Lecture 4 Notes for STT861

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1 Review

Definitions

$\mathbf{S}(\neq \emptyset)$ is the sample space.
$\mathscr{A}$: α - field on $\mathbf{S}$
A set function $P : \mathscr{A} \mapsto \mathbb{R}$ is a probability if it satisfies
i) $P(A) \geq 0 \ \forall \ A \in \mathscr{A}$
ii) $P(S) = 1$
iii) if $A_1, A_2, ... \in \mathscr{A}$ are pairwise disjoint, then
$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{infty} P(A_i)$

Consequences

a. $P(\emptyset) = 0$

b. if A and B are pairwise disjoint, then $P(A \cup B) = P(A) + P(B)$
General Form: if A_1, ... , A_n are pairwise disjoint, then
$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

c. $P(A^c) = 1 - P(A)$

d. if $A \subset B$, then $P(A) \geq P(B)$ and $P(B \setminus A) = P(B) - P(A)$

e. $P(A) \geq 1$, $\forall \ A \subset (A)$

2 Class Notes

Def A collection of sets $\{E_1, E_2, ...\}$ is called a **partition** of event A if:
i) $E_i \cap E_j = \emptyset$, $\forall \ i \neq j$ *(pairwise disjoint)*
ii) $\cup_{i=1}^{\infty} E_i = A$ *(exhaustive)*

Remark Partition $\{E_1$, ..., $E_n\}$ is a finite partition

Remark For $\mathbf{S}$, $\{A, A^c\}$ is a partition

Remark If $E_n : n \geq 1$ is a partition of A, then $P(A) = \sum_{i=1}^{\infty} P(E_i)$

More Consequences

f. Suppose A and B are two events. Then $\{A \cap B, A^c \cap B\}$ is a partition of B. Also, $P(A^c \cap B) = P(B) - P(A \cap B)$
Proof. $(A \cap B) \cap (A^c \cap B) = (A \cap A^c) \cap (B \cap B) = 0$ *(pairwise disjoint)*
$(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = B$ *(exhaustive)*

g. A and B are two events. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(This is the generalized version of b))

h. if $\{C_1, C_2, ...\}$ is a partition of $\mathbf{S}$, then
$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$

i. **Boole's Inequality**
$P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$

j. **Bonferroni's Inequality**
$P(\cap_{i=1}^{\infty} A_i) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

Remark Proving Bonferroni from Boole:
$P(\cup_{i=1}^{\infty} A_i^c) \leq \sum_{i=1}^{\infty} P(A_i^c)$
$1 - P(\cup_{i=1}^{\infty} A_i^c) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$
$= P[(\cup_{i=1}^{\infty} A_i^c)^c] \geq ...$
$= P[\cap_{i=1}^{\infty} (A_i^c)^c] \geq ...$
$= P(\cap_{i=1}^{\infty} A_i) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

Def A sequence of events $\{A_1, A_2 ...\}$ is **increasing to event** A if:
$A_1 \subset A_2 \subset ...$
and $A = \cup_{n=1}^{\infty} A_n$
Notation: $A_n \uparrow A$

Def Similarly, $B_n \downarrow B$ if $B_1 \supset B_2 \supset ...$ and $B = \cap_{n=1}^{\infty} B_n$

k. If $A_n \uparrow A$, then $P(A) = \lim_{n \to \infty} P(A_n)$

l. If $B_n \downarrow B$ then $P(B) = \lim_{n \to \infty} P(B_n)$