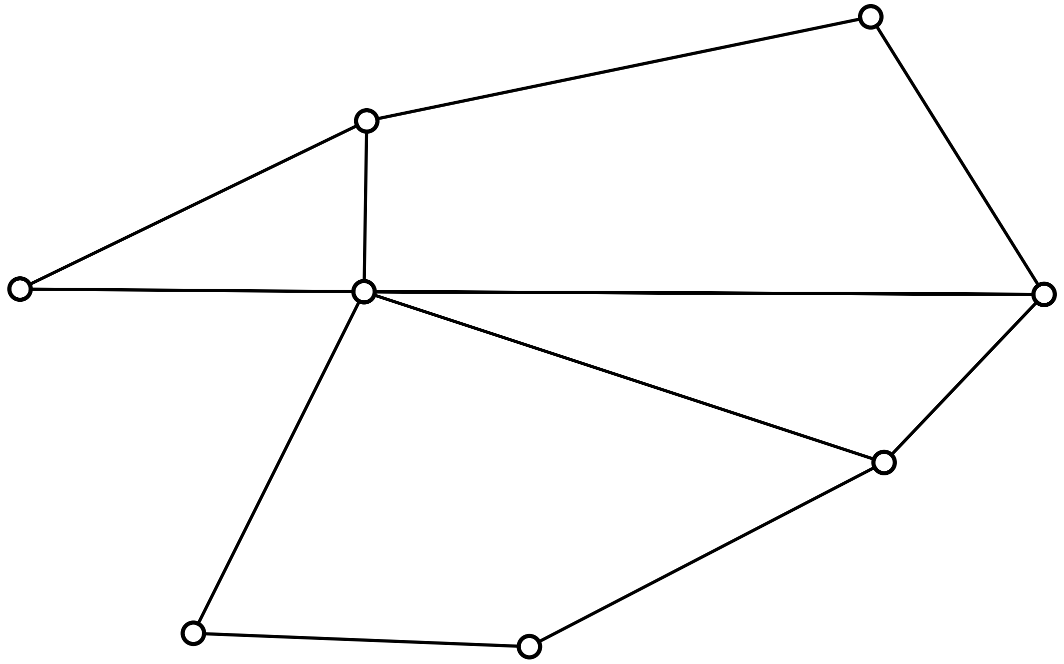
Applications of Duality

Total Unimodularity

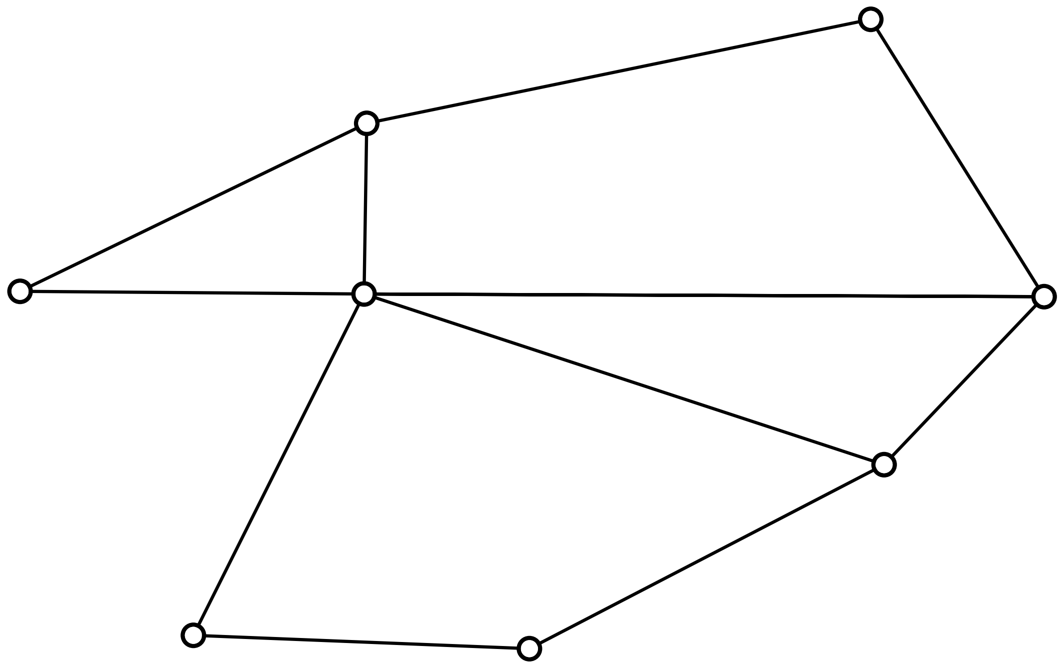
Matchings, Flows, and Shortest paths

Matchings



Matchings

An edge set $M \subseteq E$ of a graph $G = (V, E)$ is a matching if no two edges of M are adjacent (i.e., share an end vertex).



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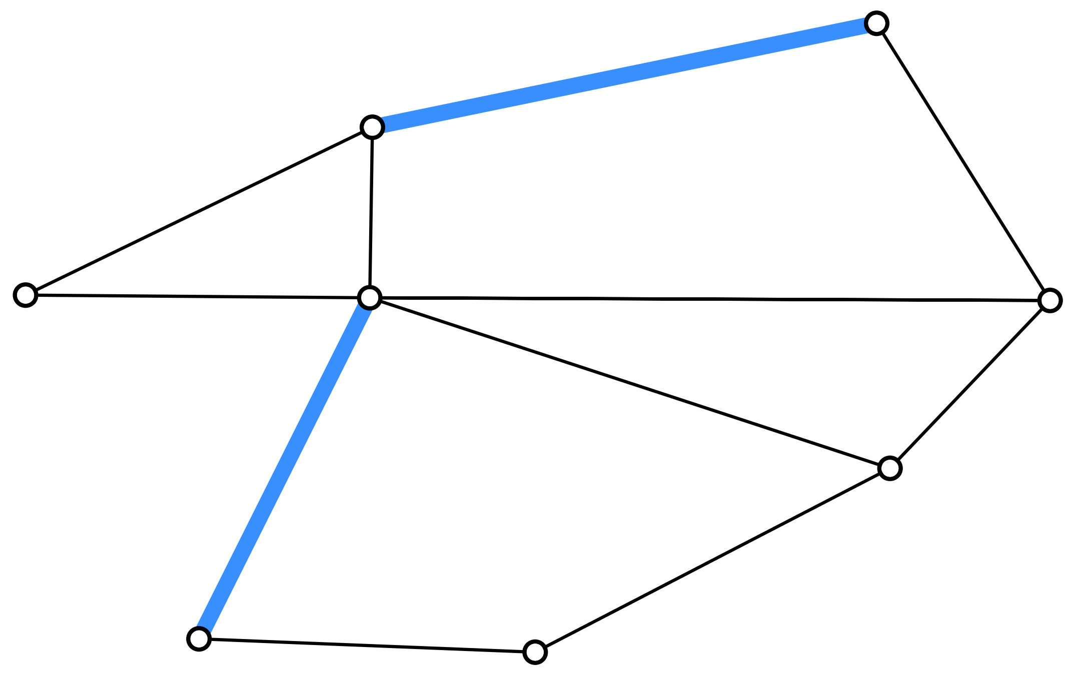

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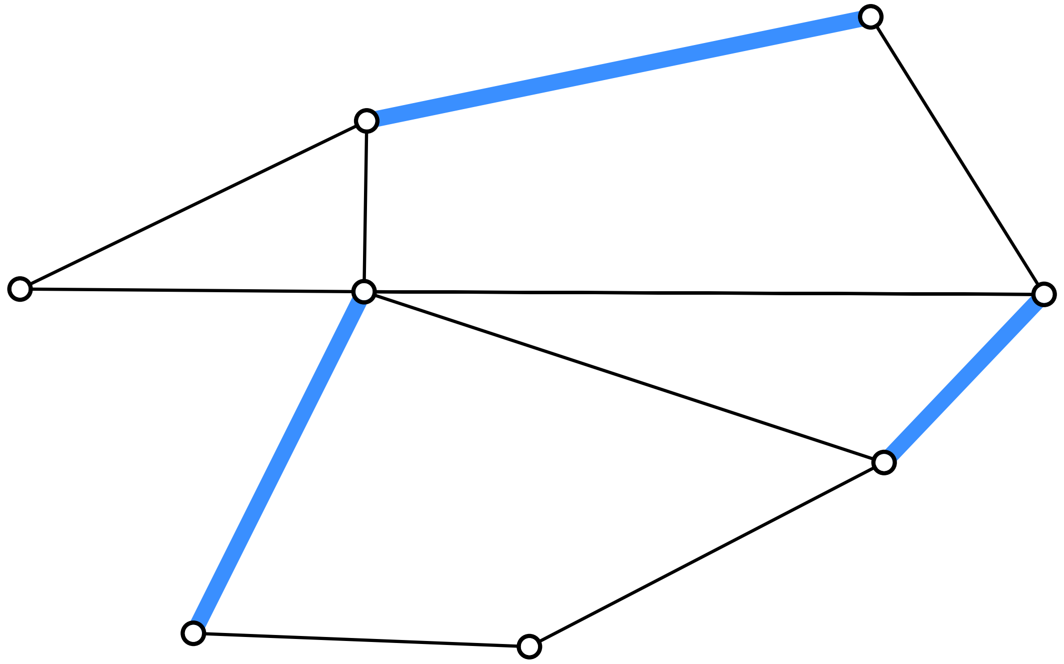

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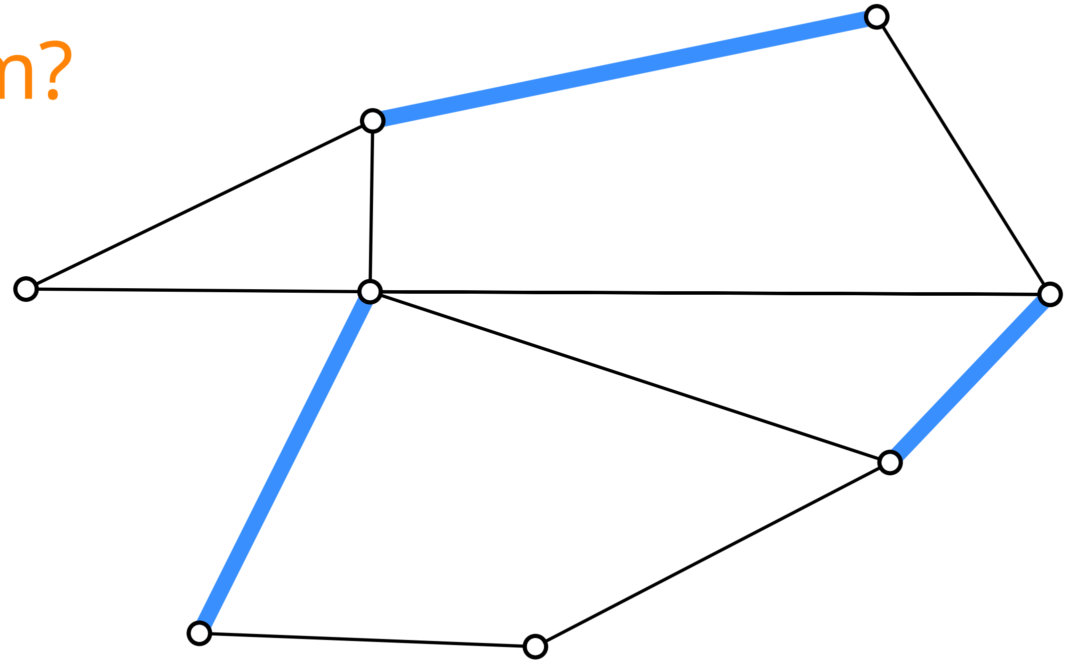
M is maximum if it has the largest number of edges among all matchings.

Is this maximum?



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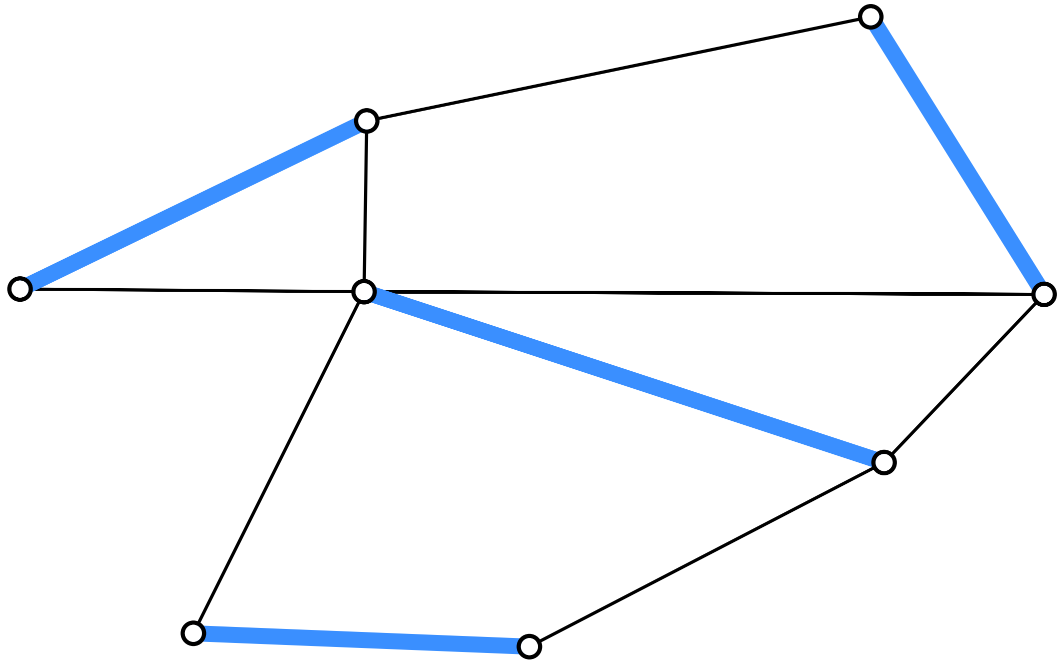

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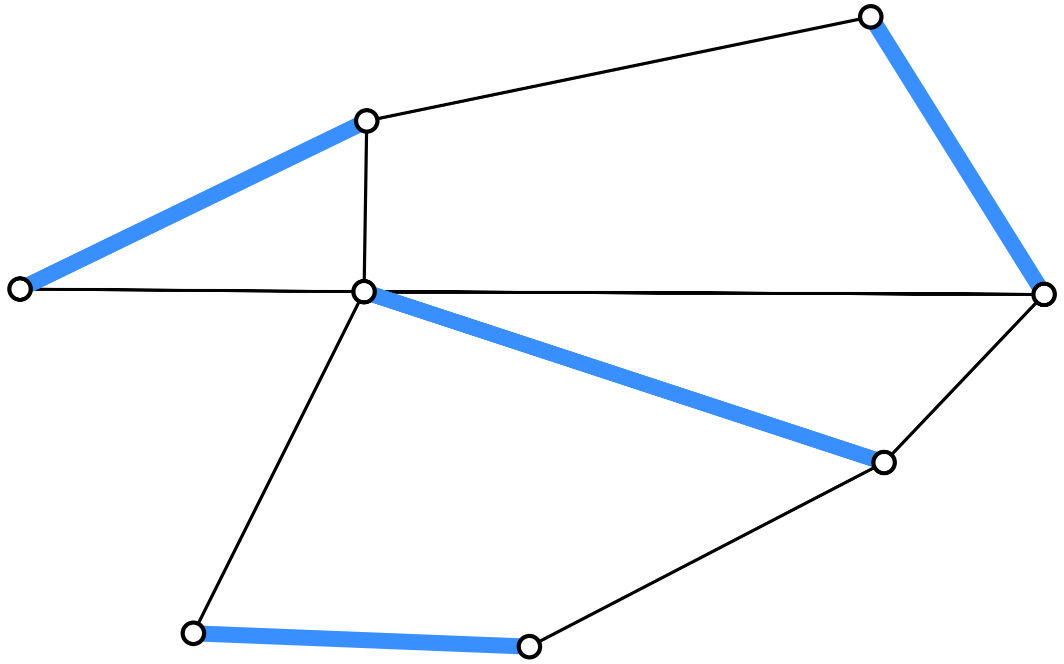
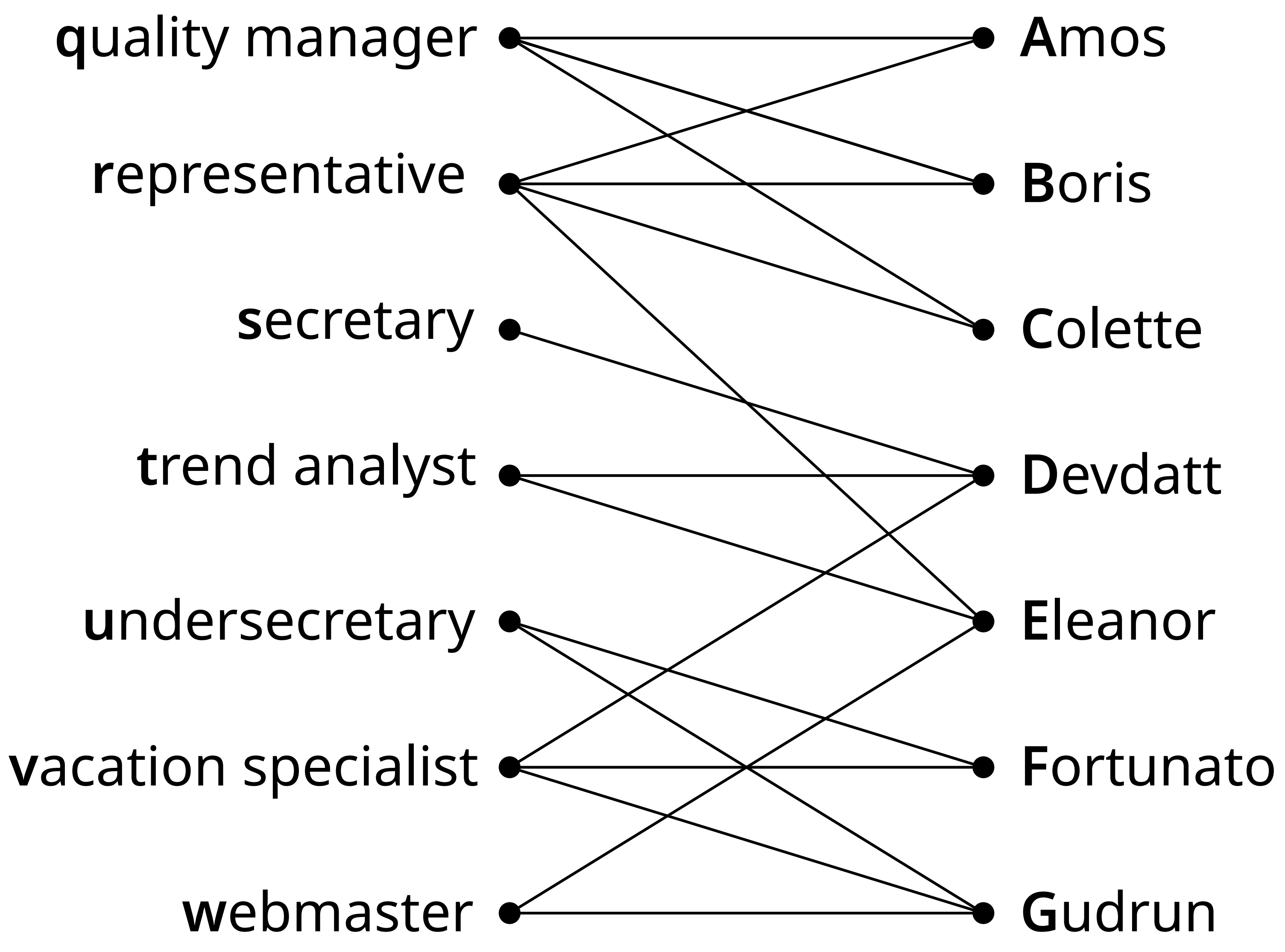
Remark: Not every matching can be extended to a maximum one.



Perfect matching

A company is assigning workers to jobs. In the bipartite graph an edge connects a worker to a job they are willing to take.



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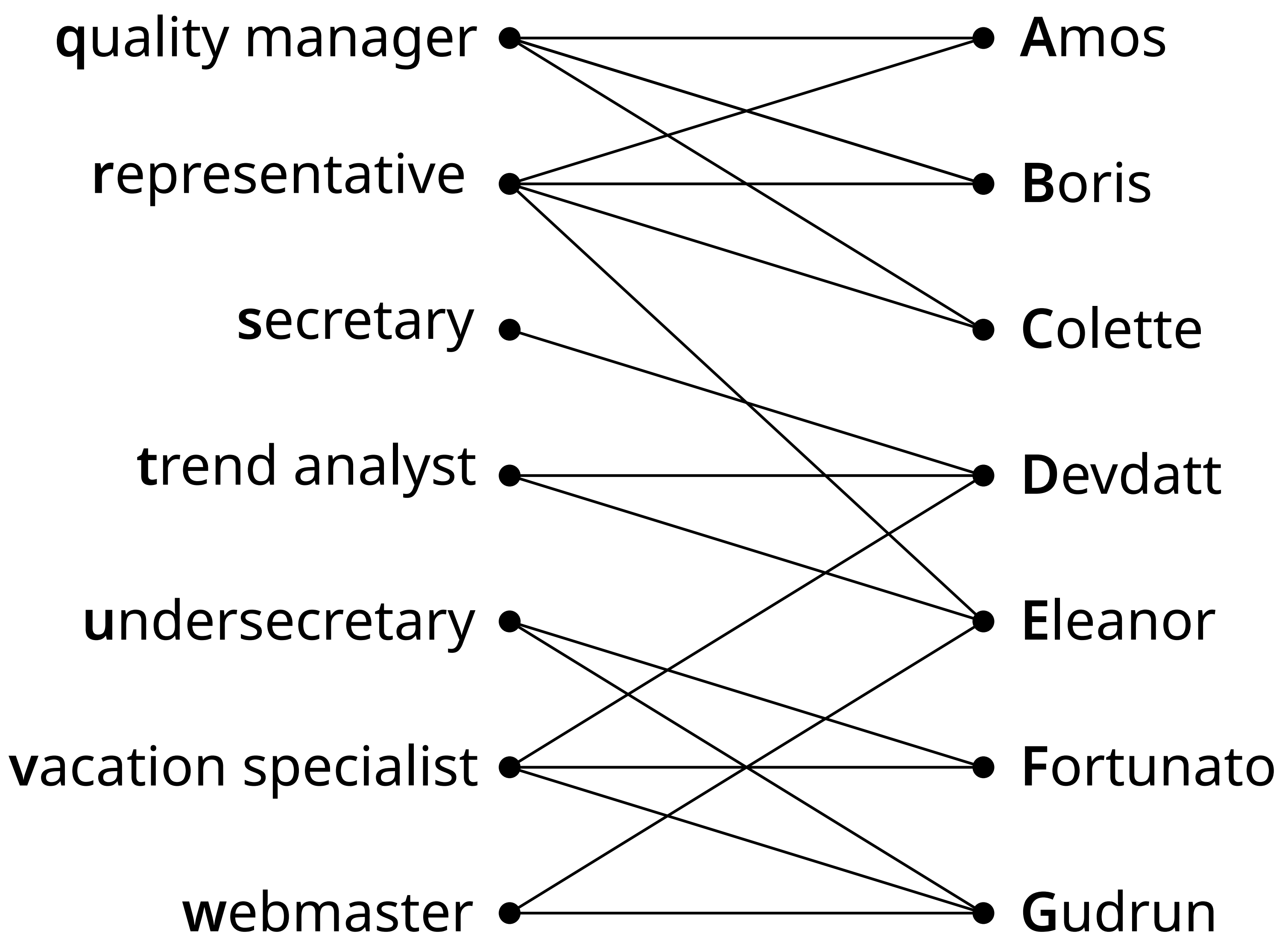

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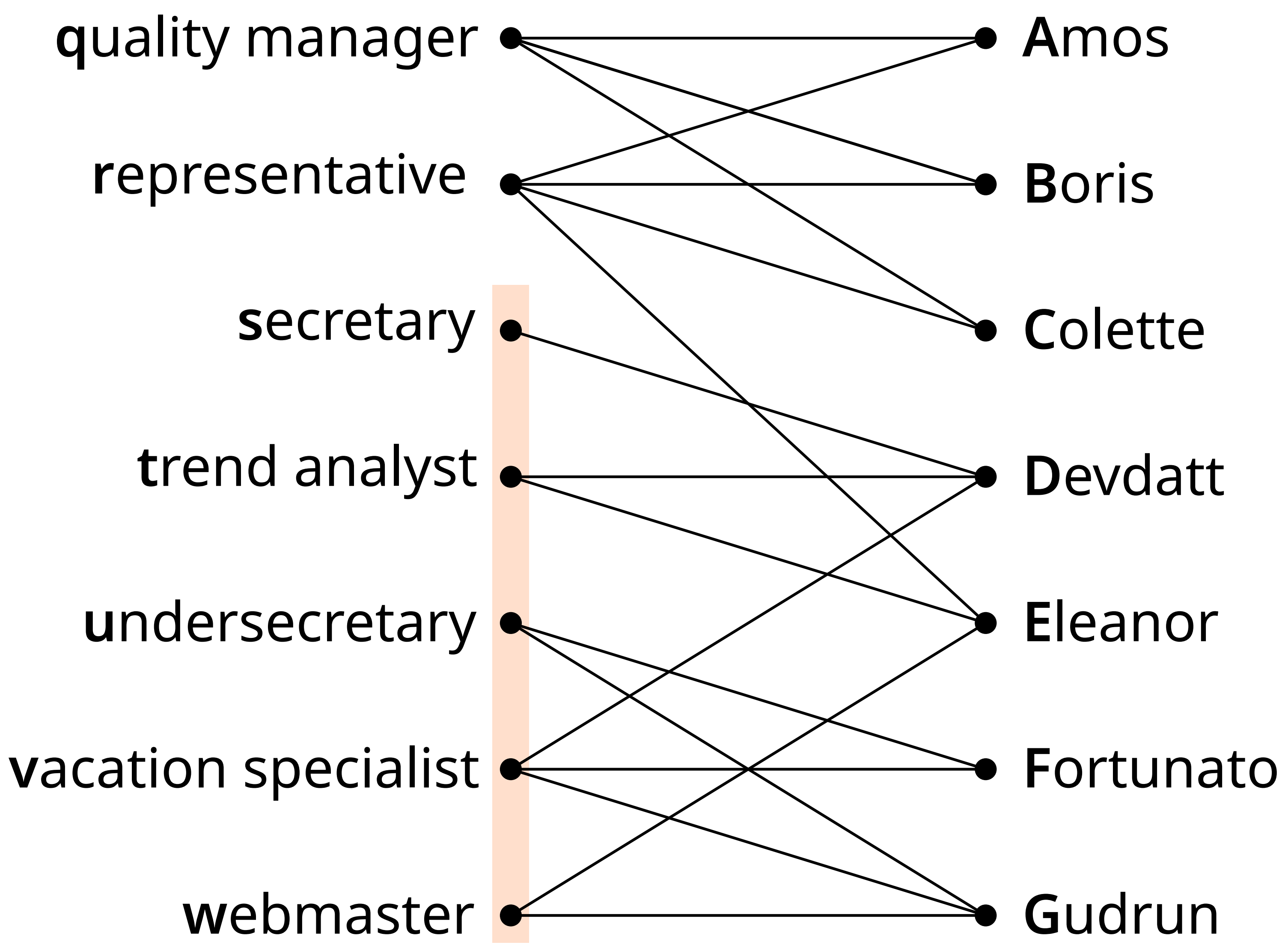

Neighborhood$\{s, t, u, v, w\} =$

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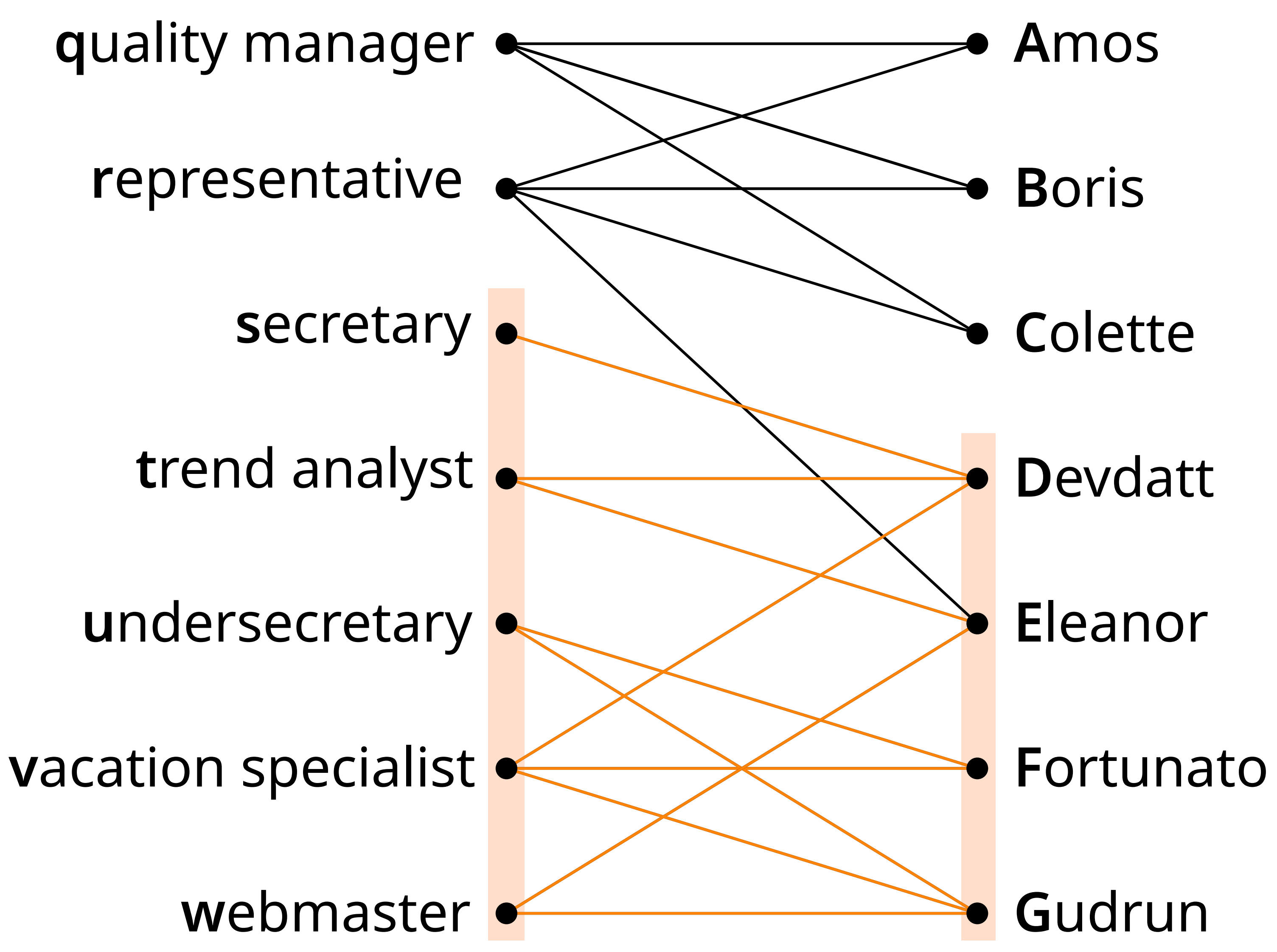

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Hall's Theorem: Let G be a bipartite graph with bipartition $V = X \cup Y$.
Then G has a matching covering X $\Leftrightarrow$ $|\text{Neighborhood}(X')| \geq |X'|$

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Proof: $\Rightarrow$ is clear.

$\Leftarrow$ will follow from König's Theorem.

Vertex Cover and König’s Theorem

König’s Theorem: In a bipartite graph the size of a minimum vertex cover equals the size of a maximum matching.

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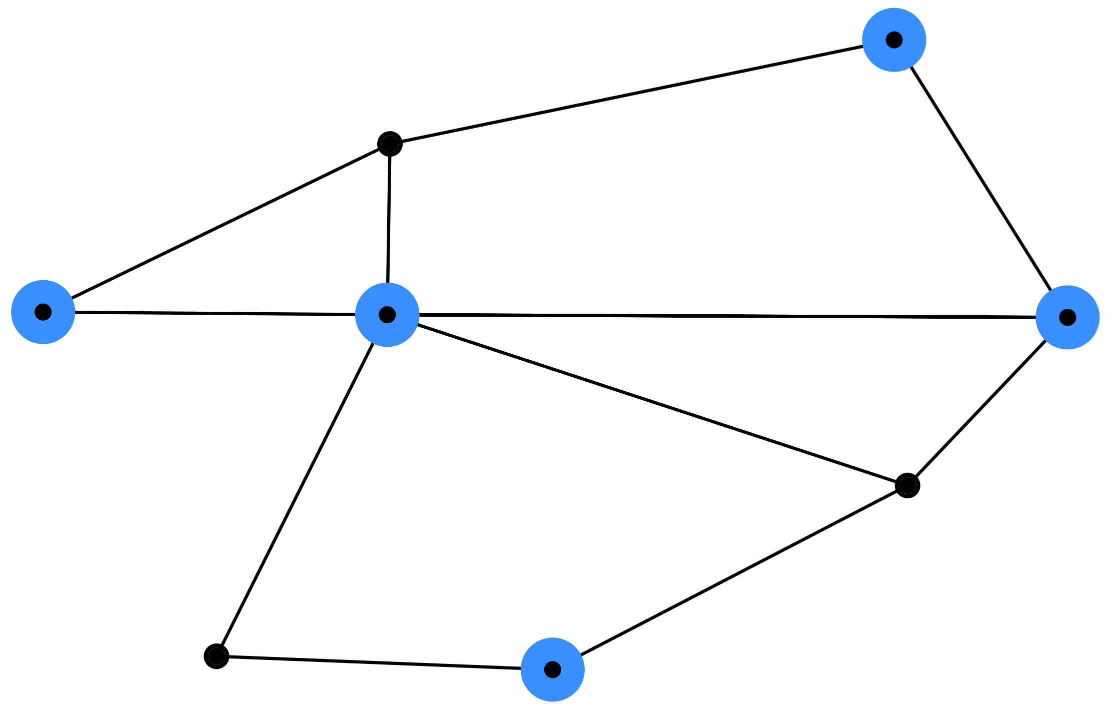

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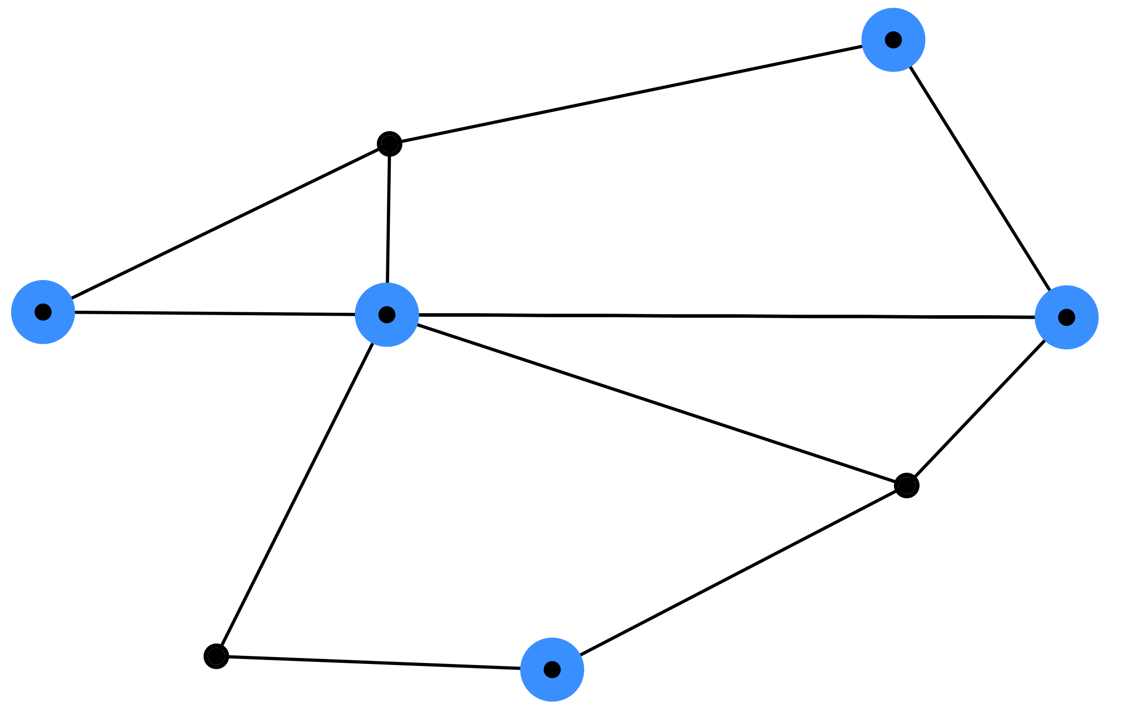
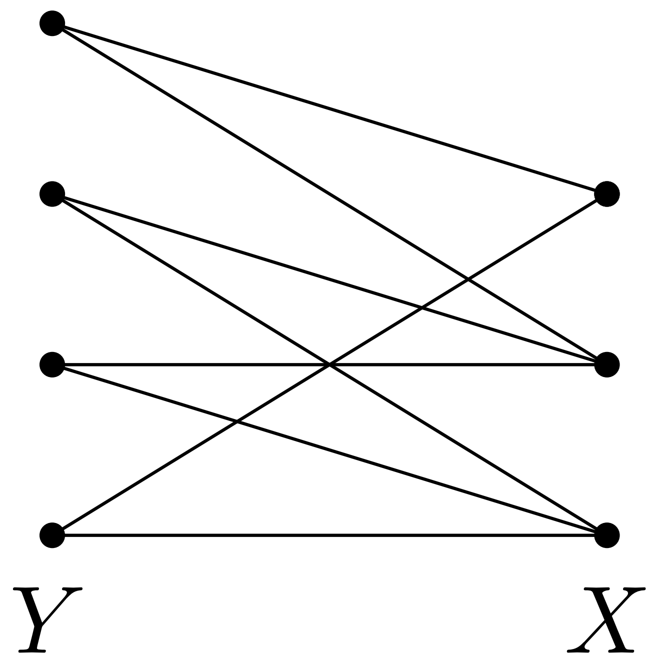
We will proof this using duality!



Proof of Hall's Theorem from König's Theorem

Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$.



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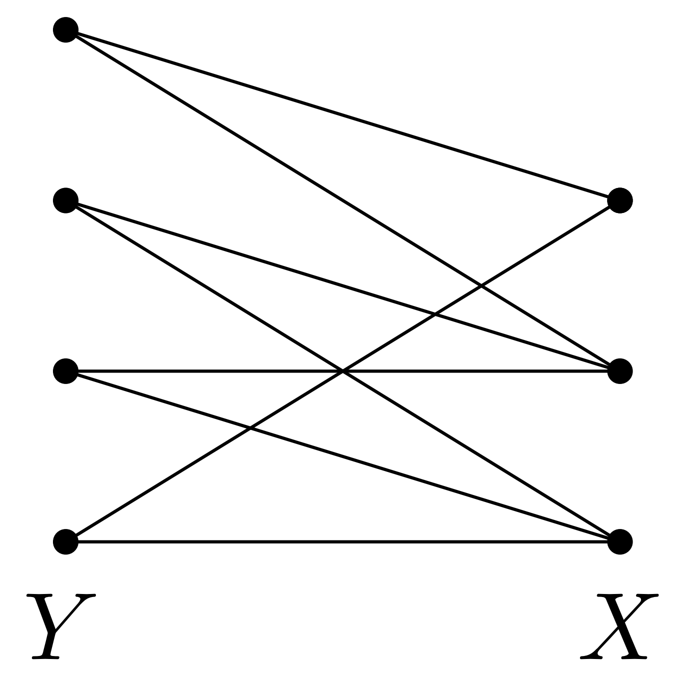

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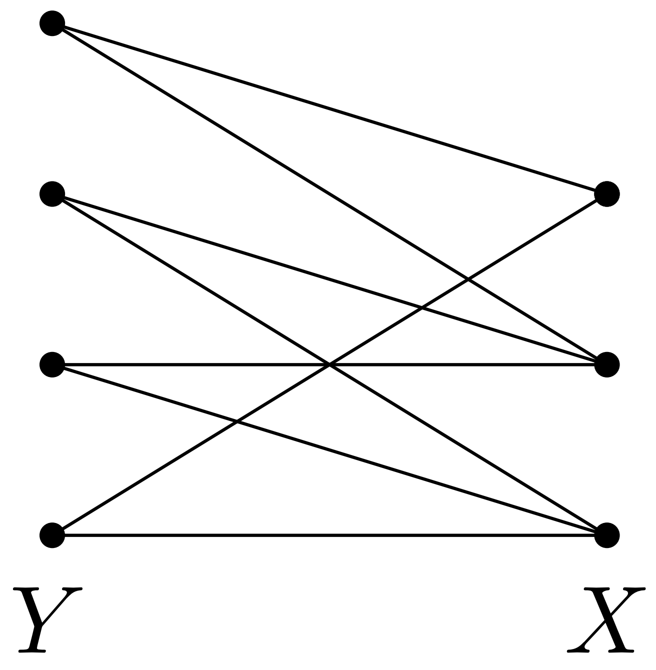

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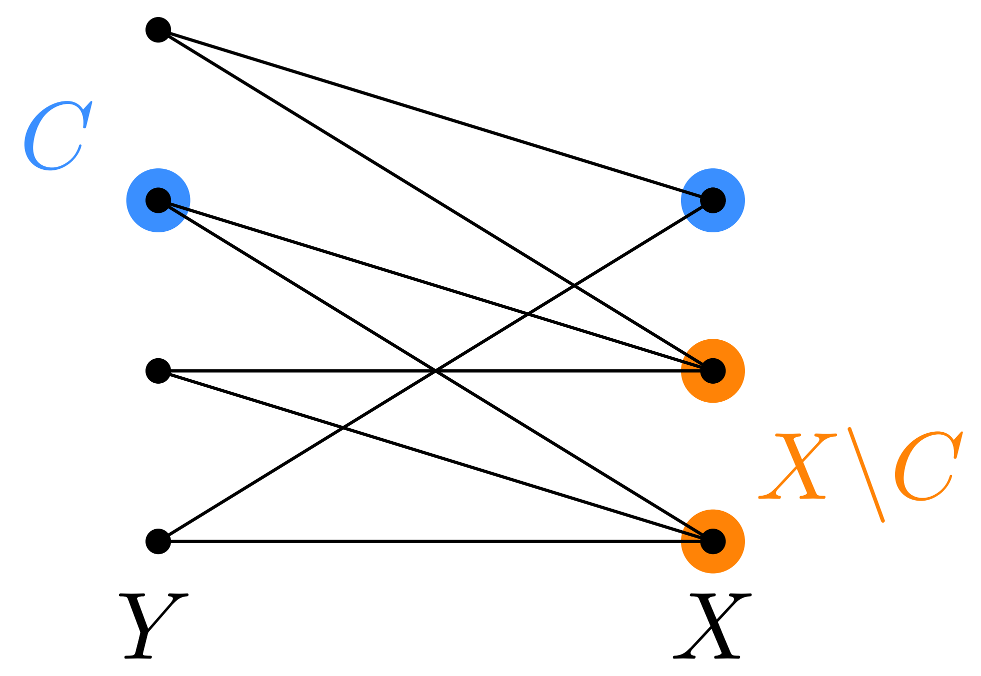
For a contradiction, suppose there is a vertex cover C with

- k vertices from X
- $< |X| - k$ vertices from Y



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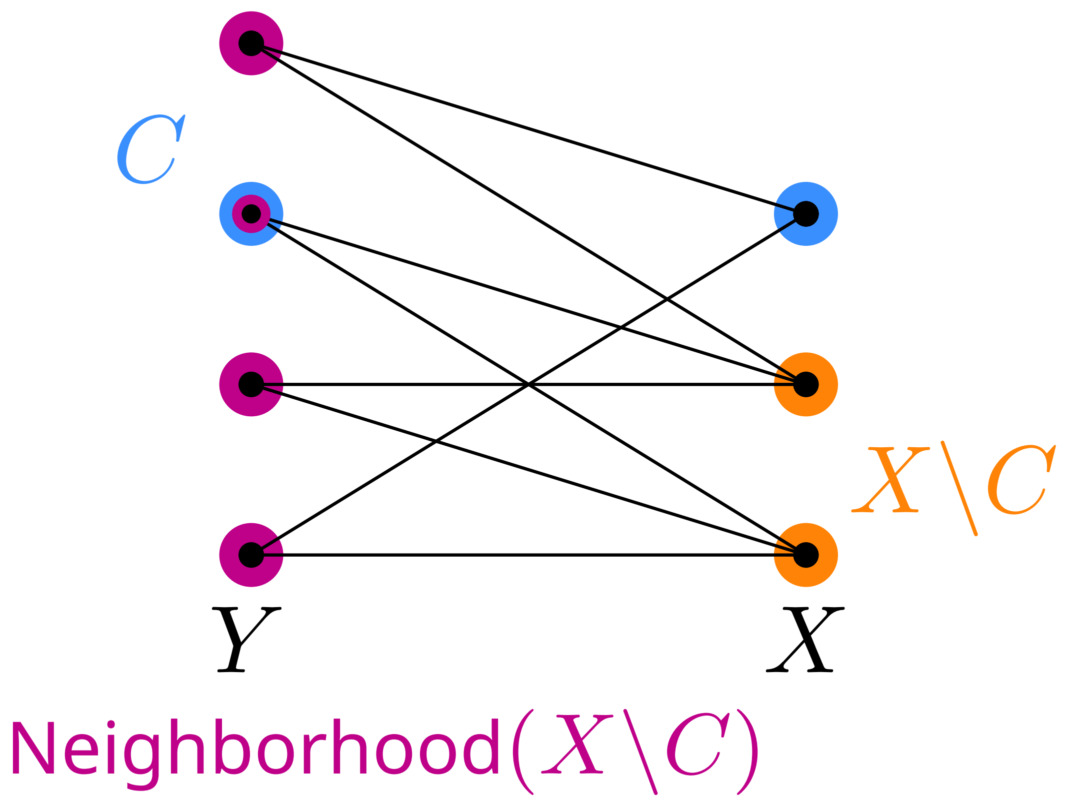

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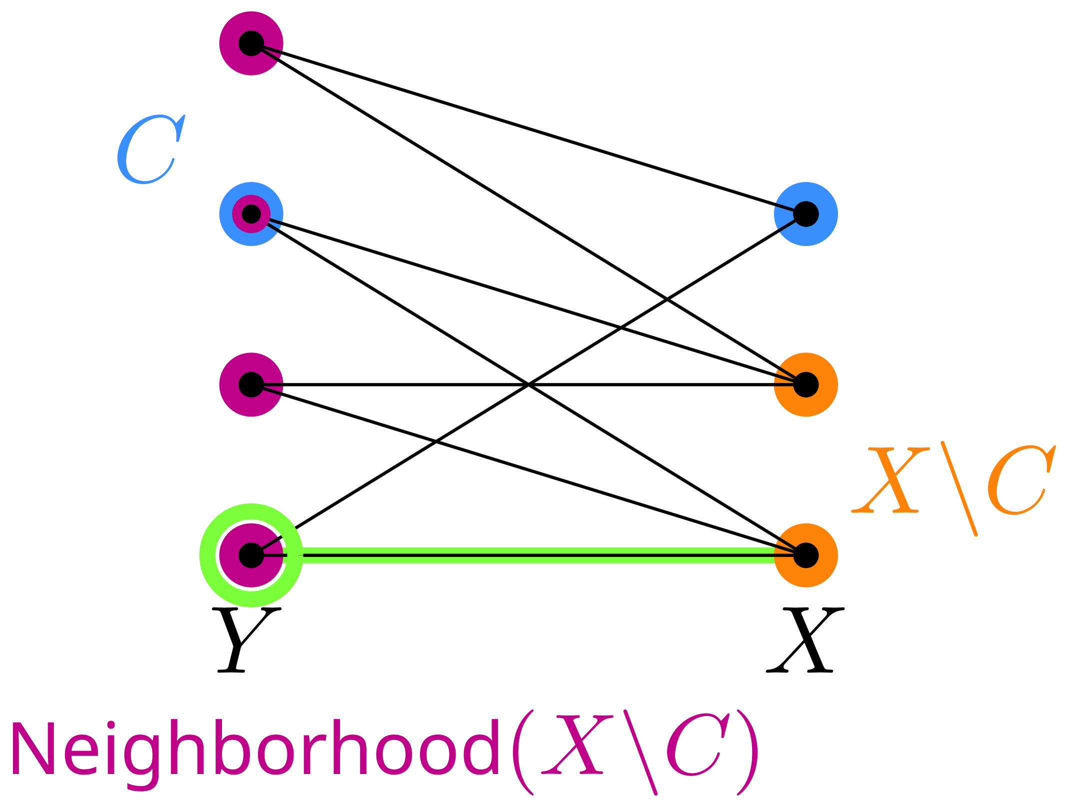
This gives a vertex in $\text{Neighborhood}(X \backslash C)$ that is not in $C \cap Y$ – showing C is not a cover.



Total Unimodular Matrices

A matrix is totally unimodular if every square submatrix has determinant $0, 1$, or -1.

Example

What does this matrix represent?

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

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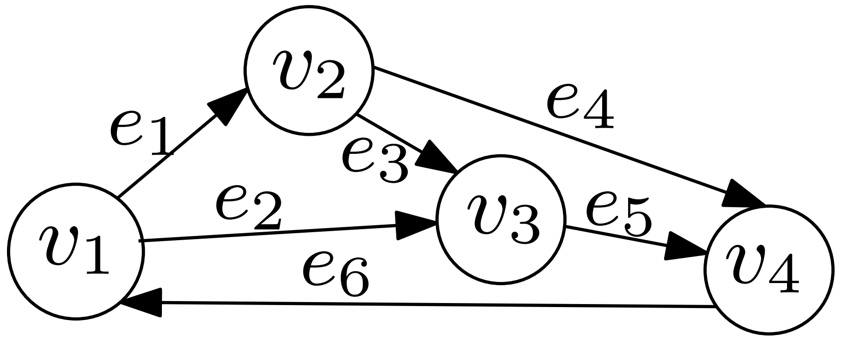

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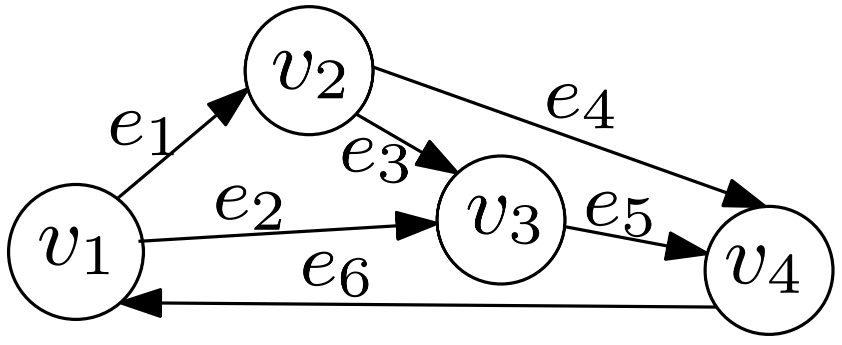

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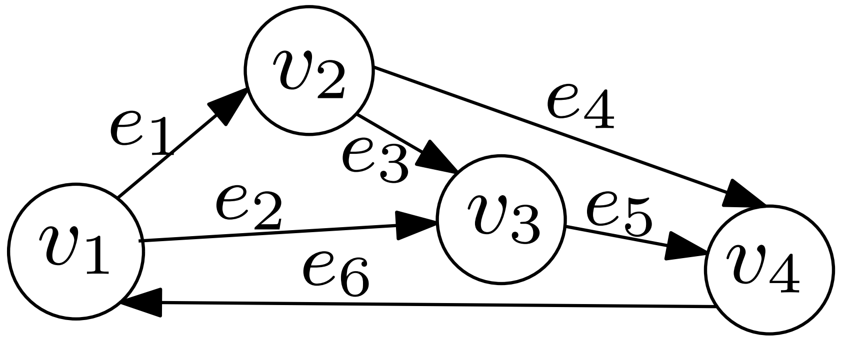

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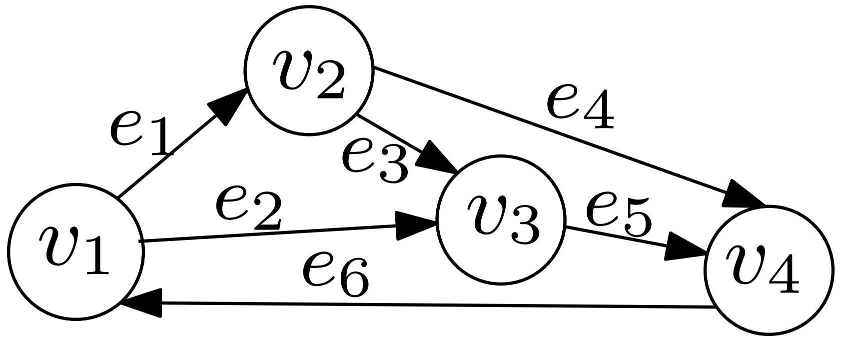

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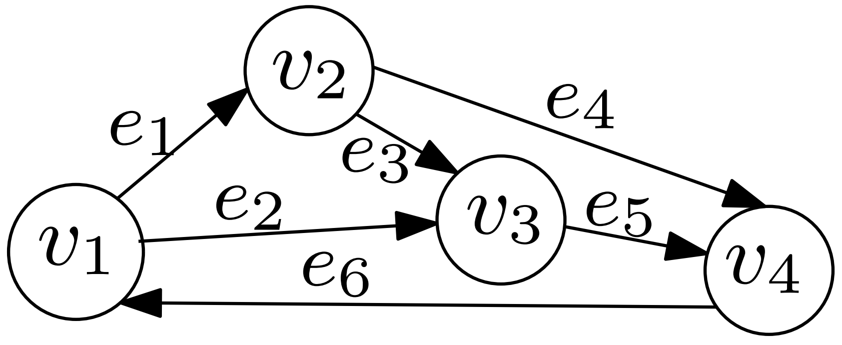

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Example If matrix A is totally unimodular, then so is $\left[\begin{array}{cc} A & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} \end{array}\right]$

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Theorem: Consider the linear program $\max c^T x$ subject to $Ax \leq b, x \geq 0$. If A is totally unimodular, if $b \in \mathbb{Z}^m$, and if there is an optimal solution, then there is an optimal integral solution $x^\star \in \mathbb{Z}^n$.

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Proof: Bring in equational form: $\bar{A}\bar{x} = b$ with $\bar{A} = (A|I)$ and $\bar{x} \in \mathbb{R}^{n+m}$.

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$\bar{A}$ totally unimodular
$\bar{A}$ nonsingular

What do we know about $\det(\bar{A})$?

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$\bar{A}$ totally unimodular $\Rightarrow$ $\det(\bar{A}_B) \in \{-1, 0, 1\}$.
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By Cramer's rule the coefficients of $\bar{x}^\star$ are rational numbers with denominator det(A_B), i.e., integer numbers! That is $\bar{x}^\star \in \mathbb{Z}^{n+m}$ and hence $x^\star \in \mathbb{Z}^n$.

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Lemma: The incidence matrix A of a bipartite graph $G = (X \cup Y, E)$ is totally unimodular.

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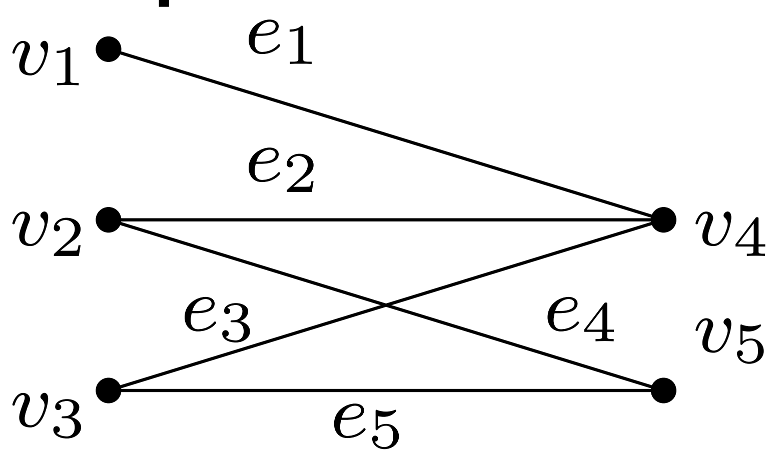

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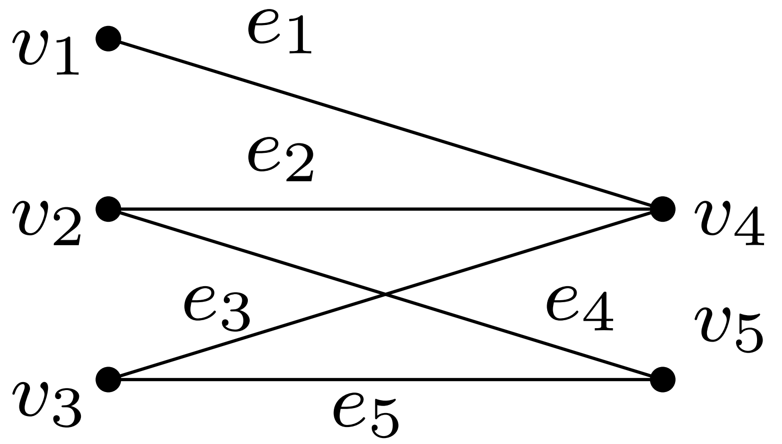

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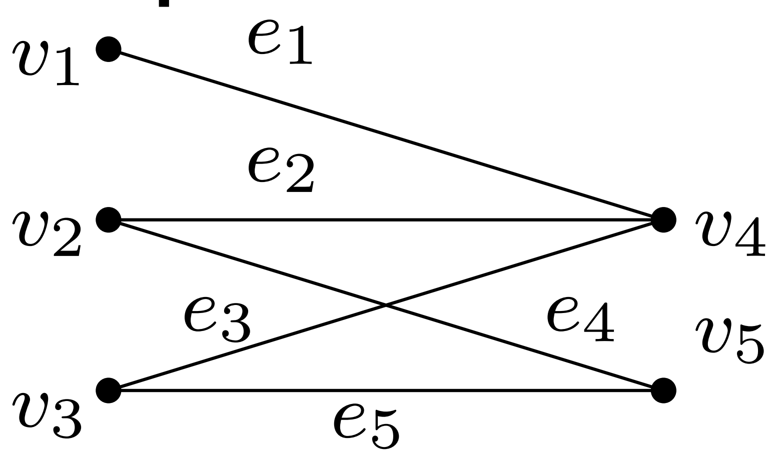

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<u>Base case</u> $\ell = 1$: Clear by definition of incidence matrix.

<u>Inductive step:</u> Assume true for $\ell - 1$.

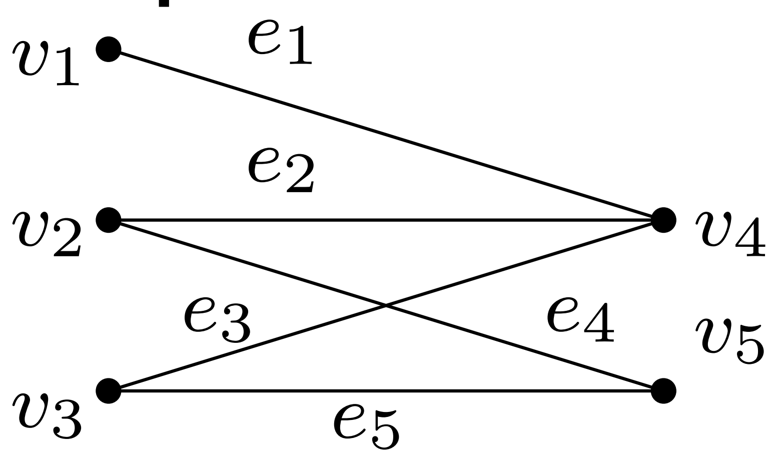
Let Q be an $\ell \times \ell$ submatrix.

If any column of Q is all zero, determinant is zero.

If any column of Q has one 1, true by induction.

Else all columns of Q have two 1.

Example



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Total Unimodularity and König's Theorem

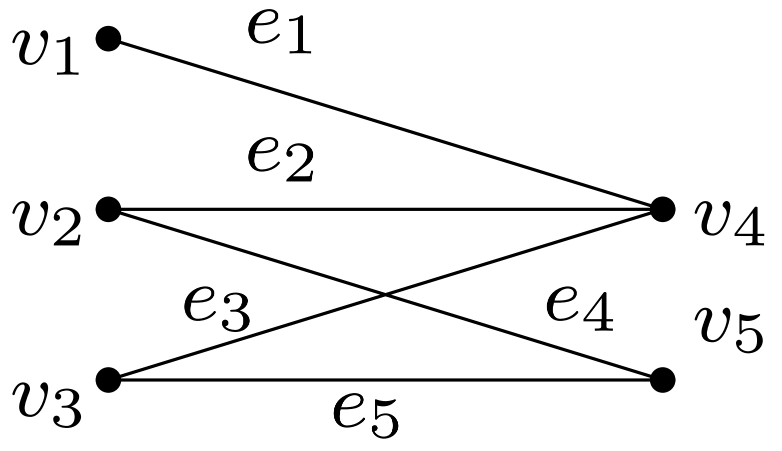
Lemma: The incidence matrix A of a bipartite graph $G = (X \cup Y, E)$ is totally unimodular.

Proof: We must show that ever $\ell \times \ell$ submatrix X of A has determinant $0, 1$, or -1. We show this by induction(ℓ).

Base case $\ell = 1$: Clear by definition of incidence matrix.

Inductive step: Assume true for $\ell - 1$.
Let Q be an $\ell \times \ell$ submatrix.
If any column of Q is all zero, determinant is zero.
If any column of Q has one 1, true by induction.
Else all columns of Q have two 1.
The sum of all rows for vertices in X gives $(1, 1, \ldots, 1)$.
The sum of all rows for vertices in Y gives $(1, 1, \ldots, 1)$.
Hence the rows of Q are linearly dep. and so $\det(Q) = 0$.

Example



$$\begin{array}{c c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

König’s Theorem

Theorem: In a bipartite graph the size of a minimum vertex cover equals the size of a maximum matching.

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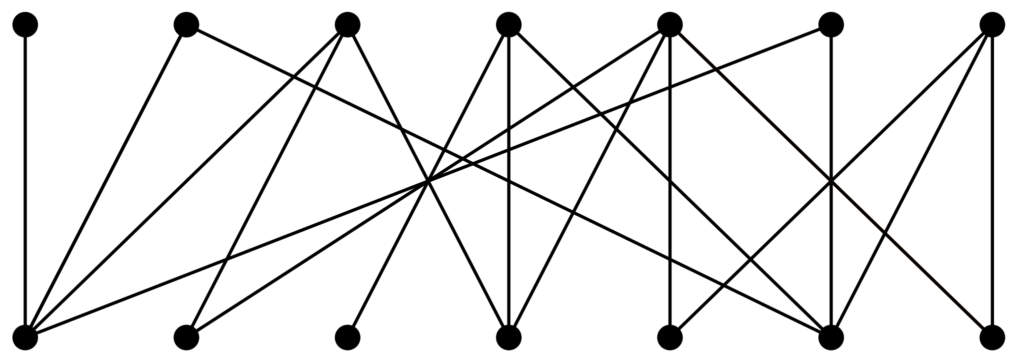

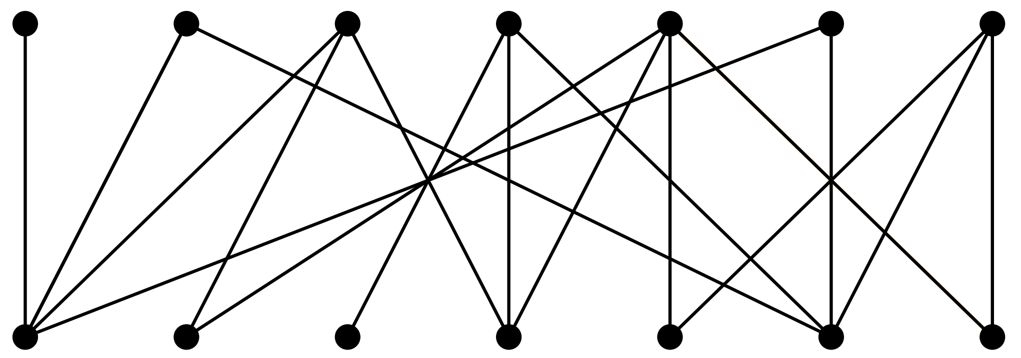

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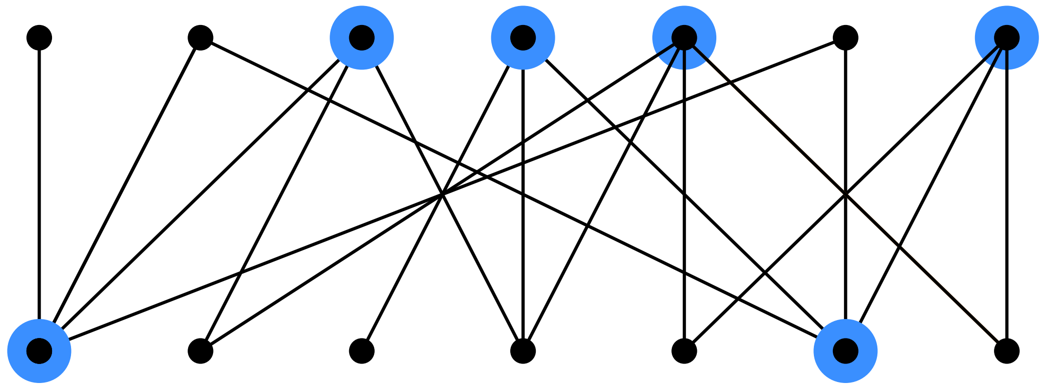
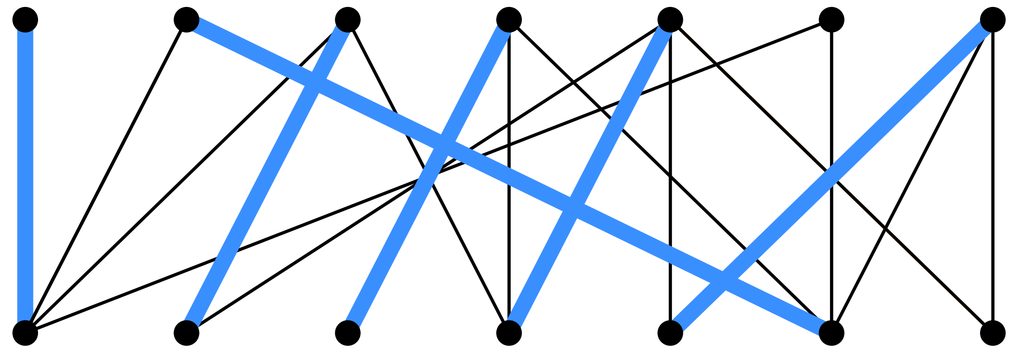


minimum vertex cover



maximum matching

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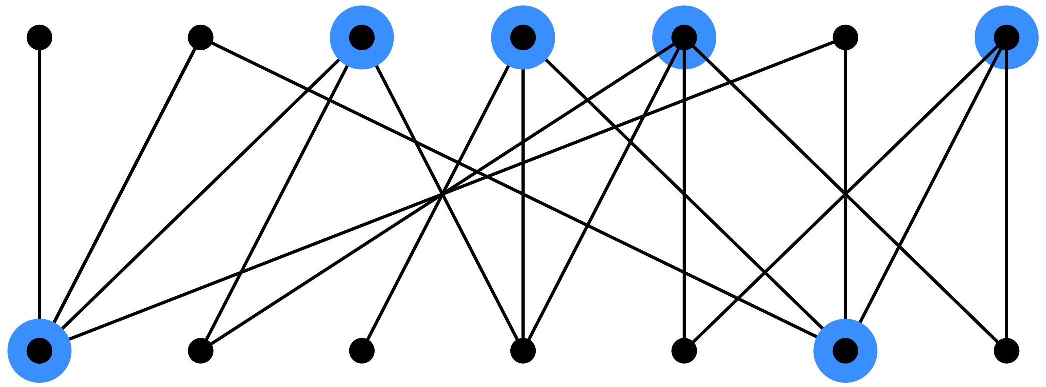
Proof:

Let A be the incidence matrix of the bipartite graph.

What are their LP’s?



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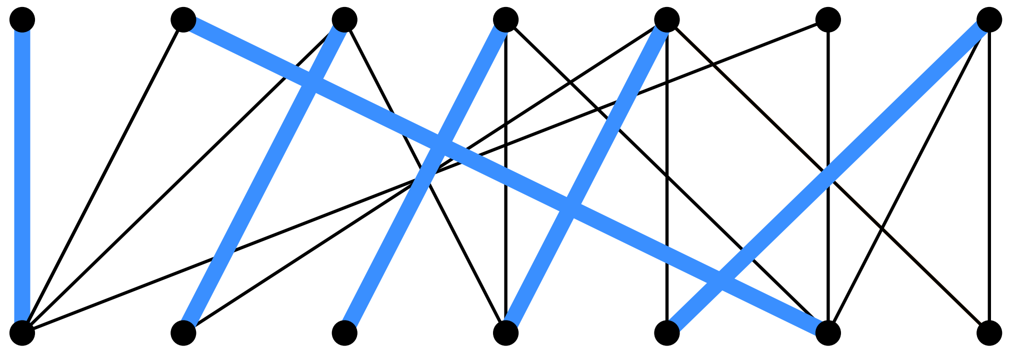

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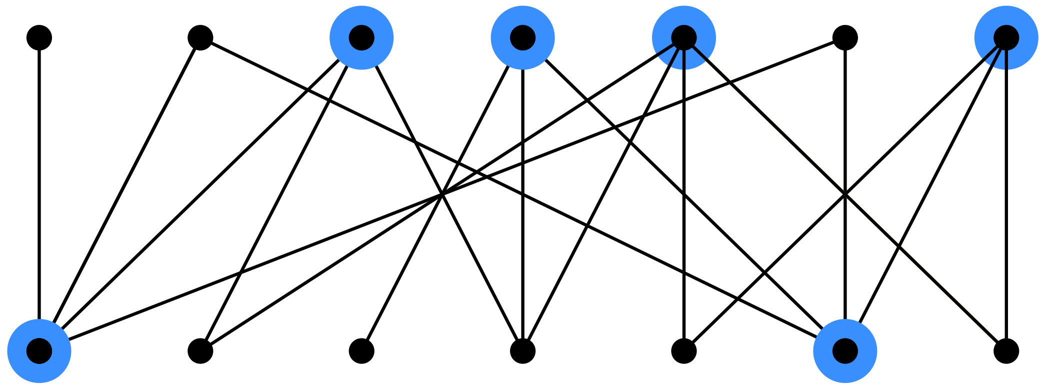
Cover		Matching	
min	$\sum y_i$	max	$\sum x_j$
subject to	$A^T y \geq 1$	subject to	$Ax \leq 1$
	$y \geq 0$		$y \geq 0$

optionally: $y_i, x_j \in \{0, 1\}$



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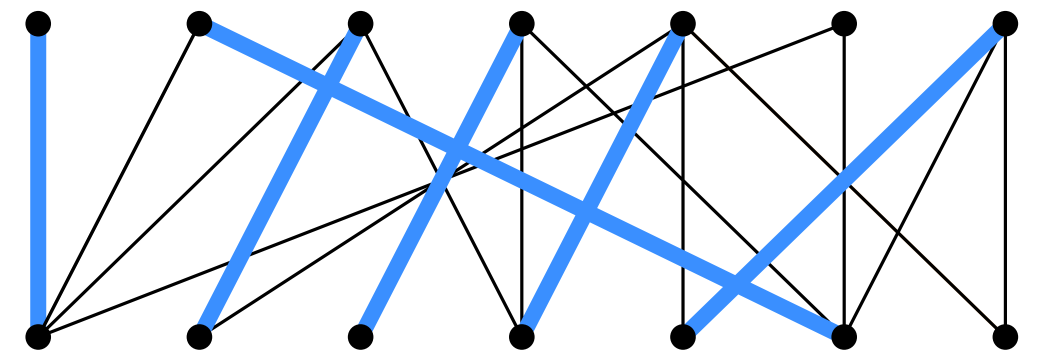

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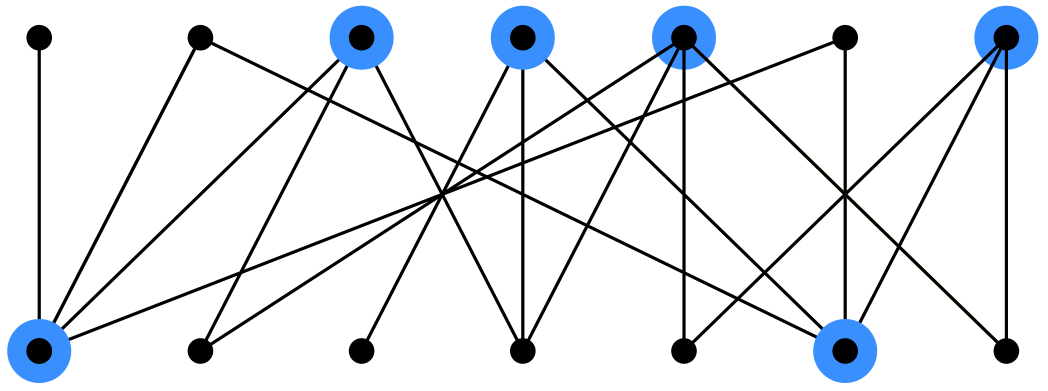

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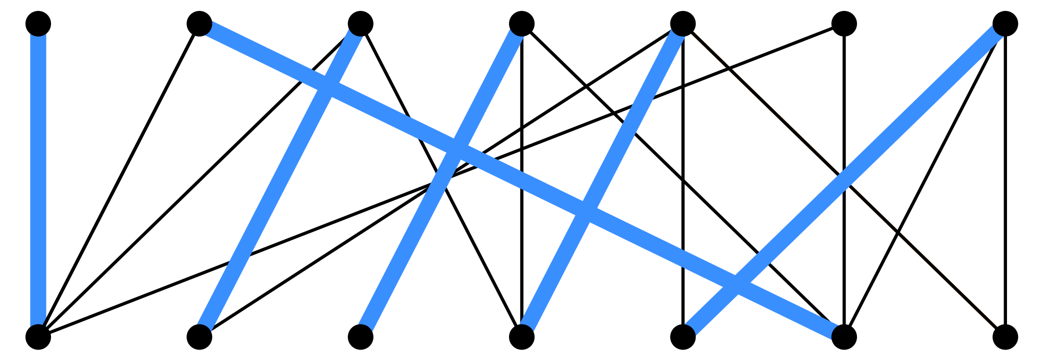

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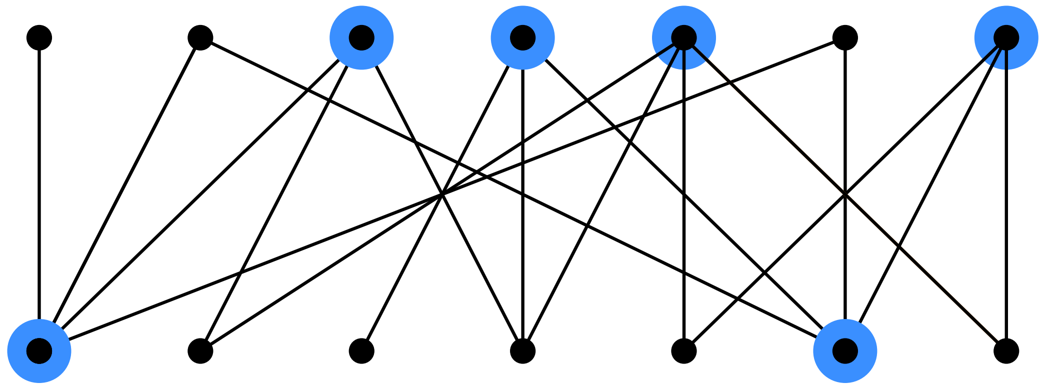
The duality of these linear programs then proves the theorem.



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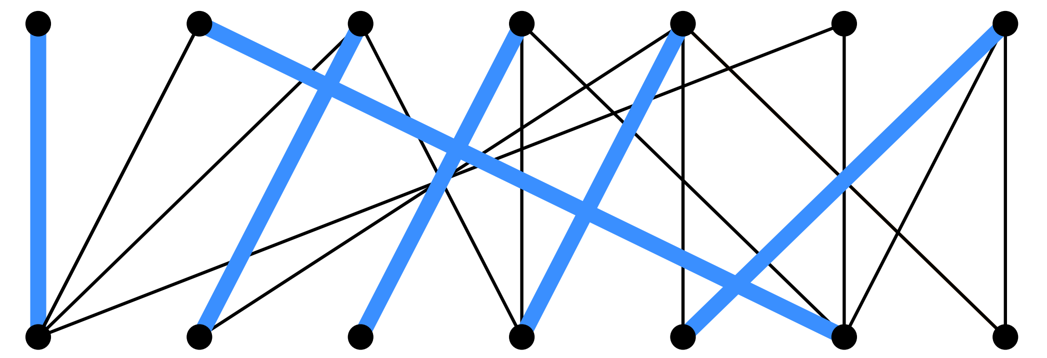
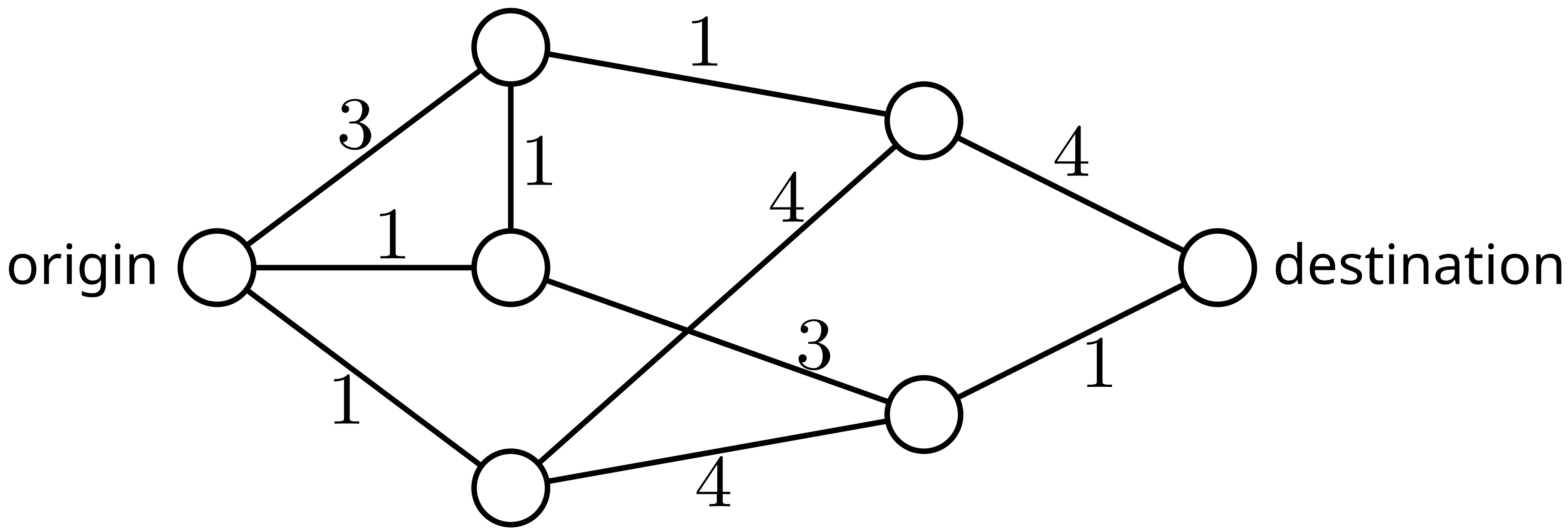


maximum matching

Duality shows Max Flow = Min Cut

Recall: Flow in a Network

How to send as much data as possible over a local network?



nodes cannot store data and links can transport in only one direction

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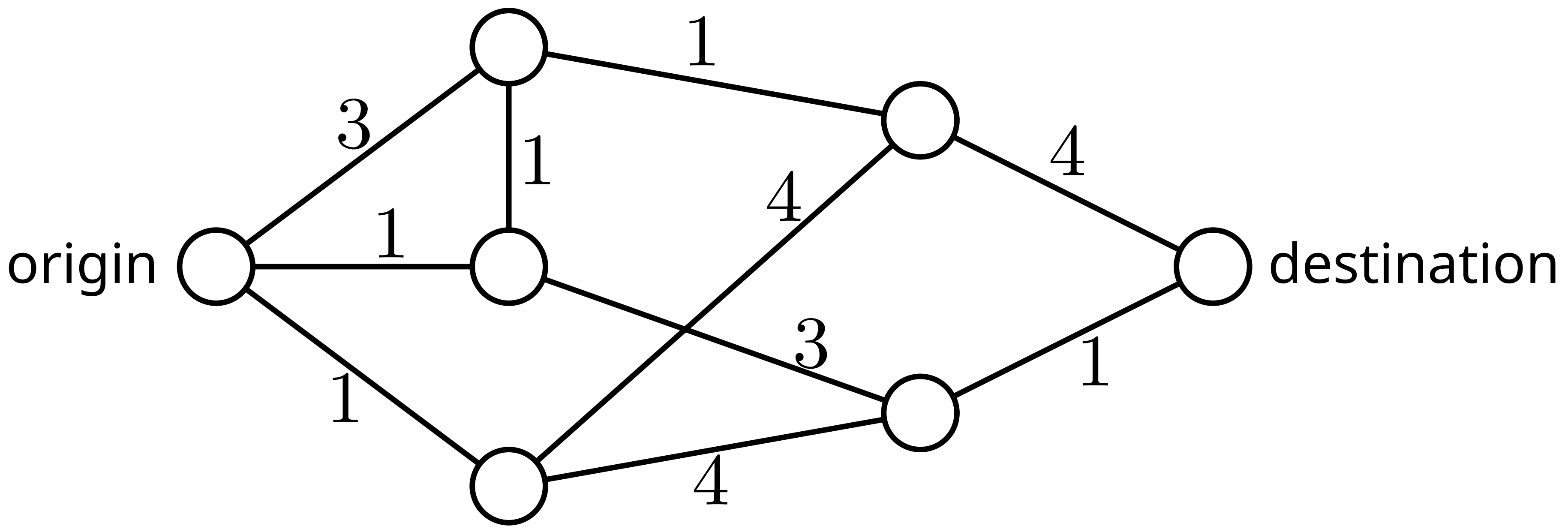

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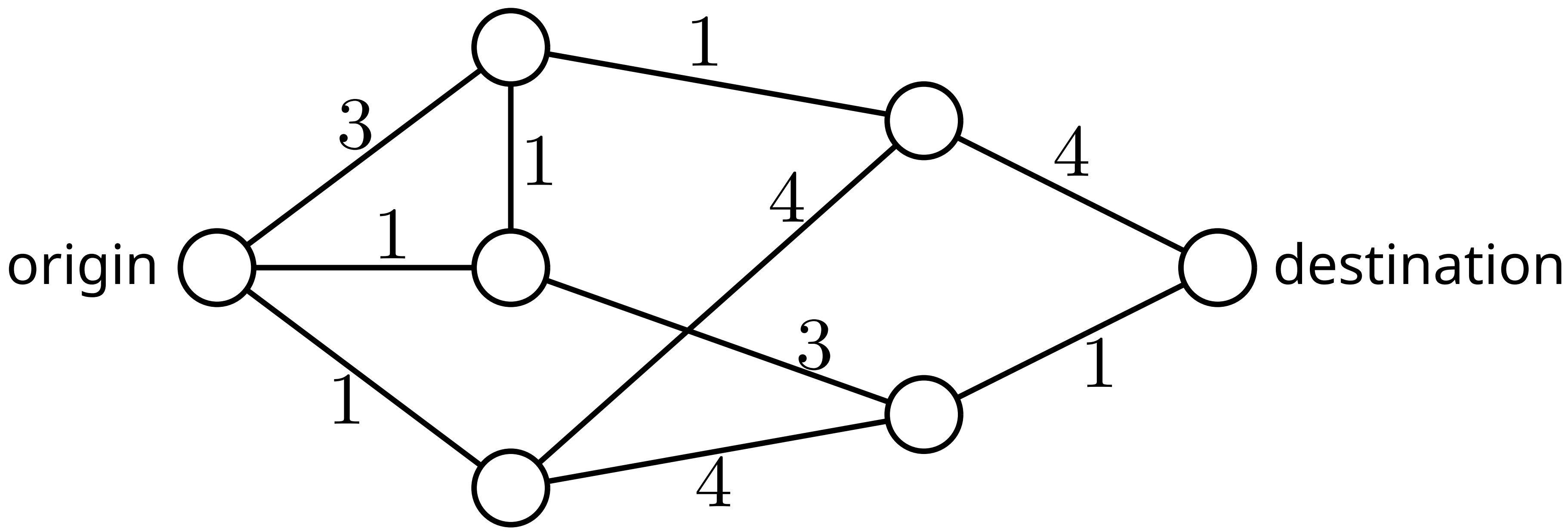

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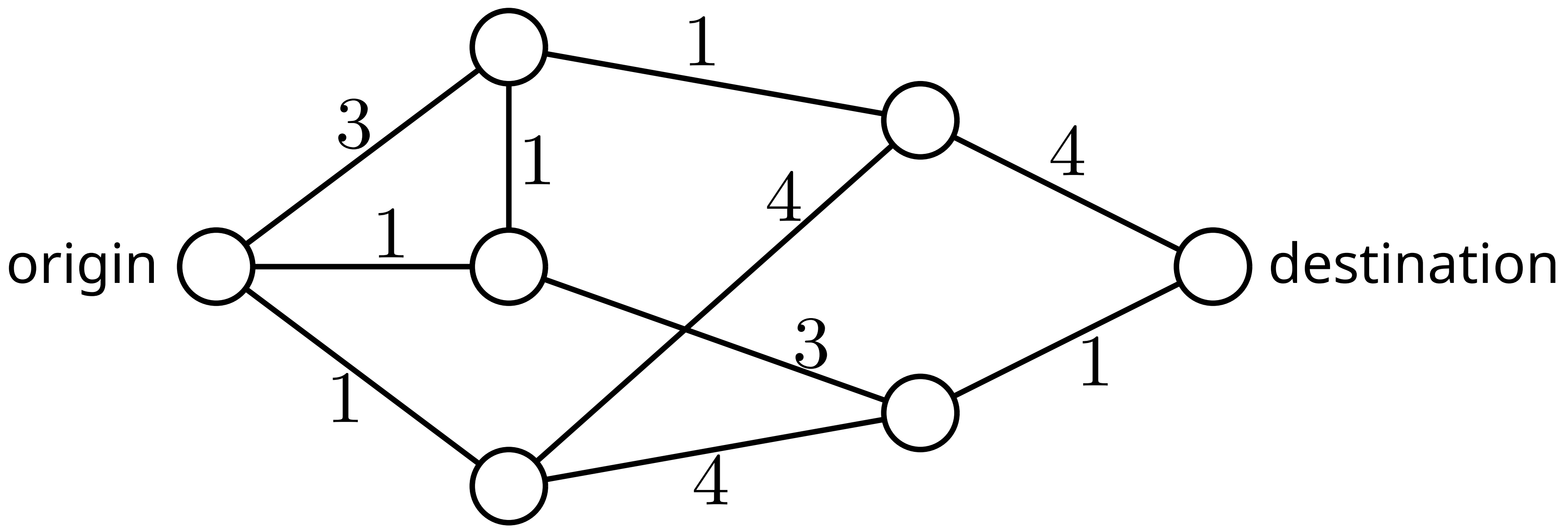

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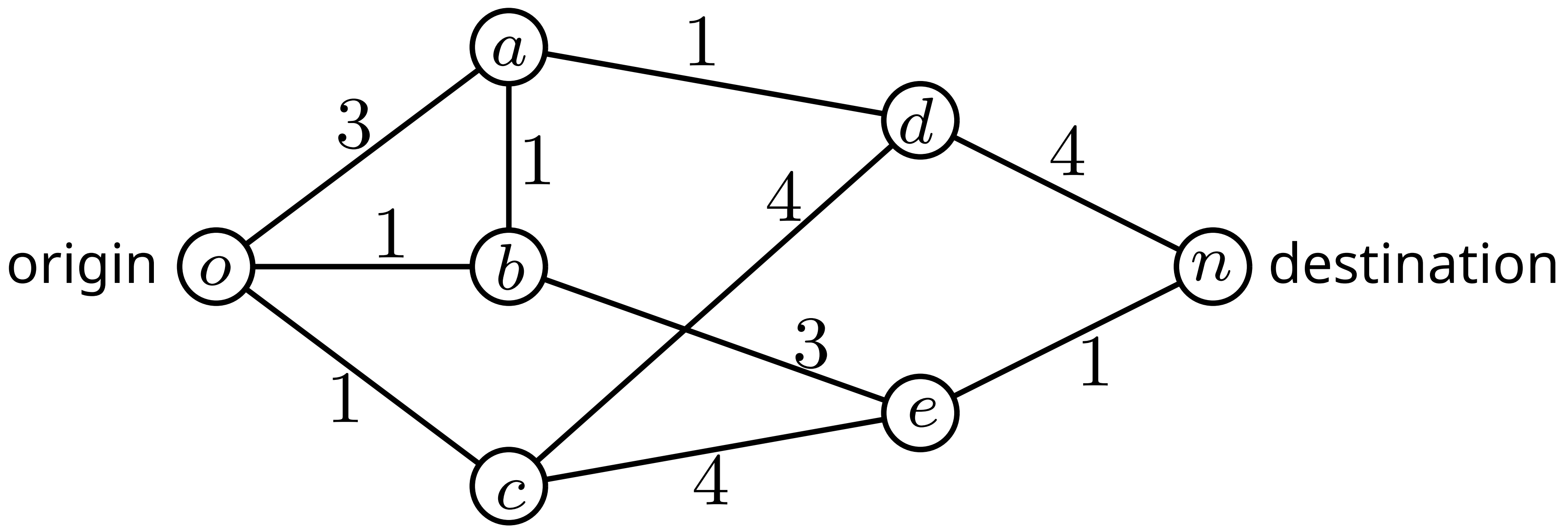

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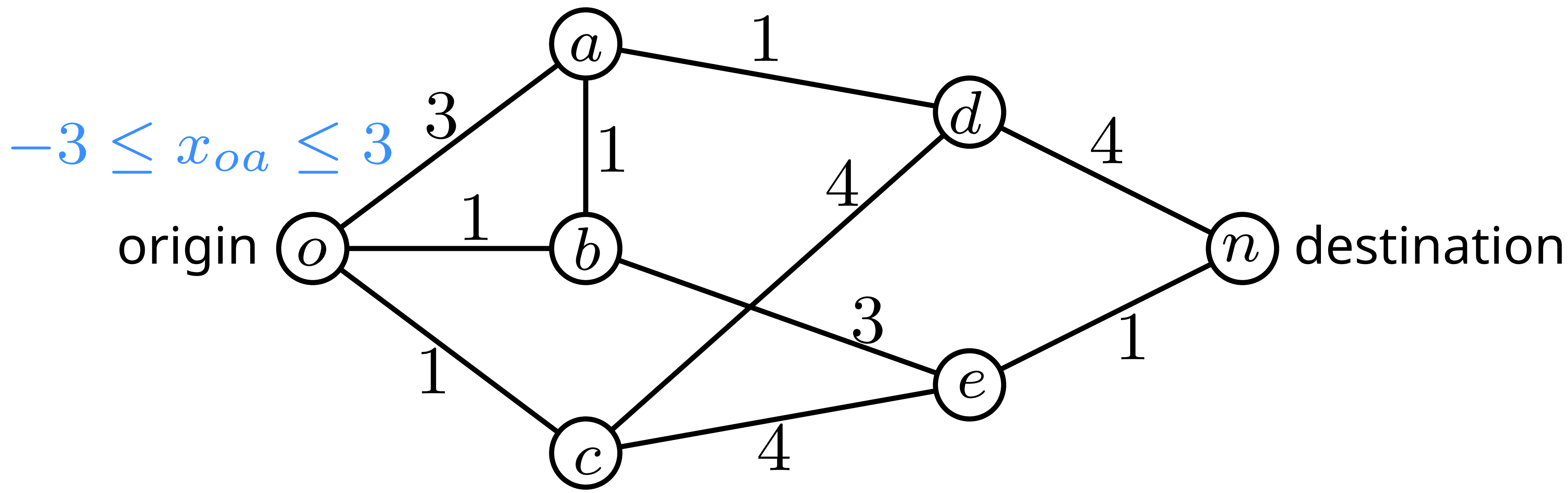

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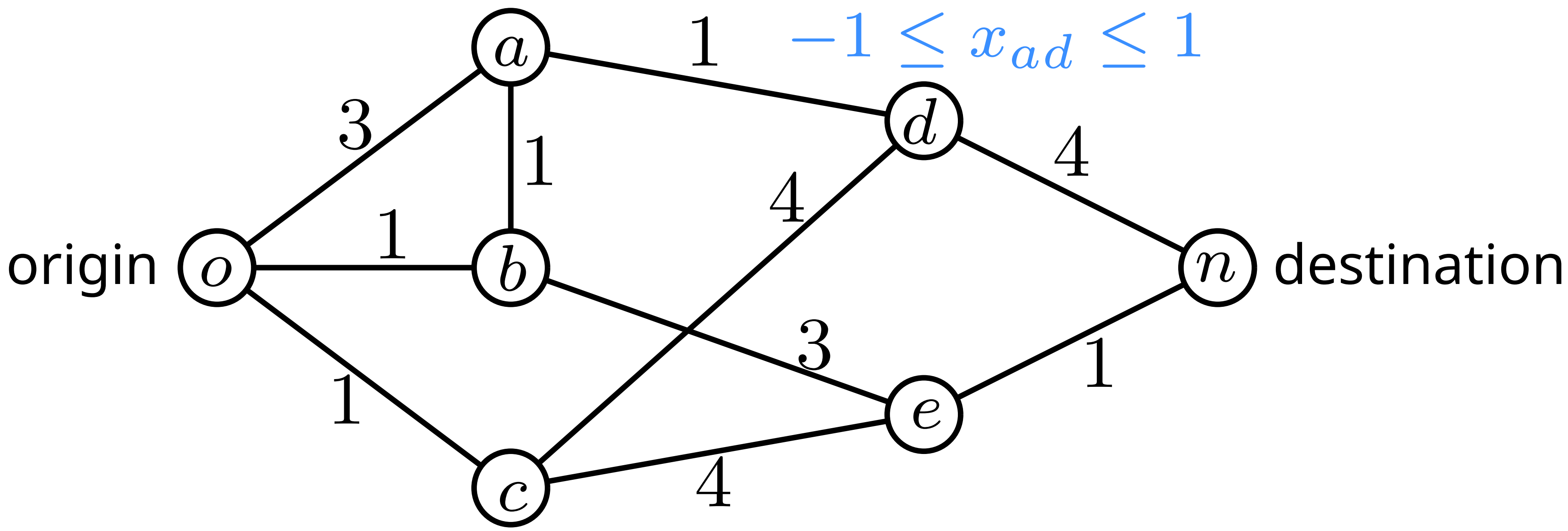

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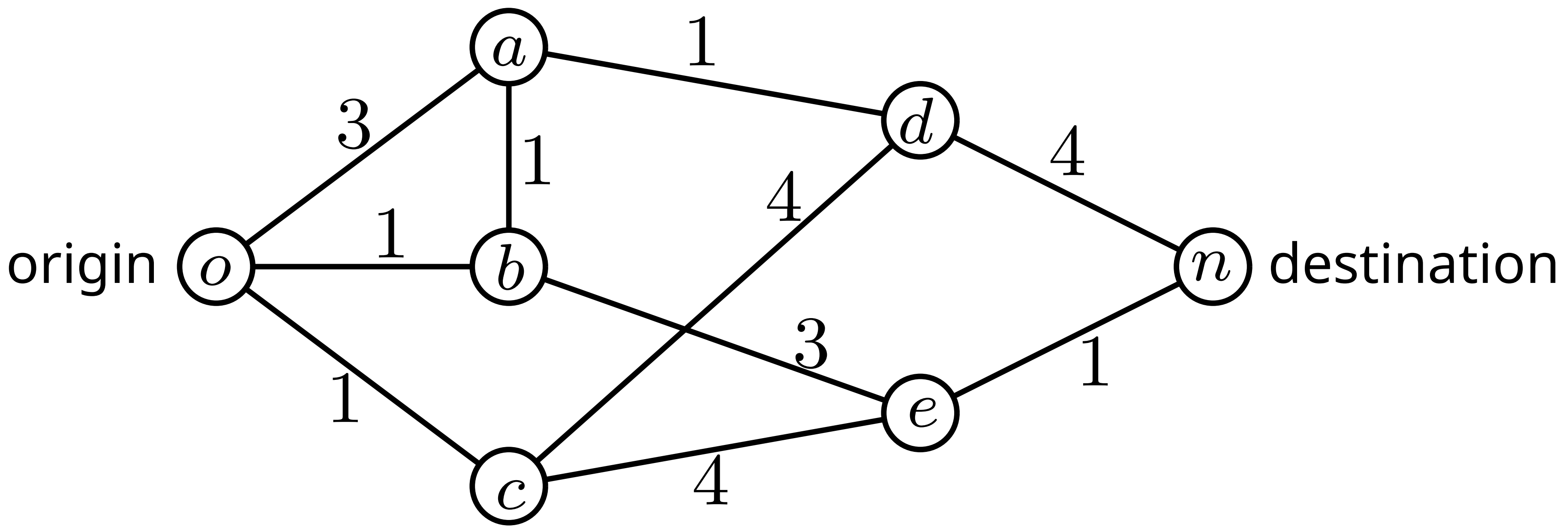

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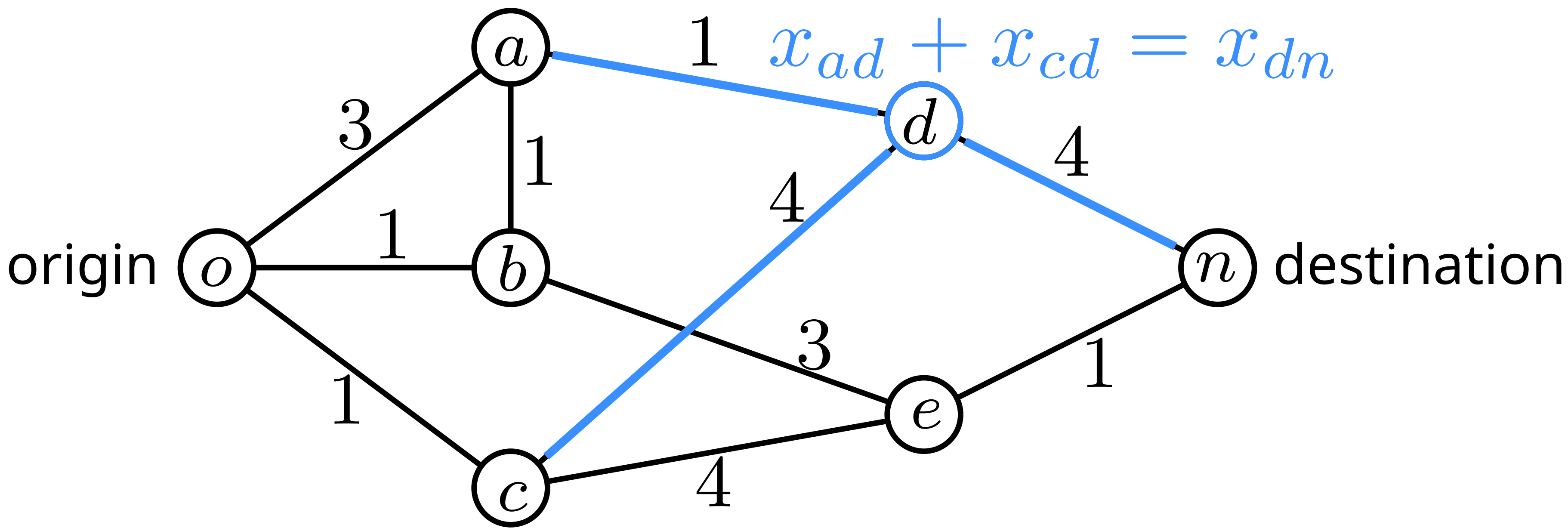

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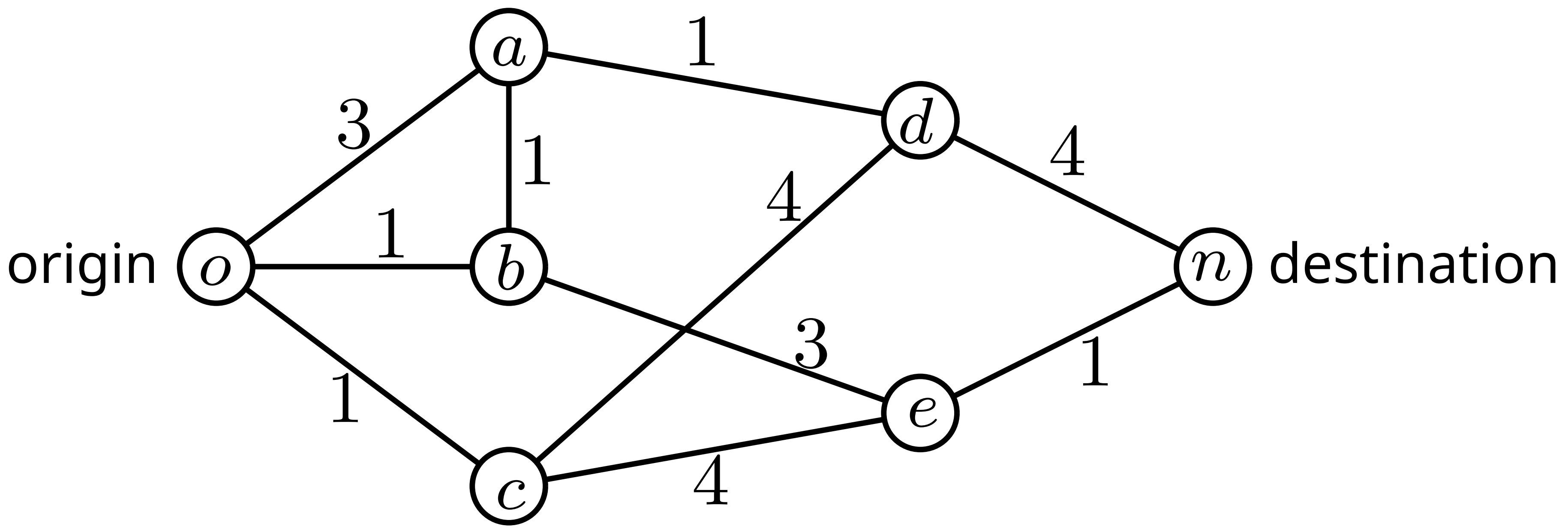

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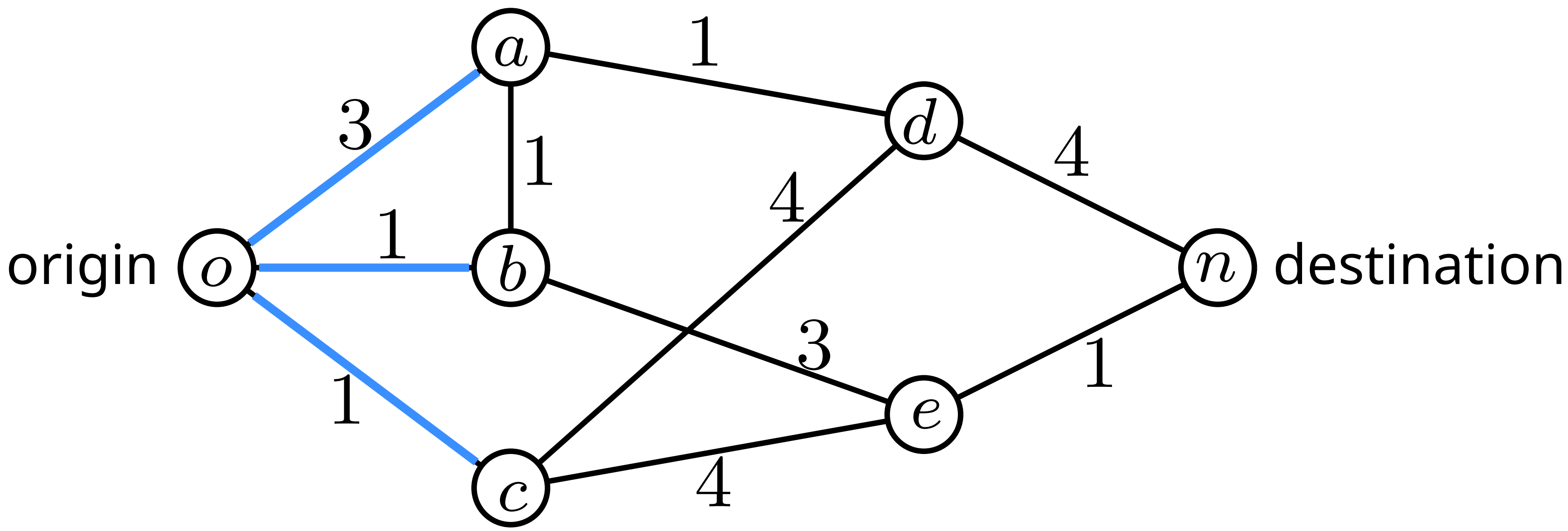

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$x_{oa} + x_{ob} + x_{oc}$

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Linear Program Formulation

maximize $x_{oa} + x_{ob} + x_{oc}$

subject to

$$-3 \leq x_{oa} \leq 3,\ -1 \leq x_{ob} \leq 1,\ -1 \leq x_{oc} \leq 1$$

$$-1 \leq x_{ab} \leq 1,\ -1 \leq x_{ad} \leq 1,\ -3 \leq x_{be} \leq 3$$

$$-4 \leq x_{cd} \leq 4,\ -4 \leq x_{ce} \leq 4,\ -4 \leq x_{dn} \leq 4$$

$$-1 \leq x_{en} \leq 1$$

$$x_{oa} = x_{ab} + x_{ad}$$

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$$x_{ad} + x_{cd} = x_{dn}$$

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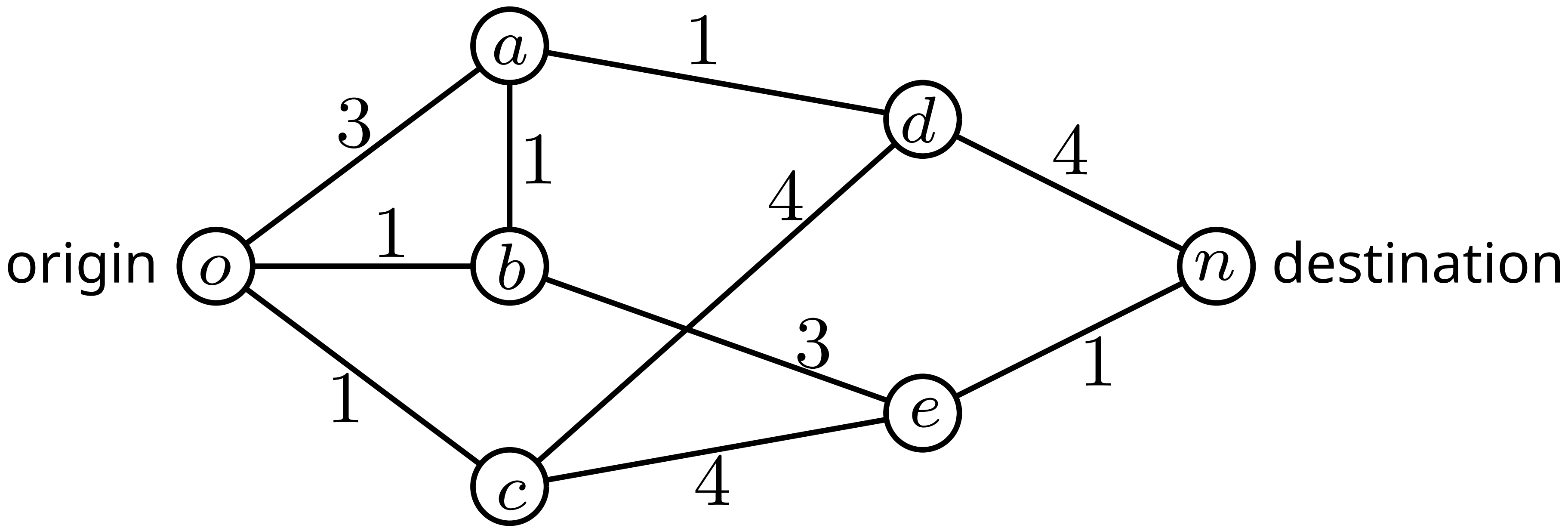

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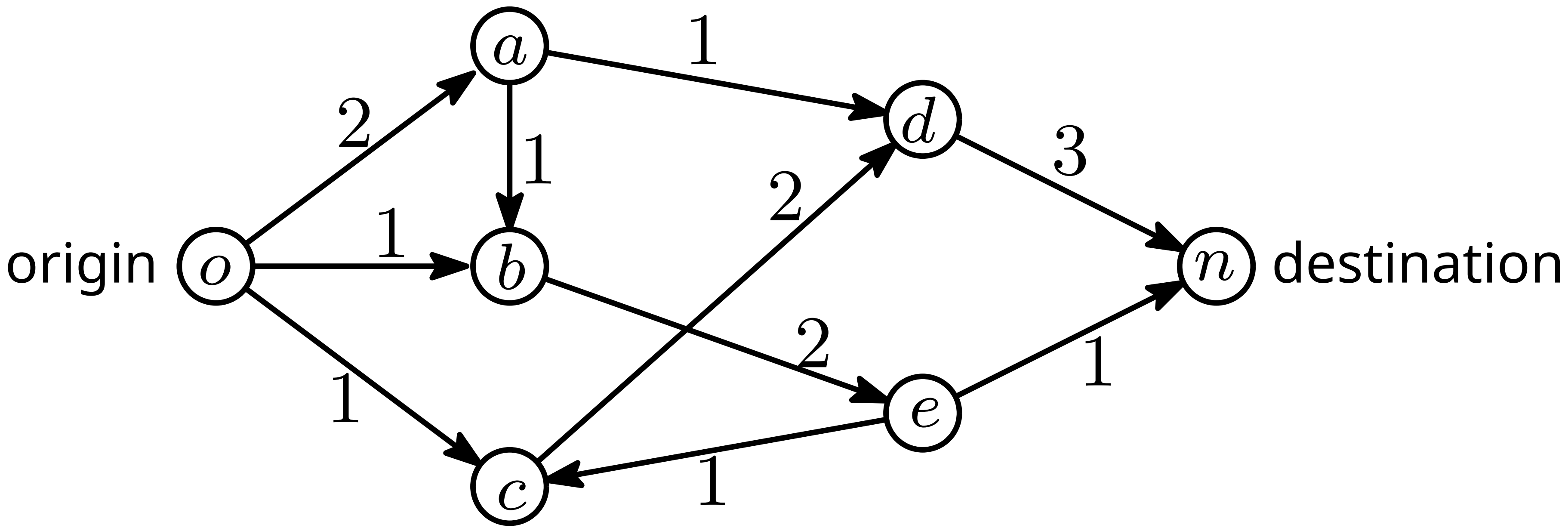
Optimal solution: 4



Recall: Flow in a Network

Linear Program Formulation

$$
\begin{aligned}
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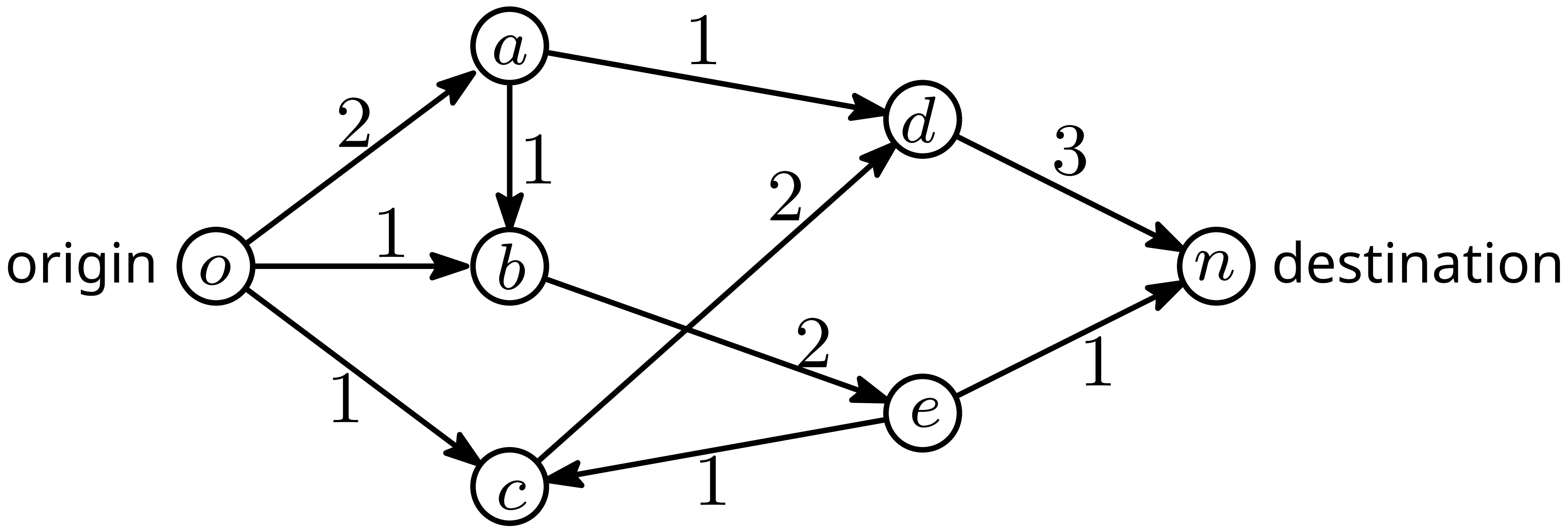

well-known “max flow = min cut” $\longrightarrow$ now via LP-duality!

Recall: Flow in a Network

Linear Program Formulation

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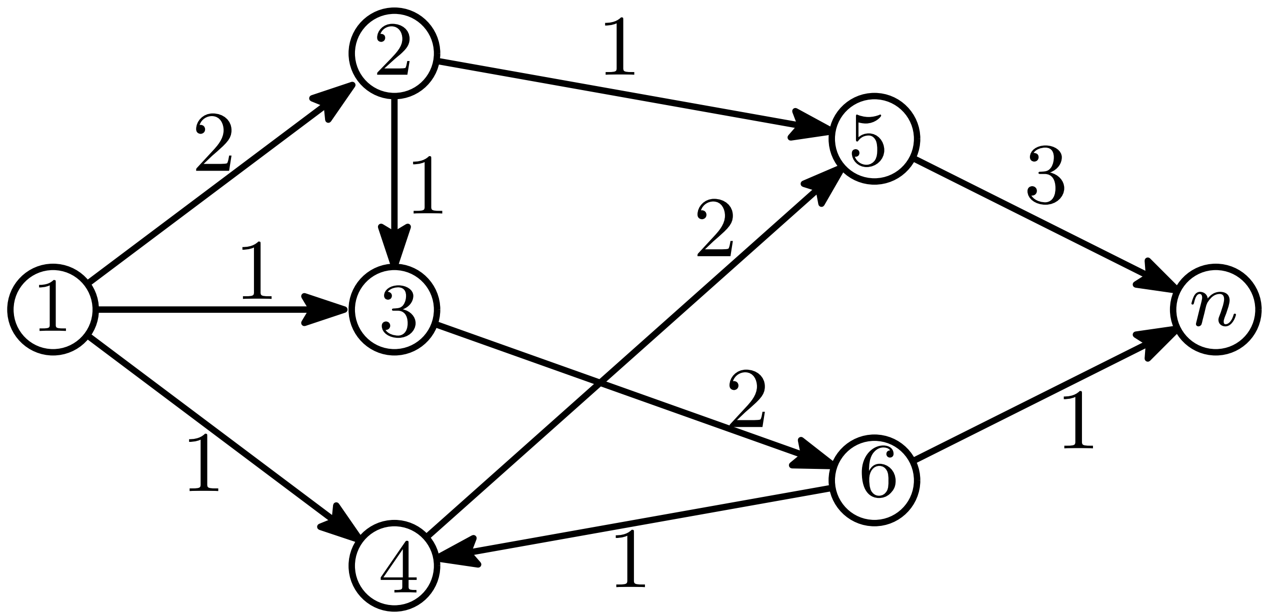
first we formulate the LP more concisely:
let the vertices be numbered $1, \dots, n$, let c_{ij} the capacity and x_{ij} the flow on directed edge (i, j), and let f be the max flow.

Optimal solution: 4



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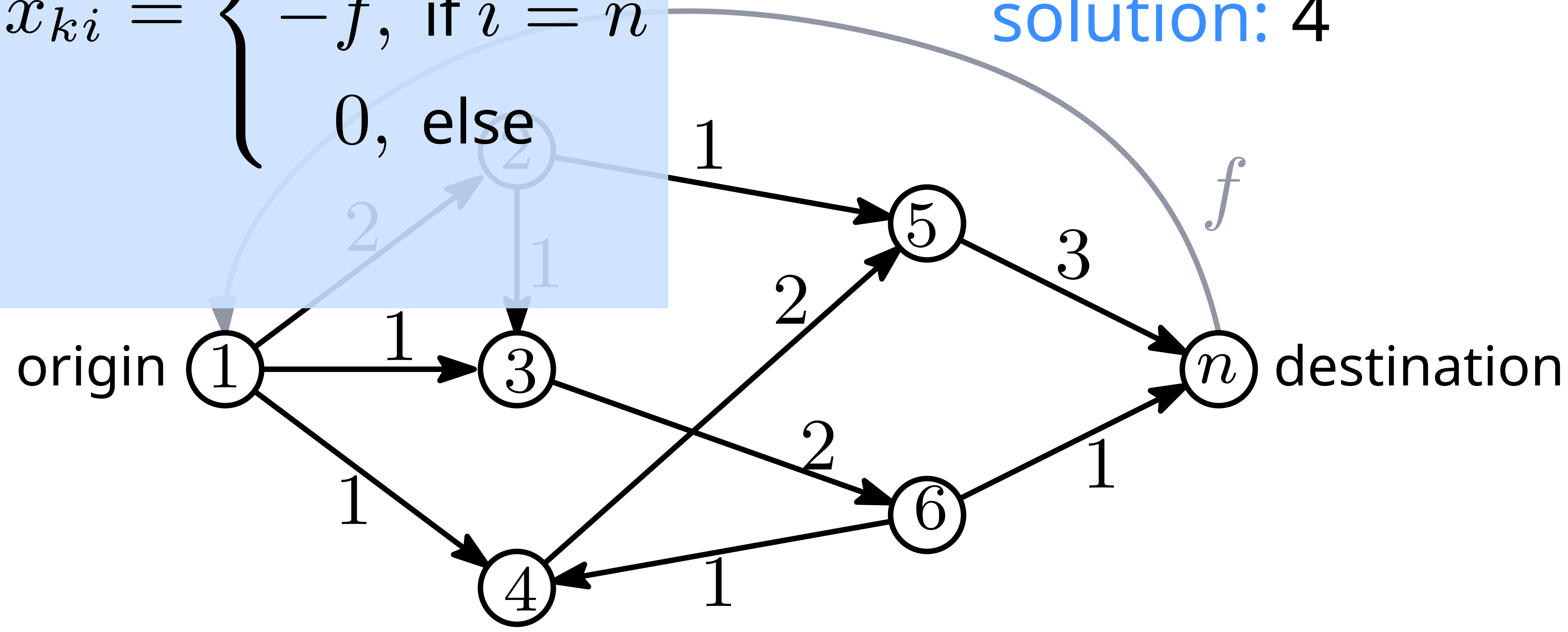
maximize f

subject to $$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} = \begin{cases} f, & \text{if } i = 1 \\ -f, & \text{if } i = n \\ 0, & \text{else} \end{cases}$$

$$0 \le x_{ij} \le c_{ij}$$



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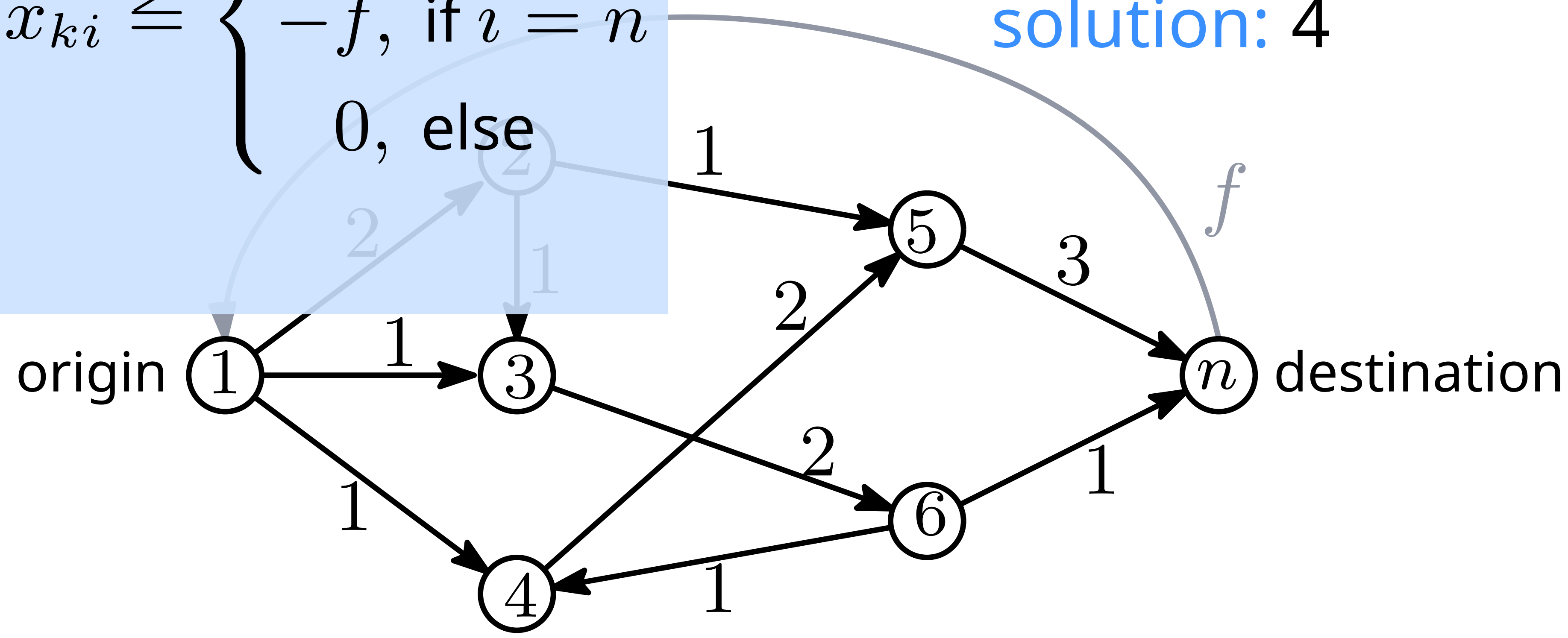

actually we can relax the constraint without changing the optimum

Recall: Flow in a Network

Linear Program Formulation

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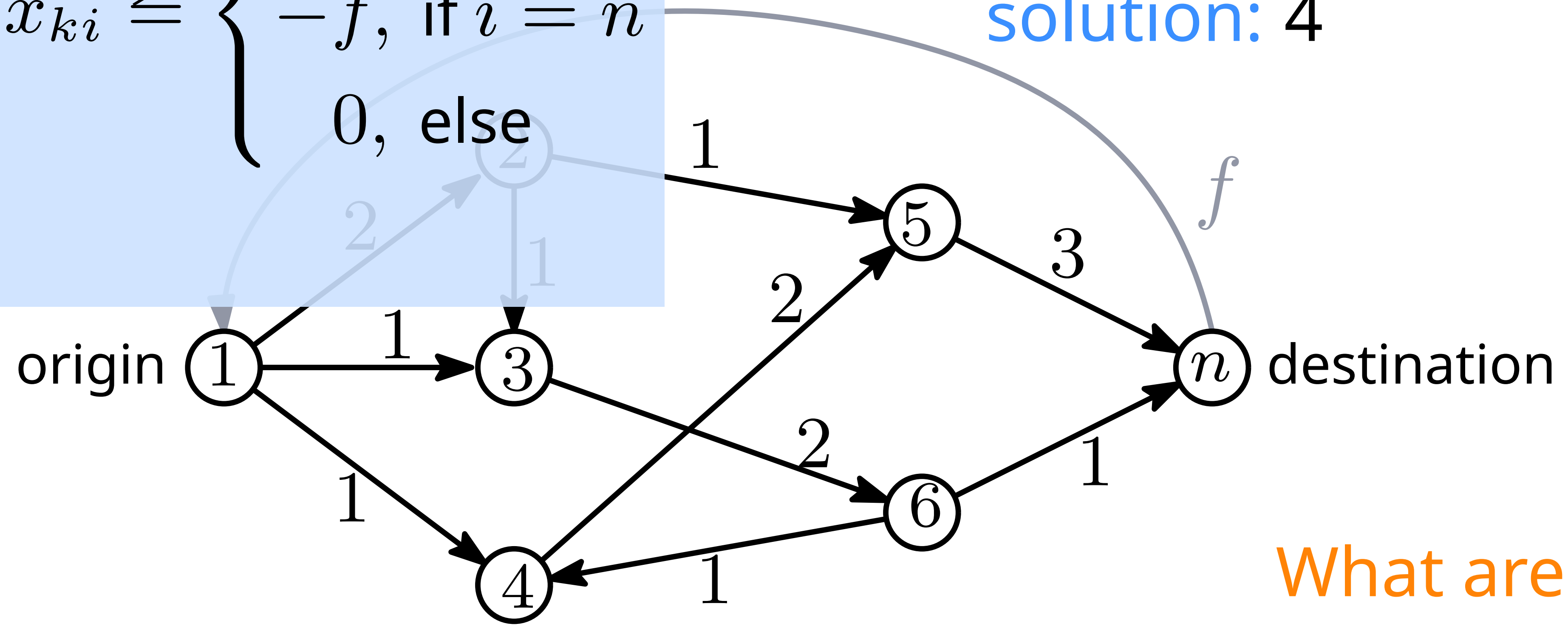

What are A, x, b, c?

Let's write this in Matrix form $\max f$ subject to $Ax \leq b, x \geq 0$.

Recall: Flow in a Network

Linear Program Formulation

in Matrix form $\max f$ subject to $Ax \le b, x \ge 0$ where

$$x = \begin{bmatrix} f \\ x_{ij} \\ .. \\ .. \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ .. \\ .. \end{bmatrix} \quad A = \left[\begin{array}{c|ccccc} -1 & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 1 & .. & .. & .. & .. \\ \hline 0 & 1 & 0 & .. & .. \\ .. & 0 & .. & .. & .. \\ .. & .. & .. & .. & 0 \\ 0 & .. & .. & 0 & 1 \end{array}\right] \quad b = \left[\begin{array}{c} 0 \\ .. \\ .. \\ 0 \\ \hline c_{ij} \\ .. \\ .. \\ .. \end{array}\right]$$

Recall: Flow in a Network

Linear Program Formulation

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every column contains exactly one -1 and one 1

Recall: Flow in a Network

Linear Program Formulation

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every column contains exactly one -1 and one 1

What is the dual?

Dual of the Max Flow LP

Linear Program Formulation

in Matrix form $\min \sum c_{ij} y_{ij}$ subject to $A^T y \geq c, x \geq 0$ where

every row contains exactly one -1 and one 1

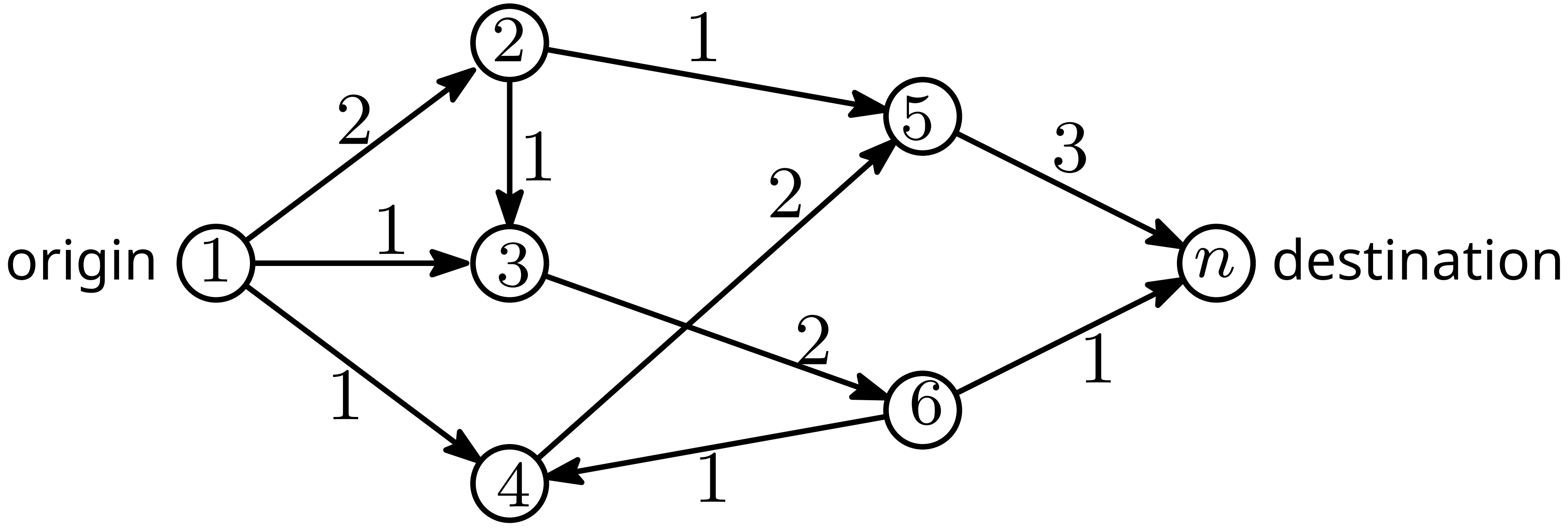
$$y = \begin{bmatrix} u_1 \\ .. \\ .. \\ u_n \\ y_{ij} \\ .. \\ .. \\ .. \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ .. \\ .. \\ 0 \\ c_{ij} \\ .. \\ .. \\ .. \end{bmatrix} \quad A^T = \left[\begin{array}{cccc|cccc} -1 & 0 & .. & 1 & 0 & .. & .. & 0 \\ \hline .. & .. & .. & .. & 1 & 0 & .. & .. \\ .. & .. & .. & .. & 0 & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & 0 & 1 \end{array}\right] \quad c = \begin{bmatrix} 1 \\ 0 \\ .. \\ .. \end{bmatrix}$$

Dual of the Max Flow LP

Linear Program Formulation

in constraint form

$$\begin{aligned} \text{minimize} \quad & \sum c_{ij} y_{ij} \\ \text{subject to} \quad & -u_1 + u_n \geq 1 \\ & u_i - u_j + y_{ij} \geq 0 \\ & u_i \geq 0 \\ & y_{ij} \geq 0 \end{aligned}$$



Dual of the Max Flow LP

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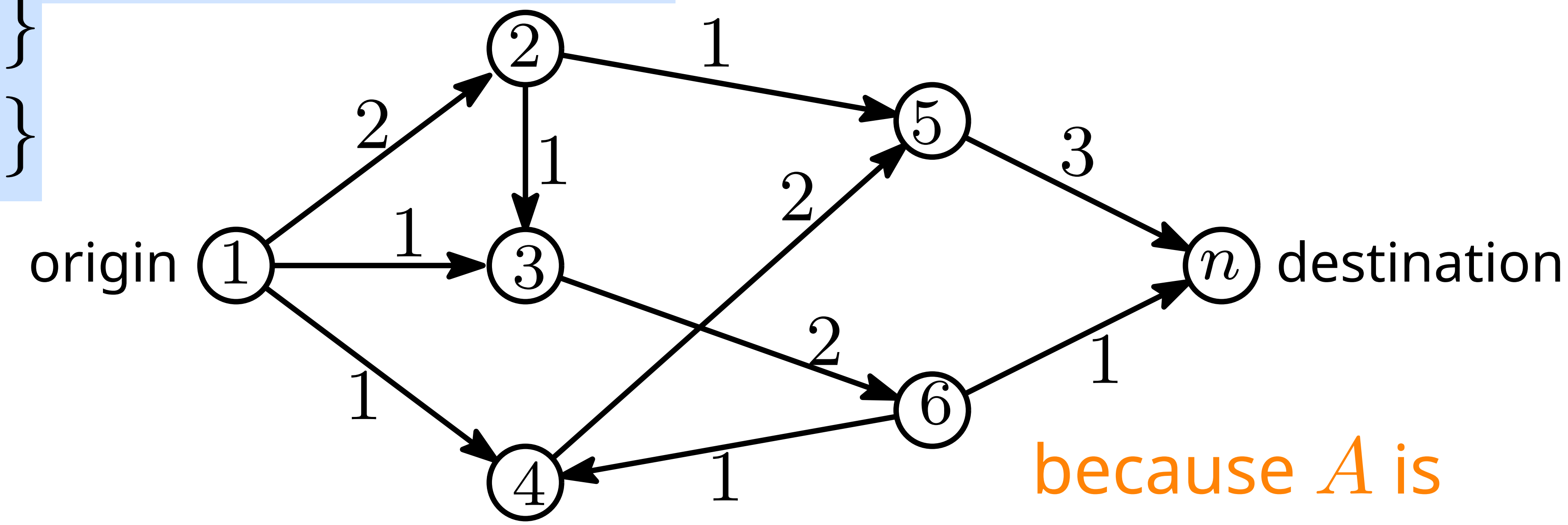

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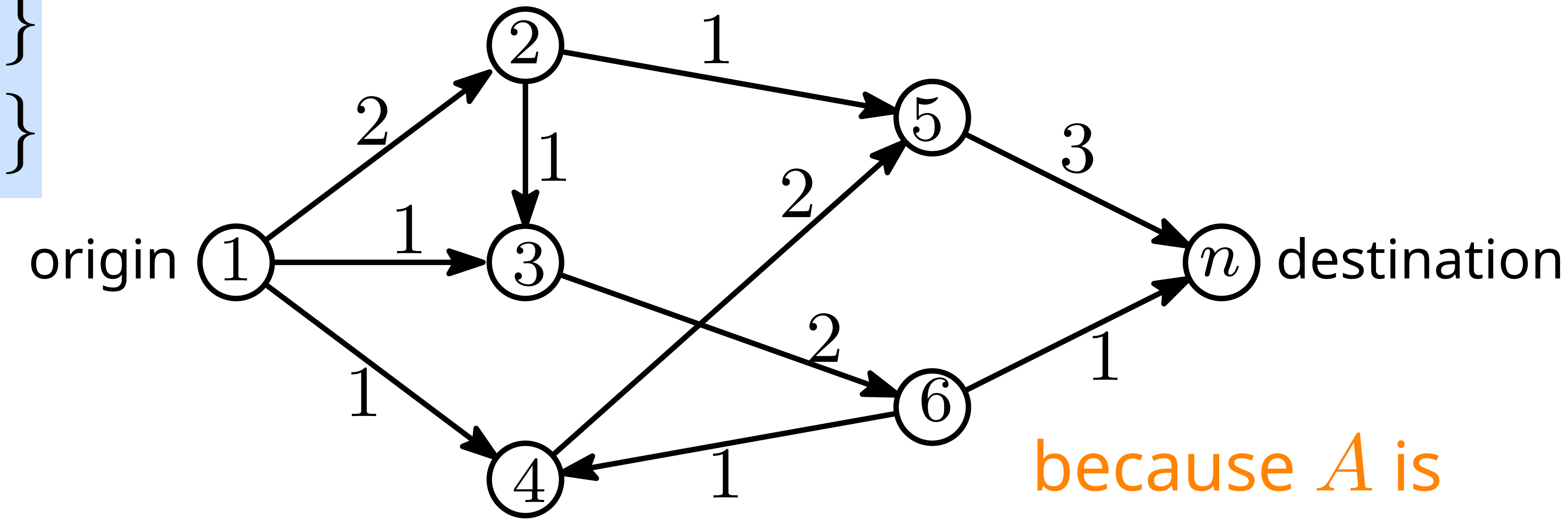
$\rightarrow$ Min Cut



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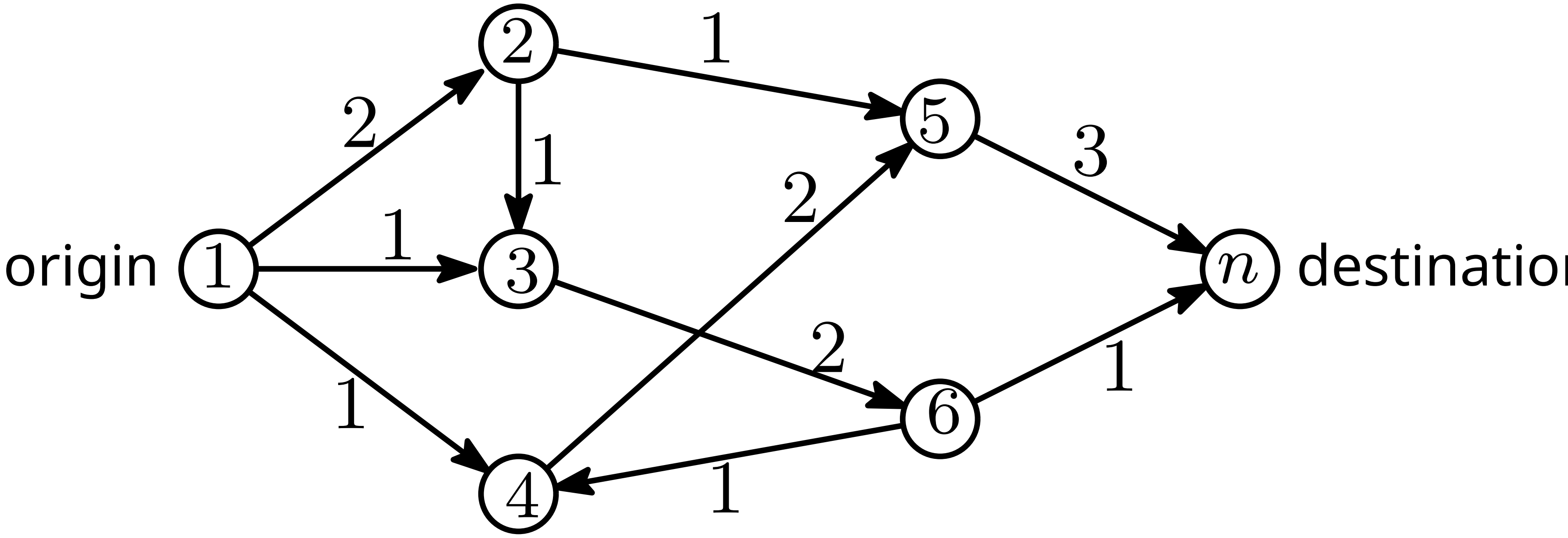
Flows and Cuts in a Network

Alternative LP

Let P be the set of all paths $q \rightsquigarrow s$
use variables x_p, for all $p \in P$



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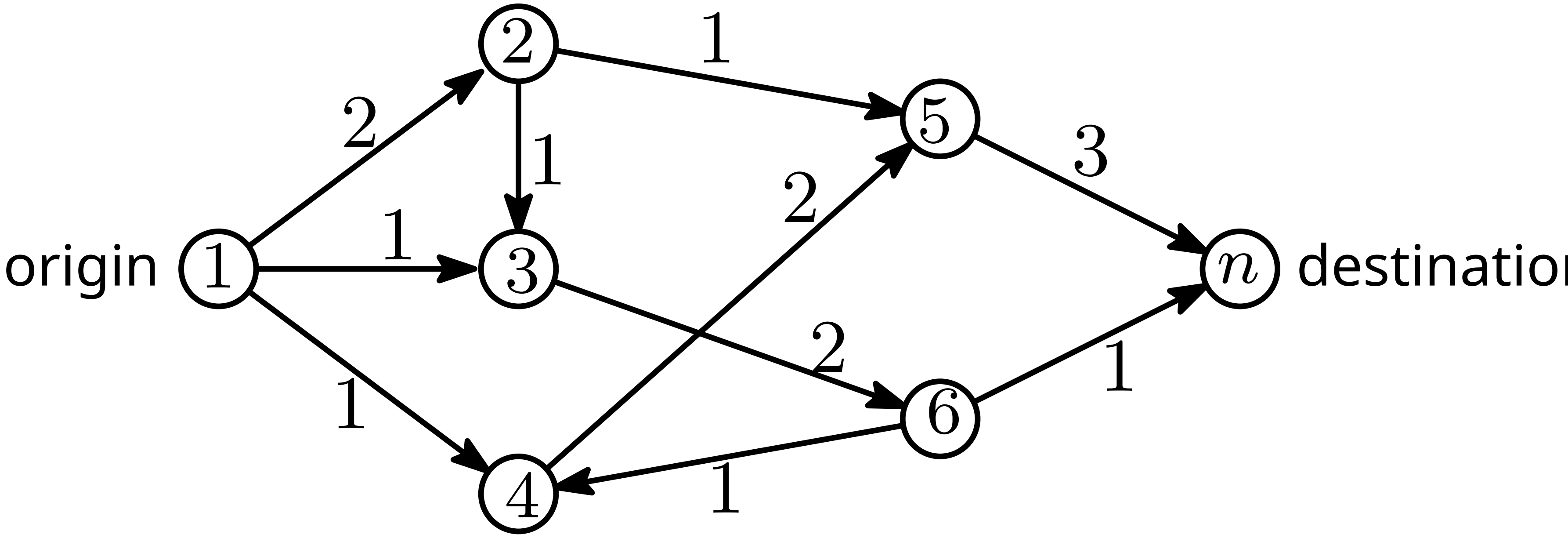

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What is the dual?

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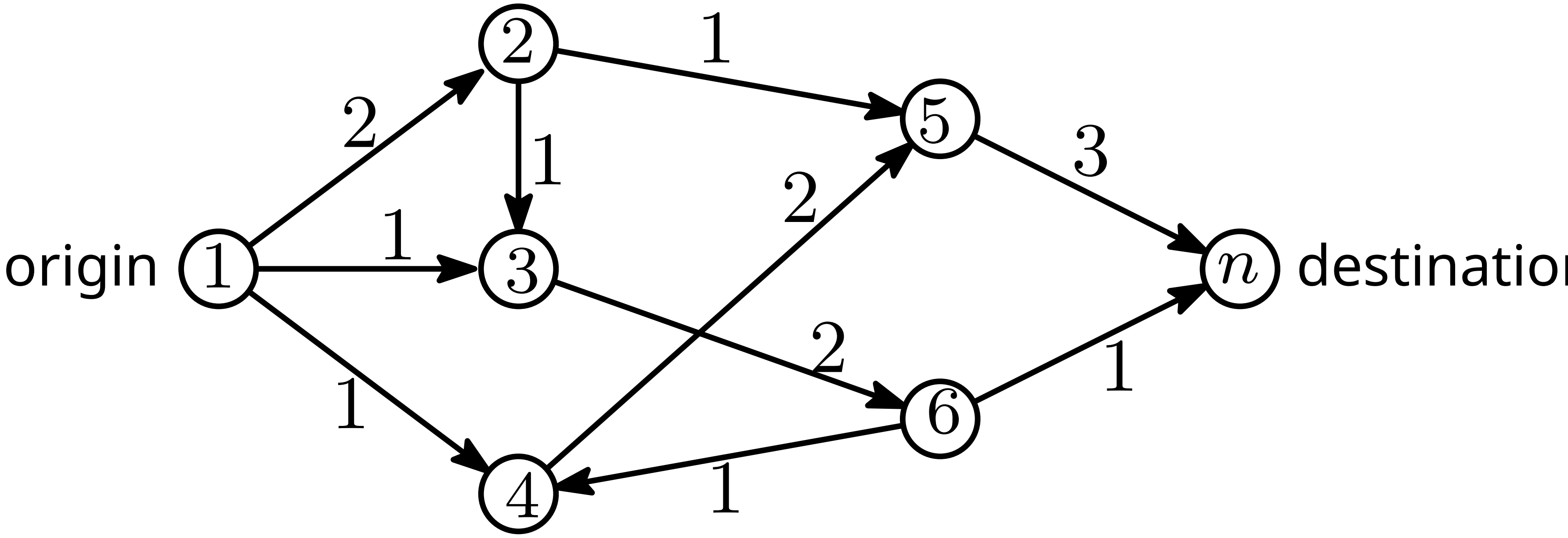

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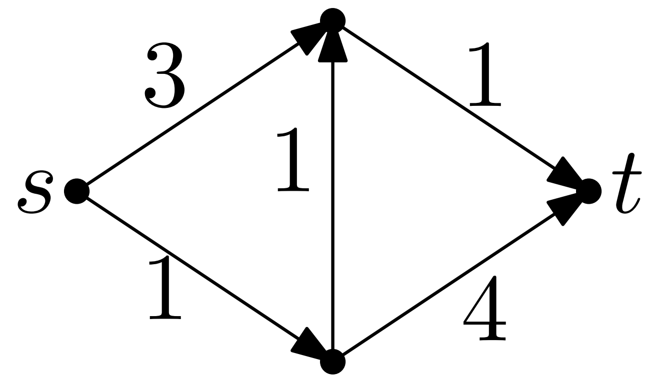
dual

Min Cut

Shortest Path Problem

Given a directed graph $G = (V, E)$ with edge weights w, we are looking for a shortest path from s to t.



Can we also model this as ILP? Yes! How?

Idea 1: use variable x_{uv} for whether edge (u, v) is used.

minimize $\displaystyle\sum_{(u,v)\in E} w(u,v)x_{uv}$

subject to $\displaystyle\sum_{(u,v)\in E} x_{uv} = \sum_{(v,w)\in E} x_{vw}$ for each vertex $v \in V \setminus \{s, t\}$, and

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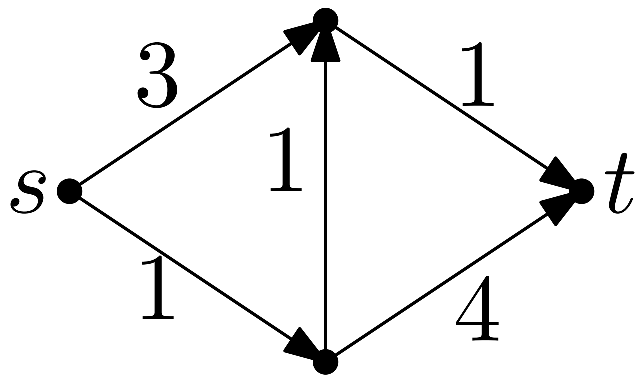

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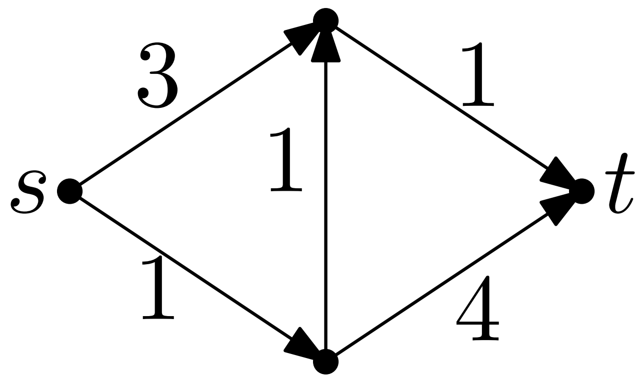

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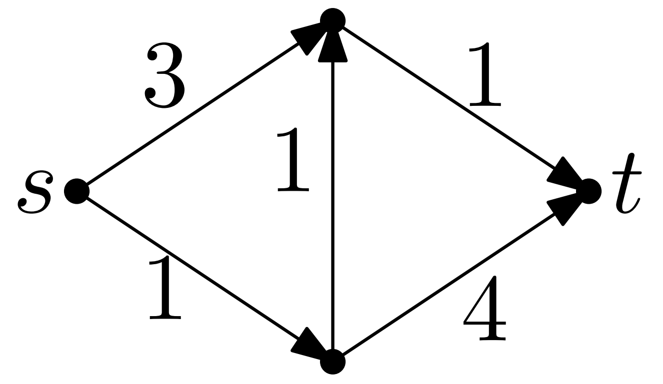

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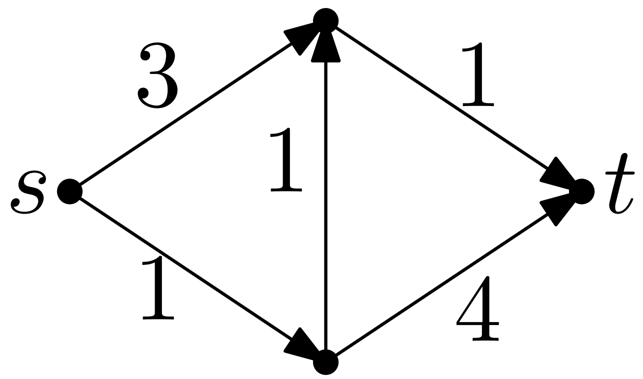

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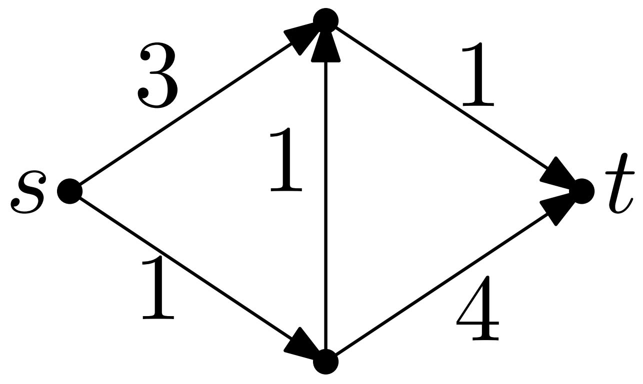

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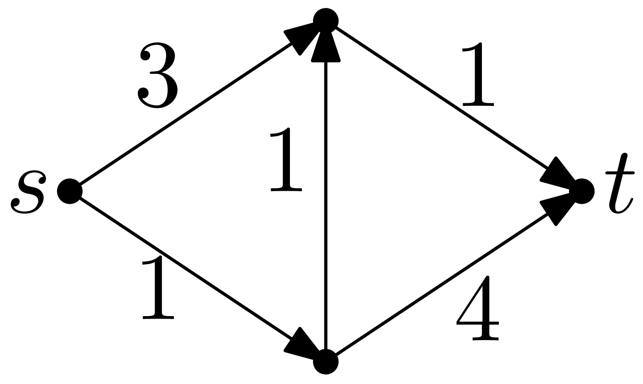

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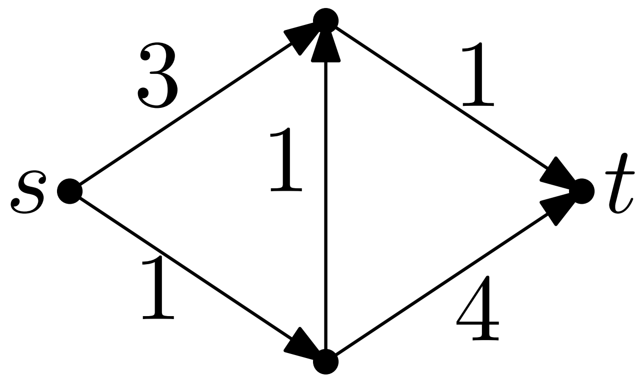

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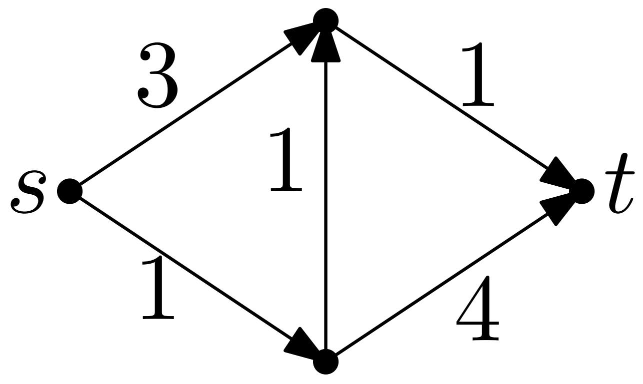

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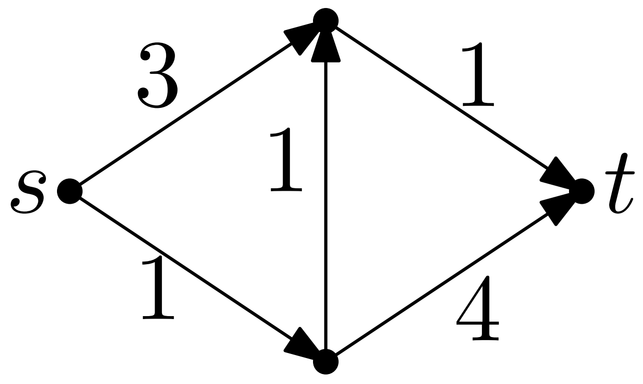

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Is there a connection between the two LPs? They are dual to each other!