The Simplex method

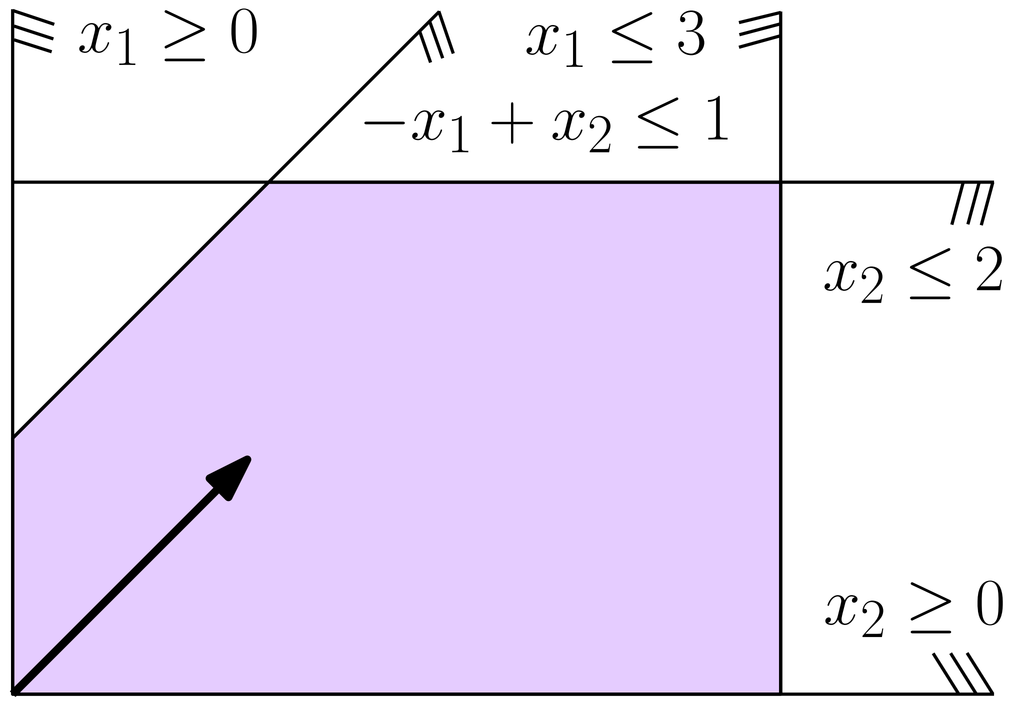
Introductory example

$$
\begin{aligned}
\text{Maximize } \; & x_1 + x_2 \\
\text{subject to } \; & -x_1 + x_2 \le 1 \\
& x_1 \le 3 \\
& x_2 \le 2 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Introductory example

Maximize $x_1 + x_2$

subject to
$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

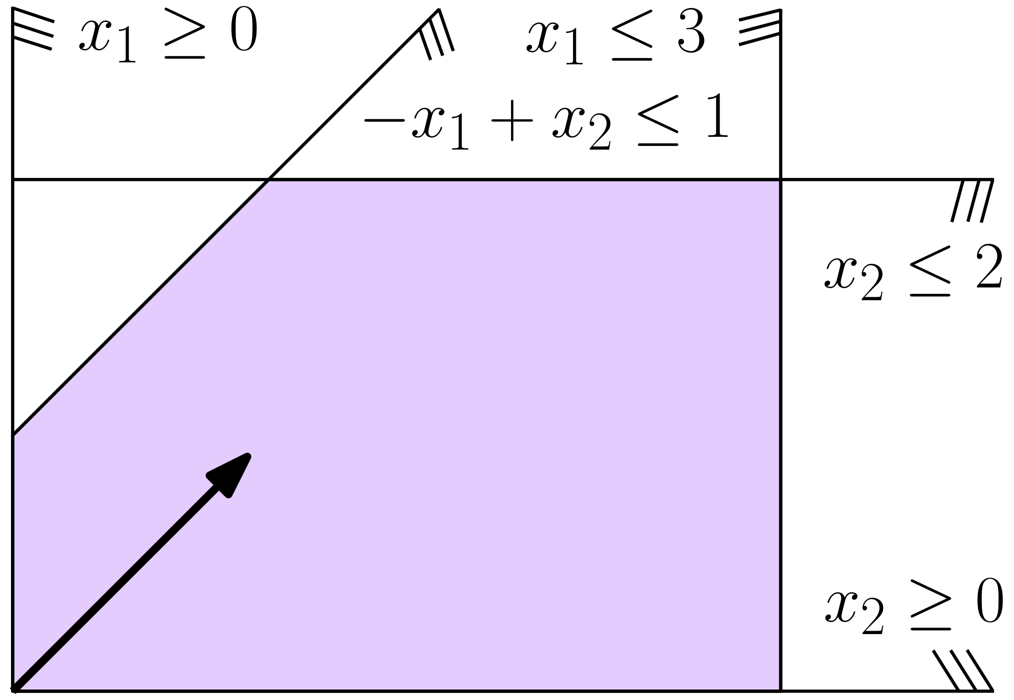


How to transform into equational form?

Introductory example

Maximize $x_1 + x_2$

subject to
$$\begin{aligned} -x_1 + x_2 &\le 1 \\ x_1 &\le 3 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$



add slack variables

Equational form

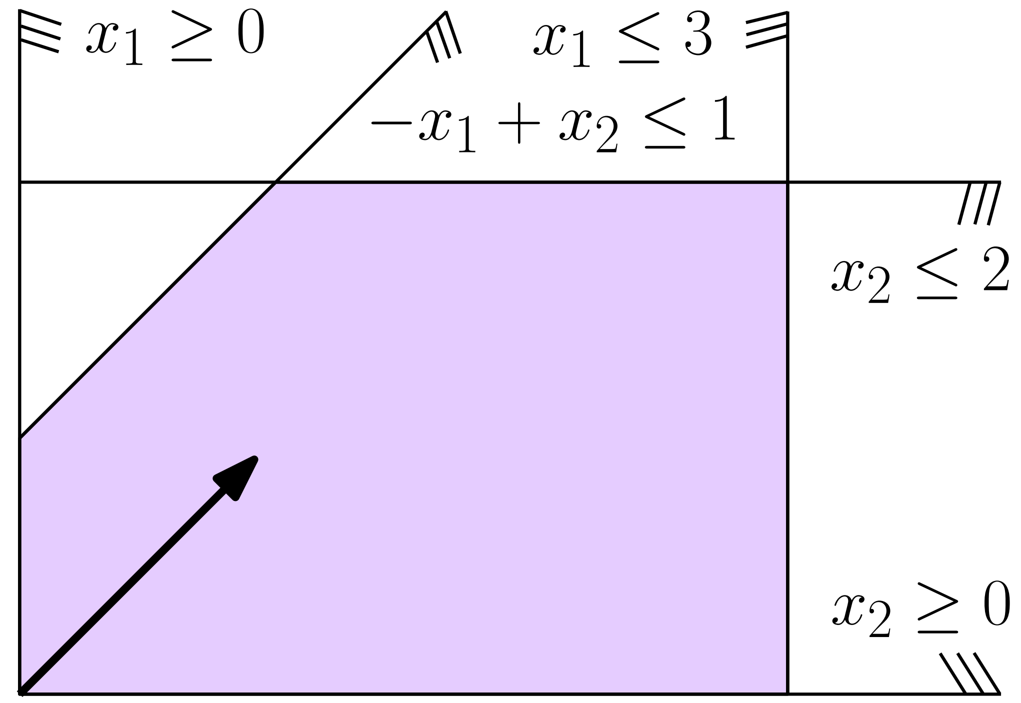
Maximize $x_1 + x_2$

subject to
$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

Introductory example

Maximize $x_1 + x_2$

subject to
$$\begin{aligned} -x_1 + x_2 &\le 1 \\ x_1 &\le 3 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$



add slack variables

Equational form

Maximize $x_1 + x_2$

subject to
$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

in matrix form

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

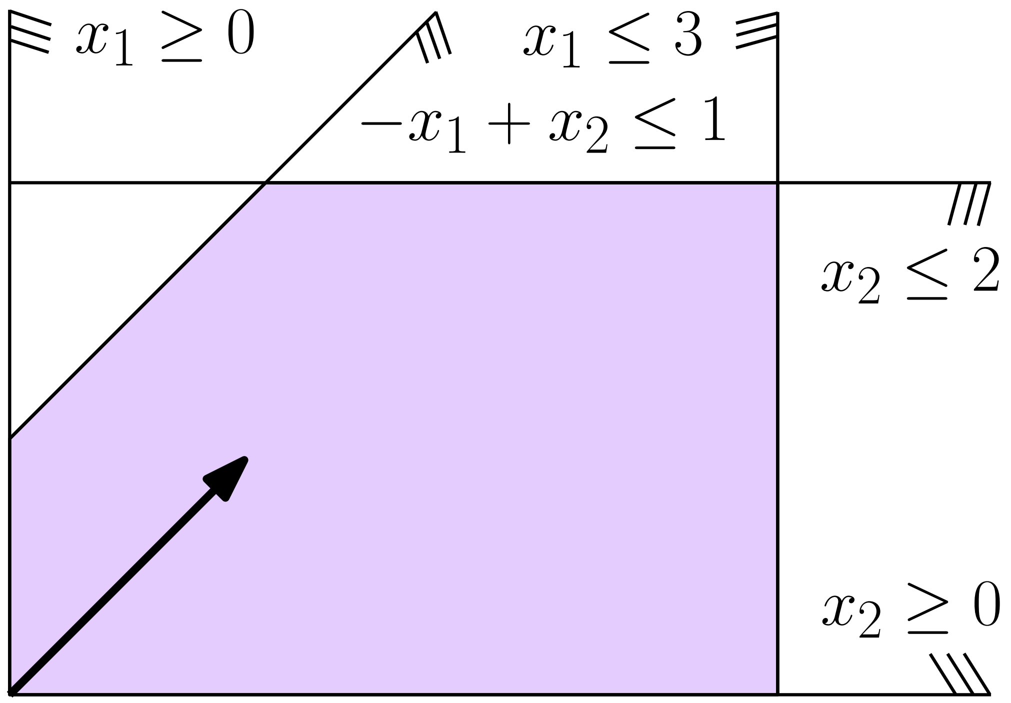
$$c^T = (1, 1, 0, 0, 0), \;\; b^T = (1, 3, 2)$$

Introductory example - Simplex tableau A

Simplex tableau A

$$\begin{array}{rcl} x_3 &=& 1 + x_1 - x_2 \\ x_4 &=& 3 - x_1 \\ x_5 &=& 2 \qquad\quad - x_2 \\ \hline z &=& \qquad x_1 + x_2 \end{array}$$

rewrite



Equational form

Maximize $x_1 + x_2$

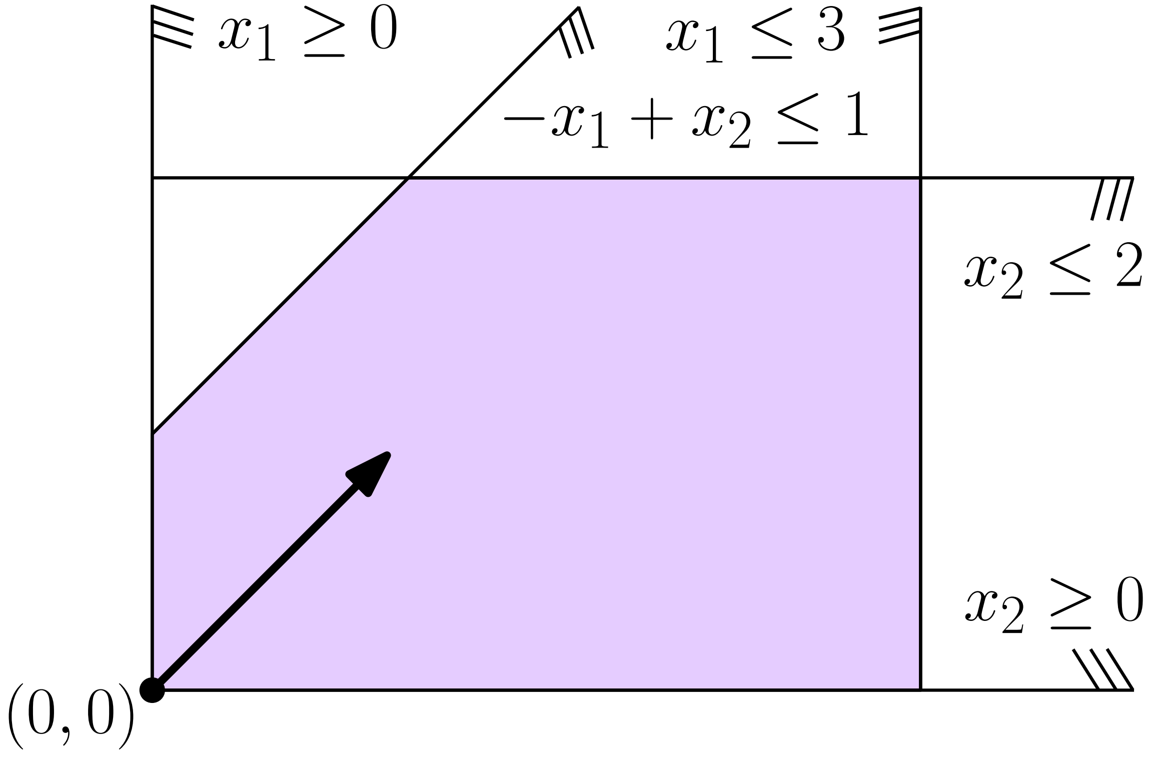
subject to
$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Introductory example - Simplex tableau A

Simplex tableau A

$$\text{basic variables}\left\{\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 \phantom{{}+x_1} - x_2 \end{aligned}\right.$$

$$z = \underbrace{x_1 + x_2}_{\text{non-basic variables}}$$



Plug in $x_1 = x_2 = 0$ to get a bfs with basis $B = \{3, 4, 5\}$ and value $z = 0$.

basic feasible solution

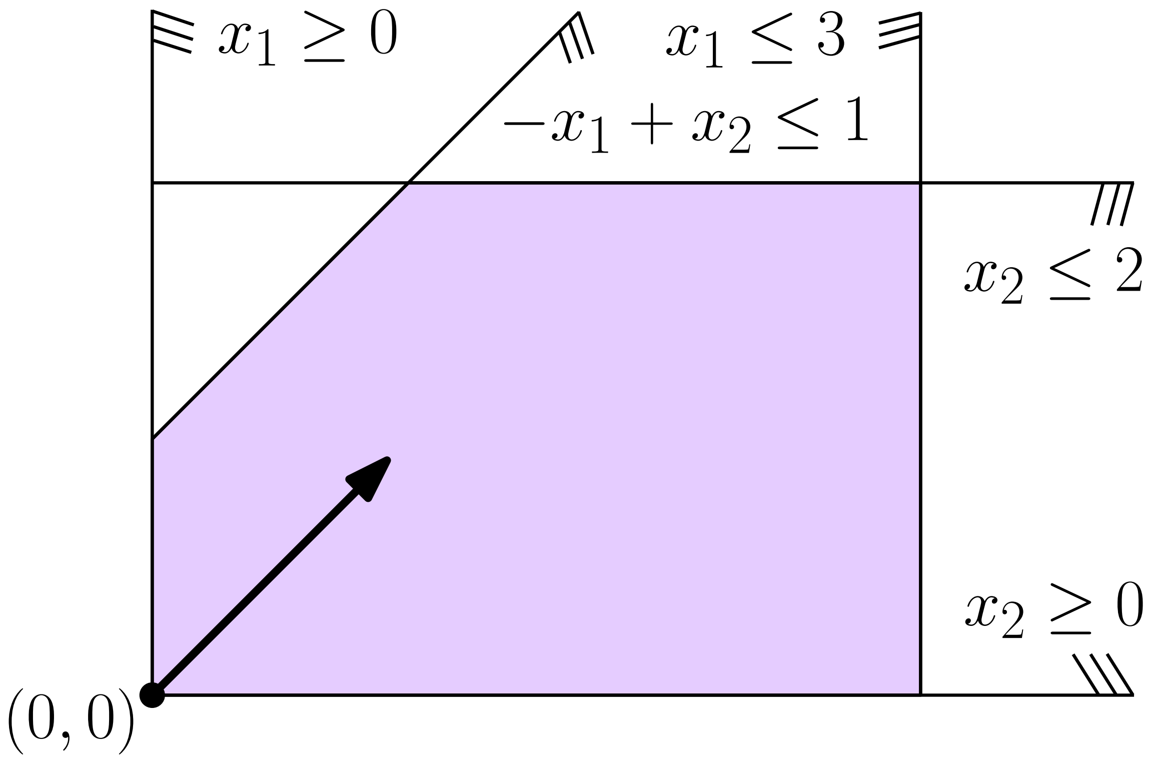
Introductory example - Simplex tableau A

Simplex tableau A

basic variables

$$\left.\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 - x_2 \end{aligned}\right.$$

$$z = \underbrace{x_1 + x_2}_{\text{non-basic variables}}$$



Plug in $x_1 = x_2 = 0$ to get a bfs with basis $B = \{3, 4, 5\}$ and value $z = 0$.

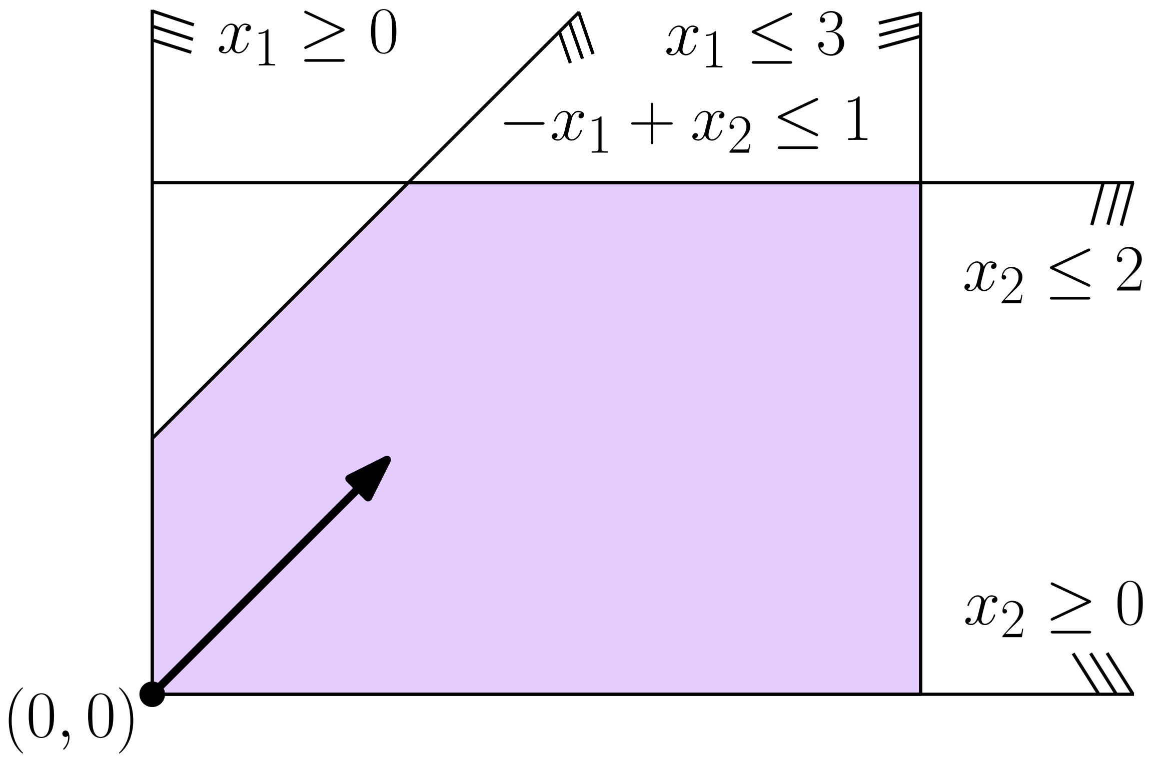
How can we increase the objective value z?

Introductory example - Simplex tableau A

Simplex tableau A

$$\text{basic variables}\left\{\begin{array}{l} x_3 = 1 + x_1 - x_2 \\ x_4 = 3 - x_1 \\ x_5 = 2 \qquad\;\; - x_2 \end{array}\right.$$

$$\overline{\quad z = \qquad \underbrace{x_1 + x_2}_{\text{non-basic variables}} \quad}$$



Plug in $x_1 = x_2 = 0$ to get a bfs with basis $B = \{3, 4, 5\}$ and value $z = 0$.

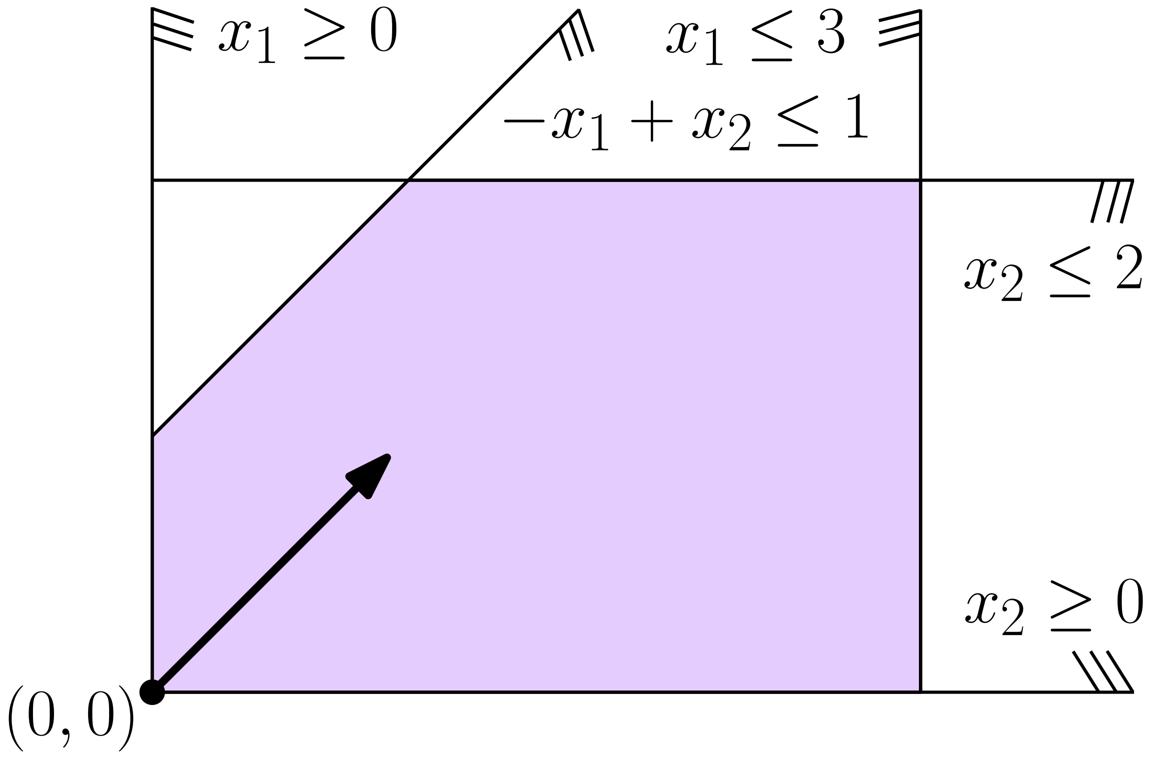
Increase z by (arbitrarily) deciding to increase x_2 while fixing $x_1 = 0$.

Introductory example - Simplex tableau A

Simplex tableau A

$$\text{basic variables}\begin{cases} x_3 = 1 + x_1 - x_2 \\ x_4 = 3 - x_1 \\ x_5 = 2 \qquad\;\; - x_2 \end{cases}$$

$$z = \underbrace{x_1 + x_2}_{\text{non-basic variables}}$$



Plug in $x_1 = x_2 = 0$ to get a bfs with basis $B = \{3, 4, 5\}$ and value $z = 0$.

Increase z by (arbitrarily) deciding to increase x_2 while fixing $x_1 = 0$.

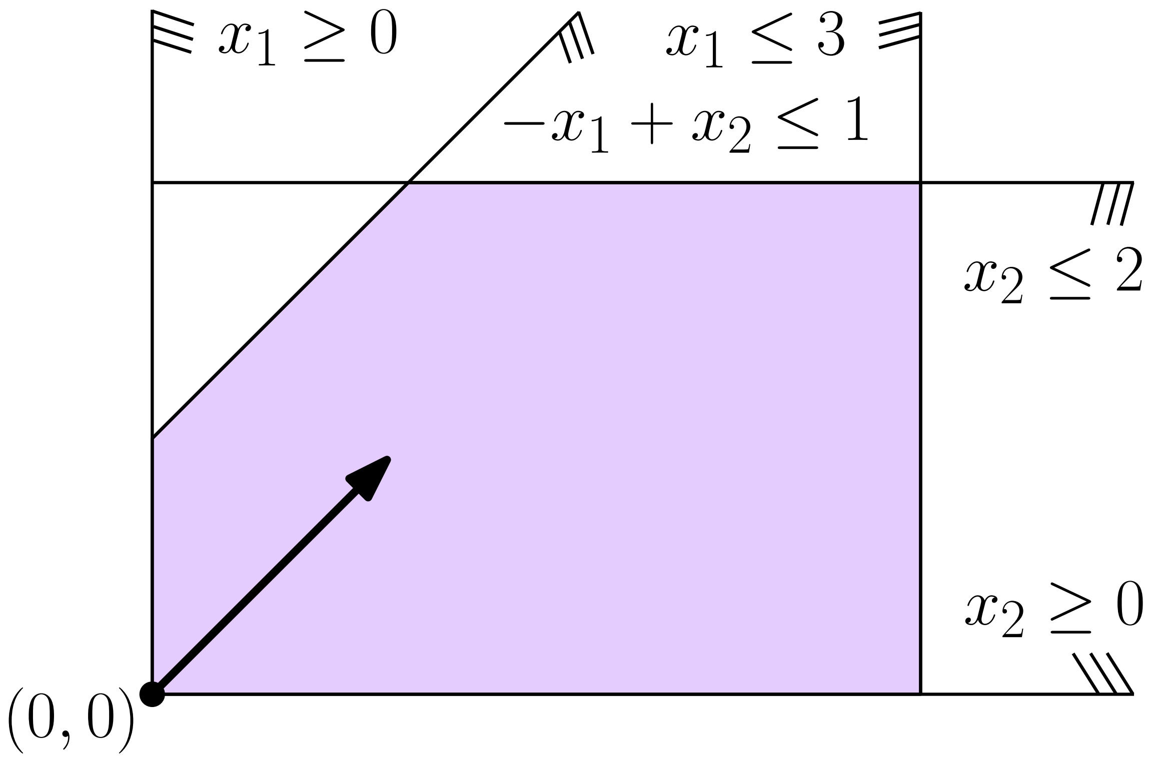
How much can we increase x_2?

Introductory example - Simplex tableau A

Simplex tableau A

basic variables

$$\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 - x_2 \\ \hline z &= \underbrace{x_1 + x_2}_{\text{non-basic variables}} \end{aligned}$$



Plug in $x_1 = x_2 = 0$ to get a bfs with basis $B = \{3, 4, 5\}$ and value $z = 0$.

Increase z by (arbitrarily) deciding to increase x_2 while fixing $x_1 = 0$.

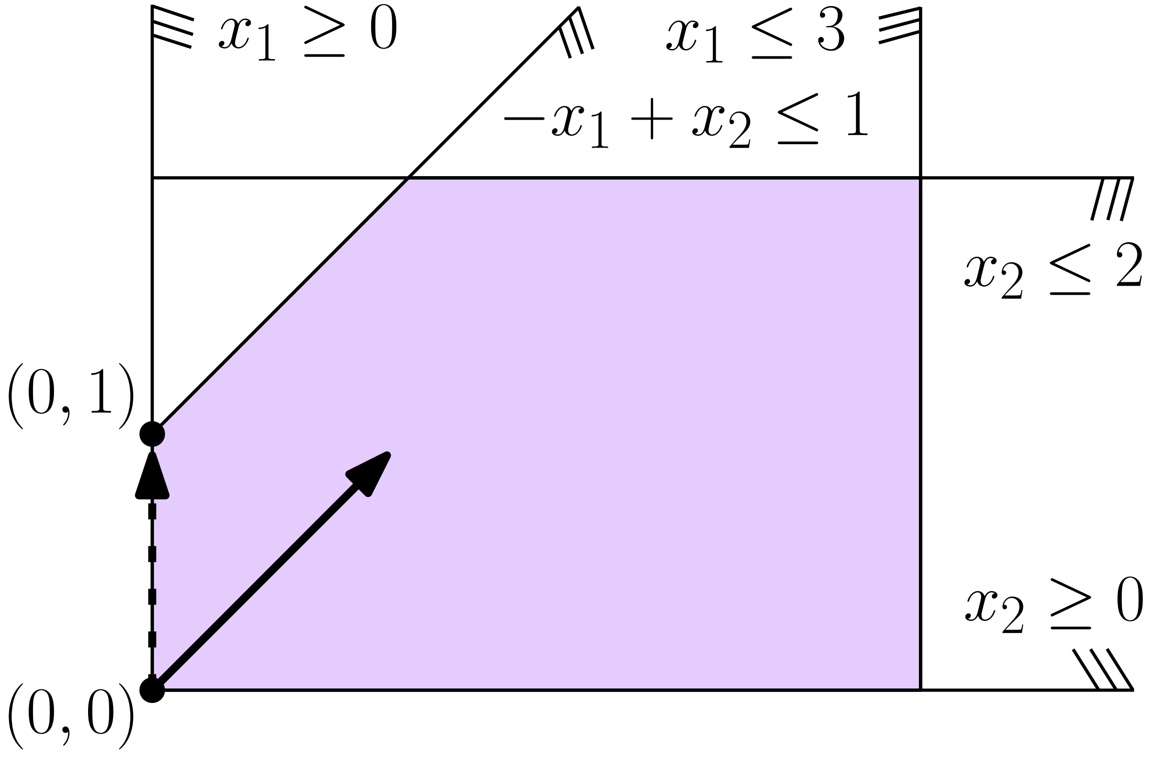
We are most limited by the equation $x_3 = 1 + x_1 - x_2$,
which we can rewrite as $x_2 = 1 + x_1 - x_3$.

Introductory example - Simplex tableau B

Simplex tableau B

$$\text{basic variables}\left\{\begin{array}{l} x_2 = 1 + x_1 - x_3 \\ x_4 = 3 - x_1 \\ x_5 = 1 - x_1 + x_3 \end{array}\right.$$

$$\overline{\quad z = 1 + \underbrace{2x_1 - x_3}_{\text{non-basic variables}}\quad}$$



Plug in $x_1 = x_3 = 0$ to get a bfs with basis $B = \{2, 4, 5\}$ and value $z = 1$.

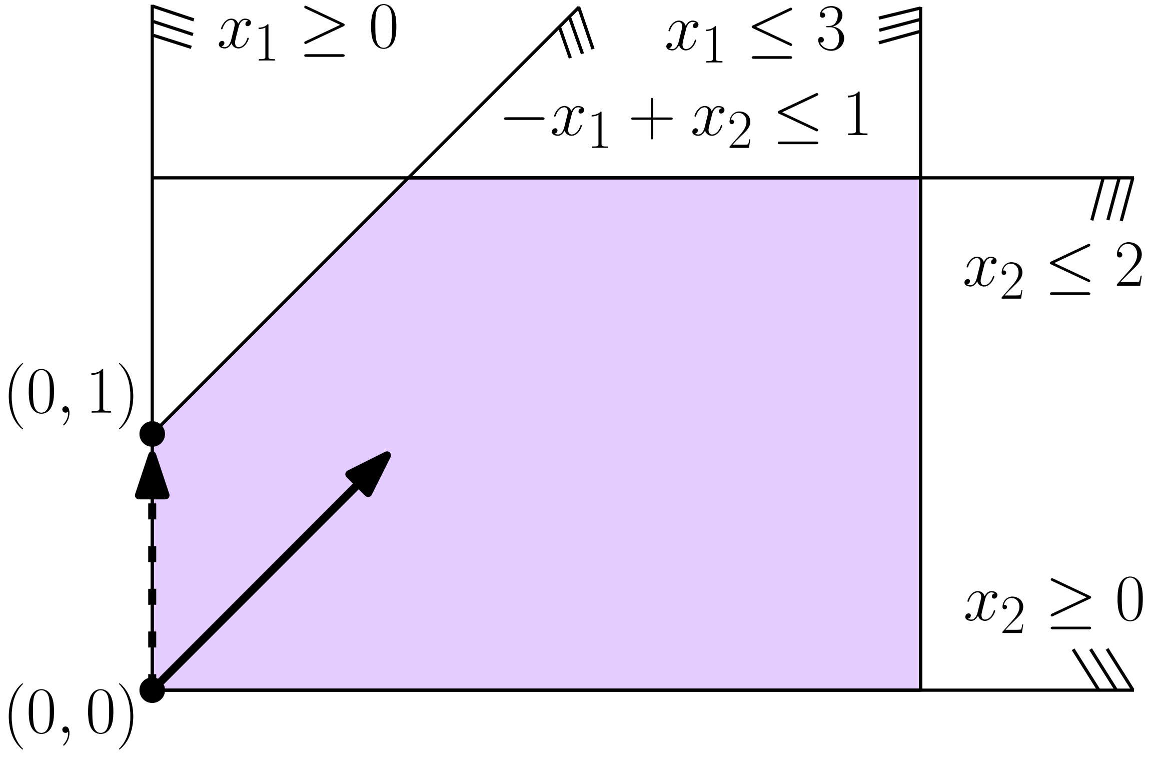
Introductory example - Simplex tableau B

Simplex tableau B

basic variables

$$\left.\begin{aligned} x_2 &= 1 + x_1 - x_3 \\ x_4 &= 3 - x_1 \\ x_5 &= 1 - x_1 + x_3 \end{aligned}\right.$$

$$z = 1 + \underbrace{2x_1 - x_3}_{\text{non-basic variables}}$$



Plug in $x_1 = x_3 = 0$ to get a bfs with basis $B = \{2, 4, 5\}$ and value $z = 1$.

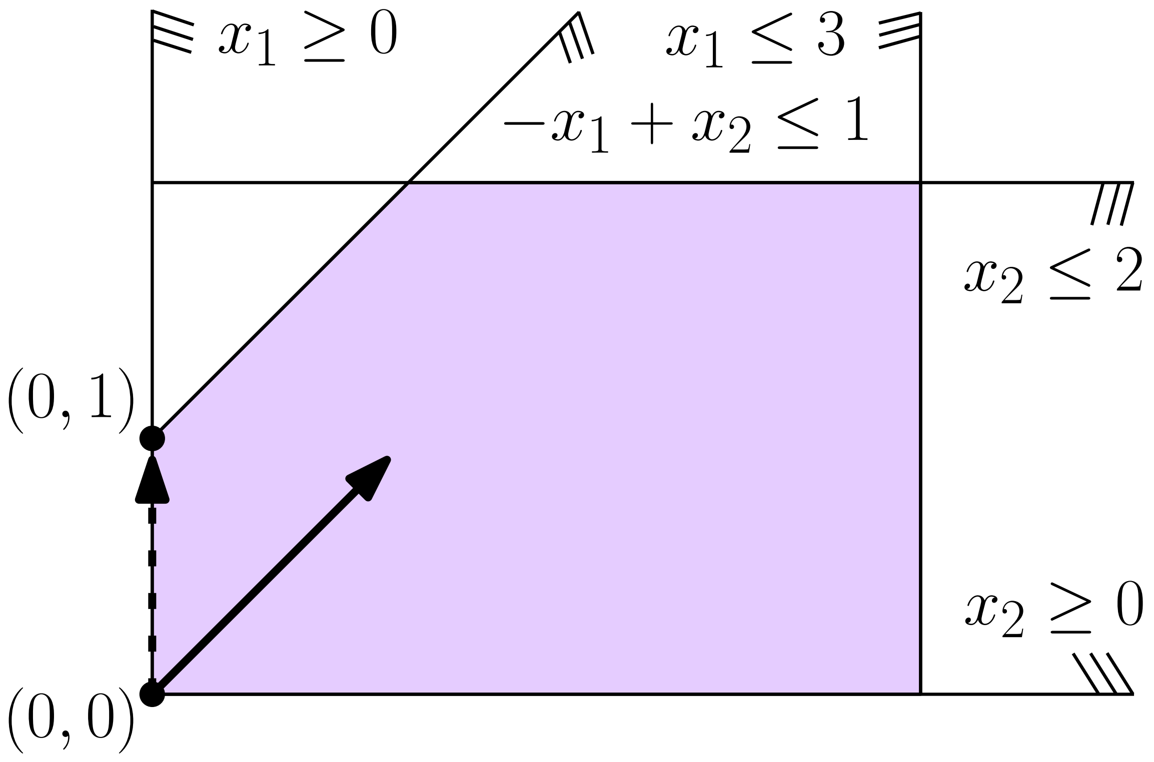
Which variable to increase next?

Introductory example - Simplex tableau B

Simplex tableau B

$$\text{basic variables}\left\{\begin{array}{l} x_2 = 1 + x_1 - x_3 \\ x_4 = 3 - x_1 \\ x_5 = 1 - x_1 + x_3 \end{array}\right.$$

$$z = 1 + \underbrace{2x_1 - x_3}_{\text{non-basic variables}}$$



Plug in $x_1 = x_3 = 0$ to get a bfs with basis $B = \{2, 4, 5\}$ and value $z = 1$.

Next increase z by increasing x_1.

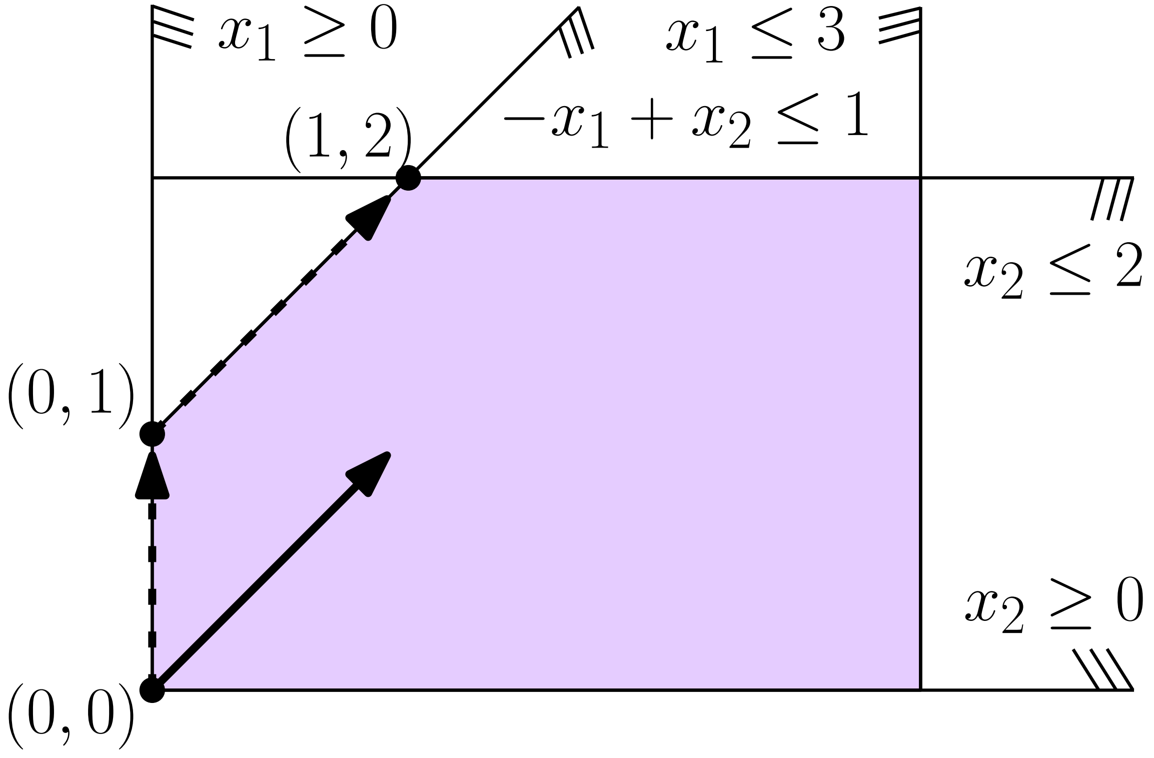
We are limited by the equation $x_5 = 1 - x_1 + x_3$,
which we can rewrite as $x_1 = 1 + x_3 - x_5$.

Introductory example - Simplex tableau C

Simplex tableau C

$$\text{basic variables}\left\{\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \end{aligned}\right.$$

$$z = 3 + \underbrace{x_3 - 2x_5}_{\text{non-basic variables}}$$



Plug in $x_3 = x_5 = 0$ to get a bfs with basis $B = \{1, 2, 4\}$ and value $z = 3$.

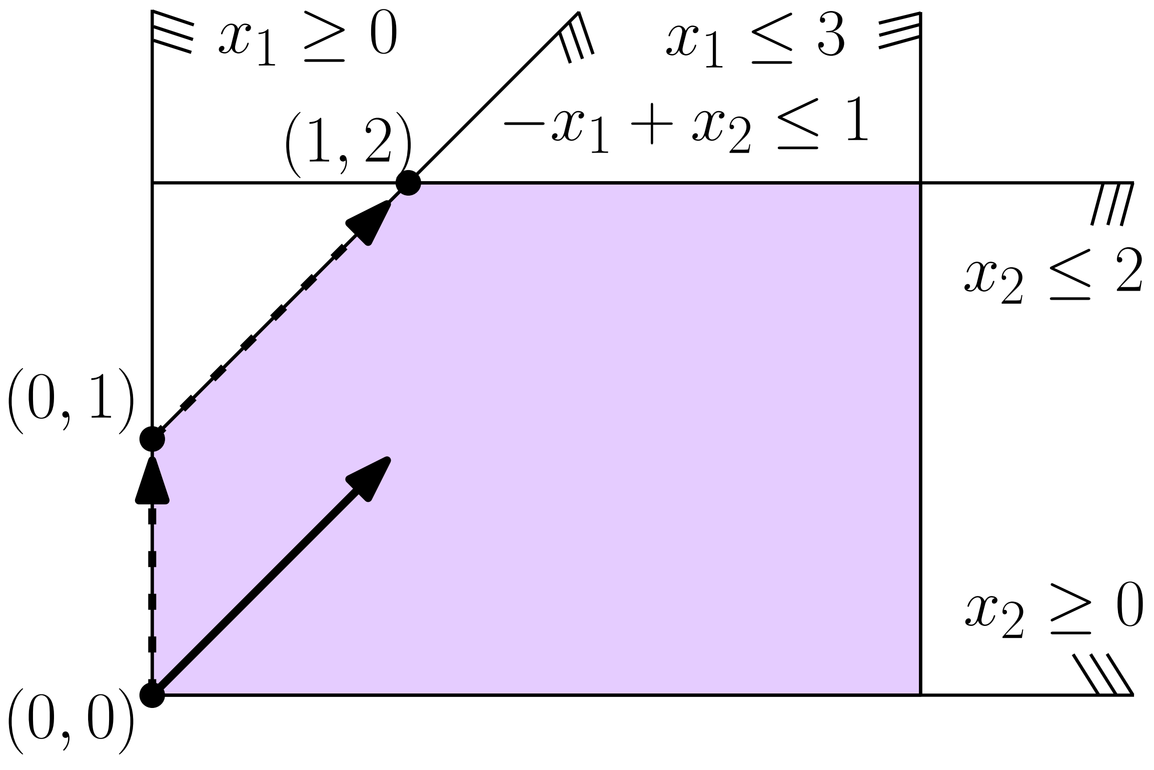
Introductory example - Simplex tableau C

Simplex tableau C

$$\text{basic variables}\left\{\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \end{aligned}\right.$$

$$\overline{}$$

$$z = 3 + \underbrace{x_3 - 2x_5}_{\text{non-basic variables}}$$



Plug in $x_3 = x_5 = 0$ to get a bfs with basis $B = \{1, 2, 4\}$ and value $z = 3$.

Increase z by increasing x_3.

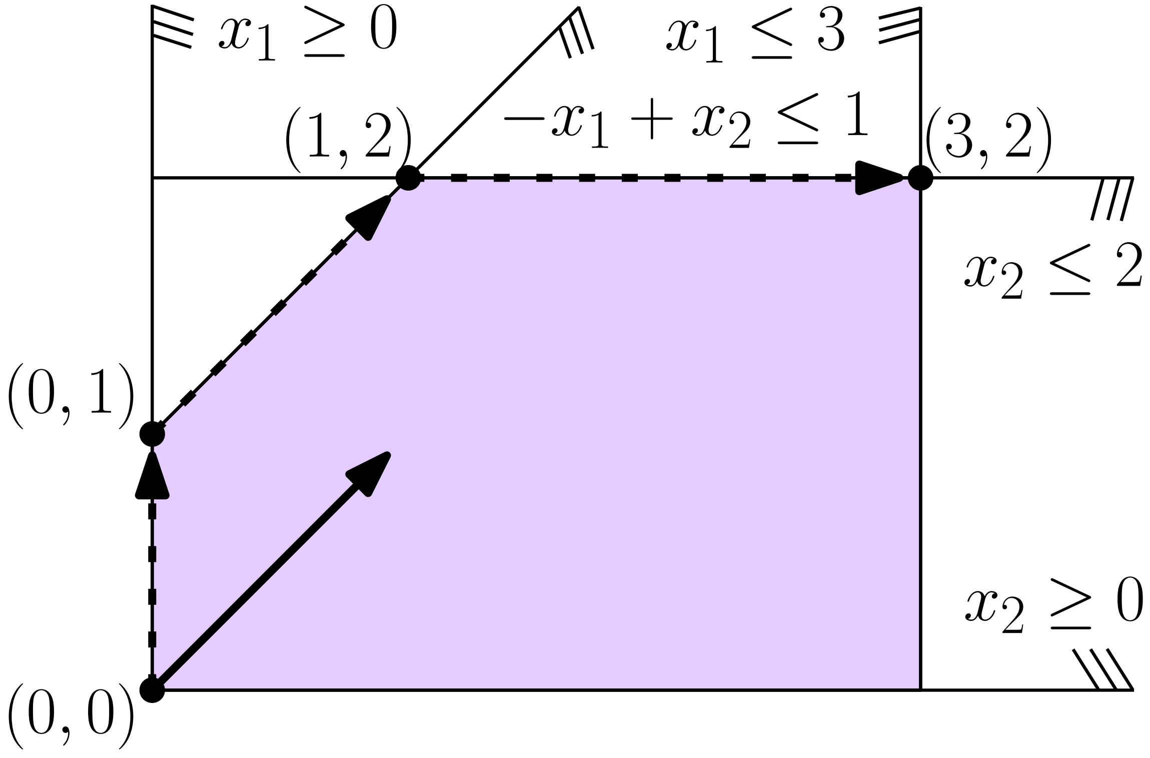
We are limited by the equation $x_4 = 2 - x_3 + x_5$,
which we can rewrite as $x_3 = 2 - x_4 + x_5$.

Introductory example - Simplex tableau D

Simplex tableau D

$$\text{basic variables}\left\{\begin{aligned} x_1 &= 3 - x_4 \\ x_2 &= 2 - x_5 \\ x_3 &= 2 - x_4 + x_5 \end{aligned}\right.$$

$$z = 5 - \underbrace{x_4 - x_5}_{\text{non-basic variables}}$$



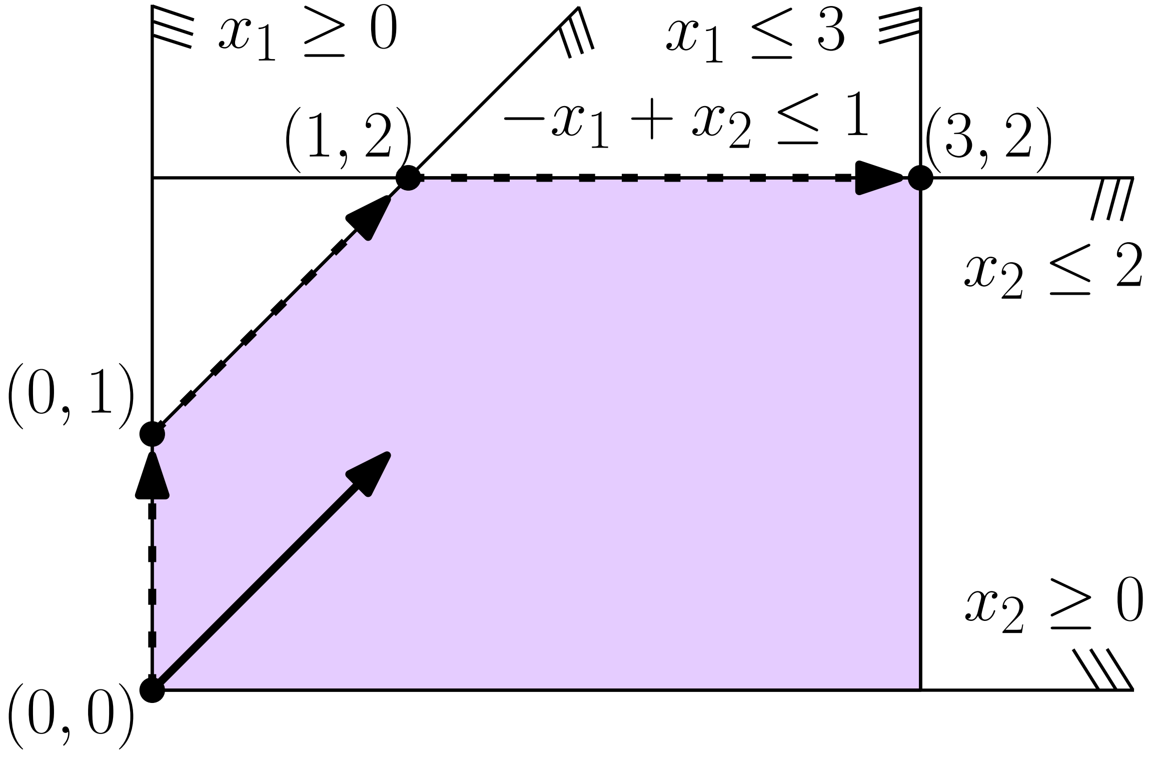
Plug in $x_4 = x_5 = 0$ to get a bfs with basis $B = \{1, 2, 3\}$ and value $z = 5$.

Introductory example - Simplex tableau D

Simplex tableau D

basic variables

$$\begin{aligned} x_1 &= 3 - x_4 \\ x_2 &= 2 - x_5 \\ x_3 &= 2 - x_4 + x_5 \\ \hline z &= 5 - \underbrace{x_4 - x_5}_{\text{non-basic variables}} \end{aligned}$$



Plug in $x_4 = x_5 = 0$ to get a bfs with basis $B = \{1,2,3\}$ and value $z = 5$.

Can we increase z further?

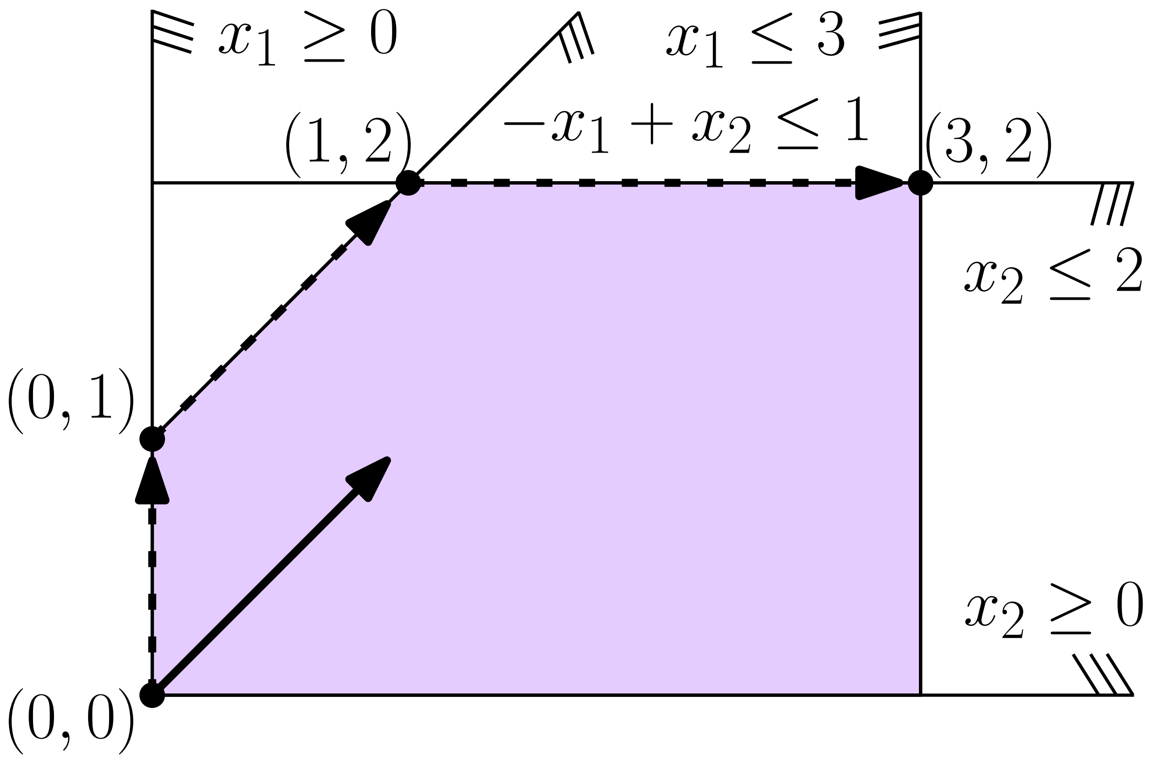
Introductory example - Simplex tableau D

Simplex tableau D

$$
\text{basic variables}\left\{
\begin{aligned}
x_1 &= 3 - x_4 \\
x_2 &= 2 - x_5 \\
x_3 &= 2 - x_4 + x_5
\end{aligned}
\right.
$$

$$
\overline{}
$$

$$
z = 5 - \underbrace{x_4 - x_5}_{\text{non-basic variables}}
$$



Plug in $x_4 = x_5 = 0$ to get a bfs with basis $B = \{1,2,3\}$ and value $z = 5$.

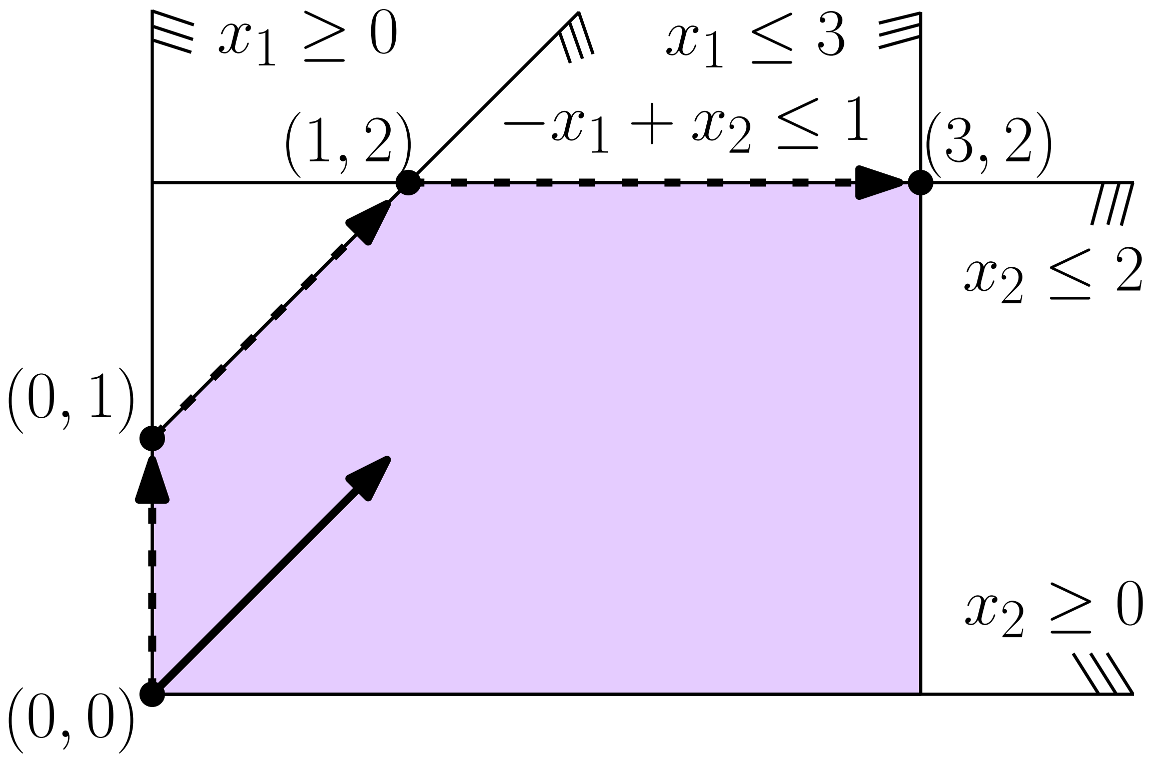
This is optimal, and moreover gives a proof of optimality, since any feasible solution satisfies $z = 5 - x_4 - x_5$ with $x_4, x_5 \geq 0$.

Introductory example - Simplex tableau D

Simplex tableau D

basic variables

$$\begin{aligned} x_1 &= 3 - x_4 \\ x_2 &= 2 - x_5 \\ x_3 &= 2 - x_4 + x_5 \\ \hline z &= 5 - \underbrace{x_4 - x_5}_{\text{non-basic variables}} \end{aligned}$$



Plug in $x_4 = x_5 = 0$ to get a bfs with basis $B = \{1, 2, 3\}$ and value $z = 5$.

This is optimal, and moreover gives a proof of optimality, since any feasible solution satisfies $z = 5 - x_4 - x_5$ with $x_4, x_5 \geq 0$.

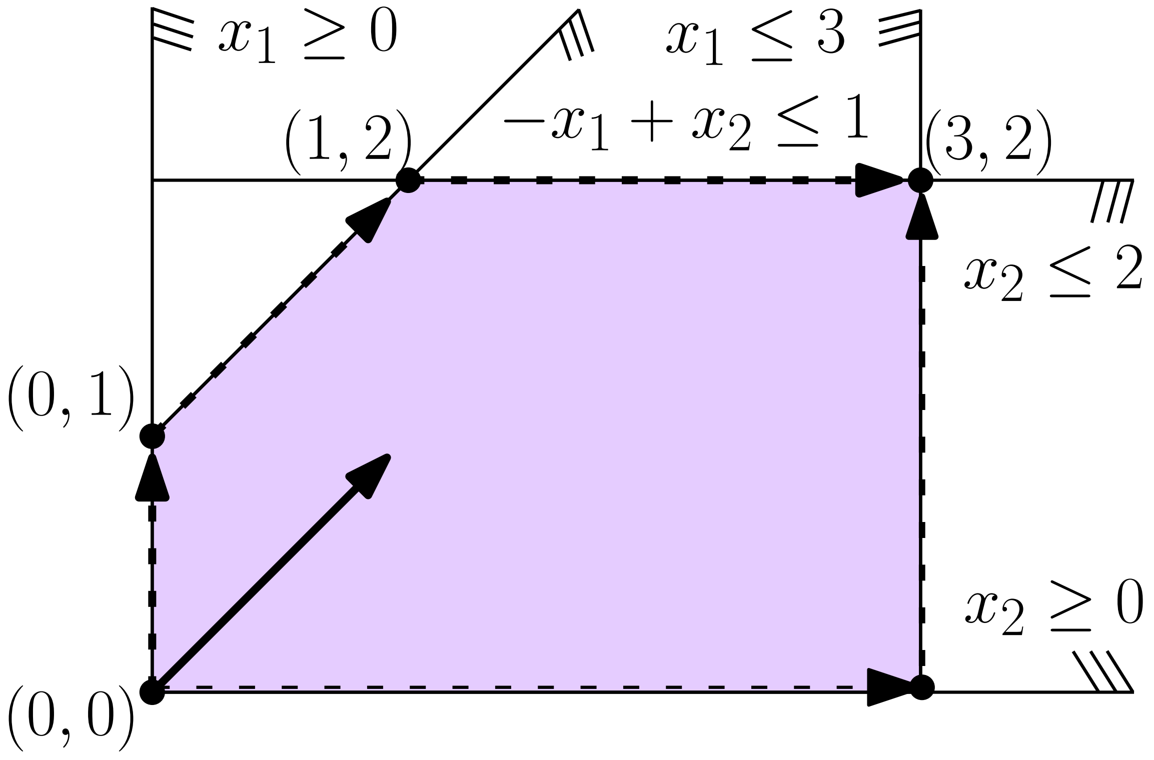
Remark: Pictures should really be in $\mathbb{R}^5$, not $\mathbb{R}^2$.

Introductory example - Simplex tableau D

Simplex tableau D

$$\text{basic variables}\left\{\begin{aligned} x_1 &= 3 - x_4 \\ x_2 &= 2 - x_5 \\ x_3 &= 2 - x_4 + x_5 \end{aligned}\right.$$

$$z = 5 - \underbrace{x_4 - x_5}_{\text{non-basic variables}}$$



Plug in $x_4 = x_5 = 0$ to get a bfs with basis $B = \{1, 2, 3\}$ and value $z = 5$.

This is optimal, and moreover gives a proof of optimality, since any feasible solution satisfies $z = 5 - x_4 - x_5$ with $x_4, x_5 \geq 0$.

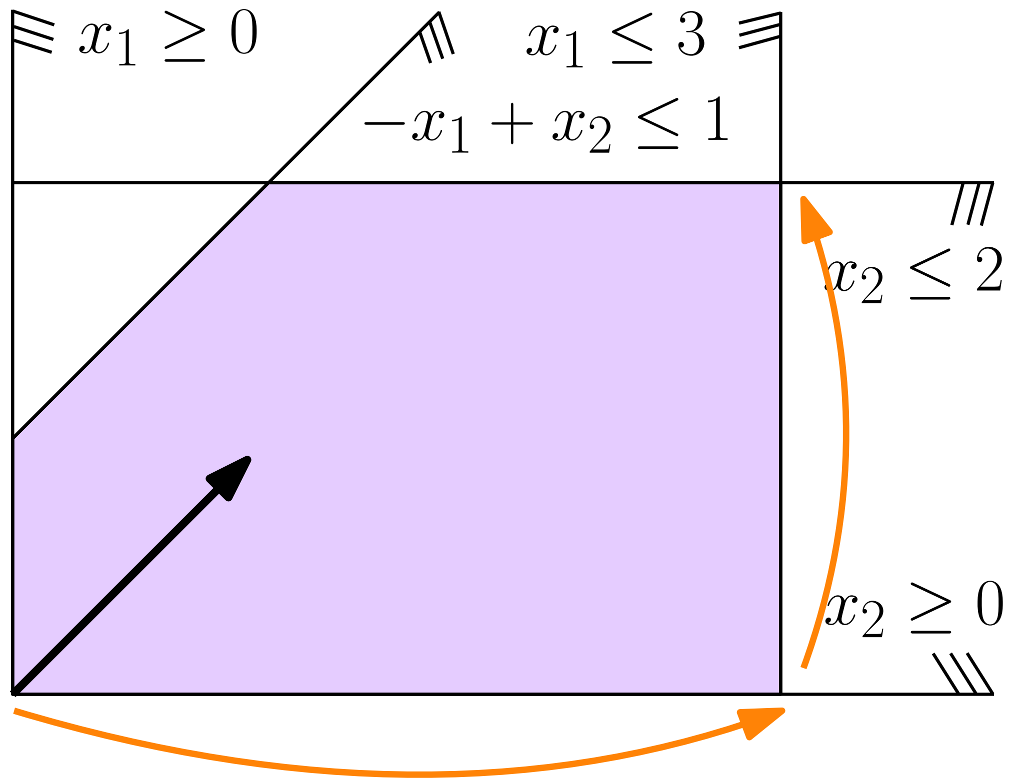
Remark: Pictures should really be in $\mathbb{R}^5$, not $\mathbb{R}^2$.

Remark: We would have needed fewer steps if we had first pivoted over x_1.

Introductory example - Question

Simplex tableau A

$$\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 \qquad\quad - x_2 \\ \hline z &= \qquad\; x_1 + x_2 \end{aligned}$$



Which Simplex tableau do we get if we increase x_1 first?

Which do we get in the second step?

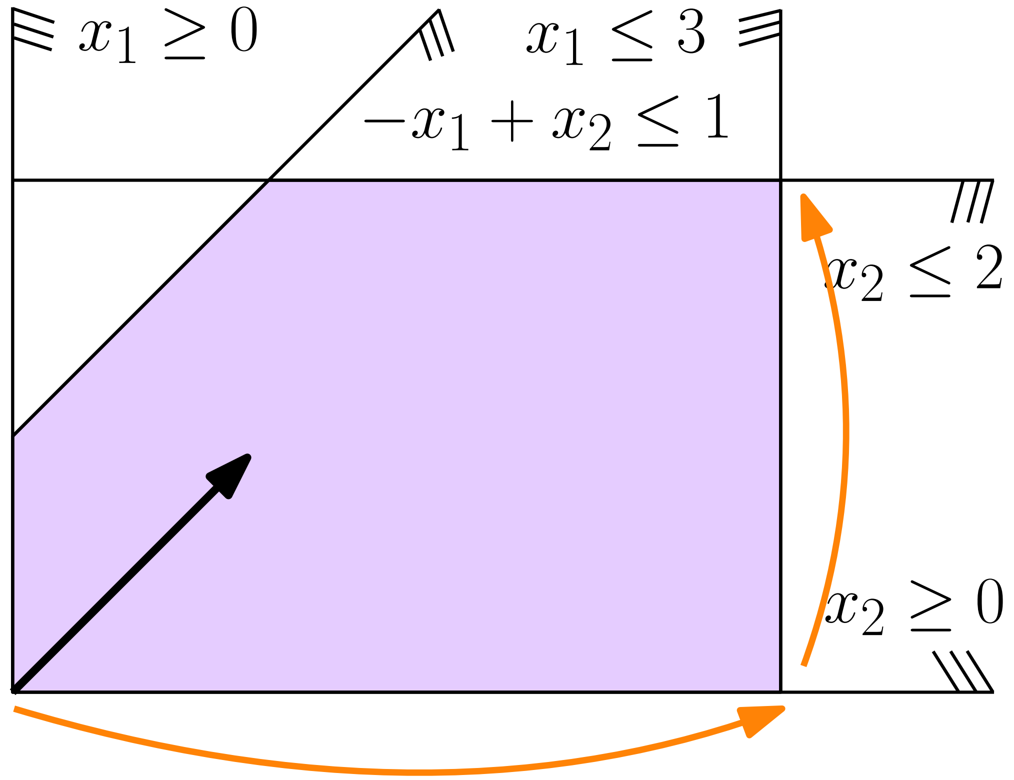
Introductory example - Question

Simplex tableau A

$$
\begin{array}{rl}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 \qquad\quad - x_2 \\
\hline
z &= \qquad\quad x_1 + x_2
\end{array}
$$

Simplex tableau B'

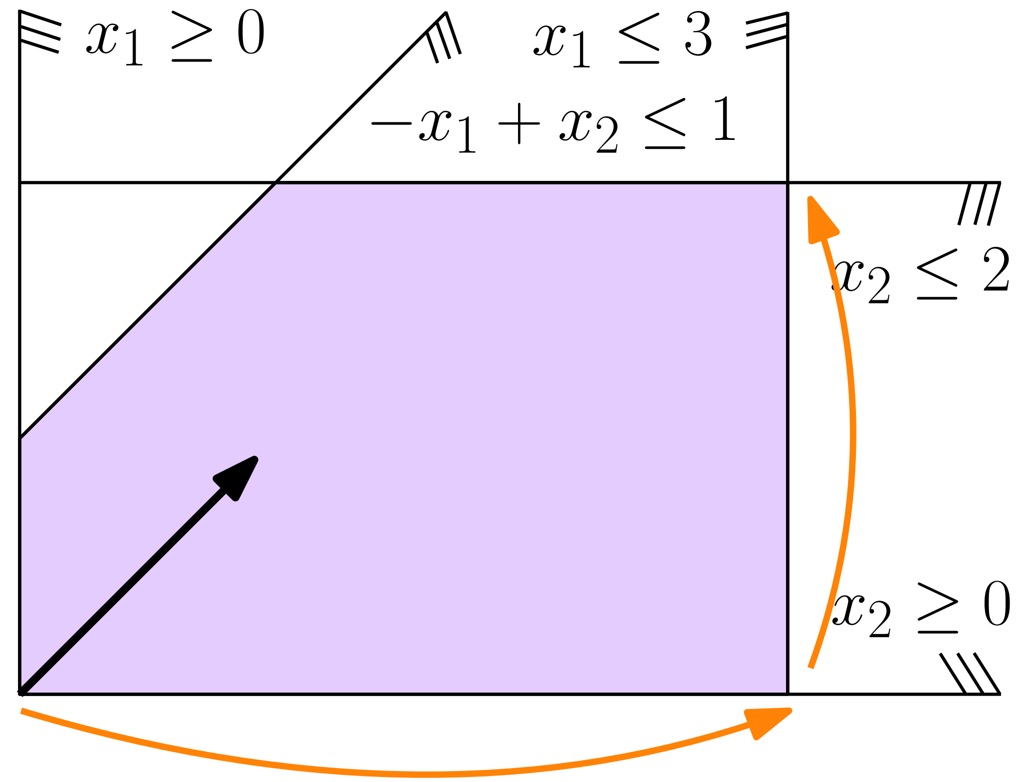
$$
\begin{array}{rl}
x_1 &= 3 - \qquad\quad x_4 \\
x_3 &= 4 - x_2 - x_4 \\
x_5 &= 2 - x_2 \\
\hline
z &= 3 + x_2 - x_4
\end{array}
$$



Introductory example - Question

Simplex tableau A

$$
\begin{aligned}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 \phantom{{}+x_1} - x_2 \\
\hline
z &= \phantom{1 +{}} x_1 + x_2
\end{aligned}
$$



Simplex tableau B'

$$
\begin{aligned}
x_1 &= 3 - \phantom{x_2 -{}} x_4 \\
x_3 &= 4 - x_2 - x_4 \\
x_5 &= 2 - x_2 \\
\hline
z &= 3 + x_2 - x_4
\end{aligned}
$$

Simplex tableau C'

$$
\begin{aligned}
x_1 &= 3 - x_4 \\
x_2 &= 2 \phantom{{}- x_4} - x_5 \\
x_3 &= 2 - x_4 + x_5 \\
\hline
z &= 5 - x_4 - x_5
\end{aligned}
$$

Exception Handling

Exception handling

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 &\le 1 \\ -x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Exception handling

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 &\le 1 \\ -x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$\xrightarrow[\text{variables}]{\text{add slack}}$

Equational form

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Exception handling

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\xrightarrow[\text{variables}]{\text{add slack}}$

Equational form

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

rewrite

Simplex tableau

$$\begin{array}{l} x_3 = 1 - x_1 + x_2 \\ x_4 = 2 + x_1 - x_2 \\ \hline z = x_1 \end{array}$$

Exception handling

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 &\le 1 \\ -x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

add slack variables

Equational form

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

rewrite

Simplex tableau

$$\begin{aligned} x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 + x_1 - x_2 \\ \hline z &= x_1 \end{aligned}$$

pivot on x_1

$$\begin{aligned} x_1 &= 1 + x_2 - x_3 \\ x_4 &= 3 - x_3 \\ \hline z &= 1 + x_2 - x_3 \end{aligned}$$

Exception handling

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

add slack variables $\longrightarrow$

Equational form

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

rewrite

Simplex tableau

$$\begin{aligned} x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 + x_1 - x_2 \\ \hline z &= x_1 \end{aligned}$$

$\xrightarrow{\text{pivot on } x_1}$

$$\begin{aligned} x_1 &= 1 + x_2 - x_3 \\ x_4 &= 3 - x_3 \\ \hline z &= 1 + x_2 - x_3 \end{aligned}$$

$\xrightarrow{\text{pivot on } x_2}$???

Exception handling: Unboundedness

Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\xrightarrow{\text{add slack variables}}$

Equational form

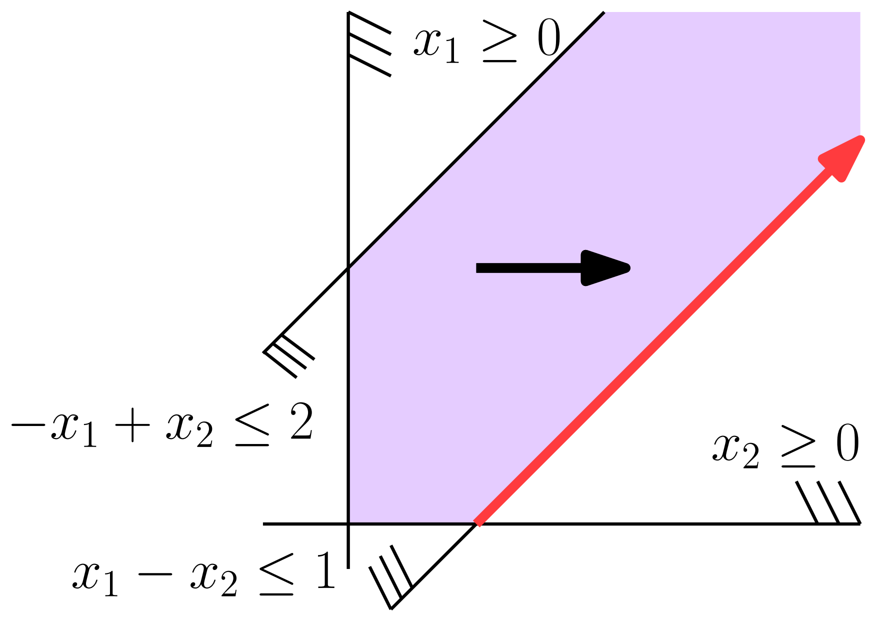
Maximize x_1

subject to:
$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

rewrite

Simplex tableau

$$\begin{array}{l} x_3 = 1 - x_1 + x_2 \\ x_4 = 2 + x_1 - x_2 \\ \hline z = x_1 \end{array} \xrightarrow{\text{pivot on } x_1} \begin{array}{l} x_1 = 1 + x_2 - x_3 \\ x_4 = 3 - x_3 \\ \hline z = 1 + x_2 - x_3 \end{array}$$



$\xrightarrow{\text{increase } x_2}$ without limit

Feasible ray $\{(1 + x_2, x_2, 0, 3) : x_2 \geq 0\}$
with unbounded objective function $1 + x_2$.

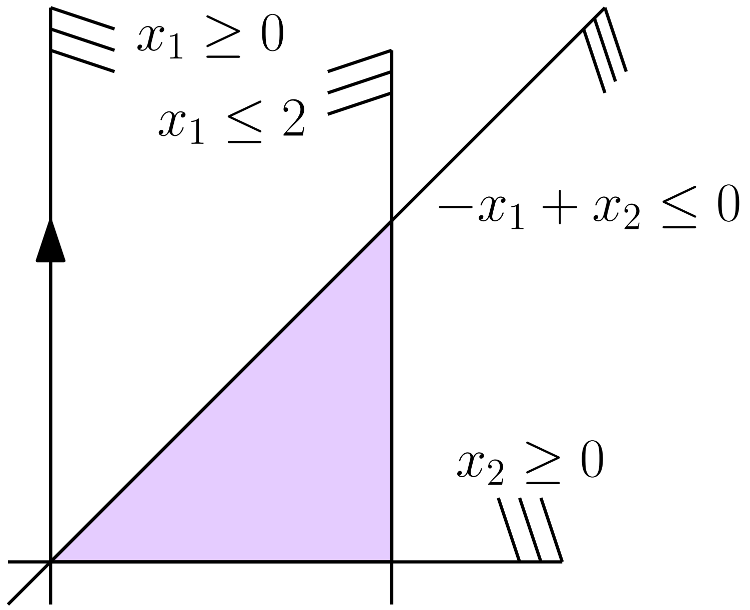
Exception handling: Degeneracy

Maximize x_2

subject to: $-x_1 + x_2 \leq 0$

$$x_1 \leq 2$$

$$x_1, x_2 \geq 0$$



Exception handling: Degeneracy

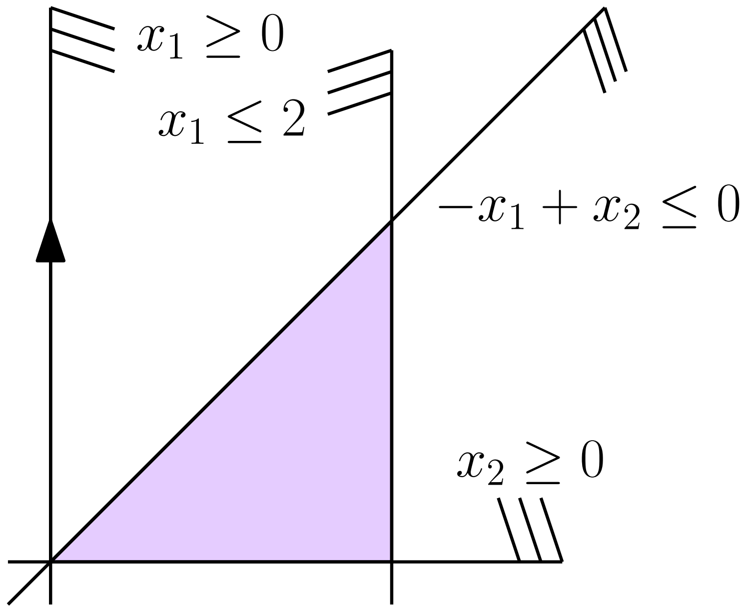
Maximize x_2

subject to:
$$\begin{aligned} -x_1 + x_2 &\leq 0 \\ x_1 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Equational form

Maximize x_2

subject to:
$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_1 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$



Exception handling: Degeneracy

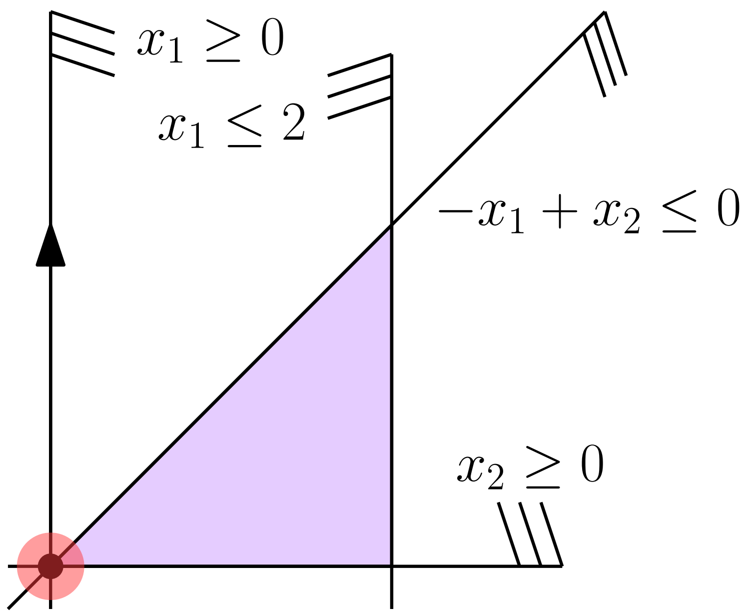
Maximize x_2

subject to:
$$\begin{aligned} -x_1 + x_2 &\leq 0 \\ x_1 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Equational form

Maximize x_2

subject to:
$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_1 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$



Simplex tableau

$$\begin{aligned} x_3 &= x_1 - x_2 \\ x_4 &= 2 - x_1 \\ \hline z &= x_2 \end{aligned}$$

Feasible solution: $(0, 0, 0, 2)$

Exception handling: Degeneracy

Maximize x_2

subject to: $-x_1 + x_2 \leq 0$

$x_1 \leq 2$

$x_1, x_2 \geq 0$

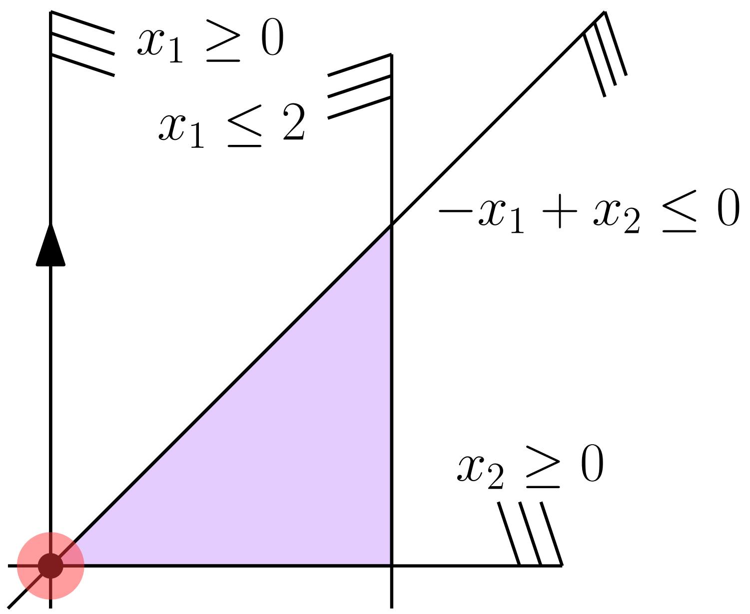
Equational form

Maximize x_2

subject to: $-x_1 + x_2 + x_3 = 0$

$x_1 + x_4 = 2$

$x_1, x_2, x_3, x_4 \geq 0$



Simplex tableau

$$\begin{array}{c} x_3 = x_1 - x_2 \\ x_4 = 2 - x_1 \\ \hline z = x_2 \end{array} \xrightarrow{\text{pivot on } x_2} \begin{array}{c} x_2 = x_1 - x_3 \\ x_4 = 2 - x_1 \\ \hline z = x_1 - x_3 \end{array}$$

Feasible solution: $(0, 0, 0, 2)$ → Feasible solution: $(0, 0, 0, 2)$

Exception handling: Degeneracy

Maximize x_2

subject to: $-x_1 + x_2 \leq 0$

$x_1 \leq 2$

$x_1, x_2 \geq 0$

Equational form

Maximize x_2

subject to: $-x_1 + x_2 + x_3 = 0$

$x_1 + x_4 = 2$

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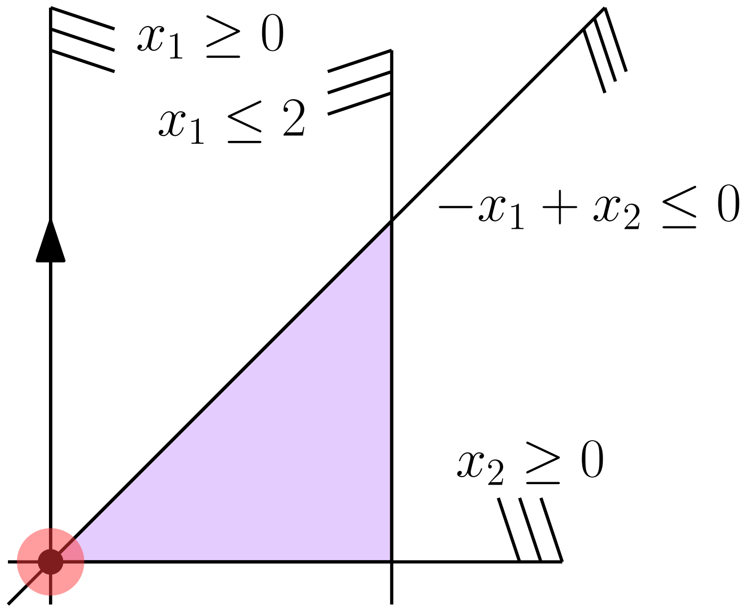


Simplex tableau

$$\begin{aligned} x_3 &= x_1 - x_2 \\ x_4 &= 2 - x_1 \\ \hline z &= x_2 \end{aligned} \xrightarrow{\text{pivot on } x_2} \begin{aligned} x_2 &= x_1 - x_3 \\ x_4 &= 2 - x_1 \\ \hline z &= x_1 - x_3 \end{aligned}$$

Feasible solution: $(0, 0, 0, 2) \rightarrow$ Feasible solution: $(0, 0, 0, 2)$

note: we changed basis for the same bfs. Can we get stuck?

Exception handling: Degeneracy

Maximize x_2

subject to: $-x_1 + x_2 \leq 0$

$x_1 \leq 2$

$x_1, x_2 \geq 0$

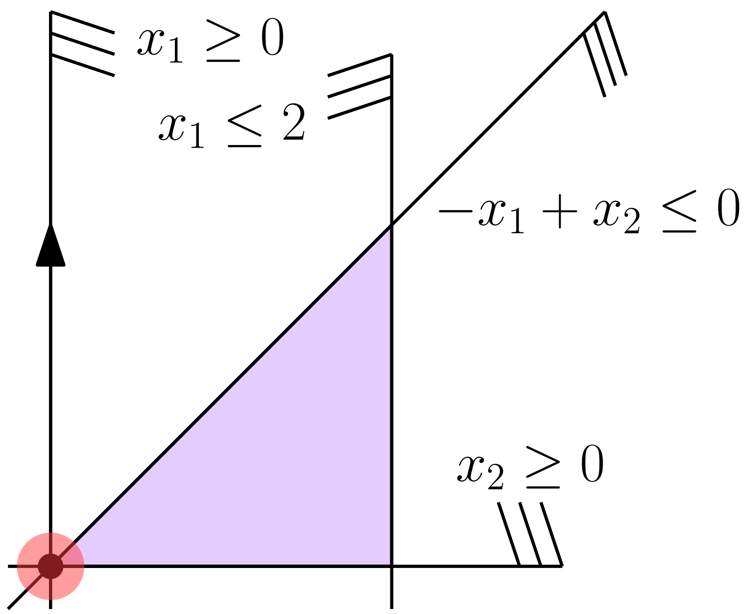
Equational form

Maximize x_2

subject to: $-x_1 + x_2 + x_3 = 0$

$x_1 + x_4 = 2$

$x_1, x_2, x_3, x_4 \geq 0$



Simplex tableau

$$\begin{array}{l} x_3 = x_1 - x_2 \\ x_4 = 2 - x_1 \\ \hline z = x_2 \end{array} \xrightarrow{\text{pivot on } x_2} \begin{array}{l} x_2 = x_1 - x_3 \\ x_4 = 2 - x_1 \\ \hline z = x_1 - x_3 \end{array}$$

Feasible solution: $(0, 0, 0, 2) \rightarrow$ Feasible solution: $(0, 0, 0, 2)$

note: we changed basis for the same bfs. There are ways to prevent cycling.

Exception handling: Degeneracy

Maximize x_2

subject to: $-x_1 + x_2 \leq 0$

$$x_1 \leq 2$$

$$x_1, x_2 \geq 0$$

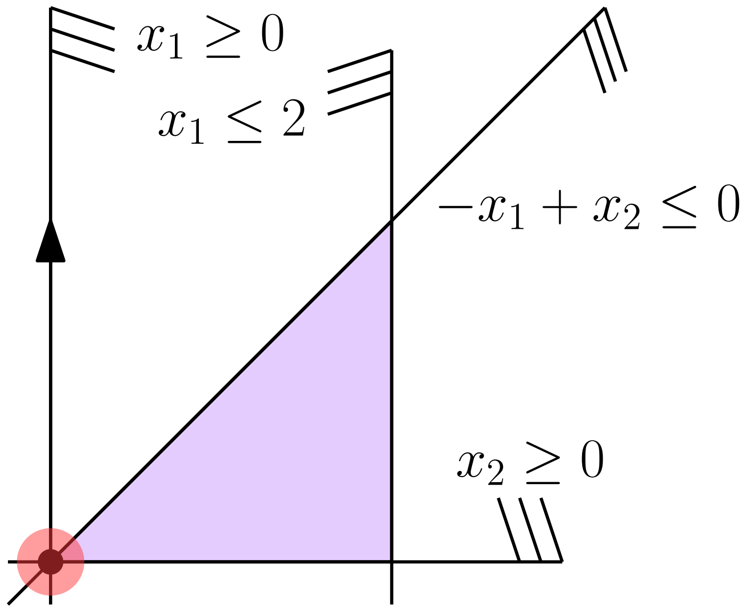
Equational form

Maximize x_2

subject to: $-x_1 + x_2 + x_3 = 0$

$$x_1 + x_4 = 2$$

$$x_1, x_2, x_3, x_4 \geq 0$$



Simplex tableau

$$\begin{array}{l} x_3 = x_1 - x_2 \\ x_4 = 2 - x_1 \\ \hline z = x_2 \end{array} \xrightarrow{\text{pivot on } x_2} \begin{array}{l} x_2 = x_1 - x_3 \\ x_4 = 2 - x_1 \\ \hline z = x_1 - x_3 \end{array} \xrightarrow{\text{pivot on } x_1} \begin{array}{l} x_1 = 2 - x_4 \\ x_2 = 2 - x_3 - x_4 \\ \hline z = 2 - x_3 - x_4 \end{array}$$

Feasible solution: $(0, 0, 0, 2)$ → Feasible solution: $(0, 0, 0, 2)$ → Optimal solution: $(2, 2, 0, 0)$

note: we changed basis for the same bfs. There are ways to prevent cycling.

Exception handling: Infeasibility of initial basis

We get a feasible basis for free in

Maximize $c^T x$ subject to $Ax \leq b,\ x \geq 0, b \geq 0$

by adding slack variables to get to equational form and then letting the basis be the slack variables.

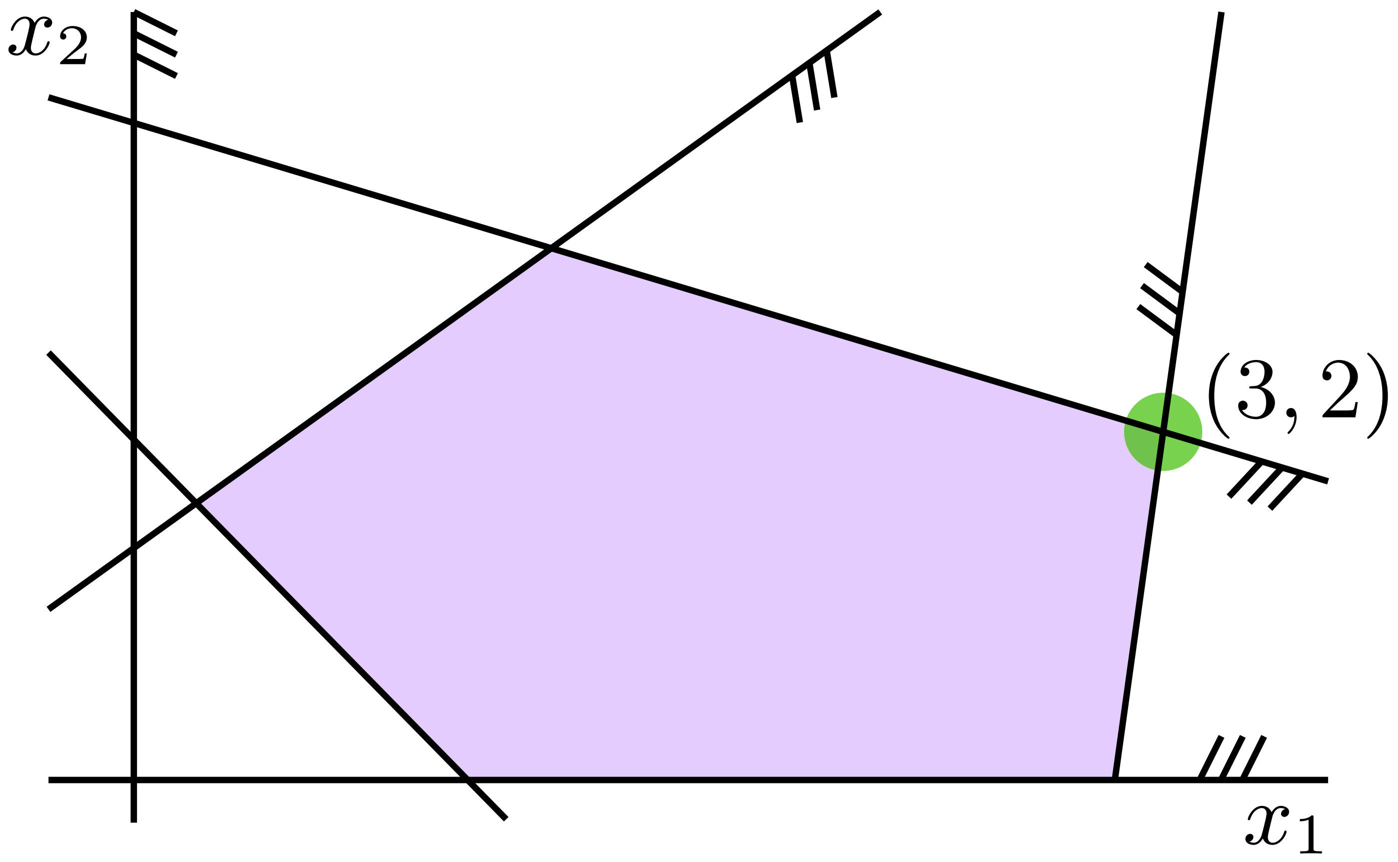
Exception handling: Infeasibility of initial basis

We get a feasible basis for free in

Maximize $c^T x$ subject to $Ax \leq b,\ x \geq 0, b \geq 0$

by adding slack variables to get to equational form and then letting the basis be the slack variables.

But in general, we might first need to find a feasible solution!



Maximize $x_1 + 2x_2$

subject to: $x_1 + 3x_2 + x_3 = 4$

$$2x_2 + x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Note: $(x_1, x_2, x_3) = (0,0,0)$ is not feasible.

Exception handling: Infeasibility of initial basis

Solution: Auxilliary problem with auxilliary variables to find a feasible solution via simplex

Maximize $-x_4 - x_5$

subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 4 \\ 2x_2 + x_3 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

The objective value is $0 \iff$ there is a feasible solution to the original problem.

Maximize $x_1 + 2x_2$

subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Exception handling: Infeasibility of initial basis

Solution: Auxilliary problem with auxilliary variables to find a feasible solution via simplex

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subject to: $$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 4 \\ 2x_2 + x_3 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

The objective value is $0 \iff$ there is a feasible solution to the original problem.

Simplex tableau

$$\begin{aligned} x_4 &= 4 - x_1 - 3x_2 - x_3 \\ x_5 &= 2 - 2x_2 - x_3 \\ \hline z &= -6 + x_1 + 5x_2 + 2x_3 \end{aligned}$$

Exception handling: Infeasibility of initial basis

Solution: Auxilliary problem with auxilliary variables to find a feasible solution via simplex

Maximize $-x_4 - x_5$
subject to: $x_1 + 3x_2 + x_3 + x_4 = 4$

$$2x_2 + x_3 + x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The objective value is $0 \iff$ there is a feasible solution to the original problem.

Simplex tableau

$$\begin{array}{c}
x_4 = 4 - x_1 - 3x_2 - x_3 \\
x_5 = 2 - 2x_2 - x_3 \\
\hline
z = -6 + x_1 + 5x_2 + 2x_3
\end{array}
\xrightarrow[\text{pivot on } x_1]{}
\begin{array}{c}
x_1 = 4 - 3x_2 - x_3 - x_4 \\
x_5 = 2 - 2x_2 - x_3 \\
\hline
z = -2 + 2x_2 + x_3 - x_4
\end{array}$$

Exception handling: Infeasibility of initial basis

Solution: Auxilliary problem with auxilliary variables to find a feasible solution via simplex

?

Simplex tableau

$$\begin{aligned} x_4 &= 4 - x_1 - 3x_2 - x_3 \\ x_5 &= 2 - 2x_2 - x_3 \\ \hline z &= -6 + x_1 + 5x_2 + 2x_3 \end{aligned} \xrightarrow[\text{pivot on } x_1]{} \begin{aligned} x_1 &= 4 - 3x_2 - x_3 - x_4 \\ x_5 &= 2 - 2x_2 - x_3 \\ \hline z &= -2 + 2x_2 + x_3 - x_4 \end{aligned}$$

Exception handling: Infeasibility of initial basis

Solution: Auxilliary problem with auxilliary variables to find a feasible solution via simplex

Simplex tableau

$$\begin{array}{l} x_4 = 4 - x_1 - 3x_2 - x_3 \\ x_5 = 2 - 2x_2 - x_3 \\ \hline z = -6 + x_1 + 5x_2 + 2x_3 \end{array}$$

pivot on x_1

$$\begin{array}{l} x_1 = 4 - 3x_2 - x_3 - x_4 \\ x_5 = 2 - 2x_2 - x_3 \\ \hline z = -2 + 2x_2 + x_3 - x_4 \end{array}$$

pivot on x_3

$$\begin{array}{l} x_1 = 2 - x_2 - x_4 + x_5 \\ x_3 = 2 - 2x_2 - x_5 \\ \hline z = -x_4 - x_5 \end{array}$$

Exception handling: Infeasibility of initial basis

Solution: Auxilliary problem with auxilliary variables to find a feasible solution via simplex

Auxilliary optimal solution $(2,0,2,0,0)$ yields the basic feasible solution $(2,0,2)$ of the original problem.

$$\begin{array}{l} x_1 = 2 - x_2 - x_4 + x_5 \\ x_3 = 2 - 2x_2 - x_5 \\ \hline z = -x_4 - x_5 \end{array}$$

$\uparrow$ pivot on x_3

Simplex tableau

$$\begin{array}{l} x_4 = 4 - x_1 - 3x_2 - x_3 \\ x_5 = 2 - 2x_2 - x_3 \\ \hline z = -6 + x_1 + 5x_2 + 2x_3 \end{array} \xrightarrow{\text{pivot on } x_1} \begin{array}{l} x_1 = 4 - 3x_2 - x_3 - x_4 \\ x_5 = 2 - 2x_2 - x_3 \\ \hline z = -2 + 2x_2 + x_3 - x_4 \end{array}$$

Exception handling: Infeasibility of initial basis

Back to the original problem:

Maximize $x_1 + 2x_2$

subject to: $x_1 + 3x_2 + x_3 = 4$

$$2x_2 + x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Simplex tableau

$$\begin{array}{rl} x_1 &= 2 - x_2 \\ x_3 &= 2 - 2x_2 \\ \hline z &= 2 + x_2 \end{array} \xrightarrow{?}$$

Exception handling: Infeasibility of initial basis

Back to the original problem:

Maximize $x_1 + 2x_2$

subject to: $x_1 + 3x_2 + x_3 = 4$

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$$x_1, x_2, x_3 \geq 0$$

Simplex tableau

$$\begin{array}{rl} x_1 &= 2 - x_2 \\ x_3 &= 2 - 2x_2 \\ \hline z &= 2 + x_2 \end{array} \xrightarrow{\text{pivot on } x_2} \begin{array}{rl} x_1 &= 1 + \frac{1}{2}x_3 \\ x_2 &= 1 - \frac{1}{2}x_3 \\ \hline z &= 3 - \frac{1}{2}x_3 \end{array}$$

Exception handling: Infeasibility of initial basis

Back to the original problem:

Maximize $x_1 + 2x_2$

subject to: $x_1 + 3x_2 + x_3 = 4$

$$2x_2 + x_3 = 2$$

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Simplex tableau

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Optimal solution: $(1, 1, 0)$ with value 3.

the simplex algorithm in general

Simplex tableaus in general

Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, with A of size $m \times n$.

Recall: a feasible basis is a m-element set $B \subseteq \{1, 2, ..., n\}$
with A_B nonsingular and the (unique) solution $A_B x_B = b$ nonnegative.

Example: Maximize $x_1 + x_2$

$$\begin{aligned} \text{subject to: } -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$c^T = (1, 1, 0, 0, 0), \;\; b^T = (1, 3, 2)$$

Simplex tableaus in general

Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, with A of size $m \times n$.

Recall: a feasible basis is a m-element set $B \subseteq \{1, 2, \ldots, n\}$
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$$c^T = (1, 1, 0, 0, 0), \;\; b^T = (1, 3, 2)$$

Def.: The simplex tableau determined by the feasible basis B is
$$x_B = p + Q x_N$$
$$z = z_0 + r^T x_N$$
where x_B is the vector of basic variables , $N = \{1, 2, \ldots, n\} \setminus B$,
x_N is the vector of nonbasic variables,
$p \in \mathbb{R}^m, r \in \mathbb{R}^{n-m}$, Q $m \times (n-m)$ matrix, $z_0 \in \mathbb{R}$.

Simplex tableaus in general

Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, with A of size $m \times n$.

Recall: a feasible basis is a m-element set $B \subseteq \{1, 2, \ldots, n\}$
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Example: Maximize $x_1 + x_2$

subject to:
$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

$$\begin{array}{rcl} x_3 &=& 1 + x_1 - x_2 \\ x_4 &=& 3 - x_1 \\ x_5 &=& 2 \qquad\;\; - x_2 \\ \hline z &=& x_1 + x_2 \end{array}$$

Def.: The simplex tableau determined by the feasible basis B is
$$x_B = p + Q x_N$$
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Simplex tableaus in general

Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, with A of size $m \times n$.

Recall: a feasible basis is a m-element set $B \subseteq \{1, 2, \ldots, n\}$
with A_B nonsingular and the (unique) solution $A_B x_B = b$ nonnegative.

Example:

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$z = 0 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 \qquad\; - x_2 \\ \hline z &= x_1 + x_2 \end{aligned}$$

Def.: The simplex tableau determined by the feasible basis B is

$$x_B = p + Qx_N$$
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where x_B is the vector of basic variables , $N = \{1, 2, \ldots, n\} \setminus B$,
x_N is the vector of nonbasic variables,
$p \in \mathbb{R}^m, r \in \mathbb{R}^{n-m}$, Q $m \times (n - m)$ matrix, $z_0 \in \mathbb{R}$.

Simplex tableaus in general

Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, with A of size $m \times n$.

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$$z = 0 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in general?

$$\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 \qquad\;\; - x_2 \\ \hline z &= x_1 + x_2 \end{aligned}$$

Def.: The simplex tableau determined by the feasible basis B is

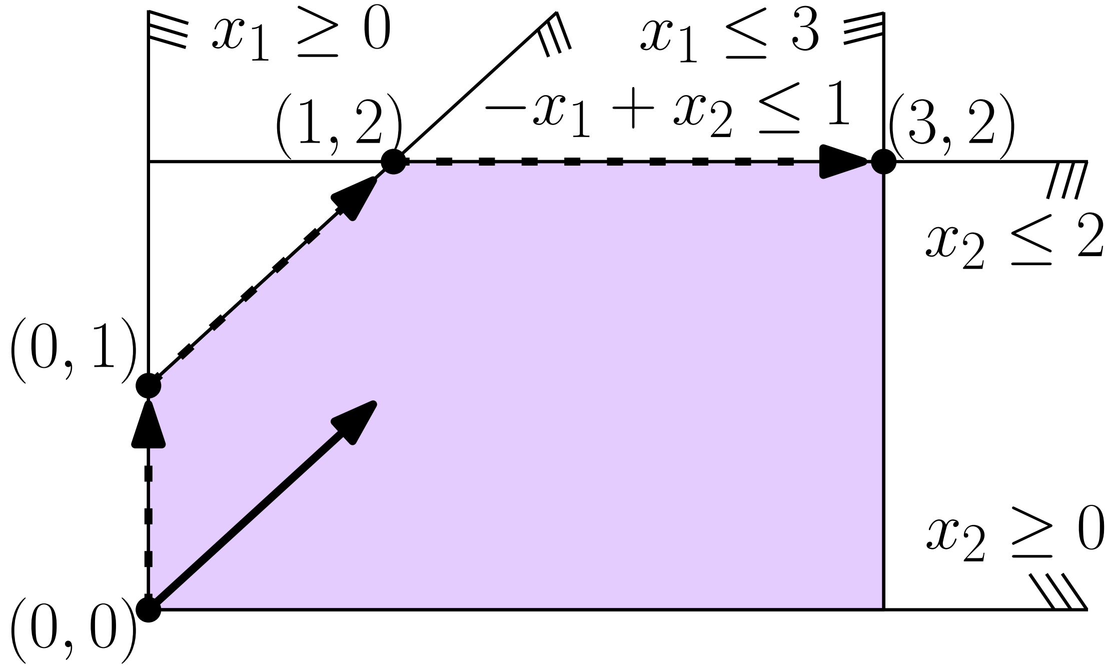
$$x_B = p + Q x_N$$
$$z = z_0 + r^T x_N$$

where x_B is the vector of basic variables , $N = \{1, 2, \ldots, n\} \setminus B$,
x_N is the vector of nonbasic variables,
$p \in \mathbb{R}^m$, $r \in \mathbb{R}^{n-m}$, Q $m \times (n-m)$ matrix, $z_0 \in \mathbb{R}$.

Simplex tableaus in general

Simplex tableau

$$\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \\ \hline z &= 3 + x_3 - 2x_5 \end{aligned}$$



Lemma 5.5.1:

$$Q = -A_B^{-1}A_N, \; p = A_B^{-1}b, \;\; z_0 = c_B^T A_B^{-1} b, \;\; r = c_N - (c_B{}^T A_B^{-1} A_N)^T$$

Remark: Don’t memorize these formulas, just know they exist and depend on A_B^{-1}.

Simplex tableaus in general

Lemma 5.5.1:
$Q = -A_B^{-1} A_N,\ p = A_B^{-1} b,\ \ z_0 = c_B^T A_B^{-1} b,\ \ r = c_N - ({c_B}^T A_B^{-1} A_N)^T$

Simplex tableaus in general

Lemma 5.5.1:

$Q = -A_B^{-1}A_N,\ \ p = A_B^{-1}b,\ \ z_0 = c_B^T A_B^{-1}b,\ \ r = c_N - ({c_B}^T A_B^{-1}A_N)^T$

$$\begin{pmatrix} x_B = p + Qx_N \\ z = z_0 + r^T x_N \end{pmatrix}$$

Proof:

Rewrite $Ax = b$ as $A_B x_B + A_N x_N = b$, or $A_B x_B = b - A_N x_N$,
giving $x_B = A_B^{-1}(b - A_N x_N)$.

Simplex tableaus in general

Lemma 5.5.1:

$Q = -A_B^{-1}A_N,\ p = A_B^{-1}b,\ \ z_0 = c_B^T A_B^{-1}b,\ \ r = c_N - ({c_B}^T A_B^{-1}A_N)^T$

Proof:

$$\begin{pmatrix} x_B = p + Qx_N \\ z = z_0 + r^T x_N \end{pmatrix}$$

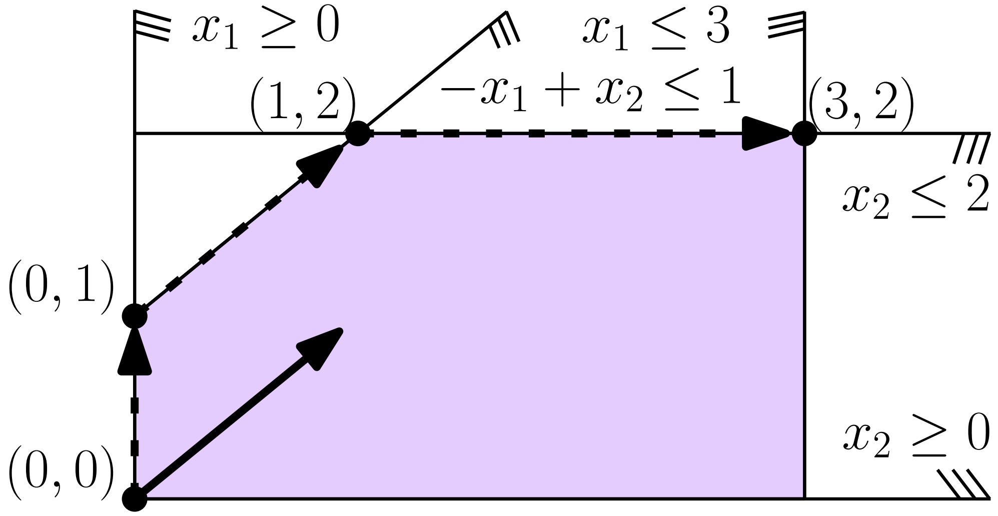
Rewrite $Ax = b$ as $A_B x_B + A_N x_N = b$, or $A_B x_B = b - A_N x_N$,
giving $x_B = A_B^{-1}(b - A_N x_N)$.

$$\begin{aligned} z = c^T x &= c_B^T x_B + c_N^T x_N \\ &= c_B^T(A_B^{-1}(b - A_N x_N)) + c_N^T x_N \\ &= c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N. \end{aligned}$$

Simplex tableaus in general

Simplex tableau

$$\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \\ \hline z &= 3 + x_3 - 2x_5 \end{aligned}$$



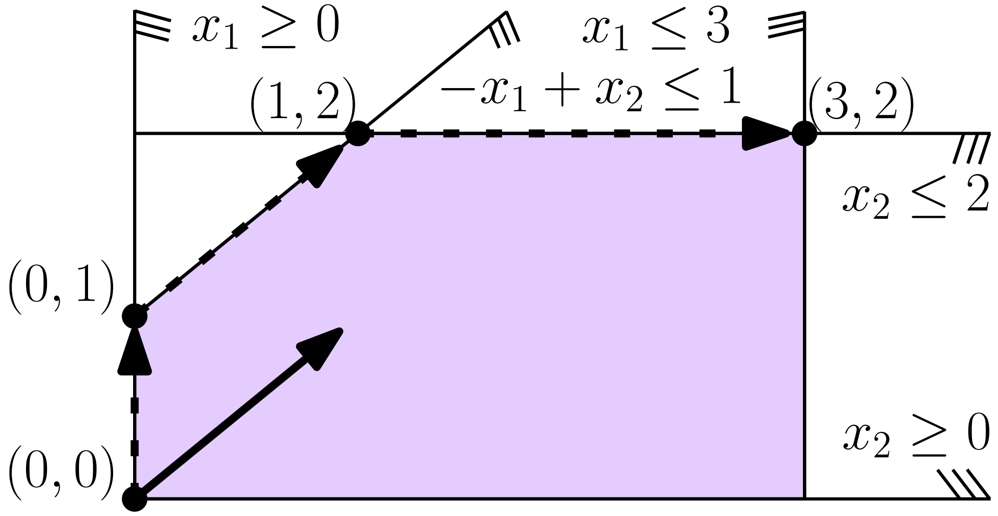
$$\begin{pmatrix} x_B = p + Qx_N \\ z = z_0 + r^T x_N \end{pmatrix}$$

Remark: If $r \leq 0$ then the corresponding bfs is ?

Simplex tableaus in general

Simplex tableau

$$\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \\ \hline z &= 3 + x_3 - 2x_5 \end{aligned}$$



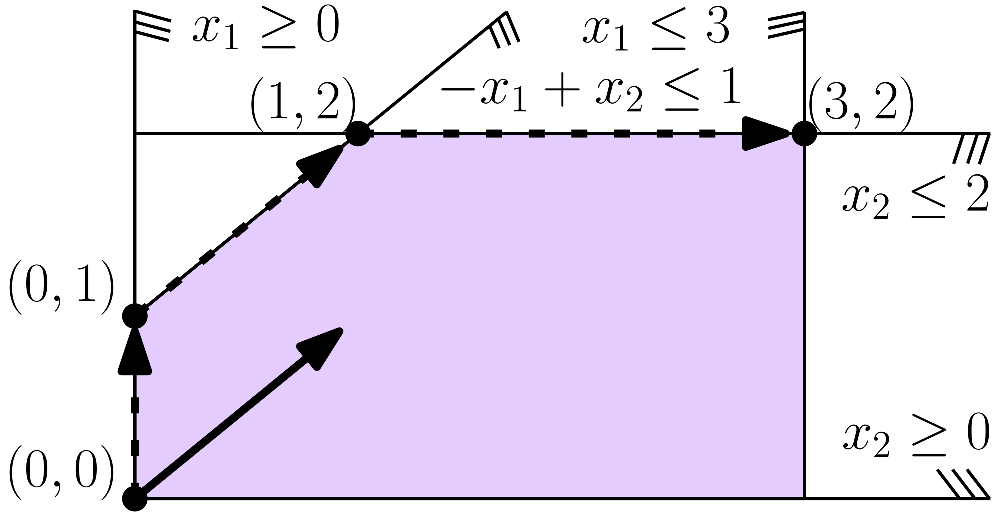
$$\begin{pmatrix} x_B = p + Qx_N \\ z = z_0 + r^T x_N \end{pmatrix}$$

Remark: If $r \leq 0$ then the corresponding bfs is optimal.

Simplex tableaus in general

Simplex tableau

$$\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \\ \hline z &= 3 + x_3 - 2x_5 \end{aligned}$$



$$\begin{pmatrix} x_B = p + Qx_N \\ z = z_0 + r^T x_N \end{pmatrix}$$

Remark: If $r \leq 0$ then the corresponding bfs is optimal.

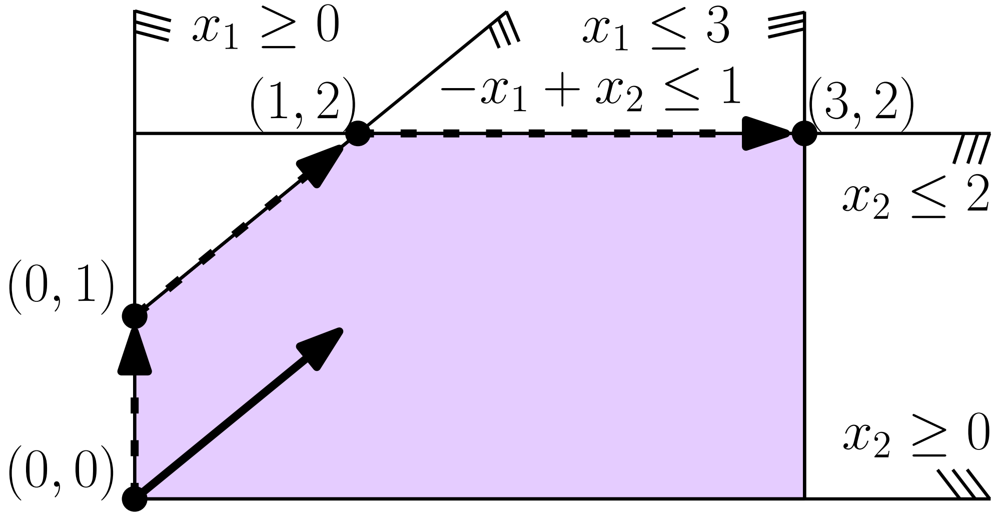
Remark: A nonbasic variable x_v may enter the basis

$\iff$ its coefficient in r is ?

Simplex tableaus in general

Simplex tableau

$$\begin{array}{rl} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \\ \hline z &= 3 + x_3 - 2x_5 \end{array}$$



$$\begin{pmatrix} x_B = p + Qx_N \\ z = z_0 + r^T x_N \end{pmatrix}$$

Remark: If $r \leq 0$ then the corresponding bfs is optimal.

Remark: A nonbasic variable x_v may enter the basis

$\iff$ its coefficient in r is positive.

Simplex tableaus in general

Simplex tableau

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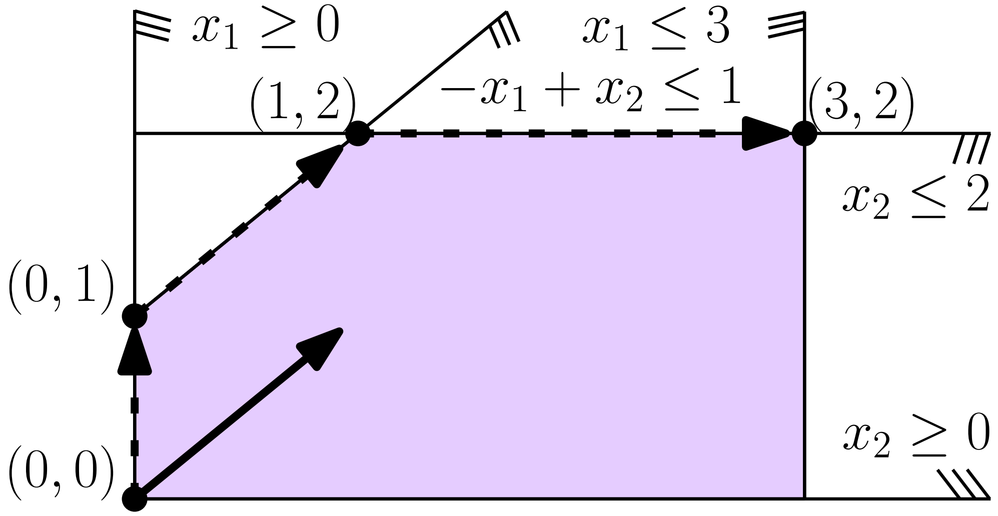

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Simplex tableaus in general

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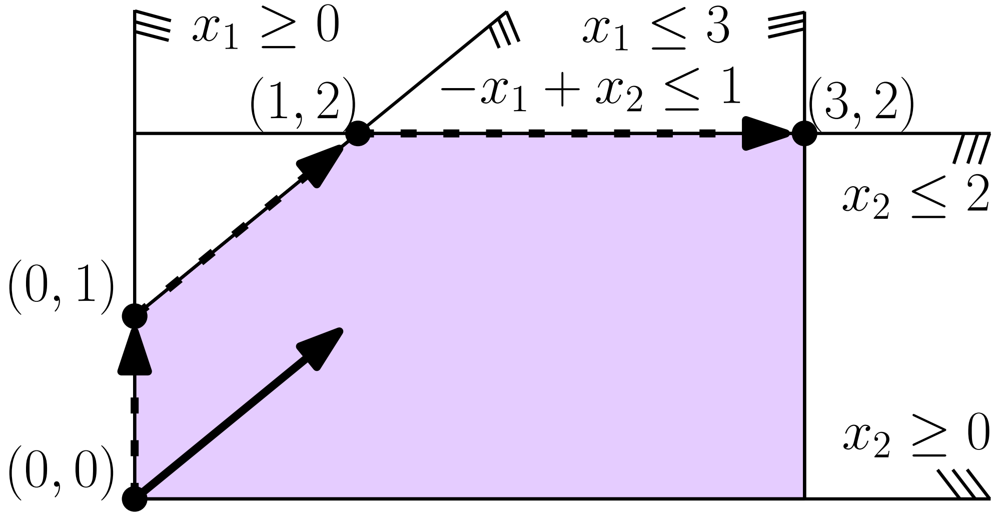

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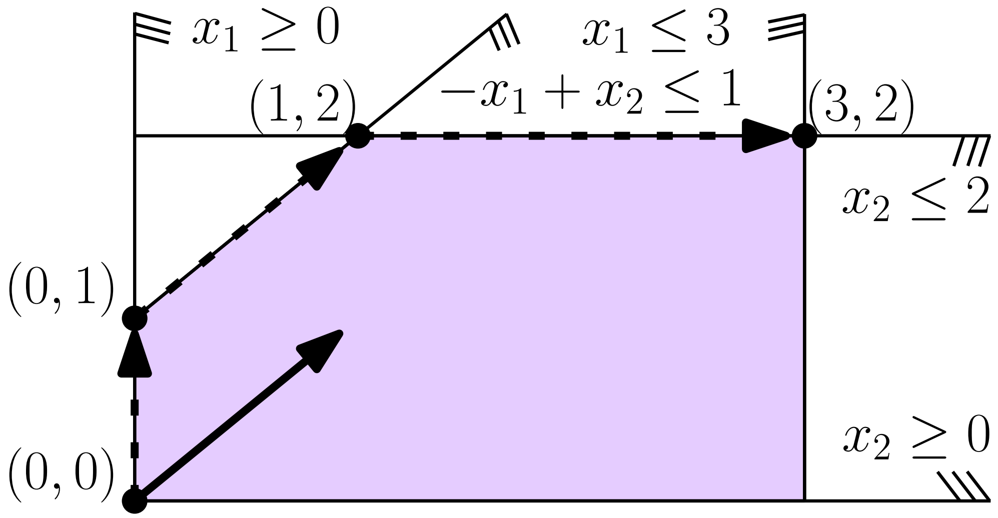

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(There could be more than one choice of the leaving variable.)

Simplex method - Correctness

Lemma 5.6.1:

If B is a feasible basis and $T(B)$ the corresponding simplex tableau, and if the entering variable x_v and the leaving variable x_u have been selected according to the above criteria (and otherwise arbitrarily), then $B' = (B \backslash \{u\}) \cup \{v\}$ is again a feasible basis.

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Proof sketch:

We need to show that

- $A_{B'}$ is nonsingular
- Basis B' is feasible

Computation and Efficiency

Optional Warm-up Exercise: Another LP

Maximize $9x_1 + 3x_2 + x_3$

subject to:
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\begin{aligned}
x_1 &\leq 1 \\
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natural choice: x_1

because largest coefficient: 9,

but is it a good choice?

Simplex method - Computations

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Remark: In this revised simplex method, a pivot step takes time $\mathcal{O}(m^2)$ instead of $\mathcal{O}(mn)$ operations with the full tableau.

Pivot rules

Simplex tableau

$$x_3 = 1 + x_1 - x_2$$
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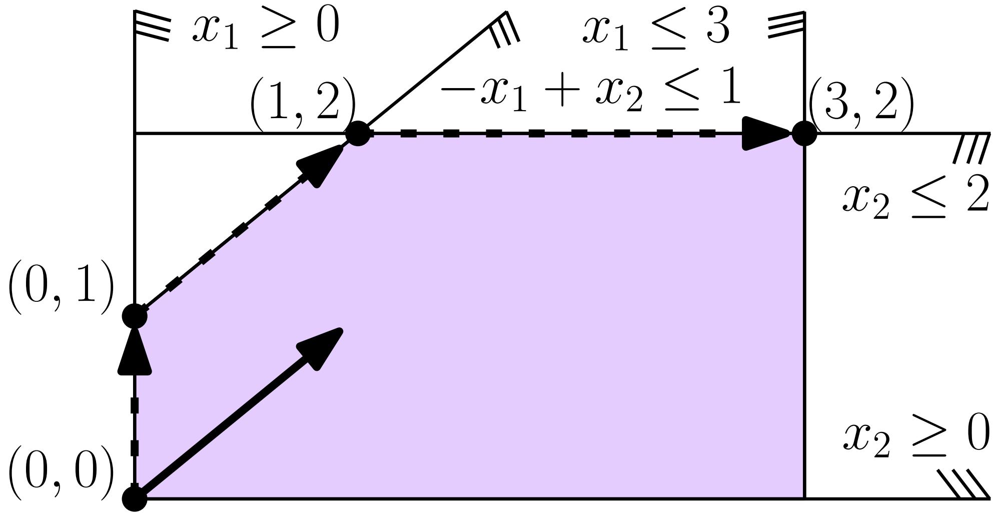

How to choose which variable to pivot on, i.e. the entering variable?

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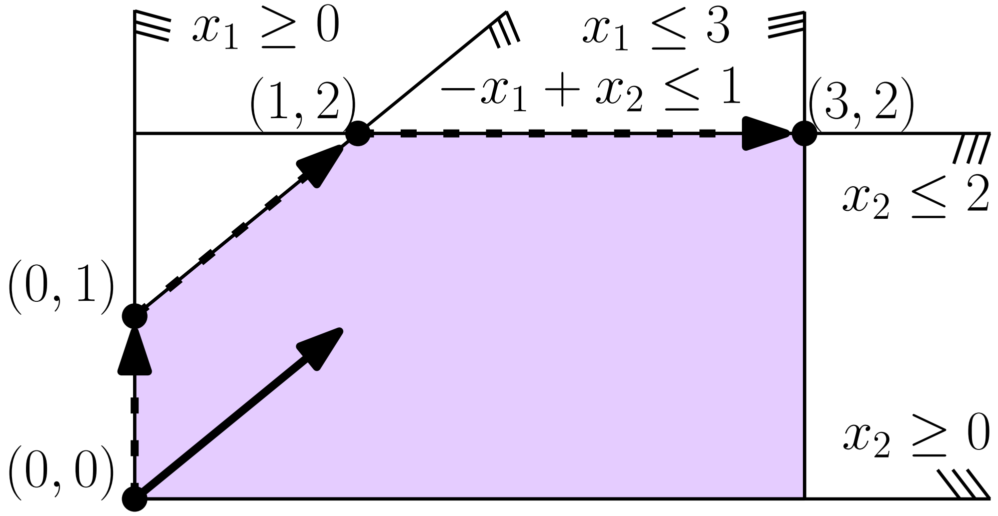

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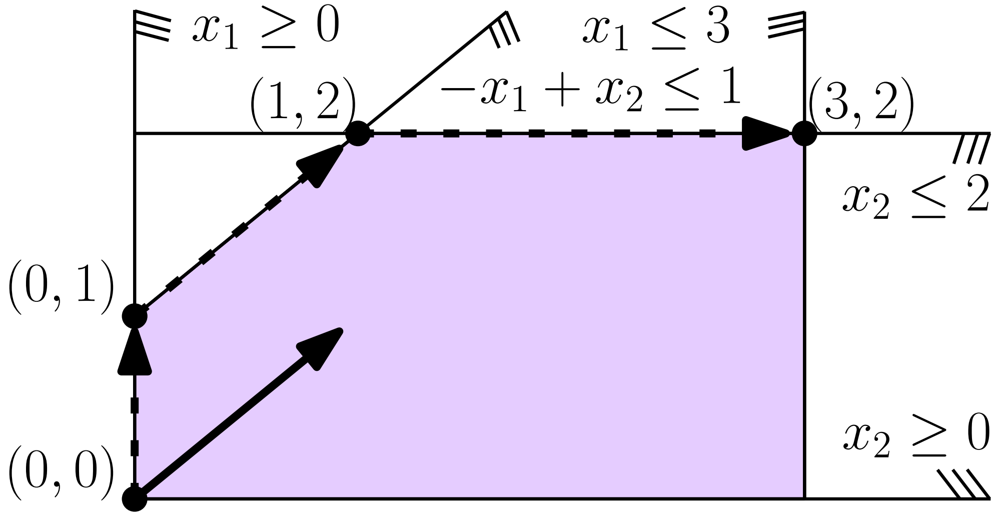

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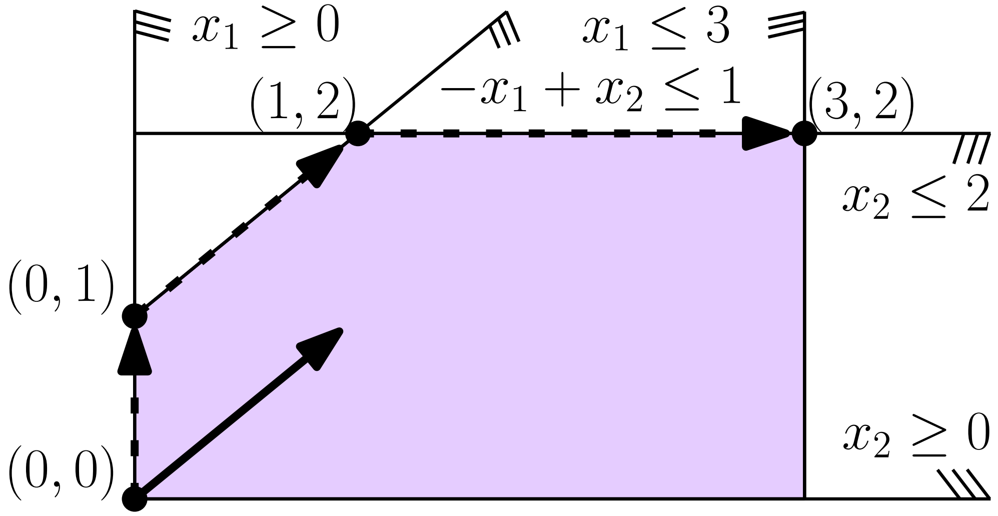

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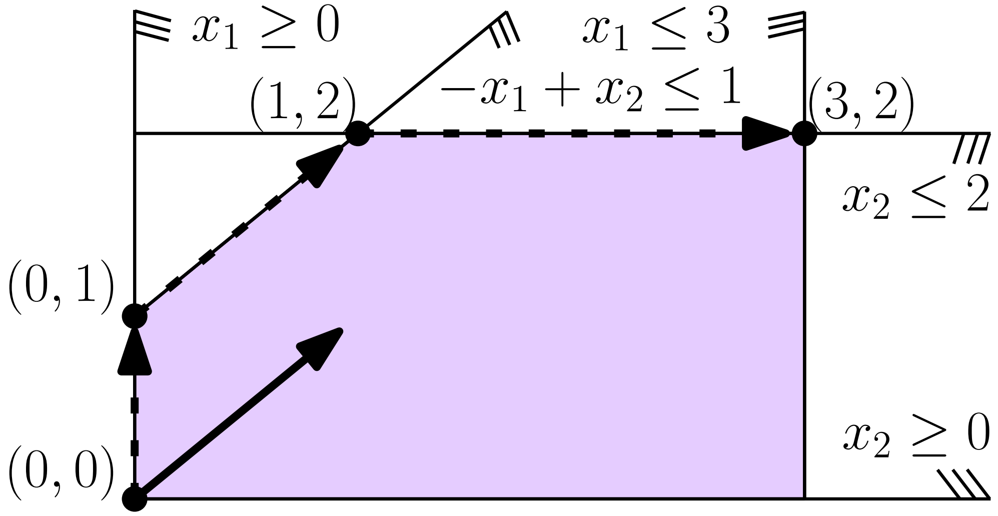

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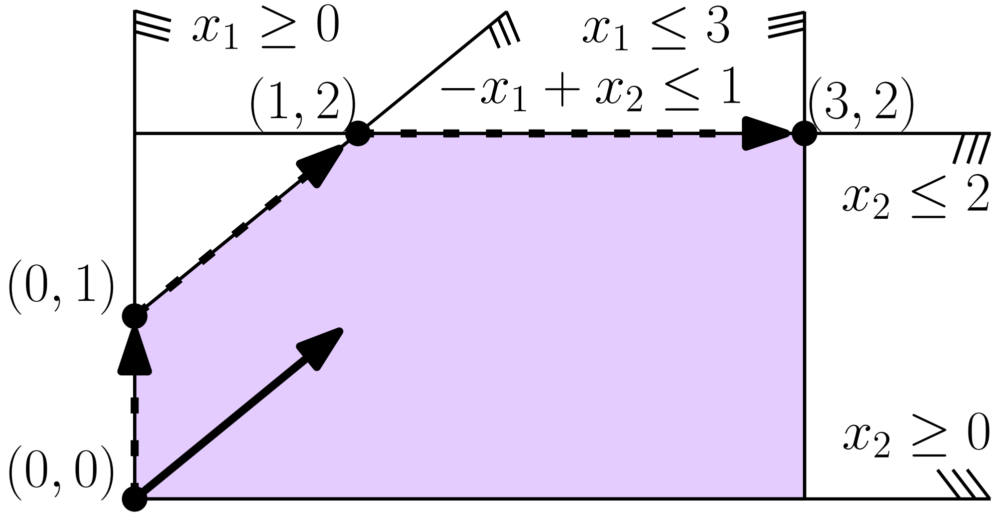

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Random edge:

Random methods lead to best provable bounds for expected simplex method efficiency.

Efficiency of the simplex method

How many pivot steps does the simplex method do?

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In the worst case, the simplex method runs in exponential time.

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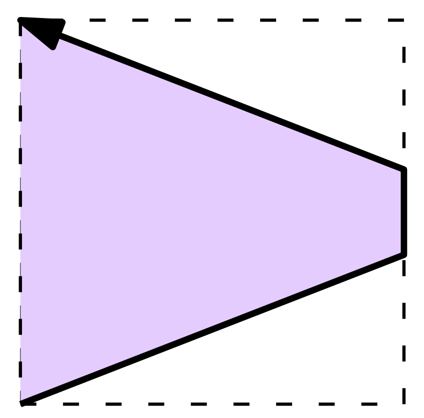

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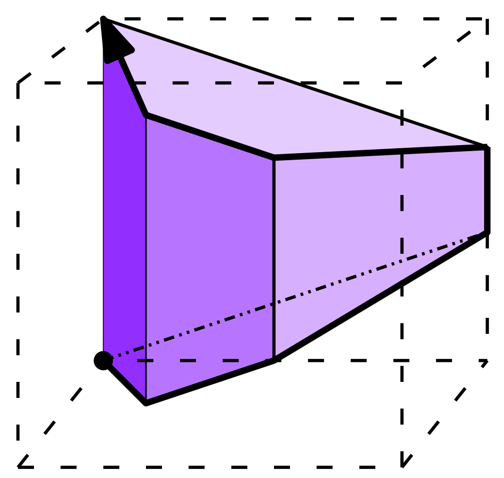

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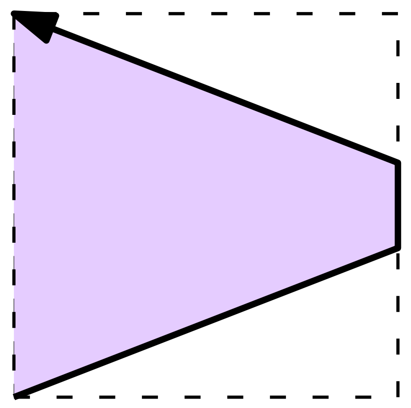

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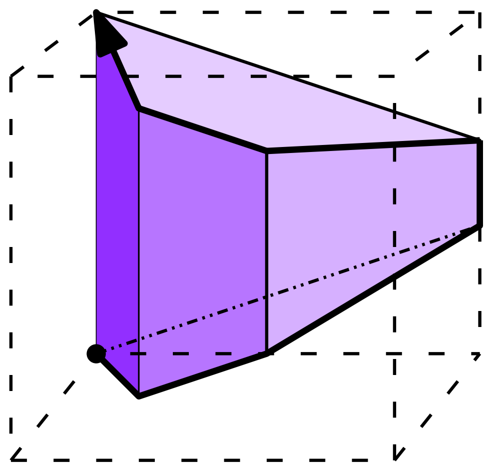

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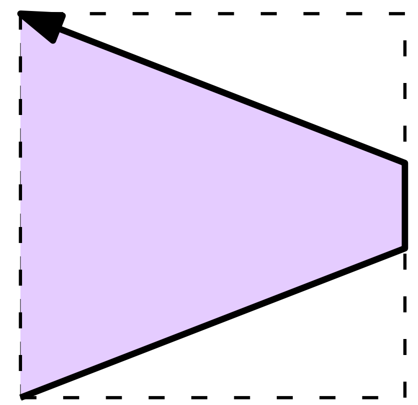

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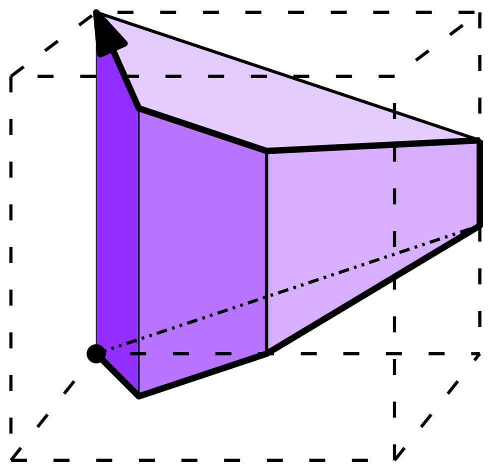

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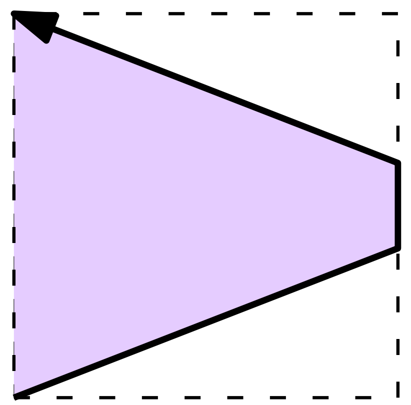

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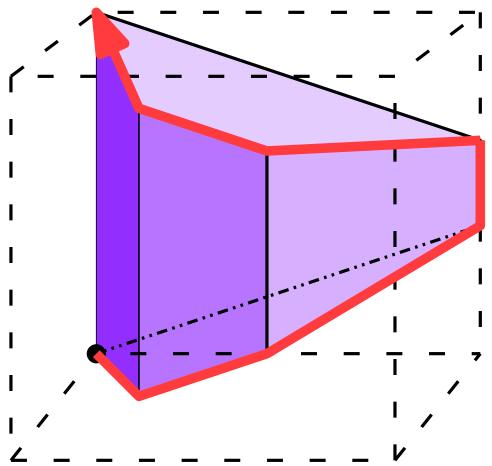

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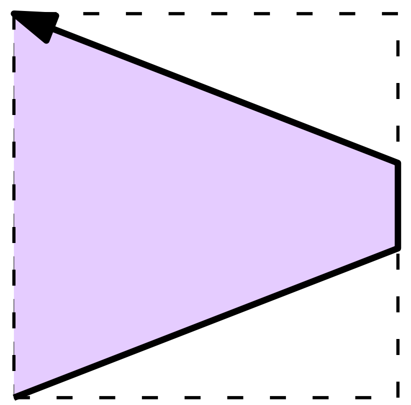

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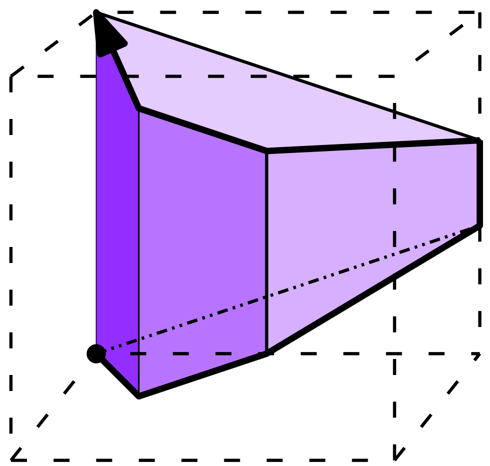

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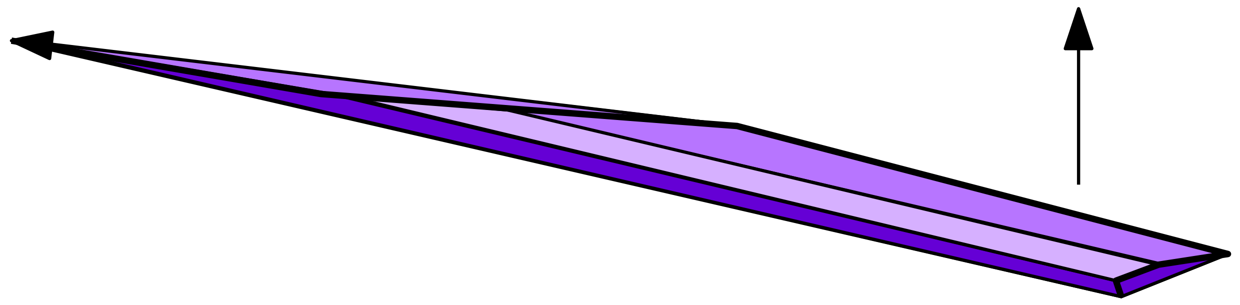

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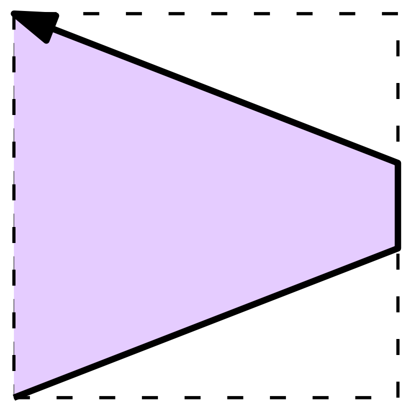

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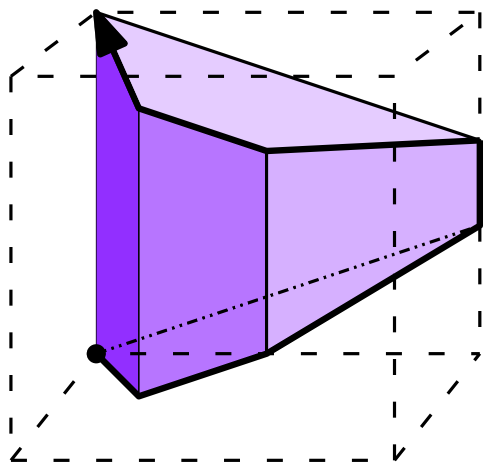

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Polynomial Hirsch conjecture: still open.

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Summary

Simplex Method

1. Convert the input linear program to equational form.
2. If a basic feasible solution is not obvious, solve the auxiliary LP to find an bfs (if such exists – if not LP is infeasible, **stop**).
3. For a feasible basis B compute the simplex tableau $T(B)$.
4. If in $T(B)$ no nonbasic variable appears positively, return an optimal solution; **stop**.
5. Otherwise, choose an entering variable using some pivot rule (if necessary).
6. If the column of the entering variable in the simplex tableau is nonnegative, the linear program is unbounded; **stop**.
7. Otherwise, choose a leaving variable using some pivot rule (if necessary).
8. Update B and $T(B)$, and go to step 4.