Amortized Analysis

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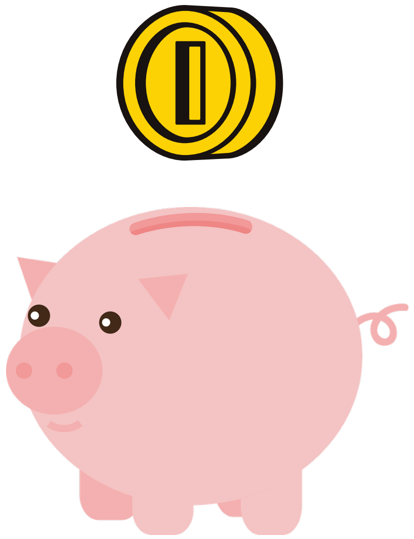

First Example: Multi-Pop Stack

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A: $\Theta(n)$

B: $\Theta(n \log n)$

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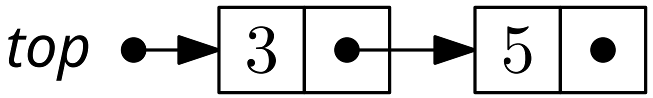
Push, Pop, *size*: $O(1)$ with simply linked list



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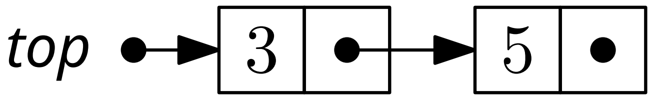

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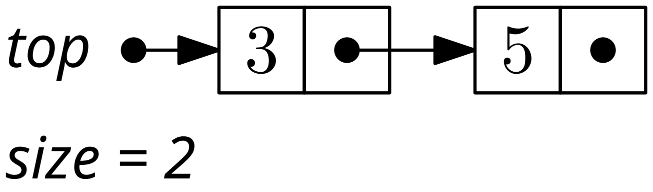

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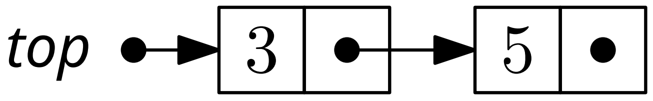

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too pessimistic!

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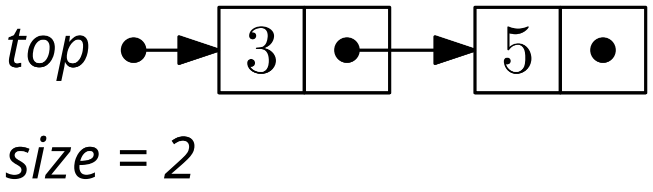

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Examples:

- Multi-pop stack
- Binary counter
- Dynamic array

Aggregate Analysis: Multi-Pop Stack

Aggregate („total sum”) Analysis:

- Compute worst-case running time $T(n)$ for sequence of n operations
- amortized cost of one operation: $\frac{T(n)}{n}$

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Given: Starting with an empty stack S, mixed sequence of n operations of PUSH(S, x), POP(S) and MULTI-POP(S, k)

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Multi-Pop(S, k)
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2        Pop(S)
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Running time $T(n)$:

- $\#$ Push Operationen $\leq n$: $O(n)$
- $\#$ Pop $\leq \#$ Push $\leq n$: $O(n)$, including Pop in Multi-Pop!

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Amortized running time per operation:

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Amortized running time per operation:

$$T(n)/n = O(n)/n = O(1)$$

$\rightarrow$ hence the $\Theta(n)$ runtime in first example

Accounting Method

Assign to each operation an amortized cost ("coins")

- $D_i =$ data structure after ith operation
- $c_i =$ actual cost of ith operation
- $\hat{c}_i =$ amortized cost of ith operation = assigned coins

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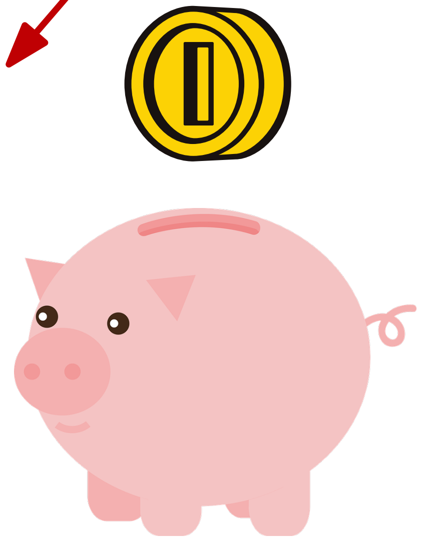

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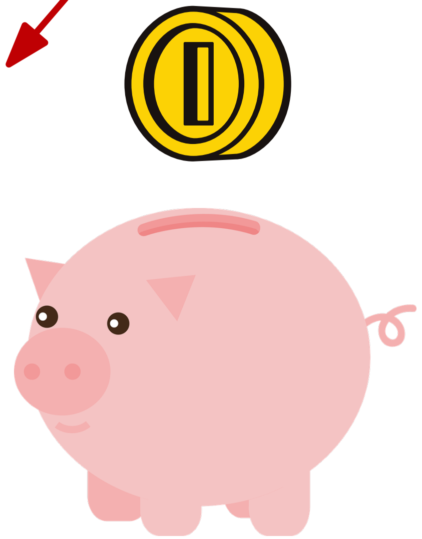

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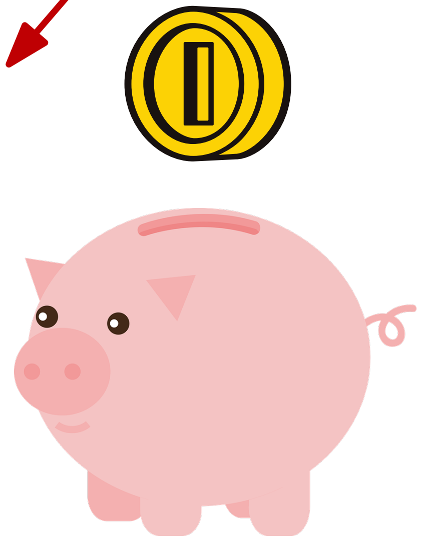

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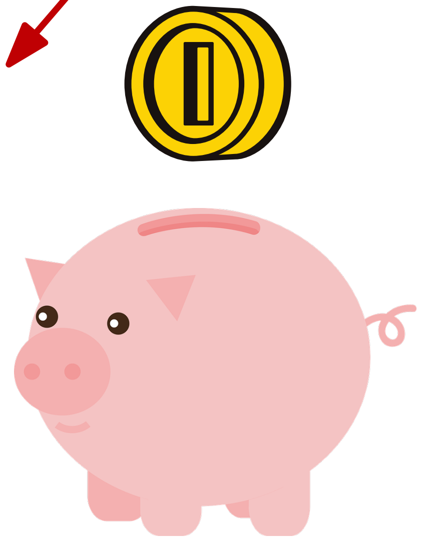
our task: choose amortized costs, such that invariant always holds

Accounting Method

Assign to each operation an amortized cost (“coins”)

- D_i $=$ data structure after ith operation
- c_i $=$ actual cost of ith operation
- $\hat{c}_i$ $=$ amortized cost of ith operation = assigned coins

larger, equal or smaller than c_i



- If $\hat{c}_i > c_i$: Save unused coins in D_i
- If $\hat{c}_i < c_i$: Pay with saved coins
- initial credit in D_0 is 0

cost invariant: saved coins ≥ 0

for all j : $\sum_{i=1}^{j} \hat{c}_i - \sum_{i=1}^{j} c_i \geq 0$

our task: choose amortized costs, such that invariant always holds

invariant implies: total cost of n operations $\leq$ sum of amortized costs

$$\boxed{\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i}$$

Accounting Method

Assign to each operation an amortized cost ("coins")

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$$\boxed{\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i}$$

intuition: measure running time in coins („time is money")

Accounting Method: Multi-Pop Stack

Actual costs per operation: 1 coin per PUSH or POP ($c_i = 1$)

k coins per MULTI-POP(S, k) ($c_i = k$)

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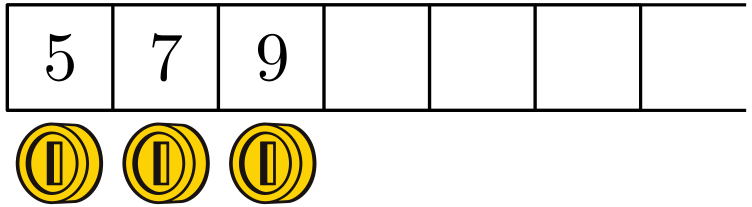

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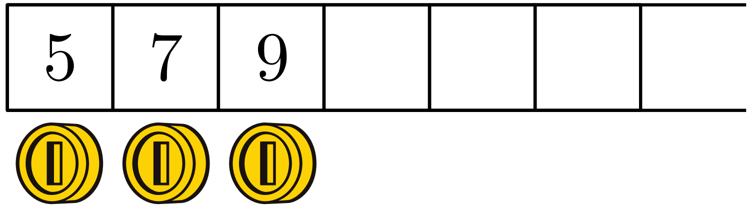

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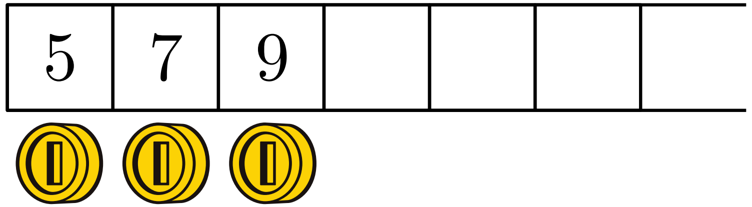

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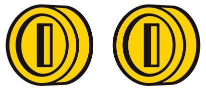

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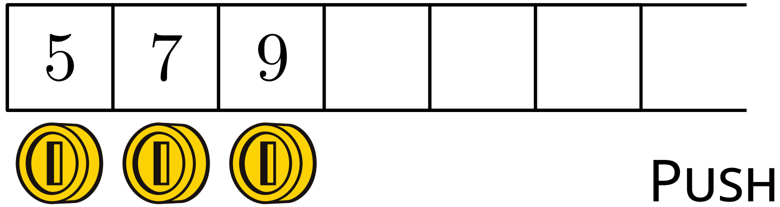

PUSH($S, 8$)

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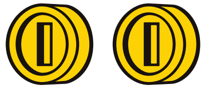

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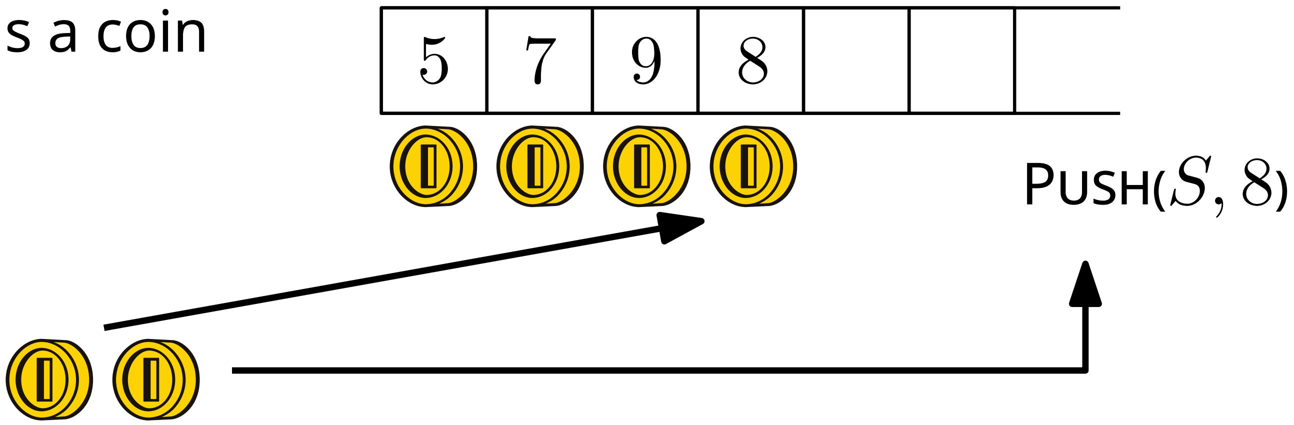

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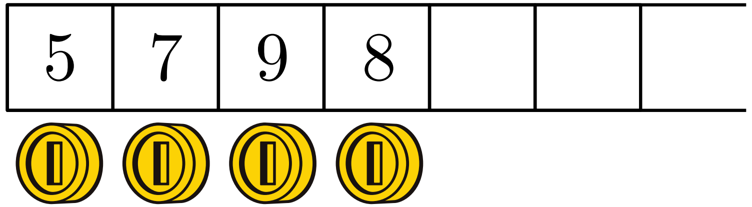

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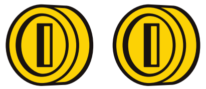

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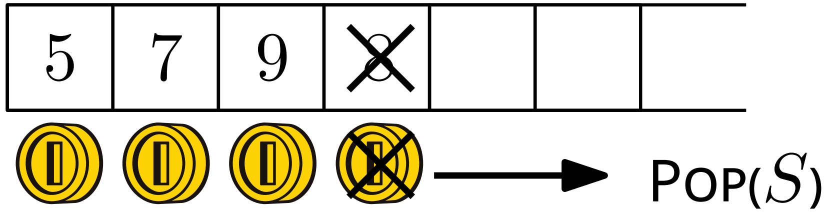

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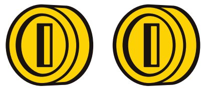

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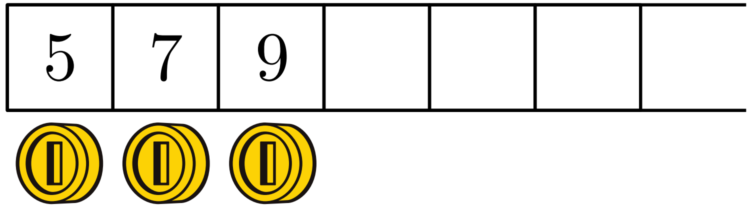

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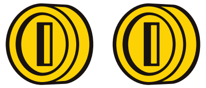

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How many coins do we need to assign to Multi-Pop?

A: 0

B: 1

C: k

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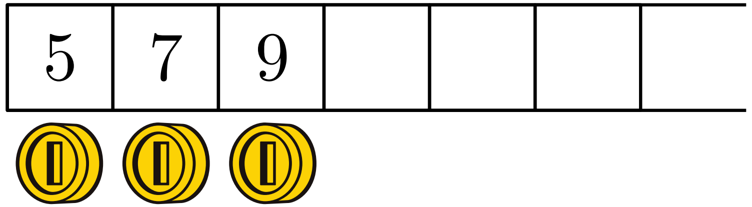

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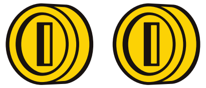

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How many coins do we need to assign to MULTI-POP?

A: 0 B: 1 C: k

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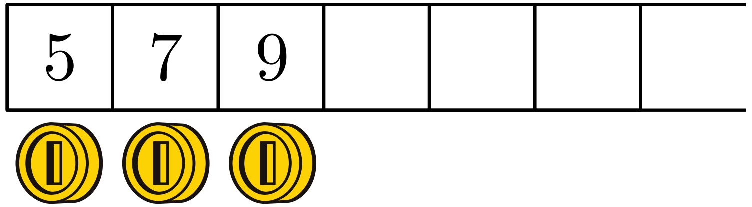

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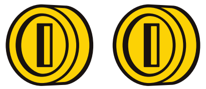

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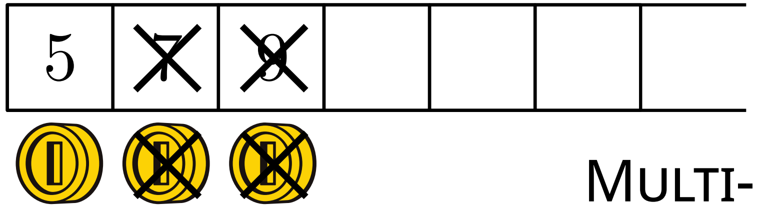

Multi-Pop($S, 2$)

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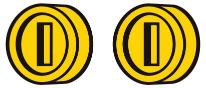

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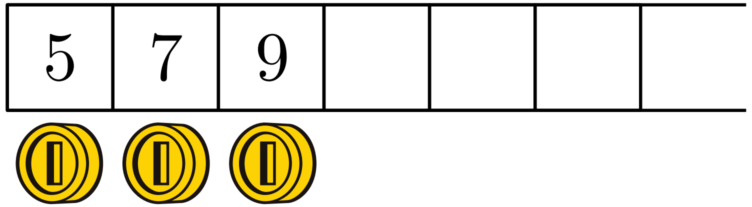

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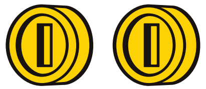

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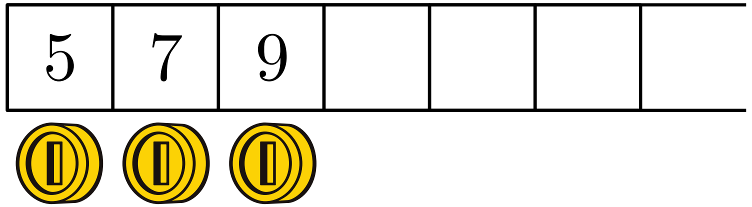

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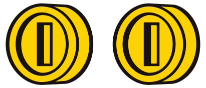

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Running time of a sequence of n Push, Pop, Multi-Pop operations (starting from an empty stack) is in $O(n)$. The amortized cost per operation is $O(\hat{c}_i) = O(1)$

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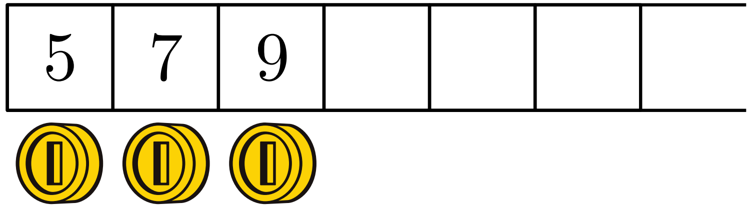

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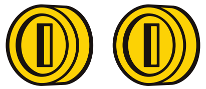

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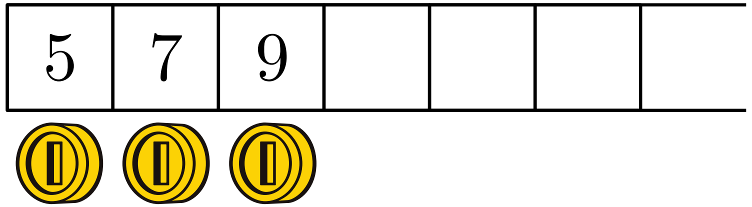

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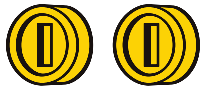

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- actual total costs $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \leq \sum_{i=1}^{n} 2 = 2n$

Overview

Amortized Analysis:

- Consider a sequence of n operations
- Take the average of the worst-case running time of the operations over the sequence

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Examples:

- multi-pop stack ✓
- binary counter
- dynamic array

Binary Counter

Algorithm. increment a k- bit binary counter

Representation as array. $A[j]$: jth least-significant bit

example: $k = 6$

value	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0
1	0	0	0	0	0	1

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5	0	0	0	1	0	1
6	0	0	0	1	1	0
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flipped bits

costs. number of bits flipped per operation ($= O(k)$)

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costs. number of bits flipped per operation ($= O(k)$)

running time. worst-case running time of sequence of n increments is $O(kn)$. ← too pessimistic!

Aggregate Analysis: Binary Counter

Aggregate Analysis. Analyze worst-case running time $T(n)$ of n increment operations.

value	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1
6	0	0	0	1	1	0
7	0	0	0	1	1	1
8	0	0	1	0	0	0

Aggregate Analysis: Binary Counter

Aggregate Analysis. Analyze worst-case running time $T(n)$ of n increment operations.

How often is $A[i]$ flipped?

value	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1
6	0	0	0	1	1	0
7	0	0	0	1	1	1
8	0	0	1	0	0	0

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1	0	0	0	0	0	1
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1
6	0	0	0	1	1	0
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1	0	0	0	0	0	1
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1
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3	0	0	0	0	1	1
4	0	0	0	1	0	0
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$$= \sum_{i=0}^{k-1} \lfloor n/2^i \rfloor$$

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$\cdots$

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total cost

$$= \sum_{i=0}^{k-1} \lfloor n/2^i \rfloor$$
$$\leq \sum_{i=0}^{k-1} n/2^i$$
$$< n \sum_{i=0}^{\infty} 1/2^i$$
$$= 2n$$

Aggregate Analysis: Binary Counter

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The worst-case running time $T(n)$ of a sequence of n increments (starting from 0) is $O(n)$.
The amortized running time of one increment is $T(n)/n = O(1)$

Accounting Method: Binary Counter

actual cost per operation: 1 coin per bit flipped

invariant: ???

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Which of the following invariants is suitable? (I.e., where do we need coins?)

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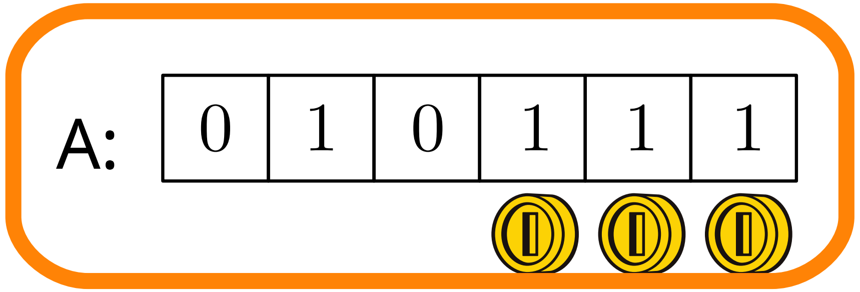

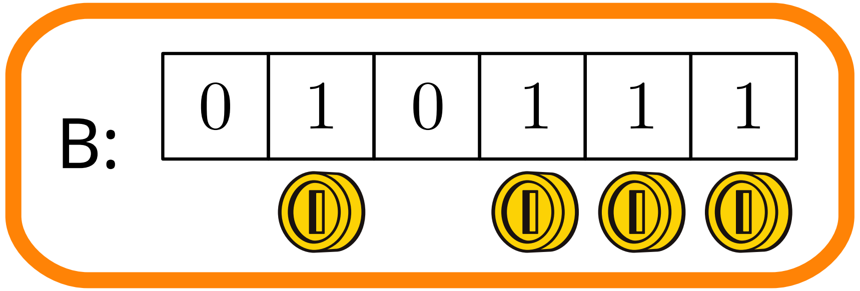

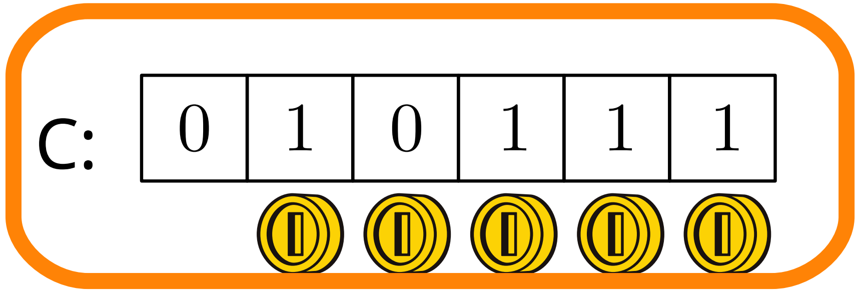

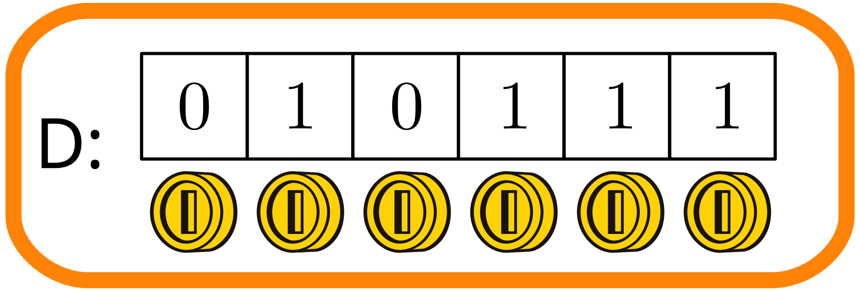

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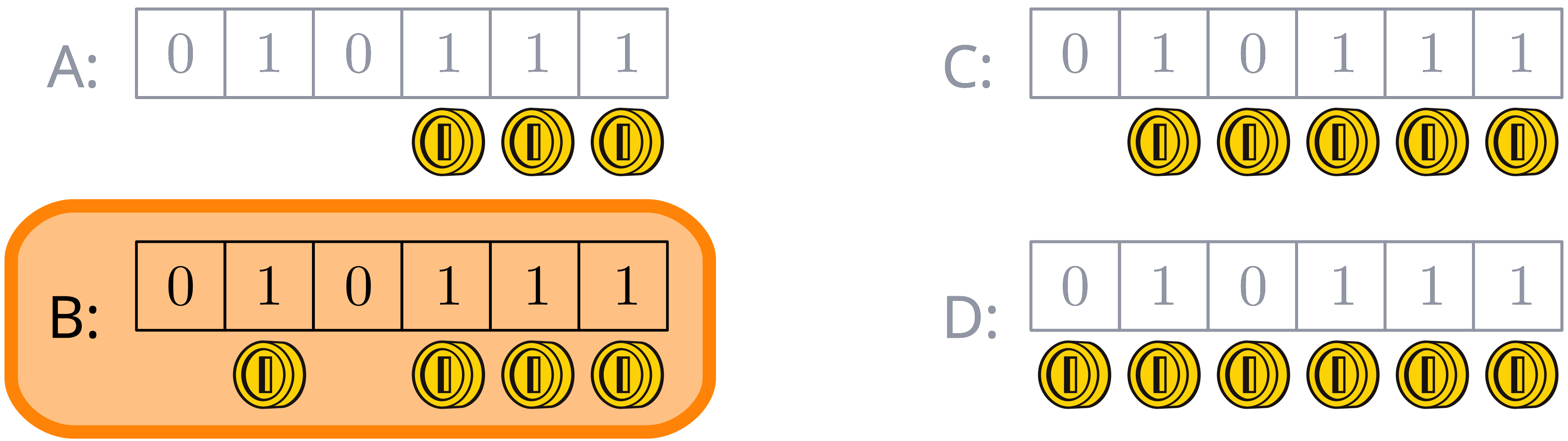

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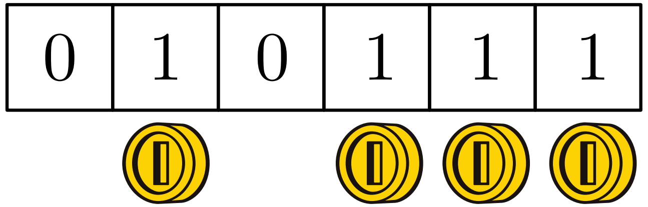

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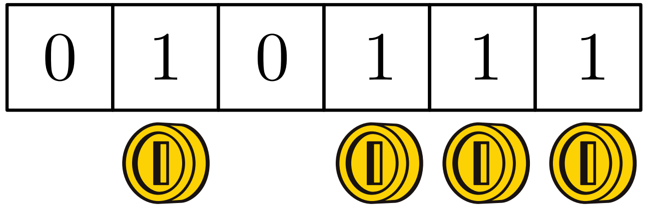

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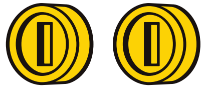

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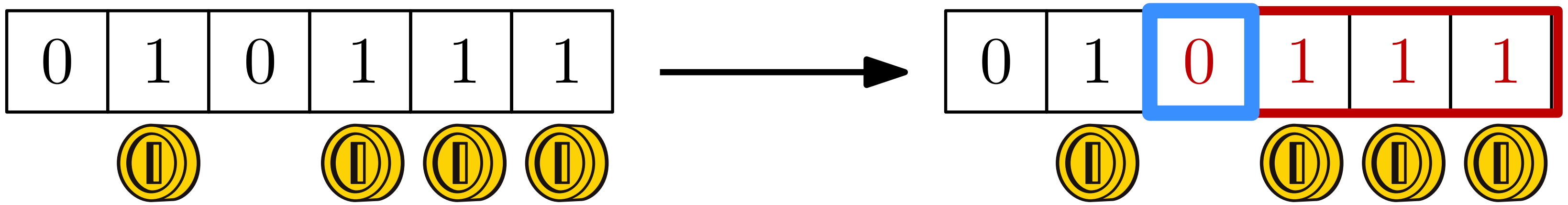

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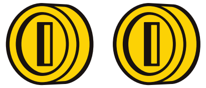
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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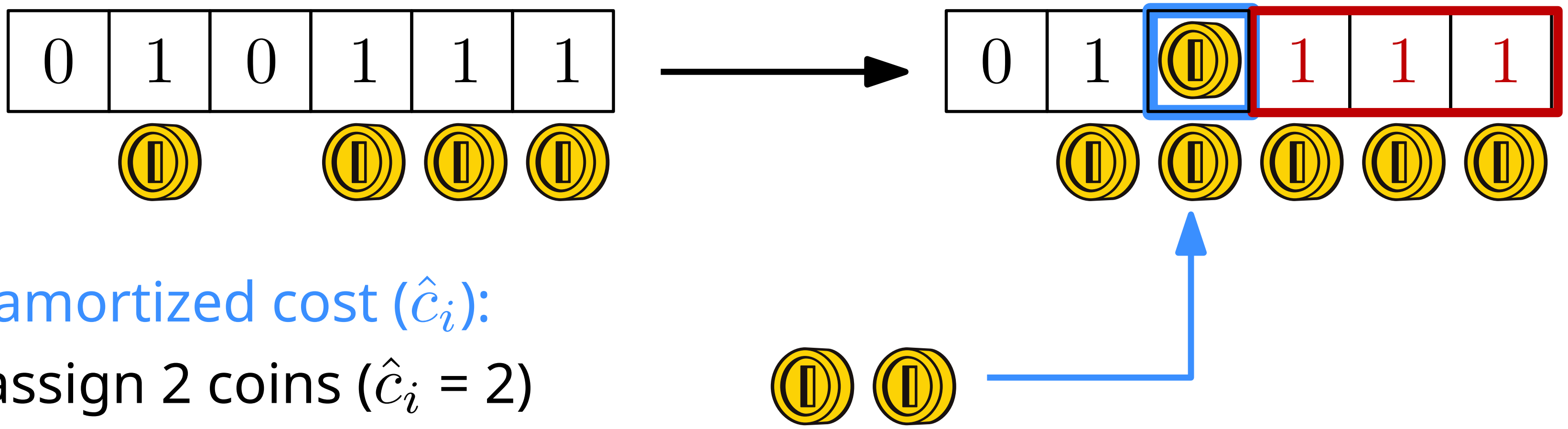

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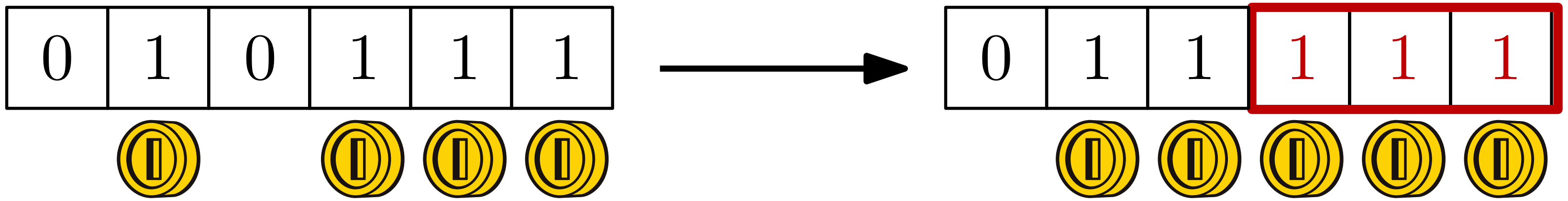

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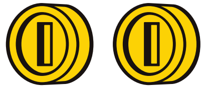
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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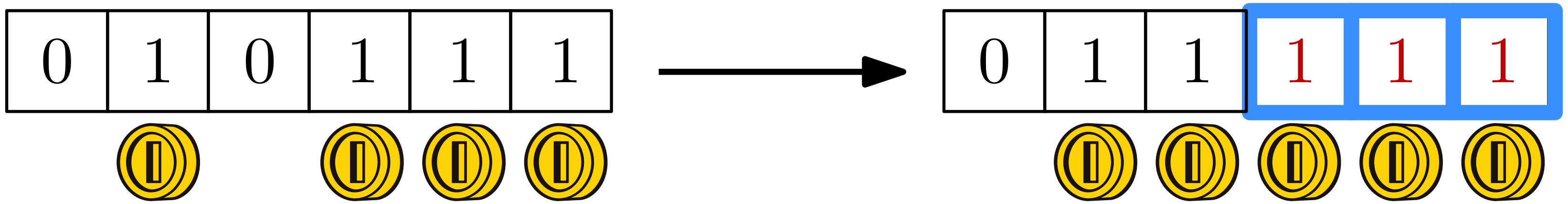

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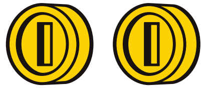

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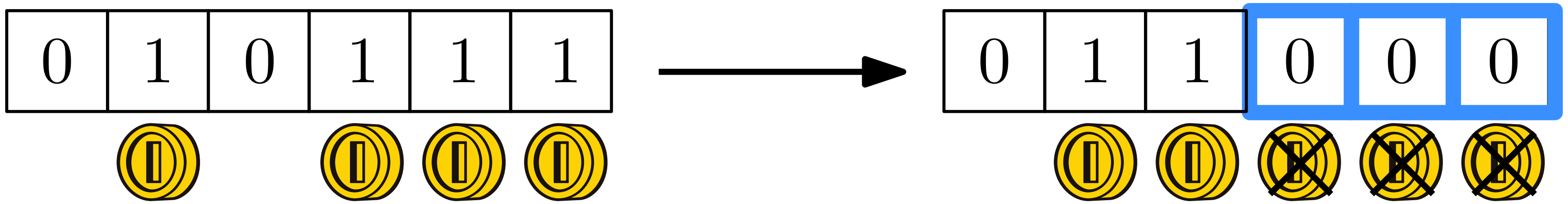

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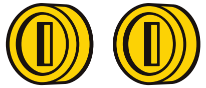
increment: assign 2 coins ($\hat{c}_i$ = 2) 

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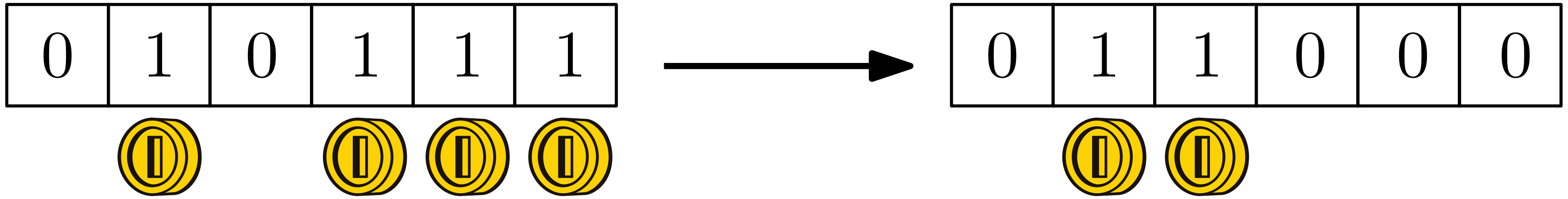

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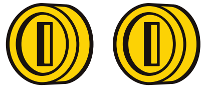

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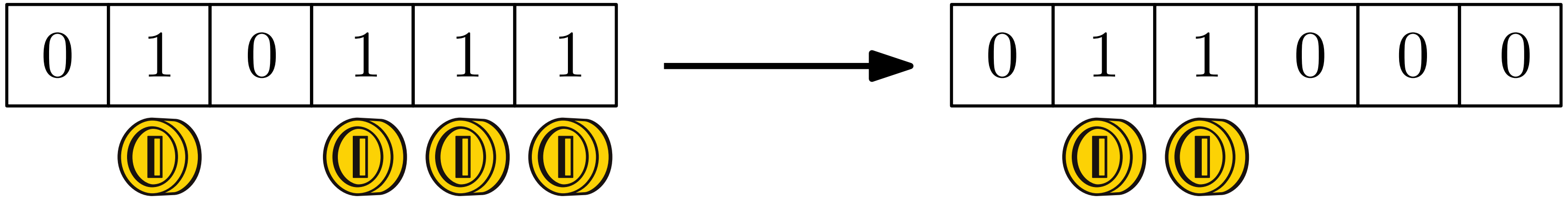

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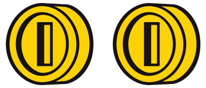

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The worst-case running time $T(n)$ of a sequence of n increments (starting from 0) is $O(n)$. The amortized running time of one increments is $O(\hat{c}_i) = O(1)$

- invariant $\Rightarrow$ number of coins in data structure $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$
- amortized cost per operation $\hat{c}_i \leq 2$
- actual total costs $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \leq \sum_{i=1}^{n} 2 = 2n$

Overview

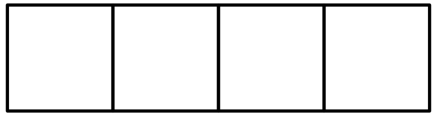
Methods:

- Aggregate analysis
- Accounting method

Examples:

- Multi-pop stack ✓
- Binary counter ✓
- dynamic array: insert, accounting method

Dynamic Array: Insert

(static) array 

+ random-access, compact

- fixed size: problem for priority queues (heaps), hash tables, . . .

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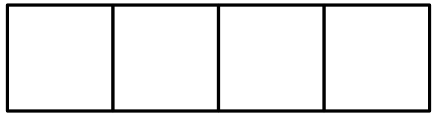
(static) array 

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dynamic array with append (insert at end)

frequently used: java ArrayList, C++ vector, Python List, . . .

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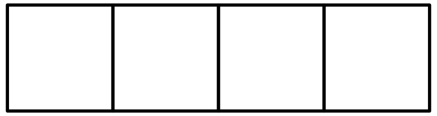

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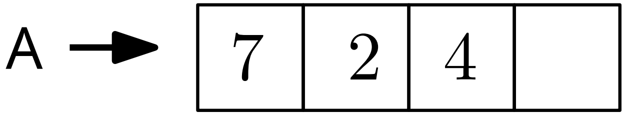
- use static array



append: $9, 3, \ldots$

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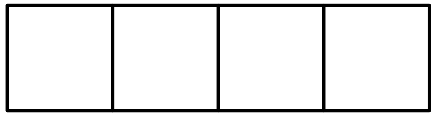

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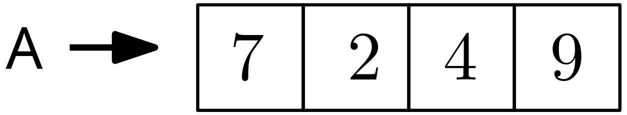

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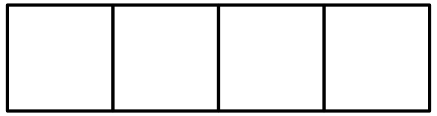

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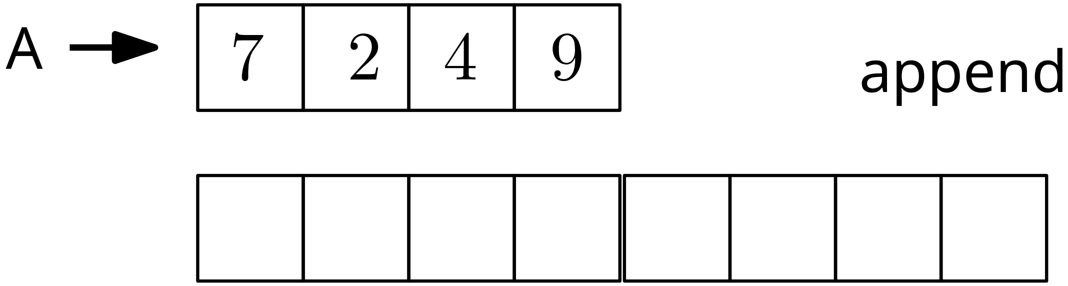

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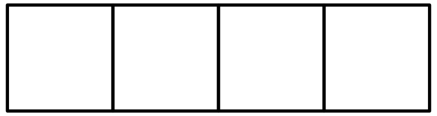

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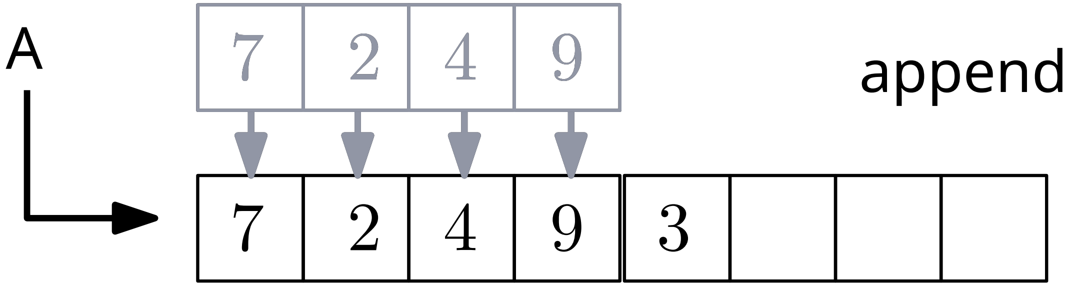

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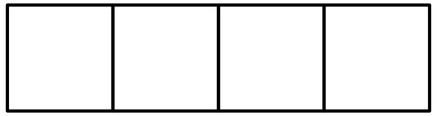
(static) array 

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cost per insertion: $O(n)$ if $n = 1+$ power of 2, otherwise $O(1)$

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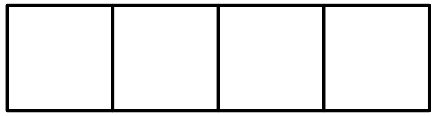
(static) array 

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- if full: create array of twice the size

  copy elements to new array (and delete old one)

too pessimistic!

cost per insertion: $O(n)$ if $n = 1+$ power of 2, otherwise $O(1)$

simple analysis: the running time of n insert operations is $O(n^2)$.

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actual cost of operation: 1 coin to insert 1 element

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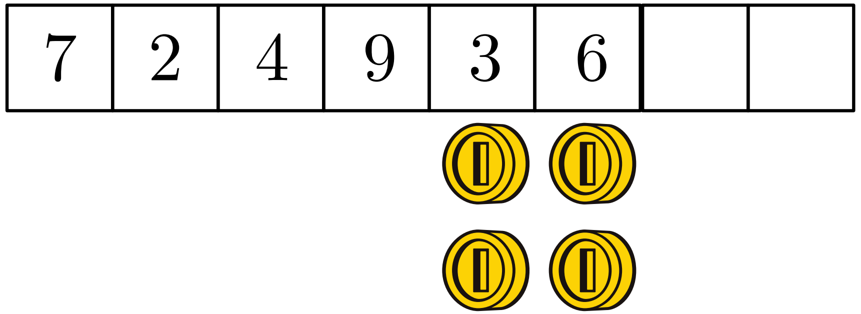

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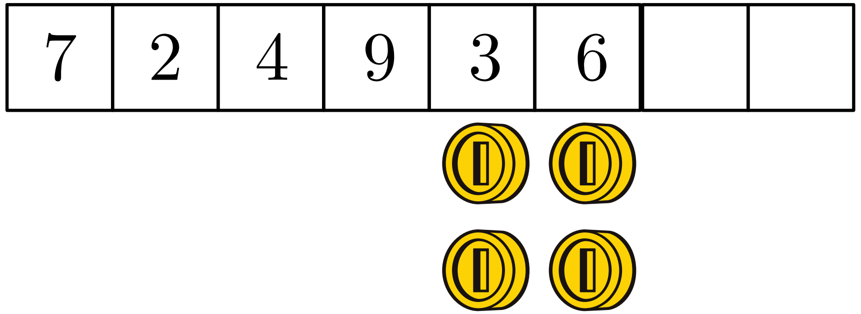

accounting/amortized costs($\hat{c}_i$):

How high do the amortized cost $\hat{c}_i$ of one append/insert need to be?

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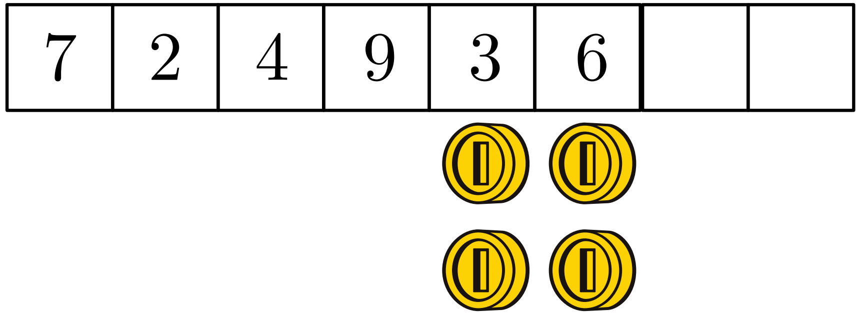

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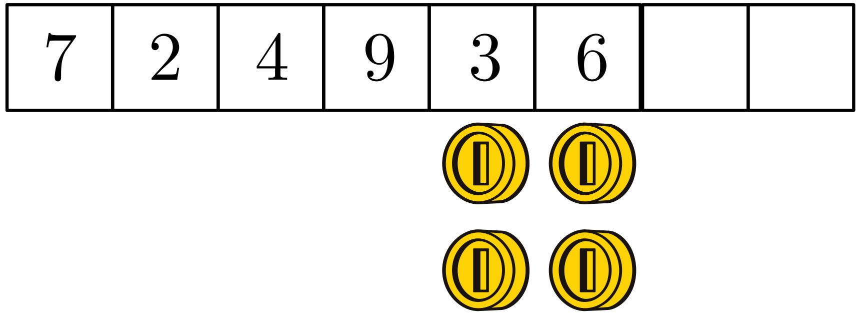

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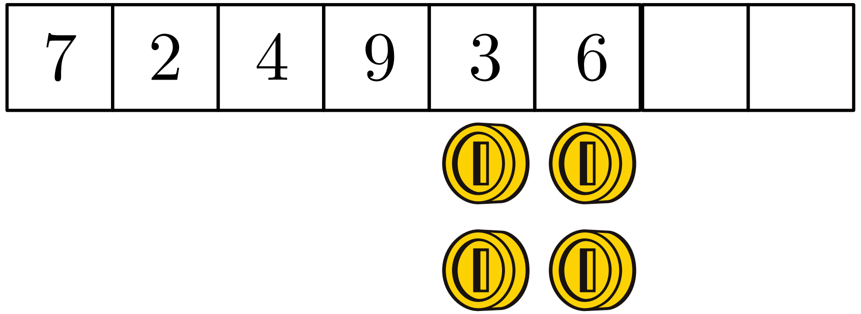

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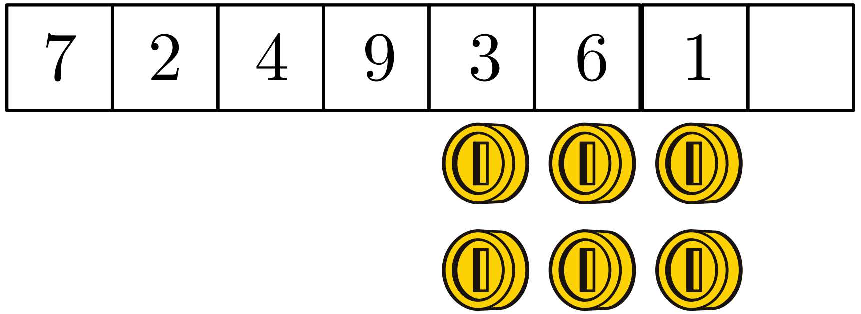

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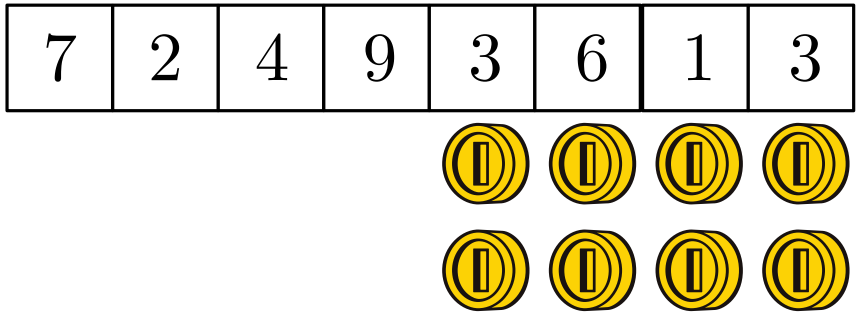

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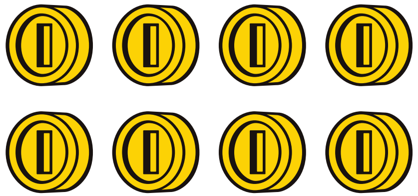

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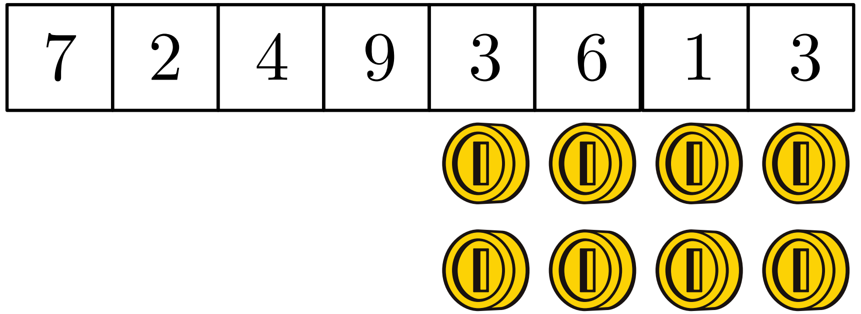

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The worst-case running time $T(n)$ of a sequence of n append operations is $O(n)$. The amortized running time of one append is $O(\hat{c}_i) = O(1)$.

Overview

Amortized analysis:

- Consider sequence of n operations
- Determine the average worst-cost of operations by averaging over the whole sequence

Methods:

- Aggregate analysis
- Accounting method

Examples:

- Multi-pop stack
- Binary counter
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now: potential method

General Setup: Accounting and Potential method

Consider sequence of data structures D_i and operations Op_i

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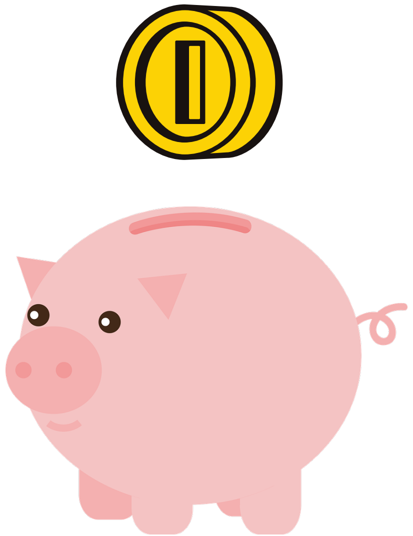

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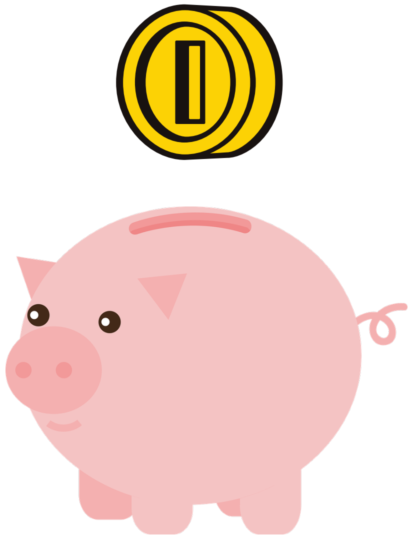

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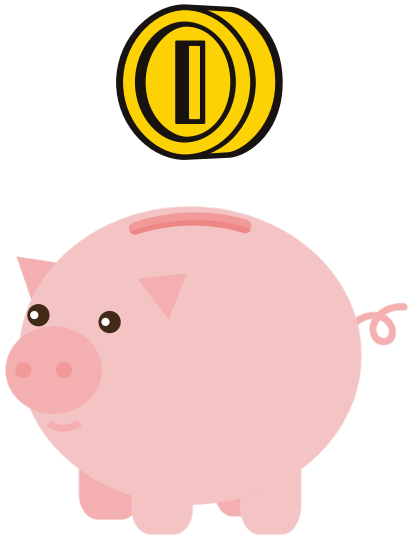

cost invariant: saved coins ≥ 0

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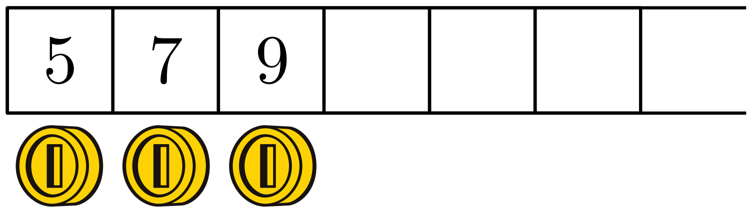
Question: How many coins should we save with a multi-pop stack?



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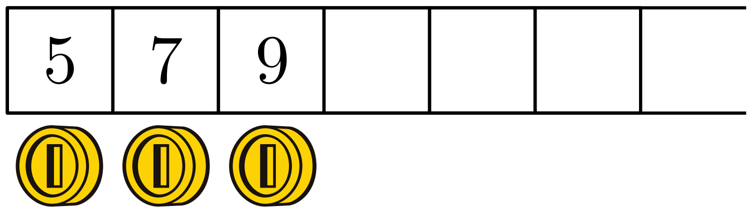

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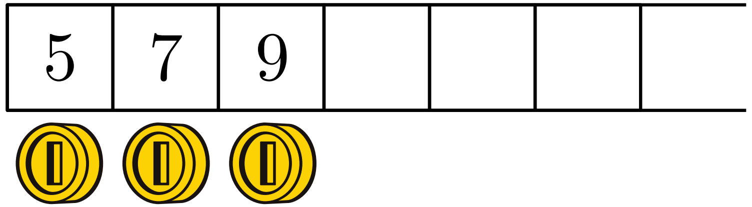

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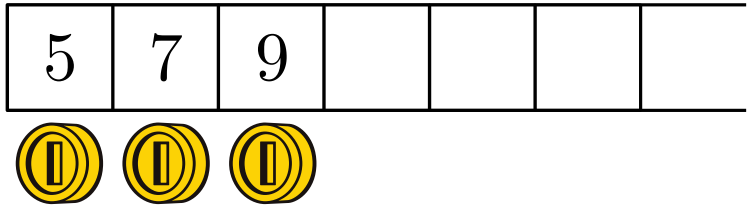

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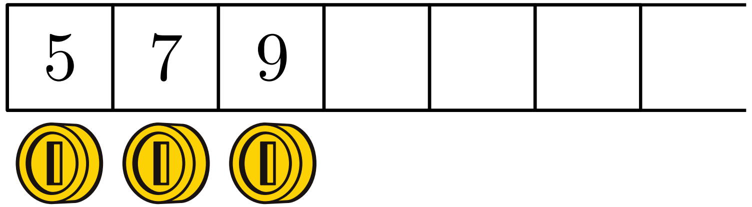

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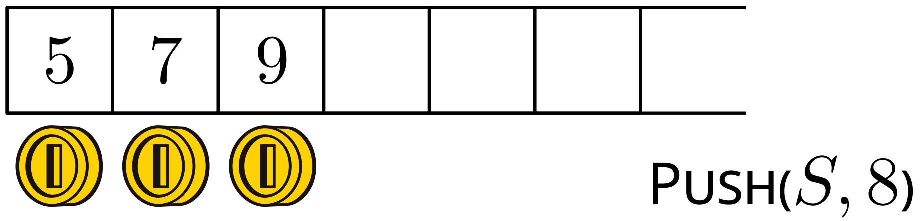

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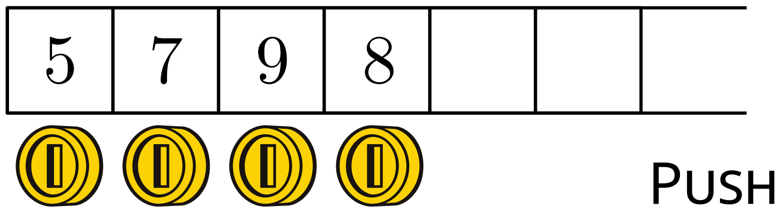

PUSH($S, 8$)

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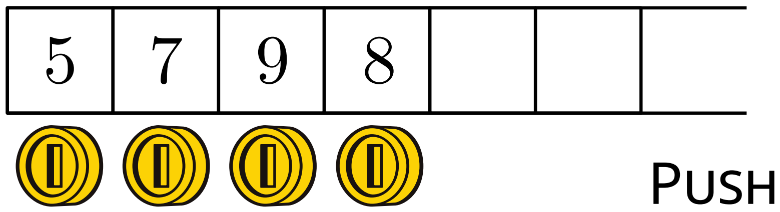
$\Phi(D_i) \geq 0 = \Phi(D_0)$



PUSH($S, 8$)

amortized costs ($\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$):

PUSH(S, x):

- $c_i = 1$ (constant-time operation)
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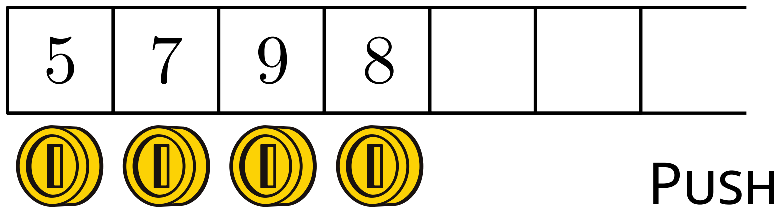

PUSH($S, 8$)

amortized costs ($\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$):

PUSH(S, x): $\hat{c}_i = 1 + 1 = 2$

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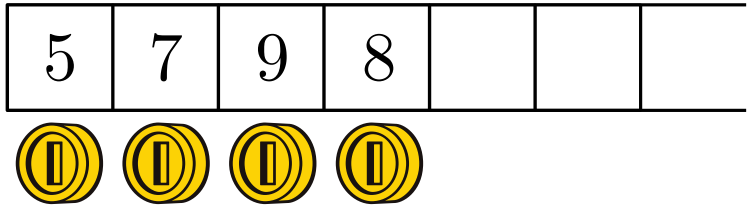

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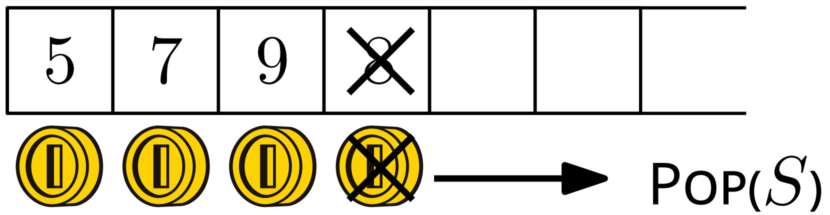

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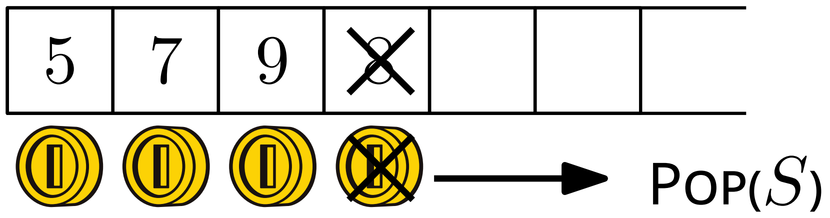

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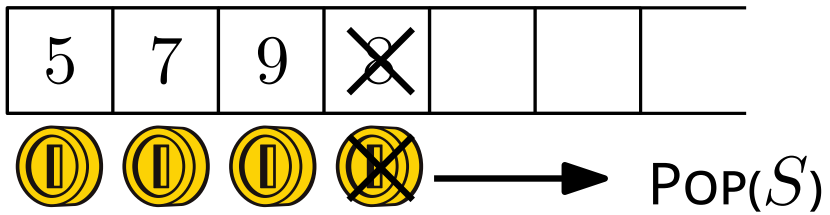

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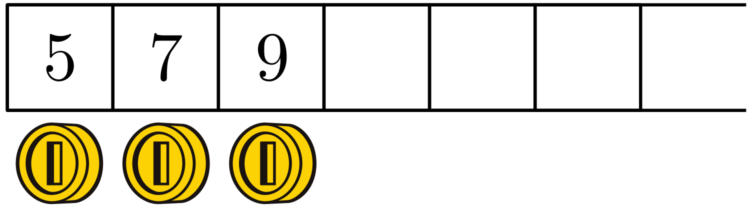

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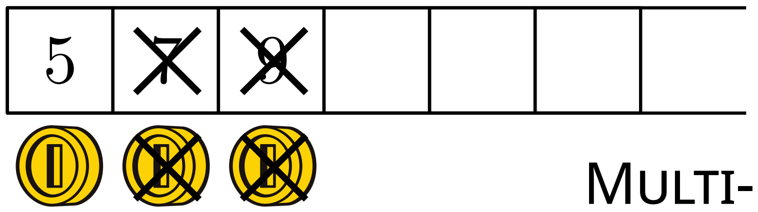

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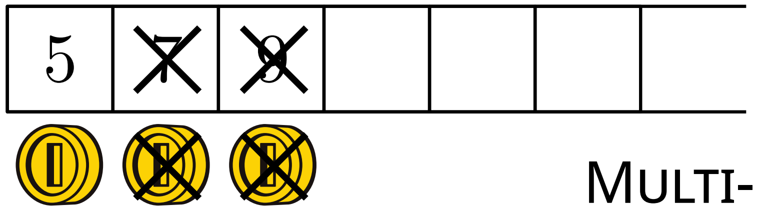

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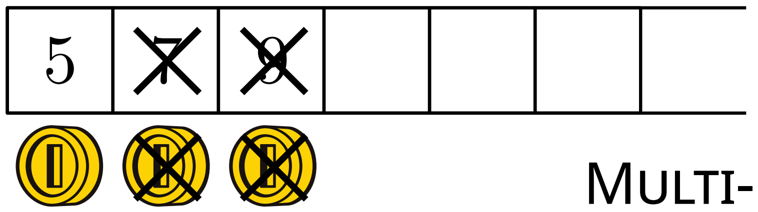

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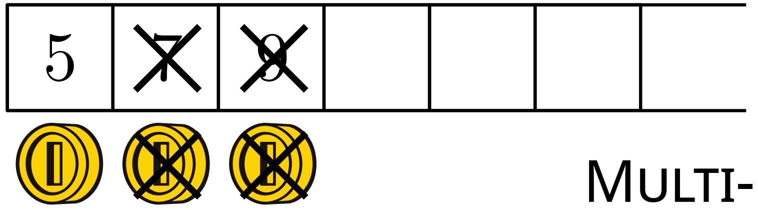

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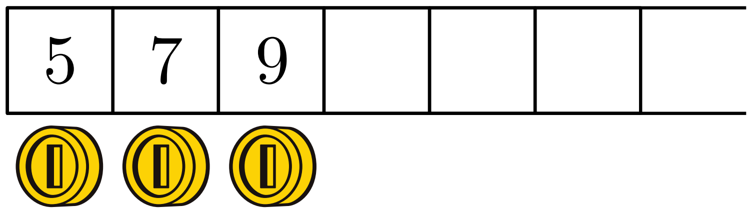

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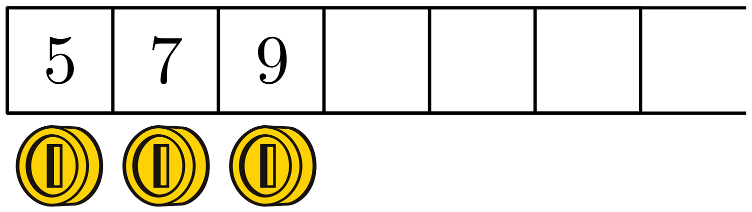

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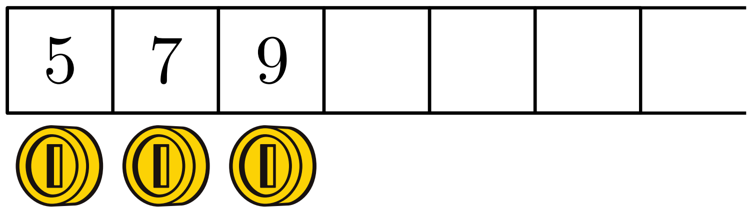

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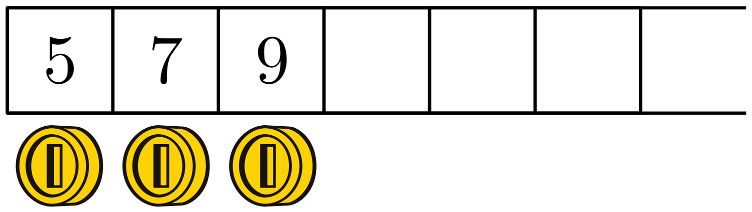

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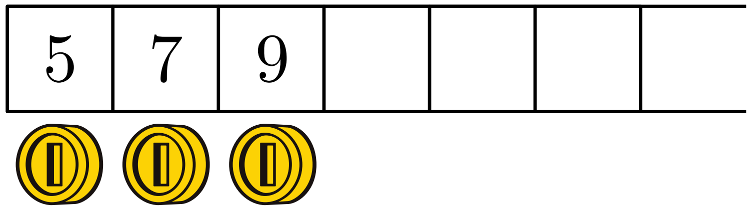

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- amortized cost per operation $\hat{c}_i \leq 2$
- actual total costs $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i$, because $\Phi(D_n) \geq \Phi(D_0) = 0$

Binary Counter (reminder)

Algorithm. increment a k- bit binary counter

Representation as array. $A[j]$: jth least-significant bit

example: $k = 6$

value	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0
1	0	0	0	0	0	1

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costs. number of bits flipped per operation ($= O(k)$)

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flipped bits

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too pessimistic!

Want to show: $O(n)$

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What makes an increment expensive?

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flipped bits

What makes an increment expensive?

What happens during such an increment?

Potential Method: Binary Counter

$\Phi(D_i) =$???

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$$
\begin{array}{r c}
 & \begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\
23: & \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 0 & 1 & 1 & 1 \\ \hline \end{array}
\end{array}
$$

Potential Method: Binary Counter

$\Phi(D_i) =$???

	5	4	3	2	1	0
$23:$	0	1	0	1	1	1

What is a good choice for $\Phi(D_{23})$?

Potential Method: Binary Counter

$\Phi(D_i) =$???

	5	4	3	2	1	0
$23:$	0	1	0	1	1	1

What is a good choice for $\Phi(D_{23})$?

A: 1

B: 3

C: 4

D: 5

Potential Method: Binary Counter

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What is a good choice for $\Phi(D_{23})$?

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Potential Method: Binary Counter

$\Phi(D_i) =$ number of 1-bits

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A: 1

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$\Phi(D_i) =$ number of 1-bits $= \sum_{j=0}^{k} A[j]$, where $A[j]$ is the jth least-significant bit

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$\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$ for all i

amortized cost ($\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$):

0	1	0	1	1	1

- c_i = number of bits flipped

  = number of bits flipped from 0 to 1 + number of bits flipped from 1 to 0

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$= ($ $1 +$ number of bits flipped from 1 to 0$) +$

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$= 2$

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$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \quad = 2$$

The worst-case running time $T(n)$ of a sequence of n increments (starting from 0) is $O(n)$. The amortized running time of one increments is $O(\hat{c}_i) = O(1)$

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- $\hat{c}_i \leq 2 = O(1)$, and actual total costs $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i = O(n)$

Summary

Amortized analysis:

- Consider sequence of n operations
- Take the average of the worst-case running time of the operations over the sequence

Methods:

- Aggregate analysis
- Accounting method
- Potential method

Examples:

- Multi-pop stack
- Binary counter
- Dynamic arrays: insert, accounting method