

1 Legislative constraint

Legislative constraint as a function of e

- I thought it would be positive at $e = 0$ and turn negative as e increases
- What does it mean that for some values it's negative at 0, becomes positive, and then goes negative again?
 - For sure I have to be careful in numerical examples

2 Numerical examples

$\delta_L = \delta_{ML} = .95$					
	E=.35	E=.4	E=.41	E=.42	E=.45
τ^{tw}				.074	.0654
e^{tw}					.00123
T = 2		.07500			.057407
T = 3		.074716	.070243	.066284	.0570802
T = 4		.074708	.070233	.066275	.0570806
T = 5		.074795	.07033	.06638	.057185
T = 6	.1080	.07492			
T = 7	.1081				.057
T = 8	.10814				

I have another sheet of notes that conflicts with the first column. It just says “ $\delta = .95$ ”:

	E=.35
τ^{tw}	.1213
e^{tw}	.006003
T = 3	.1023044
T = 4	.1022411
T = 5	.1022427
T = 6	.10227
T = 7	.102305

This one just says “ $\delta = .99$ ”:

	E=.4	.39
τ^{tw}		
e^{tw}		
T = 2	.068510	.06391
T = 3	.068261	.06374
T = 4	.068289	.06378
T = 5	.06841	.06391

This has the note, “This at least works in the direction I thought it would” with “ $\delta_L = .94$, $\delta_{ML} = .95$ ”:

	E=.4
τ^{tw}	
e^{tw}	
T = 4	.07464
T = 5	.07421
T = 6	.07481
T = 7	.07492

(“Really want to know if reducing δ_L — making future term less important — will give me the σ result I’ve been after; really, no result at all; depends on other parameters.)

Some summaries

- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .95$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07494$ at $T = 3$.
- $E = .5$, assume I kept $\delta_L = .99$, $\delta_{ML} = .95$. Optimal $\tau^a = .04864$ at $T = 3$.
- $E = .4$, $\delta_L = \delta_{ML} = .99$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07470$ at $T = 3$.
- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .5$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07802$ at $T = 2$.
- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .75$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07629$ at $T = 3$.

3 How does Optimal T vary with political strength?

Trying to understand what is really going on with constraint in terms of T

- Want to get good intuition for why T can go up as $\sigma \downarrow$.
 - Not obvious that it always does; I think it's possible that the direction of T in response to σ is indeterminant
- I think I need to show effect of σ on \bar{e} first (I have informally that $\sigma \uparrow \Rightarrow \bar{e} \downarrow$)
 - Then, impact of σ on τ^a . Next look to see if net profits at τ^a increase more than those at τ^{tw} , then lobby's future incentives are muted

Result 1. $\frac{d\bar{e}}{d\sigma} > 0$

Proof: Corollary 4 shows that $\frac{d\bar{e}}{d\gamma} < 0$. All that is left is to show that $\frac{d\gamma}{d\sigma} < 0$.

- The derivative of $\gamma = 1 + \frac{1}{\sigma}e^\sigma$ w.r.t. σ is

$$\frac{1}{\sigma} \ln \sigma e^\sigma + e^\sigma \left(-\frac{1}{\sigma^2} \right)$$

Both terms are negative given $\sigma \in (0, 1)$ and $e \geq 0$. QED.

Now for the result on τ^a . Differentiating the lobby's condition with respect to σ , we have

$$\frac{\partial \Pi}{\partial \tau^a} \frac{d\tau^a}{d\gamma} \frac{d\gamma}{d\sigma} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{d\gamma} \frac{d\gamma}{d\sigma} + \frac{\partial \Pi}{\partial \gamma} \frac{d\gamma}{d\sigma} = 0$$

$$\frac{d\tau^a}{d\gamma} \frac{d\gamma}{d\sigma} = \left[-\frac{\frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{d\gamma} + \frac{\partial \Pi}{\partial \gamma}}{\frac{\partial \Pi}{\partial \tau^a}} \right] \frac{d\gamma}{d\sigma} \quad (1)$$

We know from Corollary 5 that $\frac{d\tau^a}{d\gamma}$ is positive, and we've just shown that $\frac{d\gamma}{d\sigma}$ is negative. Thus $\frac{d\tau^a}{d\sigma} < 0$.

Write constraint:

$$\bar{e}(\tau^a) - \pi(\tau^b(\bar{e}(\tau^a))) + \pi(\tau^a) - e_a - \frac{\delta_L + \delta_L^{T+1}}{1 - \delta_L} [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a] = 0$$

For now, assume this has an interior solution so calculus works.

First, what does T do to $\bar{e}(\tau^a)$?

By the Implicit Function Theorem:

$$\frac{d\bar{e}}{dT} = -\frac{\frac{\partial \Omega}{\partial T}}{\frac{\partial \Omega}{\partial \bar{e}}} = -\frac{-\frac{\delta_{ML}^{T+1} \ln \delta_{ML}}{1-\delta_{ML}} [W_{ML}(\gamma(\bar{e}), \tau^a) - W_{ML}(\gamma(\bar{e}), \tau^{tw})]}{\frac{\delta_{ML} - \delta_{ML}^{T+1}}{1-\delta_{ML}} \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^a) - \pi(\tau^{tw})] - \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^b(\bar{e})) - \pi(\tau^a)]} > 0 \quad (2)$$

So if $T \uparrow$ then $\bar{e} \uparrow$.

Now, want to know about effect of T on τ^a .

$$\frac{\partial \Pi}{\partial \tau^a} \frac{d\tau^a}{dT} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial \Pi}{\partial T} = 0$$

Because $\frac{\partial \Pi}{\partial T} = \frac{\ln \delta_L \delta_L^{T+1}}{1-\delta_L} [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]$, we are looking for

$$\frac{d\tau^a}{dT} = -\frac{\frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial \Pi}{\partial T}}{\frac{\partial \Pi}{\partial \tau^a}} = \frac{-\left(1 - \frac{d\pi}{d\bar{e}}\right) \cdot \frac{d\bar{e}}{dT} - \frac{\ln \delta_L \delta_L^{T+1}}{1-\delta_L} [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]}{\left(1 + \frac{\delta_L - \delta_L^{T+1}}{1-\delta_L}\right) \left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right]} \quad (3)$$

The proof of Corollary 3 shows that $\left(1 - \frac{d\pi}{d\bar{e}}\right)$ is positive, and the above result shows that $\frac{d\bar{e}}{dT}$ is positive. The second term is positive since net profits are maximized at e_{tw} and $\delta_L < 1$ so that its log is negative. With the leading negative signs, the numerator has both a negative and a positive part. This is not changed by the denominator, as the arguments given in the proof of Corollary 1 show that the denominator is positive.

- Remember simplified version where actors are infinitely patient
- Can play around with trick of looking for minimum τ^a by setting this equal to zero

As $\sigma \downarrow$, $\bar{e} \downarrow$ for a given τ^a .

- Remember that change in \bar{e} is really a change in the $(\tau^a, \bar{e}(\tau^a))$ schedule that derives from legislature's constraint
- So τ^a has to be raised to satisfy lobby's constraint

- Net profits are greatest at τ^{tw} , so relative gap between net profits at τ^a and τ^{tw} (future) closes faster than that between break profits and trade agreement profits (present)
- How much τ^a adjusts depends on magnitude of $\frac{\delta_L + \delta_L^{T+1}}{1 - \delta_L}$
- $\pi(\tau^b(\bar{e}(\tau^a))) - \bar{e}(\tau^a)$ is negative, gets less negative when \bar{e} is reduced.
- $\pi(\tau^a) - e_a$ is positive, becomes larger as τ^a rises

$$0 \geq - [\bar{e}(\tau^a) - \pi(\tau^b(\bar{e}(\tau^a))) + \pi(\tau^a) - e_a] + \frac{\delta_L + \delta_L^{T+1}}{1 - \delta_L} [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]$$

Where present part of constraint is on left and future part is on right. Remember present part must be negative.

Let $\gamma(e) = 1 + \frac{1}{1-\theta}e^{1-\theta}$. Or $\gamma(e) = 1 + \frac{1}{\sigma}e^\sigma$.

- $\frac{\partial \gamma}{\partial e} = e^{\sigma-1} > 0 \forall \sigma$
- $\frac{\partial^2 \gamma}{\partial \sigma \partial e} = \ln e \cdot e^{\sigma-1} < 0$ for $e < 1$

i.e. as $\sigma \downarrow$, $\frac{\partial \gamma}{\partial e} \uparrow$.

Can I show that the optimal T can go either way when σ changes? That is, a counterexample to my quasi-result?

- My result says that as lobby gets stronger ($\frac{\partial \gamma}{\partial e} \uparrow$, so $\sigma \downarrow$), T should have to decrease.
- If optimal T increases when σ decreases (i.e. if T is decreasing in σ), this is a counterexample.
- I think it's possible that both cases can happen depending on other parameters, like δ .

Look at legislature's constraint:

$$\frac{\delta_{ML} - \delta_{ML}^{T+1}}{1 - \delta_{ML}} [W_{ML}(\gamma(e_b), \tau^a) - W_{ML}(\gamma(e_b), \tau^{tw})] \geq W_{ML}(\gamma(e_b), \tau^b(e_b), \tau^{*a}) - W_{ML}(\gamma(e_b), \tau^a)$$

If T is too small, future gap can be smaller than current-period gap. So if T gets too short, can't enforce on legislature.

Note that changing σ changes τ^{tw}

- e_{tw} is solution to $\frac{\partial \pi}{\partial \tau} \frac{\partial \tau}{\partial \gamma} \frac{\partial \gamma}{\partial e} = 1$; or $\frac{\partial \pi}{\partial \tau} \frac{\partial \tau}{\partial \gamma} = \frac{1}{\frac{\partial \gamma}{\partial e}}$
- When $\frac{\partial \gamma}{\partial e} \uparrow$, RHS \downarrow , so LHS must go down.

4 Optimal T result with perfect patience

November 13, 2015

- I know that $\frac{\partial \tau^a}{\partial T}$ has both a negative and positive part.
 - Shown for the general case where δ is anything.
- Now I want to know if/when $\frac{\partial^2 \tau^a}{\partial T^2}$ is positive so that I could set $\frac{\partial \tau^a}{\partial T} = 0$ and optimal T (the one that minimizes τ^a)
 - Would need to be careful of corner solutions
 - Will do this for case where $\delta \rightarrow 1$
 - * This means $\frac{\delta - \delta^{T+1}}{1 - \delta} \rightarrow T$ and $-\frac{\delta^{T+1} \ln \delta}{1 - \delta} \rightarrow 1$

Start with simplifying result for $\frac{d\tau^a}{dT}$.

Again:

$$\frac{\partial \Pi}{\partial \tau^a} \frac{d\tau^a}{dT} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial \Pi}{\partial T} = 0 \quad (4)$$

$$\frac{d\tau^a}{dT} = -\frac{\frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial \Pi}{\partial T}}{\frac{\partial \Pi}{\partial \tau^a}} = \frac{-\left(1 - \frac{d\pi}{d\bar{e}}\right) \cdot \frac{d\bar{e}}{dT} + [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]}{(1 + T) \left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a} \right]} \quad (5)$$

- $\left(1 - \frac{d\pi}{d\bar{e}}\right)$ is positive
- $\frac{d\bar{e}}{dT}$ is positive:

$$\frac{d\bar{e}}{dT} = -\frac{\frac{\partial \Omega}{\partial T}}{\frac{\partial \Omega}{\partial \bar{e}}} = \frac{W_{ML}(\gamma(\bar{e}), \tau^a) - W_{ML}(\gamma(\bar{e}), \tau^{tw})}{T \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^{tw}) - \pi(\tau^a)] + \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^b(\bar{e})) - \pi(\tau^a)]} > 0 \quad (6)$$

- denominator is positive (proof of Corollary 1)

Now, on to $\frac{d^2 \tau^a}{dT^2}$

$$\begin{aligned} \frac{\partial}{\partial T} \left[\frac{\partial \Pi}{\partial \tau^a} \frac{d\tau^a}{dT} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial \Pi}{\partial T} \right] &= 0 \\ \frac{\partial \Pi}{\partial \tau^a} \frac{d^2 \tau^a}{dT^2} + \frac{\partial^2 \Pi}{\partial T \partial \tau^a} \frac{d\tau^a}{dT} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d^2 \bar{e}}{dT^2} + \frac{\partial^2 \Pi}{\partial T \partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial^2 \Pi}{\partial T^2} &= 0 \end{aligned} \quad (7)$$

Going to need:

$$\frac{\partial^2 \Pi}{\partial T \partial \tau^a} = \frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a} > 0$$

$$\frac{\partial^2 \Pi}{\partial T \partial \bar{e}} = 0$$

$$\frac{\partial^2 \Pi}{\partial T^2} = 0$$

To get $\frac{d^2 \bar{e}}{dT^2}$, have to do a little more work.

$$\frac{\partial}{\partial T} \left[\frac{\partial \Omega}{\partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial \Omega}{\partial T} \right] = 0$$

$$\frac{\partial \Omega}{\partial \bar{e}} \frac{d^2 \bar{e}}{dT^2} + \frac{\partial^2 \Omega}{\partial T \partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial^2 \Omega}{\partial T^2} = 0$$

$$\frac{\partial^2 \Omega}{\partial T^2} = \frac{\partial}{\partial T} \left(\frac{\partial \Omega}{\partial T} \right) = 0$$

$$\frac{\partial^2 \Omega}{\partial T \partial \bar{e}} = \frac{\partial}{\partial T} \left(\frac{\partial \Omega}{\partial \bar{e}} \right) = \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^a) - \pi(\tau^{tw})] < 0$$

So,

$$\frac{d^2 \bar{e}}{dT^2} = \frac{-\frac{\partial^2 \Omega}{\partial T \partial \bar{e}} \frac{d\bar{e}}{dT} - \frac{\partial^2 \Omega}{\partial T^2}}{\frac{\partial \Omega}{\partial \bar{e}}} = \frac{-\frac{\partial^2 \Omega}{\partial T \partial \bar{e}} \frac{d\bar{e}}{dT}}{\frac{\partial \Omega}{\partial \bar{e}}} < 0$$

Now we have everything we need for $\frac{d^2 \tau^a}{dT^2}$. Rearranging Equation 7, we have

$$\frac{d^2 \tau^a}{dT^2} = -\frac{\frac{\partial^2 \Pi}{\partial T \partial \tau^a} \frac{d\tau^a}{dT} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d^2 \bar{e}}{dT^2} + \frac{\partial^2 \Pi}{\partial T \partial \bar{e}} \frac{d\bar{e}}{dT} + \frac{\partial^2 \Pi}{\partial T^2}}{\frac{\partial \Pi}{\partial \tau^a}}$$

Elements of the last two terms in the denominator have been shown to be zero when $\delta \rightarrow 1$, so we have

$$\frac{d^2 \tau^a}{dT^2} = -\frac{\frac{\partial^2 \Pi}{\partial T \partial \tau^a} \frac{d\tau^a}{dT} + \frac{\partial \Pi}{\partial \bar{e}} \frac{d^2 \bar{e}}{dT^2}}{\frac{\partial \Pi}{\partial \tau^a}}$$

The denominator is positive, so as goes the numerator, so goes the whole thing. So let's write out the numerator:

$$\left(\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a} \right) \left[\frac{\left(1 - \frac{d\pi}{d\bar{e}} \right) \cdot \frac{d\bar{e}}{dT} - [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]}{(1+T) \left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a} \right]} \right] -$$

$$\left(1 - \frac{d\pi}{d\bar{e}} \right) \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^{tw}) - \pi(\tau^a)] \frac{W_{\text{ML}}(\gamma(\bar{e}), \boldsymbol{\tau}^a) - W_{\text{ML}}(\gamma(\bar{e}), \boldsymbol{\tau}^{tw})}{\left[T \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^{tw}) - \pi(\tau^a)] + \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^b(\bar{e})) - \pi(\tau^a)] \right]^2} \quad (8)$$

$$\frac{\left(1 - \frac{d\pi}{d\bar{e}}\right) \cdot \frac{d\bar{e}}{dT} - [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]}{(1 + T)} - \frac{\left(1 - \frac{d\pi}{d\bar{e}}\right) \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^{tw}) - \pi(\tau^a)] [W_{ML}(\gamma(\bar{e}), \boldsymbol{\tau}^a) - W_{ML}(\gamma(\bar{e}), \boldsymbol{\tau}^{tw})]}{\left[T \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^{tw}) - \pi(\tau^a)] + \frac{\partial \gamma}{\partial \bar{e}} [\pi(\tau^b(\bar{e})) - \pi(\tau^a)]\right]^2}} \quad (9)$$

4.1 Solutions for optimal T

On the solution(s) to $\frac{d\tau^a}{dT}$:

- $\frac{d^2\tau^a}{dT^2}$ has one term that is positive, one term that reverses the sign of $\frac{d\tau^a}{dT}$
- this means if $\frac{d\tau^a}{dT} = 0$, it can't be a global max
 - The slope would have to start positive, end negative.
 - The second term in $\frac{d^2\tau^a}{dT^2}$ would start negative, end positive, so whole term would be positive as T gets large
 - This means SOC CAN'T be negative everywhere. A contradiction
- It's perfectly consistent to be a global min
 - Let slope start negative, end positive.
 - The second term in $\frac{d^2\tau^a}{dT^2}$ would start positive, end negative, so whole term would start positive. As T gets large, either stay positive (global min), or turn negative
 - If start positive and turn negative, would be an inflection point and slope doesn't actually turn positive
- If $\frac{d^2\tau^a}{dT^2}$ positive everywhere and $\frac{d\tau^a}{dT}(0) < 0$ then interior min
- We know the negative part of $\frac{d\tau^a}{dT}$ gets smaller as $T \uparrow$
 - If negative to start out, eventually becomes positive
 - If positive to start out, gets more positive.
 - * In this case, want T as small as possible.