Theory of Linear Programming

- Convex Polyhedra
- Equational Form
- Basic Feasible Solutions

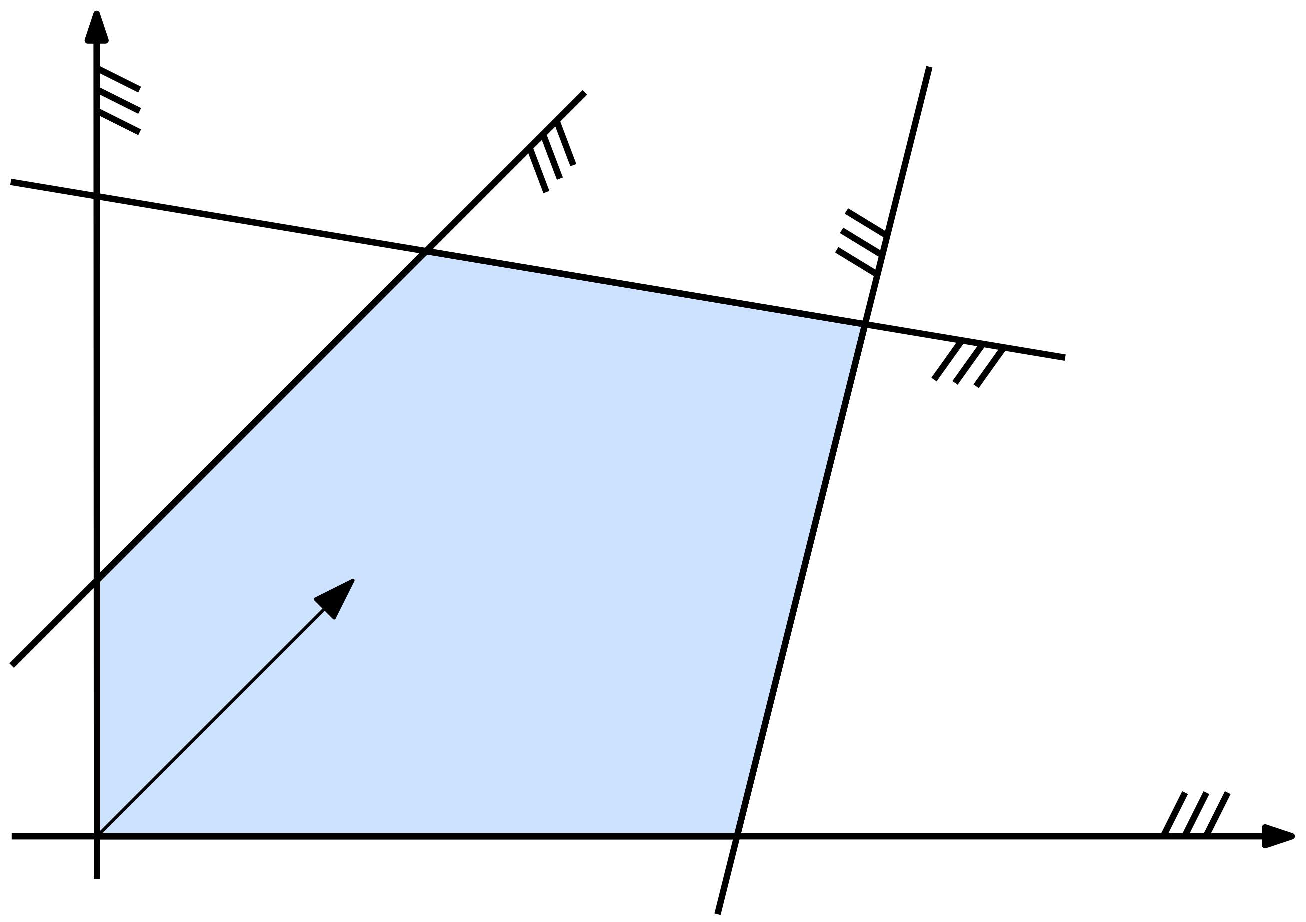
Analogy

Linear programming

maximize $x_1 + x_2$

for $x_1, x_2 \in \mathbb{R}$ satisfying

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ -x_1 + x_2 &\leq 1 \\ x_1 + 6x_2 &\leq 15 \\ 4x_1 - x_2 &\leq 10 \end{aligned}$$



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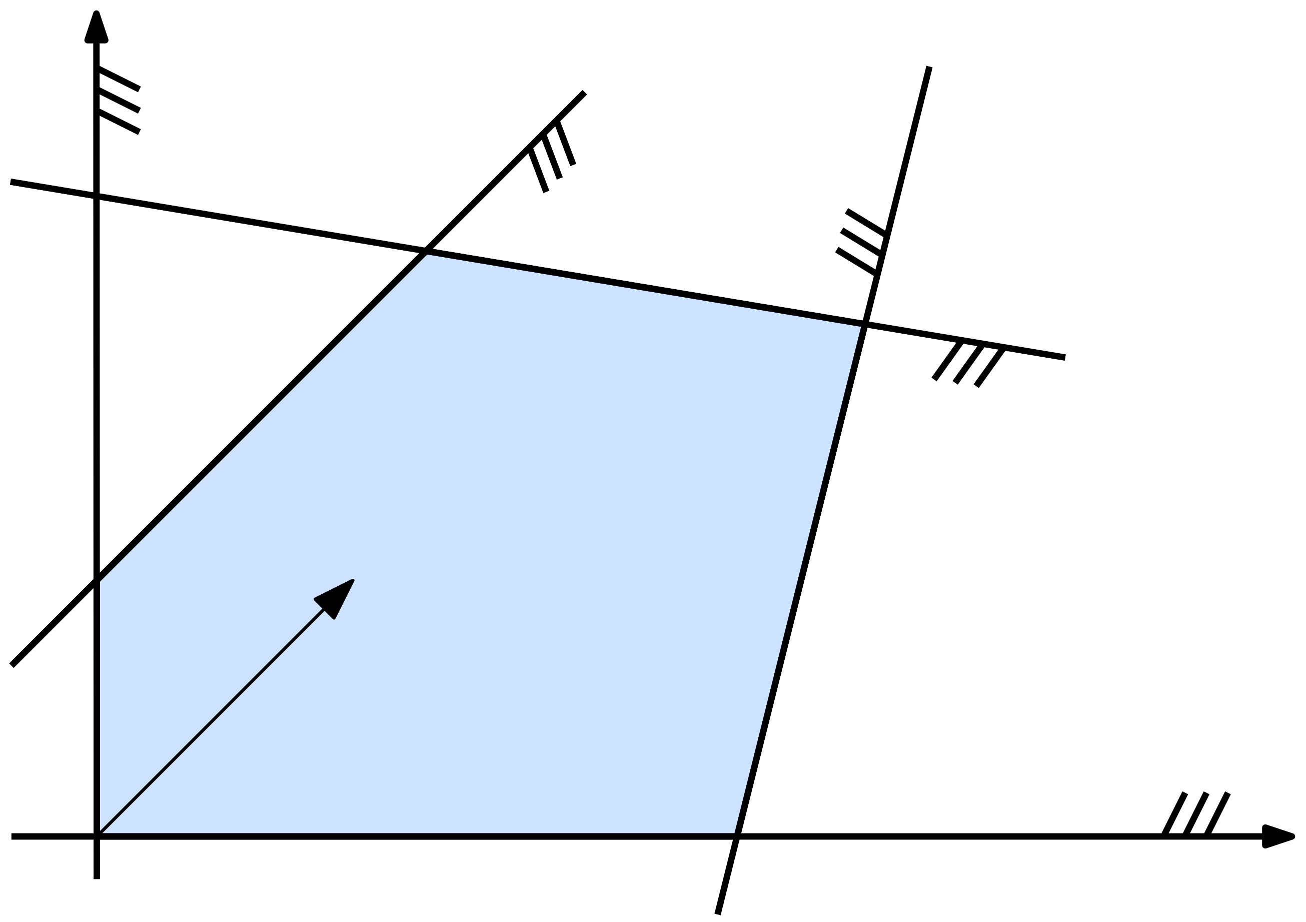
Linear programming is like linear algebra over $\mathbb{R}^n_{\geq 0}$.

	Basic problem
Linear algebra	Linear equations: $Ax = b$
Linear programming	Linear equations: $Ax \leq b$



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Important differences:

1.) convex polyhedra can be very complex

2.) objective function: LP only needs to compute one (optimal) solution, not the whole solution set

Convex Polytopes

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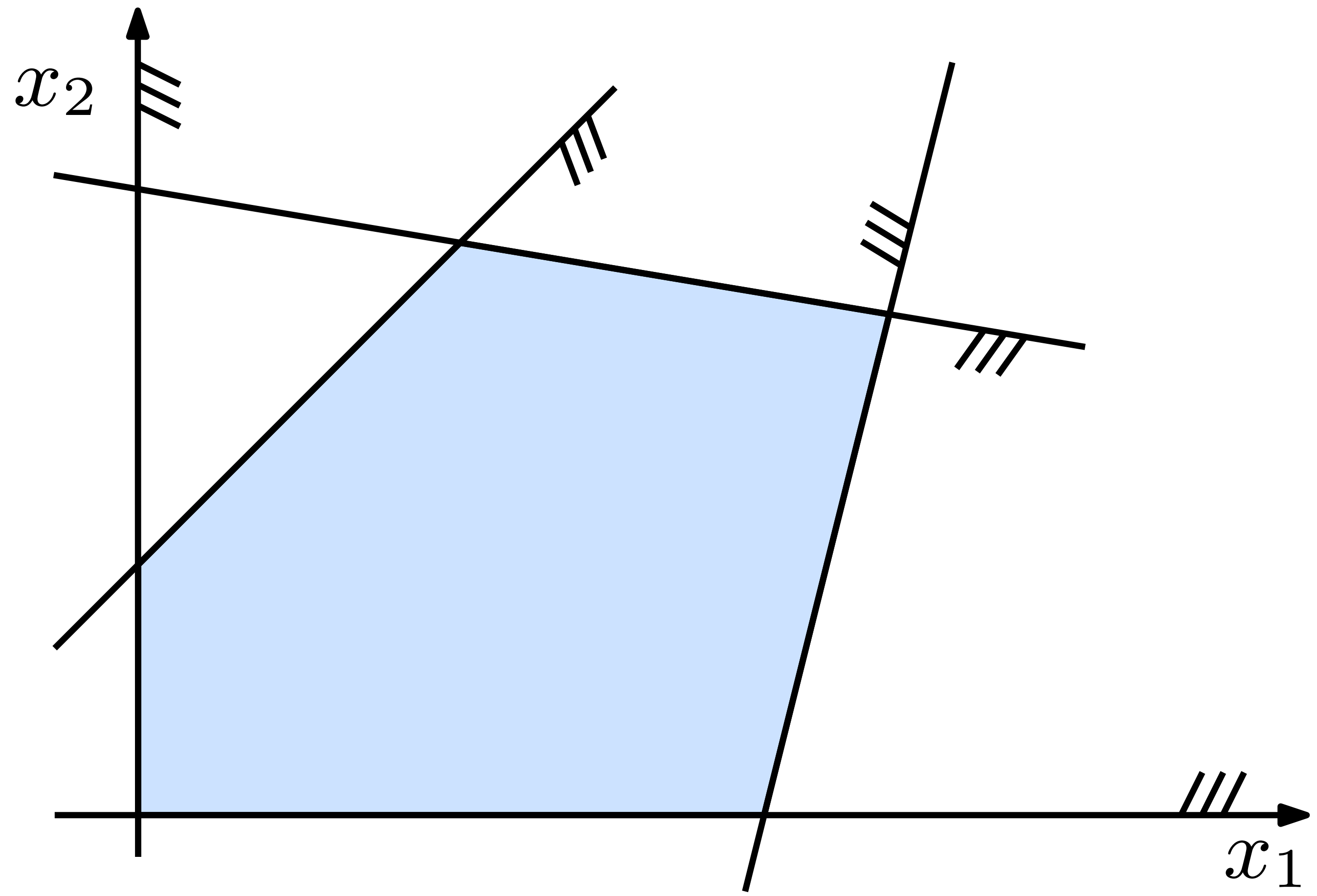
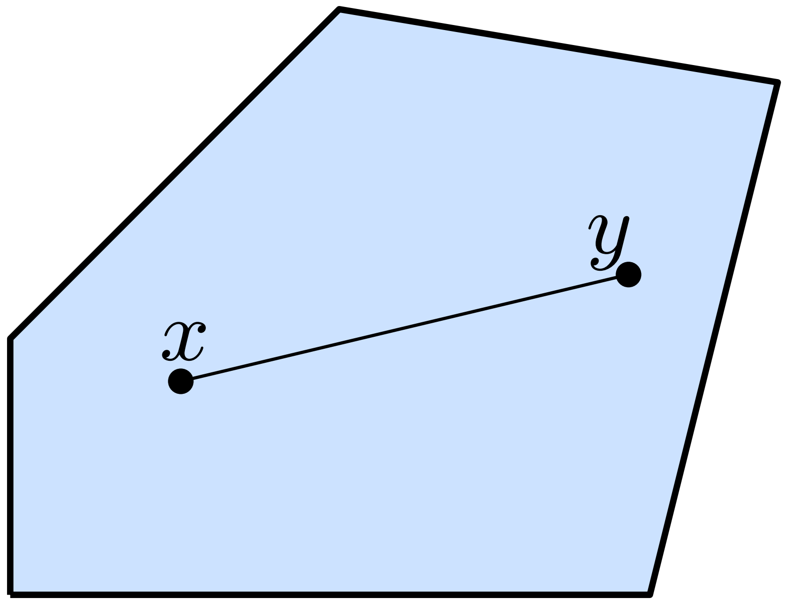


Convex Polytopes

Def.: $X \subseteq \mathbb{R}^n$ is convex if for every two $x, y \in X$: X also contains the line segment xy.

Why is the solution space convex?



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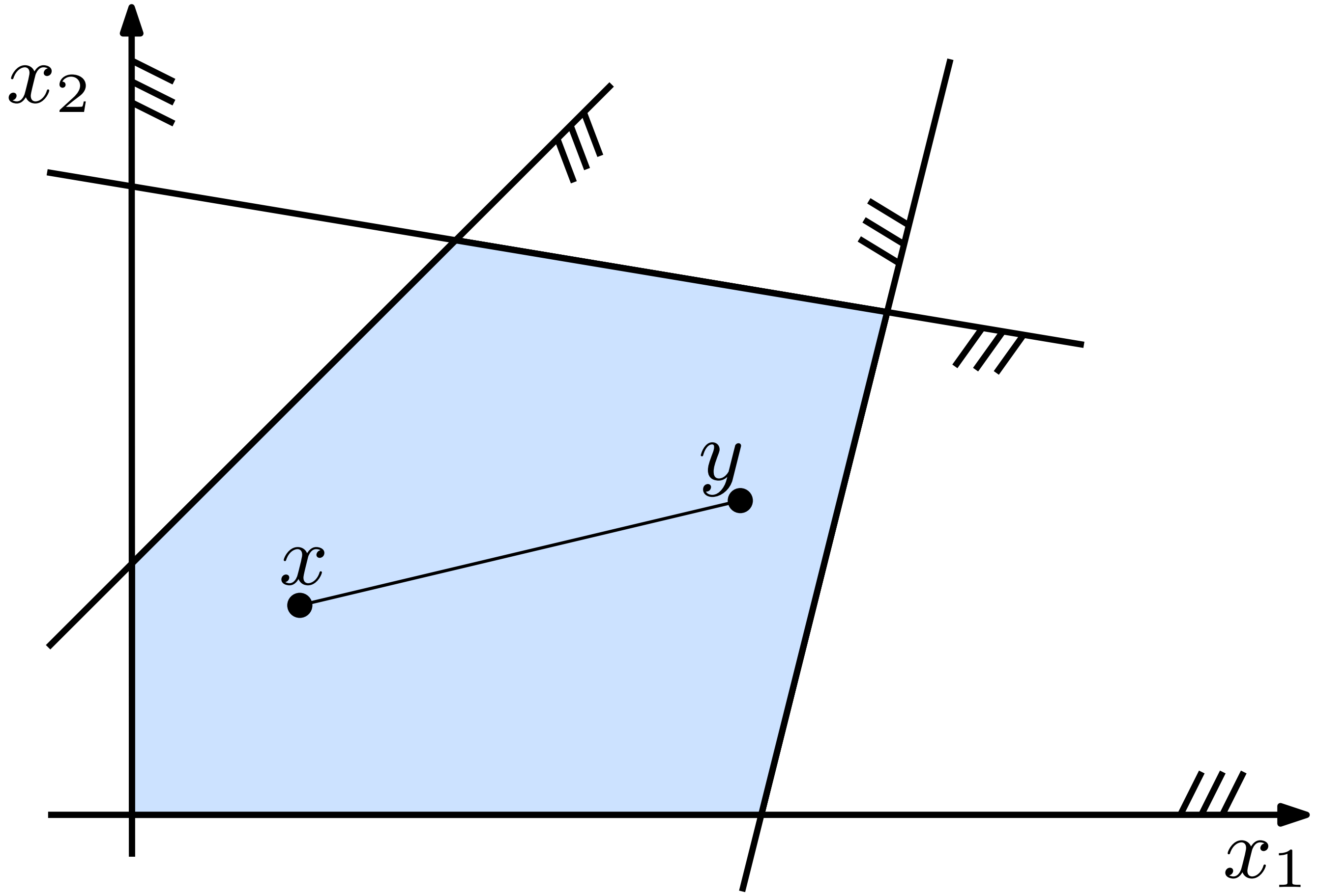

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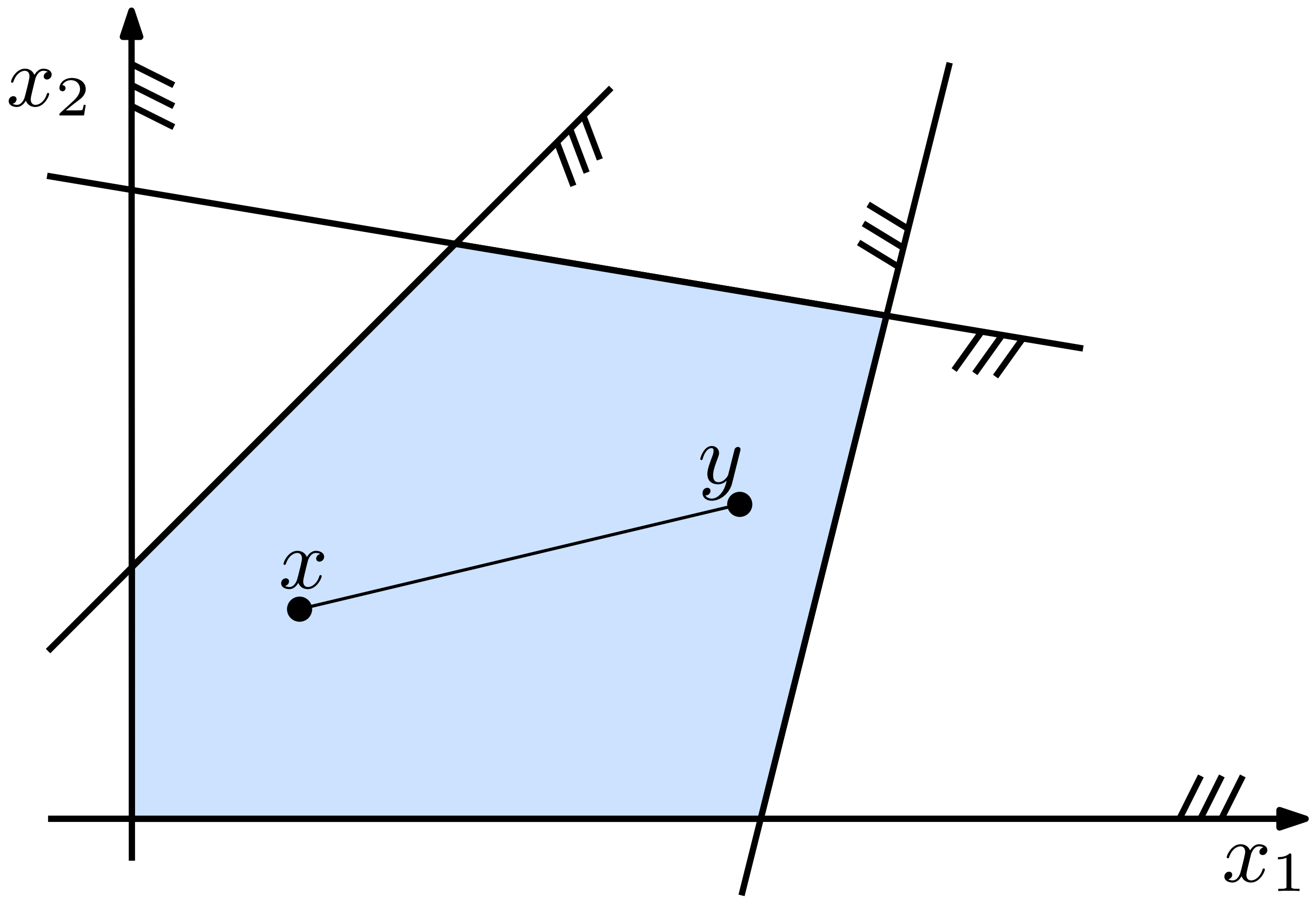

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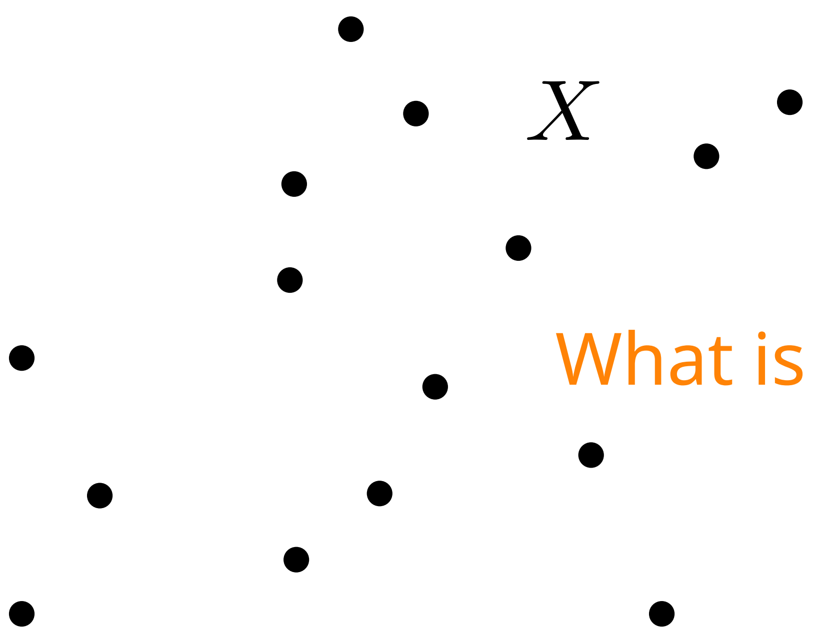

What is the convex hull of X?

Convex Polytopes

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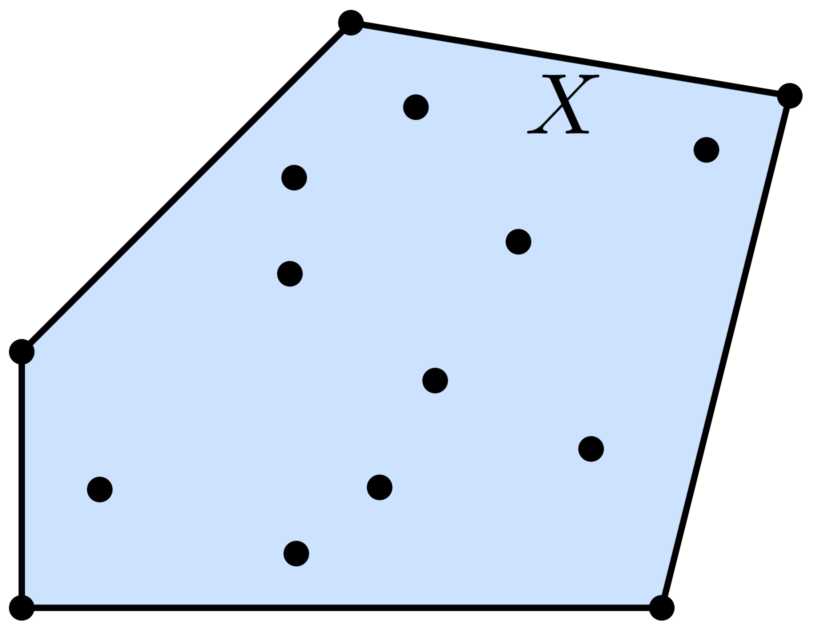

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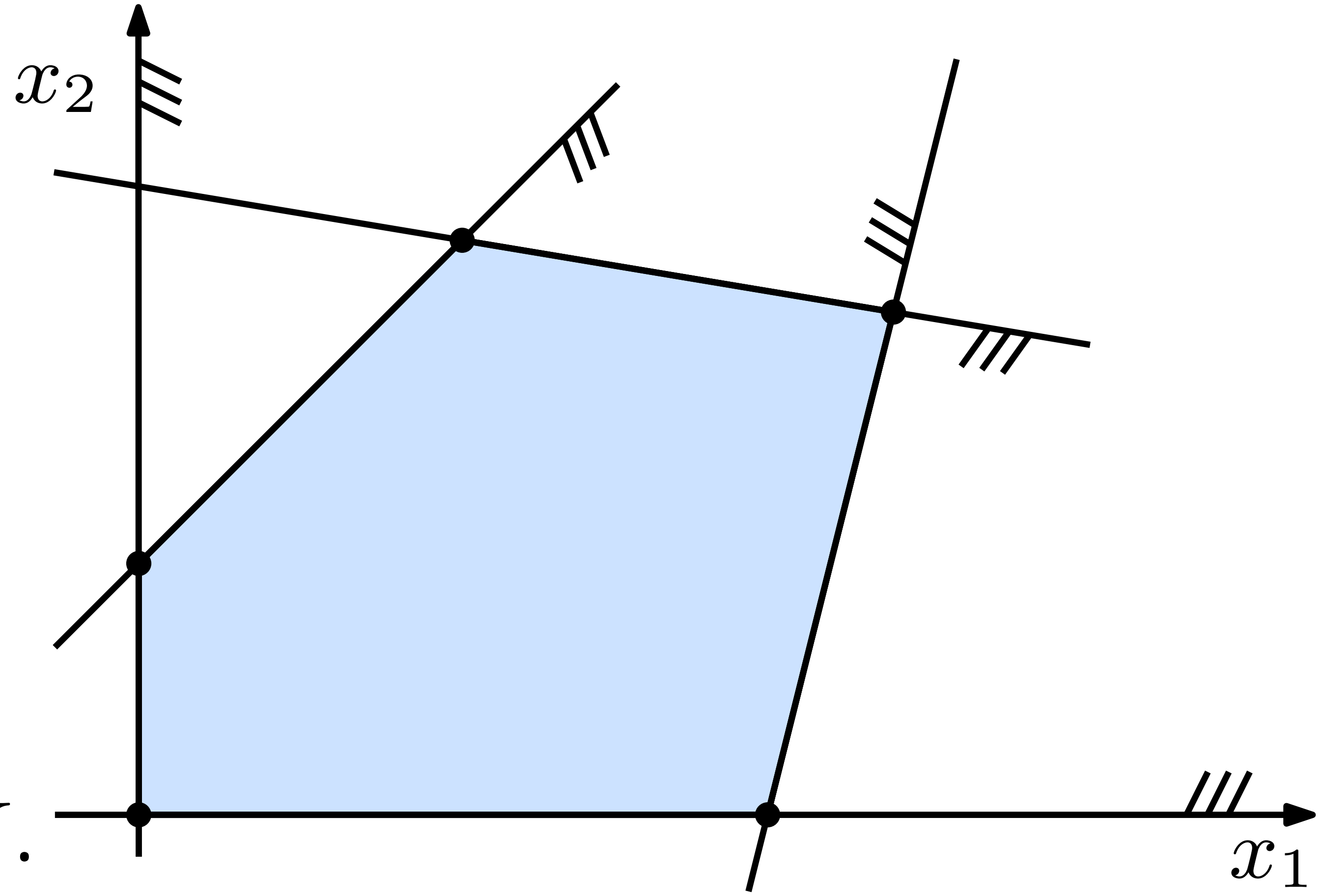

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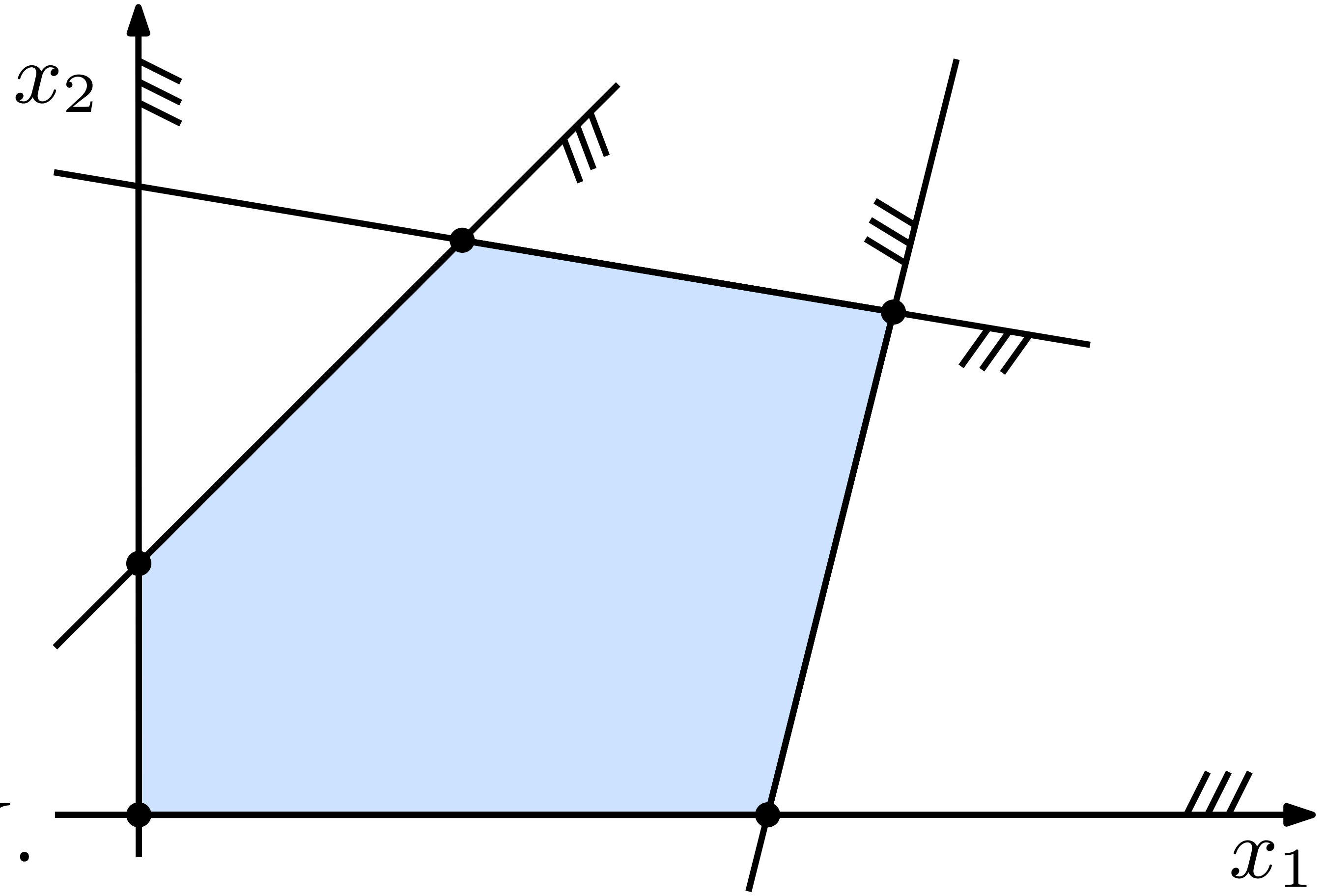

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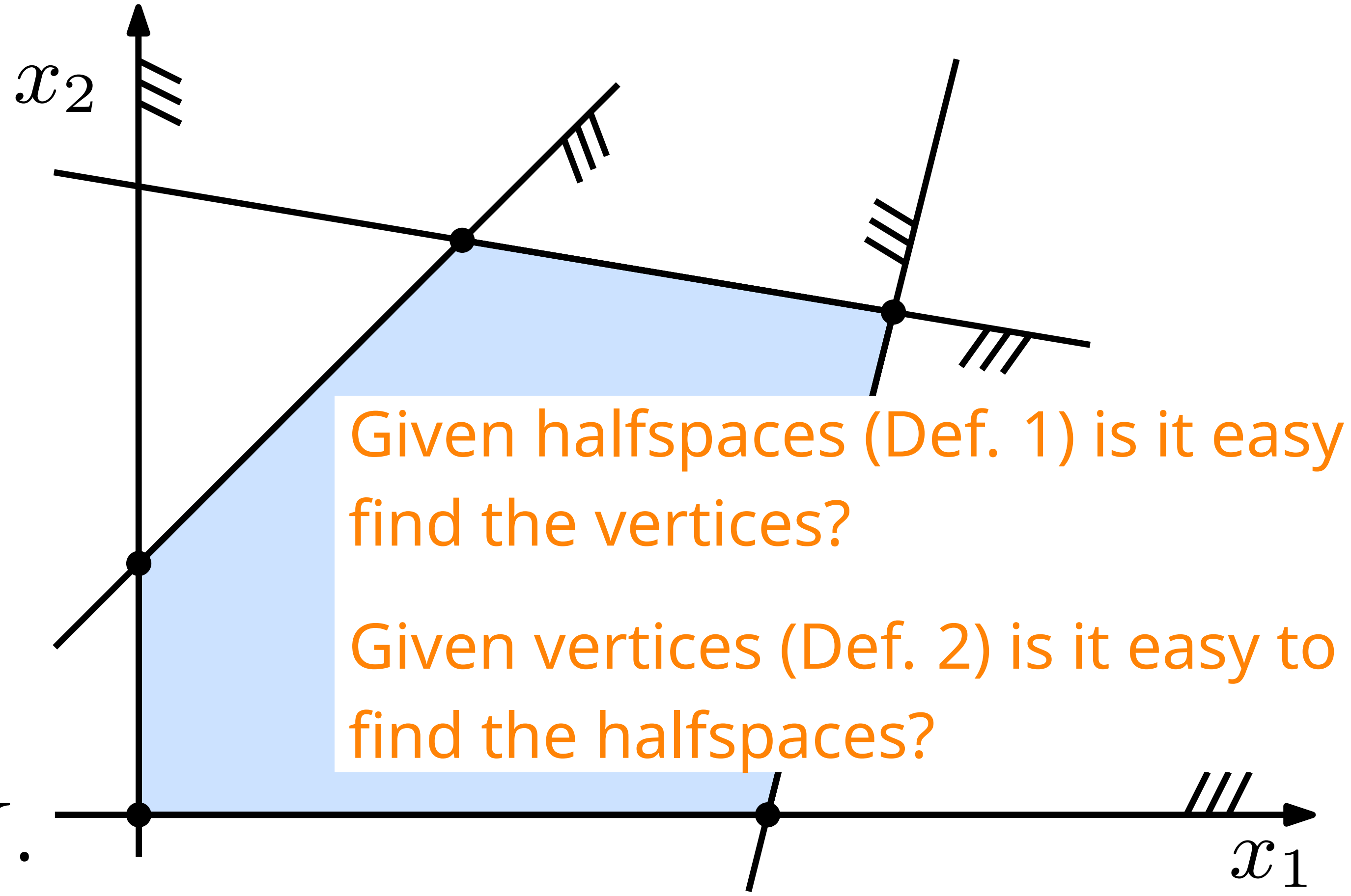

Given halfspaces (Def. 1) is it easy to find the vertices?

Given vertices (Def. 2) is it easy to find the halfspaces?

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Cubes and cross-polytopes

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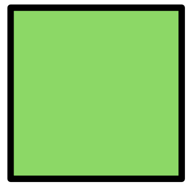
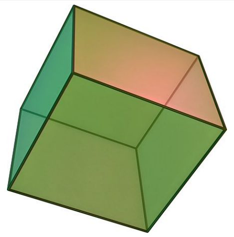
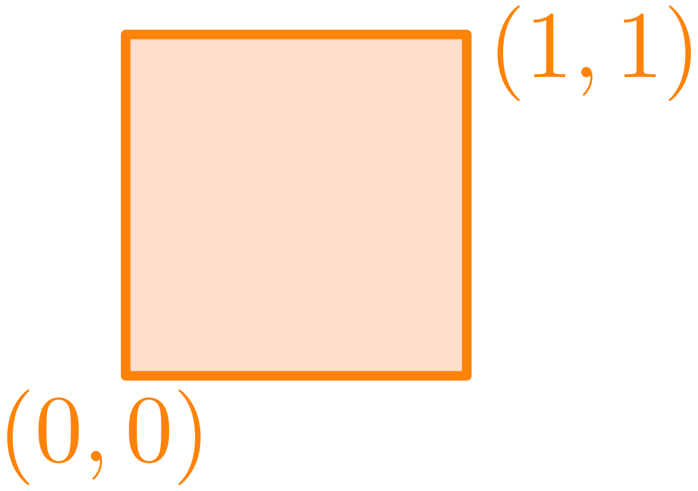
Cube			

How can we write the cube $[0,1]^n$ as intersection of halfspaces?

How many halfspaces?

How many vertices does it have?



Cubes and cross-polytopes

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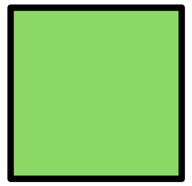
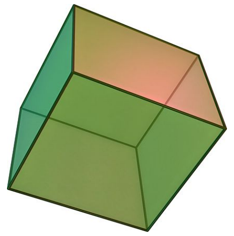
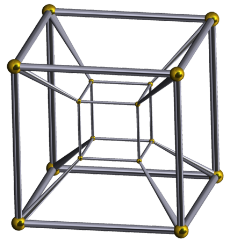
Cube					
vertices	2	4	8	16	2^n
$(n-1)$-faces	2	4	6	8	$2n$

n-dimensional cube: $\{x \in \mathbb{R}^d : \ \max\{|x_1|, |x_2|, \dots, |x_n|\} \leq 1\}$

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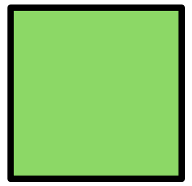
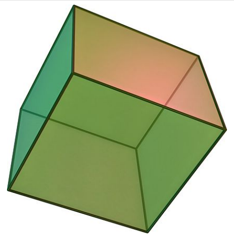
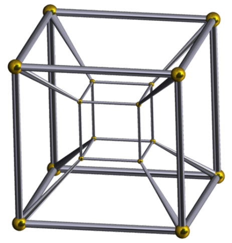
Cube					
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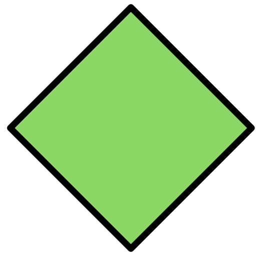
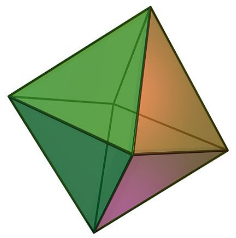
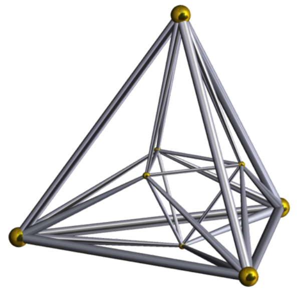
Cross-polytope					

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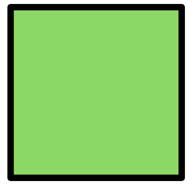
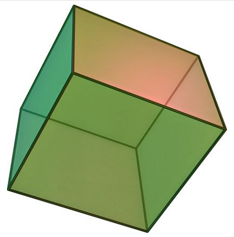
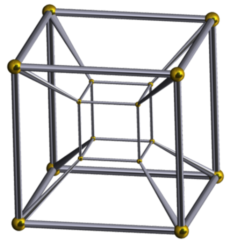
Cube					
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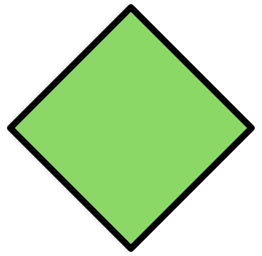
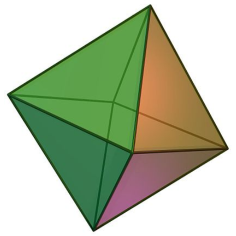
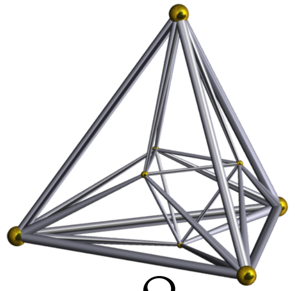
Cross-polytope					
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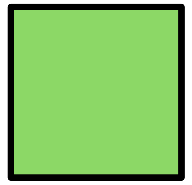
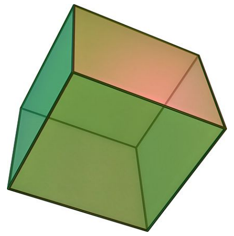
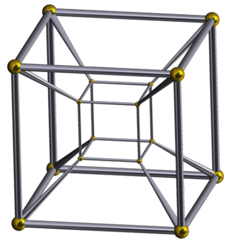
Cube					
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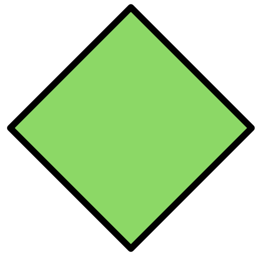
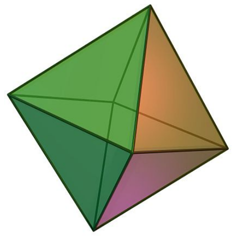
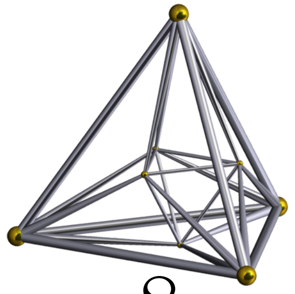
Cross-polytope					
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The cube has exponentially many vertices compared to faces.
Difficult in general to go from faces to vertices.

The cross-polytope has exponentially many faces compared to vertices.
Difficult in general to go from vertices to faces.

Equational Form

Non-Equational Forms of a Linear Program

Any linear program can be rewritten as

$$\text{maximize } c^T x$$
$$\text{subject to } Ax \leq b$$

Not equational form

with $c \in \mathbb{R}^n, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

Here $n = \#$variables and $m = \#$constraints.

Non-Equational Forms of a Linear Program

Example:

$$\begin{array}{ll} \text{minimize} & 3x_1 + 4x_2 \quad \leftarrow \\ \text{subject to} & 2x_1 - x_2 \geq 2 \\ & x_1 + x_2 = 3 \end{array}$$

becomes

$$\begin{array}{ll} \text{maximize} & ??? \\ \text{subject to} & \end{array}$$

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becomes

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with $c = \begin{bmatrix} -3 \\ -4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix},$

$n = 2, m = 3.$

Equational Form of a Linear Program

The simplex method requires a different form, called standard or equational form:

$$\begin{aligned} \text{maximize } & c^T x \\ \text{subject to } & Ax = b \\ & x \geq 0 \end{aligned}$$

Equational form

Equational Form of a Linear Program

Example:

$$\begin{aligned} \text{maximize } & 3x_1 + 4x_2 \\ \text{subject to } & 2x_1 - x_2 \leq 4 \\ & x_1 + 3x_2 \geq 5 \\ & x_2 \geq 0 \end{aligned}$$

Equational Form of a Linear Program

Example:

maximize $3x_1 + 4x_2$

subject to $2x_1 - x_2 \leq 4$

$x_1 + 3x_2 \geq 5$

$x_2 \geq 0$

(1) $2x_1 - x_2 \leq 4$ becomes $2x_1 - x_2 + x_3 = 4$

slact variable $\longrightarrow$ $x_3 \geq 0$

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slack variable $\longrightarrow$ $x_3 \geq 0$

(2) $x_1 + 3x_2 \geq 5$ becomes ???

Equational Form of a Linear Program

Example: maximize $3x_1 + 4x_2$

subject to $2x_1 - x_2 \leq 4$

$x_1 + 3x_2 \geq 5$

$x_2 \geq 0$

(1) $2x_1 - x_2 \leq 4$ becomes $2x_1 - x_2 + x_3 = 4$

slack variable $\longrightarrow$ $x_3 \geq 0$

(2) $x_1 + 3x_2 \geq 5$ becomes $-x_1 - 3x_2 \leq -5$

and then $-x_1 - 3x_2 + x_4 = -5$

slack variable $\longrightarrow$ $x_4 \geq 0$

Equational Form of a Linear Program

Example (updated):

maximize $3x_1 + 4x_2$

subject to $2x_1 - x_2 + x_3 = 4$

$$-x_1 - 3x_2 + x_4 = -5$$

$$x_2, x_3, x_4 \geq 0$$

Equational Form of a Linear Program

Example (updated):

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subject to $2x_1 - x_2 + x_3 = 4$

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$$x_2, x_3, x_4 \geq 0$$

missing: $x_1 \geq 0$

How can we add it?

Equational Form of a Linear Program

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How can we add it?

(3) To handle the "missing" nonnegativity constraint, let $x_1 = x_1' - x_1''$ with $x_1' \geq 0, x_1'' \geq 0$.

Equational Form of a Linear Program

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Result:

maximize $3x_1' - 3x_1'' + 4x_2$

subject to $2x_1' - 2x_1'' - x_2 + x_3 = 4$

$$-x_1' + x_1'' - 3x_2 + x_4 = -5$$

$$x_1' \geq 0, x_1'' \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

Equational Form of a Linear Program

Example (updated):

maximize $3x_1 + 4x_2$

subject to $2x_1 - x_2 + x_3 = 4$

$$-x_1 - 3x_2 + x_4 = -5$$

$$x_2, x_3, x_4 \geq 0$$

missing: $x_1 \geq 0$

How can we add it?

(3) To handle the "missing" nonnegativity constraint, let $x_1 = x_1' - x_1''$ with $x_1' \geq 0, x_1'' \geq 0$.

(4) Then relabel $x_1', x_1'', x_2, x_3, x_4$ as x_1, x_2, x_3, x_4, x_5

Result: maximize $3x_1 - 3x_2 + 4x_3$

subject to $2x_1 - 2x_2 - x_3 + x_4 = 4$

$$-x_1 + x_2 - 3x_3 + x_5 = -5$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

Equational Form of a Linear Program

Remark

This translation takes us from n variables and m constraints ($\leq$, $\geq$, or $=$) to:

- at most $m + 2n$ variables
- m equations
- all nonnegativity constraints

Further Requirements

We consider only linear programs in equational form

$$\begin{aligned} \text{maximize } & c^T x \\ \text{subject to } & Ax = b \\ & x \geq 0 \end{aligned} \qquad (A \text{ size } m \times n)$$

such that

- $Ax = b$ has at least one solution
- the rows of A are linearly independent

What if this does not hold?

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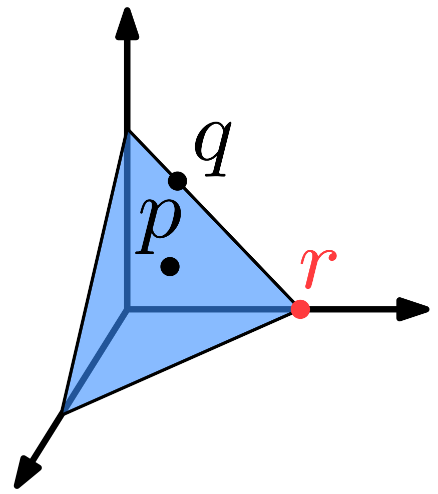
Gaussian elimination

Basic Feasible Solutions

Basic feasible solution

Intuition:

As optimal solutions of an LP only corners are possible.



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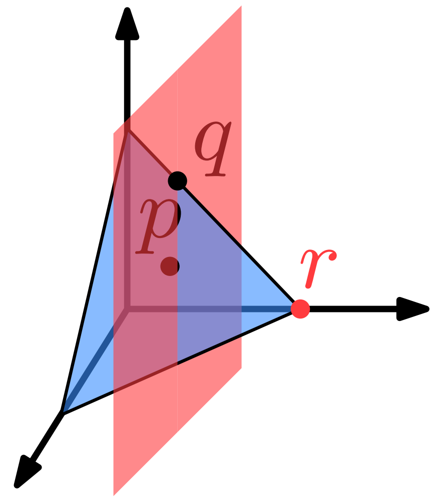

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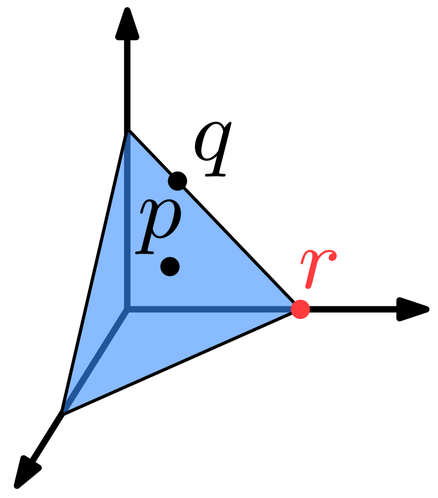

corner:

- affine space is $(n-m)$-dimensional
- cutting with $n-m$ sides of positive octant gives a corner
- at least $n-m$ coordinates 0

$\rightarrow$ all coordinates 0 except for m many

m = # constraints, n = # variables

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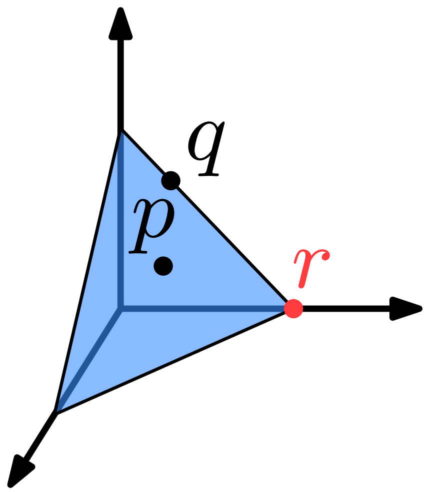

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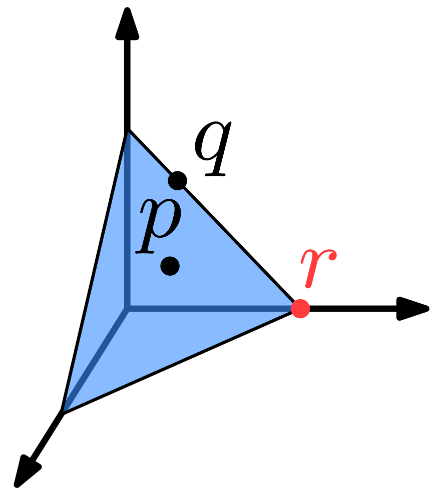
basic feasible solutions formalize this.

Basic feasible solution

Definition

A feasible solution $x \in \mathbb{R}^n$ is basic if there is an m-element set $B \subseteq \{1, 2, \ldots, n\}$ such that

- the square matrix A_B is nonsingular, i.e., the columns indexed by B are independent
- $x_j = 0$ for all $j \notin B$



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full rank

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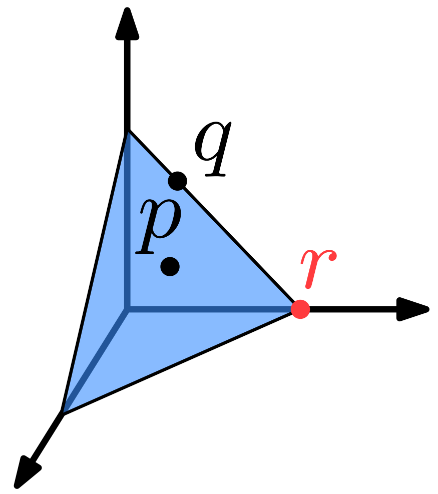

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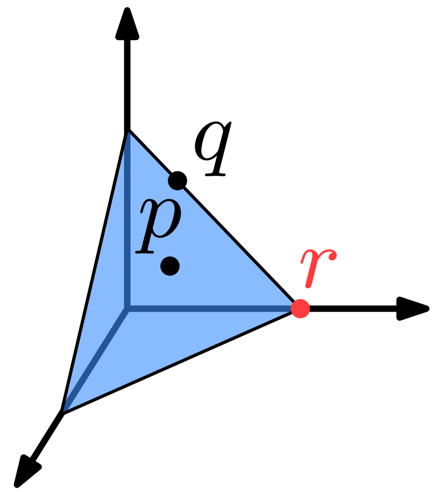

Example:

If $A = \begin{bmatrix} 1 & \boxed{\begin{matrix} 5 \\ 1 \end{matrix}} & 3 & \boxed{\begin{matrix} 4 \\ 5 \end{matrix}} & 6 \\ 0 & & 3 & & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$, then $x = [0, 2, 0, 1, 0]$

is a basic feasible solution with basis $B = \{2, 4\}$

Basic feasible solution

Example:

If $A = \begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, then $x = [0, 2, 0, 0, 0]$ is a basic feasible solution with four different choices for B: $B = \{1, 2\}$, $\{2, 3\}$, $\{2, 4\}$, or $\{2, 5\}$.

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$B = \{1, 2\}$, $\{2, 3\}$, $\{2, 4\}$, or $\{2, 5\}$.

Moral: A basic feasible solution (bfs) x does not determine the basis B.

Basic feasible solutions – Algebra

By contrast

Proposition 4.2.2

A basis B determines at most one bfs x.

Basic feasible solutions – Algebra

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A basis B determines at most one bfs x.

Why? In the above example, if we set $B = \{1, 4\}$, then the bfs x must satisfy $x = [x_1, 0, 0, x_4, 0]$.

$$\begin{bmatrix} 10 \\ 2 \end{bmatrix} = b = Ax = A_B \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_4 \\ 5x_4 \end{bmatrix}$$

Basic feasible solutions – Algebra

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$\Rightarrow \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42/5 \\ 2/5 \end{bmatrix}$, using the fact that A_B is invertible.

So $x = [42/5, 0, 0, 2/5, 0]$.

Basic feasible solutions – Algebra

Example:

If $A = \begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, then does $B = \{3, 4\}$ yield a bfs x?

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Answer: Consider $x = [0, 0, x_3, x_4, 0]$

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$$\Rightarrow \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42/3 \\ -8 \end{bmatrix}.$$

No, $B = \{3, 4\}$ does not yield a bfs since the corresponding $x = [0, 0, 42/3, -8, 0]$ is not nonnegative, i.e., it is not feasible.

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Example: Does $B = \{3, 5\}$ yield a bfs x?

Answer: No, B is not even a basis since $A_B = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix}$ is singular.

Optimal $\to$ Basic feasible solution

Theorem 4.2.3
If an optimal solution exists to maximize $c^T x$ subject to $Ax = b,\ x \geq 0$ then there is also a bfs that is optimal.

Proof 1 Follows from the proof of correctness of the simplex method.

Proof 2 Follows since each vertex of the feasible region corresponds to a bfs.

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Impractical algorithm for solving linear programs

Consider all $\binom{n}{m}$ subsets $B \subseteq \{1, ..., n\}$ of size m, see if B corresponds to a bfs x, take the max over all $c^T x$.

Optimal solution vs Vertex of Convex Polyhedron

Definition A feasible solution $x \in \mathbb{R}^n$ is basic if there is an m-element set $B \subseteq \{1, 2, ..., n\}$ such that:

- the square matrix A_B is nonsingular
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Proposition 4.2.2 A basis B determines at most one bfs x.

Theorem 4.2.3 If an optimal solution exists, then an optimal bfs exists.

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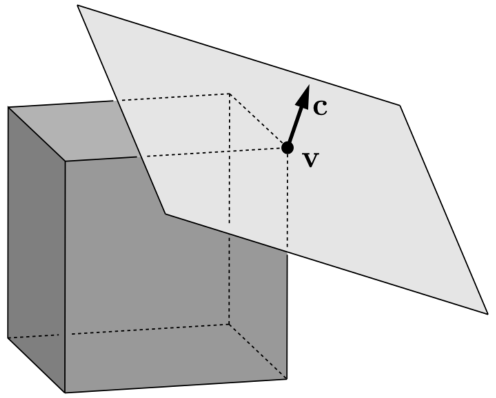

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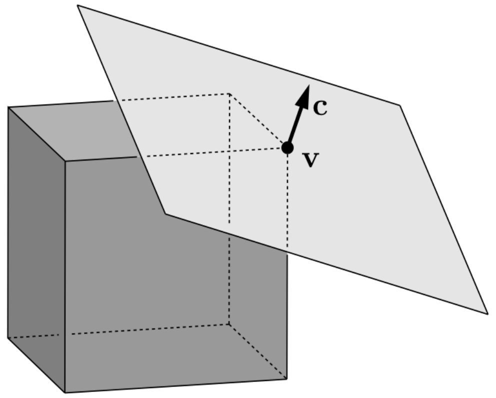

$$c^T y = c^T x$$

Definition x is a vertex of a convex polyhedron $P \subseteq \mathbb{R}^n$ if there is some $c \in \mathbb{R}^n$ with $c^T x > c^T y$ for all $y \in P \setminus \{x\}$.

Basic feasible solution $\leftrightarrow$ Vertex

Theorem 4.4.1

Given a linear program in equational form, x is a vertex of the feasible region if and only if x is a bfs.

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Basic feasible solution $\leftrightarrow$ Vertex

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Given a linear program in equational form, x is a vertex of the feasible region if and only if x is a bfs.

Proof:

($\Rightarrow$) Follows from Theorem 4.2.3, with c being the vector showing x is a vertex.

($\Leftarrow$) Let x be a bfs with basis B.

Define $c \in \mathbb{R}^n$ by $$c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$$

Note $c^T x = 0$, and proposition 4.2.2 implies $c^T y < 0$ for all feasible $y \neq x$.

Hence x is a vertex of the feasible region.

Summary

The set of feasible solutions of an LP is a convex polyhedron

Every LP can be written in equational form.

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next:
simplex algorithm: finds an optimal bfs in a clever way.