

STT 861 Compendium

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Definitions

Def Suppose A and B are two sets. $A \subset B$ (A is a **subset** of B) if $x \in A$ implies $x \in B$. If $A \subset B$ and $B \subset A$ then $A = B$.

Def A and B are **disjoint** if $A \cap B = \emptyset$

Remark Suppose for some set A , $P(A) = 1$ Does this imply $A = S$?

Does $P(B) = 0$ imply $B \neq \emptyset$?

Does $P(A \cap B) = 0$ imply A, B are disjoint?

Not necessarily for all above!

Properties of set theory

Commutative

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

Associative

$$(A \cup B) \cup C = A \cup (B \cup C) =: A \cup B \cup C$$

$$(A \cap B) \cap C = A \cap (B \cap C) =: A \cap B \cap C$$

Distributive

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Def Set Difference

$$A \setminus B = \{x : x \in A, \text{ but } x \notin B\}$$

Def \mathcal{A} is a collection of subsets of $S[\neq \emptyset]$ satisfying:

i) $S \in \mathcal{A}$

ii) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

iii) if $A_1, A_2, \dots \in \mathcal{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$

We call \mathcal{A} a σ - algebra (or σ - field)

Any domain should be a σ - field

Def Given sample space $S(\neq \emptyset)$, and the measurable space (S, \mathcal{A})

A function $P : \mathcal{A} \mapsto \mathbb{R}$ is called probability if it satisfies:

a. $P(A) \geq 0$ for any $A \in \mathcal{A}$

b. $P(S) = 1$

c. if A_1, A_2, \dots are disjoint sets from \mathcal{A} , then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Remark Desired Properties of $P(\cdot)$

a. $P(\emptyset) = 0$

b. If A and B are disjoint then $P(A \cup B) = P(A) + P(B)$

- c. $P(A^c) = 1 - P(A)$
- d. If $A \subset B$ then $P(A) \leq P(B)$
- e. $P(A) \leq 1$

Def A collection of sets $\{E_1, E_2, \dots\}$ is called a **partition** of event A if:

- i) $E_i \cap E_j = \emptyset, \forall i \neq j$ (*pairwise disjoint*)
- ii) $\cup_{i=1}^{\infty} E_i = A$ (*exhaustive*)
- a. Suppose A and B are two events. Then $\{A \cap B, A^c \cap B\}$ is a partition of B . Also, $P(A^c \cap B) = P(B) - P(A \cap B)$
Proof. $(A \cap B) \cap (A^c \cap B) = (A \cap A^c) \cap (B \cap B) = \emptyset$ (*pairwise disjoint*)
 $(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = B$ (*exhaustive*)
- b. A and B are two events. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(This is the generalized version of b))
- c. if $\{C_1, C_2, \dots\}$ is a partition of **S**, then
 $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$
- d. **Boole's Inequality**
 $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$
- e. **Bonferroni's Inequality**
 $P(\cap_{i=1}^{\infty} A_i) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$
- a. If $A_n \uparrow A$, then $P(A) = \lim_{n \rightarrow \infty} P(A_n)$
- b. If $B_n \downarrow B$ then $P(B) = \lim_{n \rightarrow \infty} P(B_n)$

Def A sequence of events $\{A_1, A_2, \dots\}$ is **increasing to event A** if:

$A_1 \subset A_2 \subset \dots$
and $A = \cup_{n=1}^{\infty} A_n$
Notation: $A_n \uparrow A$

Def Similarly, $B_n \downarrow B$ if $B_1 \supset B_2 \supset \dots$ and $B = \cap_{n=1}^{\infty} B_n$

- a. $P(A^c|B) = 1 - P(A|B)$
- b. If $A \subseteq B$ the $P(A|B) = \frac{P(A)}{P(B)}$ and $P(B|A) = 1$
- c. If E_1, E_2, \dots, E_n is partition of S then
 $P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^n P(E_j)P(A|E_j)}$

Def Counting Methods

	WOR	WR
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\frac{n!}{r!(n-r)!} = \binom{n}{r}$	$\binom{n+r-1}{r}$

Def Conditional Probability - if two events A, B with $P(B) > 0$, then the conditional probability of $A|B$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If $P(B) = 0$, $P(A|B)$ is undefined

$$P(A \cap B) = P(A)P(A|B) = P(B)P(B|A)$$

Def Multiplication Rule - If E_1, E_2, \dots, E_n are events,
 $P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2) \dots P(E_n|E_1 \cap \dots \cap E_{n-1})$

Def Bayes' Rule

If E_1, E_2, \dots, E_n is partition of S then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^n P(E_j)P(A|E_j)}$$

Def Two events A, B are **Independent** if $P(A \cap B) = P(A)P(B)$
This is equivalent to $P(A) = P(A|B)$ and $P(B) = P(B|A)$

Def A collection of events A_1, A_2, \dots, A_n are **mutually independent** if for any subcollection A_{i_1}, \dots, A_{i_k} , we have:

$$P(\cap_{j=1}^k A_{i_j}) = \prod_{j=1}^k P(A_{i_j})$$

Remark If A, B are disjoint, then A, B are independent if either $P(A) = 0, P(B) = 0$ or both.

Remark If A and B are independent events, then the following pairs are also independent:

- A and B^c
- A^c and B
- A^c and B^c

Def A **random variable** X is a measurable real valued function defined on the sample space $X : S \mapsto \mathbb{R}$

Def A function F is called the cumulative distribution function iff:

- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$
- $F(x)$ is a nondecreasing fn of x
- $F(x)$ is right-continuous; that is for every number $x_0, \lim_{x \downarrow x_0} F(x) = F(x_0)$

Def A PMF of a discrete r.v. x is $f_X(x) = P_X(X = x)$:

- $0 \leq f_X(x) \leq 1 \quad \forall x$
- $\sum_X f_X(x) = 1$

Def A PDF of a continuous r.v. x is $f_X(x) = \frac{d}{dx} F_X(x)$:

- $f_X(x) \geq 0 \quad \forall x$
- $\int_{-\infty}^{\infty} f_X(x) dx = 1$
- $F_X(x) = \int_{-\infty}^x f_X(s) ds$

Remark Theorem 2.1.3 Let X have cdf $F_X(x)$, let $Y = g(X)$, and let \mathcal{X} and \mathcal{Y} be defined.

- if g is an increasing fn on \mathcal{X} , $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$
- if g is a decreasing fn on \mathcal{X} and X is a continuous r.v., $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$

Remark Theorem 2.1.5 Let X have pdf $f_X(x)$ and let $Y = g(x)$, where g is a monotone fn. Let \mathcal{X} and \mathcal{Y} be defined. Suppose that $f_X(x)$ is continuous on \mathcal{X} and that $g^{-1}(y)$ has continuous derivative on \mathcal{Y} . Then the pdf of Y is given by:

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

Remark Theorem 2.1.8 Let X have pdf $f_X(x)$ and let $Y = g(x)$, and define sample space \mathcal{X} . Suppose there exists a partition A_0, A_1, \dots, A_k of \mathcal{X} such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i . Further, suppose there exist functions $g_1(x), \dots, g_k(x)$ defined on A_1, \dots, A_k , respectively, satisfying

- a. $g(x) = g_i(x)$ for $x \in A_i$
- b. $g_i(x)$ is monotone on A_i
- c. the set $\mathcal{Y} = \{y : y = g_i(x) \text{ for some } x \in A_i \text{ is the same for each } i = 1, \dots, k\}$
- d. $g_i^{-1}(y)$ has a continuous derivative on \mathcal{Y} , for each $i = 1, \dots, k$

Then:

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$