Miscellaneous notes on JIE R&R of SOP_Repeated

Legislative constraint as a function of e

- I thought it would be positive at $e = 0$ and turn negative as e increases
- What does it mean that for some values it's negative at 0, becomes positive, and then goes negative again?
  - For sure I have to be careful in numerical examples

Numerical examples

$\delta_L = \delta_{ML} = .95$

	E=.35	E=.4	E=.41	E=.42	E=.45
τ^{tw}				.074	.0654
e^{tw}					.00123
T = 2		.07500			.057407
T = 3		.074716	.070243	.066284	**.0570802**
T = 4		**.074708**	**.070233**	**.066275**	.0570806
T = 5		.074795	.07033	.06638	.057185
T = 6	.1080	.07492			
T = 7	.1081				.057
T = 8	.10814				

I have another sheet of notes that conflicts with the first column. It just says "$\delta = .95$":

	E=.35
τ^{tw}	.1213
e^{tw}	.006003
T = 3	.1023044
T = 4	**.1022411**
T = 5	.1022427
T = 6	.10227
T = 7	.102305

This one just says "$\delta = .99$":

	E=.4	.39
τ^{tw}		
e^{tw}		
T = 2	.068510	.06391
T = 3	**.068261**	**.06374**
T = 4	.068289	.06378
T = 5	.06841	.06391

This has the note, "This at least works in the direction I thought it would" with "$\delta_L = .94$, $\delta_{ML} = .95$":

	E=.4
τ^{tw}	
e^{tw}	
T = 4	.07464
T = 5	**.07421**
T = 6	.07481
T = 7	.07492

("Really want to know if reducing δ_L — making future term less important — will give me the σ result I've been after; really, no result at all; depends on other parameters.)

Some summaries

- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .95$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07494$ at $T = 3$.
- $E = .5$, assume I kept $\delta_L = .99$, $\delta_{ML} = .95$. Optimal $\tau^a = .04864$ at $T = 3$.
- $E = .4$, $\delta_L = \delta_{ML} = .99$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07470$ at $T = 3$.
- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .5$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07802$ at $T = 2$.
- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .75$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07629$ at $T = 3$.

Trying to understand what is really going on with constraint in terms of T

- Want to get good intuition for why T can go up as $\sigma \downarrow$.
  - Not obvious that it always does; I think it's possible that the direction of T in response to σ is indeterminant
- I think I need to show effect of σ on $\bar{e}$ first (I have informally that $\sigma \uparrow \Rightarrow \bar{e} \downarrow$)
  - Then, impact of σ on τ^a. Next look to see if net profits at τ^a increase more than those at τ^{tw}, then lobby's future incentives are muted

Result 1. $\frac{\mathrm{d}\bar{e}}{\mathrm{d}\sigma} > 0$

Proof: Corollary 4 shows that $\frac{\mathrm{d}\bar{e}}{\mathrm{d}\gamma} < 0$. All that is left is to show that $\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} < 0$.

- The derivative of $\gamma = 1 + \frac{1}{\sigma}e^{\sigma}$ w.r.t. σ is

$$\frac{1}{\sigma}\ln\sigma e^{\sigma} + e^{\sigma}\left(-\frac{1}{\sigma^2}\right)$$

  Both terms are negative given $\sigma \in (0, 1)$ and $e \geq 0$. QED.

Now for the result on τ^a. Differentiating the lobby's condition with respect to σ, we have

$$\frac{\partial \Pi}{\partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} + \frac{\partial \Pi}{\partial \bar{e}}\frac{\mathrm{d}\bar{e}}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} + \frac{\partial \Pi}{\partial \gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} = 0$$

$$\frac{\mathrm{d}\tau^a}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} = \left[-\frac{\frac{\partial \Pi}{\partial \bar{e}}\frac{\mathrm{d}\bar{e}}{\mathrm{d}\gamma} + \frac{\partial \Pi}{\partial \gamma}}{\frac{\partial \Pi}{\partial \tau^a}}\right]\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} \tag{1}$$

We know from Corollary 5 that $\frac{\mathrm{d}\tau^a}{\mathrm{d}\gamma}$ is positive, and we've just shown that $\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma}$ is negative. Thus $\frac{\mathrm{d}\tau^a}{\mathrm{d}\sigma} < 0$.

Write constraint:

$$\bar{e}(\tau^a) - \pi(\tau^b(\bar{e}(\tau^a))) + \pi(\tau^a) - e_a - \frac{\delta_{\mathrm{L}} + \delta_{\mathrm{L}}^{T+1}}{1 - \delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right] = 0$$

For now, assume this has an interior solution so calculus works.

First, what does T do to $\overline{e}(\tau^a)$?
By the Implicit Function Theorem:

$$\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} = -\frac{\frac{\partial \Omega}{\partial T}}{\frac{\partial \Omega}{\partial \overline{e}}} = -\frac{-\frac{\delta_{\mathrm{ML}}^{T+1}\ln \delta_{\mathrm{ML}}}{1-\delta_{\mathrm{ML}}}\left[W_{\mathrm{ML}}(\gamma(\overline{e}),\boldsymbol{\tau^a})-W_{\mathrm{ML}}(\gamma(\overline{e}),\boldsymbol{\tau^{tw}})\right]}{\frac{\delta_{\mathrm{ML}}-\delta_{\mathrm{ML}}^{T+1}}{1-\delta_{\mathrm{ML}}}\frac{\partial \gamma}{\partial \overline{e}}[\pi(\tau^a)-\pi(\tau^{tw})]-\frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^b(\overline{e}))-\pi(\tau^a)\right]} > 0 \tag{2}$$

So if $T \uparrow$ then $\overline{e} \uparrow$.

Now, want to know about effect of T on τ^a.

$$\frac{\partial \Pi}{\partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Pi}{\partial T} = 0$$

Because $\frac{\partial \Pi}{\partial T} = \frac{\ln \delta_{\mathrm{L}} \delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]$, we are looking for

$$\frac{\mathrm{d}\tau^a}{\mathrm{d}T} = -\frac{\frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Pi}{\partial T}}{\frac{\partial \Pi}{\partial \tau^a}} = \frac{-\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\cdot \frac{\mathrm{d}\overline{e}}{\mathrm{d}T} - \frac{\ln \delta_{\mathrm{L}} \delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]}{\left(1 + \frac{\delta_{\mathrm{L}}-\delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\right)\left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right]} \tag{3}$$

The proof of Corollary 3 shows that $\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)$ is positive, and the above result shows that $\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}$ is positive. The second term is positive since net profits are maximized at e_{tw} and $\delta_L < 1$ so that its log is negative. With the leading negative signs, the numerator has both a negative and a positive part. This is not changed by the denominator, as the arguments given in the proof of Corollary 1 show that the denominator is positive.

- Remember simplified version where actors are infinitely patient
- Can play around with trick of looking for minimum τ^a by setting this equal to zero

As $\sigma \downarrow$, $\overline{e} \downarrow$ for a given τ^a.

- Remember that change in $\overline{e}$ is really a change in the $(\tau^a, \overline{e}(\tau^a))$ schedule that derives from legislature's constraint
- So τ^a has to be raised to satisfy lobby's constraint

  - Net profits are greatest at τ^{tw}, so relative gap between net profits at τ^a and τ^{tw} (future) closes faster than that between break profits and trade agreement profits (present)

- How much τ^a adjusts depends on magnitude of $\frac{\delta_{\mathrm{L}}+\delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}$
- $\pi(\tau^b(\bar{e}(\tau^a))) - \bar{e}(\tau^a)$ is negative, gets less negative when $\bar{e}$ is reduced.
- $\pi(\tau^a) - e_a$ is positive, becomes larger as τ^a rises

$$0 \geq -\left[\bar{e}(\tau^a) - \pi(\tau^b(\bar{e}(\tau^a))) + \pi(\tau^a) - e_a\right] + \frac{\delta_{\mathrm{L}} + \delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]$$

Where present part of constraint in on left and future part is on right. Remember present part must be negative.

Let $\gamma(e) = 1 + \frac{1}{1-\theta}e^{1-\theta}$. Or $\gamma(e) = 1 + \frac{1}{\sigma}e^{\sigma}$.

- $\frac{\partial\gamma}{\partial e} = e^{\sigma-1} > 0 \; \forall\sigma$
- $\frac{\partial^2\gamma}{\partial\sigma\partial e} = \ln e \cdot e^{\sigma-1} < 0$ for $e < 1$

i.e. as $\sigma \downarrow$, $\frac{\partial\gamma}{\partial e} \uparrow$.

Can I show that the optimal T can go either way when σ changes? That is, a counterexample to my quasi-result?

- My result says that as lobby gets stronger ($\frac{\partial\gamma}{\partial e} \uparrow$, so $\sigma \downarrow$), T should have to decrease.
- If optimal T increases when σ decreases (i.e. if T is decreasing in σ), this is a counterexample.
  - I think it's possible that both cases can happen depending on other parameters, like δ.

Look at legislature's constraint:

$$\frac{\delta_{\mathrm{ML}} - \delta_{\mathrm{ML}}^{T+1}}{1-\delta_{\mathrm{ML}}}\left[W_{\mathrm{ML}}(\gamma(e_b), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(e_b), \boldsymbol{\tau^{tw}})\right] \geq$$
$$W_{\mathrm{ML}}(\gamma(e_b), \tau^b(e_b), \tau^{*a}) - W_{\mathrm{ML}}(\gamma(e_b), \boldsymbol{\tau^a})$$

If T is too small, future gap can be smaller than current-period gap. So if T gets too short, can't enforce on legislature.

Note that changing σ changes τ^{tw}

- e_{tw} is solution to $\frac{\partial \pi}{\partial \tau}\frac{\partial \tau}{\partial \gamma}\frac{\partial \gamma}{\partial e} = 1$; or $\frac{\partial \pi}{\partial \tau}\frac{\partial \tau}{\partial \gamma} = \frac{1}{\frac{\partial \gamma}{\partial e}}$
- When $\frac{\partial \gamma}{\partial e} \uparrow$, RHS $\downarrow$, so LHS must go down.

November 13, 2015

- I know that $\frac{\partial\tau^a}{\partial T}$ has both a negative and positive part.
  - Shown for the general case where δ is anything.
- Now I want to know if/when $\frac{\partial^2\tau^a}{\partial T^2}$ is positive so that I could set $\frac{\partial\tau^a}{\partial T}=0$ and optimal T (the one that minimizes τ^a)
  - Would need to be careful of corner solutions
  - Will do this for case where $\delta \to 1$
    * This means $\frac{\delta-\delta^{T+1}}{1-\delta} \to T$ and $-\frac{\delta^{T+1}\ln\delta}{1-\delta} \to 1$

Start with simplifying result for $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$.
Again:

$$\frac{\partial\Pi}{\partial\tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T}+\frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}+\frac{\partial\Pi}{\partial T}=0 \tag{4}$$

$$\frac{\mathrm{d}\tau^a}{\mathrm{d}T}=-\frac{\frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}+\frac{\partial\Pi}{\partial T}}{\frac{\partial\Pi}{\partial\tau^a}}=\frac{-\left(1-\frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\cdot\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}+\left[\pi(\tau^{tw})-e_{tw}-\pi(\tau^a)+e_a\right]}{(1+T)\left[\frac{\partial\pi(\tau^a)}{\partial\tau^a}-\frac{\partial e_a}{\partial\tau^a}\right]} \tag{5}$$

- $\left(1-\frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)$ is positive
- $\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}$ is positive:

$$\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}=-\frac{\frac{\partial\Omega}{\partial T}}{\frac{\partial\Omega}{\partial\overline{e}}}=\frac{W_{\mathrm{ML}}(\gamma(\overline{e}),\boldsymbol{\tau^a})-W_{\mathrm{ML}}(\gamma(\overline{e}),\boldsymbol{\tau^{tw}})}{T\frac{\partial\gamma}{\partial\overline{e}}\left[\pi(\tau^{tw})-\pi(\tau^a)\right]+\frac{\partial\gamma}{\partial\overline{e}}\left[\pi(\tau^b(\overline{e}))-\pi(\tau^a)\right]}>0 \tag{6}$$

- denominator is positive (proof of Corollary 1)

Now, on to $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$

$$\frac{\partial}{\partial T}\left[\frac{\partial\Pi}{\partial\tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T}+\frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}+\frac{\partial\Pi}{\partial T}\right]=0$$

$$\frac{\partial\Pi}{\partial\tau^a}\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}+\frac{\partial^2\Pi}{\partial T\partial\tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T}+\frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2}+\frac{\partial^2\Pi}{\partial T\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}+\frac{\partial^2\Pi}{\partial T^2}=0 \tag{7}$$

Going to need:

$$\frac{\partial^2\Pi}{\partial T\partial\tau^a}=\frac{\partial\pi(\tau^a)}{\partial\tau^a}-\frac{\partial e_a}{\partial\tau^a}>0$$

$$\frac{\partial^2\Pi}{\partial T\partial\overline{e}}=0$$

$$\frac{\partial^2 \Pi}{\partial T^2} = 0$$

To get $\frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}T^2}$, have to do a little more work.

$$\frac{\partial}{\partial T}\left[\frac{\partial \Omega}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Omega}{\partial T}\right] = 0$$

$$\frac{\partial \Omega}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} + \frac{\partial^2 \Omega}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Omega}{\partial T^2} = 0$$

$$\frac{\partial^2 \Omega}{\partial T^2} = \frac{\partial}{\partial T}\left(\frac{\partial \Omega}{\partial T}\right) = 0$$

$$\frac{\partial^2 \Omega}{\partial T \partial \overline{e}} = \frac{\partial}{\partial T}\left(\frac{\partial \Omega}{\partial \overline{e}}\right) = \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^a) - \pi(\tau^{tw})\right] < 0$$

So,

$$\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} = \frac{-\frac{\partial^2 \Omega}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} - \frac{\partial^2 \Omega}{\partial T^2}}{\frac{\partial \Omega}{\partial \overline{e}}} = \frac{-\frac{\partial^2 \Omega}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}}{\frac{\partial \Omega}{\partial \overline{e}}} < 0$$

Now we have everything we need for $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$. Rearranging Equation 7, we have

$$\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2} = -\frac{\frac{\partial^2 \Pi}{\partial T \partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} + \frac{\partial^2 \Pi}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Pi}{\partial T^2}}{\frac{\partial \Pi}{\partial \tau^a}}$$

Elements of the last two terms in the denominator have been shown to be zero when $\delta \to 1$, so we have

$$\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2} = -\frac{\frac{\partial^2 \Pi}{\partial T \partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2}}{\frac{\partial \Pi}{\partial \tau^a}}$$

The denominator is positive, so as goes the numerator, so goes the whole thing. So let's write out the numerator:

$$\left(\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right)\left[\frac{\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\cdot\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} - \left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]}{(1+T)\left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right]}\right] -$$
$$\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^{tw}) - \pi(\tau^a)\right]\frac{W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^{tw}})}{\left[T\frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^{tw}) - \pi(\tau^a)\right] + \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^b(\overline{e})) - \pi(\tau^a)\right]\right]^2} \quad (8)$$