Fundamental Algorithms & Data Structures

basic concepts
Dijkstra’s algorithm and priority queues
dynamic programming for the Traveling Salesperson Problem

Plan for this week

revisit preliminaries

today
- fundamental concepts for algorithms and data structures
- Dijkstras algorithm & priority queues for Shortest Paths Computation
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next lecture

- amortized analysis

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If you need to refresh any of the concepts discussed today,
this week is a good time for it!

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Which of these do you know?

Which lectures on algorithms have you taken?

Asymptotic Analysis

O-Notation: $O(f(n)) = \{g(n) \mid \exists c > 0 \; \exists n_0 \geq 1 \; \forall n \geq n_0 : g(n) \leq c \cdot f(n)\}$

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x $\leftarrow$ dummy-alg$(a, i + 0, i + \lceil n/2 \rceil)$
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Master-Theorem:

$$T(n) = \Theta(n^{\log_2(3)})$$

Data structures

Abstract Data Types: Priority Queues, Sorted Sequences, Dictionaries

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Questions:

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- What is the difference between chaining and open addressing?

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Quiz: what paradigm is insertion sort ?

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Quiz: what paradigm is selection sort ?

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Quiz: what paradigm is merge sort ?

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  - $\rightarrow$ selection sort

Quiz: what paradigm is quick sort ?

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  $\rightarrow$ merge sort , quick sort
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Quiz: what paradigm is Dijkstras algorithm ?

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  $\rightarrow$ Dijkstra's algorithm (also Bellman-Ford and Floyd-Warshall)
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Quiz: what paradigm is Jarnik-Prim and Kruskal's algorithm ?

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what is an optimal cutting here?

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how to compute this efficiently?

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how to do this faster?

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how to do this faster? $\rightarrow$ memoization
$\rightarrow$ bottom-up

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Algorithm BOTTOM-UP-CUT-ROD (p, n)

```
r[0, n] new array
r[0] = 0
for j = 1 to n do
    q = −∞
    for i = 1 to j do
        q = max{q, p_i + r(j − i)}
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Dynamic Programs apply to problems that have overlapping optimal substructures

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graph representations: adjacency matrix; adjacency list

graph algorithms: traversal (BFS, DFS), shortest paths, minimum spanning tree

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- How fast can one compute a minimum spanning tree?

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- Which algorithm is used for topological sorting? DFS
- How many edges does a tree on n vertices have? $n-1$
- How fast can one compute a minimum spanning tree?
  - Kruskal: $O(|E|\log|V|)$
  - Jarnik-Prim: $O(|E|+|V|\log|V|)$ with Fibonacci heaps
  - faster algorithms exist

NP-hardness

complexity class P: all problems that can be solved in polynomial time by a deterministic turing machine

complexity class NP: all problems that can be solved in polynomial time by a non-deterministic turing machine

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complexity class NP: all problems that can be solved in polynomial time by a non-deterministic turing machine

famous open problem: $P \subsetneq NP$?

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- Which NP-complete problems do you know?

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Questions:

- What is an NP-hard problem? What is an NP-complete problem?
- Which NP-complete problems do you know?
- How do you prove NP-hardness of a problem?
- How to solve NP-hard problems?

Two well-known problems

shortest path

What is the shortest path from Hamburg to Munich?

shortest round trip

What is the shortest round trip through these major German cities?

Shortest Path Problem

Common Algorithms

- Dijkstras Algorithm
- Bellmann-Ford
- Floyd-Warshall

Wikipedia

Shortest Path Problem

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What are advantages and disadvantages of the algorithms?

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What else?

- many more approaches in practice
- shortest path queries make heavy use of data structures

Wikipedia

Dijkstras Algorithm

Algorithm

unmark all nodes;

init arrays d and $parent$;

while there is unmarked node u with $d[u] < \infty$ **do**

  $u \leftarrow$ such a node with minimal distance $d[u]$

  for all outgoing edges (u, v)

    check and possibly update distance to v

  mark u as *finished*

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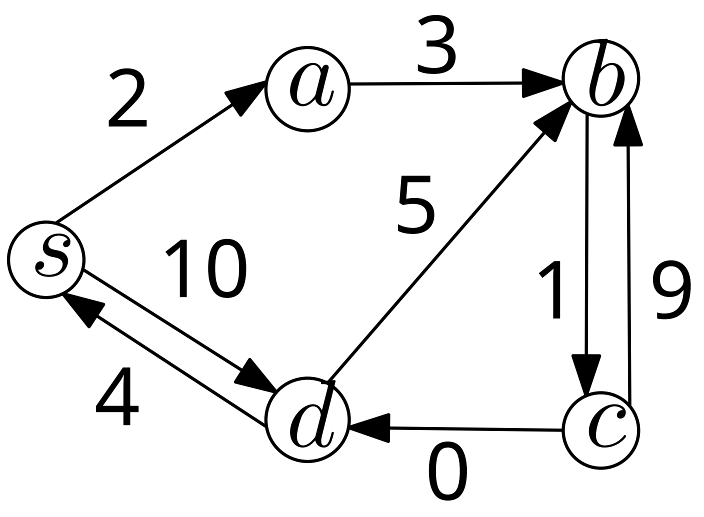
store these nodes in priority queue Q



Q

	d	parent
s		
a		
b		
c		
d		

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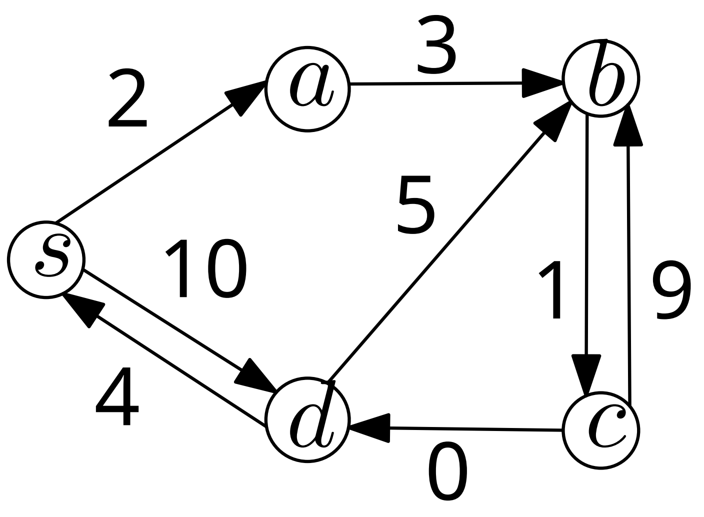

Q
$(s, 0)$

init

	d	parent
s	0	s
a	∞	$\perp$
b	∞	$\perp$
c	∞	$\perp$
d	∞	$\perp$

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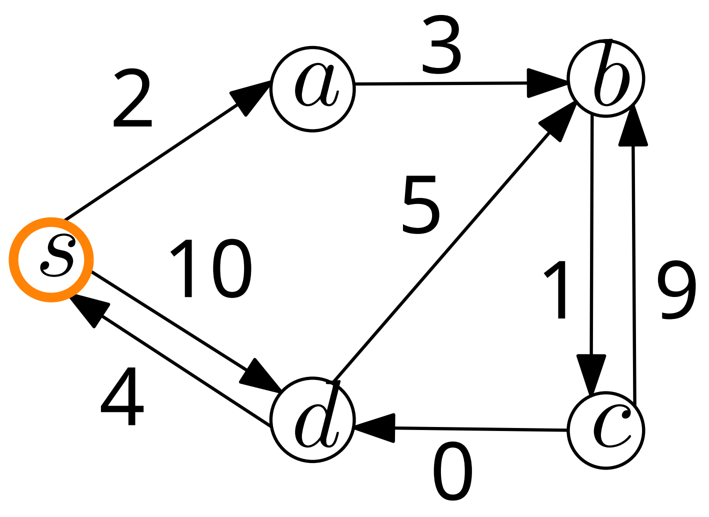

Q
$(a, 2)$
$(d, 10)$

$u \leftarrow s$

	d	parent
s	0	s
a	2	s
b	∞	$\perp$
c	∞	$\perp$
d	10	s

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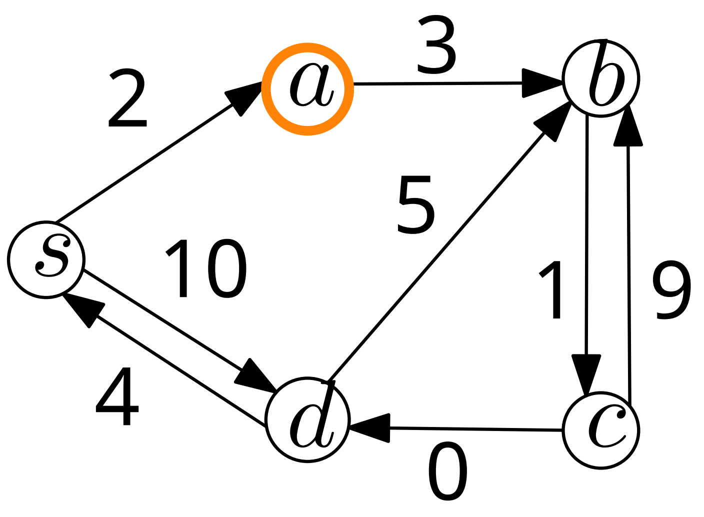

Q
$(b, 5)$
$(d, 10)$

$u \leftarrow a$

	d	parent
s	0	s
a	2	s
b	5	a
c	∞	$\perp$
d	10	s

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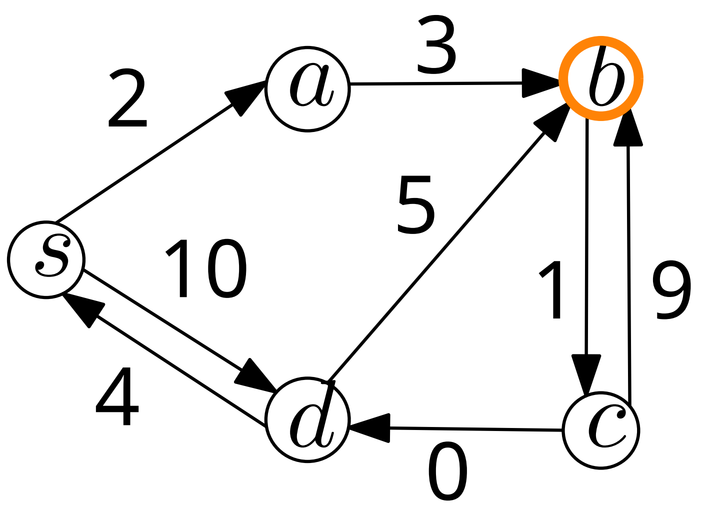

Q
$(c, 6)$
$(d, 10)$

$u \leftarrow b$

	d	parent
s	0	s
a	2	s
b	5	a
c	6	b
d	10	s

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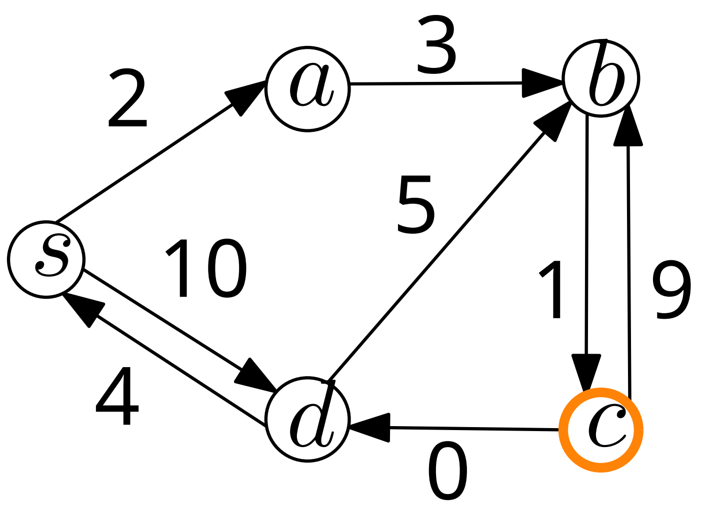

Q
$(d, 6)$

$u \leftarrow c$

	d	parent
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a	2	s
b	5	a
c	6	b
d	6	c

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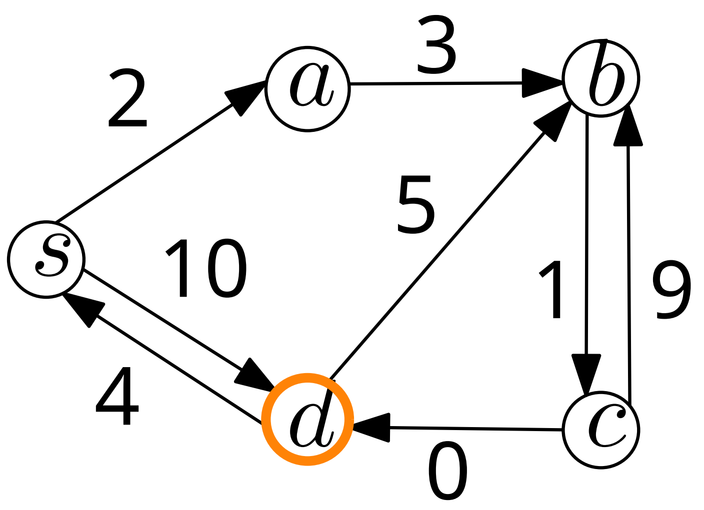

Q

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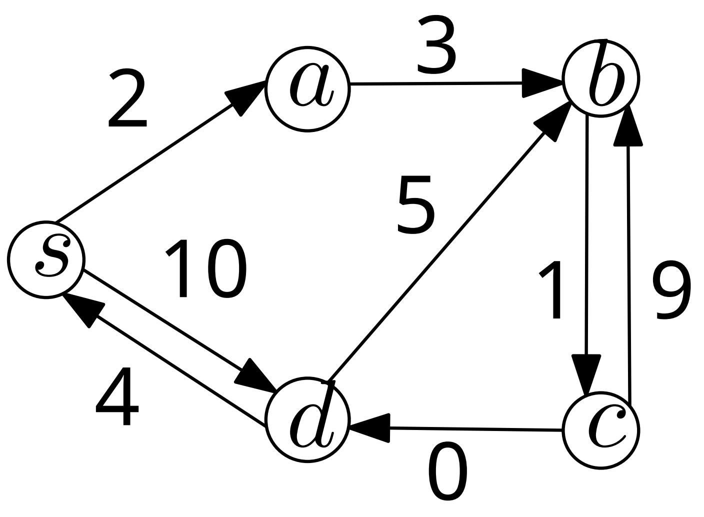

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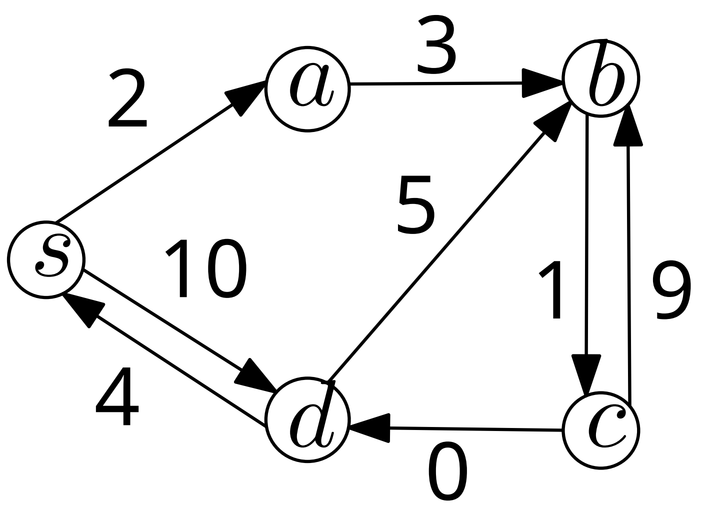

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What is the runtime?

$$O(m \cdot \underbrace{T_{decreaseKey}(n)}_{\text{each edge may decrease the distance of a node in } Q} + n \cdot (\underbrace{T_{deleteMin}(n) + T_{insert}(n)}_{\text{each node is inserted and deleted once in } Q}))$$

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depends on implementation of priority queue Q

Priority Queues

Abstract data typ: manage a set of elements with keys (their priority) under the operations *insert, minimum, deleteMinimum*, and optionally *decreaseKey, remove* and *merge*.

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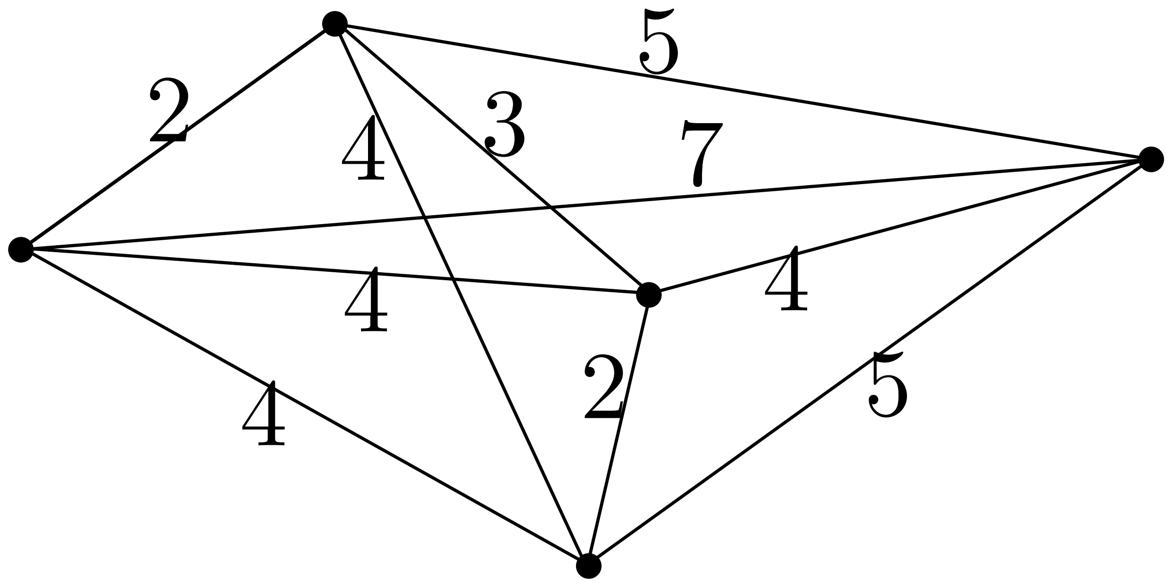
Runtime for Dijkstras Algorithm: $O(m + n \log n)$ using Fibonacci heaps

Traveling Salesperson Problem

Traveling Salesperson Problem

Given an undirected complete Graph $G = (V, E)$ with edge weights $c\colon E \to \mathbb{R}$, find a shortest Hamilton circuit in G.



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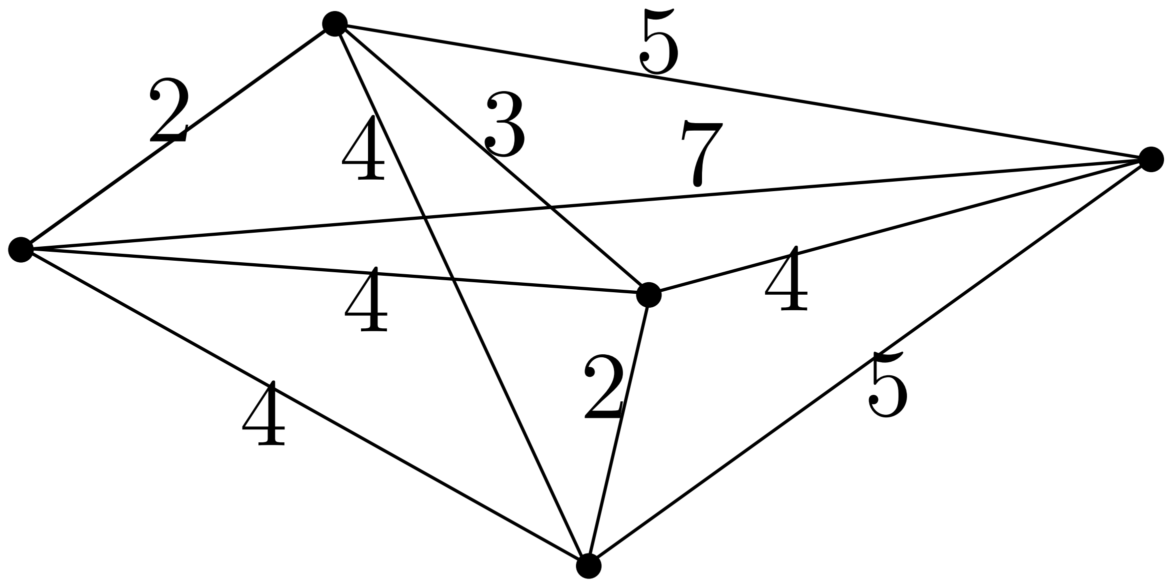

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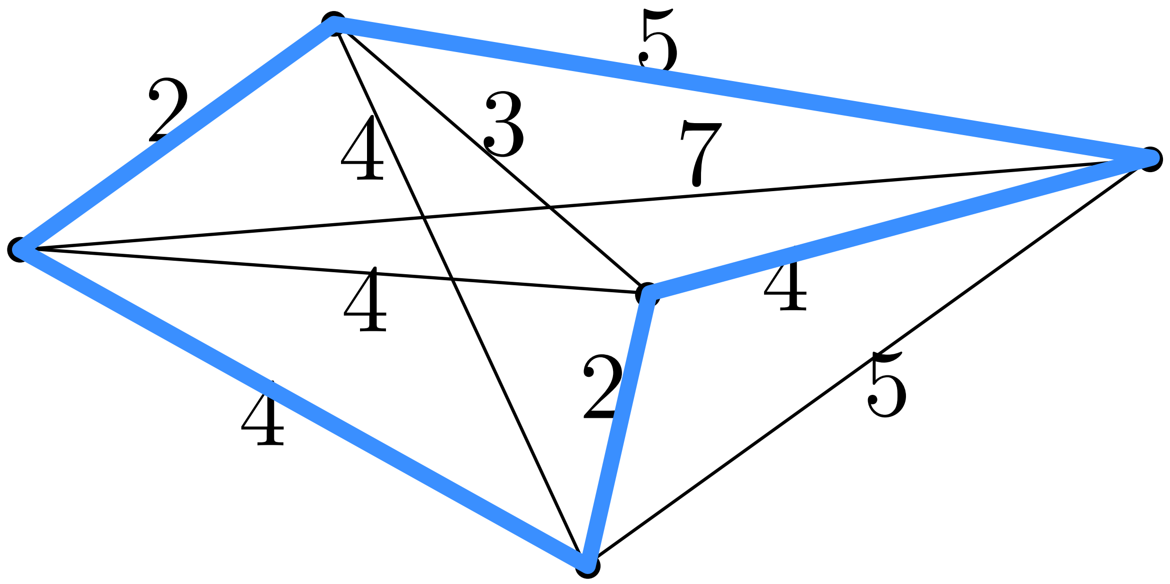

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- for approximating general version
- for deciding metric version

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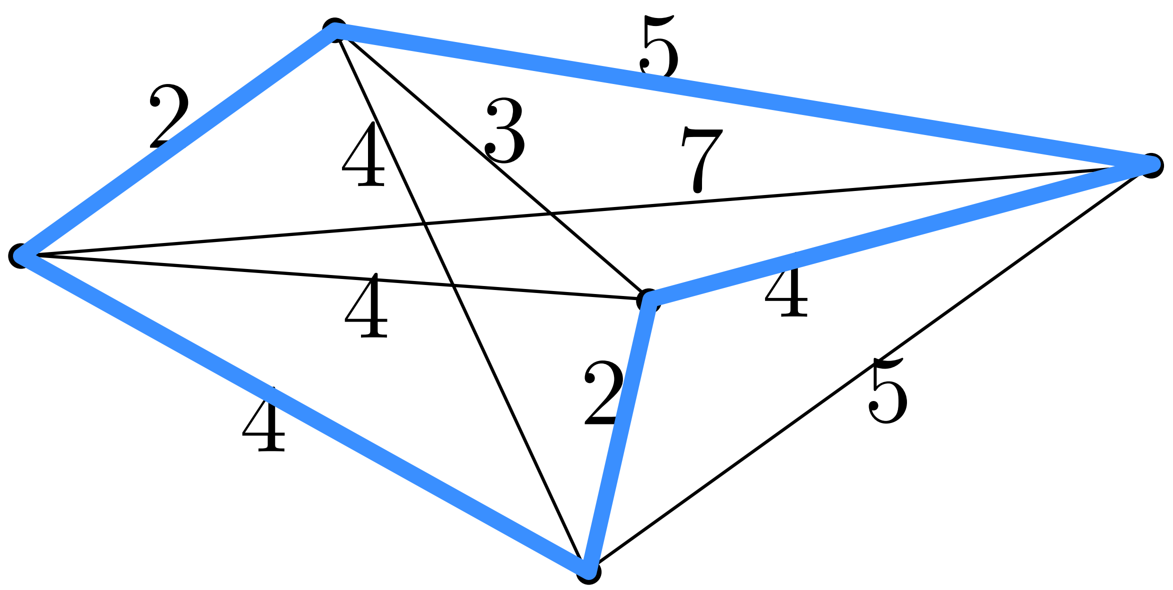

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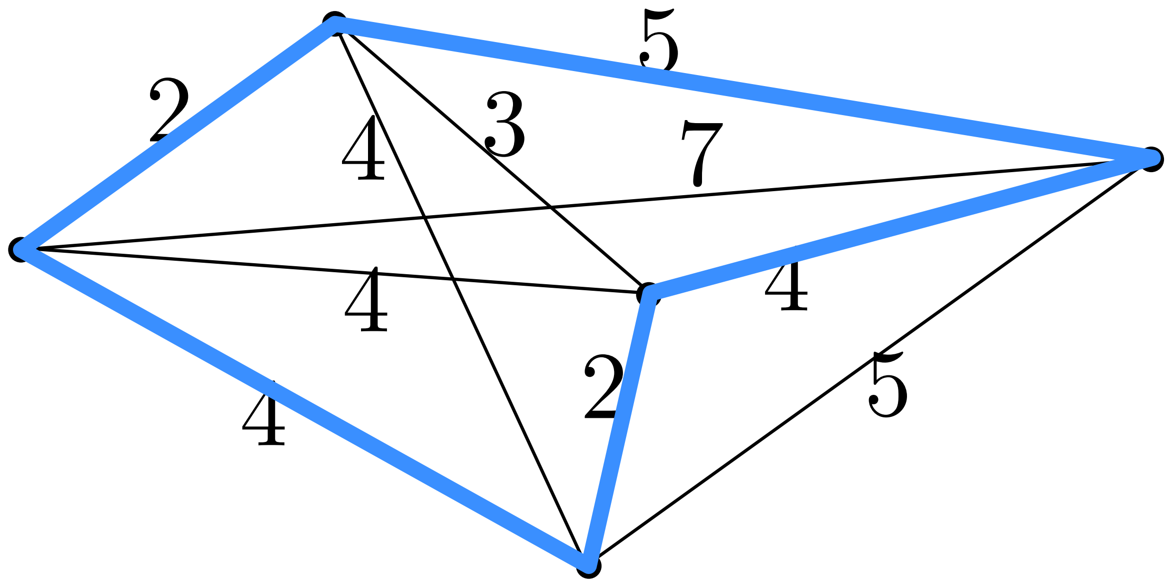

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Exact Algorithms

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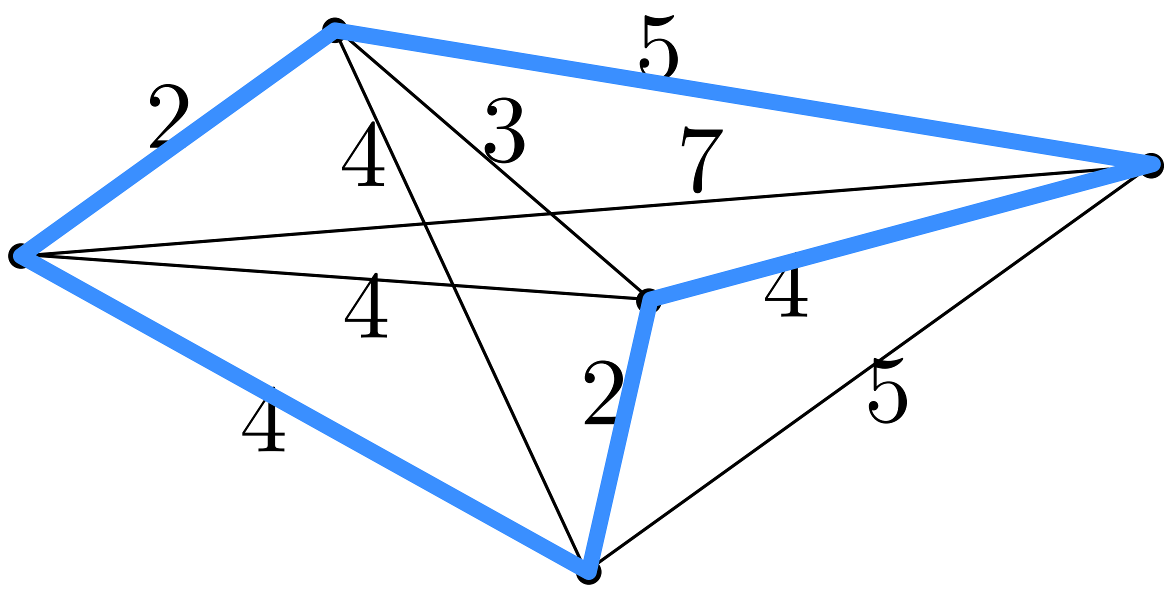

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- Held-Karp algorithm (dynamic program)
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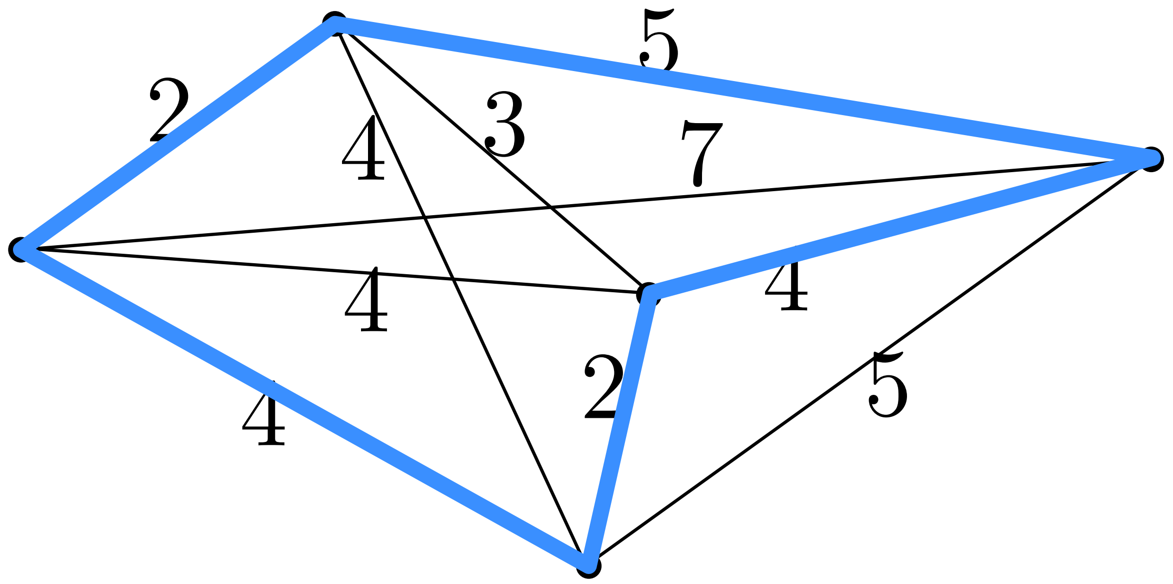

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- 3/2-Approx for Metric TSP (Christofides)
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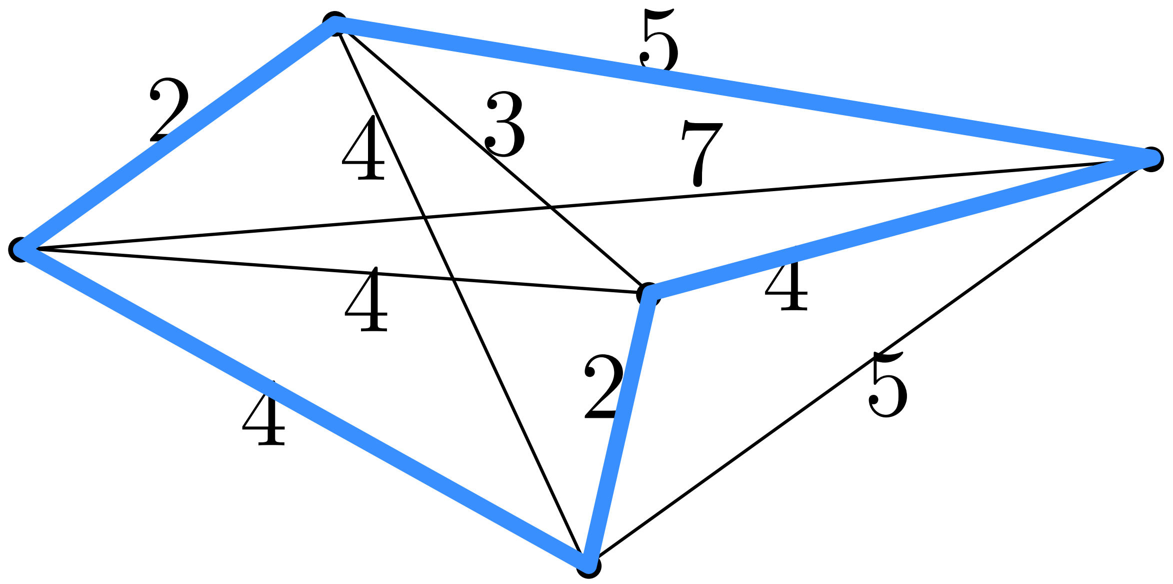

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Heuristics

- Nearest Neighbor
- 2-OPT and many more

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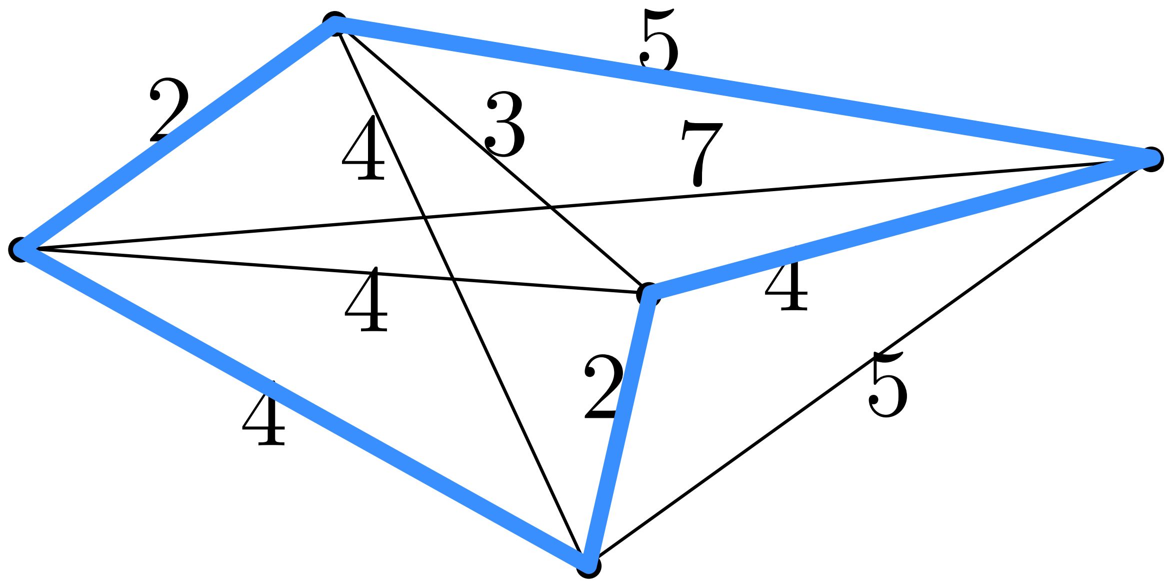
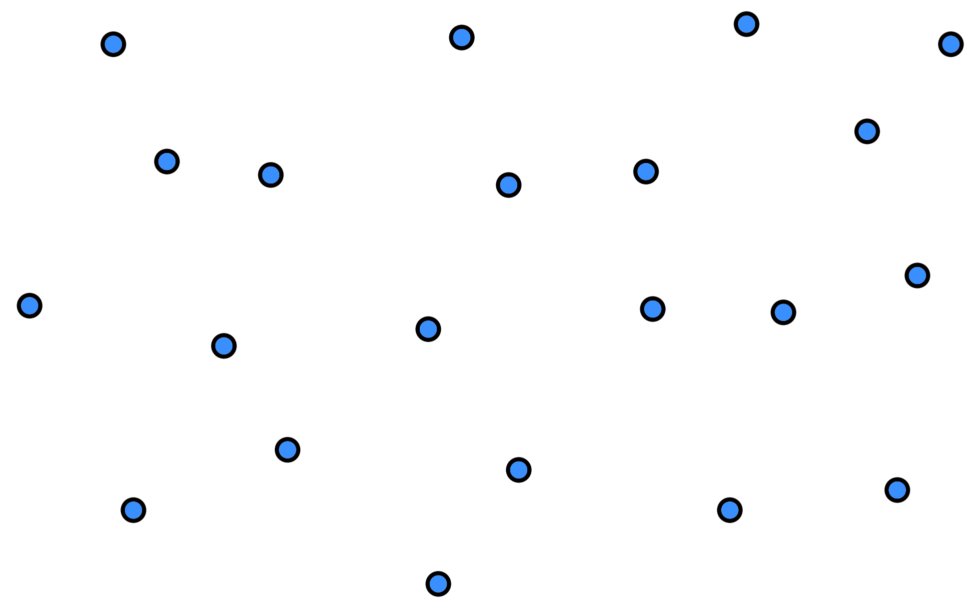


Heuristics for TSP

Example: Nearest-Neighbor Heuristic

- start at an arbitrary city
- always go the nearest unvisited city
- at the end: go back to starting city



Heuristics for TSP

Example: Nearest-Neighbor Heuristic

Algorithm Nearest-Neighbor Heuristic$(G = (V, E), c)$

Initialize tour as empty

Select arbitrary $s \in V$ (as starting vertex)

Set $v = s$ and $U = V - s$

while $U \neq \emptyset$ **do**

  Select $u \in U$ with $c(v, u) = \min_{w \in U} c(v, w)$

  Add edge {v, u} to tour

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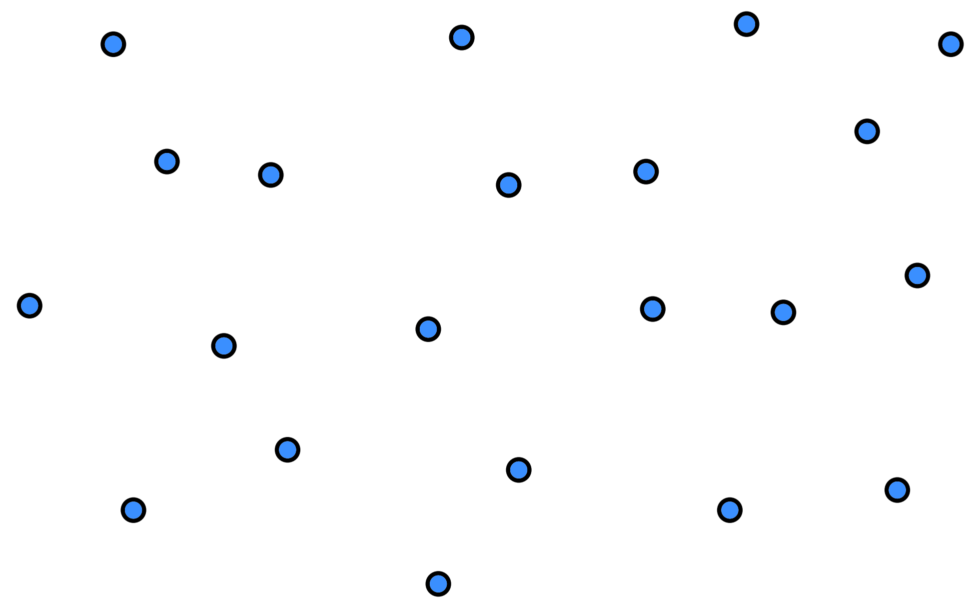
Add edge $\{v, s\}$ to tour

return tour



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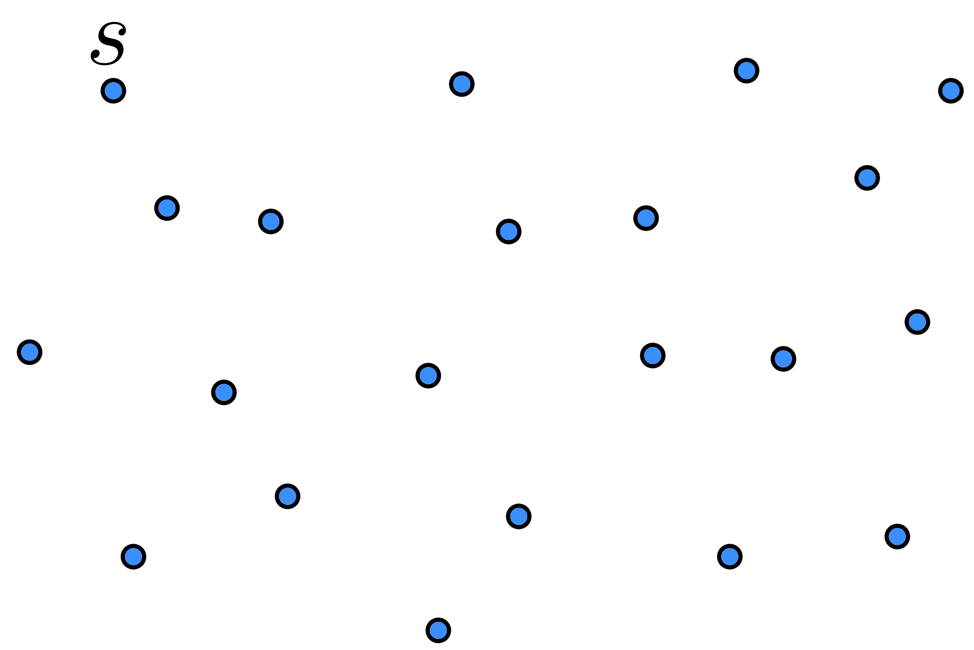

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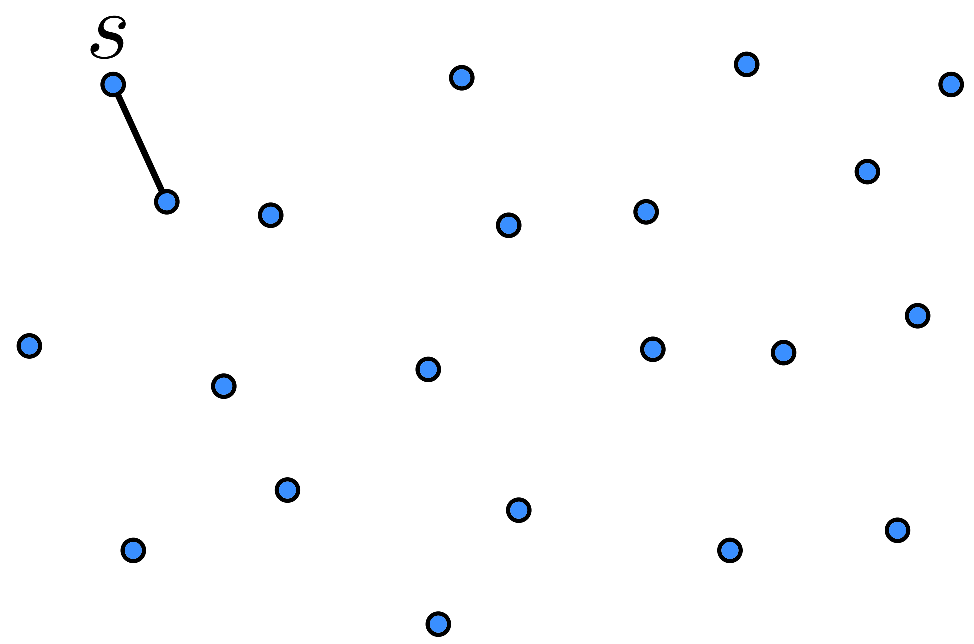

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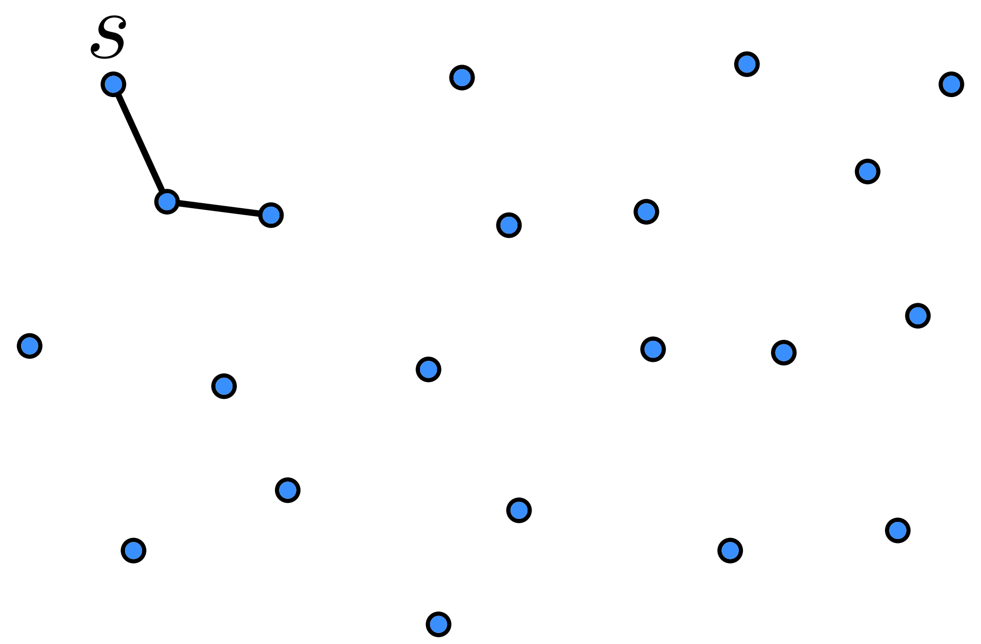

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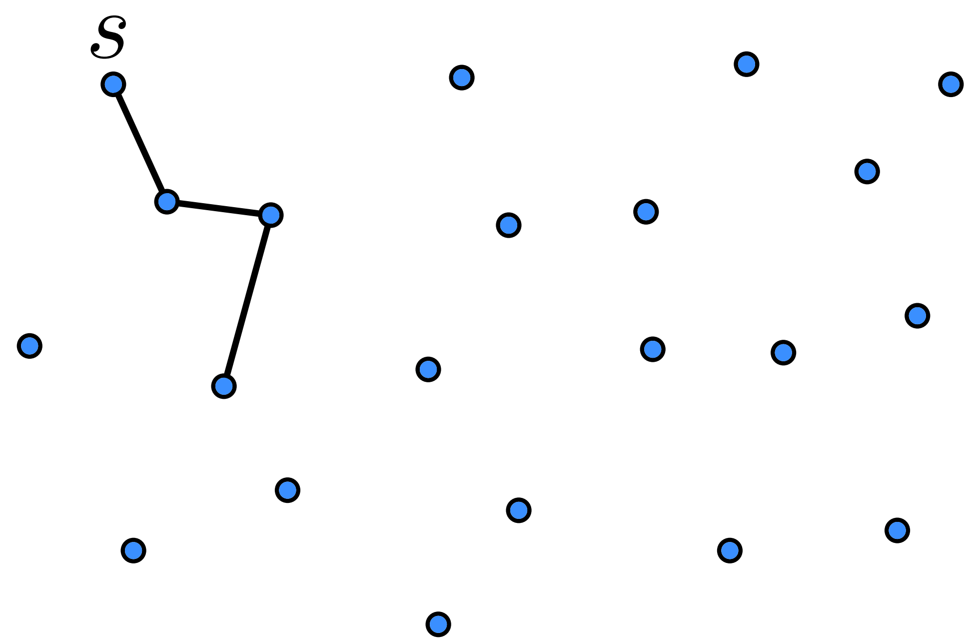

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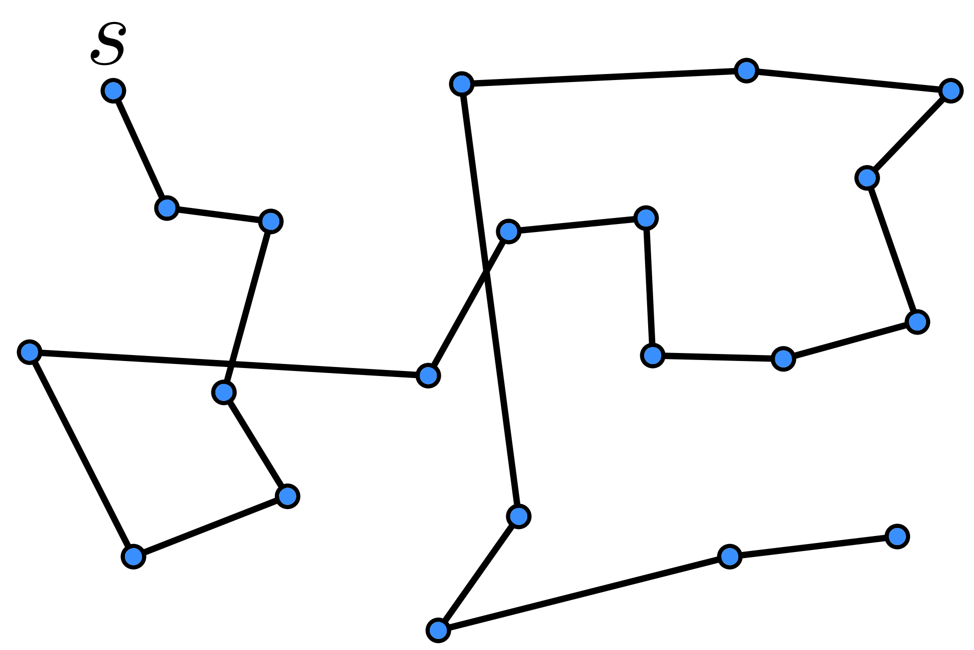

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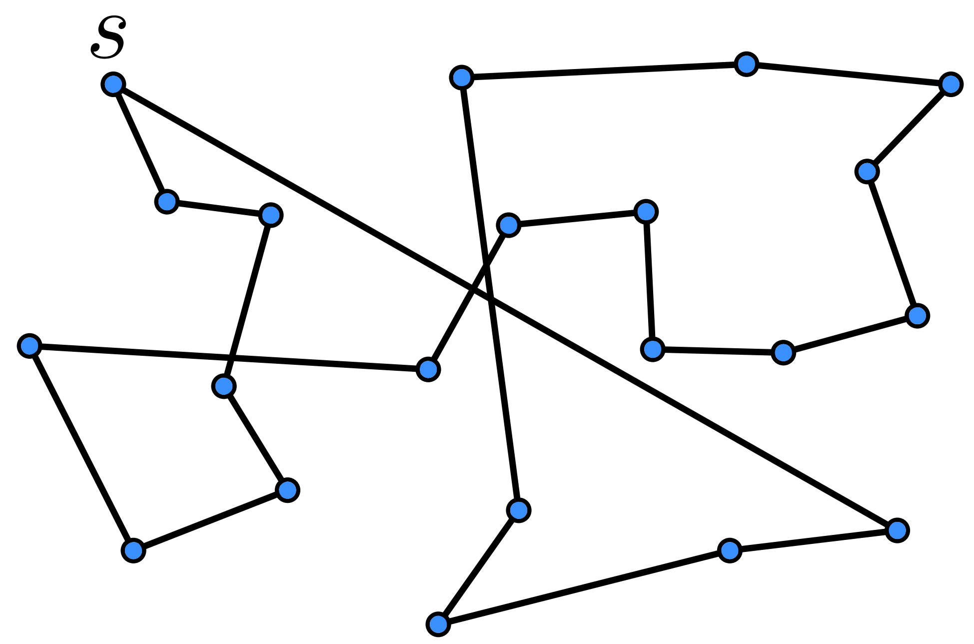

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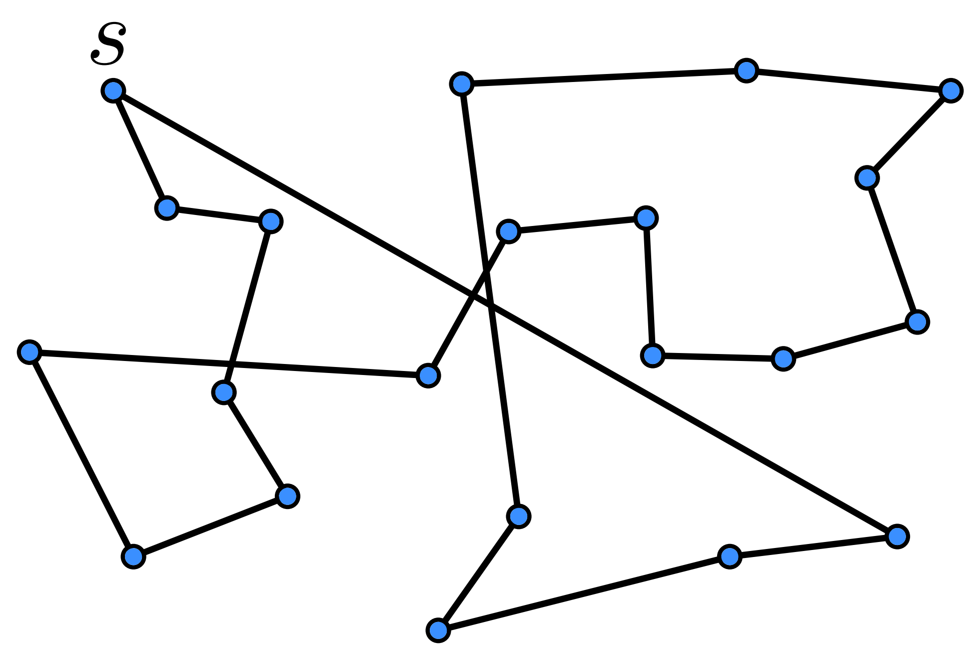
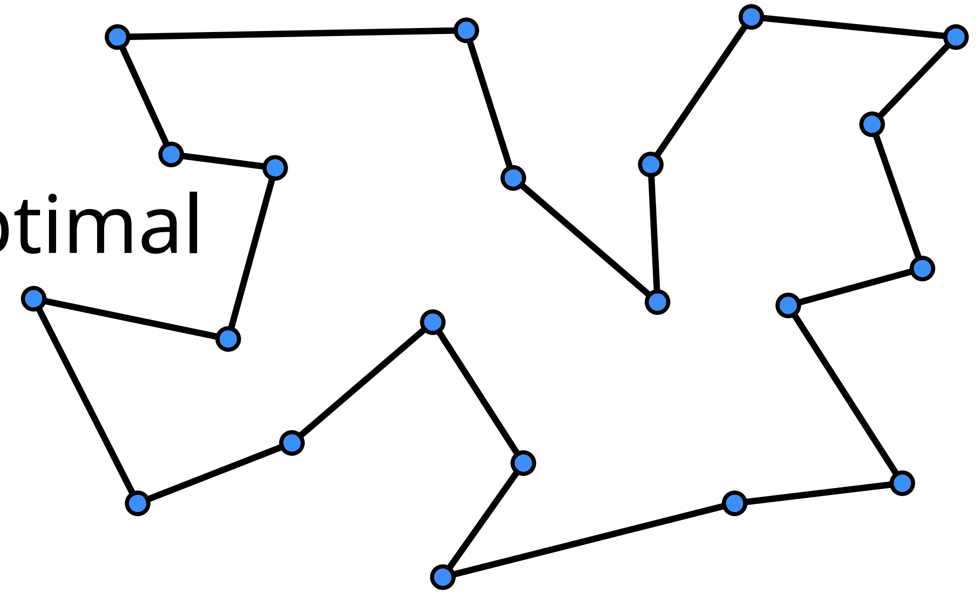
  Select $u \in U$ with $c(v, u) = \min_{w \in U} c(v, w)$
  Add edge {v, u} to tour
  Set $v = u$ and $U = U - u$
Add edge $\{v, s\}$ to tour
return tour



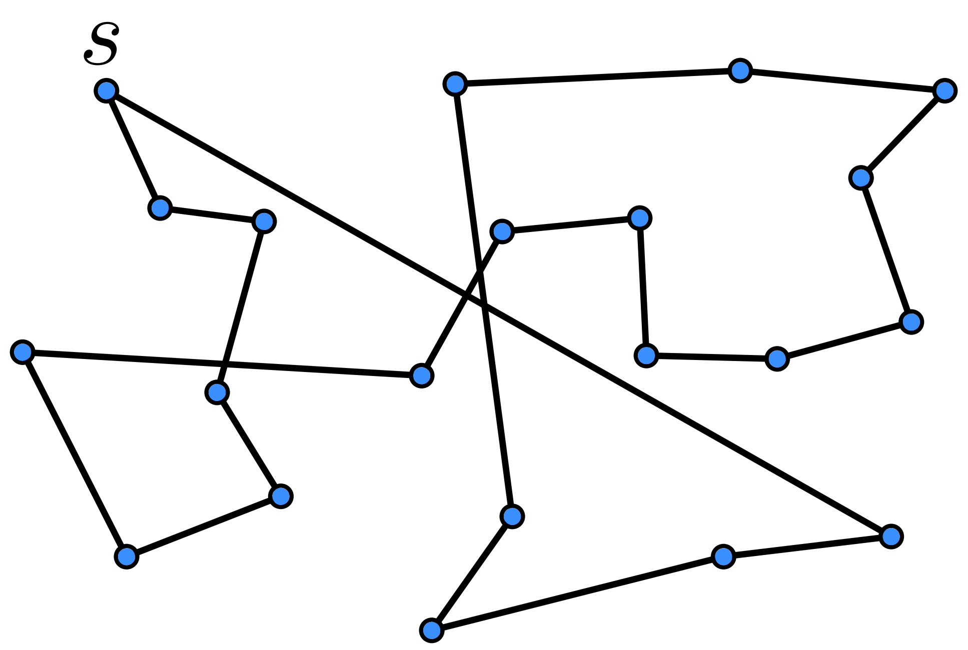


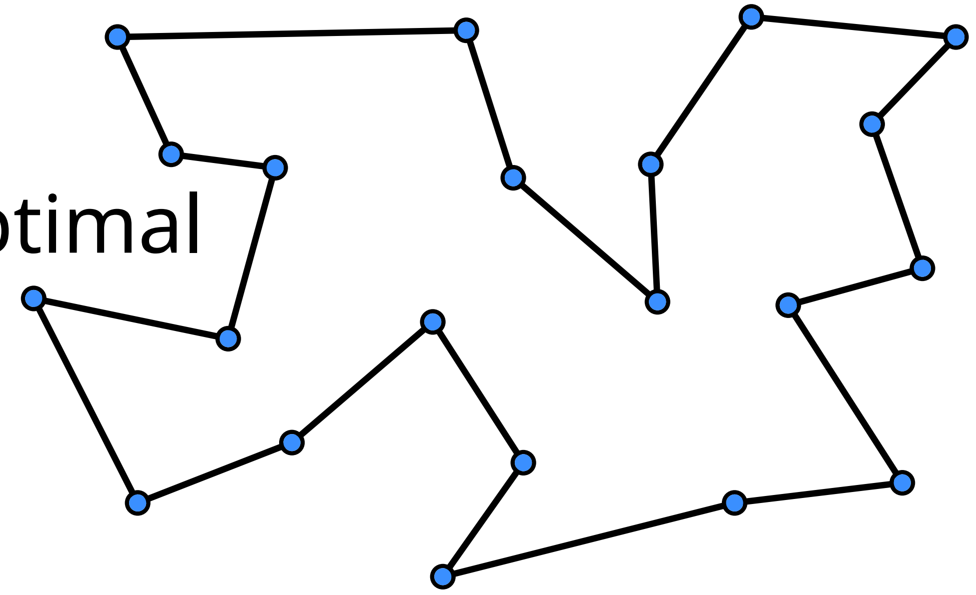
Heuristics for TSP

Example: Nearest-Neighbor Heuristic

Algorithm Nearest-Neighbor Heuristic$(G = (V, E), c)$

- Initialize tour as empty
- Select arbitrary $s \in V$ (as starting vertex)
- Set $v = s$ and $U = V - s$
- **while** $U \neq \emptyset$ **do**
  - Select $u \in U$ with $c(v, u) = \min_{w \in U} c(v, w)$
  - Add edge {v, u} to tour
  - Set $v = u$ and $U = U - u$
- Add edge $\{v, s\}$ to tour
- **return** tour



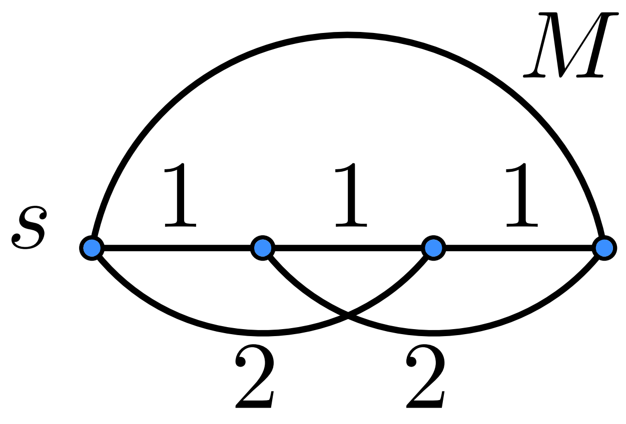


optimal

How bad can the Nearest-Neighbor Heuristic get?

Nearest-Neighbor Heuristic – Lower Bound

Consider the following graph with large M



Nearest-Neighbor Heuristic – Lower Bound

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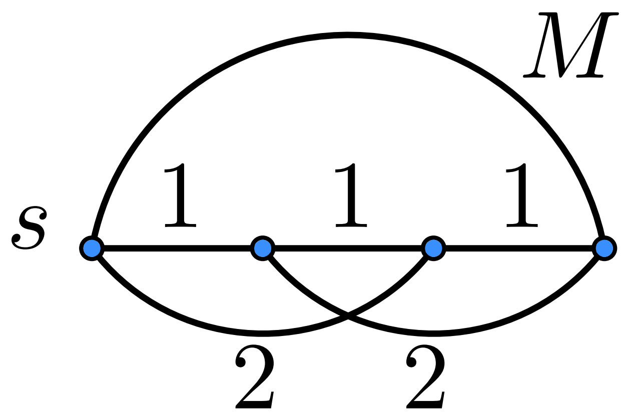
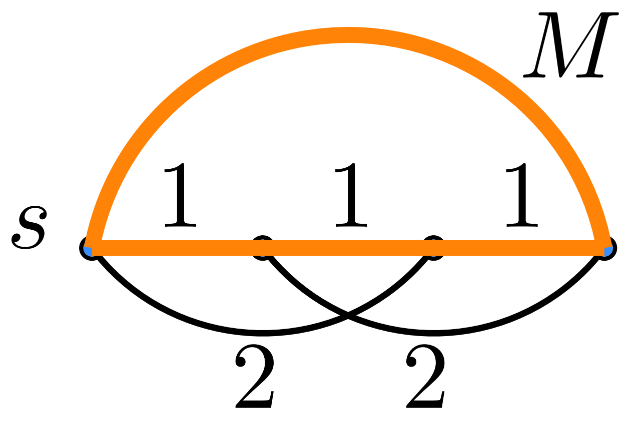


How long is the tour of the Nearest-Neighbor Heuristic?

Nearest-Neighbor Heuristic – Lower Bound

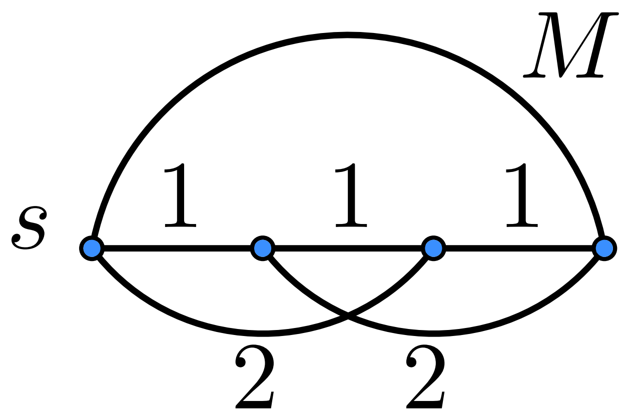
Consider the following graph with large M



How long is the tour of the Nearest-Neighbor Heuristic? $3 + M$

Nearest-Neighbor Heuristic – Lower Bound

Consider the following graph with large M

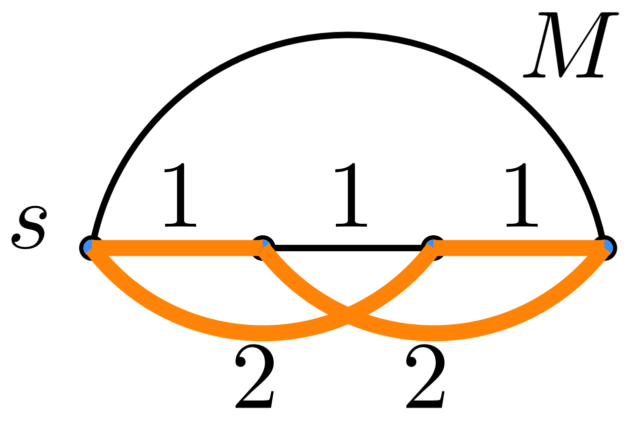


How long is the tour of the Nearest-Neighbor Heuristic? $3 + M$

How long is the optimal tour?

Nearest-Neighbor Heuristic – Lower Bound

Consider the following graph with large M

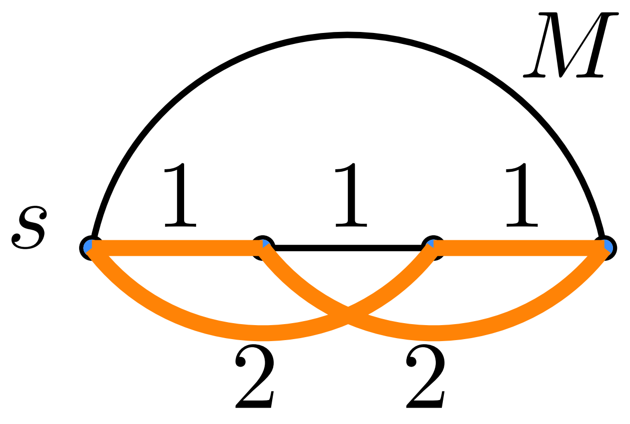


How long is the tour of the Nearest-Neighbor Heuristic? $3 + M$

How long is the optimal tour? 6

Nearest-Neighbor Heuristic – Lower Bound

Consider the following graph with large M



How long is the tour of the Nearest-Neighbor Heuristic? $3 + M$

How long is the optimal tour? 6

- can be arbitrary bad by making M arbitrary large

Nearest-Neighbor Heuristic – Lower Bound

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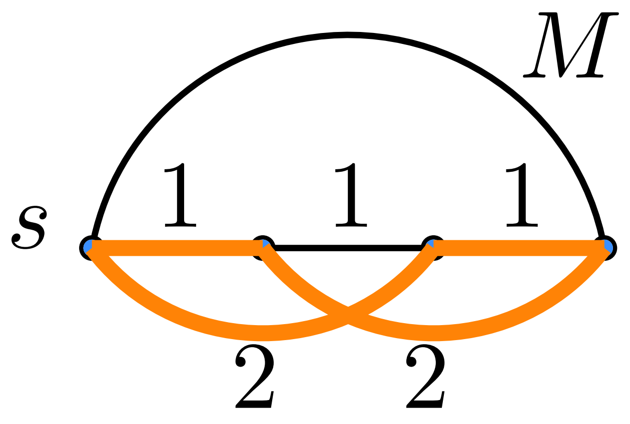

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Nearest-Neighbor Heuristic – Lower Bound

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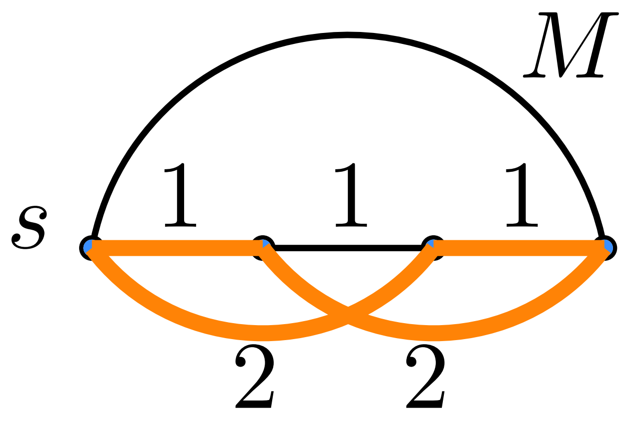

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- for metric TSP (triangle inequality): at most a factor $\Theta(\log n)$ from optimal

Nearest-Neighbor Heuristic – Lower Bound

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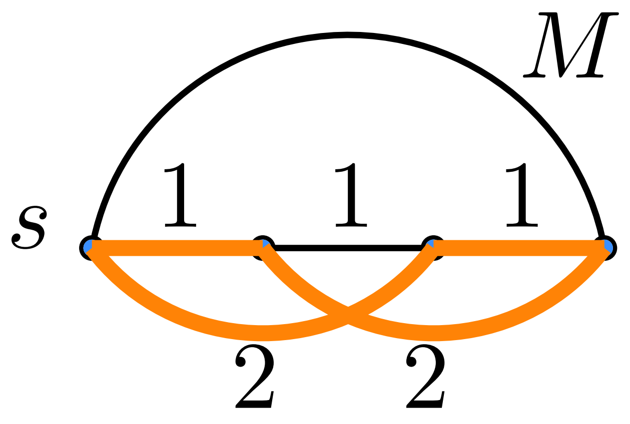


How long is the tour of the Nearest-Neighbor Heuristic? $3 + M$

How long is the optimal tour? 6

- can be arbitrary bad by making M arbitrary large
- but for large n, any polynomial-time algorithm is arbitrarily bad if $P \neq NP$
- for metric TSP (triangle inequality): at most a factor $\Theta(\log n)$ from optimal
- fast but other heuristics give better results

Dynamic Program for TSP

TSP can easily be solved in exponential time by enumerating all solutions, but that is too expensive already for small instances.

Can we do better using dynamic programming?

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

How can the problem be solved recursively?

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

How can the problem be solved recursively?

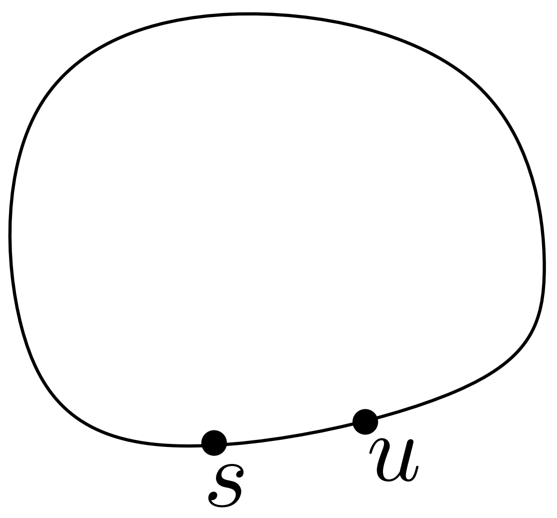
What are the optimal substructures?

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

How can the problem be solved recursively?
What are the optimal substructures?

Choose a starting vertex $s \in V$ and decompose the tour at s:



- path from s through all nodes in $V - s$ ending in some $u \in V - s$
- edge from u to s

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

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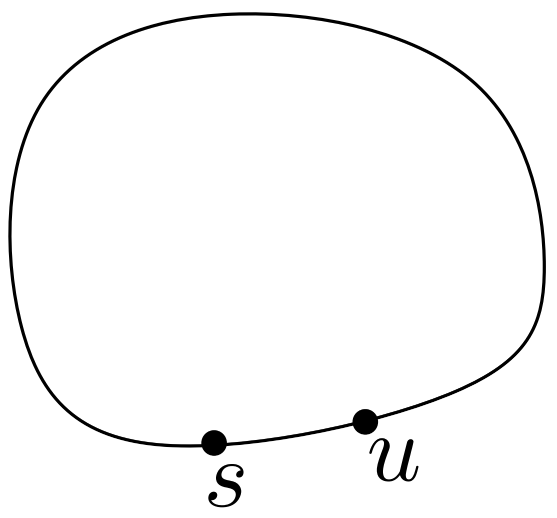

- path from s through all nodes in $V - s$ ending in some $u \in V - s$
- edge from u to s

short for $V \setminus \{s\}$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

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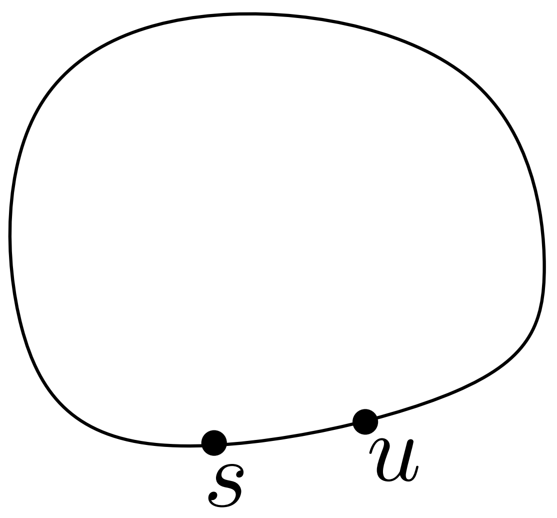
What are the optimal substructures?

Choose a starting vertex $s \in V$ and decompose the tour at s:



- path from s through all nodes in $V - s$ ending in some $u \in V - s$
- edge from u to s

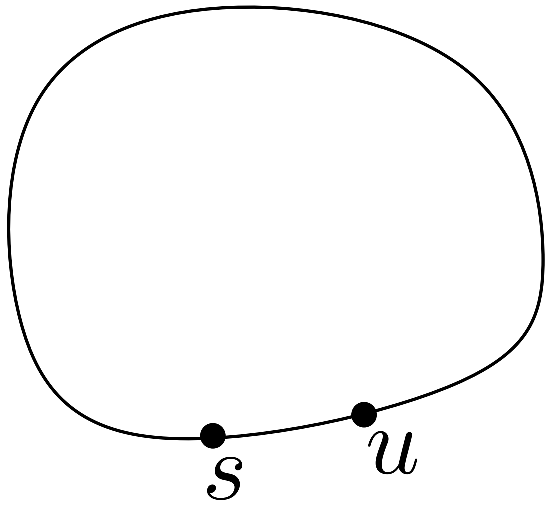
minimize the sum of both lengths!

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

How can the problem be solved recursively?
What are the optimal substructures?

Choose a starting vertex $s \in V$ and decompose the tour at s:



- path from s through all nodes in $V - s$ ending in some $u \in V - s$ — compute recursively
- edge from u to s — fixed

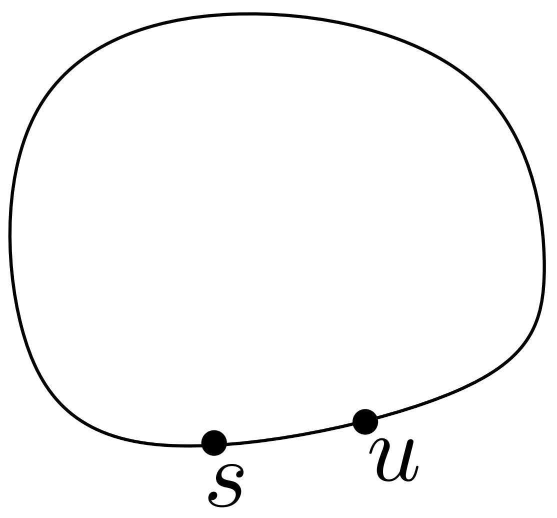
minimize the sum of both lengths!

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

How can the problem be solved recursively?
What are the optimal substructures?

Choose a starting vertex $s \in V$ and decompose the tour at s:



- path from s through all nodes in $V - s$ ending in some $u \in V - s$
- edge from u to s

compute recursively

fixed

minimize the sum of both lengths!

Parameter

Dynamic Program for TSP

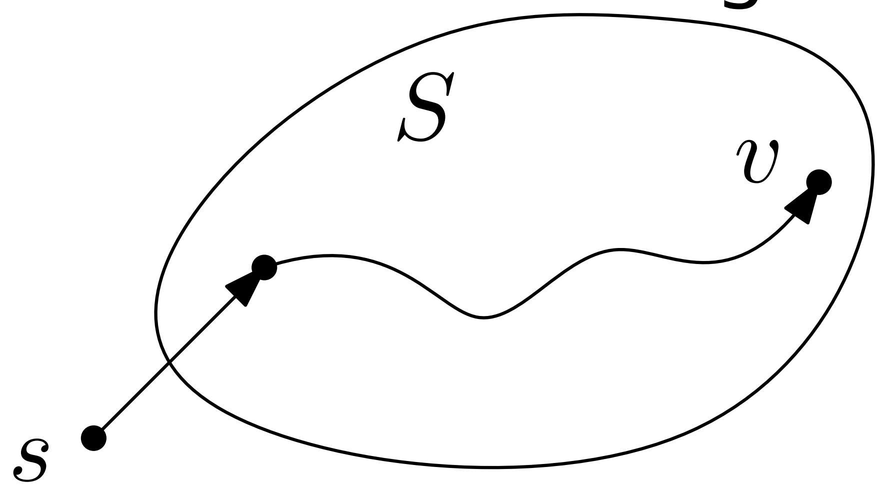
Algorithm of Bellmann, Held and Karp

How can the problem be solved recursively?
What are the optimal substructures?

Choose a starting vertex $s \in V$ and decompose the tour at s:

For $S \subseteq V - s$ and $v \in S$ let:

$\text{OPT}[S, v] =$ length of shortest s-v-path,
visiting all vertices in $S \cup \{s\}$.



Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Start of recursion: $S = \{v\}$ is simply:

$$\text{OPT}[\{v\}, v] = c(s, v).$$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

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For $|S| \geq 2$, compute $\text{OPT}[S, v]$ recursively:

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Start of recursion: $S = \{v\}$ is simply:

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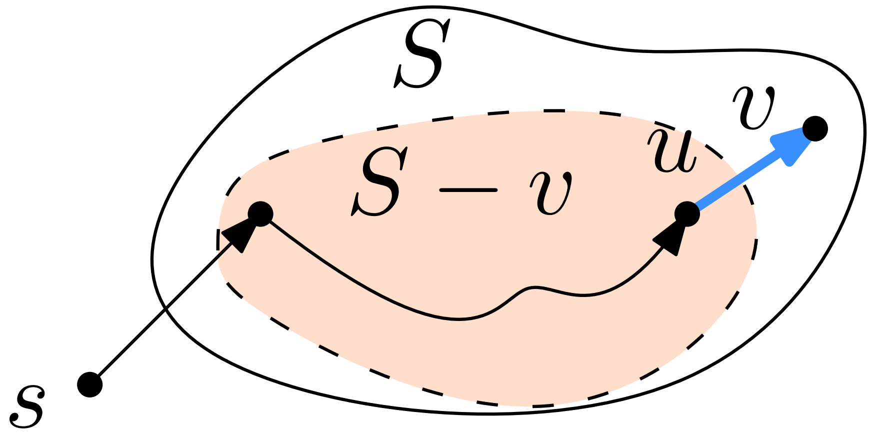
For $|S| \geq 2$, compute $\text{OPT}[S, v]$ recursively:

$$\text{OPT}[S, v] = \min\{\ \text{OPT}[S - v, u] + c(u, v) \mid u \in S - v\ \}$$



Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

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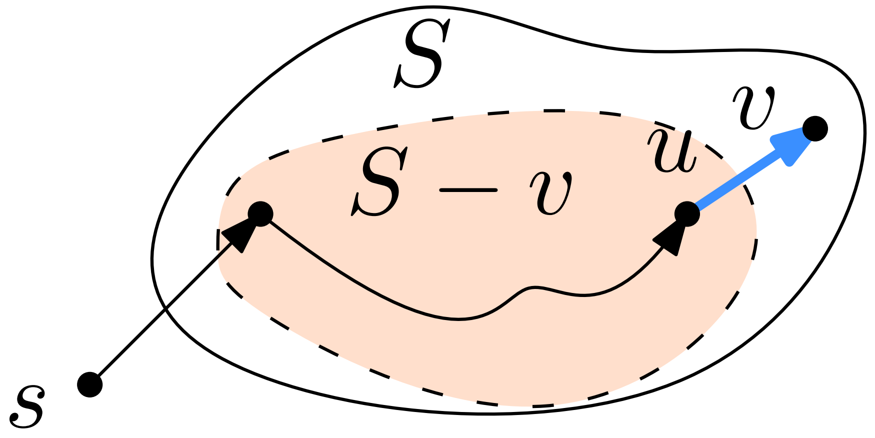

The optimal TSP-tour can then be easily obtained as

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Start of recursion: $S = \{v\}$ is simply:

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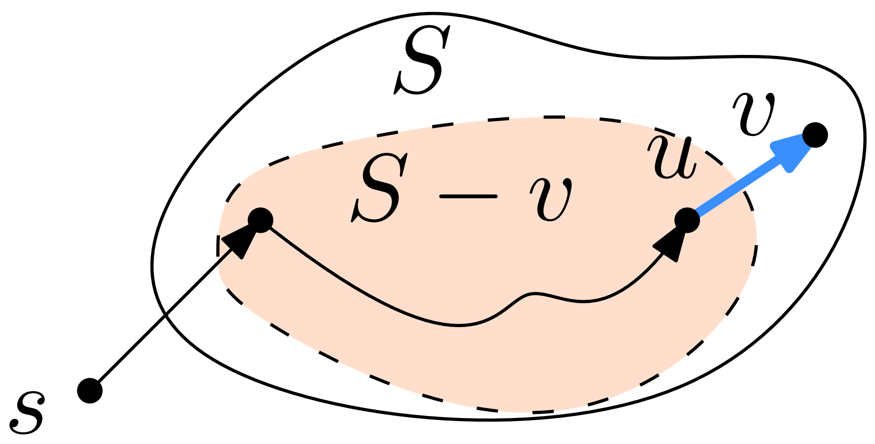

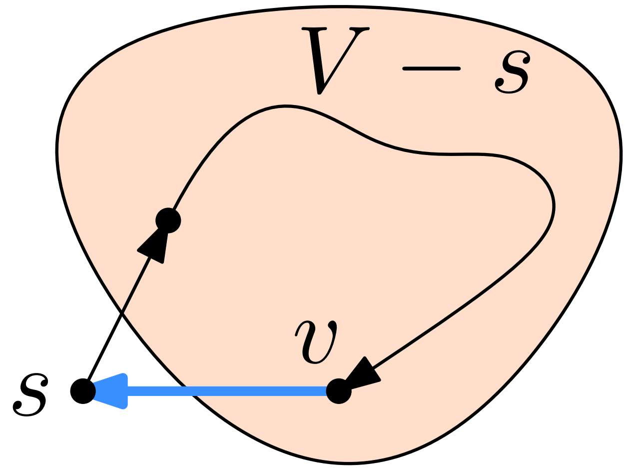

The optimal TSP-tour can then be easily obtained as

$$\text{OPT} = \min\{\ \text{OPT}[V - s, v] + c(v, s)\ \mid v \in V - s\,\}$$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Algorithm Bellmann-Held-Karp(G, c)

foreach $v \in V - s$ **do**
- OPT$[\{v\}, v] = c(s, v)$

for $j = 2$ **to** $n - 1$ **do**
- **foreach** $S \subseteq V - s$ with $|S| = j$ **do**
  - **foreach** $v \in S$ **do**
    - OPT$[S, v] = \min\{$ OPT$[S - v, u] + c(u, v) \mid u \in S - v \}$

return $\min\{$ OPT$[V - s, v] + c(v, s) \mid v \in V - s \}$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

```
Algorithm Bellmann-Held-Karp(G, c)

  foreach v ∈ V − s do
    OPT[{v}, v] = c(s, v)

  for j = 2 to n − 1 do
    foreach S ⊆ V − s with |S| = j do
      foreach v ∈ S do
        OPT[S, v] = min{ OPT[S − v, u] + c(u, v) | u ∈ S − v }

  return min{ OPT[V − s, v] + c(v, s) | v ∈ V − s }
```

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

```
Algorithm Bellmann-Held-Karp(G, c)

  foreach v ∈ V − s do
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  for j = 2 to n − 1 do
    foreach S ⊆ V − s with |S| = j do
      foreach v ∈ S do
        OPT[S, v] = min{ OPT[S − v, u] + c(u, v) | u ∈ S − v }

  return min{ OPT[V − s, v] + c(v, s) | v ∈ V − s }
```

Runtime: ???

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Algorithm Bellmann-Held-Karp(G, c)

foreach $v \in V - s$ **do** $\quad O(n)$
  OPT$[\{v\}, v] = c(s, v)$

for $j = 2$ **to** $n - 1$ **do** $\quad \big\} \; O(2^n)$
  foreach $S \subseteq V - s$ with $|S| = j$ **do**
    foreach $v \in S$ **do** $\quad \} \; O(j)$
      OPT$[S, v] = \min\{$ OPT$[S - v, u] + c(u, v) \mid u \in S - v \}$ $\quad O(j)$

return $\min\{$ OPT$[V - s, v] + c(v, s) \mid v \in V - s \}$ $\quad O(n)$

Runtime: $O(2^n \cdot n^2)$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Algorithm Bellmann-Held-Karp(G, c)

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return $\min\{$ OPT$[V - s, v] + c(v, s) \mid v \in V - s \}$ $\quad O(n)$

Runtime: $O(2^n \cdot n^2)$
Space: $\Theta(2^n \cdot n)$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Algorithm Bellmann-Held-Karp(G, c)

foreach $v \in V - s$ **do** $\qquad O(n)$
  OPT$[\{v\}, v] = c(s, v)$

for $j = 2$ **to** $n - 1$ **do** $\qquad \Big\} \; O(2^n)$
  foreach $S \subseteq V - s$ with $|S| = j$ **do**
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      OPT$[S, v] = \min\{$ OPT$[S - v, u] + c(u, v) \mid u \in S - v \}$ $\qquad O(j)$

return $\min\{$ OPT$[V - s, v] + c(v, s) \mid v \in V - s \}$ $\quad O(n)$

Runtime: $O(2^n \cdot n^2)$
Space: $\Theta(2^n \cdot n)$

do we need this much space, even though for value j we only need the values for $j - 1$?

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Algorithm Bellmann-Held-Karp(G, c)

foreach $v \in V - s$ **do** $\quad O(n)$
  OPT$[\{v\}, v] = c(s, v)$

for $j = 2$ **to** $n - 1$ **do**
  foreach $S \subseteq V - s$ with $|S| = j$ **do** $\quad \Big\} \; O(2^n)$
    foreach $v \in S$ **do** $\quad \} \; O(j)$
      OPT$[S, v] = \min\{\,$OPT$[S - v, u] + c(u, v) \mid u \in S - v\,\}$ $\quad O(j)$

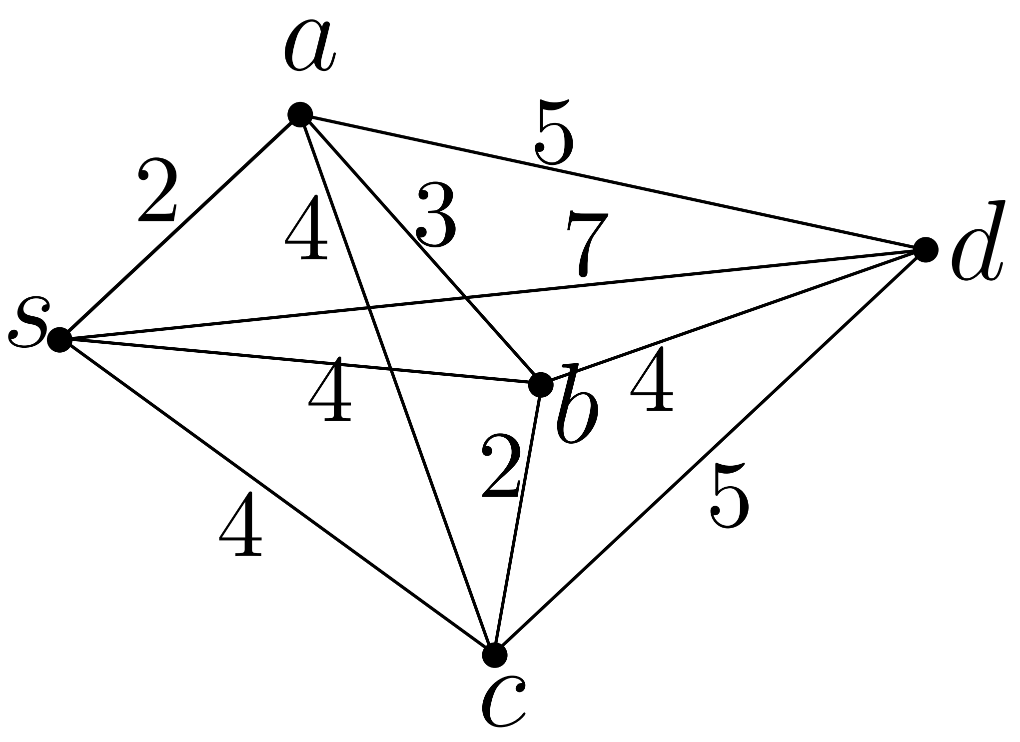
return $\min\{\,$OPT$[V - s, v] + c(v, s) \mid v \in V - s\,\}$ $\quad O(n)$

Runtime: $O(2^n \cdot n^2)$
Space: $\Theta(2^n \cdot n)$
A shortest tour can be found through backtracking in the table.

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

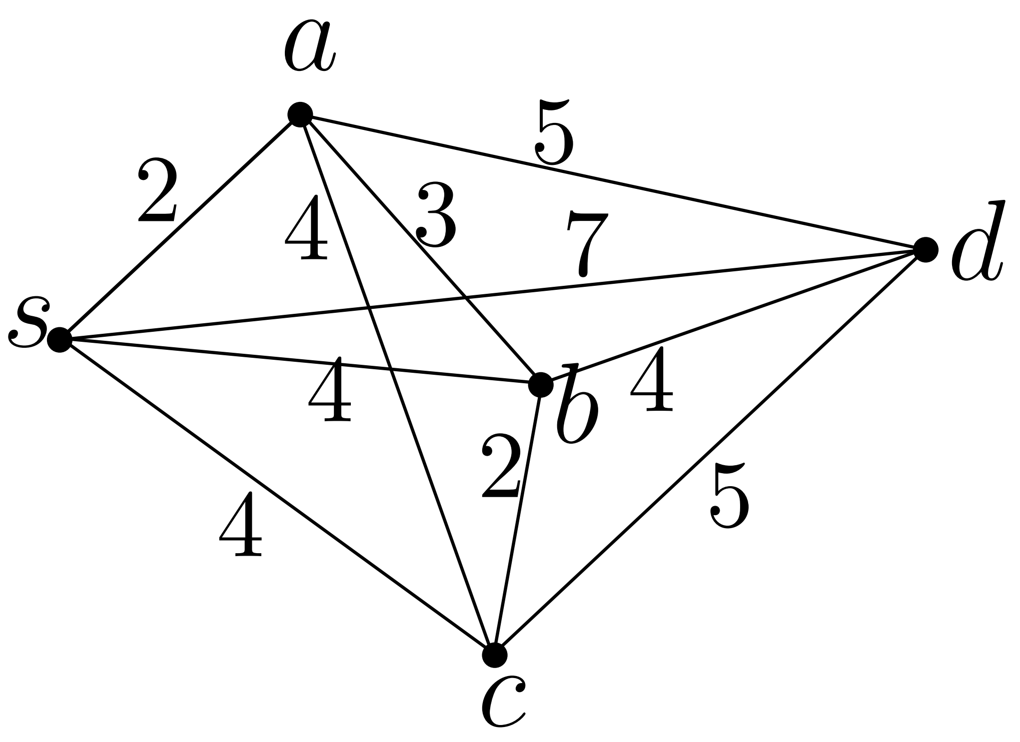


Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

$j \backslash v$	$S - v \downarrow$ a	b	c	d	S
1					
2					
3					
4					
$\sum$					



OPT$[S, v]$

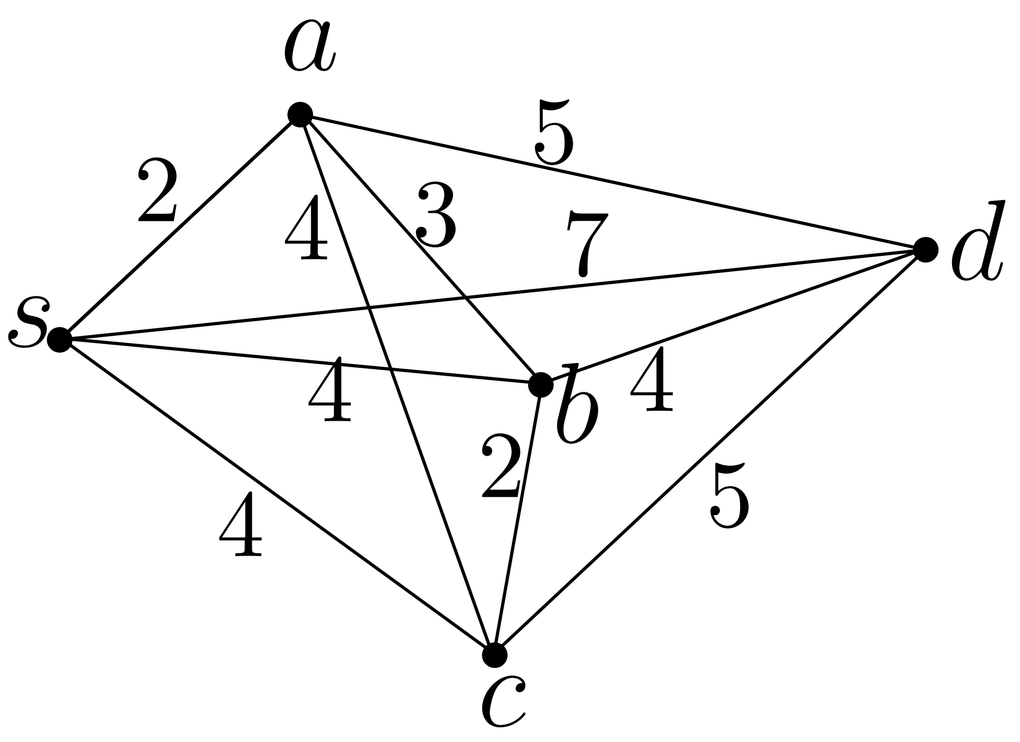
OPT$[V - s, v] + c(v, s)$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

v j	$S - v$ ↓ a	b	c	d	S
1	– 2	– 4	– 4	– 7	$\{a\}, \{b\},$ $\{c\}, \{d\}$
2					
3					
4					
$\sum$					



$\mathsf{OPT}[S, v]$

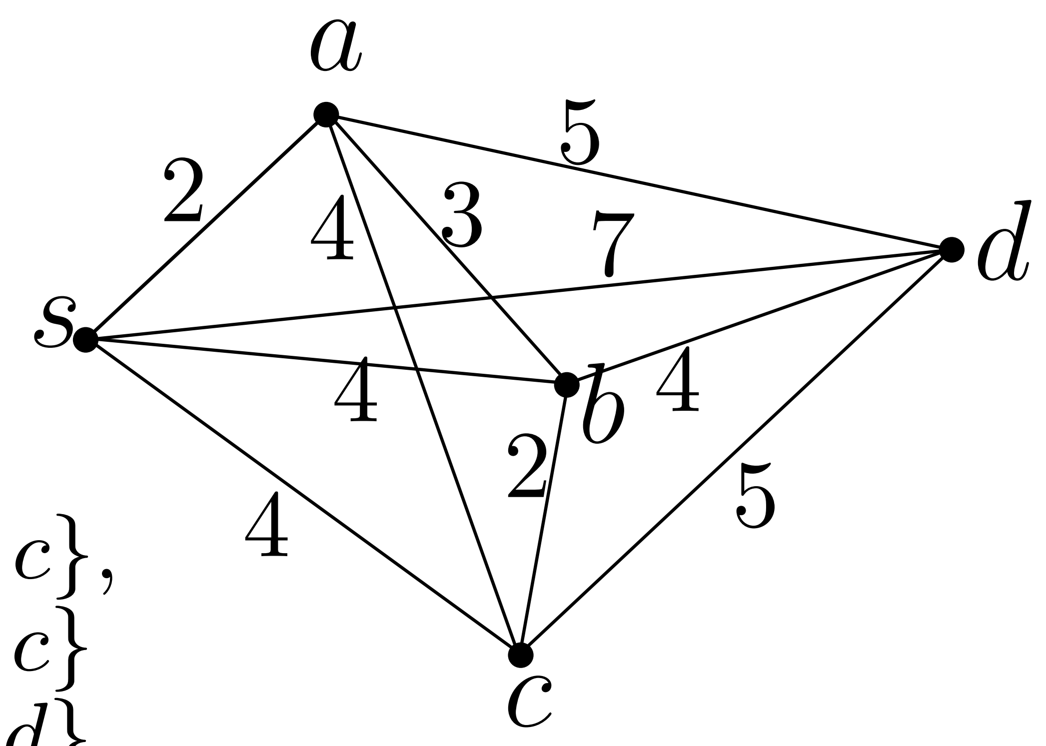
$\mathsf{OPT}[V - s, v] + c(v, s)$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

$j \backslash v$	$S-v$ ↓	a		b		c		d	S
1	–	2	–	4	–	4	–	7	$\{a\}, \{b\}, \{c\}, \{d\}$
2	b c d	7 8 12	a c d	5 6 11	a b d	6 6 12	a b c	7 8 9	$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$
3									
4									
$\sum$									



OPT$[S, v]$

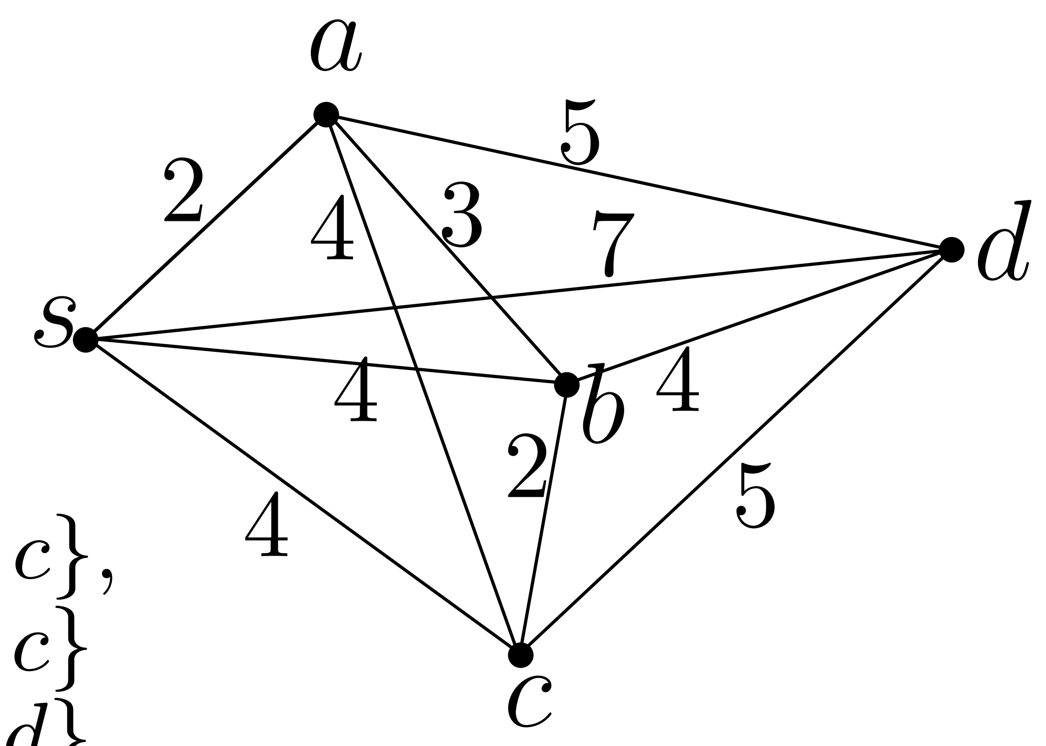
OPT$[V - s, v] + c(v, s)$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

$j \setminus v$	$S - v$ ↓	a		b		c		d	S
1	–	2	–	4	–	4	–	7	$\{a\}, \{b\}, \{c\}, \{d\}$
2	b c d	7 8 12	a c d	5 6 11	a b d	6 6 12	a b c	7 8 9	$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$
3	b,c b,d c,d	9 13 14	a,c a,d c,d	8 11 13	a,b a,d b,d	7 12 13	a,b a,c b,c	9 11 10	$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$
4									
$\sum$									



OPT$[S, v]$

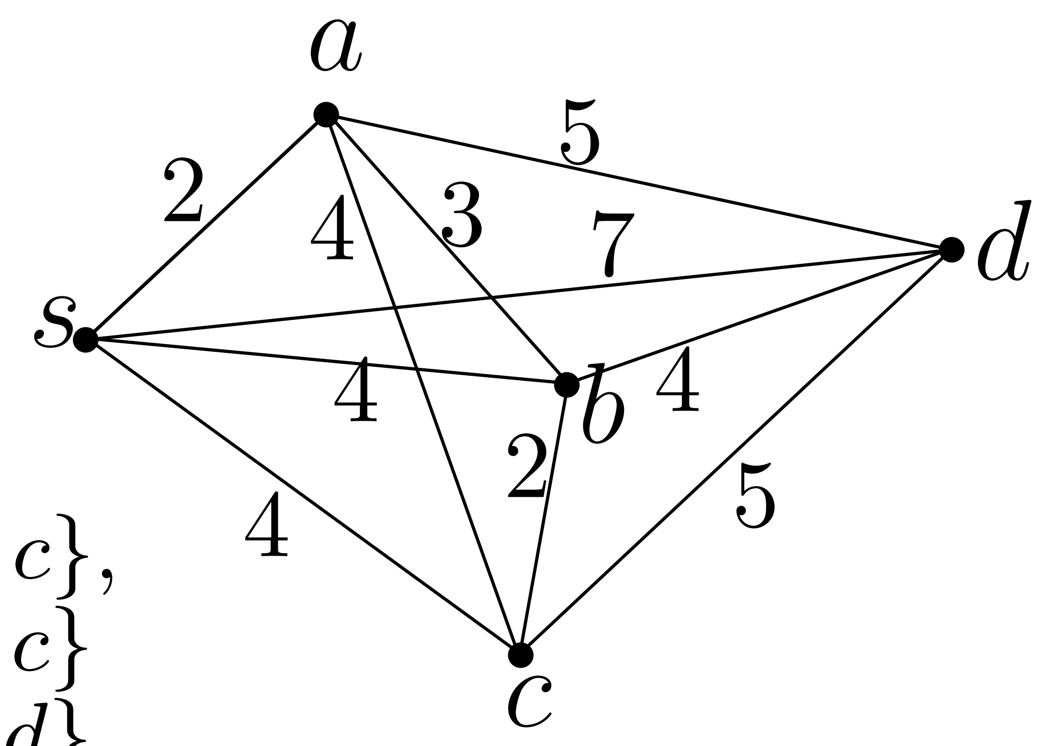
OPT$[V - s, v] + c(v, s)$

Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

j \ v	$S-v$ ↓	a		b		c		d	S
1	–	2	–	4	–	4	–	7	$\{a\}, \{b\}, \{c\}, \{d\}$
2	b c d	7 8 12	a c d	5 6 11	a b d	6 6 12	a b c	7 8 9	$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$
3	b,c b,d c,d	9 13 14	a,c a,d c,d	8 11 13	a,b a,d b,d	7 12 13	a,b a,c b,c	9 11 10	$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$
4	b,c,d	15	a,c,d	14	a,b,d	13	a,b,c	12	$\{a,b,c,d\}$
$\sum$									



$\mathrm{OPT}[S, v]$

$\mathrm{OPT}[V - s, v] + c(v, s)$

Dynamic Program for TSP

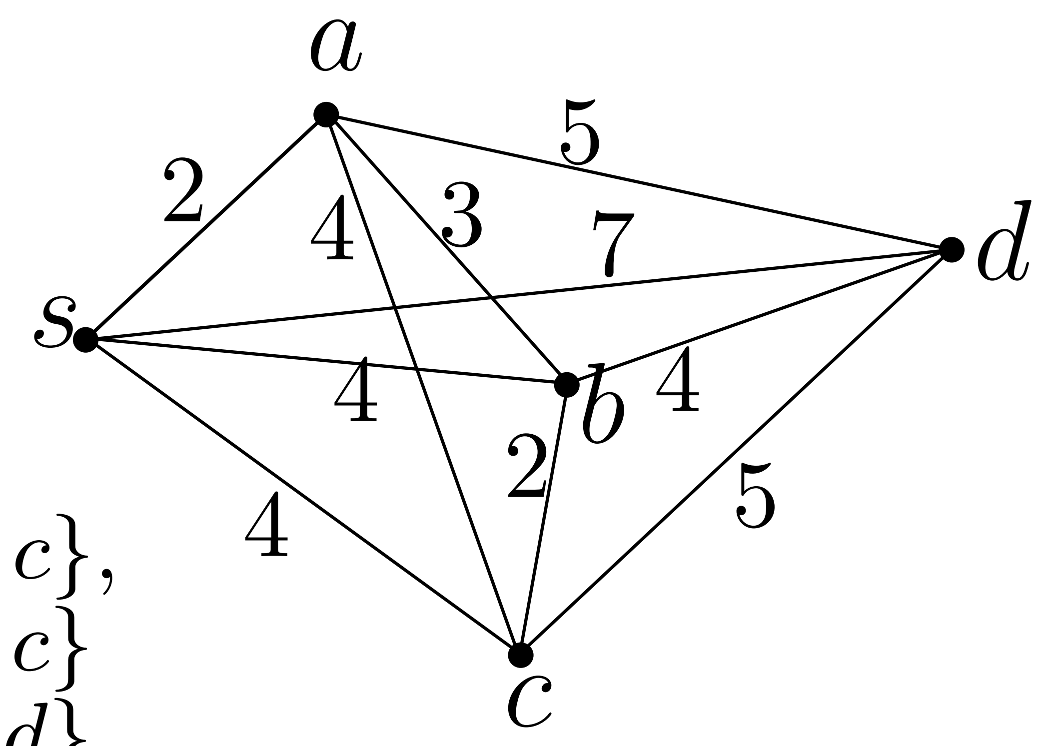
Algorithm of Bellmann, Held and Karp

Example

$j \setminus v$	$S-v \downarrow$	a		b		c		d	S
1	–	2	–	4	–	4	–	7	$\{a\}, \{b\}, \{c\}, \{d\}$
2	b c d	7 8 12	a c d	5 6 11	a b d	6 6 12	a b c	7 8 9	$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$
3	b,c b,d c,d	9 13 14	a,c a,d c,d	8 11 13	a,b a,d b,d	7 12 13	a,b a,c b,c	9 11 10	$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$
4	b,c,d	15	a,c,d	14	a,b,d	13	a,b,c	12	$\{a,b,c,d\}$
$\sum$		17		18		17		19	

OPT$[S, v]$

OPT$[V-s, v] + c(v, s)$



Dynamic Program for TSP

Algorithm of Bellmann, Held and Karp

Example

$j \setminus v$	$S-v$ ↓	a		b		c		d	S
1	–	2	–	4	–	**4**	–	7	$\{a\}, \{b\},$ $\{c\}, \{d\}$
2	b c d	7 8 12	a c d	5 **6** 11	a b d	6 6 12	a b c	7 8 9	$\{a,b\}, \{a,c\},$ $\{a,d\}, \{b,c\}$ $\{b,d\}, \{c,d\}$
3	b,c b,d c,d	9 13 14	a,c a,d c,d	8 11 13	a,b a,d b,d	7 12 13	a,b a,c b,c	9 11 **10**	$\{a,b,c\},$ $\{a,b,d\},$ $\{a,c,d\},$ $\{b,c,d\}$
4	b,c,d	**15**	a,c,d	14	a,b,d	13	a,b,c	12	$\{a,b,c,d\}$
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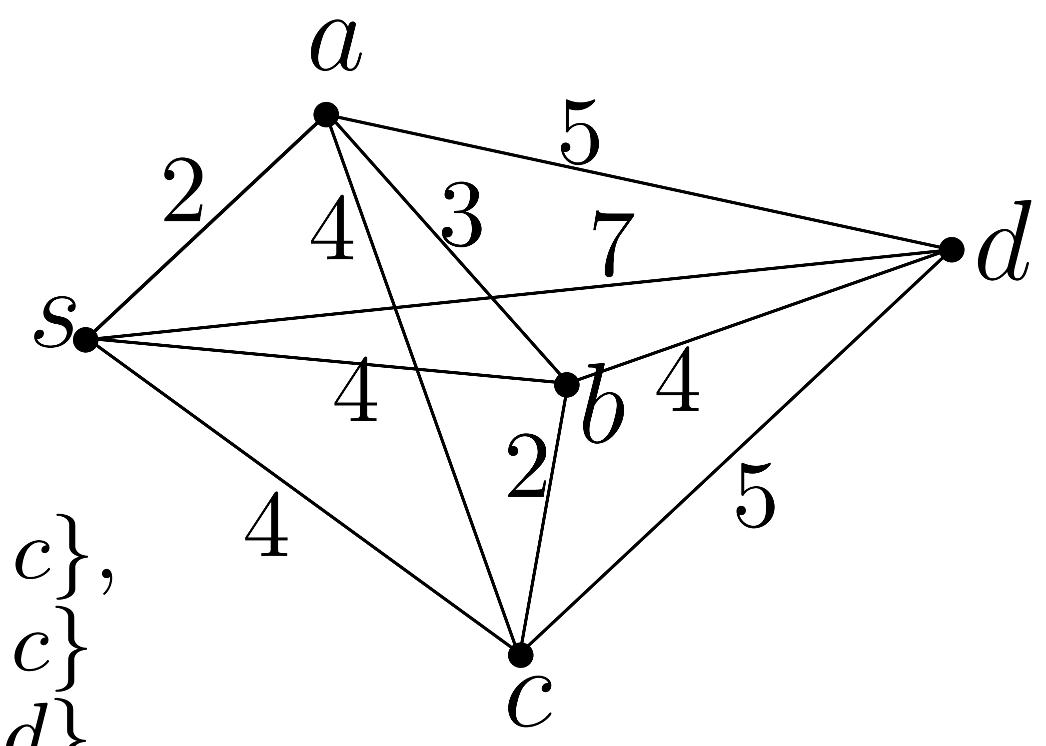

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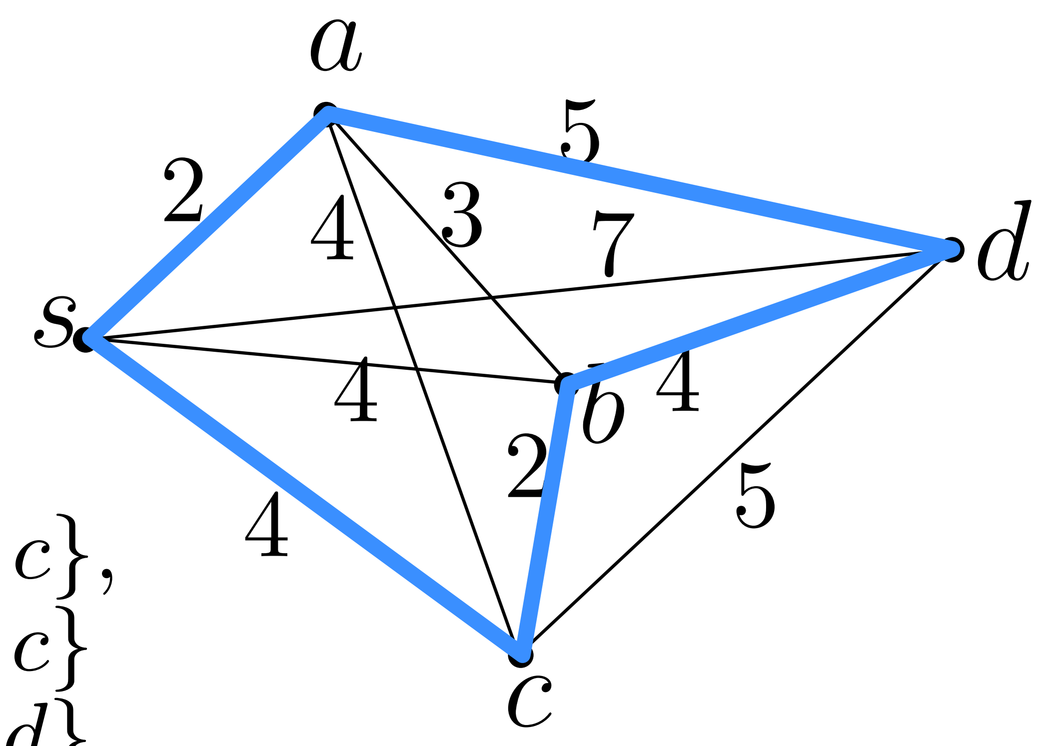
3	b,c b,d c,d	9 13 14	a,c a,d c,d	8 11 13	a,b a,d b,d	7 12 13	a,b a,c b,c	9 11 $\mathbf{10}$	$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$
4	b,c,d	$\mathbf{15}$	a,c,d	14	a,b,d	13	a,b,c	12	$\{a,b,c,d\}$
$\sum$		$\mathbf{17}$		18		17		19	

OPT$[S, v]$

OPT$[V - s, v] + c(v, s)$



Summary

shortest path

- efficiently solvable
- heaps with $O(1)$ decreaseKey

shortest round trip

- NP-hard
- many algorithmic approaches
- dynamic program: exponential-time algorithm

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fast algorithms for hard problems

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- parameterized