

Neutrino - DM Scattering

Kiara Carloni

February 2, 2023

The models and calculations covered here feature in the following reference works:

- Constraining Dark Matter Neutrino Interaction with High Energy Neutrinos, Dec. 2016
- Imaging Galactic Dark Matter with High-Energy Cosmic Neutrinos, Mar. 2021
- Dark Matter-Neutrino Scattering in the Galactic Centre with IceCube, Adam McMullen (Thesis, May 2021)

1 Kinematics:

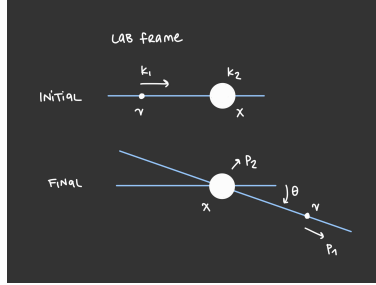
Neutrino - DM scattering is analogous to Compton scattering ($\gamma + e^- \rightarrow \gamma + e^-$) if we treat the neutrino as massless. We can therefore follow the derivation in Peskin+Schroeder.

From P+S (4.79), p.106, we get the formula for the differential cross-section for $2 \rightarrow n$ scattering with the final state phase space integral:

$$d\sigma = \frac{1}{2E_A 2E_B} \frac{1}{|k_A/E_A - k_B/E_B|} d\Pi_n |M|^2$$

We will work in the lab frame (the dark matter rest frame). Here, we can write the initial and final four-momenta as:

$$\begin{aligned} k_1 &= (E, 0, 0, E) \\ k_2 &= (m_\chi, 0, 0, 0) \\ p_1 &= (E', 0, -E' \sin \theta, E' \cos \theta) \\ p_2 &= (\sqrt{m_\chi^2 + p'^2}, 0, p' \sin \psi, p' \cos \psi) \end{aligned}$$



We can then solve for E' as a function of E, θ by imposing four-momentum conservation, resulting in:

$$\begin{aligned} EE'(1 - \cos \theta) &= m_\chi(E - E') \\ E' &= \frac{E}{1 + E/m_\chi(1 - \cos \theta)} \end{aligned}$$

By completing the phase-space integral (see P+S p.163) we find that

$$\begin{aligned} \frac{d\sigma}{d\cos \theta} &= \frac{1}{2m_\chi} \frac{1}{2E} \left(\frac{1}{8\pi} \frac{E'^2}{Em_\chi} \right) |M|^2 \\ &= \frac{1}{32\pi} \frac{E'^2}{E^2 m_\chi^2} |M|^2 \end{aligned}$$

2 Matrix Elements:

Note: In this section, we will frequently use the Mandelstam variables s, t, u .

The matrix elements depend on the particular form of the interaction. We have considered four "simplified" models thus far:

2.1 Scalar Dark Matter χ , Scalar Mediator ϕ

In this model we have two interactions:

- a scalar-fermion-fermion interaction $-g\phi\bar{\nu}\nu$
(if we use a pseudo-scalar mediator instead of a scalar, this would be $-ig\phi\bar{\nu}\gamma^5\nu$)
- a triple scalar interaction $-g'\phi\chi^2$
note: this is not a renormalizable interaction, and the coupling g' is not dimensionless!
– it has units of energy.

With these two interactions, at tree level we can only build a **t-channel** diagram. We can now immediately write the scattering matrix element:

$$iM = [\bar{u}(p_1)(-ig)u(k_1)] \frac{i}{t - m_\phi^2} [g']$$

We follow the standard procedure to get the spin-sum/averaged matrix amplitude squared:

- avg./sum over initial/final spins
- use contractions to write the expression in terms of a trace
- apply trace identities

We get:

$$\begin{aligned} \langle |M|^2 \rangle &= \frac{1}{2} \frac{g^2 g'^2}{(t - m_\phi^2)^2} \text{Tr} [\not{p}_1 \not{k}_1] \\ &= \frac{1}{2} \frac{g^2 g'^2 t}{(t - m_\phi^2)^2} \end{aligned}$$

2.2 Fermion Dark Matter χ , Scalar Mediator ϕ

In this model we have two (?) interactions:

- a scalar-fermion-fermion interaction $-g\phi\bar{\nu}\nu$
- another scalar-fermion-fermion interaction $-g'\phi\bar{\chi}\chi$
- (? not sure why we don't consider this) $\phi\bar{\chi}\nu$ or $\phi\bar{\nu}\chi$

Using these two interactions, at tree level we can only build a **t-channel** diagram.

$$iM = [\bar{u}(p_1)(-ig)u(k_1)] \frac{i}{t - m_\phi^2} [\bar{u}(p_2)(-ig')u(k_2)]$$

With the same manipulations as above, we can straightforwardly calculate $\langle |M|^2 \rangle$:

$$\begin{aligned} \langle |M|^2 \rangle &= g^2 g'^2 \frac{1}{(t - m_\phi^2)^2} \frac{1}{4} \text{Tr} [\not{p}_1 \not{k}_1] \text{Tr} [(\not{p}_2 + m_\chi)(\not{k}_2 + m_\chi)] \\ &= g^2 g'^2 \frac{1}{(t - m_\phi^2)^2} \frac{1}{4} (-2t) (2(-t + 2m_\chi^2) + 4m_\chi^2) \\ &= g^2 g'^2 \frac{-t}{(t - m_\phi^2)^2} (4m_\chi^2 - t) \end{aligned}$$

2.3 Fermion Dark Matter χ , Vector Mediator ϕ

In this model we have two (?) interactions:

- A vector-fermion-fermion interaction, $-g\bar{\nu}\gamma^\mu\phi_\mu\nu$ (?) or should it be $\gamma^\mu(1-\gamma^5)$
- Another vector-fermion-fermion interaction, $-g\bar{\chi}\gamma^\mu\phi_\mu\chi$
- (?) what about something like $\phi\bar{\nu}\chi$ or $\phi\bar{\chi}\nu$?

Using these two interactions, at tree level we can only build a **t-channel** diagram.

$$iM = [g\bar{u}(p_1)\gamma^\mu u(k_1)] \frac{g_{\mu\nu} - q_\mu q_\nu / m_\phi^2}{q^2 - m_\phi^2} [g'\bar{u}(p_2)\gamma^\nu u(k_2)]$$

As above, we use the standard procedure to calculate $\langle |M|^2 \rangle$. In this case, we also make the assumption that the mediator momentum $q = (p_1 - k_1)$ has much smaller magnitude than the mediator mass: $q^2 \ll m_\phi^2$, which allows us to neglect the second term in the vector propagator:

$$\frac{g_{\mu\nu} - q_\mu q_\nu / m_\phi^2}{q^2 - m_\phi^2} \approx \frac{g_{\mu\nu}}{q^2 - m_\phi^2}$$

Thus we only have one term to evaluate when we square M :

$$\begin{aligned} iM &= \frac{gg'}{t - m_\phi^2} [\bar{u}(p_1)\gamma^\mu u(k_1)\bar{u}(p_2)\gamma_\mu u(k_2)] \\ \langle |M|^2 \rangle &= \frac{g^2 g'^2}{4(t - m_\phi^2)} \text{Tr} [\gamma^\mu \not{k}_1 \gamma^\nu \not{p}_1] \text{Tr} [\gamma_\mu (\not{k}_2 + m_\chi) \gamma_\nu (\not{p}_2 + m_\chi)] \end{aligned}$$

Let's evaluate the product of the traces:

$$\begin{aligned} &= 4 (k_1^\mu p_1^\nu - g^{\mu\nu}(-t/2) + p_1^\mu k_1^\nu) [4(k_{2\mu} p_{2\nu} - g_{\mu\nu}(m_\chi^2 - t/2) + k_{2\nu} p_{2\mu}) + 4g_{\mu\nu} m_\chi^2] \\ &= 16 \left((k_1 \cdot k_2)(p_1 \cdot p_2) - (k_1 \cdot p_1)(m_\chi^2 - t/2) + (k_1 \cdot p_2)(k_2 \cdot p_1) + (k_1 \cdot p_1)m_\chi^2 \right. \\ &\quad \left. - (k_2 \cdot p_2)(-t/2) + 4(-t/2)(m_\chi^2 - t/2) - (k_2 \cdot p_2)(-t/2) - 4(-t/2)m_\chi^2 \right. \\ &\quad \left. + (p_1 \cdot k_2)(p_2 \cdot k_1) - (p_1 \cdot k_1)(m_\chi^2 - t/2) + (k_1 \cdot k_2)(p_1 \cdot p_2) + (p_1 \cdot k_1)m_\chi^2 \right) \\ &= 16 \left(2(k_1 \cdot k_2)(p_1 \cdot p_2) + 2(p_1 \cdot k_2)(p_2 \cdot k_1) + (k_1 \cdot p_1)t + 4(t/2)^2 + (k_2 \cdot p_2)t \right) \\ &= 8 \left((s - m_\chi^2)^2 + (u - m_\chi^2)^2 + 2m_\chi^2 t \right) \\ &= 8 \left(s^2 + u^2 + 2m_\chi^4 + 2m_\chi^2(t - s - u) \right) \end{aligned}$$

So we have in conclusion:

$$\langle |M|^2 \rangle = 2g^2 g'^2 \frac{s^2 + u^2 + 2m_\chi^4 + 2m_\chi^2(t - s - u)}{(t - m_\phi^2)^2}$$

2.4 Scalar Dark Matter χ , Fermion Mediator ϕ

In this model we have only one interaction,

- a mixed scalar-fermion-fermion interaction, $-g\chi\phi\nu$.

With this interaction, we can write both s-channel and u-channel diagrams at tree level. The two diagrams result in matrix elements with parallel form, except for the value of the mediator momentum q . For clarity $q_s = k_1 + k_2$, while $q_u = k_1 - p_2$, such that $q_s^2 = s$, $q_u^2 = u$.

$$iM = [-gu(k_1)] \left(\frac{i(\not{q}_s + m_\phi)}{s - m_\phi^2} + \frac{i(\not{q}_u + m_\phi)}{u - m_\phi^2} \right) [-g\bar{u}(p_1)]$$

After squaring and taking the sum/average over spins, we are left with four trace terms in the numerator: an $s - s$ term, $u - u$ term, and $s - u$ and $u - s$ terms. The general form of the trace is the same for all of them, just the value of each q is different in each:

$$\text{Tr}[(\not{q} + m_\phi)\not{p}_1(\not{q} + m_\phi)\not{k}_1] = 4[2(q \cdot q_1)(q \cdot k_1) - q^2(p \cdot k_1) + (p_1 \cdot k_1)m_\phi^2]$$

We can now evaluate this trace for each of the four terms. We will use the Mandelstam variable property $s + t + u = \sum_i m_i^2$ ie. $= 2m_\chi^2$ in our case.

- $s - s$ term: $q = (k_1 + k_2)$ for both qs .

$$\begin{aligned} &= 4 \left[\frac{1}{2}(-t + m_\chi^2 - u)(s - m_\chi^2) - s(-t/2) + (-t/2)m_\phi^2 \right] \\ &= 2 \left[-ts + tm_\chi^2 + sm_\chi^2 - us + um_\chi^2 + st - tm_\phi^2 \right] \\ &= 2 \left[-su + m_\chi^4 - tm_\phi^2 \right] \end{aligned}$$

- $u - u$ term: $q = (k_1 - p_2)$ for both qs .

$$\begin{aligned} &= 4 \left[\frac{1}{2}(-t + m_\chi^2 - s)(u - m_\chi^2) - u(-t/2) + (-t/2)m_\phi^2 \right] \\ &= 2 \left[-su + m_\chi^4 - tm_\phi^2 \right] \end{aligned}$$

- $u - s$ and $s - u$ terms:

$$\begin{aligned} &= 4 \left[\frac{1}{2}(-t + m_\chi^2 - u)(u - m_\chi^2) + \frac{1}{2}(-t + m_\chi^2 - s)(s - m_\chi^2) - 2(-m_\chi^2)(-t/2) + 2(-t/2)m_\phi^2 \right] \\ &= 2 \left[-ut - u^2 + tm_\chi^2 + 2um_\chi^2 - m_\chi^4 \right. \\ &\quad \left. - st - s^2 + tm_\chi^2 + 2sm_\chi^2 - m_\chi^4 - 2m_\chi^2 t - 2m_\phi^2 t \right] \\ &= 2 \left[2us - 2m_\chi^4 - 2tm_\phi^2 \right] \end{aligned}$$

(?) the below disagrees with the previous calculation by a sign error...

Putting it all together, we find that:

$$\langle |M|^2 \rangle = g^4 \left[- (su + tm_\phi^2 - m_\chi^4) \left(\frac{1}{(s - m_\phi^2)^2} + \frac{1}{(u - m_\phi^2)^2} \right) + \frac{2(us - tm_\phi^2 - m_\chi^4)}{(s - m_\phi^2)(u - m_\phi^2)} \right]$$

(?) previous calculation result:

$$\langle |M|^2 \rangle = g^4 \left[- (su + tm_\phi^2 - m_\chi^4) \left(\frac{1}{(s - m_\phi^2)} + \frac{1}{(u - m_\phi^2)} \right)^2 \right]$$

3 Evaluating Differential Cross-Sections in the Lab Frame

Before we begin, we can establish some useful variable definitions and equivalences:

Mandelstam variables and combinations:

$$\begin{aligned}
x &= \cos \theta \\
E' &= E \frac{m_\chi}{m_\chi + E(1-x)} = \frac{Em_\chi}{B} \\
\frac{d\sigma}{dx} &= \frac{1}{32\pi} \frac{E'^2}{E^2 m_\chi^2} \langle |M|^2 \rangle \\
&= \frac{1}{32\pi} \frac{\langle |M|^2 \rangle}{B^2} \\
s &= 2Em_\chi + m_\chi^2 \\
u &= -2E'm_\chi + m_\chi^2 \\
t &= -2EE'(1-x) = -2m_\chi(E-E') \\
&= -2m_\chi E^2 \frac{1-x}{B} \\
s+t+u &= 2m_\chi^4 \\
su &= m_\chi^4 + 2m_\chi^2 m_\chi(E-E') - 4EE'm_\chi^2 \\
&= m_\chi^4 + 2m_\chi^2 m_\chi(E-E') \left(1 - \frac{2}{1-x}\right) \\
&= m_\chi^4 - m_\chi^2 t \left(1 - \frac{2}{1-x}\right)
\end{aligned}$$

3.1 Scalar Dark Matter χ , Scalar Mediator ϕ

$$\langle |M|^2 \rangle = g^2 g'^2 \frac{t}{(t - m_\phi^2)^2}$$

We get the differential cross-section straightforwardly:

$$\begin{aligned}
\frac{d\sigma}{dx} &= \frac{1}{32\pi} \frac{E'^2}{E^2 m_\chi^2} \langle |M|^2 \rangle \\
&= \frac{g^2 g'^2}{32\pi} \frac{1}{B^2} * \frac{2m_\chi E^2 (1-x)}{B} \frac{B^2}{(m_\phi^2 B + 2m_\chi E^2 (1-x))^2} \\
&= \frac{g^2 g'^2}{32\pi} \frac{2m_\chi E^2 (1-x)}{(m_\chi + E(1-x)) \left(m_\phi^2 (m_\chi + E(1-x)) + 2m_\chi E^2 (1-x) \right)^2}
\end{aligned}$$

3.2 Fermion Dark Matter χ , Scalar Mediator ϕ

The following equations are copied in from Imaging Galactic Dark Matter... and have not been re-checked.

3.3 Fermion Dark Matter χ , Vector Mediator ϕ

(?) The differential cross-section published in Imaging Galactic Dark Matter... must have some typo, because the dimensions are wrong. I think the calculation below is correct. The integrated cross-section that was published does not have any dimension issues.

$$\langle |M|^2 \rangle = 2g^2 g'^2 \frac{s^2 + u^2 + 2m_\chi^4 + 2m_\chi^2(t - s - u)}{(t - m_\phi^2)^2}$$

We can first evaluate pieces of the matrix element:

$$\begin{aligned} s^2 + u^2 + 2m_\chi^4 &= (s + u)^2 - 2su + 2m_\chi^4 \\ &= (2m_\chi^2 - t)^2 + 2m_\chi^2 t \left(1 - \frac{2}{1-x}\right) \\ &= 4m_\chi^4 + t^2 + 2m_\chi^2 t \left(-1 - \frac{2}{1-x}\right) \end{aligned}$$

$$s + u - t = 2m_\chi^2 - 2t$$

Thus the numerator of the matrix element can be simplified:

$$\begin{aligned} &= \left[4m_\chi^4 + t^2 + 2m_\chi^2 t \left(-1 - \frac{2}{1-x}\right)\right] - 2m_\chi^2(2m_\chi^2 - 2t) \\ &= t^2 + 2m_\chi^2 t \left(\frac{-(1+x)}{1-x}\right) \end{aligned}$$

We can now evaluate the differential cross-section:

$$\begin{aligned} \frac{d\sigma}{dx} &= \frac{2g^2 g'^2}{32\pi} \frac{1}{B^2} \left[\frac{(2m_\chi E^2(1-x))^2}{B^2} - 2m_\chi^2 \frac{-2m_\chi E^2(1-x)}{B} \frac{1+x}{1-x} \right] \frac{B^2}{(m_\phi^2 B + 2m_\chi E^2(1-x))^2} \\ &= \frac{2g^2 g'^2}{32\pi} \left[(2m_\chi E^2(1-x))^2 + 4m_\chi^3 E^2(1+x)B \right] \frac{1}{B^2} \frac{1}{(m_\phi^2 B + 2m_\chi E^2(1-x))^2} \end{aligned}$$

Which we can write out in full form:

$$\frac{d\sigma}{dx} = \frac{2g^2 g'^2}{32\pi} \frac{4m_\chi^2 \left(E^4(1-x)^2 + m_\chi^2 E^2(1+x) + m_\chi E^3(1-x^2) \right)}{\left[m_\chi + E(1-x) \right]^2 \left(m_\phi^2 (m_\chi + E(1-x)) + 2m_\chi E^2(1-x) \right)^2}$$

Note that $\frac{d\sigma}{dx}$ should have units of $\frac{1}{E^2}$, which it does.

3.4 Scalar Dark Matter χ , Fermion Mediator ϕ

The following equations are copied in from Imaging Galactic Dark Matter... and have not been re-checked.