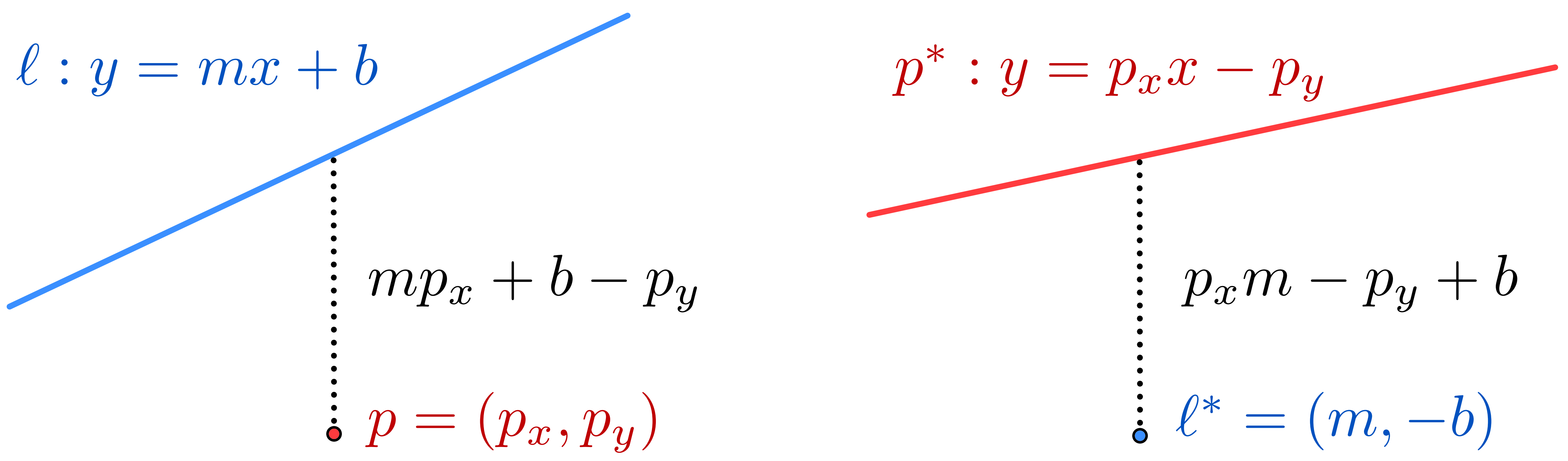

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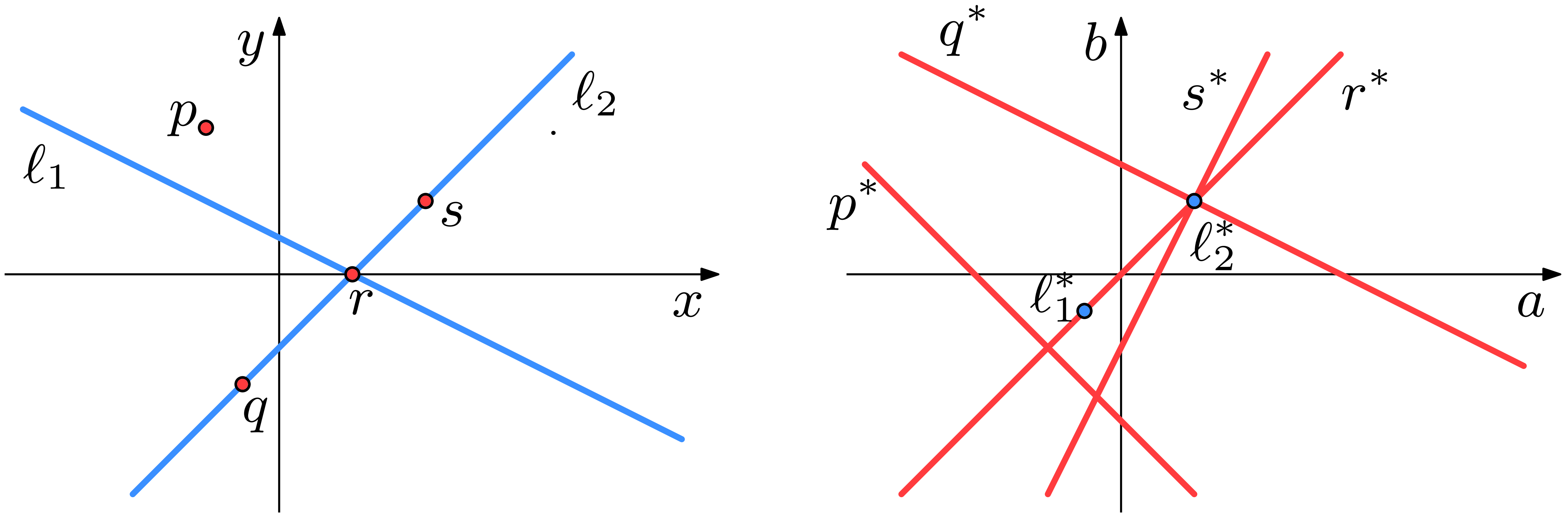

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- ℓ_1 and ℓ_2 intersect in $r \Leftrightarrow r^*$ goes through ℓ_1^* and ℓ_2^*
- q, r, s collinear $\Leftrightarrow q^*, r^*, s^*$ intersect in a point

Properties

What is the dual of a line segment $s = \overline{pq}$?

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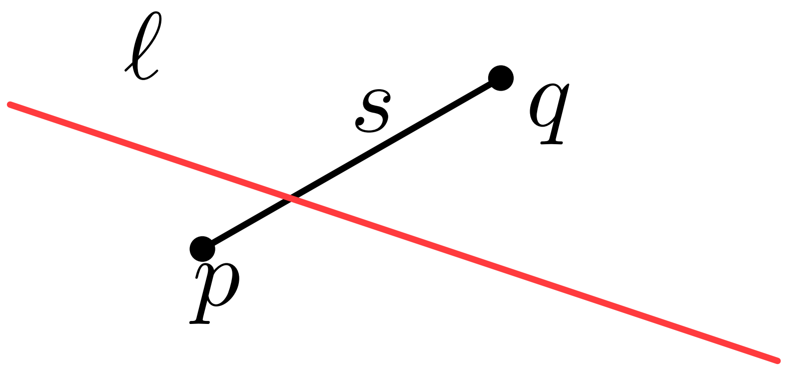

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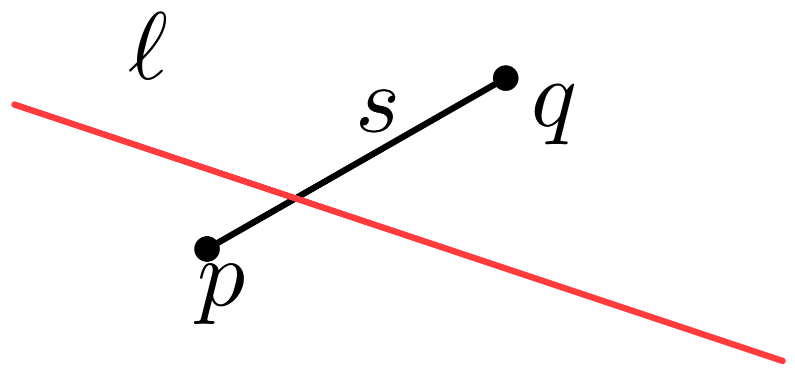
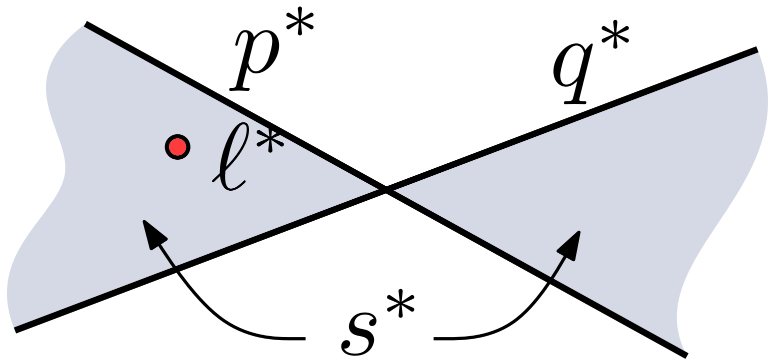
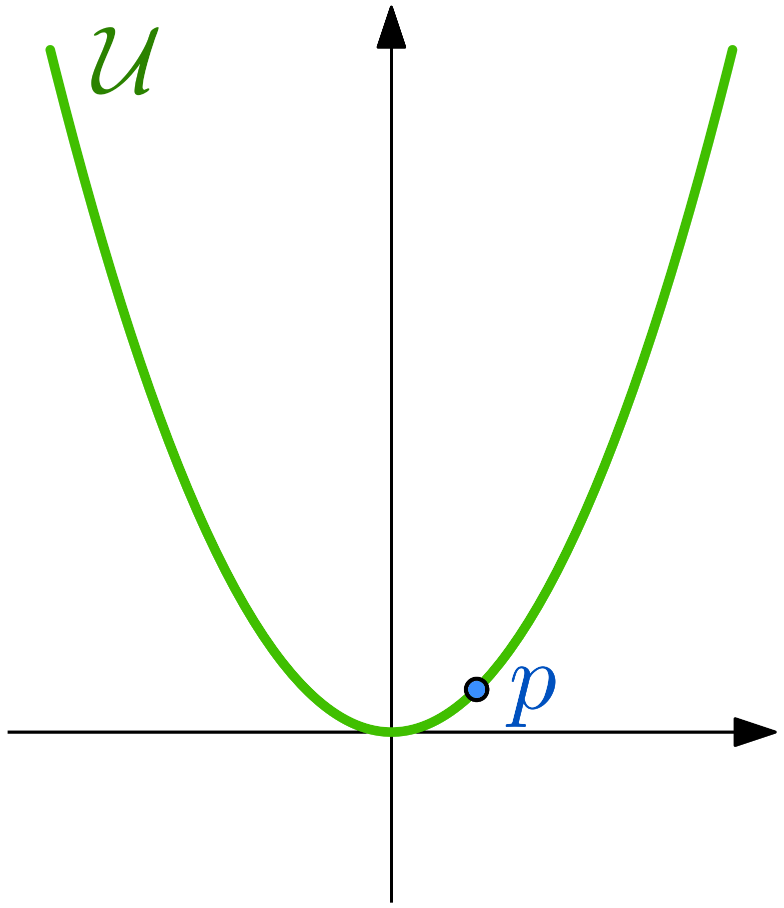




Geometric interpretation

Given parabola $\mathcal{U}$: $y = x^2/2$ and $p = (p_x, p_y)$ point on $\mathcal{U}$



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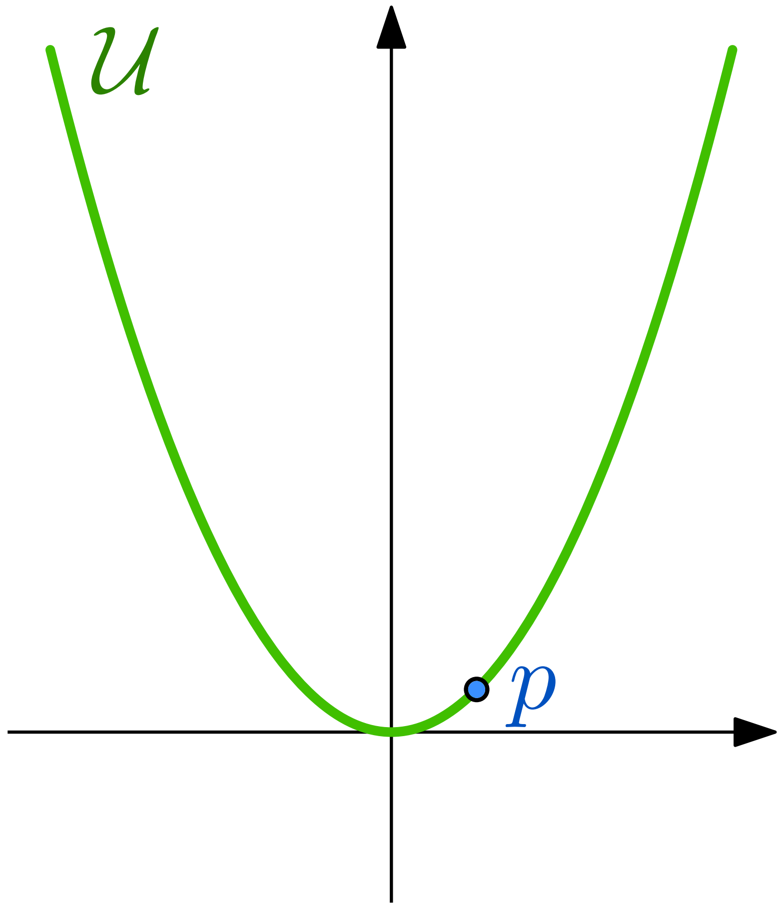
Question: Tangent to $\mathcal{U}$ at p?



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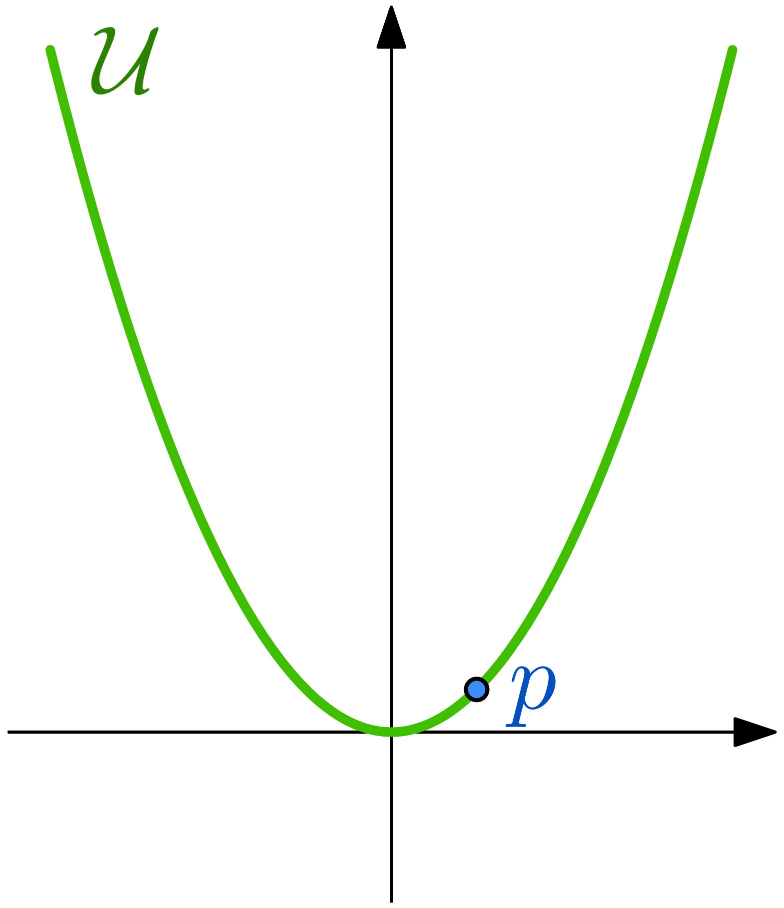

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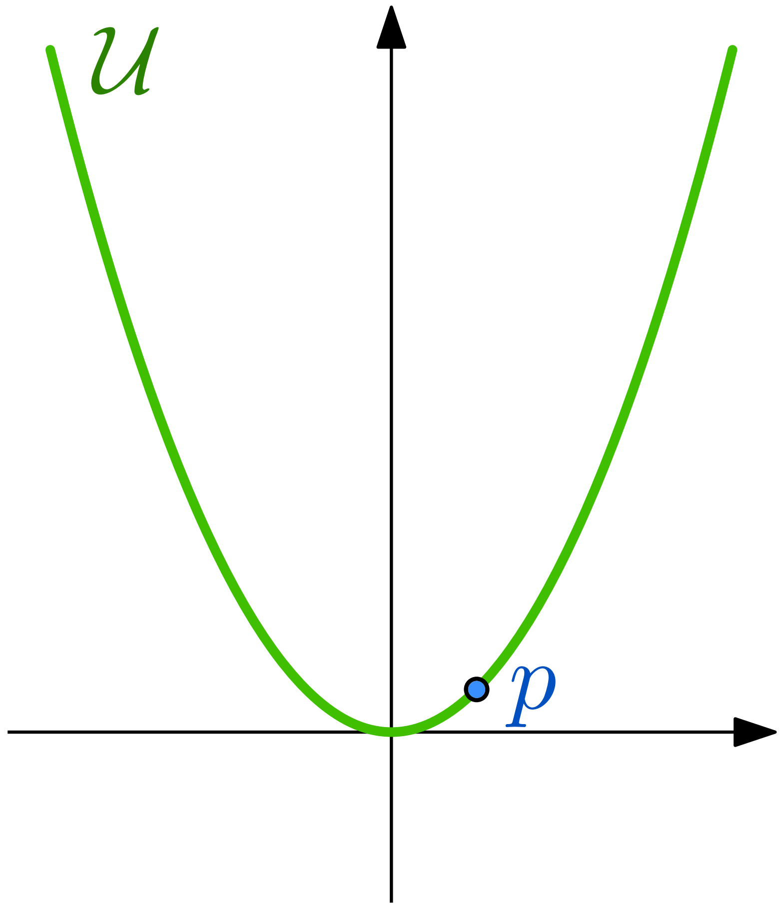

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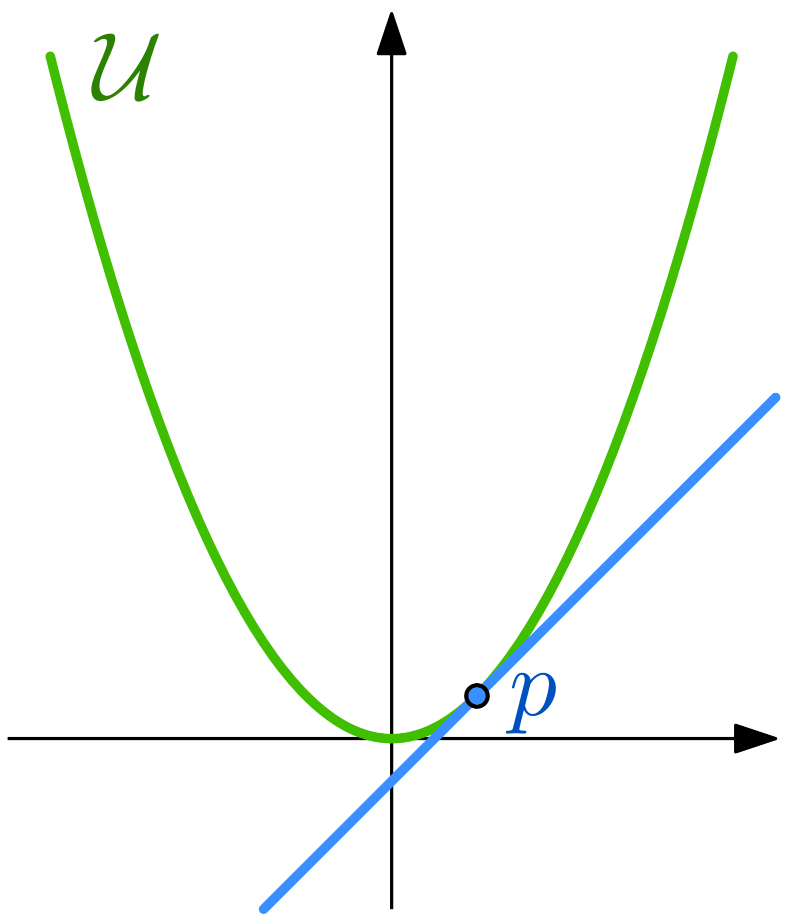

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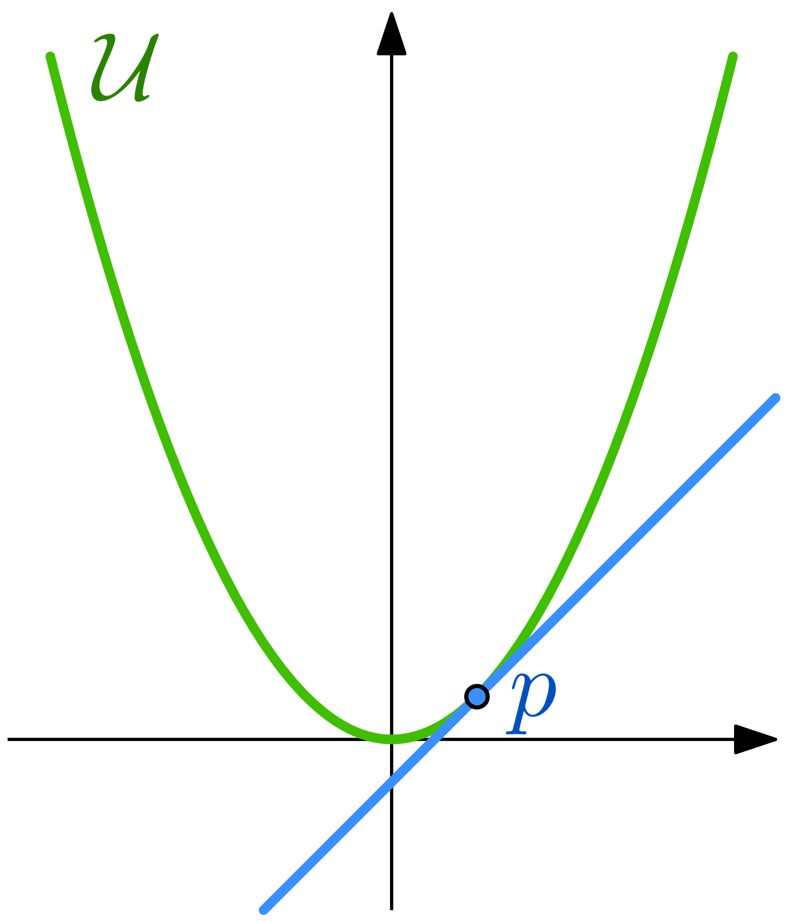

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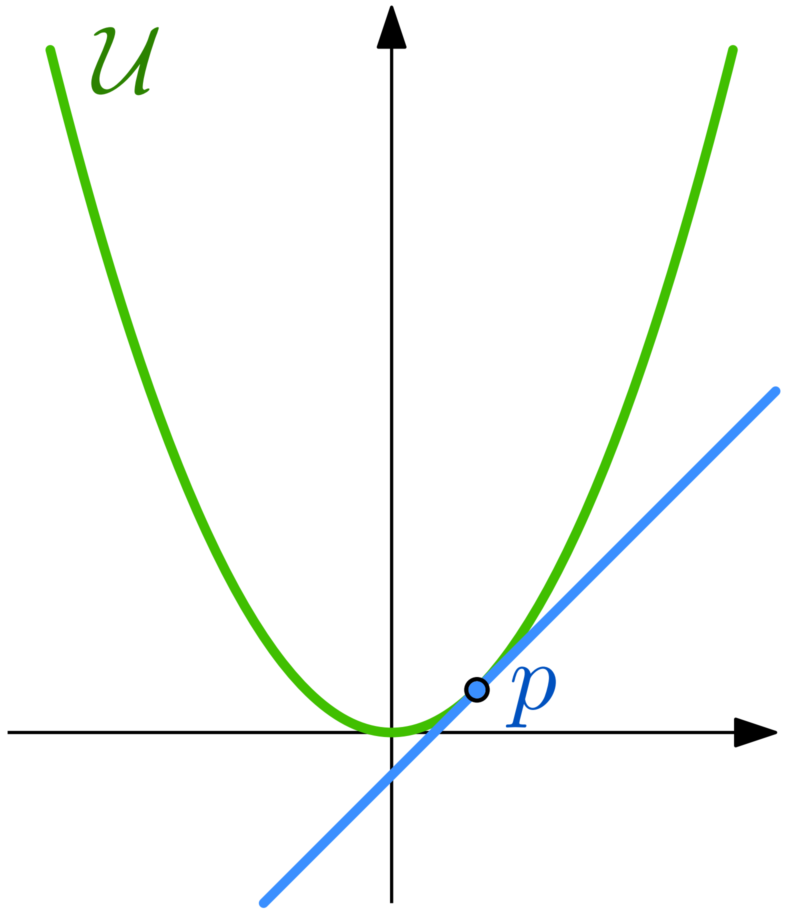

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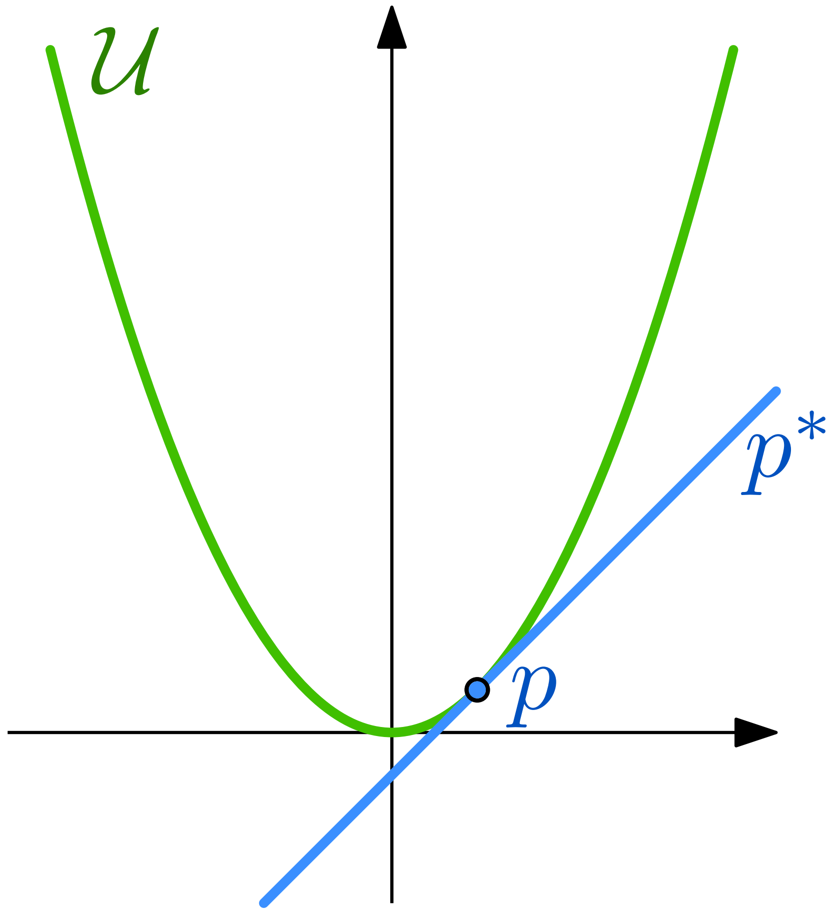

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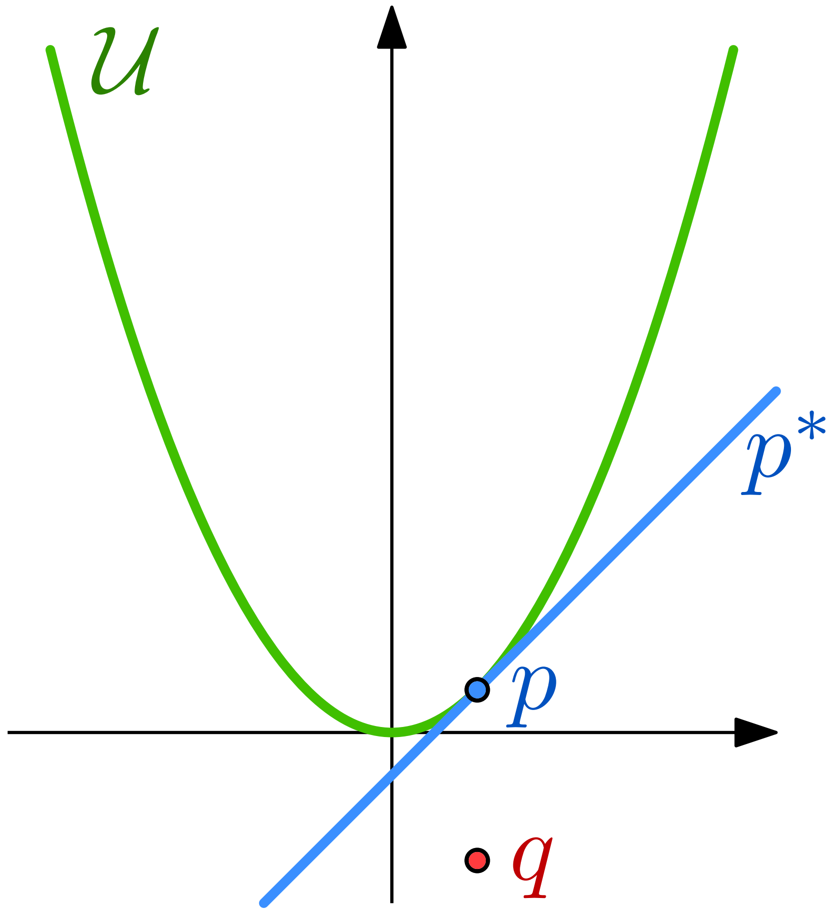

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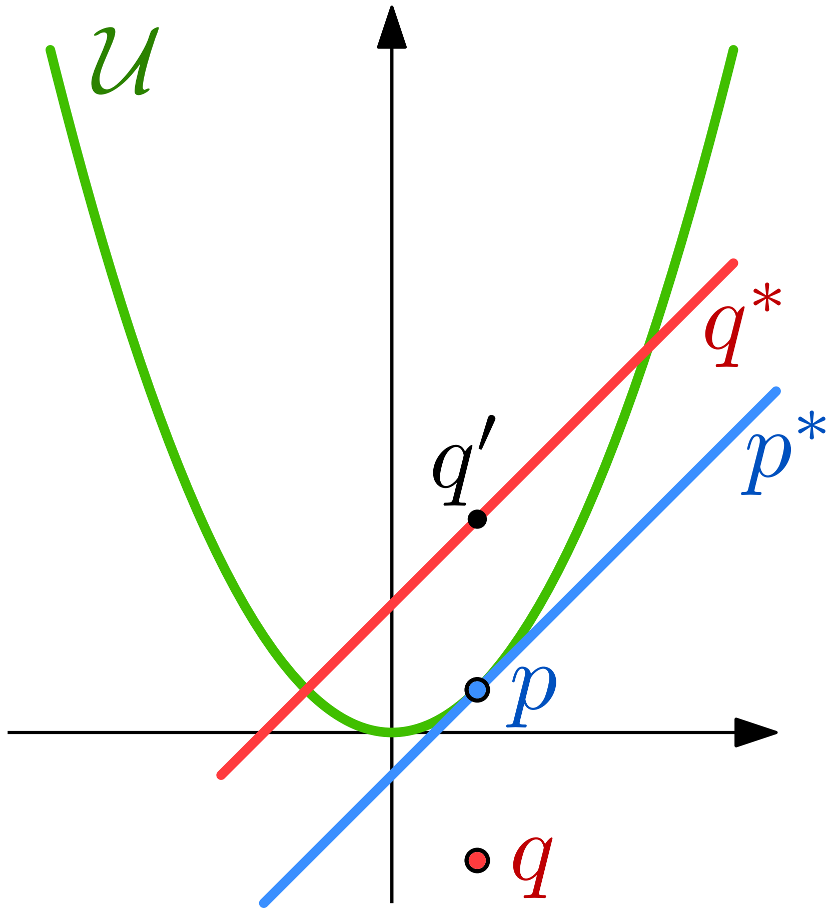

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- q^* has same slope as p^*
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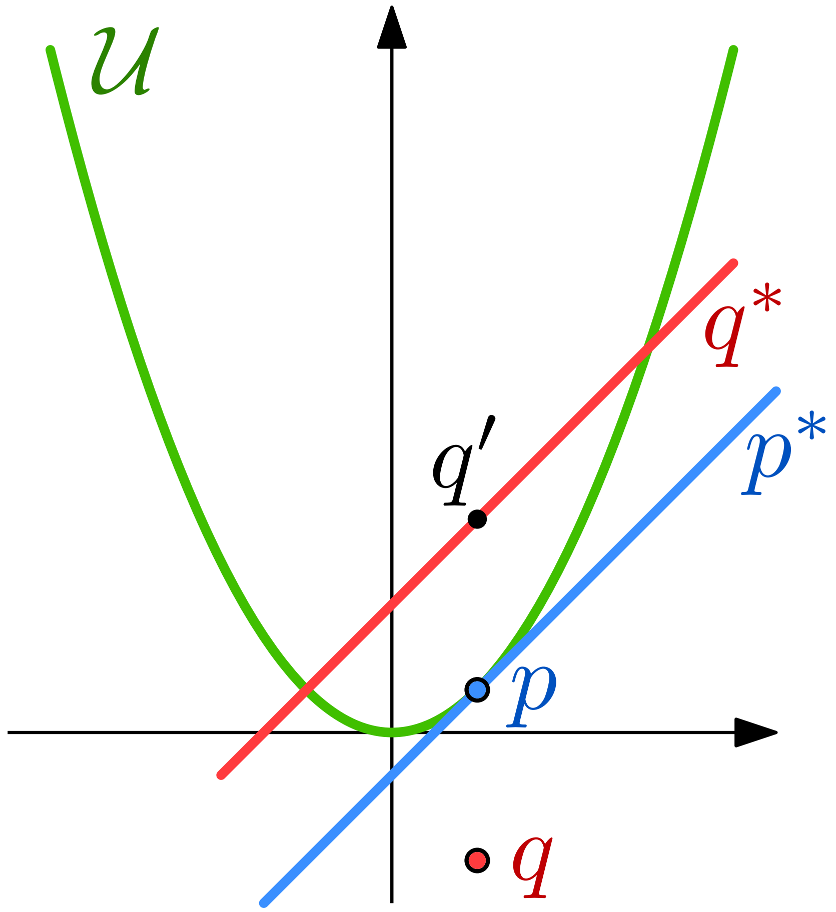

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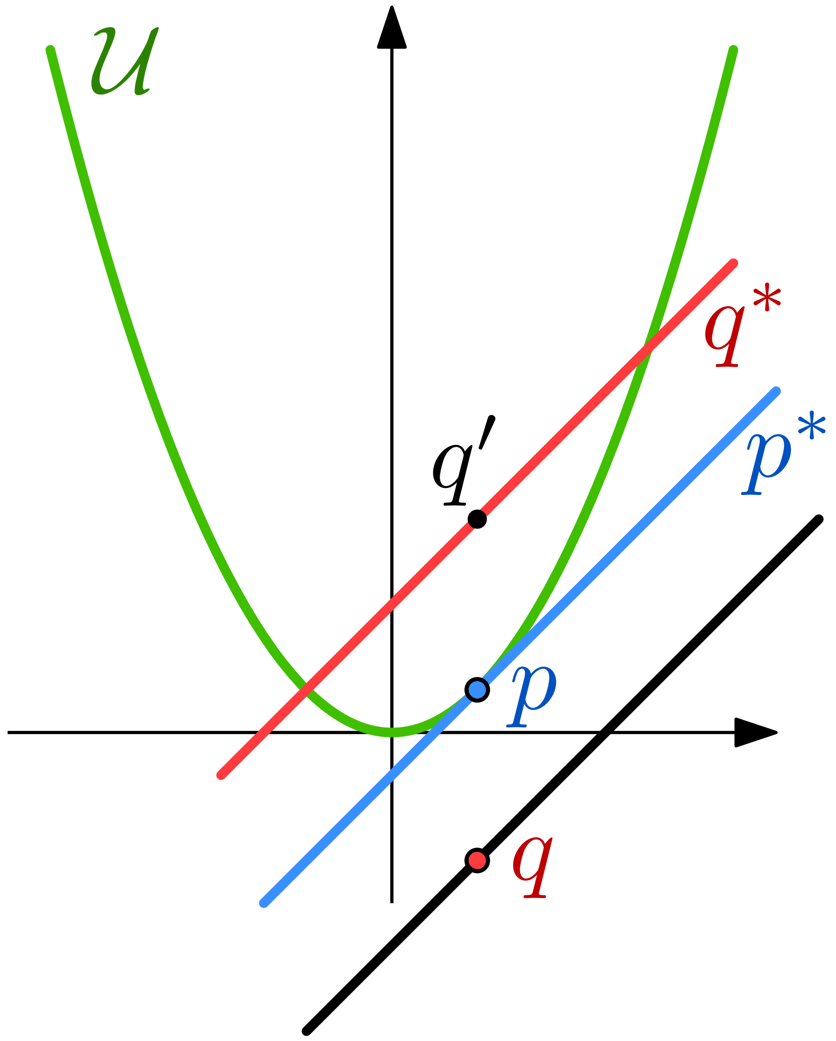

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Why duality?

Duality provides new perspective!

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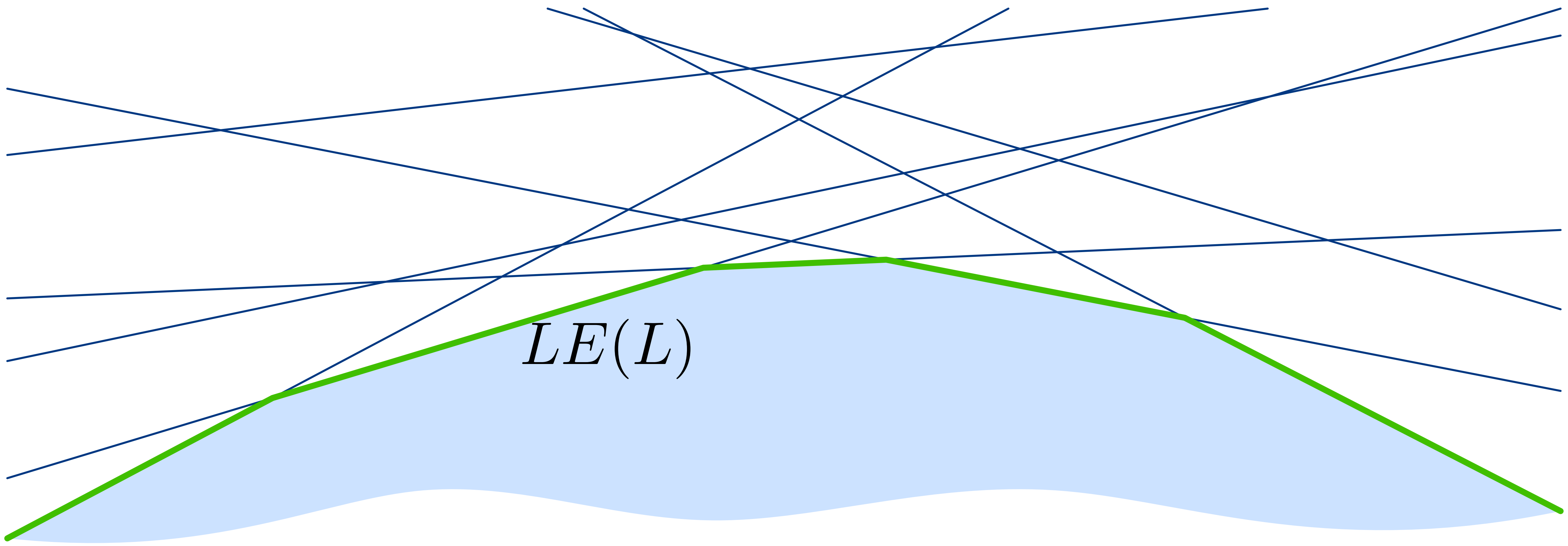
Examples:

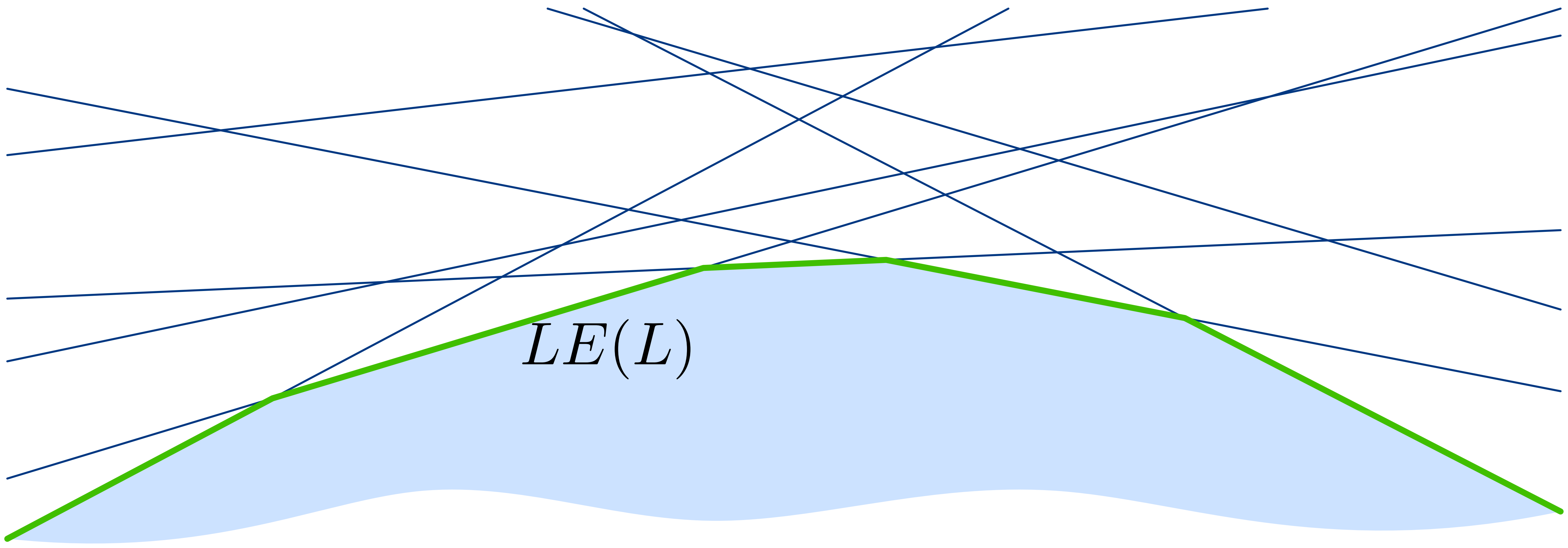
- three points on a line
- smallest-area triangle in a set of points
- angular order
- upper/lower envelope of a set of lines

Lower envelope



Definition: The lower envelope $LE(L)$ of a set L of lines is the set of points from $\cup_{\ell \in L} \ell$ that do not have lines of L below them.

Lower envelope



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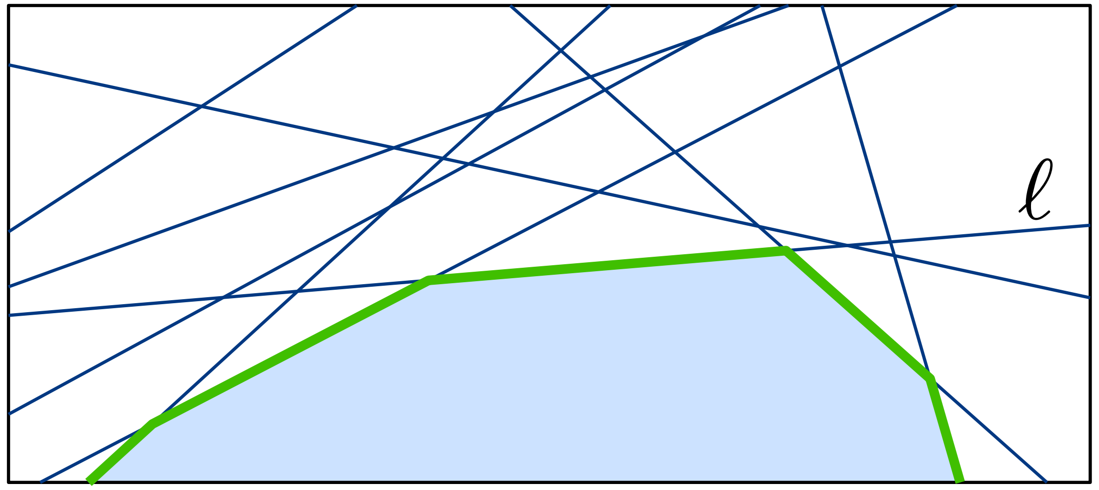
How to compute it:

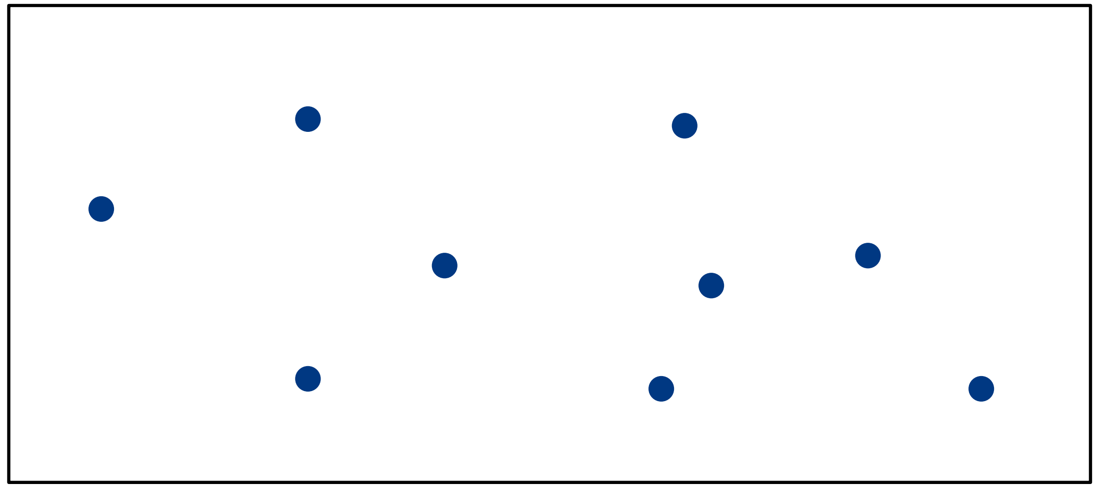
- dual problem on the set of points $L^* = \{\ell^* \mid \ell \in L\}$?

Envelopes and duality

primal

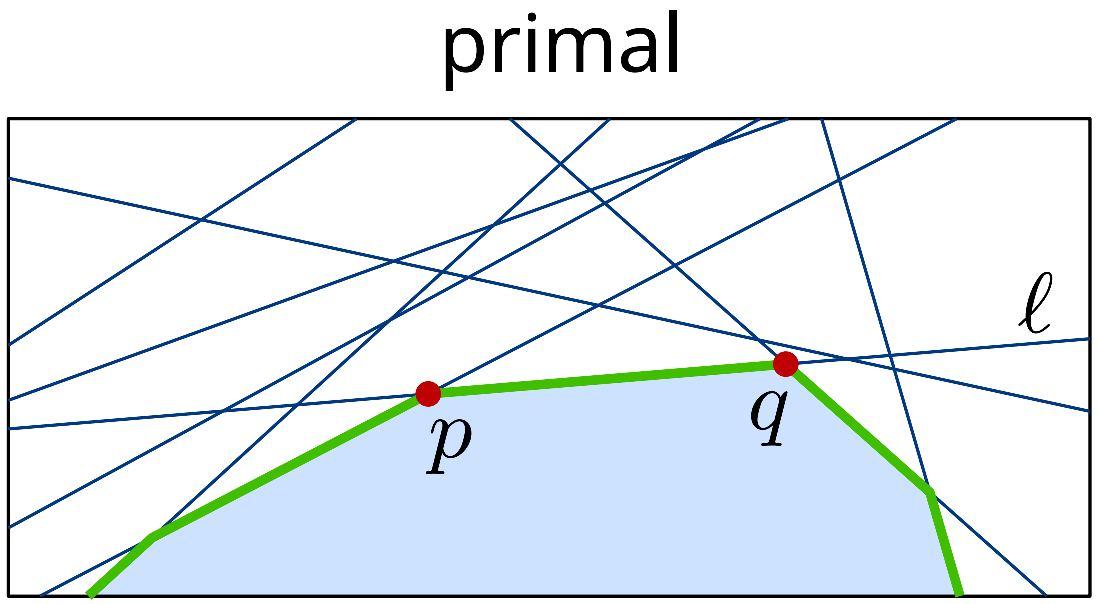


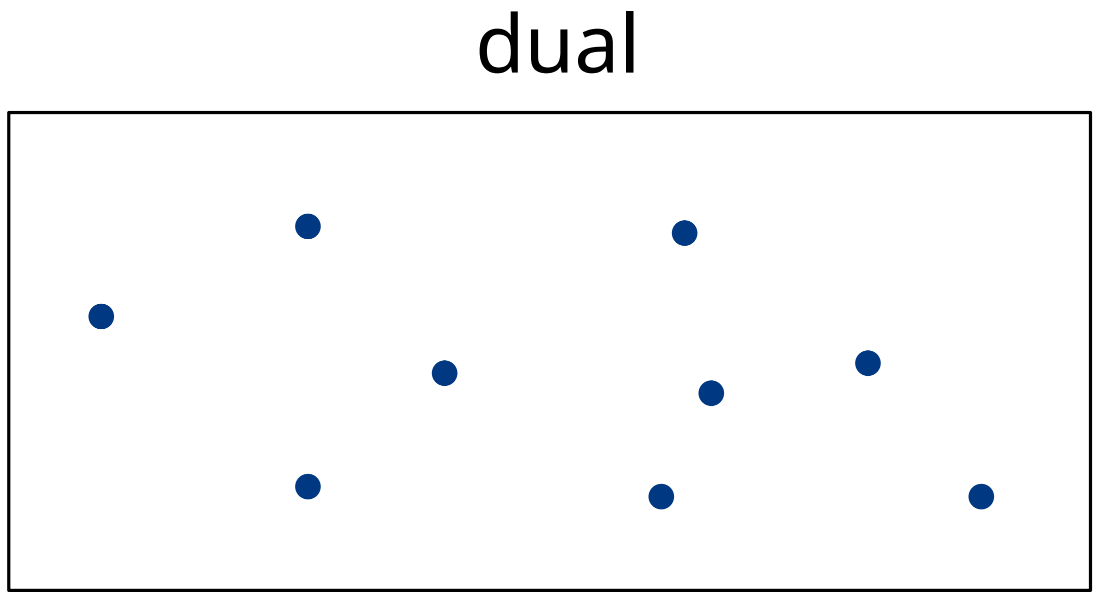
dual



Question: Which lines support segments on $LE(L)$?

Envelopes and duality





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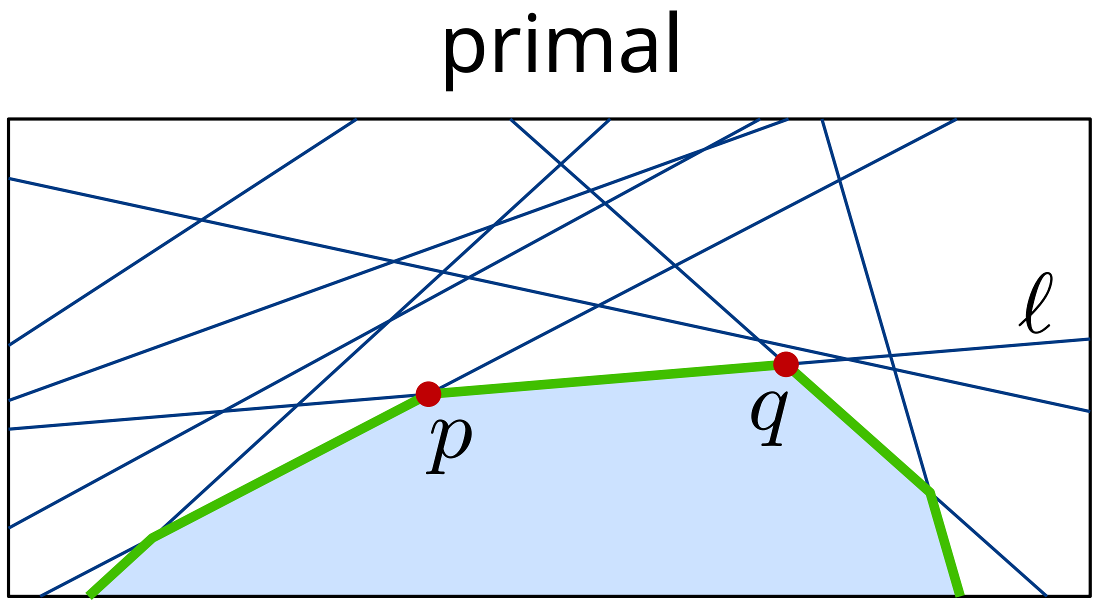
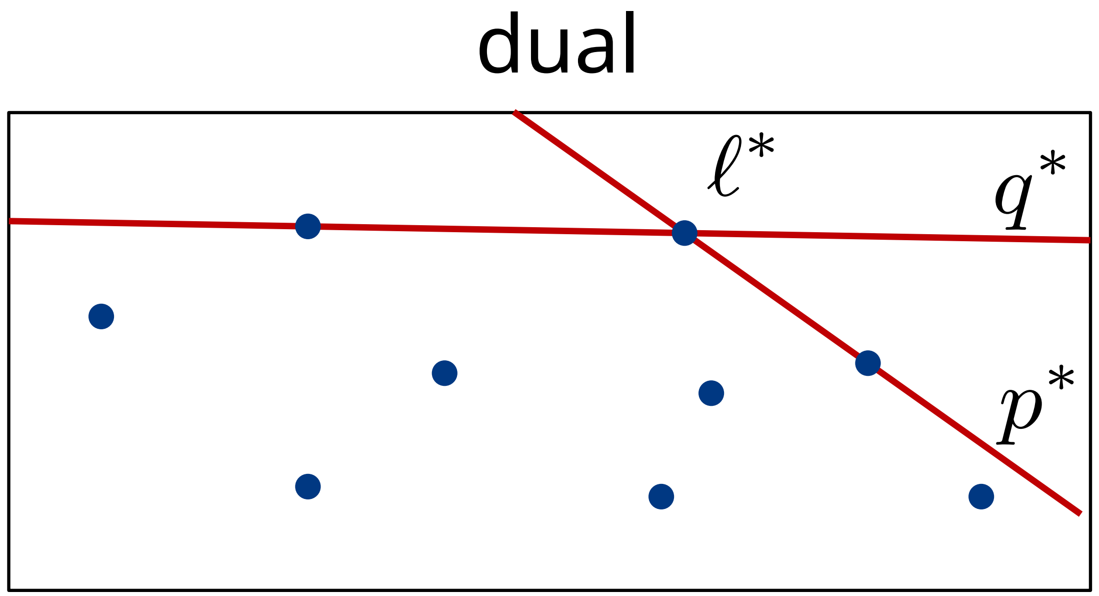
Envelopes and duality





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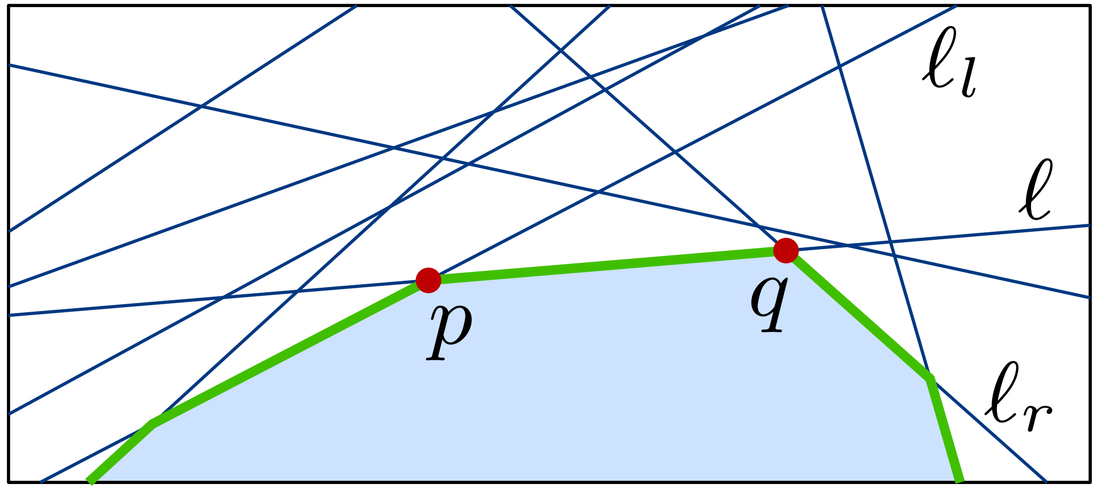
Envelopes and duality

primal



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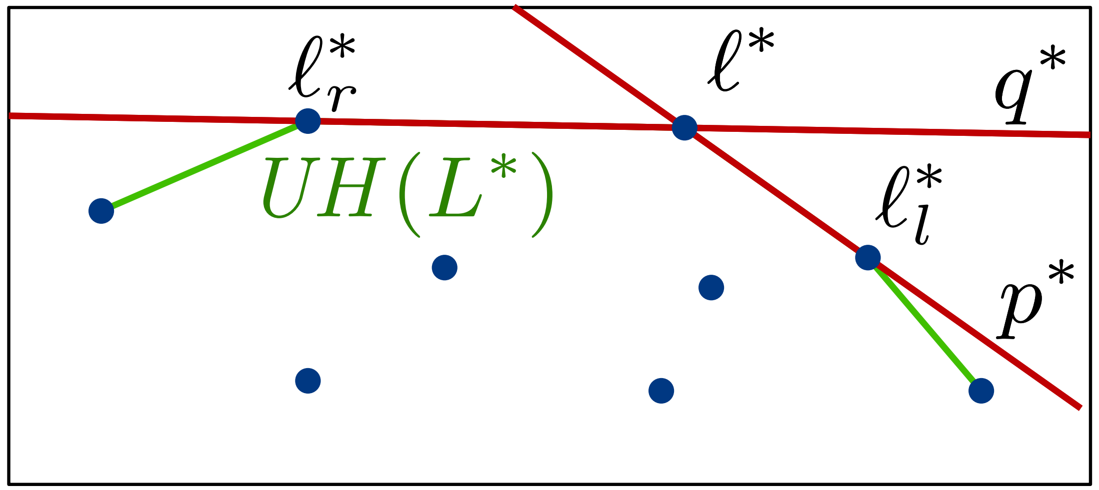


Question: Which lines support segments on $LE(L)$?

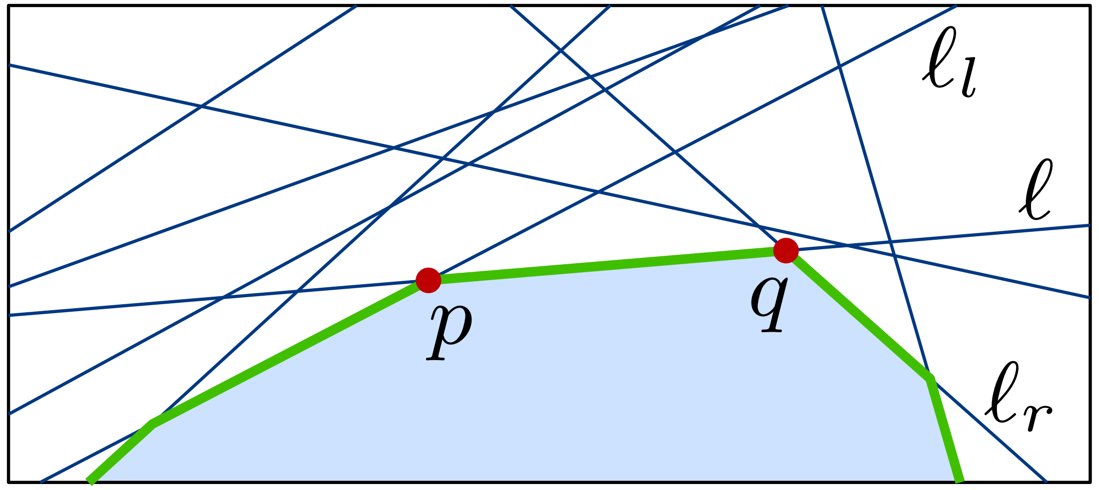
- p and q lie below all lines of L
- p^* and q^* lie above all points of L^*

$\Rightarrow$ they support neighboring edges on upper hull $UH(L^*)$

$\Rightarrow$ p^* and q^* intersect in a vertex ℓ^* of $UH(L^*)$

Envelopes and duality

primal



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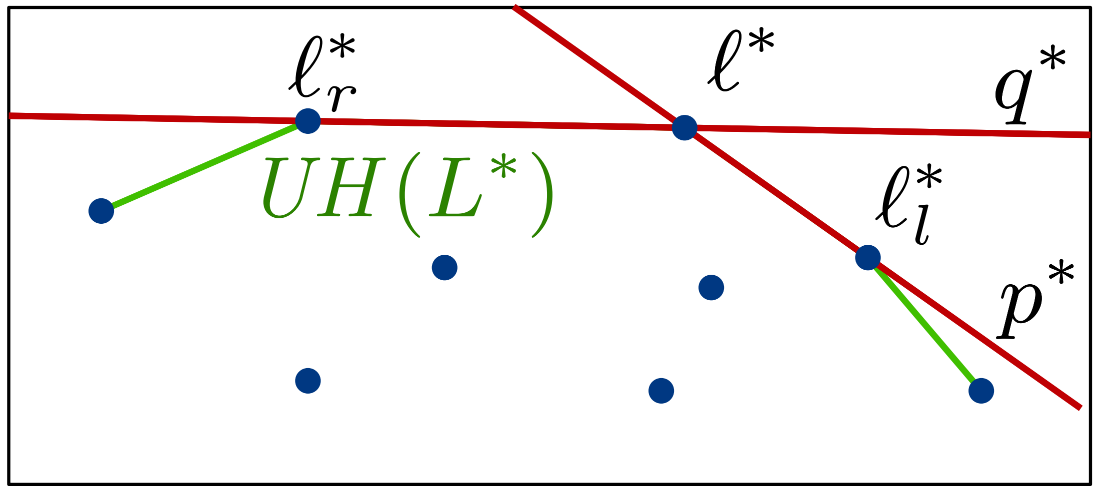

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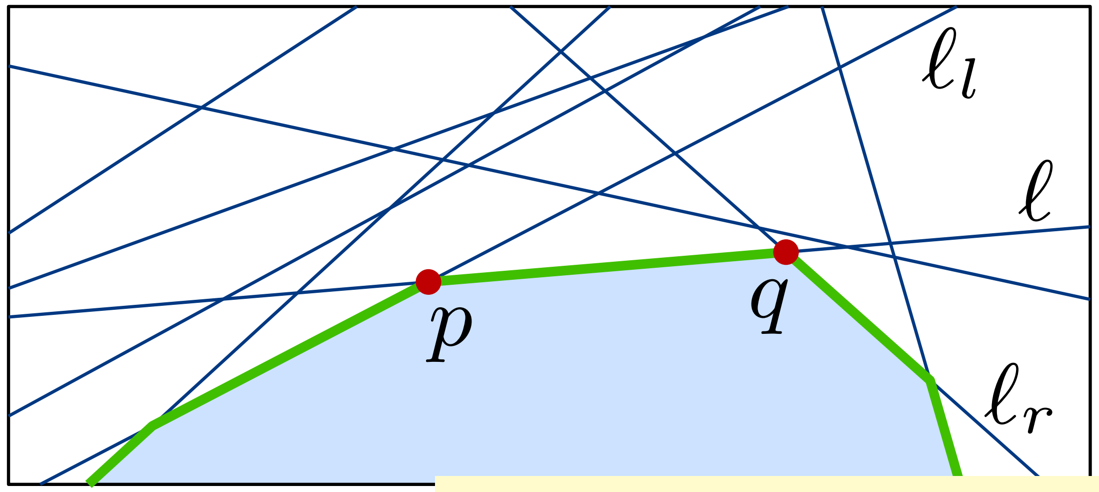
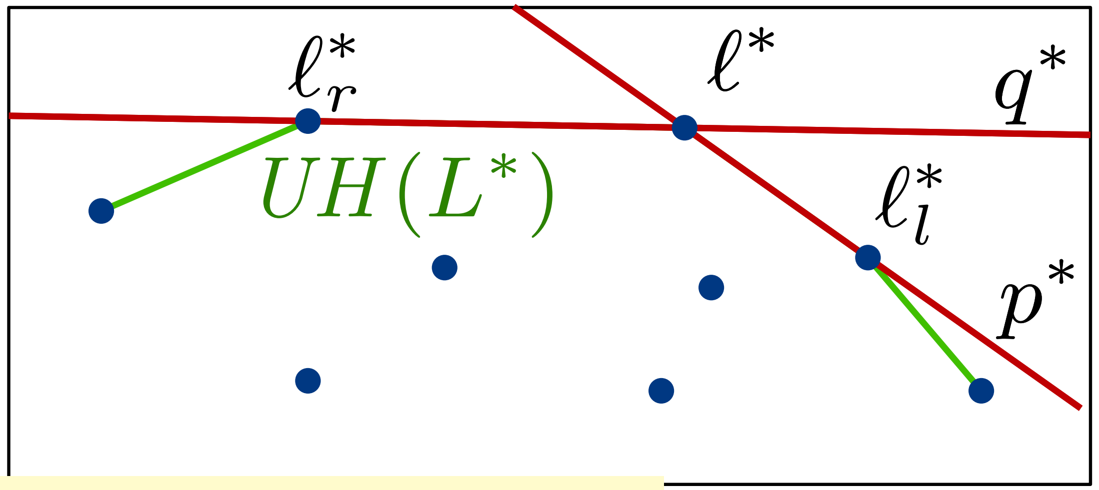
Lemma 2: The lines of $LE(L)$ from right to left correspond to the vertices of $UH(L^*)$ from left to right.

Envelopes and duality





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$\Rightarrow$ they support neighboring edges on upper hull $UH(L^*)$

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We can compute intersection of half-spaces by computing convex hulls, and vice versa!

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problem on set of lines $L \rightarrow$ problem on set of points L^*

- upper/lower envelope of a set of lines

problem on set of points $P \rightarrow$ problem on set of lines P^*

- three points on a line
- smallest-area triangle in a set of points

Why duality?

Duality provides new perspective!

problem on set of lines $L \rightarrow$ problem on set of points L^*

- upper/lower envelope of a set of lines ✔

problem on set of points $P \rightarrow$ problem on set of lines P^*

- three points on a line
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- three points
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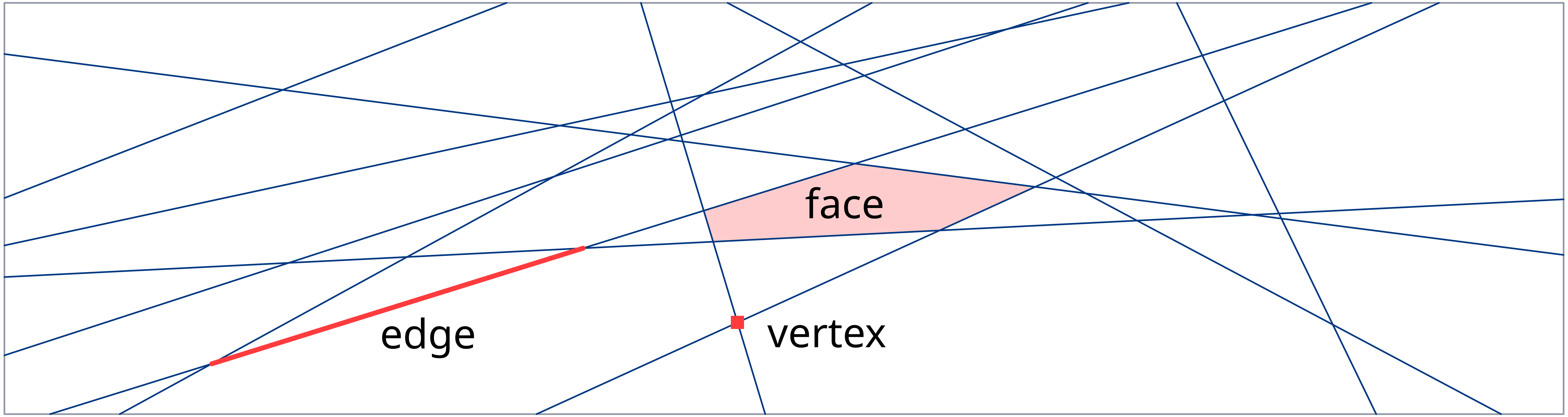
We will want to compute the planar subdivision induced by these lines, called arrangement of lines

Arrangement of lines

Definition: Arrangement $\mathcal{A}(L)$ of a set of lines L is a subdivision of the plane induced by the lines in L.



$\mathcal{A}(L)$ is called simple if no three points intersect in a point

Complexity of $\mathcal{A}(L)$

The combinatorial complexity of $\mathcal{A}(L)$ is the number of vertices, edges and faces.

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Theorem 1: Let $\mathcal{A}(L)$ be a simple arrangement of n lines. Then $\mathcal{A}(L)$ has ? vertices, ? edges and ? faces.

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Theorem 1: Let $\mathcal{A}(L)$ be a simple arrangement of n lines. Then $\mathcal{A}(L)$ has $n(n-1)/2$ vertices, n^2 edges and $n(n+1)/2+1$ faces.

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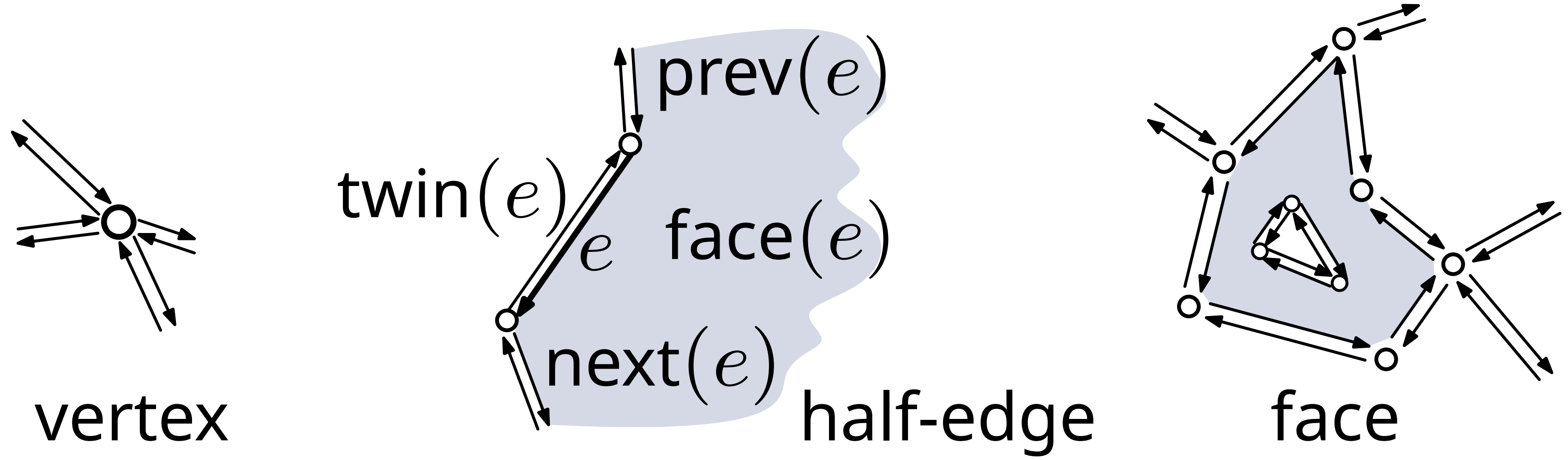
Data structure for $\mathcal{A}(L)$:

- insert a bounding box of all line intersections
  $\Rightarrow$ planar straight-line graph G
- doubly-connected edge list for G



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Can we do better?

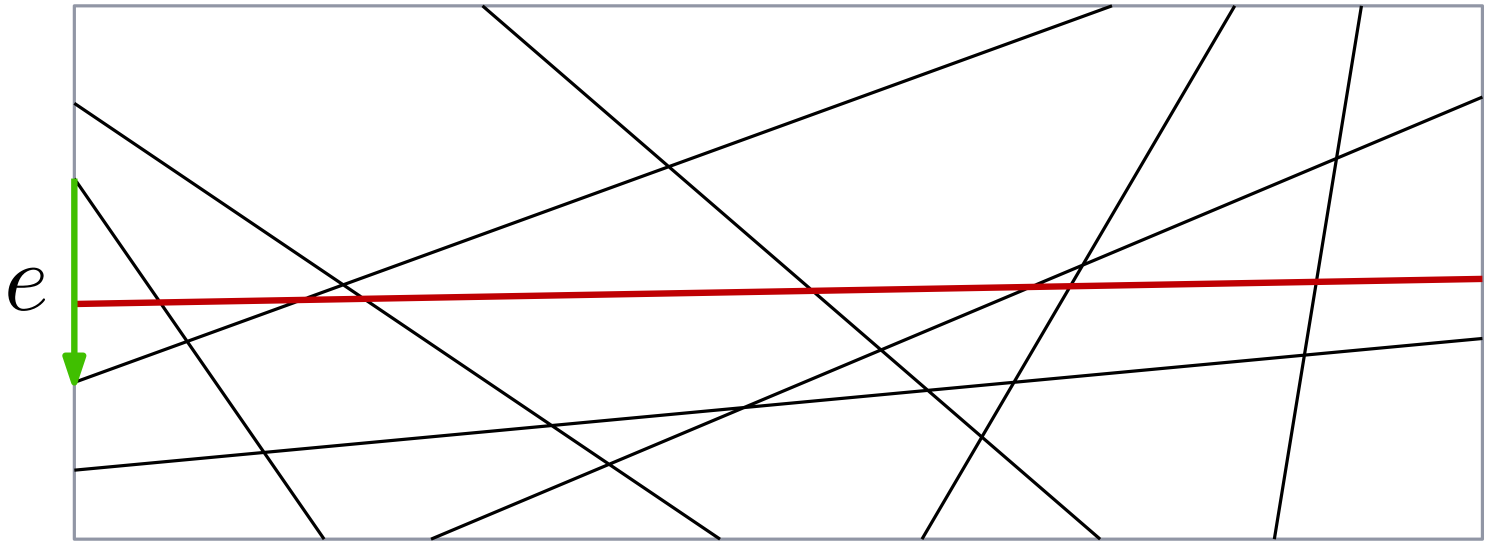
Incremental construction of $\mathcal{A}(L)$

Algorithm ConstructArrangement(L)

Input: lines $L = \{\ell_1, \ldots, \ell_n\}$

Output: DCEL $\mathcal{D}$ for $\mathcal{A}(L)$

```
initialize D for bounding box B of the vertices of A(L)
for i ← 1 to n do
    find left-most intersection of ℓ_i with some edge e on B
    f ← inner face incident to e
    while f ≠ outer face do
        partition f, update D and set f to the next intersected face
```



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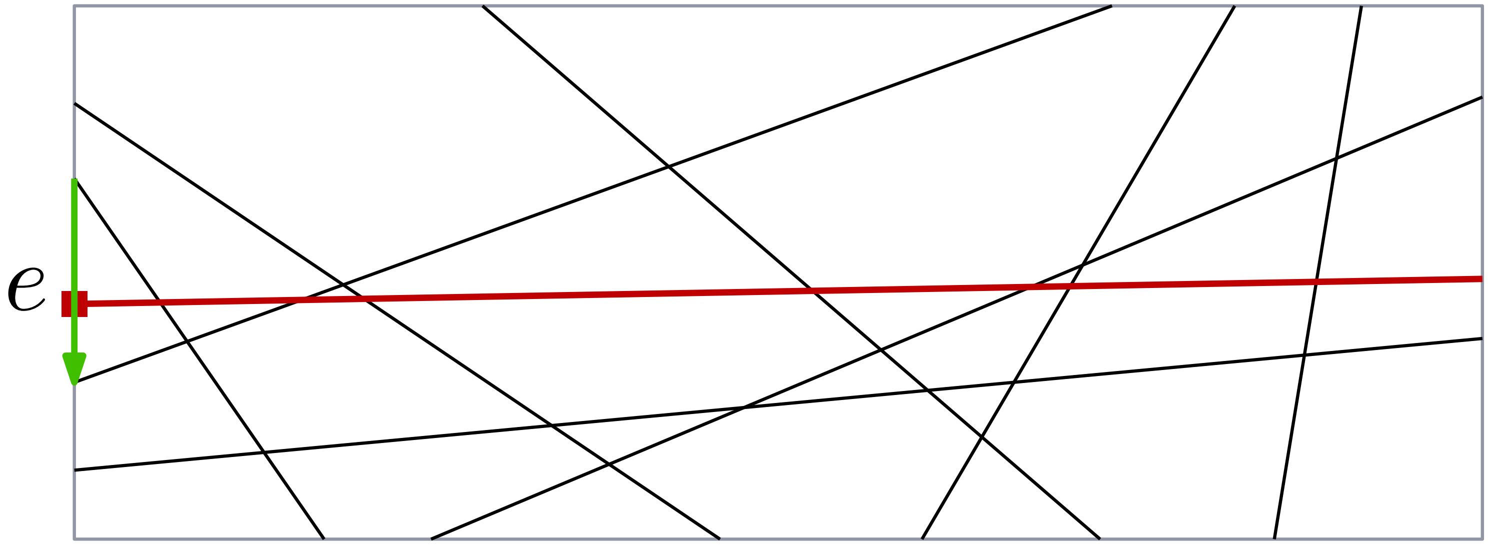

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Input: lines $L = \{\ell_1, \ldots, \ell_n\}$

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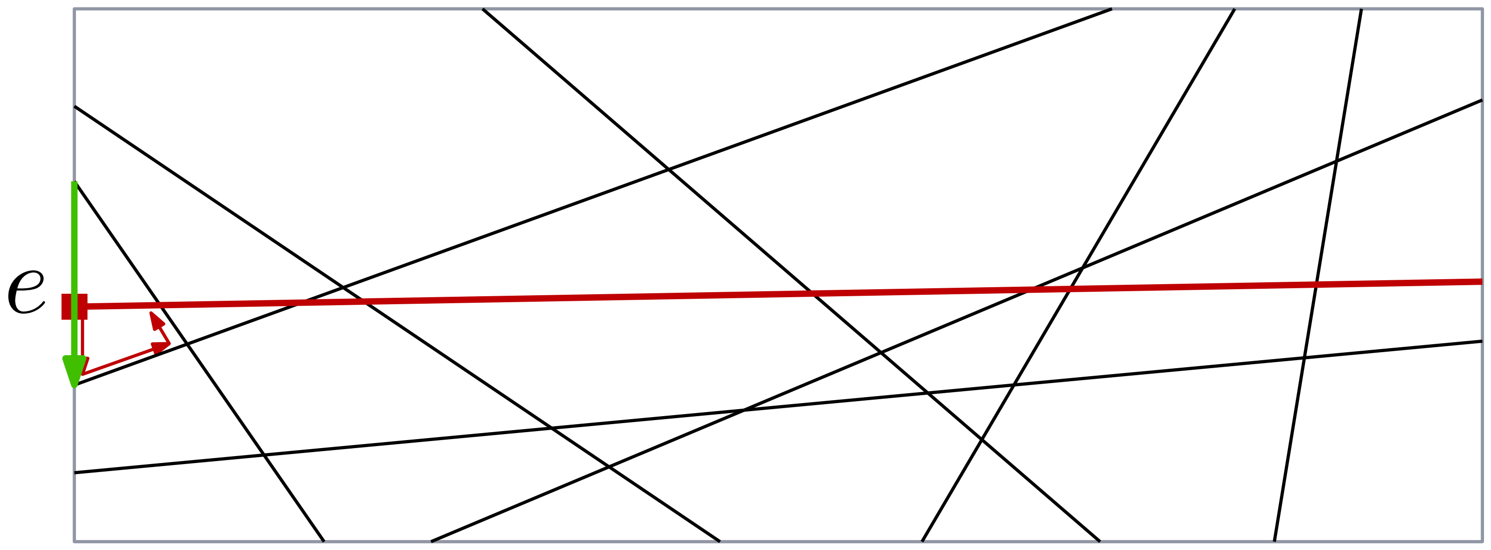

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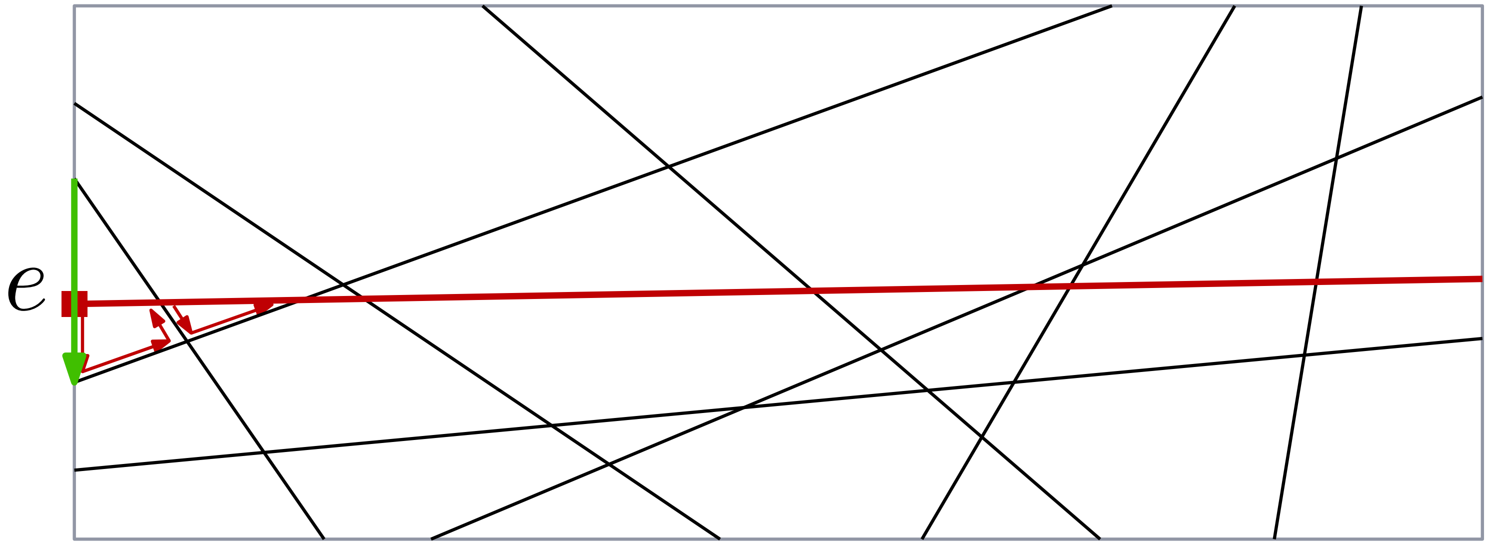

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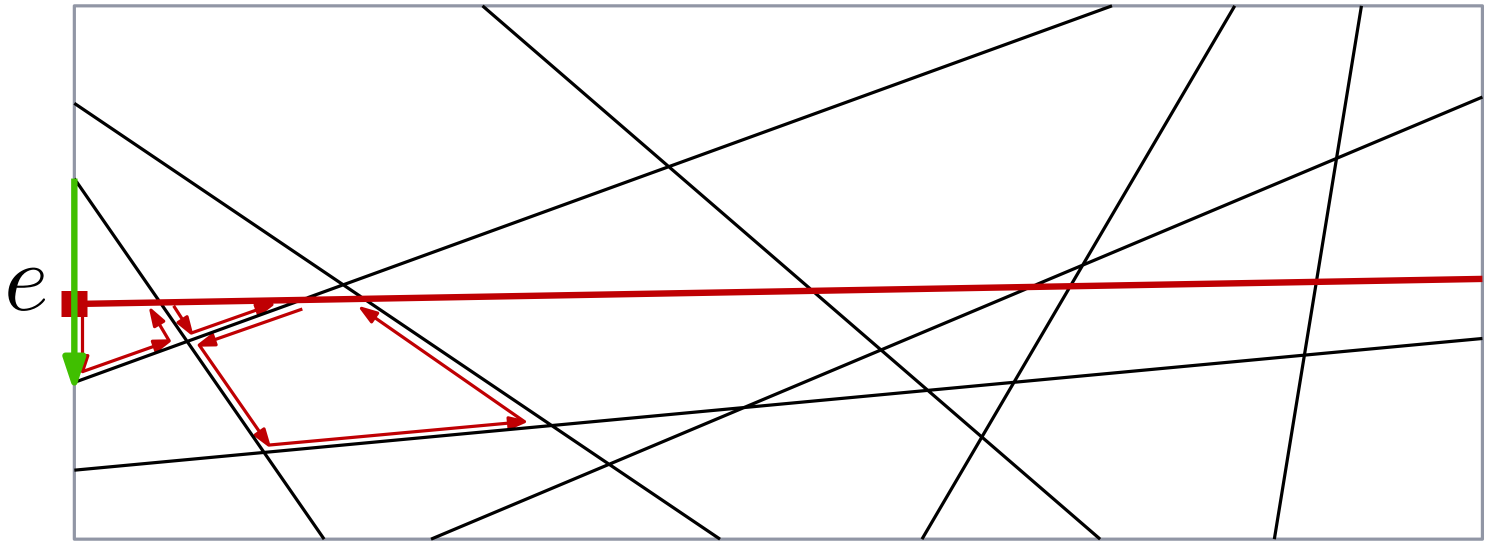

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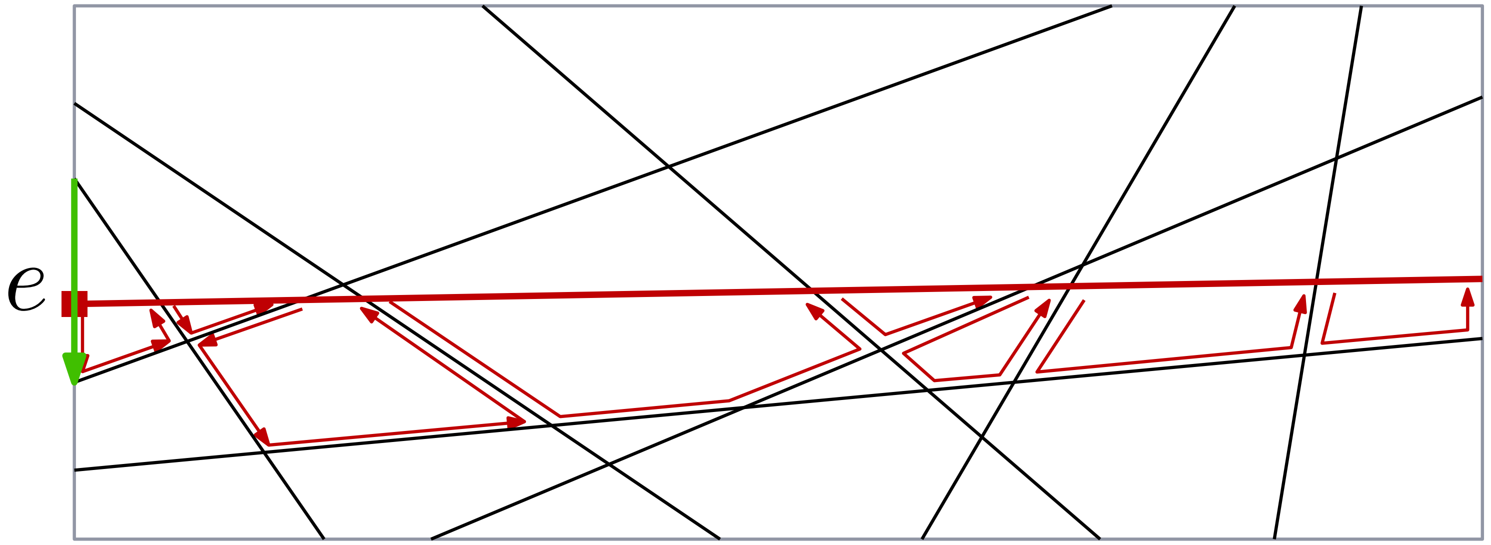

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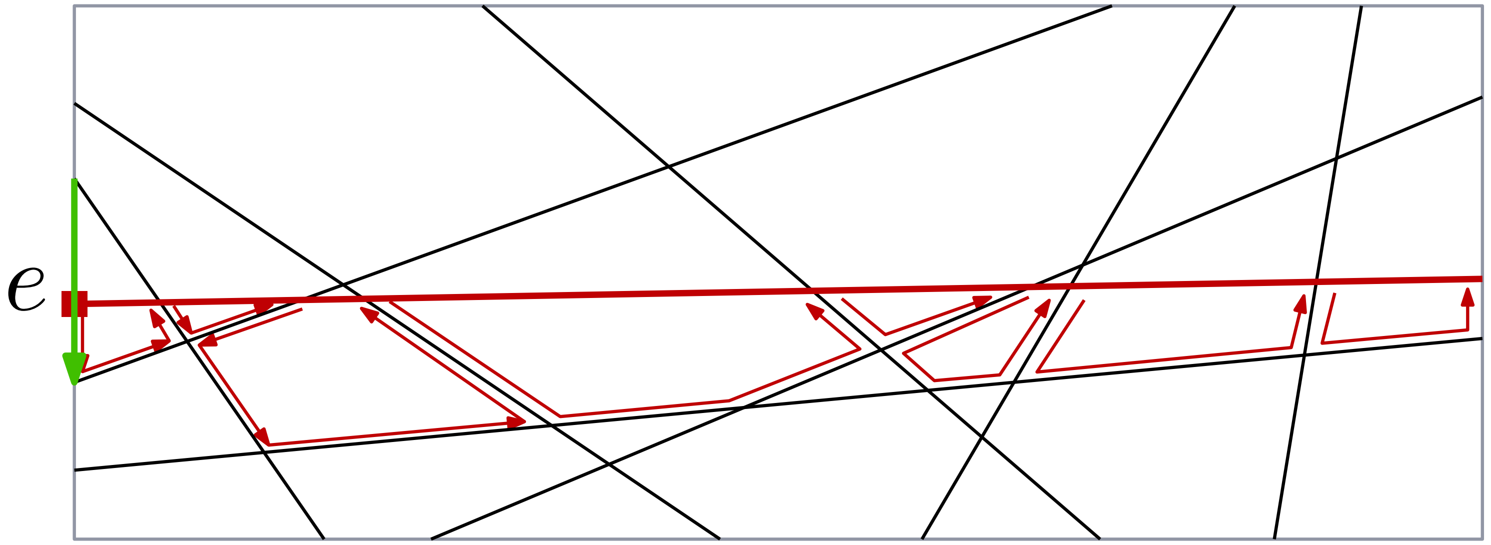

Running time?

Incremental construction of $\mathcal{A}(L)$

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Input: lines $L = \{\ell_1, \ldots, \ell_n\}$

Output: DCEL $\mathcal{D}$ for $\mathcal{A}(L)$

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2: **for** $i \leftarrow 1$ **to** n **do**
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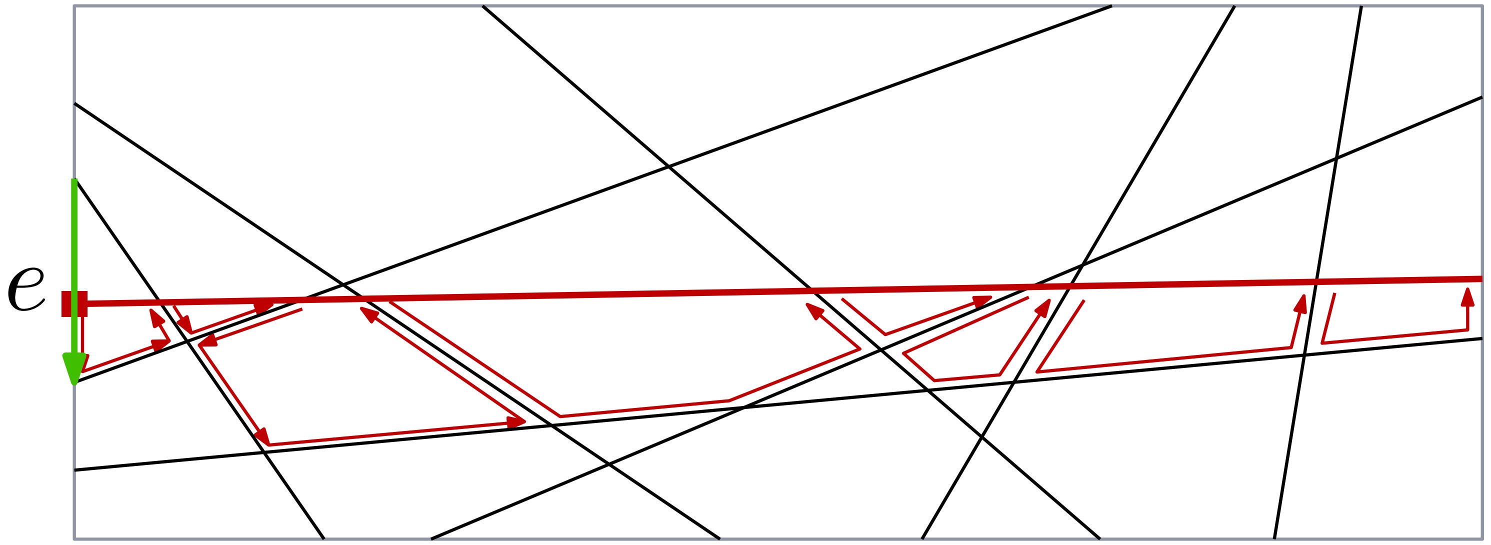

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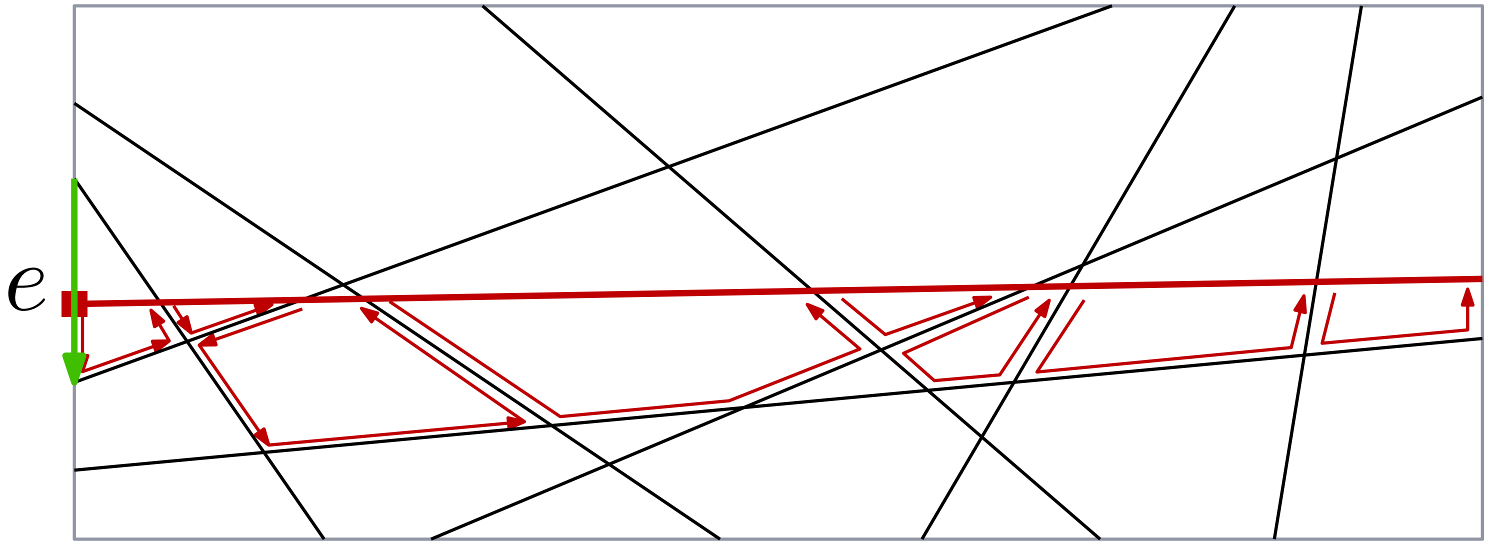

Running time?

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Input: lines $L = \{\ell_1, \dots, \ell_n\}$

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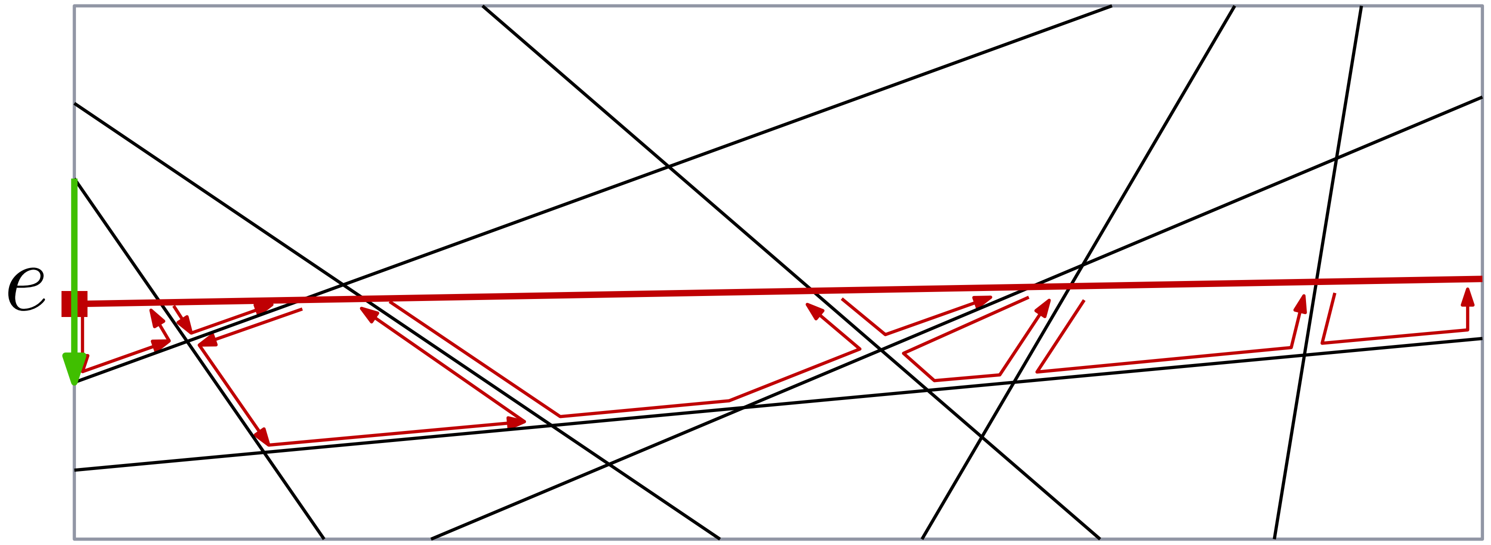
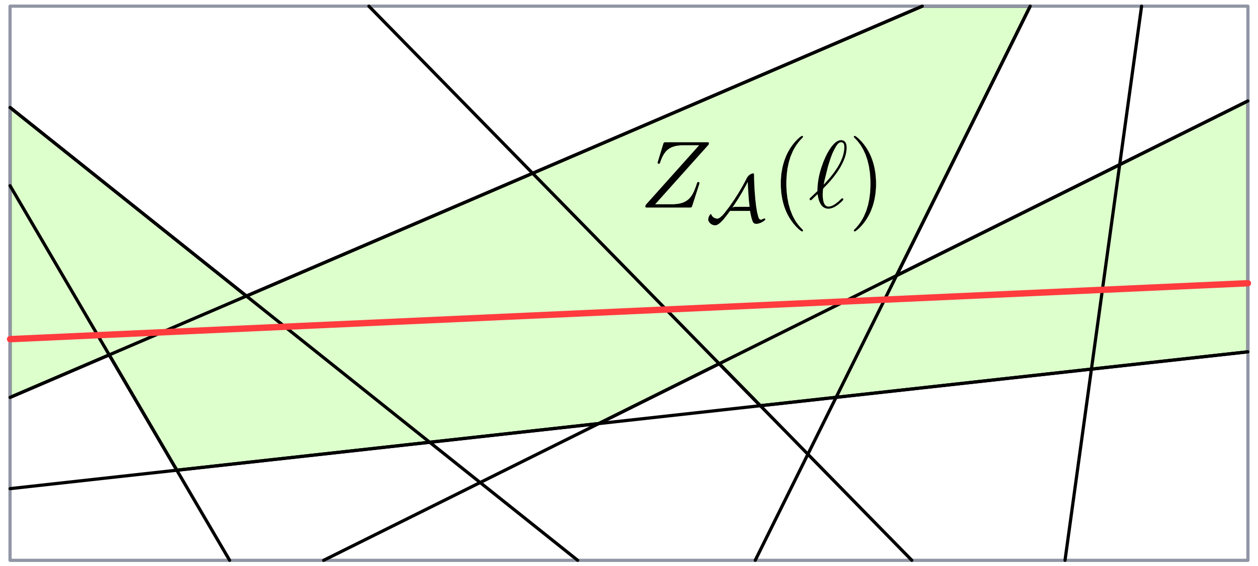


Running time?

Zone theorem

Definition: For arrangement $\mathcal{A}(L)$ and line $\ell \notin L$ the zone $Z_{\mathcal{A}}(\ell)$ is the set of all faces of $\mathcal{A}(L)$ whose closure intersect ℓ.



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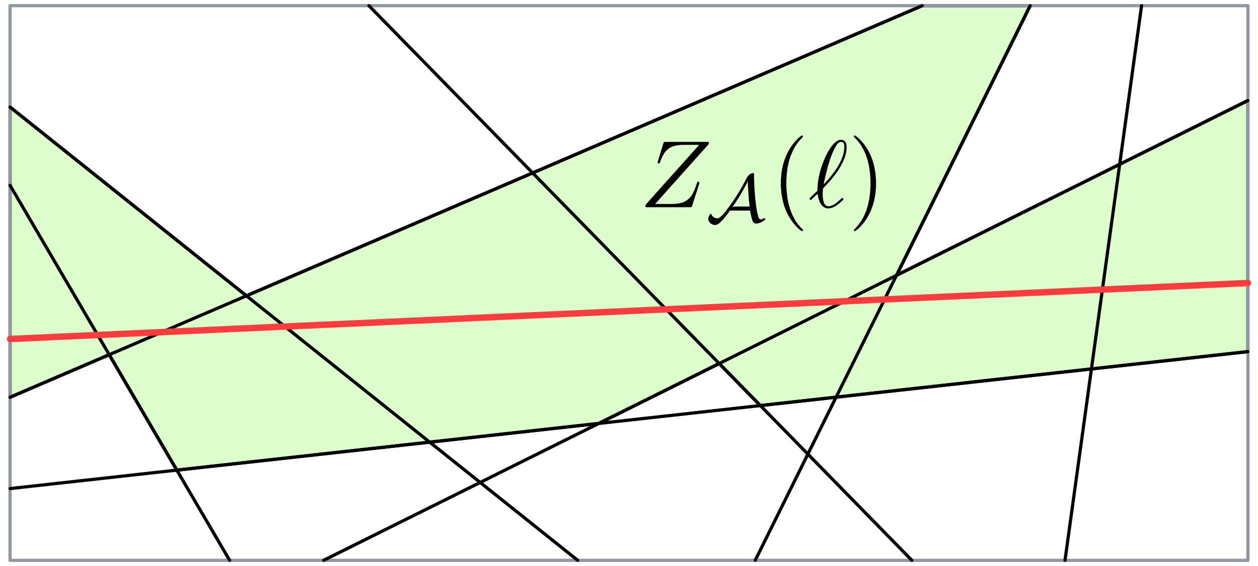

How many cells in $Z_{\mathcal{A}}(\ell)$?

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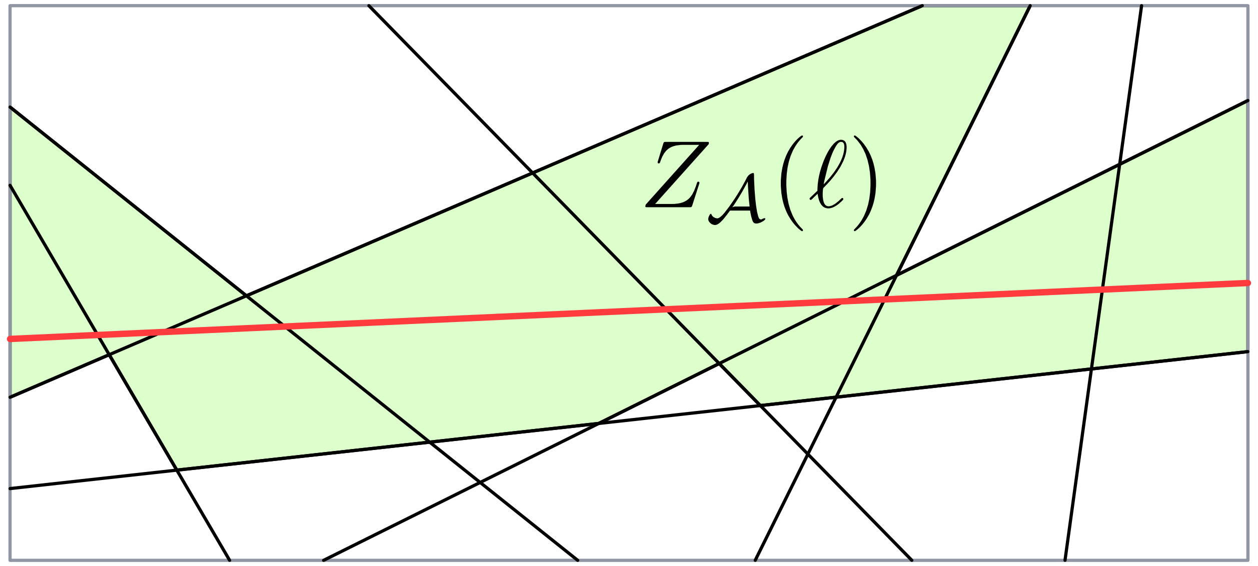

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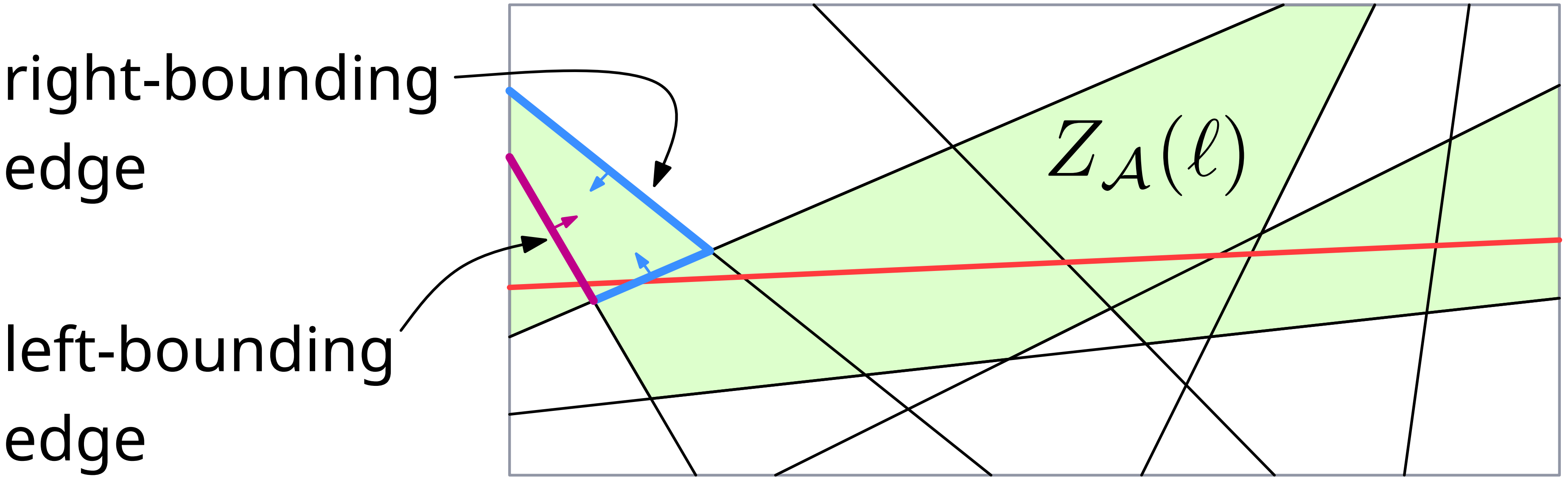

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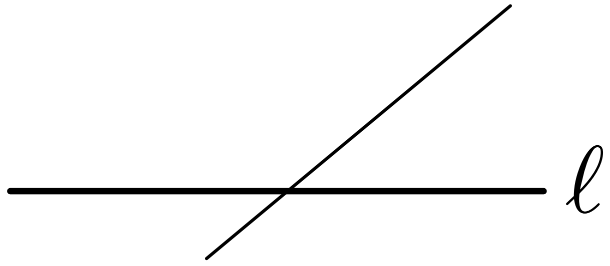
Proof:
- assume ℓ horizontal
- count *left-bounding* edges
- “insert” edges from left to right
- an insertion adds ≤ 5 left-bound edges



$n = 1$

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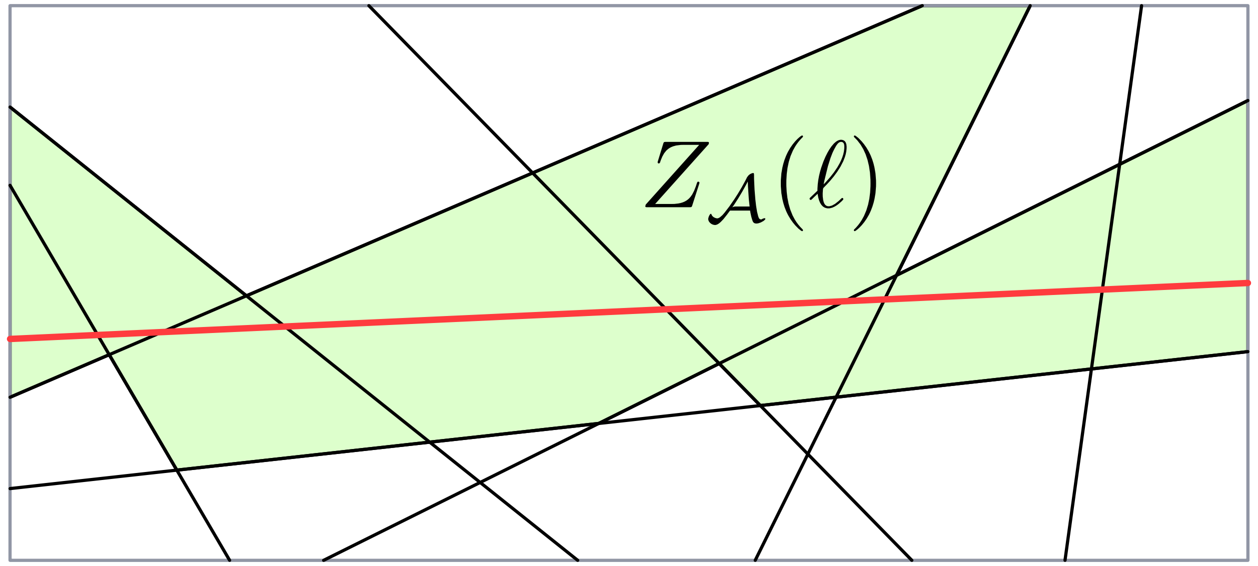

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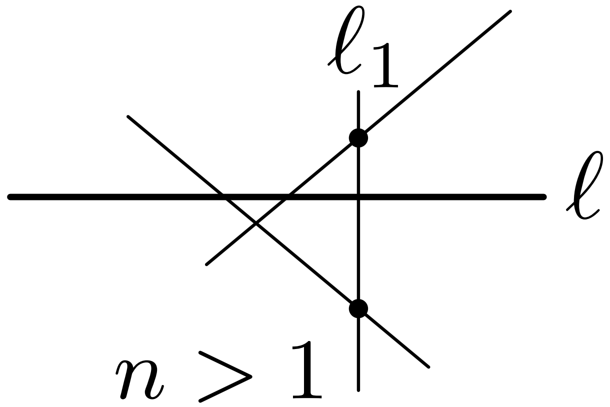

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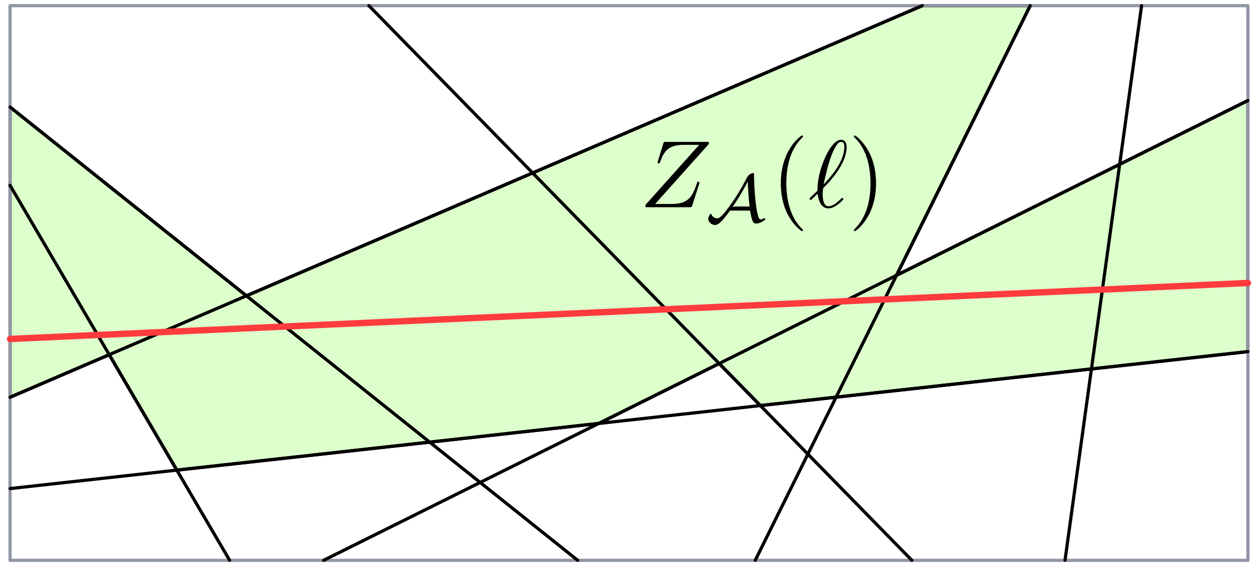


Theorem 2: For an arrangement $\mathcal{A}(L)$ of n lines and a line $\ell \notin \mathcal{A}(L)$ the zone $Z_{\mathcal{A}}(\ell)$ has at most $10n$ edges.

Theorem 3: The arrangement $\mathcal{A}(L)$ of n lines can be computed in $O(n^2)$ time and space.

Summary so far

Duality

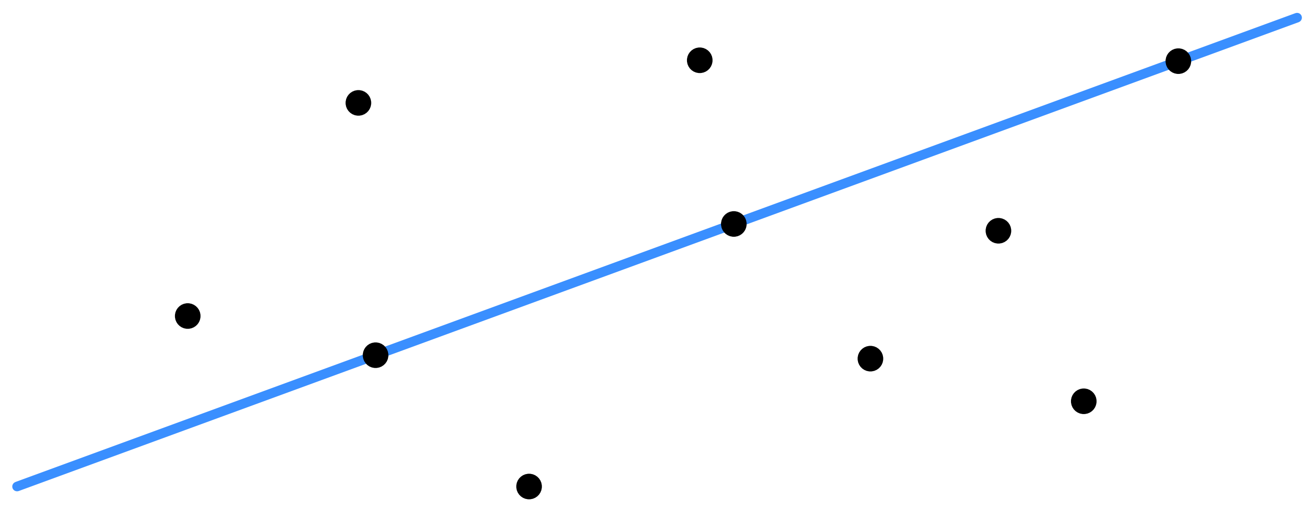
- preserves important properties
- gives new perspective

Arrangements of n lines

- $O(n^2)$ complexity
- $O(n^2)$ construction time
- $O(n)$ complexity of a zone

3 points on a line

Problem: Given a set P of n points in $\mathbb{R}^2$, determine whether there are three points on a line.



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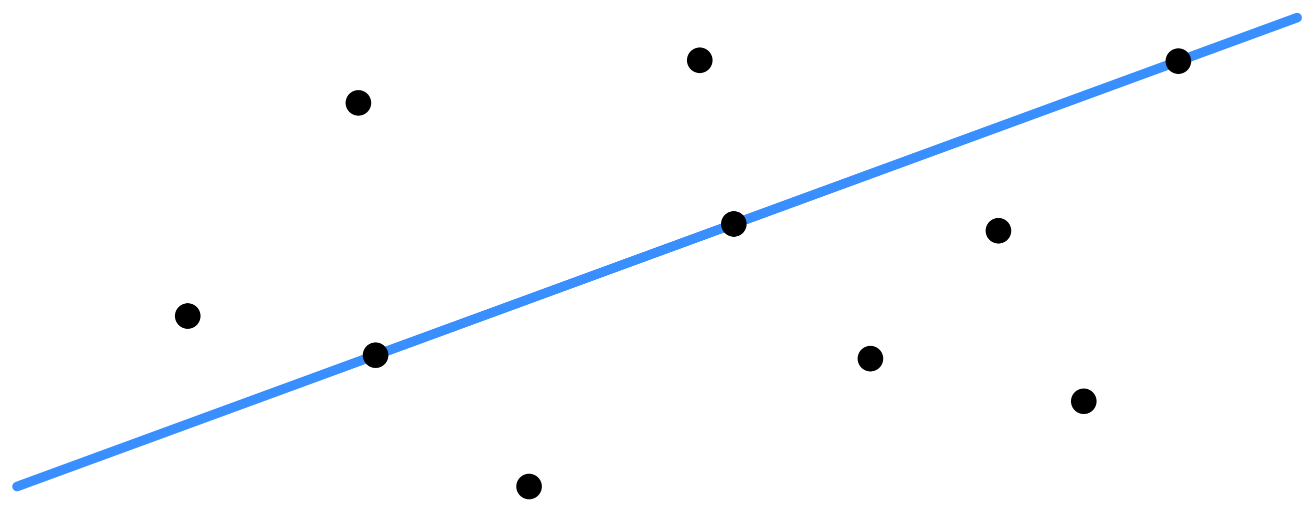

Algorithm:

- incrementally compute arrangement of dual lines
- stop when three lines go through a point

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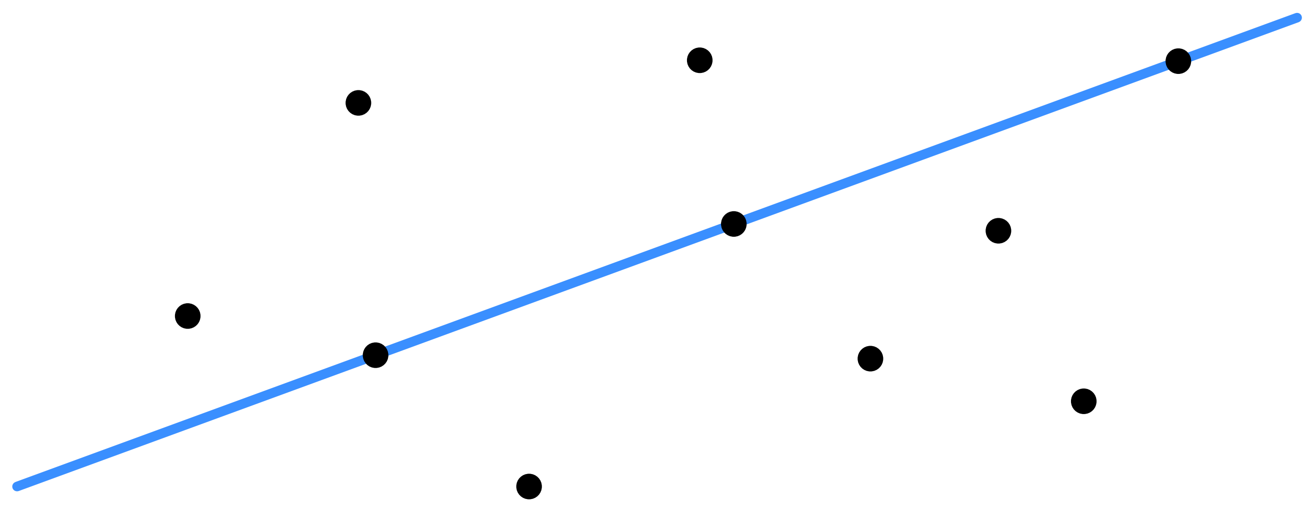

Algorithm:

- incrementally compute arrangement of dual lines
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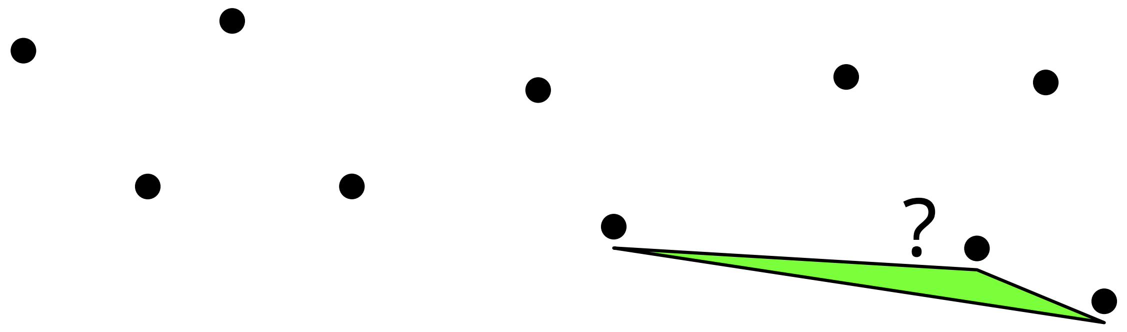
Running time: $O(n^2)$

Min-area triangle

Problem: Given a set P of n points in $\mathbb{R}^2$, find the smallest-area triangle with corners $p, q, r \in P$.



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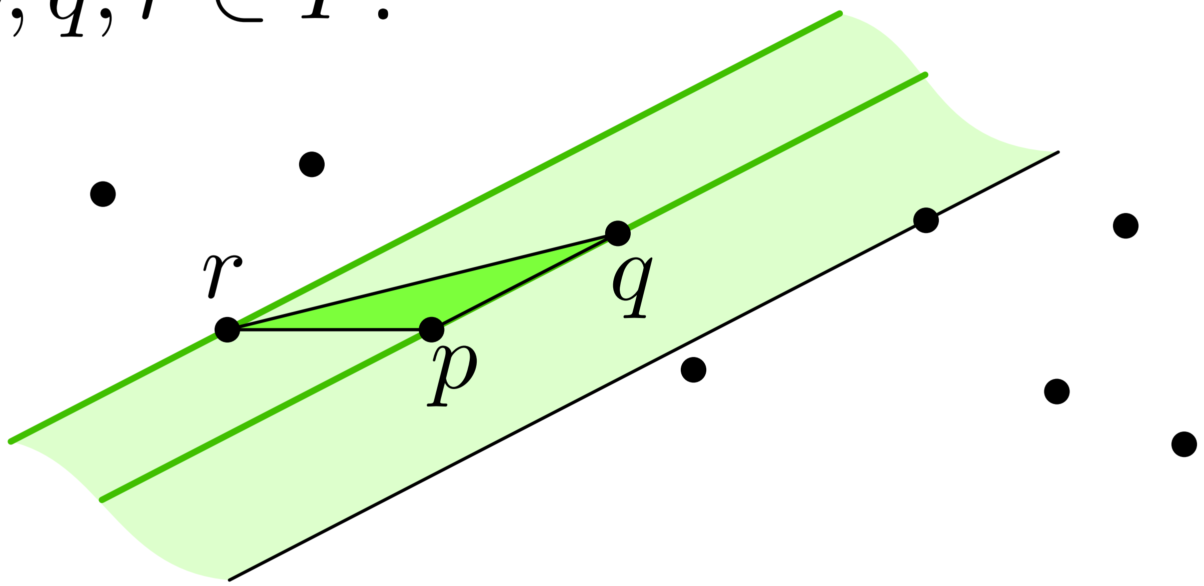

For $p, q \in P$, the point $r \in P \setminus \{p, q\}$ that minimizes $\triangle pqr$ lies on the boundary of the largest empty corridor for pq:

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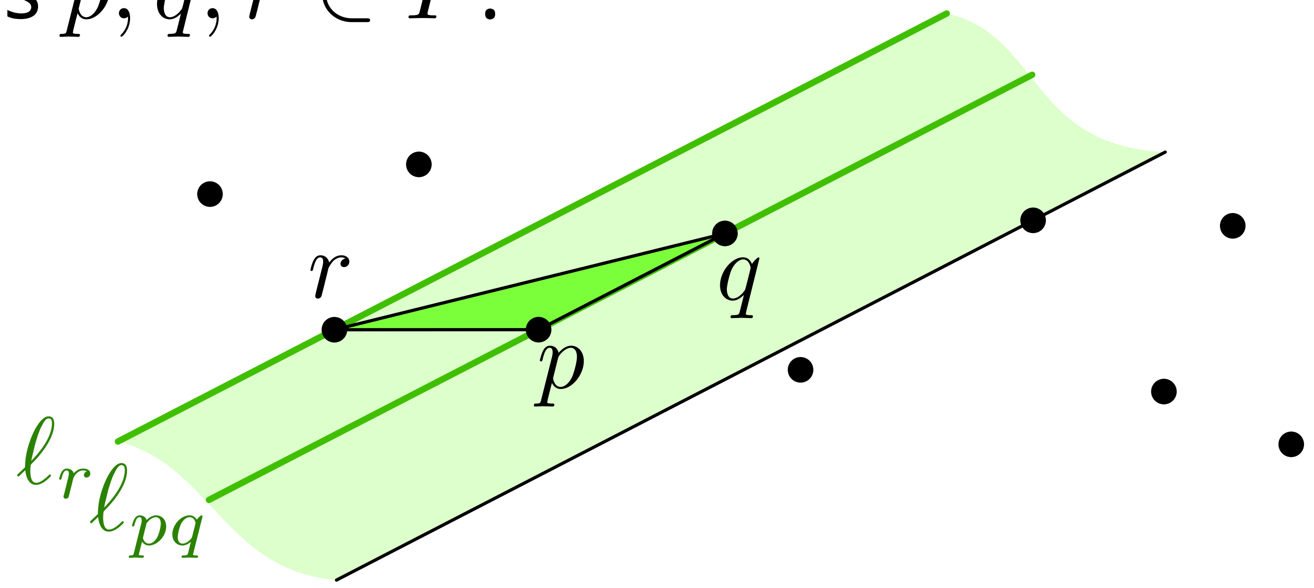

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- there is no point from P between ℓ_{pq} and ℓ_r

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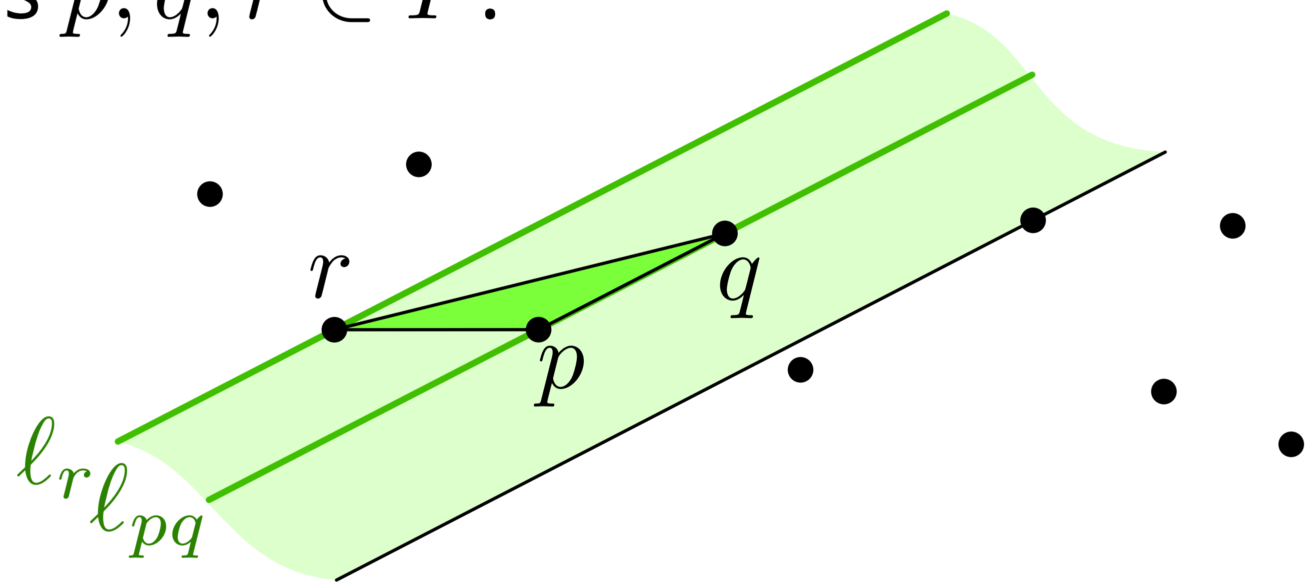

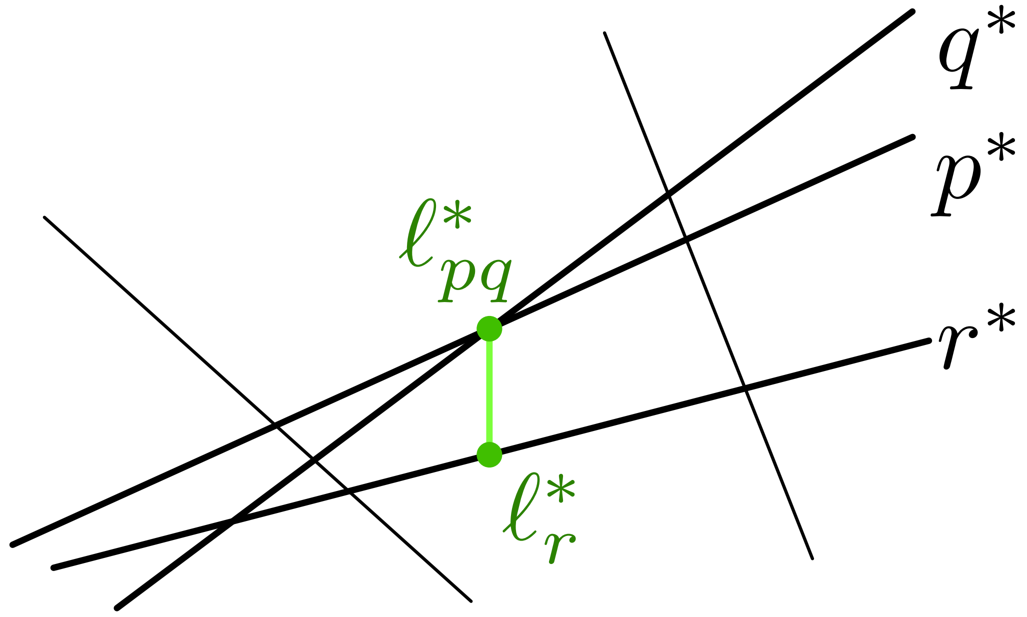

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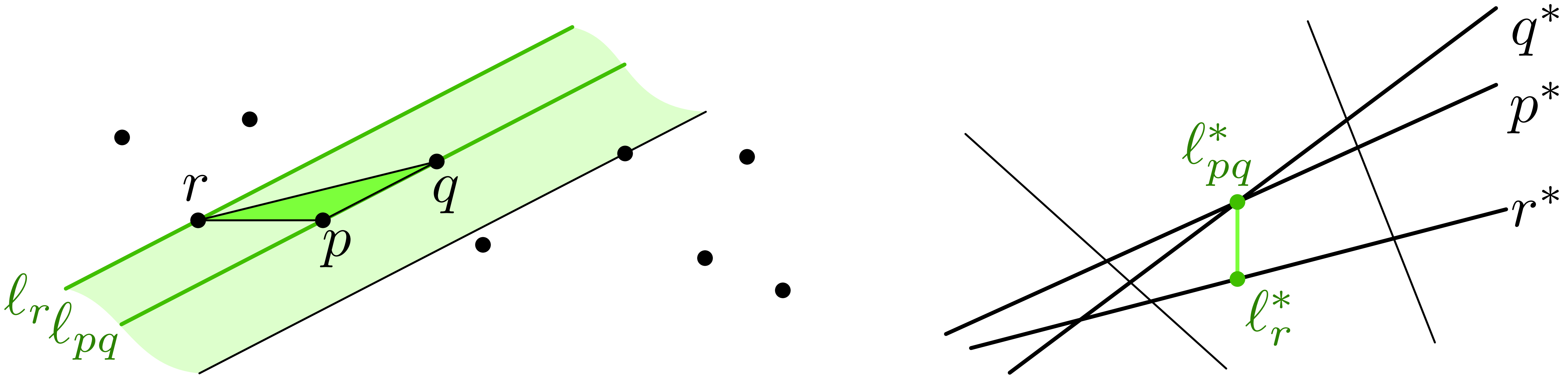
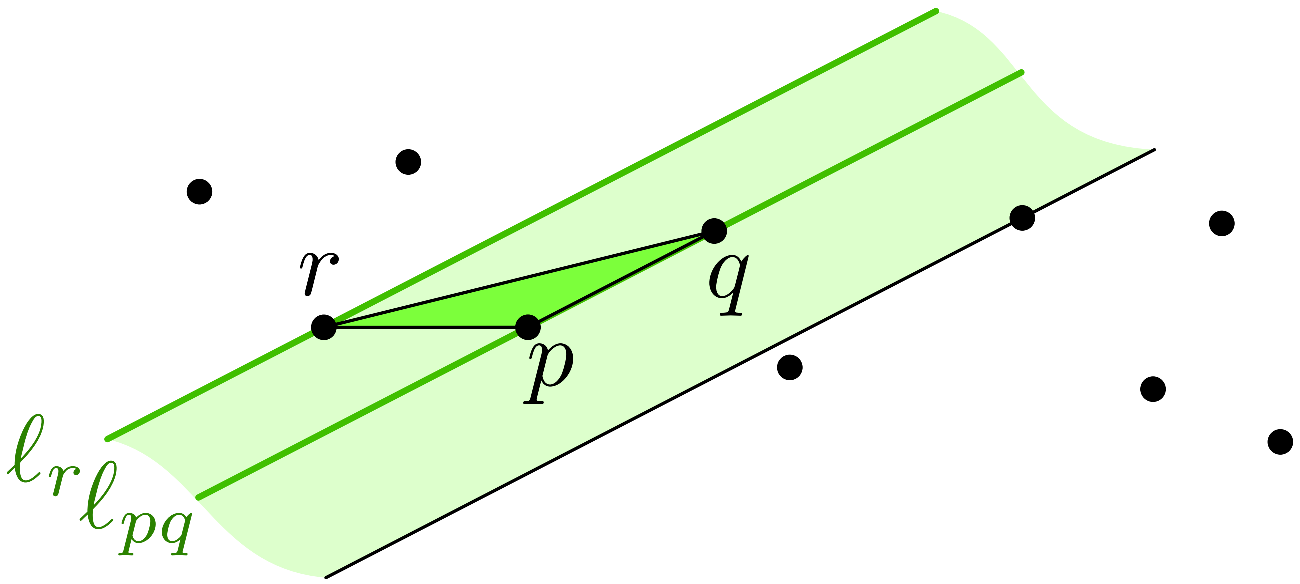
Dual:
- ℓ^*_r lies on r^*
- ℓ^*_r and ℓ^*_{pq} have the same x-coordinate
- no line $p^* \in P^*$ intersects $\overline{\ell^*_r \ell^*_{pq}}$

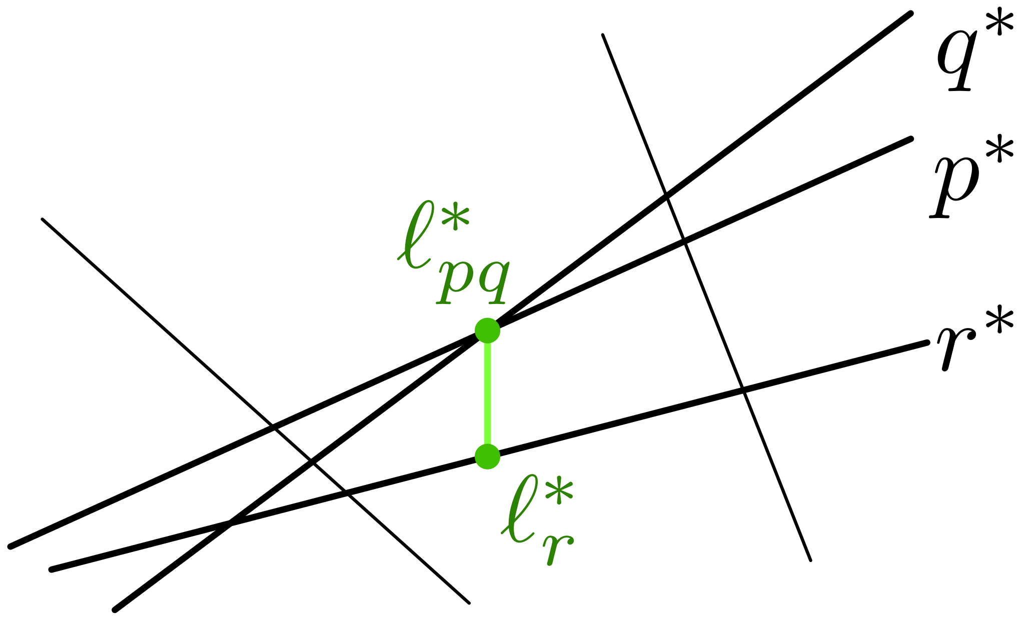
Min-area triangle



- ℓ_r^* lies vertically above or below ℓ_{pq}^* in a face $\mathcal{A}(P^*) \Rightarrow$ two candidates per ℓ_{pq}^*

Min-area triangle





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- compute arrangement $\mathcal{A}(P^*)$ in $O(n^2)$ time

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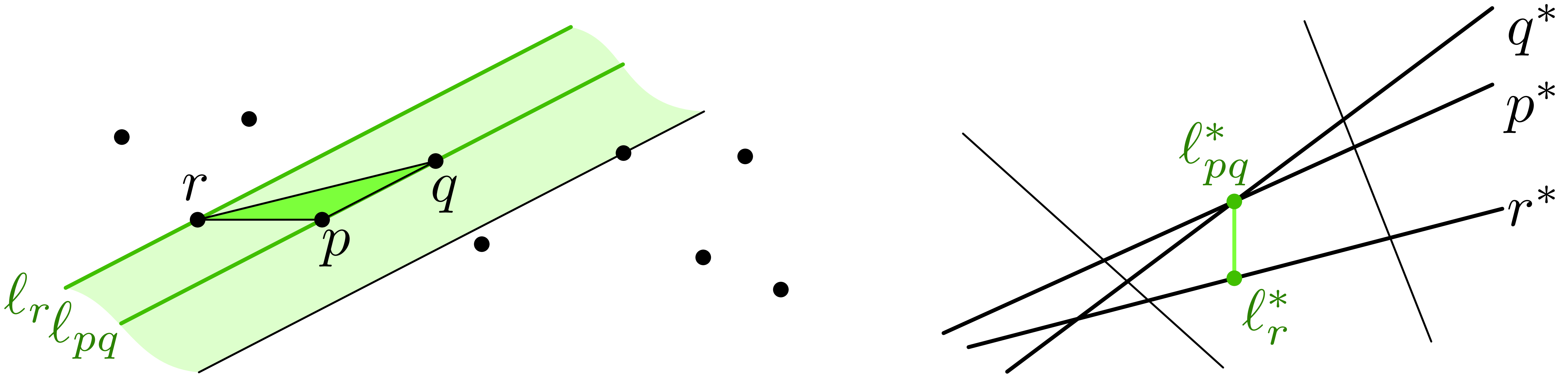
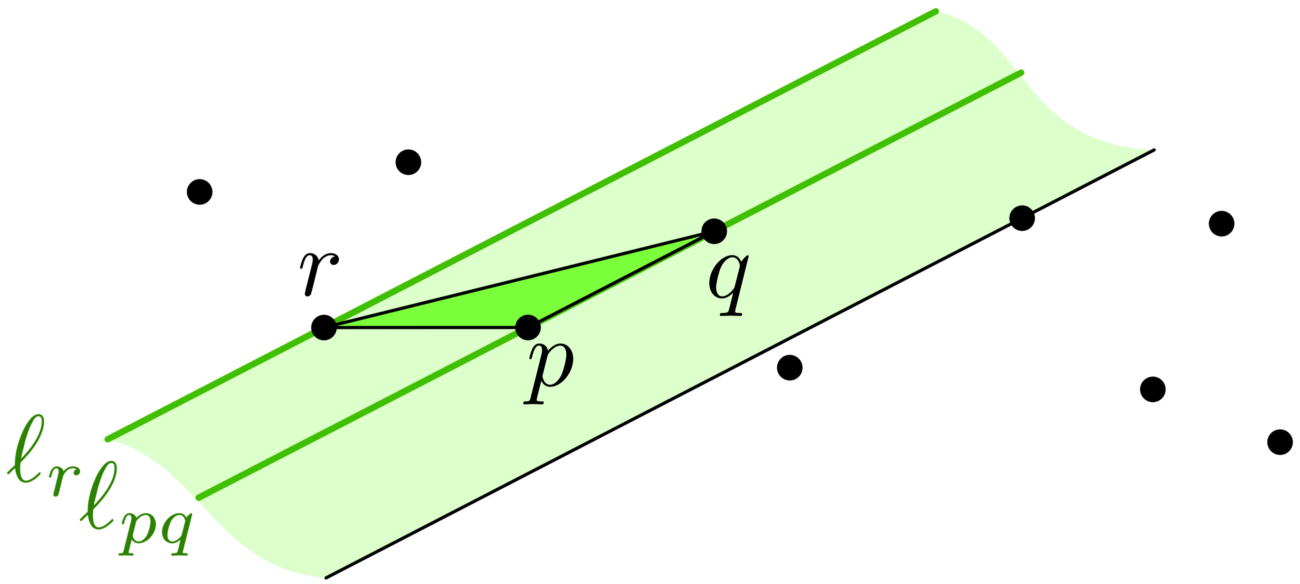
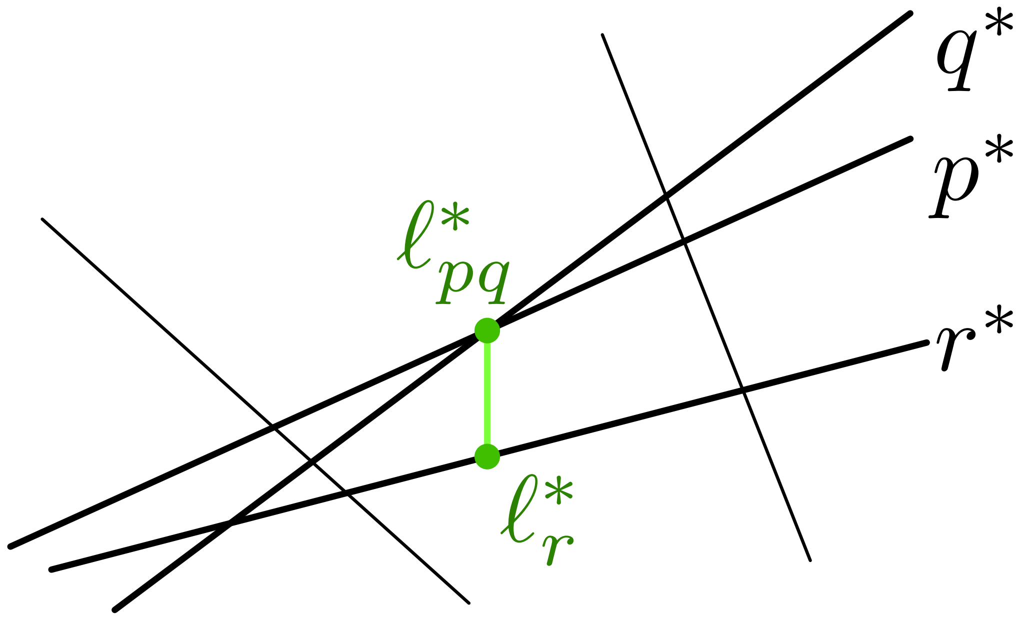


- ℓ^*_r lies vertically above or below ℓ^*_{pq} in a face $\mathcal{A}(P^*)$ $\Rightarrow$ two candidates per ℓ^*_{pq}
- compute arrangement $\mathcal{A}(P^*)$ in $O(n^2)$ time
- in every face compute vertical neighbor of all vertices
  $\Rightarrow$ time linear in complexity of the face

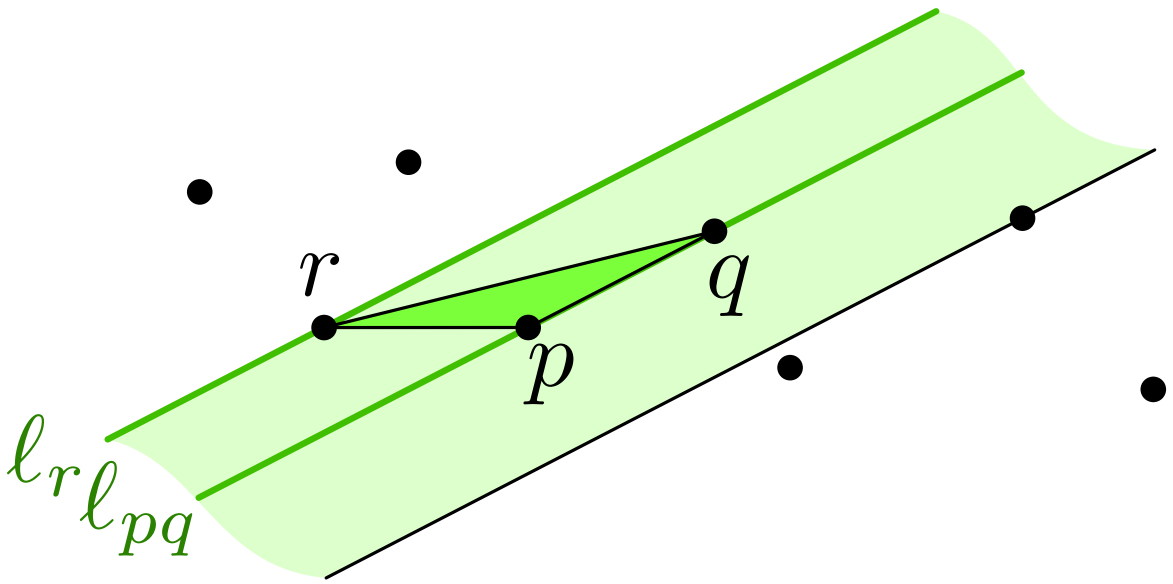
Min-area triangle

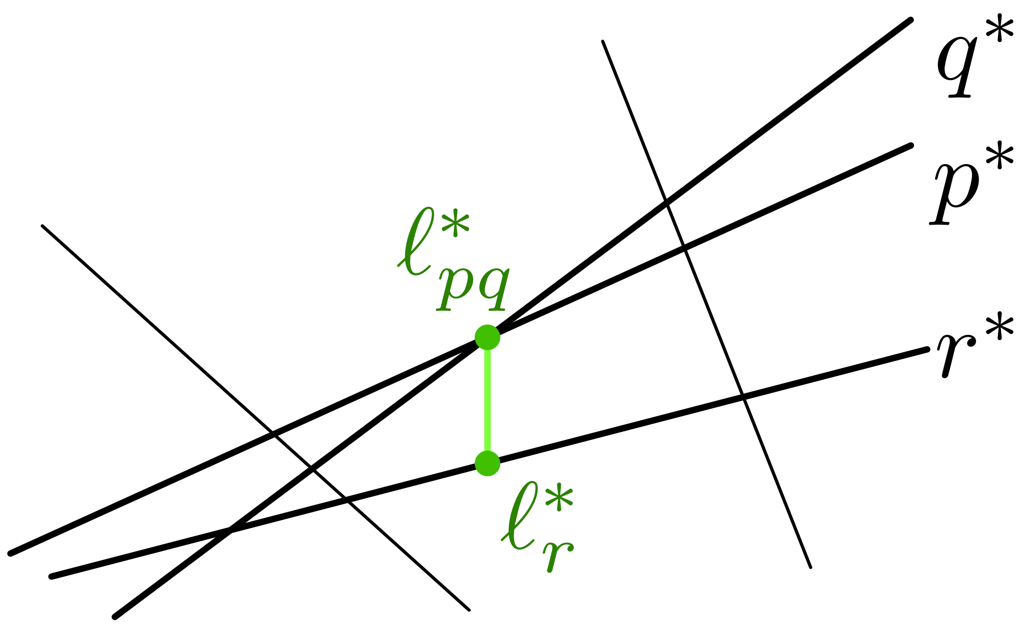




- ℓ^*_r lies vertically above or below ℓ^*_{pq} in a face $\mathcal{A}(P^*)$ $\Rightarrow$ two candidates per ℓ^*_{pq}
- compute arrangement $\mathcal{A}(P^*)$ in $O(n^2)$ time
- in every face compute vertical neighbor of all vertices
  $\Rightarrow$ time linear in complexity of the face
- for all $O(n^2)$ candidate pairs $\ell^*_{pq}\ell^*_r$:
  compute the area $\triangle pqr$ in $O(1)$ time

Min-area triangle





- ℓ_r^* lies vertically above or below ℓ_{pq}^* in a face $\mathcal{A}(P^*) \Rightarrow$ two candidates per ℓ_{pq}^*
- compute arrangement $\mathcal{A}(P^*)$ in $O(n^2)$ time
- in every face compute vertical neighbor of all vertices
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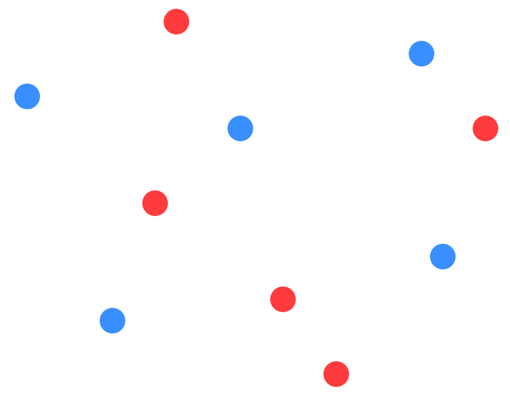
- for all $O(n^2)$ candidate pairs $\ell_{pq}^* \ell_r^*$:
  compute the area $\triangle pqr$ in $O(1)$ time
- find minimum in $O(n^2)$ time

Ham-sandwich cut

Problem: Given a set P of n blue and red points in $\mathbb{R}^2$, is there are line that splits both point sets evenly?



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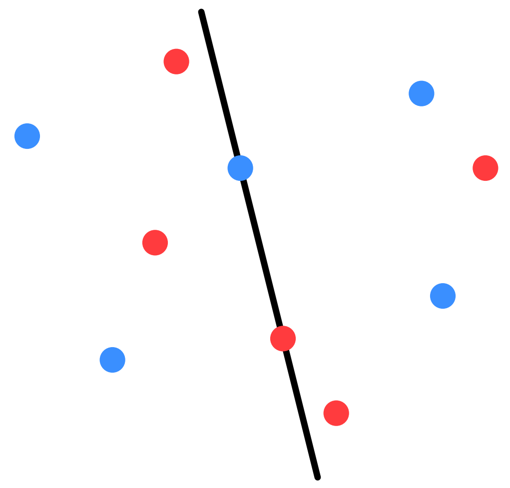

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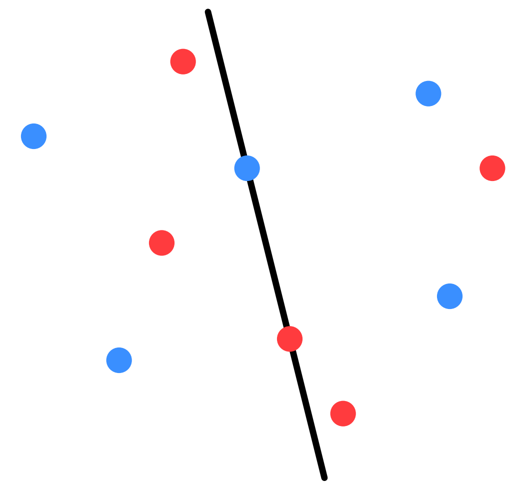

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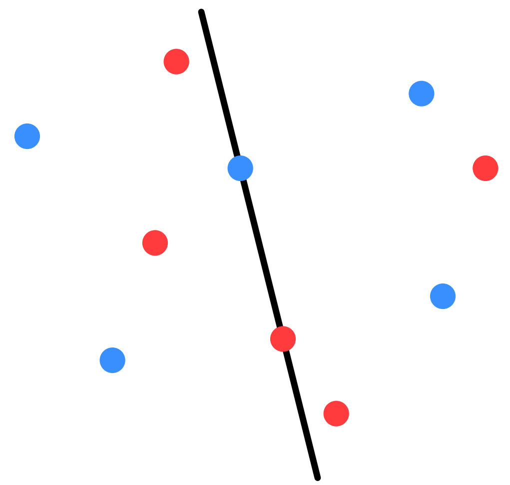

Solution:

- consider first one of the colors
- assume odd number of points, if not remove one
- for every slope there is a separating line

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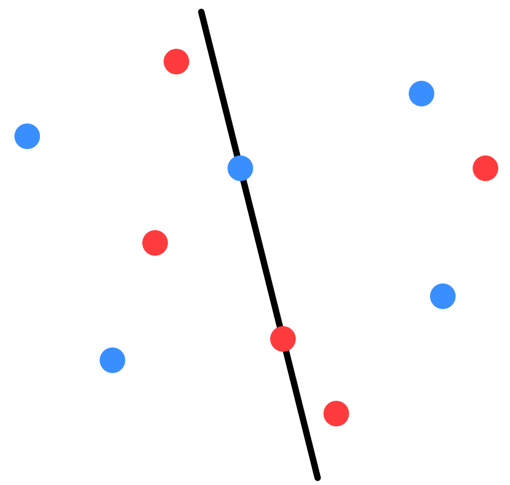
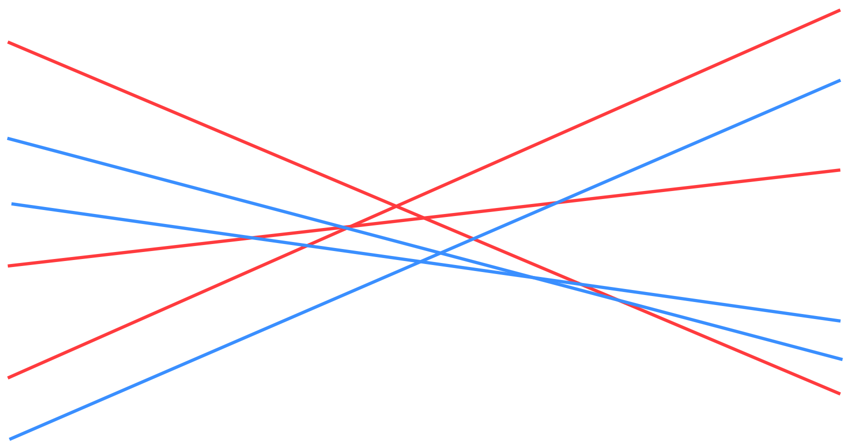


Solution:

- dual: the bisecting lines in the primal are points on the $n/2$-level of the arrangement of the dual lines
- points on the $n/2$-level have $n/2$ lines below them; $n/2$-level starts and ends on the line of median slope
- the $n/2$-levels of the two colors necessarily intersect (since median lines intersect)
- the dual of this intersection point splits both sets evenly



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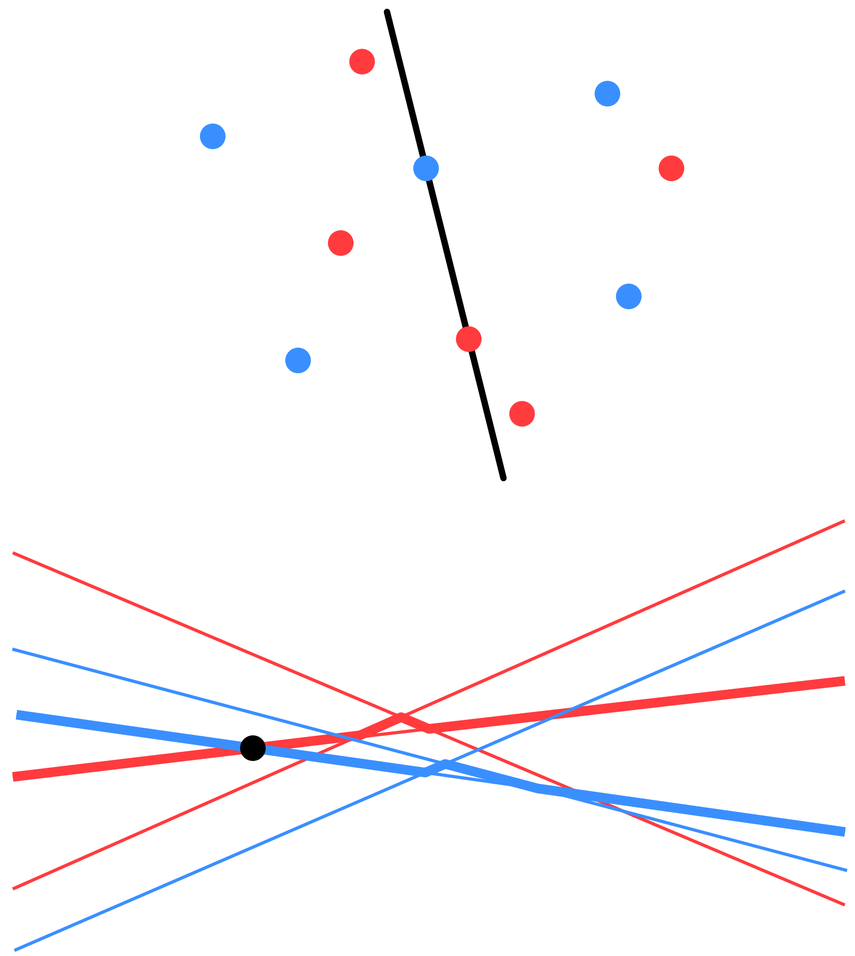

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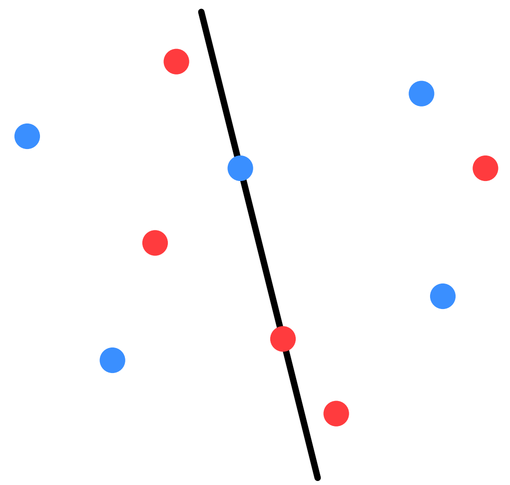

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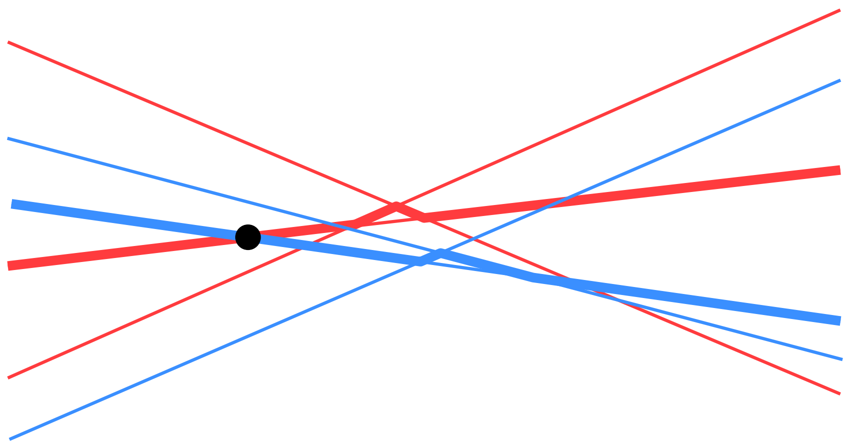

Note: $O(n^2)$-time algorithm, but $O(n)$ time algorithm is known

Discussion

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in discrete and computational geometry!

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Are there arrangements of objects other than lines?

Yes, for instance the arrangement of n line segments.
In it even a single cell can have super-linear complexity
$\Theta(n\alpha(n))$ [Sharir, Agarwal, 1995].