Miscellaneous notes on JIE R&R of SOP_Repeated

1 Legislative constraint

Legislative constraint as a function of e

- I thought it would be positive at $e = 0$ and turn negative as e increases
- What does it mean that for some values it's negative at 0, becomes positive, and then goes negative again?
  - For sure I have to be careful in numerical examples

2 Numerical examples

$\delta_L = \delta_{ML} = .95$

	E=.35	E=.4	E=.41	E=.42	E=.45
τ^{tw}				.074	.0654
e^{tw}					.00123
T = 2		.07500			.057407
T = 3		.074716	.070243	.066284	**.0570802**
T = 4		**.074708**	**.070233**	**.066275**	.0570806
T = 5		.074795	.07033	.06638	.057185
T = 6	.1080	.07492			
T = 7	.1081				.057
T = 8	.10814				

I have another sheet of notes that conflicts with the first column. It just says "$\delta = .95$":

	E=.35
τ^{tw}	.1213
e^{tw}	.006003
T = 3	.1023044
T = 4	**.1022411**
T = 5	.1022427
T = 6	.10227
T = 7	.102305

This one just says "$\delta = .99$":

	E=.4	.39
τ^{tw}		
e^{tw}		
T = 2	.068510	.06391
T = 3	**.068261**	**.06374**
T = 4	.068289	.06378
T = 5	.06841	.06391

This has the note, "This at least works in the direction I thought it would" with "$\delta_L = .94$, $\delta_{ML} = .95$":

	E=.4
τ^{tw}	
e^{tw}	
T = 4	.07464
T = 5	**.07421**
T = 6	.07481
T = 7	.07492

("Really want to know if reducing δ_L — making future term less important — will give me the σ result I've been after; really, no result at all; depends on other parameters.)

Some summaries

- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .95$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07494$ at $T = 3$.
- $E = .5$, assume I kept $\delta_L = .99$, $\delta_{ML} = .95$. Optimal $\tau^a = .04864$ at $T = 3$.
- $E = .4$, $\delta_L = \delta_{ML} = .99$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07470$ at $T = 3$.
- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .5$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07802$ at $T = 2$.
- $E = .4$, $\delta_L = .99$, $\delta_{ML} = .75$, $e_{tw} = .00232$, $\tau^{tw} = .08185$. Optimal $\tau^a = .07629$ at $T = 3$.

3 How does Optimal T vary with political strength?

Trying to understand what is really going on with constraint in terms of T

- Want to get good intuition for why T can go up as $\sigma \downarrow$.
  - Not obvious that it always does; I think it's possible that the direction of T in response to σ is indeterminant
- I think I need to show effect of σ on $\overline{e}$ first (I have informally that $\sigma \uparrow \Rightarrow \overline{e} \downarrow$)
  - Then, impact of σ on τ^a. Next look to see if net profits at τ^a increase more than those at τ^{tw}, then lobby's future incentives are muted

Result 1. $\frac{\mathrm{d}\overline{e}}{\mathrm{d}\sigma} > 0$

Proof: Corollary 4 shows that $\frac{\mathrm{d}\overline{e}}{\mathrm{d}\gamma} < 0$. All that is left is to show that $\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} < 0$.

- The derivative of $\gamma = 1 + \frac{1}{\sigma}e^{\sigma}$ w.r.t. σ is

$$\frac{1}{\sigma}\ln\sigma e^{\sigma} + e^{\sigma}\left(-\frac{1}{\sigma^2}\right)$$

  Both terms are negative given $\sigma \in (0, 1)$ and $e \geq 0$. QED.

Now for the result on τ^a. Differentiating the lobby's condition with respect to σ, we have

$$\frac{\partial\Pi}{\partial\tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} + \frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} + \frac{\partial\Pi}{\partial\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} = 0$$

$$\frac{\mathrm{d}\tau^a}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} = \left[-\frac{\frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}\gamma} + \frac{\partial\Pi}{\partial\gamma}}{\frac{\partial\Pi}{\partial\tau^a}}\right]\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma} \tag{1}$$

We know from Corollary 5 that $\frac{\mathrm{d}\tau^a}{\mathrm{d}\gamma}$ is positive, and we've just shown that $\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma}$ is negative. Thus $\frac{\mathrm{d}\tau^a}{\mathrm{d}\sigma} < 0$.

Write constraint:

$$\overline{e}(\tau^a) - \pi(\tau^b(\overline{e}(\tau^a))) + \pi(\tau^a) - e_a - \frac{\delta_{\mathrm{L}} + \delta_{\mathrm{L}}^{T+1}}{1 - \delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right] = 0$$

For now, assume this has an interior solution so calculus works.

First, what does T do to $\overline{e}(\tau^a)$?
By the Implicit Function Theorem:

$$\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} = -\frac{\frac{\partial\Omega}{\partial T}}{\frac{\partial\Omega}{\partial\overline{e}}} = -\frac{-\frac{\delta_{\mathrm{ML}}^{T+1}\ln\delta_{\mathrm{ML}}}{1-\delta_{\mathrm{ML}}}\left[W_{\mathrm{ML}}(\gamma(\overline{e}),\boldsymbol{\tau^a})-W_{\mathrm{ML}}(\gamma(\overline{e}),\boldsymbol{\tau^{tw}})\right]}{\frac{\delta_{\mathrm{ML}}-\delta_{\mathrm{ML}}^{T+1}}{1-\delta_{\mathrm{ML}}}\frac{\partial\gamma}{\partial\overline{e}}[\pi(\tau^a)-\pi(\tau^{tw})]-\frac{\partial\gamma}{\partial\overline{e}}\left[\pi(\tau^b(\overline{e}))-\pi(\tau^a)\right]} > 0 \tag{2}$$

So if $T\uparrow$ then $\overline{e}\uparrow$.

Now, want to know about effect of T on τ^a.

$$\frac{\partial\Pi}{\partial\tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial\Pi}{\partial T} = 0$$

Because $\frac{\partial\Pi}{\partial T} = \frac{\ln\delta_{\mathrm{L}}\delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]$, we are looking for

$$\frac{\mathrm{d}\tau^a}{\mathrm{d}T} = -\frac{\frac{\partial\Pi}{\partial\overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial\Pi}{\partial T}}{\frac{\partial\Pi}{\partial\tau^a}} = \frac{-\left(1-\frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\cdot\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} - \frac{\ln\delta_{\mathrm{L}}\delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]}{\left(1+\frac{\delta_{\mathrm{L}}-\delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\right)\left[\frac{\partial\pi(\tau^a)}{\partial\tau^a} - \frac{\partial e_a}{\partial\tau^a}\right]} \tag{3}$$

The proof of Corollary 3 shows that $\left(1-\frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)$ is positive, and the above result shows that $\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}$ is positive. The second term is positive since net profits are maximized at e_{tw} and $\delta_L < 1$ so that its log is negative. With the leading negative signs, the numerator has both a negative and a positive part. This is not changed by the denominator, as the arguments given in the proof of Corollary 1 show that the denominator is positive.

- Remember simplified version where actors are infinitely patient
- Can play around with trick of looking for minimum τ^a by setting this equal to zero

As $\sigma\downarrow$, $\overline{e}\downarrow$ for a given τ^a.

- Remember that change in $\overline{e}$ is really a change in the $(\tau^a, \overline{e}(\tau^a))$ schedule that derives from legislature's constraint
- So τ^a has to be raised to satisfy lobby's constraint

  - Net profits are greatest at τ^{tw}, so relative gap between net profits at τ^a and τ^{tw} (future) closes faster than that between break profits and trade agreement profits (present)

- How much τ^a adjusts depends on magnitude of $\frac{\delta_{\mathrm{L}}+\delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}$
- $\pi(\tau^b(\bar{e}(\tau^a))) - \bar{e}(\tau^a)$ is negative, gets less negative when $\bar{e}$ is reduced.
- $\pi(\tau^a) - e_a$ is positive, becomes larger as τ^a rises

$$0 \geq -\left[\bar{e}(\tau^a) - \pi(\tau^b(\bar{e}(\tau^a))) + \pi(\tau^a) - e_a\right] + \frac{\delta_{\mathrm{L}} + \delta_{\mathrm{L}}^{T+1}}{1-\delta_{\mathrm{L}}}\left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]$$

Where present part of constraint in on left and future part is on right. Remember present part must be negative.

Let $\gamma(e) = 1 + \frac{1}{1-\theta}e^{1-\theta}$. Or $\gamma(e) = 1 + \frac{1}{\sigma}e^{\sigma}$.

- $\frac{\partial \gamma}{\partial e} = e^{\sigma-1} > 0 \ \forall \sigma$
- $\frac{\partial^2 \gamma}{\partial \sigma \partial e} = \ln e \cdot e^{\sigma-1} < 0$ for $e < 1$

i.e. as $\sigma \downarrow$, $\frac{\partial \gamma}{\partial e} \uparrow$.

Can I show that the optimal T can go either way when σ changes? That is, a counterexample to my quasi-result?

- My result says that as lobby gets stronger ($\frac{\partial \gamma}{\partial e} \uparrow$, so $\sigma \downarrow$), T should have to decrease.
- If optimal T increases when σ decreases (i.e. if T is decreasing in σ), this is a counterexample.
  - I think it's possible that both cases can happen depending on other parameters, like δ.

Look at legislature's constraint:

$$\frac{\delta_{\mathrm{ML}} - \delta_{\mathrm{ML}}^{T+1}}{1-\delta_{\mathrm{ML}}}\left[W_{\mathrm{ML}}(\gamma(e_b), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(e_b), \boldsymbol{\tau^{tw}})\right] \geq$$
$$W_{\mathrm{ML}}(\gamma(e_b), \tau^b(e_b), \tau^{*a}) - W_{\mathrm{ML}}(\gamma(e_b), \boldsymbol{\tau^a})$$

If T is too small, future gap can be smaller than current-period gap. So if T gets too short, can't enforce on legislature.

Note that changing σ changes τ^{tw}

- e_{tw} is solution to $\frac{\partial \pi}{\partial \tau}\frac{\partial \tau}{\partial \gamma}\frac{\partial \gamma}{\partial e} = 1$; or $\frac{\partial \pi}{\partial \tau}\frac{\partial \tau}{\partial \gamma} = \frac{1}{\frac{\partial \gamma}{\partial e}}$
- When $\frac{\partial \gamma}{\partial e} \uparrow$, RHS $\downarrow$, so LHS must go down.

4 Optimal T result with perfect patience

November 13, 2015

- I know that $\frac{\partial \tau^a}{\partial T}$ has both a negative and positive part.
  - Shown for the general case where δ is anything.
- Now I want to know if/when $\frac{\partial^2 \tau^a}{\partial T^2}$ is positive so that I could set $\frac{\partial \tau^a}{\partial T} = 0$ and optimal T (the one that minimizes τ^a)
  - Would need to be careful of corner solutions
  - Will do this for case where $\delta \to 1$
    * This means $\frac{\delta - \delta^{T+1}}{1-\delta} \to T$ and $-\frac{\delta^{T+1}\ln\delta}{1-\delta} \to 1$

Start with simplifying result for $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$.
Again:

$$\frac{\partial \Pi}{\partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Pi}{\partial T} = 0 \tag{4}$$

$$\frac{\mathrm{d}\tau^a}{\mathrm{d}T} = -\frac{\frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Pi}{\partial T}}{\frac{\partial \Pi}{\partial \tau^a}} = \frac{-\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\cdot\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + [\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a]}{(1+T)\left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right]} \tag{5}$$

- $\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)$ is positive
- $\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}$ is positive:

$$\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} = -\frac{\frac{\partial \Omega}{\partial T}}{\frac{\partial \Omega}{\partial \overline{e}}} = \frac{W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^{tw}})}{T\frac{\partial\gamma}{\partial\overline{e}}[\pi(\tau^{tw}) - \pi(\tau^a)] + \frac{\partial\gamma}{\partial\overline{e}}\left[\pi(\tau^b(\overline{e})) - \pi(\tau^a)\right]} > 0 \tag{6}$$

- denominator is positive (proof of Corollary 1)

Now, on to $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$

$$\frac{\partial}{\partial T}\left[\frac{\partial \Pi}{\partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Pi}{\partial T}\right] = 0$$

$$\frac{\partial \Pi}{\partial \tau^a}\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2} + \frac{\partial^2 \Pi}{\partial T \partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} + \frac{\partial^2 \Pi}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Pi}{\partial T^2} = 0 \tag{7}$$

Going to need:

$$\frac{\partial^2 \Pi}{\partial T \partial \tau^a} = \frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a} > 0$$

$$\frac{\partial^2 \Pi}{\partial T \partial \overline{e}} = 0$$

$$\frac{\partial^2 \Pi}{\partial T^2} = 0$$

To get $\frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}T^2}$, have to do a little more work.

$$\frac{\partial}{\partial T}\left[\frac{\partial \Omega}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Omega}{\partial T}\right] = 0$$

$$\frac{\partial \Omega}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} + \frac{\partial^2 \Omega}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Omega}{\partial T^2} = 0$$

$$\frac{\partial^2 \Omega}{\partial T^2} = \frac{\partial}{\partial T}\left(\frac{\partial \Omega}{\partial T}\right) = 0$$

$$\frac{\partial^2 \Omega}{\partial T \partial \overline{e}} = \frac{\partial}{\partial T}\left(\frac{\partial \Omega}{\partial \overline{e}}\right) = \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^a) - \pi(\tau^{tw})\right] < 0$$

So,

$$\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} = \frac{-\frac{\partial^2 \Omega}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} - \frac{\partial^2 \Omega}{\partial T^2}}{\frac{\partial \Omega}{\partial \overline{e}}} = \frac{-\frac{\partial^2 \Omega}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}}{\frac{\partial \Omega}{\partial \overline{e}}} < 0$$

Now we have everything we need for $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$. Rearranging Equation 7, we have

$$\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2} = -\frac{\frac{\partial^2 \Pi}{\partial T \partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2} + \frac{\partial^2 \Pi}{\partial T \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Pi}{\partial T^2}}{\frac{\partial \Pi}{\partial \tau^a}}$$

Elements of the last two terms in the denominator have been shown to be zero when $\delta \to 1$, so we have

$$\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2} = -\frac{\frac{\partial^2 \Pi}{\partial T \partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}^2\overline{e}}{\mathrm{d}T^2}}{\frac{\partial \Pi}{\partial \tau^a}}$$

The denominator is positive, so as goes the numerator, so goes the whole thing. So let's write out the numerator:

$$\left(\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right)\left[\frac{\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\cdot\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} - \left[\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right]}{(1+T)\left[\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right]}\right] -$$
$$\left(1 - \frac{\mathrm{d}\pi}{\mathrm{d}\overline{e}}\right)\frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^{tw}) - \pi(\tau^a)\right]\frac{W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^{tw}})}{\left[T\frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^{tw}) - \pi(\tau^a)\right] + \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^b(\overline{e})) - \pi(\tau^a)\right]\right]^2} \quad (8)$$

$$\frac{\left(1-\frac{\mathrm{d}\pi}{\mathrm{d}\bar{e}}\right)\cdot\frac{\mathrm{d}\bar{e}}{\mathrm{d}T}-\left[\pi(\tau^{tw})-e_{tw}-\pi(\tau^{a})+e_{a}\right]}{(1+T)}-$$

$$\frac{\left(1-\frac{\mathrm{d}\pi}{\mathrm{d}\bar{e}}\right)\frac{\partial\gamma}{\partial\bar{e}}\left[\pi(\tau^{tw})-\pi(\tau^{a})\right]\left[W_{\mathrm{ML}}(\gamma(\bar{e}),\boldsymbol{\tau^{a}})-W_{\mathrm{ML}}(\gamma(\bar{e}),\boldsymbol{\tau^{tw}})\right]}{\left[T\frac{\partial\gamma}{\partial\bar{e}}\left[\pi(\tau^{tw})-\pi(\tau^{a})\right]+\frac{\partial\gamma}{\partial\bar{e}}\left[\pi(\tau^{b}(\bar{e}))-\pi(\tau^{a})\right]\right]^{2}} \quad (9)$$

4.1 Solutions for optimal T

On the solution(s) to $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$:

- $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$ has one term that is positive, one term that reverses the sign of $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$
- this means if $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}=0$, it can't be a global max
  - The slope would have to start positive, end negative.
  - The second term in $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$ would start negative, end positive, so whole term would be positive as T gets large
  - This means SOC CAN'T be negative everywhere. A contradiction
- It's perfectly consistent to be a global min
  - Let slope start negative, end positive.
  - The second term in $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$ would start positive, end negative, so whole term would start positive. As T gets large, either stay positive (global min), or turn negative
  - If start positive and turn negative, would be an inflection point and slope doesn't actually turn positive
- If $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}T^2}$ positive everywhere and $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}(0)<0$ then interior min
- We know the negative part of $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$ gets smaller as $T\uparrow$
  - If negative to start out, eventually becomes positive
  - If positive to start out, gets more positive.
    - In this case, want T as small as possible.

In line with last point, must fully explore corner solutions

- What is TRUE? (not just what would be convenient for me)

5 Cross Partial w.r.t. σ

$$\frac{\partial \Pi}{\partial \tau^a}\frac{\mathrm{d}^2\tau^a}{\mathrm{d}\sigma \mathrm{d}T} + \frac{\partial^2 \Pi}{\partial \sigma \partial \tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} + \frac{\partial \Pi}{\partial \overline{e}}\frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}\sigma \mathrm{d}T} + \frac{\partial^2 \Pi}{\partial \sigma \partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Pi}{\partial \sigma \partial T} = 0 \tag{10}$$

- Solving for second term
- Already know first (+), fourth (probably convex), fifth (+), eighth (+)
- Leaves four new terms to solve for. Third, sixth, seventh, ninth.

Ninth. Need:

$$\frac{\partial}{\partial \sigma}\left[\frac{\partial \Pi}{\partial T}\right] = \frac{\partial}{\partial \sigma}\left[-\left(\pi(\tau^{tw}) - e_{tw} - \pi(\tau^a) + e_a\right)\right]$$

- σ changes τ^{tw}
- leave τ^a alone as it's an equilibrium object
- also changes how much lobby has to pay for τ^{tw} and τ^a

- Conjecture:
  - When $\sigma \uparrow$, the lobby weakens $\left(\frac{\partial \gamma}{\partial e} \downarrow\right)$. This means $\pi(\tau^{tw}) - e_{tw} \downarrow$
  - So probably overall gap goes down because lobby can't get as much in the trade war
    * i.e. when lobby gets weaker, its future gain is less because it can't exert as much influence in the future (this makes sense)
  - REMEMBER: this enters negatively
    * smaller gap means tighter constraint
    * a larger gap would mean a tighter constraint

Third. Need:

$$\frac{\partial}{\partial \sigma}\left[\frac{\partial \Pi}{\partial \tau^a}\right] = \frac{\partial}{\partial \sigma}\left[(1+T)\left(\frac{\partial \pi(\tau^a)}{\partial \tau^a} - \frac{\partial e_a}{\partial \tau^a}\right)\right]$$

- For any given τ, σ doesn't affect $\pi(\tau)$.
- But it *does* affect how much has to paid to get τ.
- When $\sigma \uparrow$, have to pay more and more. This is the first derivative.

- As $\tau^a \uparrow$, $e_a \uparrow$, so $-e_a \downarrow$. As $\sigma \uparrow$, $e_a \uparrow$ further, so $-e_a \downarrow$ further. So this term must be negative overall.

Seventh. Note that

$$\frac{\partial \Pi}{\partial \overline{e}} = 1 - \frac{\partial \pi}{\partial \overline{e}} = 1 - \frac{\partial \pi}{\partial \tau}\frac{\partial \tau^b}{\partial \gamma}\frac{\partial \gamma}{\partial \overline{e}}$$

Need

$$\frac{\partial}{\partial \sigma}\left[\frac{\partial \Pi}{\partial \overline{e}}\right] = \frac{\partial}{\partial \sigma}\left[1 - \frac{\partial \pi}{\partial \tau}\frac{\partial \tau^b}{\partial \gamma}\frac{\partial \gamma}{\partial \overline{e}}\right] = -\frac{\partial}{\partial \sigma}\left[\frac{\partial \pi}{\partial \tau}\frac{\partial \tau^b}{\partial \gamma}\frac{\partial \gamma}{\partial \overline{e}}\right]$$
$$= \frac{\partial \pi}{\partial \tau}\left[\frac{\partial \tau^b}{\partial \gamma}\frac{\partial^2 \gamma}{\partial \sigma \partial \overline{e}} + \frac{\partial^2 \tau^b}{\partial \sigma \partial \gamma}\frac{\partial \gamma}{\partial \overline{e}}\right] + \frac{\partial^2 \pi}{\partial \sigma \partial \tau}\frac{\partial \tau^b}{\partial \gamma}\frac{\partial \gamma}{\partial \overline{e}}$$

- Since σ is not involved in how τ maps to profits, $\frac{\partial^2 \pi}{\partial \sigma \partial \tau} = 0$ and the second additive term is zero
- We know that $\frac{\partial \pi}{\partial \tau} > 0$, so the sign of the seventh term is determined by the sign of what's inside the square brackets x (-1)
- $\frac{\partial^2 \tau^b}{\partial \sigma \partial \gamma} = 0$: once γ is determined, it maps directly into τ given the form of the welfare function. σ only controls how e maps into γ.
- So we're left with $\frac{\partial \tau^b}{\partial \gamma}\frac{\partial^2 \gamma}{\partial \sigma \partial \overline{e}}$. We know the first term is positive, so just need to investigate $\frac{\partial^2 \gamma}{\partial \sigma \partial \overline{e}}$.
  - $\frac{\partial^2 \gamma}{\partial \sigma \partial e} = \ln e \cdot e^{\sigma - 1} < 0$ for $e < 1$
  - This means that the positive slope becomes larger as $\sigma \uparrow$. That is, the influence increases
  - $\frac{\partial}{\partial \sigma}\left[\frac{\partial \pi}{\partial e}\right] < 0$, so $\frac{\partial}{\partial \sigma}\left[1 - \frac{\partial \pi}{\partial e}\right] > 0$

Sixth

$$\frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}\sigma \mathrm{d}T}$$

$$\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} = -\frac{\frac{\partial \Omega}{\partial T}}{\frac{\partial \Omega}{\partial \overline{e}}} = \frac{W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^{tw}})}{T\frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^{tw}) - \pi(\tau^a)\right] + \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^b(\overline{e})) - \pi(\tau^a)\right]} > 0$$
$$\frac{\partial}{\partial \sigma}\left[\frac{\partial \Omega}{\partial \overline{e}}\frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial \Omega}{\partial T}\right] = 0$$

$$\frac{\partial \Omega}{\partial \overline{e}} \frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}\sigma \mathrm{d}T} + \frac{\partial^2 \Omega}{\partial \sigma \partial \overline{e}} \frac{\mathrm{d}\overline{e}}{\mathrm{d}T} + \frac{\partial^2 \Omega}{\partial \sigma \partial T} = 0$$

We're solving for $\frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}\sigma \mathrm{d}T}$. Know that $\frac{\partial \Omega}{\partial T}$ is positive.

- Know that $\frac{\mathrm{d}\overline{e}}{\mathrm{d}T}$ is positive.
- Need two terms:
  - $\frac{\partial^2 \Omega}{\partial \sigma \partial T} = \frac{\partial}{\partial \sigma}\left(\frac{\partial \Omega}{\partial T}\right) = \frac{\partial}{\partial \sigma}\left(T \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^a) - \pi(\tau^{tw})\right] + \frac{\partial \gamma}{\partial \overline{e}}\left[\pi(\tau^a) - \pi(\tau^b(\overline{e}))\right]\right)$

    $$= T\frac{\partial^2 \gamma}{\partial \sigma \partial \overline{e}}\left[\pi(\tau^a) - \pi(\tau^{tw})\right] - T\frac{\partial \gamma}{\partial \overline{e}}\frac{\partial \pi(\tau^{tw})}{\partial \sigma} + \frac{\partial^2 \gamma}{\partial \sigma \partial \overline{e}}\left[\pi(\tau^a) - \pi(\tau^b(\overline{e}))\right] > 0$$

    Every term is negative except the T and the $\frac{\partial \gamma}{\partial \overline{e}}$, so the whole expression is positive.
  - $\frac{\partial^2 \Omega}{\partial \sigma \partial \overline{e}} = \frac{\partial}{\partial \sigma}\left(\frac{\partial \Omega}{\partial \overline{e}}\right) = \frac{\partial}{\partial \sigma}\left(W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^a}) - W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^{tw}})\right)$
    * Get $\frac{\partial \gamma}{\partial \sigma}\left[\pi_x(\tau^a) - \pi_x(\tau^{tw})\right]$ term, which is negative times negative so positive
    * And get $-\frac{\mathrm{d}}{\mathrm{d}\sigma}W_{\mathrm{ML}}(\gamma(\overline{e}), \boldsymbol{\tau^{tw}})$ term, which is overall negative. But can be split into social welfare part and $(\gamma - 1)\left(-\frac{\partial \pi_x(\tau^{tw})}{\partial \tau^{tw}}\frac{\partial \tau^{tw}}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma}\right)$ part. First one is clearly positive (remember, $\sigma \uparrow$ makes $\gamma \downarrow$), second is negative (these include leading negative sign).
    * Concavity of $\pi(\cdot)$ implies that $\frac{\partial \pi_x(\tau^{tw})}{\partial \tau^{tw}} \geq \pi_x(\tau^{tw}) - \pi_x(\tau^a)$. Can use this to show that the whole term is positive as long as

      $$\pi_x(\tau^a) - \pi_x(\tau^{tw}) \geq \frac{\partial \pi_x(\tau^{tw})}{\partial \tau^{tw}}\frac{\partial \tau^{tw}}{\partial \gamma}(\gamma - 1)$$
- As long as $\frac{\partial^2 \Omega}{\partial \sigma \partial \overline{e}} \geq 0$, $\frac{\mathrm{d}^2 \overline{e}}{\mathrm{d}\sigma \mathrm{d}T} > 0$.

What is intuition?

- Intuition for $\frac{\partial \overline{e}}{\partial T}$ result: as T $\uparrow$, the legisltaure doesn't want to break the agreement. So $\overline{e}$ has to be raised to make it indifferent again because that makes it value profits more: make current gain larger through increased weight on profits AND through larger τ^b; make future loss smaller by increasing γ.
  - Now, if $\sigma \uparrow$, lobby gets weaker, so legislature cares less about profits, all else equal.
    * Will have to increase e by even more to get a big enough increase in γ and τ^b.
    * BUT have to be careful, because τ^{tw} also falls, which reduces the future loss and somewhat reduces the required e to create indifference. Without some conditions, it's possible that this effect could be stronger (at least I haven't ruled it out).

- Alternatively, from the point of view of $\frac{\partial}{\partial T}\frac{\partial \bar{e}}{\partial \sigma}$: as lobby gets weaker, cutoff e goes up. And the increase in $\bar{e}$ gets larger as $T\ \uparrow$

Now putting it all together

- If $\gamma \uparrow$, T^* should get smaller.
  - Requires slope of $\frac{\partial \tau^a}{\partial T}$ to increase when γ increases
  - i.e. $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}\gamma \mathrm{d}T} > 0$
- If $\sigma \uparrow$, T^* should get larger.
  - Because $\sigma \uparrow \Rightarrow \gamma \downarrow$
  - Requires slope of $\frac{\partial \tau^a}{\partial T}$ to decrease when σ increases
  - i.e. $\frac{\mathrm{d}^2\tau^a}{\mathrm{d}\gamma \mathrm{d}T} < 0$

$$\frac{\mathrm{d}^2\tau^a}{\mathrm{d}\sigma \mathrm{d}T} = \frac{-\frac{\partial^2\Pi}{\partial\sigma\partial\tau^a}\frac{\mathrm{d}\tau^a}{\mathrm{d}T} - \frac{\partial\Pi}{\partial\bar{e}}\frac{\mathrm{d}^2\bar{e}}{\mathrm{d}\sigma\mathrm{d}T} - \frac{\partial^2\Pi}{\partial\sigma\partial\bar{e}}\frac{\mathrm{d}\bar{e}}{\mathrm{d}T} - \frac{\partial^2\Pi}{\partial\sigma\partial T}}{\frac{\partial\Pi}{\partial\tau^a}}$$

I have every individual term positive except $\frac{\partial^2\Pi}{\partial\sigma\partial\tau^a}$ and $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$.

- $\frac{\partial^2\Pi}{\partial\sigma\partial\tau^a}$ is negative
- $\frac{\mathrm{d}\tau^a}{\mathrm{d}T}$ is probably convex. If so, taken with two negative signs, this makes the negative slope more negative and positive slope more positive. At any rate, everywhere the slope is negative becomes more negative.

At some point, may want to verify via Young's theorem (take derivative of lobby's condition w.r.t. σ first, then w.r.t. T):

$$\frac{\partial \Pi}{\partial \tau^a}\frac{\partial}{\partial T}\left[\frac{\partial \tau^a}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma}\right]+\frac{\partial^2 \Pi}{\partial T \partial \tau^a}\left[\frac{\partial \tau^a}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma}\right]+\frac{\partial \Pi}{\partial \overline{e}}\frac{\partial}{\partial T}\left[\frac{\partial \overline{e}}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma}\right]+\frac{\partial^2 \Pi}{\partial T \partial \overline{e}}\left[\frac{\partial \overline{e}}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma}\right]+\frac{\partial \Pi}{\partial \gamma}\frac{\partial^2 \gamma}{\partial T \partial \sigma}+\frac{\partial^2 \Pi}{\partial T \partial \gamma}\frac{\partial \gamma}{\partial \sigma}=0$$