Neutrino - DM Scattering

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The models and calculations covered here feature in the following reference works:

- Constraining Dark Matter Neutrino Interaction with High Energy Neutrinos, Dec. 2016
- Imaging Galactic Dark Matter with High-Energy Cosmic Neutrinos, Mar. 2021
- Dark Matter-Neutrino Scattering in the Galactic Centre with IceCube, Adam McMullen (Thesis, May 2021)

1 Kinematics:

x

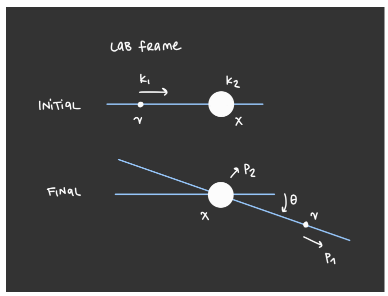
Neutrino - DM scattering is analogous to Compton scattering ($\gamma + e^- \to \gamma + e^-$) if we treat the neutrino as massless. We can therefore follow the derivation in Peskin+Schroeder.

From P+S (4.79), p.106, we get the formula for the differential cross-section for $2 \to\ n$ scattering with the final state phase space integral:

$$d\sigma = \frac{1}{2E_A 2E_B}\frac{1}{|k_A/E_A - k_B/E_B|} d\Pi_n \, |M|^2$$

We will work in the lab frame (the dark matter rest frame). Here, we can write the initial and final four-momenta as:

$$\begin{aligned}
k_1 &= (E, 0, 0, E) \\
k_2 &= (m_\chi, 0, 0, 0) \\
p_1 &= (E', 0, -E'\sin\theta, E'\cos\theta) \\
p_2 &= (\sqrt{m_\chi^2 + p'^2}, 0, p'\sin\psi, p'\cos\psi)
\end{aligned}$$



We can then solve for E' as a function of E, θ by imposing four-momentum conservation, resulting in:

$$EE'(1-\cos\theta) = m\chi(E - E')$$

$$E' = \frac{E}{1 + E/m_\chi(1-\cos\theta)}$$

Note: For fixed values of the initial energy E, there is a one-to-one relationship between the scattering angle θ and the final energy E'.

By completing the phase-space integral (see P+S p.163) we find that

$$\begin{aligned}
\frac{d\sigma}{d\cos\theta} &= \frac{1}{2m_\chi}\frac{1}{2E}\left(\frac{1}{8\pi}\frac{E'^2}{Em_\chi}\right)|M|^2 \\
&= \frac{1}{32\pi}\frac{E'^2}{E^2 m_\chi^2}\,|M|^2
\end{aligned}$$

2 Matrix Elements:

Note: In this section, we will frequently use the Mandelstam variables s, t, u.

The matrix elements depend on the particular form of the interaction. We have considered four "simplified" models thus far:

2.1 Scalar Dark Matter χ, Scalar Mediator ϕ

In this model we have two interactions:

- a scalar-fermion-fermion interaction $-g\phi\bar{\nu}\nu$
  (if we use a pseudo-scalar mediator instead of a scalar, this would be $-ig\phi\bar{\nu}\gamma^5\nu$)
- a triple scalar interaction $-g'\phi\chi^2$
  note: this is not a renormalizable interaction, and the coupling g' is not dimensionless! – it has units of energy.

With these two interactions, at tree level we can only build a **t-channel** diagram. We can now immediately write the scattering matrix element:

$$iM = [\bar{u}(p_1)(-ig)u(k_1)]\,\frac{i}{t - m_\phi^2}\,[g']$$

We follow the standard procedure to get the spin-sum/averaged matrix amplitude squared:

- avg./sum over initial/final spins
- use contractions to write the expression in terms of a trace
- apply trace identities

We get:

$$\begin{aligned}\left\langle |M|^2 \right\rangle &= \frac{1}{2}\frac{g^2 g'^2}{(t - m_\phi^2)^2}\mathrm{Tr}\left[\not{p}_1 \not{k}_1\right] \\ &= \frac{1}{2}\frac{g^2 g'^2 t}{(t - m_\phi^2)^2}\end{aligned}$$

2.2 Fermion Dark Matter χ, Scalar Mediator ϕ

In this model we have two (?) interactions:

- a scalar-fermion-fermion interaction $-g\phi\bar{\nu}\nu$
- another scalar-fermion-fermion interaction $-g'\phi\bar{\chi}\chi$
- (? not sure why we don't consider this) $\phi\bar{\chi}\nu$ or $\phi\bar{\nu}\chi$

Using these two interactions, at tree level we can only build a **t-channel** diagram.

$$iM = [\bar{u}(p_1)(-ig)u(k_1)]\,\frac{i}{t-m_\phi^2}\,[\bar{u}(p_2)(-ig')u(k_2)]$$

With the same manipulations as above, we can straightforwardly calculate $\langle|M|^2\rangle$:

$$\begin{aligned}
\langle|M|^2\rangle &= g^2 g'^2 \frac{1}{(t-m_\phi^2)^2}\frac{1}{4}\mathrm{Tr}\,[\not{p}_1\not{k}_1]\,\mathrm{Tr}\,[(\not{p}_2+m_\chi)(\not{k}_2+m_\chi)] \\
&= g^2 g'^2 \frac{1}{(t-m_\phi^2)^2}\frac{1}{4}(-2t)\Big(2(-t+2m_\chi^2)+4m_\chi^2\Big) \\
&= g^2 g'^2 \frac{-t}{(t-m_\phi^2)^2}\Big(4m_\chi^2-t\Big)
\end{aligned}$$

2.3 Fermion Dark Matter χ, Vector Mediator ϕ

In this model we have two (?) interactions:

- A vector-fermion-fermion interaction, $-g\bar{\nu}\gamma^{\mu}\phi_{\mu}\nu$ (?) or should it be $\gamma^{\mu}(1-\gamma^{5})$
- Another vector-fermion-fermion interaction, $-g\bar{\chi}\gamma^{\mu}\phi_{\mu}\chi$
- (?) what about something like $\phi\bar{\nu}\chi$ or $\phi\bar{\chi}\nu$?

Using these two interactions, at tree level we can only build a **t-channel** diagram.

$$iM = [g\bar{u}(p_1)\gamma^{\mu}u(k_1)]\,\frac{g_{\mu\nu} - q_{\mu}q_{\nu}/m_{\phi}^2}{q^2 - m_{\phi}^2}\,\left[g'\bar{u}(p_2)\gamma^{\nu}u(k_2)\right]$$

As above, we use the standard procedure to calculate $\langle |M|^2\rangle$. In this case, we also make the assumption that the mediator momentum $q = (p_1 - k_1)$ has much smaller magnitude than the mediator mass: $q^2 << m_{\phi}^2$, which allows us to neglect the second term in the vector propagator:

$$\frac{g_{\mu\nu} - q_{\mu}q_{\nu}/m_{\phi}^2}{q^2 - m_{\phi}^2} \approx \frac{g_{\mu\nu}}{q^2 - m_{\phi}^2}$$

Thus we only have one term to evaluate when we square M:

$$iM = \frac{gg'}{t - m_{\phi}^2}\,[\bar{u}(p_1)\gamma^{\mu}u(k_1)\bar{u}(p_2)\gamma_{\mu}u(k_2)]$$

$$\langle |M|^2\rangle = \frac{g^2 g'^2}{4(t - m_{\phi}^2)}\mathrm{Tr}\,[\gamma^{\mu}\not{k}_1\gamma^{\nu}\not{p}_1]\;\mathrm{Tr}\,[\gamma_{\mu}(\not{k}_2 + m_{\chi})\gamma_{\nu}(\not{p}_2 + m_{\chi})]$$

Let's evaluate the product of the traces:

$$\begin{aligned}
&= 4\,(k_1^{\mu}p_1^{\nu} - g^{\mu\nu}(-t/2) + p_1^{\mu}k_1^{\nu})\,[4(k_{2\mu}p_{2\nu} - g_{\mu\nu}(m_{\chi}^2 - t/2) + k_{2\nu}p_{2\mu}) + 4g_{\mu\nu}m_{\chi}^2]\\
&= 16\Big((k_1\cdot k_2)(p_1\cdot p_2) - (k_1\cdot p_1)(m_{\chi}^2 - t/2) + (k_1\cdot p_2)(k_2\cdot p_1) + (k_1\cdot p_1)m_{\chi}^2\\
&\qquad - (k_2\cdot p_2)(-t/2) + 4(-t/2)(m_{\chi}^2 - t/2) - (k_2\cdot p_2)(-t/2) - 4(-t/2)m_{\chi}^2\\
&\qquad + (p_1\cdot k_2)(p_2\cdot k_1) - (p_1\cdot k_1)(m_{\chi}^2 - t/2) + (k_1\cdot k_2)(p_1\cdot p_2) + (p_1\cdot k_1)m_{\chi}^2\Big)\\
&= 16\Big(2(k_1\cdot k_2)(p_1\cdot p_2) + 2(p_1\cdot k_2)(p_2\cdot k_1) + (k_1\cdot p_1)t + 4(t/2)^2 + (k_2\cdot p_2)t\Big)\\
&= 8\,\Big((s - m_{\chi}^2)^2 + (u - m_{\chi}^2)^2 + 2m_{\chi}^2 t\Big)\\
&= 8\,\Big(s^2 + u^2 + 2m_{\chi}^4 + 2m_{\chi}^2(t - s - u)\Big)
\end{aligned}$$

So we have in conclusion:

$$\langle |M|^2\rangle = 2g^2g'^2\frac{s^2 + u^2 + 2m_{\chi}^4 + 2m_{\chi}^2(t - s - u)}{(t - m_{\phi}^2)^2}$$

2.4 Scalar Dark Matter χ, Fermion Mediator ϕ

In this model we have only one interaction,

- a mixed scalar-fermion-fermion interaction, $-g\chi\phi\nu$.

With this interaction, we can write both s-channel and u-channel diagrams at tree level. The two diagrams result in matrix elements with parallel form, except for the value of the mediator momentum q. For clarity $q_s = k_1 + k_2$, while $q_u = k_1 - p_2$, such that $q_s^2 = s$, $q_u^2 = u$.

$$iM = [-gu(k_1)]\left(\frac{i(\not{q}_s + m_\phi)}{s - m_\phi^2} + \frac{i(\not{q}_u + m_\phi)}{u - m_\phi^2}\right)[-g\bar{u}(p_1)]$$

After squaring and taking the sum/average over spins, we are left with four trace terms in the numerator: an $s-s$ term, $u-u$ term, and $s-u$ and $u-s$ terms. The general form of the trace is the same for all of them, just the value of each q is different in each:

$$\mathrm{Tr}\big[(\not{q} + m_\phi)\not{p}_1(\not{q} + m_\phi)\not{k}_1\big] = 4\big[2(q\cdot q_1)(q\cdot k_1) - q^2(p\cdot k_1) + (p_1\cdot k_1)m_\phi^2\big]$$

We can now evaluate this trace for each of the four terms. We will use the Mandelstam variable property $s + t + u = \sum_i m_i^2$ ie. $= 2m_\chi^2$ in our case.

- $s-s$ term: $q = (k_1 + k_2)$ for both qs.

$$\begin{aligned}
&=4\left[\frac{1}{2}(-t + m_\chi^2 - u)(s - m_\chi^2) - s(-t/2) + (-t/2)m_\phi^2\right]\\
&=2\left[-ts + tm_\chi^2 + sm_\chi^2 - us + um_\chi^2 + st - tm_\phi^2\right]\\
&=2\left[-su + m_\chi^4 - tm_\phi^2\right]
\end{aligned}$$

- $u-u$ term: $q = (k_1 - p_2)$ for both qs.

$$\begin{aligned}
&=4\left[\frac{1}{2}(-t + m_\chi^2 - s)(u - m_\chi^2) - u(-t/2) + (-t/2)m_\phi^2\right]\\
&=2\left[-su + m_\chi^4 - tm_\phi^2\right]
\end{aligned}$$

- $u-s$ and $s-u$ terms:

$$\begin{aligned}
&=4\big[\frac{1}{2}(-t + m_\chi^2 - u)(u - m_\chi^2) + \frac{1}{2}(-t + m_\chi^2 - s)(s - m_\chi^2) - 2(-m_\chi^2)(-t/2) + 2(-t/2)m_\phi^2\big]\\
&=2\big[-ut - u^2 + tm_\chi^2 + 2um_\chi^2 - m_\chi^4\\
&\qquad - st - s^2 + tm_\chi^2 + 2sm_\chi^2 - m_\chi^4 - 2m_\chi^2 t - 2m_\phi^2 t\big]\\
&=2\left[2us - 2m_\chi^4 - 2tm_\phi^2\right]
\end{aligned}$$

(?) the below disagrees with the previous calculation by a sign error...
Putting it all together, we find that:

$$\langle|M|^2\rangle = g^4\Big[-(su+tm_\phi^2-m_\chi^4)\big(\frac{1}{(s-m_\phi^2)^2}+\frac{1}{(u-m_\phi^2)^2}\big)+\frac{2(us-tm_\phi^2-m_\chi^4)}{(s-m_\phi^2)(u-m_\phi^2)}\Big]$$

(?) previous calculation result:

$$\langle|M|^2\rangle = g^4\Big[-(su+tm_\phi^2-m_\chi^4)\Big(\frac{1}{(s-m_\phi^2)}+\frac{1}{(u-m_\phi^2)}\Big)^2\Big]$$

3 Evaluating Differential Cross-Sections in the Lab Frame

Before we begin, we can establish some useful variable definitions and equivalences:

$$x = \cos\theta$$

$$E' = E\frac{m_\chi}{m_\chi + E(1-x)} = \frac{Em_\chi}{B}$$

$$\frac{d\sigma}{dx} = \frac{1}{32\pi}\frac{E'^2}{E^2 m_\chi^2}\langle|M|^2\rangle = \frac{1}{32\pi}\frac{\langle|M|^2\rangle}{B^2}$$

Mandelstam variables and combinations:

$$s = 2Em_\chi + m_\chi^2$$
$$u = -2E'm_\chi + m_\chi^2$$
$$t = -2EE'(1-x) = -2m_\chi(E-E') = -2m_\chi E^2\frac{1-x}{B}$$

$$s + t + u = 2m_\chi^4$$

$$\begin{aligned} su &= m_\chi^4 + 2m_\chi^2 m_\chi(E-E') - 4EE'm_\chi^2 \\ &= m_\chi^4 + 2m_\chi^2 m_\chi(E-E')(1-\frac{2}{1-x}) \\ &= m_\chi^4 - m_\chi^2 t(1-\frac{2}{1-x}) \end{aligned}$$

3.1 Scalar Dark Matter χ, Scalar Mediator ϕ

$$\left\langle|M|^2\right\rangle = g^2 g'^2\frac{t}{(t-m_\phi^2)^2}$$

We get the differential cross-section straightforwardly:

$$\begin{aligned} \frac{d\sigma}{dx} &= \frac{1}{32\pi}\frac{E'^2}{E^2 m_\chi^2}\langle|M|^2\rangle \\ &= \frac{g^2g'^2}{32\pi}\frac{1}{B^2} * \frac{2m_\chi E^2(1-x)}{B}\frac{B^2}{(m_\phi^2 B + 2m_\chi E^2(1-x))^2} \\ &= \frac{g^2g'^2}{32\pi}\frac{2m_\chi E^2(1-x)}{(m_\chi + E(1-x))\Big(m_\phi^2(m_\chi + E(1-x)) + 2m_\chi E^2(1-x)\Big)^2} \end{aligned}$$

3.2 Fermion Dark Matter χ, Scalar Mediator ϕ

The following equations are copied in from Imaging Galactic Dark Matter... and have not been re-checked.

3.3 Fermion Dark Matter χ, Vector Mediator ϕ

(?) The differential cross-section published in Imaging Galactic Dark Matter... must have some typo, because the dimensions are wrong. I think the calculation below is correct. The integrated cross-section that was published does not have any dimension issues.

$$\langle |M|^2 \rangle = 2g^2 g'^2 \frac{s^2 + u^2 + 2m_\chi^4 + 2m_\chi^2(t - s - u)}{(t - m_\phi^2)^2}$$

We can first evaluate pieces of the matrix element:

$$\begin{aligned} s^2 + u^2 + 2m_\chi^4 &= (s+u)^2 - 2su + 2m_\chi^4 \\ &= (2m_\chi^2 - t)^2 + 2m_\chi^2 t\Big(1 - \frac{2}{1-x}\Big) \\ &= 4m_\chi^4 + t^2 + 2m_\chi^2 t\Big(-1 - \frac{2}{1-x}\Big) \end{aligned}$$

$$s + u - t = 2m_\chi^2 - 2t$$

Thus the numerator of the matrix element can be simplified:

$$\begin{aligned} &= \Big[4m_\chi^4 + t^2 + 2m_\chi^2 t(-1 - \frac{2}{1-x})\Big] - 2m_\chi^2(2m_\chi^2 - 2t) \\ &= t^2 + 2m_\chi^2 t\Big(\frac{-(1+x)}{1-x}\Big) \end{aligned}$$

We can now evaluate the differential cross-section:

$$\begin{aligned} \frac{d\sigma}{dx} &= \frac{2g^2g'^2}{32\pi}\frac{1}{B^2}\Big[\frac{(2m_\chi E^2(1-x))^2}{B^2} - 2m_\chi^2 \frac{-2m_\chi E^2(1-x)}{B}\frac{1+x}{1-x}\Big]\frac{B^2}{(m_\phi^2 B + 2m_\chi E^2(1-x))^2} \\ &= \frac{2g^2g'^2}{32\pi}\Big[(2m_\chi E^2(1-x))^2 + 4m_\chi^3 E^2(1+x)B\Big]\frac{1}{B^2}\frac{1}{(m_\phi^2 B + 2m_\chi E^2(1-x))^2} \end{aligned}$$

Which we can write out in full form:

$$\frac{d\sigma}{dx} = \frac{2g^2g'^2}{32\pi}\frac{4m_\chi^2\Big(E^4(1-x)^2 + m_\chi^2 E^2(1+x) + m_\chi E^3(1-x^2)\Big)}{\Big[m_\chi + E(1-x)\Big]^2\Big(m_\phi^2(m_\chi + E(1-x)) + 2m_\chi E^2(1-x)\Big)^2}$$

Note that $\frac{d\sigma}{dx}$ should have units of $\frac{1}{E^2}$, which it does.

3.4 Scalar Dark Matter χ, Fermion Mediator ϕ

The following equations are copied in from Imaging Galactic Dark Matter... and have not been re-checked.

These equations are quite lengthy, so as a shorthand I will define:

$$A = 2E + m_\chi \qquad B = 2E - m_\chi \qquad y = 1 - x$$

Also, as above, E here refers to the initial neutrino energy.

$$\frac{d\sigma}{dx} = \frac{g^4}{4\pi} \frac{E^4 m_\chi^5 (1+x)\big(yE + 2m_\chi\big)^2}{(m_\chi(2E + m_\chi) - m_\phi^2)^2 * (yE + m_\chi)^3 * \Big(E\big(-ym_\phi^2 - (1+x)m_\chi^2\big) + m_\chi(m_\chi^2 - m_\phi^2)\Big)^2}$$

$$\sigma = \frac{g^4}{64\pi}\Bigg[\frac{8E^2 m_\chi}{A(m_\chi A - m_\phi^2)^2} + \frac{4}{m_\chi B - m_\phi^2} + \frac{8}{m_\chi A - m_\phi^2} \\ + \Big(\frac{3}{E^2} - \frac{6m_\phi^2 + 2m_\chi B}{Em_\chi(m_\chi A - m_\phi^2)}\Big)\log\Big(1 + \frac{4E^2 m_\chi}{m_\phi^2 A - m_\chi^3}\Big)\Bigg]$$

4 Visualizing and Checking the Cross-Sections and their Differentials

4.1 Scalar Dark Matter χ, Scalar Mediator ϕ

4.2 Fermion Dark Matter χ, Scalar Mediator ϕ

4.3 Fermion Dark Matter χ, Vector Mediator ϕ

4.4 Scalar Dark Matter χ, Fermion Mediator ϕ

The most significant feature of the Scalar-Fermion cross-section is the presence of a resonance when the center-of-mass energy equals the mediator mass squared, $s = m_\phi^2$. This feature is a direct result of the s-channel diagram.

Since $s = 2Em_\chi + m_\chi^2 \approx 2Em_\chi$ if $E \gg m_\chi$, this resonance appears as a kink at $E_{\text{kink}} \approx m_\phi^2/2m_\chi$. In order to see the effects of this kink in IceCube data, we need $E_{\text{kink}} \sim 1$ TeV, which roughly fixes the ratio between relevant values of m_χ, m_ϕ. For GeV scale mediators, the dark matter mass range is keV to MeV in scale.

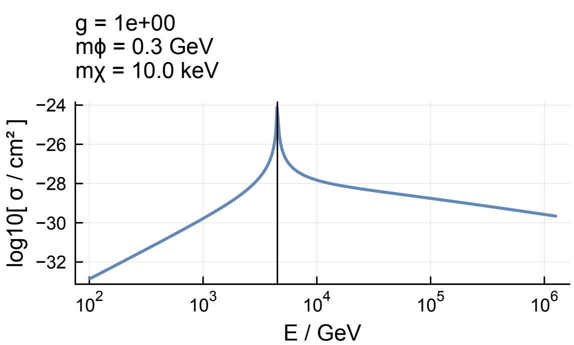


Figure 1: The Scalar-Fermion cross-section as a function of energy, for fixed model parameters. As expected, the scale of the cross-section is roughly equal to that of neutrino DIS scattering at 1 TeV, $\sim 10^{-30} \frac{1}{\text{cm}^2}$.

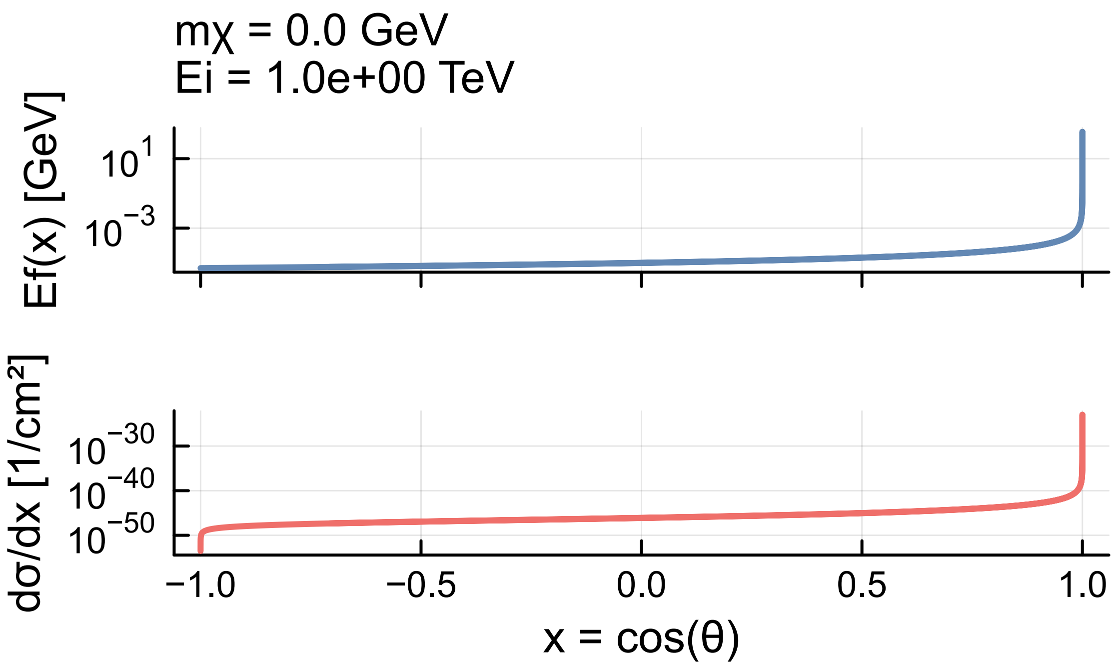


Figure 2: The cross-section is highly forward peaked. As expected, the scale of the cross-section at its peak is roughly equal to $\sim 10^{-30}\frac{1}{\text{cm}^2}$.

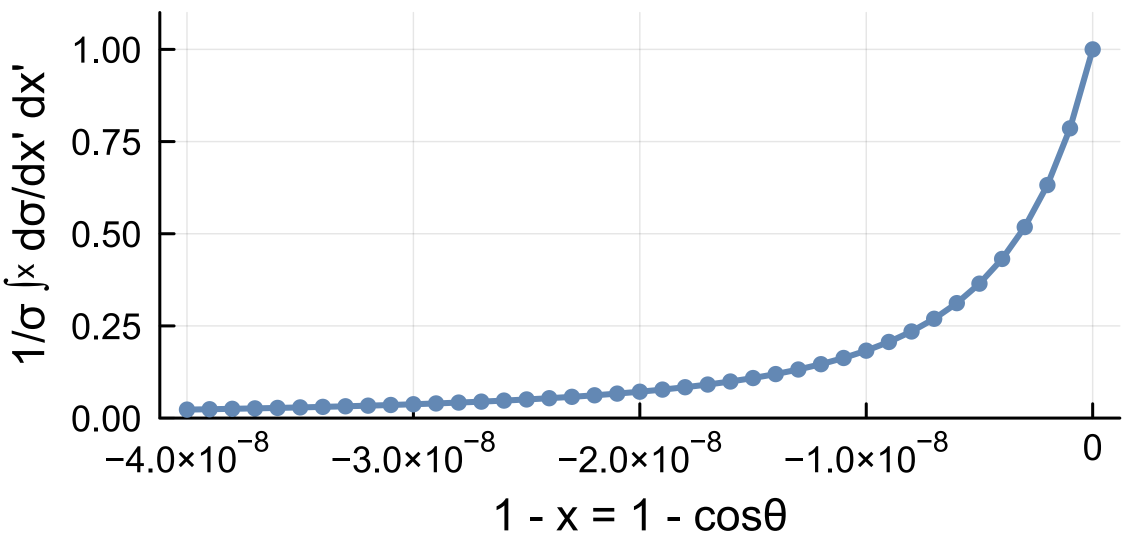


Figure 3: Partial integrals of $\frac{d\sigma}{d\cos\theta}$, in the region very close to $\cos\theta = 1$.