Equilibrium Strategy Profiles (For JLEO Revision)

First, need to get timing clear for evolution of state variable. Take state variable to be q_{st}.

- q_{s1}: beginning of the world (or, $Q_{s0} - \mu = q_{s1}$)
- $Q_{s1} = q_{s1} + R_{p1} + R_{c1}$
  - $q_{s2} = Q_{s1} - \mu$

Strategies (this is for the most interesting state variable, but could write it out for the other five)

- For patron: function of q_{st}
- For c: function of $q_{st} + R_{pt}$
- For s: function of $Q_{st} = q_{st} + R_{pt} + R_{ct}$
- For g: not a function of q_{st} at all

According to Mailath and Samuelson (2006) p. 177.

- The strategy profile σ is a stationary Markov strategy if for any two ex post histories $\tilde{h}^t$ and $\tilde{h}^\tau$ (of equal or different lengths) terminating in the same state, $\sigma\left(\tilde{h}^t\right) = \sigma\left(\tilde{h}^\tau\right)$.
- The strategy profile σ is a stationary Markov equilibrium if σ is a stationary Markov strategy profile and a subgame-perfect equilibrium.

We have six state variables, so $s = (q_s, q_g, l_s, l_g, w_s, w_g)$.

- Strategies are a function of all six state variables
  - We can restrict attention to relevant ranges of the state variables. CAN I CHANGE PROP 1 PARTS 1 AND 2 TO BE INITIAL VALUES OF STATE VARIABLES? If so, then these conditions and what is necessary in each period to make the game continue.

- Strategies for the patron and c are also how much to invest in each of the six state variables. Some can be ruled out by preference assumptions:
  - c dislikes war, so will never invest in w_s or w_g. It would also not want to make the government lose, so won't invest in l_g either.
  - Because the patron's preferences are aligned with the secessionists and against the government, it never invests in w_g or l_s.

This leaves four state varibles in which the patron may invest: q_s, q_g, l_g and w_s. Three in which c may invest: q_s, q_g, l_s.

- Also, Gov't / Secessionists: Choose unilateral, simultaneous best responses depending on magnitudes of Q_{i1}, L_{i1} and ω_{i1}
  - Game only continues if (SQ,SQ) or (Cede, Cede) was played

Assumptions

- $Q_{s0} \geq L_{s0} \Rightarrow q_{s1} + \mu \geq l_{s1}$
- $Q_{g0} \geq L_{g0} \Rightarrow q_{g1} \geq l_{g1}$

Period 1

1. Patron would like $Q_{s1} \geq L_{s1}$ to prevent c from incentivizing secessionists to Cede
   - Equivalent to $q_{s1} + R_{p1} \geq l_{s1} + R_{c1}$ / $Q_{s0} + \mu + R_{p1} \geq L_{s0} + R_{c1}$ / $R_{p1} \geq R_{c1} - (Q_{s0} - \mu - L_{s0})$
     - $R_{c1} \leq \frac{\beta}{1-\delta} \Rightarrow R_{p1} \geq \frac{\beta}{1-\delta} - (Q_{s0} - \mu - L_{s0})$ allows the patron to ensure the original inequality
     - Initial assumption means $R_{p1} \geq \frac{\beta}{1-\delta} + \mu$ is a tighter condition
     - Since patron is willing to pay up to $R_{p1} = \frac{\alpha}{1-\delta}$, when $\frac{\alpha}{1-\delta} \geq \frac{\beta}{1-\delta} + \mu$, it will invest $R_{p1} = \frac{\beta}{1-\delta} - (Q_{s0} - \mu - L_{s0})$ to augment q_{s1} if this is greater than 0. Else, $R_{p1} = 0$.
       (a) If $R_{p1} + (Q_{s0} - \mu - L_{s0}) \geq \frac{\beta}{1-\delta}$, $R_{c1} = 0$. (SQ,SQ) is played and game continues.
       (b) If instead $R_{p1} + (Q_{s0} - \mu - L_{s0}) < \frac{\beta}{1-\delta}$ (i.e. assumption doesn't hold), $R_{c1} = l_{s1} - (q_{s1} + R_{p1}) + \varepsilon$ to augment l_{s1}. Note optimal R_{p1} in this case is 0.

* (SQ,Cede) is played and game ends

2. Patron would like $L_{g1} > Q_{g1}$ to achieve recognition directly. Must invest to augment l_{g1}

   - Equivalent to $l_{g1} + R_{p1} > q_{g1} + R_{c1}$ / $L_{g0} + R_{p1} > Q_{g0} + R_{c1}$ / $R_{p1} > R_{c1} + (Q_{g0} - L_{g0})$
     - $R_{c1} \leq \frac{\nu}{1-\delta} \Rightarrow R_{p1} > \frac{\nu}{1-\delta} + (Q_{g0} - L_{g0})$ allows the patron to ensure the original inequality
     - Since patron is willing to pay up to $R_{p1} = \frac{\lambda+\mu+\beta}{1-\delta}$, when $\frac{\lambda+\mu+\beta}{1-\delta} \leq \frac{\nu}{1-\delta}$, patron will invest 0 in l_{g1}. $R_{c1} = 0$ as well.
     - If instead assumption 4 did not hold and $R_{p1} > \frac{\nu}{1-\delta} + (Q_{g0} - L_{g0})$, it will invest this amount to augment l_{g1}. Again, $R_{c1} = 0$. (Cede, SQ) is played and game ends.

3. Patron might want to instigate fighting to have a p_{1s} probability of achieving recognition indirectly. If $\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta}$ is positive, invest in w_{s1} to change ω_{s1}. Will not invest when the quantity is non-positive. Inequality is $\omega_{s1} > Q_{s1}$

   - $\omega_{s1} = -\zeta_s(1-\delta) + W_{s1}p_{1s} + L_{s1}(1-p_{1s}) > Q_{s1}$
   - $-\zeta_s(1-\delta) + (W_{s0} + R_{p1})p_{1s} + L_{s0}(1-p_{1s}) > Q_{s0} - \mu + R_{c1}$
   - $R_{p1}p_{1s} > R_{c1} - \mu + Q_{s0} - (-\zeta_s(1-\delta) + W_{s0}p_{1s} + L_{s0}(1-p_{1s}))$
   - $R_{c1} \leq \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta}$, so if $R_{p1}p_{1s} > \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu + Q_{s0} - (-\zeta_s(1-\delta) + W_{s0}p_{1s} + L_{s0}(1-p_{1s}))$, $R_{c1} = 0$ and (SQ,Fight) is played in third stage. Outcome depends on war lottery.
   - Patron will spend at most $\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu$
     - So need

$$
\begin{aligned}
p_{1s}\left[\tfrac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu\right] &< \tfrac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu \leq \tfrac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu \\
&\leq \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu + Q_{s0} - (-\zeta_s(1-\delta) + W_{s0}p_{1s} + L_{s0}(1-p_{1s})) \quad (1)
\end{aligned}
$$

4. Patron could also invest in q_g to counter investment by c in government's payoffs. But this would only been needed if c had incentive to invest in g playing Fight, and we've ruled that out.

Now generalize profiles for all t

1. Patron would like $Q_{st} \geq L_{st}$ to prevent c from incentivizing secessionists to Cede

   - Equivalent to $q_{st} + R_{pt} \geq l_{st} + R_{ct}$ / $Q_{s,t-1} + \mu + R_{pt} \geq L_{s,t-1} + R_{ct}$ / $R_{pt} \geq R_{ct} - (Q_{s,t-1} - \mu - L_{s,t-1})$

- $R_{ct} \leq \frac{\beta}{1-\delta} \Rightarrow R_{pt} \geq \frac{\beta}{1-\delta} - (Q_{s,t-1} - \mu - L_{s,t-1})$ allows the patron to ensure the original inequality
- Assumption 1 combined with equilibrium play (equivalently, if $L_{s,t-1} > Q_{s,t-1}$, the game would have ended before the start of period t) means $R_{pt} \geq \frac{\beta}{1-\delta} + \mu$ is a tighter condition
- Since patron is willing to pay up to $R_{pt} = \frac{\alpha}{1-\delta}$, when $\frac{\alpha}{1-\delta} \geq \frac{\beta}{1-\delta} + \mu$, it will invest $R_{pt} = \frac{\beta}{1-\delta} - (Q_{s,t-1} - \mu - L_{s,t-1})$ to augment q_{st} if this is greater than 0. Else, $R_{pt} = 0$.
  (a) If $R_{pt} + (Q_{s,t-1} - \mu - L_{s,t-1}) \geq \frac{\beta}{1-\delta}$, $R_{ct} = 0$. (SQ,SQ) is played and game continues.
  (b) If instead $R_{pt} + (Q_{s,t-1} - \mu - L_{s,t-1}) < \frac{\beta}{1-\delta}$ (i.e. assumption 3 doesn't hold), $R_{ct} = l_{st} - (q_{st} + R_{pt}) + \varepsilon$ where ε is small to augment l_{st}. Note optimal R_{pt} in this case is 0.
    * (SQ,Cede) is played and game ends

2. Patron would like $L_{gt} > Q_{gt}$ to achieve recognition directly. Must invest to augment l_{gt}

- Equivalent to $l_{gt} + R_{pt} > q_{gt} + R_{ct}$ / $L_{g,t-1} + R_{pt} > Q_{g,t-1} + R_{ct}$ / $R_{pt} > R_{ct} + (Q_{g,t-1} - L_{g,t-1})$
  - $R_{ct} \leq \frac{\nu}{1-\delta} \Rightarrow R_{pt} > \frac{\nu}{1-\delta} + (Q_{g,t-1} - L_{g,t-1})$ allows the patron to ensure the original inequality
  - Since patron is willing to pay up to $R_{pt} = \frac{\lambda+\mu+\beta}{1-\delta} - \mu$, when $\frac{\lambda+\mu+\beta}{1-\delta} - \mu \leq \frac{\nu}{1-\delta}$, patron will invest 0 in l_{gt}. $R_{ct} = 0$ as well.
  - If instead assumption 4 did not hold and $R_{pt} > \frac{\nu}{1-\delta} + (Q_{g,t-1} - L_{g,t-1})$, patron will invest this amount to augment l_{gt}. Again, $R_{ct} = 0$. (Cede, SQ) is played and game ends.

3. Patron might want to instigate fighting to have a p_{1s} probability of achieving recognition indirectly. If $\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu$ is positive, invest in w_{st} to change ω_{st}. Will not invest when the quantity is non-positive. Inequality is $\omega_{st} > Q_{st}$

- $\omega_{st} = -\zeta_s(1-\delta) + W_{st}p_{1s} + L_{st}(1-p_{1s}) > Q_{st}$
- $-\zeta_s(1-\delta) + (W_{s,t-1} + R_{pt})p_{1s} + L_{s,t-1}(1-p_{1s}) > Q_{s,t-1} - \mu + R_{ct}$
- $R_{pt}p_{1s} > R_{ct} - \mu + Q_{s,t-1} - (-\zeta_s(1-\delta) + W_{s,t-1}p_{1s} + L_{s,t-1}(1-p_{1s}))$
- $R_{ct} \leq \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta}$, so if $R_{pt}p_{1s} > \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu + Q_{s,t-1} - (-\zeta_s(1-\delta) + W_{s,t-1}p_{1s} + L_{s,t-1}(1-p_{1s}$ $R_{ct} = 0$ and (SQ,Fight) is played in third stage. Outcome depends on war lottery.
- Patron will spend at most $\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu$

- So need

$$p_{1s}\left[\tfrac{\lambda p_{1s}-\alpha(1-p_{1s})+\mu+\beta}{1-\delta}-\mu\right] < \tfrac{\lambda p_{1s}-\alpha(1-p_{1s})+\mu+\beta}{1-\delta}-\mu \leq \tfrac{\nu p_{1s}-\beta(1-p_{1s})}{1-\delta}-\mu$$
$$\leq \frac{\nu p_{1s}-\beta(1-p_{1s})}{1-\delta}-\mu+Q_{s,t-1}-(-\zeta_s(1-\delta)+W_{s,t-1}p_{1s}+L_{s,t-1}(1-p_{1s})) \tag{2}$$

4. Patron could also invest in q_g to counter investment by c in government's payoffs. But this would only been needed if c had incentive to invest in g playing Fight, and we've ruled that out.

Prose version of equilibrium strategy profiles:

In period t, the patron invests $R_{pt} = \max\left\{\frac{\beta}{1-\delta} - \left(Q_{s,t-1} - \mu - L_{s,t-1}\right), 0\right\}$ to augment the status quo payoffs of the secessionists, q_{st}. Player c then chooses $R_{ct} = 0$ so that playing status quo is a best response for both inside players. (SQ,SQ) is played in the third stage and the game continues to period $t+1$.
If instead $R_{pt} < \frac{\beta}{1-\delta} - (Q_{s,t-1} - \mu - L_{s,t-1})$ (i.e. assumption 3 doesn't hold), player c will invest $R_{ct} = l_{st} - (q_{st} + R_{pt}) + \varepsilon$, for ε small, to augment l_{st}. In this case, status quo remains a best response for the government but cede becomes the best response for the secessionists. The game ends with the secessionist territory being reunited.

1. Patron would like $L_{gt} > Q_{gt}$ to achieve recognition directly. Must invest to augment l_{gt}
   - Equivalent to $l_{gt} + R_{pt} > q_{gt} + R_{ct}$ / $L_{g,t-1} + R_{pt} > Q_{g,t-1} + R_{ct}$ / $R_{pt} > R_{ct} + (Q_{g,t-1} - L_{g,t-1})$
     - $R_{ct} \leq \frac{\nu}{1-\delta} \Rightarrow R_{pt} > \frac{\nu}{1-\delta} + (Q_{g,t-1} - L_{g,t-1})$ allows the patron to ensure the original inequality
     - Since patron is willing to pay up to $R_{pt} = \frac{\lambda+\mu+\beta}{1-\delta} - \mu$, when $\frac{\lambda+\mu+\beta}{1-\delta} - \mu \leq \frac{\nu}{1-\delta}$, patron will invest 0 in l_{gt}. $R_{ct} = 0$ as well.
     - If instead assumption 4 did not hold and $R_{pt} > \frac{\nu}{1-\delta} + (Q_{g,t-1} - L_{g,t-1})$, patron will invest this amount to augment l_{gt}. Again, $R_{ct} = 0$. (Cede, SQ) is played and game ends.

2. Patron might want to instigate fighting to have a p_{1s} probability of achieving recognition indirectly. If $\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu$ is positive, invest in w_{st} to change ω_{st}. Will not invest when the quantity is non-positive. Inequality is $\omega_{st} > Q_{st}$
   - $\omega_{st} = -\zeta_s(1-\delta) + W_{st}p_{1s} + L_{st}(1-p_{1s}) > Q_{st}$
   - $-\zeta_s(1-\delta) + (W_{s,t-1} + R_{pt})p_{1s} + L_{s,t-1}(1-p_{1s}) > Q_{s,t-1} - \mu + R_{ct}$
   - $R_{pt}p_{1s} > R_{ct} - \mu + Q_{s,t-1} - (-\zeta_s(1-\delta) + W_{s,t-1}p_{1s} + L_{s,t-1}(1-p_{1s}))$
   - $R_{ct} \leq \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta}$, so if $R_{pt}p_{1s} > \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu + Q_{s,t-1} - (-\zeta_s(1-\delta) + W_{s,t-1}p_{1s} + L_{s,t-1}(1-p_{1s}$ $R_{ct} = 0$ and (SQ,Fight) is played in third stage. Outcome depends on war lottery.
   - Patron will spend at most $\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu$
     - So need

$$
\begin{aligned}
&p_{1s}\left[\frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu\right] < \frac{\lambda p_{1s} - \alpha(1-p_{1s}) + \mu + \beta}{1-\delta} - \mu \leq \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu \\
&\leq \frac{\nu p_{1s} - \beta(1-p_{1s})}{1-\delta} - \mu + Q_{s,t-1} - (-\zeta_s(1-\delta) + W_{s,t-1}p_{1s} + L_{s,t-1}(1-p_{1s}))
\end{aligned} \tag{3}
$$

3. Patron could also invest in q_g to counter investment by c in government's payoffs. But this would only been needed if c had incentive to invest in g playing Fight, and we've ruled that out.