acceleration of — N-body — distance between

$$\vec{a}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} G m_j \frac{\vec{r}_j - \vec{r}_i}{(|\vec{r}_j - \vec{r}_i|^2 + \epsilon^2)^{3/2}}$$ ← gravity

i = particle 1
j = particle 2

softening parameter → $\epsilon \approx 0.01 \times \boxed{\frac{R_{cluster}}{N^{1/3}}}$ mean inter-particle spacing

$$r_{ij}^2 = (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2$$ ← distance

use $M_\odot$, AU, yr !

↓

$$G = 4\pi^2 (AU^3 M_\odot^{-1} yr^{-2})$$

Monitor + plot :

- total KE
- total grav. PE = W
- total E = KE + W
- virial ratio

$$Q = \frac{|2KE + W|}{|W|}$$

should be < 0.01

* count each pair only once ($i < j$)

Integration methods :

general structure :

$$\dot{\vec{y}} = \vec{f}(t, \vec{y}) = [\vec{v}, \vec{a}]$$

$$\vec{y} = [\vec{r}, \vec{v}]$$

- eulers method

$$\underbrace{\vec{y}_{n+1} = \vec{y}_n + \Delta t}_{\text{same}} \cdot \boxed{\vec{f}(t_n, \vec{y}_n)}$$

- runge-kutta method
  - RK2

$$\vec{k}_1 = \vec{f}(t_n, \vec{y}_n)$$

$$\vec{k}_2 = \boxed{\vec{f}\left(t_n + \frac{\Delta t}{2}, \vec{y}_n + \frac{\Delta t}{2}\vec{k}_1\right)}$$ same thing

$$\underbrace{\vec{y}_{n+1} = \vec{y}_n + \Delta t}_{\text{same}} \cdot \vec{k}_2$$

  - RK4

$$\vec{k}_1 = \vec{f}(t_n, \vec{y}_n)$$

$$\vec{k}_2 = \vec{f}\left(t_n + \frac{\Delta t}{2}, \vec{y}_n + \frac{\Delta t}{2}\vec{k}_1\right)$$ same thing

$$\vec{k}_3 = \vec{f}\left(t_n + \frac{\Delta t}{2}, \vec{y}_n + \frac{\Delta t}{2}k_2\right)$$

$$\vec{k}_4 = \vec{f}(t_n + \Delta t, \vec{y}_n + \Delta t \cdot \vec{k}_3)$$

$$\underbrace{\vec{y}_{n+1} = \vec{y}_n + \Delta t}_{\text{same}} \frac{1}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)$$

- symplectic integrators = leapfrog

*similar: $\vec{v}_{n+1/2} = \vec{v}_n + \Delta t \cdot \frac{1}{2} \cdot \vec{a}(\vec{r}_n)$

similar: $\vec{r}_{n+1} = \vec{r}_n + \Delta t \cdot \vec{v}_{n+1/2}$

similar: $\vec{v}_{n+1} = \vec{v}_{n+1/2} + \Delta t \cdot \frac{1}{2} \cdot \vec{a}(\vec{r}_{n+1})$

Sampling:

- np.random.uniform (low, high, array shape)

- kroupa IMF:

$$\alpha = \begin{cases} 1.3 & \text{for } 0.08 \le m < 0.5 \\ 2.3 & \text{for } 0.5 \le m \end{cases}$$

$$\xi = \begin{cases} A_1 m_1^{-\alpha_1} \\ A_2 m_2^{-\alpha_2} \end{cases}$$

$$I = \begin{cases} (m_{0.5}^{1-\alpha_1} - m_{min}^{1-\alpha_1})/(1-\alpha_1) \\ (m_{max}^{1-\alpha_2} - m_{0.5}^{1-\alpha_2})/(1-\alpha_2) \end{cases}$$

$$A = \begin{cases} N/(I_1 + m_{0.5}^{\alpha_2-\alpha_1} \cdot I_2) \\ A_1 \cdot m_{0.5}^{\alpha_2-\alpha_1} \end{cases}$$

$$P = \begin{cases} A_1 \cdot I_1 / N \\ A_2 \cdot I_2 / N \end{cases}$$

* m on [a,b], $\alpha \neq 1$ ← do per segment

$c = (m_b^{1-\alpha} - m_a^{1-\alpha})/(1-\alpha)$

$u \sim U(0,1)$ ← uniform distribution

$m = \left(m_a^{1-\alpha} + u(m_b^{1-\alpha} - m_a^{1-\alpha})\right)^{1/(1-\alpha)}$

Normalize ξ_1 and ξ_2

Calculate P_1, P_2 from ξ

$s \sim U(0,1)$ ← random.uniform

If $s < P_1$, sample from seg. 1 : α_1, $m \in [min, 0.5]$

else, sample from seg. 2 : α_2, $m \in [0.5, max]$

Testing:

all 4 methods

- Sun + Earth
  $r = 1$ AU, $v_{circular} = \sqrt{\frac{GM}{r}}$
  10 yrs, $dt = 0.01$

RK4, leapfrog

- Sun + Earth + Jupiter
  sun at origin
  Earth at 1 AU ($3 \times 10^{-6} M_\odot$)
  Jup at 5.2 AU ($9.55 \times 10^{-4} M_\odot$)
  24 yrs

P1

- N-body
  uniform mass $1 M_\odot$
  random positions
  Radius = 100 AU
  $\vec{v}_i = 0$

then leapfrog

P2

Kroupa IMF sampling
$m \in [0.08, 150] M_\odot$

Plummer sphere positions
sphere radius a

virial equilibrium v
$v_i \sim \mathcal{N}(0, \sigma_{1D}(r_i))$

test with loops of
$N = 10, 20, 50, 100$

then

P3

rewrite force eq. with
Numpy broadcasting
No loops!

array operations
r_{ij} = shape (N, N, 3)

test with
$N = 10, 20, 50, 100,$
$200, 500, 1000, 2000,$
5000, maybe 10_000

Required Plots:

- 2 body
  - 2x2 comparison of
    1. orbital trajectories (line styles)
    2. total E vs time (colors)
    3. relative E error (log)
    4. virial ratio
- 3 body
  - same as above, just only 2 methods
  - can split orbital into own plot
- N-body
  - P1
    - energy diagnostics only
      ↳ compare RK4 to leapfrog
  - P2
    - initial cond. validation ($N = 100$)
      - log-log mass histogram w/ kroupa slopes (-1.3, -2.3)
      - radial density profile showing plummer $\rho \propto \left(1 + \frac{r^2}{a^2}\right)^{-5/2}$
    - cluster evol. snapshots at $\frac{1}{4}$ intervals through time
  - P3
    - verify E comp. for loop vs. vectorized ($N = 100$)
    - preformance analysis
      - log log time vs N
      - overlay $O(N^2)$ ref. line
      - speed-up factor $\left(\frac{t_{loop}}{t_{vec}}\right)$ vs N
    - large N showcase
      - 3x2 evol. snapshots at $\frac{1}{5}$ intervals
      - color by mass
      - energy diagnostics

- plummer sphere positions

$$\rho = \frac{3M_0}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-5/2}$$

M_0 : total mass
a : plummer radius

$$\Phi = \frac{-GM_0}{\sqrt{r^2 + a^2}}$$

velocity dispersion

$$\sigma^2 = \frac{GM_0}{6\sqrt{r^2 + a^2}}$$

$$f = \frac{24\sqrt{2}}{7\pi^3} \frac{a^2}{G^3 M_0^4} (-E)^{7/2}$$ ← isotropic distribution function

If $E < 0$ and $f = 0$
else $E = \frac{1}{2}v^2 + \Phi$ ← specific energy

$$M(<r) = \frac{M_0 r^3}{(r^2 + a^2)^{3/2}}$$ ← mass within a radius

core radius → $r_c = a\sqrt{\sqrt{2} - 1}$

half-mass radius → $r_h = \left(\frac{1}{0.5^{2/3}} - 1\right)^{-0.5} \cdot a$

virial radius → $r_v = \frac{16}{3\pi} a$

$x_e \to y_s$

$x_J \to x_s$

$x_J \to x_e$

$$ps = \begin{bmatrix} x_s & y_s & z_s \\ x_e & y_e & z_e \\ x_j & y_j & z_j \end{bmatrix} = \text{shape}[N, d]$$

$$ps[\text{none}, :, :] = \text{shape}\underbrace{[1, N, d]}$$

$$ps[:, \text{None}, :] = \underbrace{\text{shape}[N, 1, d]}$$

$$\text{diff} = \text{shape}[N, N, d]$$

$$m = \text{shape}[1, N, 1]$$

$$= \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 \\ x_{es} & y_{es} & z_{es} \\ x_{js} & y_{js} & z_{js} \end{bmatrix} & \begin{array}{l} \\ \cdot m_e \\ \cdot m_j \end{array} \end{array} \cdots \begin{array}{cc} \begin{bmatrix} x_{se} & y_{se} & z_{se} \\ 0 & 0 & 0 \\ x_{je} & y_{je} & z_{je} \end{bmatrix} & \begin{array}{l} \cdot m_s \\ \\ \cdot m_j \end{array} \end{array} \cdots \begin{array}{cc} \begin{bmatrix} x_{sj} & y_{sj} & z_{sj} \\ x_{ej} & y_{ej} & z_{ej} \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \cdot m_s \\ \cdot m_e \\ \\ \end{array} \end{array}$$

D2

D3

D1

recieving force

acting force

dimension

$$\text{diff}_{sx} = \text{diff}[:, 0, 0]$$

$$\vec{a}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} G m_j \frac{\vec{r}_j - \vec{r}_i}{(|\vec{r}_j - \vec{r}_i|^2 + \epsilon^2)^{3/2}}$$

$$m_j \cdot \vec{r}_j - \vec{r}_i$$

M_0 = total mass

a = scale radius (given)

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2}$$

$$M(<r) = M_0 \frac{r^3}{(r^2+a^2)^{3/2}}$$

$$F(r) = M(<r)/M \in [0,1)$$

$$F(r) = u \quad , \quad u \sim U(10^{-12}, 1-10^{-12})$$

$$r = a\left(u^{-2/3}-1\right)^{-1/2}$$

$$\phi = 2\pi v \quad , \quad v \sim U(0,1)$$

$$\cos\theta = 1-2\omega \quad , \quad \omega \sim U(0,1)$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r_h = \frac{a}{\sqrt{2^{2/3}-1}}$$

$$a = r_h\sqrt{2^{2/3}-1}$$

$\vec{y} = [\vec{r}, \vec{v}]$

Modules

- constants ✓
- integration methods
  - eulers $(\vec{y}_n, \Delta t, t_n, \vec{f} \rightarrow \vec{y}_{n+1})$
  - RK $(\vec{y}_n, \Delta t, t_n, \vec{f} \rightarrow \vec{y}_{n+1})$
  - leapfrog $(\vec{y}_n, \Delta t, t_n, \vec{f} \rightarrow \vec{y}_{n+1})$
- sampling methods
  - random (cluster radius, N → pos)
  - kroupa IMF (N, m_min, m_max → mass)
  - plummer sphere (cluster radius, N, mass_total → pos)
- plotting
  - orbits (pos, t, N, ax, fig → ax, fig)
  - —— vs time (——, t, N, ax, fig → ax, fig)
  - —— vs N (——, t, N, ax, fig → ax, fig)
  - evol. screenshots (pos, t_i, ax, fig → ax, fig)
- testing
  - 2 body
  - 3 body
  - N body
    - P1
    - P2
    - P3
- acting forces
  - gravity $(m_j, \text{separation}, \epsilon \rightarrow \vec{a}_i)$
  - ϵ (cluster radius, N → ϵ)

pos = [x, y, z] ↓ N

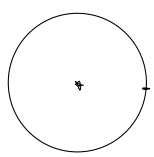
vel = [vx, vy, vz] ↓ N

acc = [ax, ay, az] ↓ N

mass = [m] ↓ N

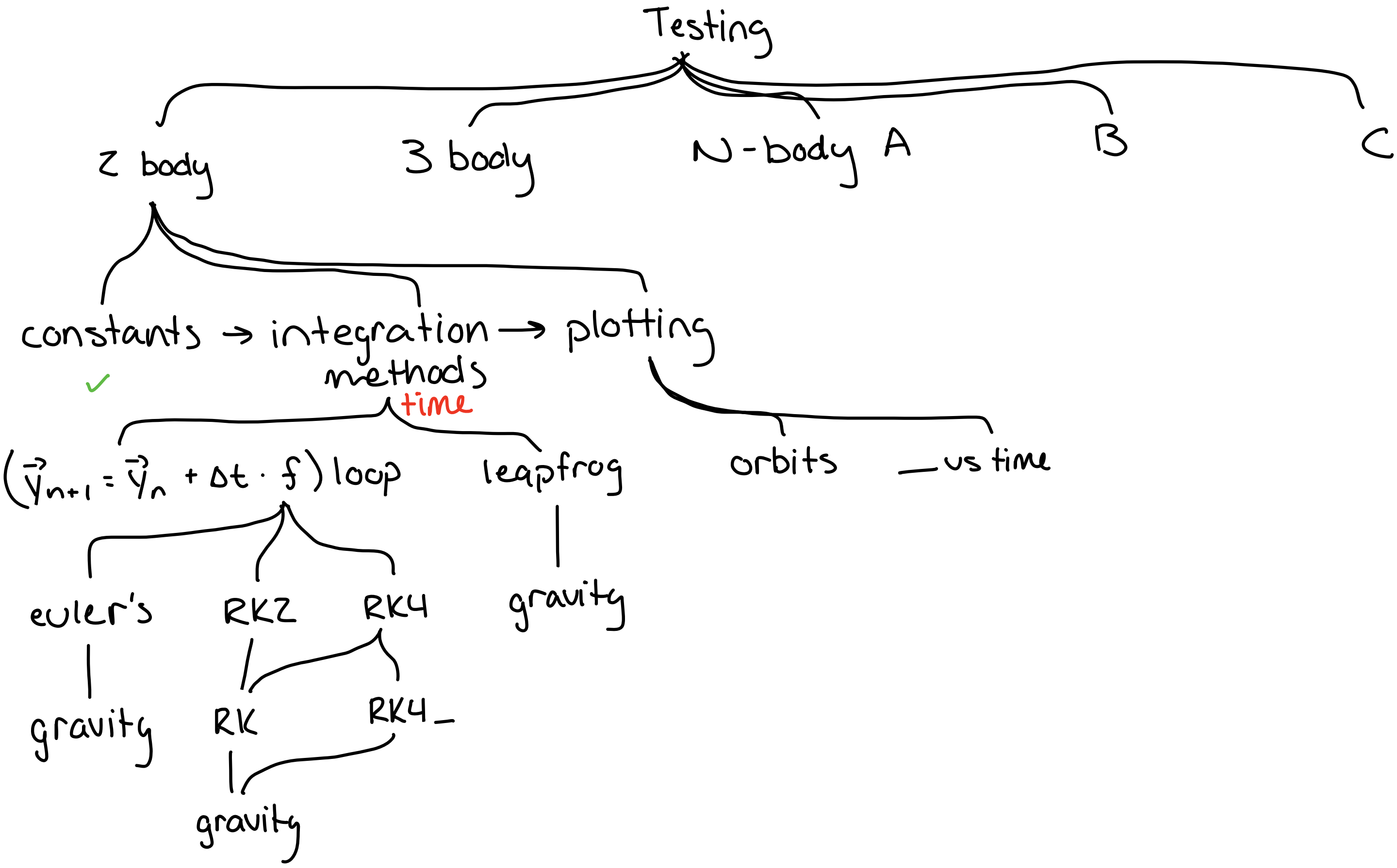
system_t = [system, t] ↓ steps

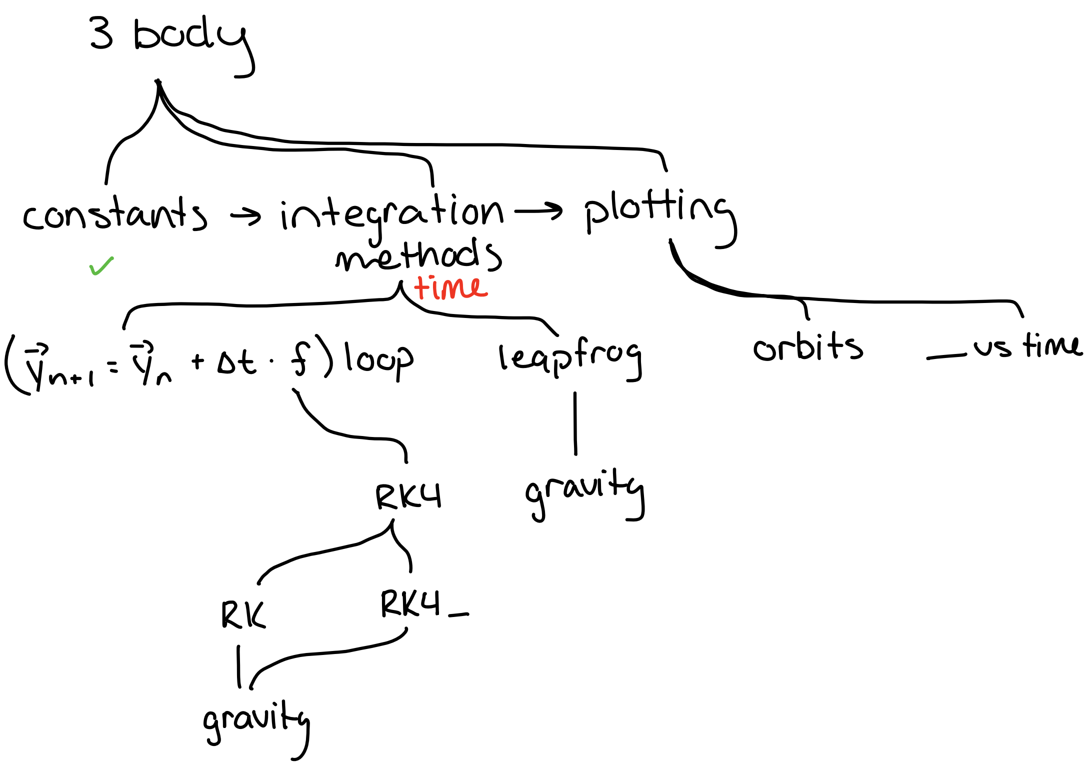
system = N_body



$v = \frac{d}{t} = \frac{2\pi(1\,AU)}{1\,yr} = \frac{2\pi(5.2\,AU)}{12\,yr}$

2π


Testing
2 body
3 body
N-body
A
B
C
constants → integration → plotting
methods
time
✓
$(\vec{y}_{n+1} = \vec{y}_n + \Delta t \cdot f)$ loop
leapfrog
orbits
_ vs time
euler's
RK2
RK4
gravity
RK
RK4_
gravity
gravity


3 body
constants → integration → plotting
methods
time
✓
$(\vec{y}_{n+1} = \vec{y}_n + \Delta t \cdot f)$ loop
leapfrog
orbits
_ vs time
RK4
gravity
RK
RK4_
gravity

N-body A

constants → sampling → integration methods → plotting

✓

random

time

$(\vec{y}_{n+1} = \vec{y}_n + \Delta t \cdot f)$ loop?

leapfrog

RK4

gravity

RK

RK4_

gravity

class NbodyIntegrator

pos:

x	y	z
x2	y2	z2
x3	y3	z3

masses
positions
velocities
ϵ
method

functions:

init

acceleration

step

energy