STT 861 Compendium

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Set Theory

<u>**Def**</u> Suppose A and B are two sets. $A \subset B$ (A is a **subset** of B) if $x \in A$ implies $x \in B$. If $A \subset B$ and $B \subset A$ then $A = B$.

<u>**Def**</u> A and B are **disjoint** if $A \cap B = \emptyset$

Remark Suppose for some set $A, P(A) = 1$ Does this imply $A = S$?
Does $P(B) = 0$ imply $B \neq \emptyset$?
Does $P(A \cap B) = 0$ imply A, B are disjoint?
Not necessarily for all above!

<u>**Properties of set theory**</u>
Commutative
$A \cup B = B \cup A$ and $A \cap B = B \cap A$
Associative
$(A \cup B) \cup C = A \cup (B \cup C) =: A \cup B \cup C$
$(A \cap B) \cap C = A \cap (B \cap C) =: A \cap B \cap C$
Distributive
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
De Morgan's Law
$(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$

<u>**Def**</u> **Set Difference**
$A \setminus B = \{x : x \in A, \text{ but } x \notin B\}$

Probability

Remark Geometric Sum

$$\sum_{i=0}^{\infty} r^n = \frac{1}{1-r}$$

<u>**Def**</u> $\mathscr{A}$ is a collection of subsets of $\mathbf{S}[\neq \emptyset]$ satisfying:
i) $\mathbf{S} \in \mathscr{A}$
ii) if $A \subset \mathscr{A}$, then $A^c \in \mathscr{A}$
iii) if $A_1, A_2, ... \in \mathscr{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathscr{A}$
We call $\mathscr{A}$ a σ - algebra (or σ - field)
Any domain should be a σ - field

<u>**Def**</u> Given sample space $\mathbf{S}(\neq \emptyset)$, and the measurable space $(\mathbf{S}, \mathscr{A})$
A function $P : \mathscr{A} \mapsto \mathbb{R}$ is called probability if it satisfies:

a. $P(A) \geq 0$ for any $A \in \mathscr{A}$

b. $P(\mathbf{S}) = 1$

c. if $A_1, A_2 ...$ are disjoint sets from $\mathscr{A}$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Remark Desired Properties of P(.)

a. $P(\emptyset) = 0$

b. If A and B are disjoint then $P(A \cup B) = P(A) + P(B)$

c. $P(A^c) = 1 - P(A)$

d. If $A \subset B$ then $P(A) \leq P(B)$

e. $P(A) \leq 1$

Def A collection of sets $\{E_1, E_2, ...\}$ is called a **partition** of event A if:
i) $E_i \cap E_j = \emptyset$, $\forall\, i \neq j$ *(pairwise disjoint)*
ii) $\cup_{i=1}^{\infty} E_i = A$ *(exhaustive)*

a. Suppose A and B are two events. Then $\{A \cap B, A^c \cap B\}$ is a partition of B. Also, $P(A^c \cap B) = P(B) - P(A \cap B)$

Proof. $(A \cap B) \cap (A^c \cap B) = (A \cap A^c) \cap (B \cap B) = 0$ *(pairwise disjoint)*
$(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = B$ *(exhaustive)*

b. A and B are two events. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

c. if $\{C_1, C_2, ...\}$ is a partition of **S**, then
$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$

d. **Boole's Inequality**
$P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$

e. **Bonferroni's Inequality**
$P(\cap_{i=1}^{\infty} A_i) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

f. If $A_n \uparrow A$, then $P(A) = \lim_{n \to \infty} P(A_n)$

g. If $B_n \downarrow B$ then $P(B) = \lim_{n \to \infty} P(B_n)$

Def A sequence of events $\{A_1, A_2 ...\}$ is **increasing to event** A if:
$A_1 \subset A_2 \subset ...$
and $A = \cup_{n=1}^{\infty} A_n$
Notation: $A_n \uparrow A$

Def Similarly, $B_n \downarrow B$ if $B_1 \supset B_2 \supset ...$ and $B = \cap_{n=1}^{\infty} B_n$

Def Counting Methods

	WOR	**WR**
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\frac{n!}{r!(n-r)!} = \binom{n}{r}$	$\binom{n+r-1}{r}$

Conditional Probability and Independence

Def Conditional Probability - if two events A, B with $P(B) >= 0$, then the conditional probability of $A|B$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If $P(B) = 0$, $P(A|B)$ is undefined
$P(A \cap B) = P(A)P(A|B) = P(B)P(B|A)$

a. $P(A^c|B) = 1 - P(A|B)$

b. If $A \subseteq B$ then $P(A|B) = \frac{P(A)}{P(B)}$ and $P(B|A) = 1$

Def **Multiplication Rule -** If E_1, E_2, ... E_n are events,
$P(E_1 \cap E_2 \cap ... \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)...\ P(E_n|E_1 \cap\ ... \cap E_{n-1})$

Def **Bayes' Rule**
If $E_1, E_2 ... E_n$ is partition of S then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$$

Def Two events A, B are **Independent** if $P(A \cap B) = P(A)P(B)$
This is equivalent to $P(A) = P(A|B)$ and $P(B) = P(B|A)$

Def A collection of events $A_1, A_2, ..., A_n$ are **mutually independent** if for any subcollection $A_{i1}, ..., A_{ik}$, we have:

$$P(\cap_{j=1}^{k} A_{i_j}) = \prod_{j=1}^{k} P(A_{i_j})$$

Remark If A, B are disjoint, then A, B are independent if either $P(A) = 0$, $P(B) = 0$ or both.

Remark If A and B are independent events, then the following pairs are also independent:

a. A and B^c

b. A^c and B

c. A^c and B^c

Random Variables and Distribution Functions

Def A **random variable** X is a measurable real valued function defined on the sample space $X : S \mapsto \mathbb{R}$

Def A function F is called the cumulative distribution function iff:

a. $lim_{x \to -\infty} F(x) = 0$ and $lim_{x \to \infty} = 1$

b. $F(x)$ is a nondecreasing fn of x

c. $F(x)$ is right-continuous; that is for every number $x_0, lim_{x \downarrow x_0} F(x) = F(x_0)$

Density and Mass Functions

Def A PMF of a discrete r.v. x is $f_X(x) = P_X(X = x)$:

a. $0 \leq f_X(x) \leq 1\ \forall x$

b. $\sum_X f_X(x) = 1$

Def A PDF of a continuous r.v. x is $f_X(x) = \frac{d}{dx} F_X(x)$:

a. $f_X(x) \geq 0\ \forall x$

b. $\int_{-\infty}^{\infty} f_X(x) dx = 1$

c. $F_X(x) = \int_{-\infty}^{x} f_X(s) ds$

Distributions of Functions of a Random Variable

<u>Def</u> **Sample Spaces**:
$\mathscr{X} = x : f_X(x) > 0$ and $\mathscr{Y} = y : y = g(x)\ for\ some\ x \in \mathscr{X}$

Remark Theorem 2.1.3 Let X have cdf $F_X(x)$, let $Y = g(X)$, and let $\mathscr{X}$ and $\mathscr{Y}$ be defined.

a. if g is an increasing fn on $\mathscr{X}$, $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathscr{Y}$

b. if g is a decreasing fn on $\mathscr{X}$ and X is a continuous r.v., $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathscr{Y}$

Remark Theorem 2.1.5 Let X have pdf $f_X(x)$ and let $Y = g(x)$, where g is a monotone fn. Let $\mathscr{X}$ and $\mathscr{Y}$ be defined. Suppose that $f_X(x)$ is continuous on $\mathscr{X}$ and that $g^{-1}(y)$ has continuous derivative on $\mathscr{Y}$. Then the pdf of Y is given by:

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \dfrac{d}{dy} g^{-1}(y) \right| & y \in \mathscr{Y} \\ 0 & otherwise \end{cases}$$

Remark Theorem 2.1.8 Let X have pdf $f_X(x)$ and let $Y = g(x)$, and define sample space $\mathscr{X}$. Suppose there exists a partition $A_0, A_1, ..., A_k$ of $\mathscr{X}$ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further, suppose there exist functions $g_1(x), ..., g_k(x)$ defined on $A_1, ..., A_k$, respectively, satisfying

a. $g(x) = g_i(x)$ for $x \in A_i$

b. $g_i(x)$ is monotone on A_i

c. the set $\mathscr{Y} = y : y = g_i(x)\ for\ some\ x \in A_i$ is the same for each $i = 1, ..., k)$

d. $g_i^{-1}(y)$ has a continuous derivative on $\mathscr{Y}$, for each $i = 1, ..., k$
Then:

$$f_Y(y) = \begin{cases} \displaystyle\sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \dfrac{d}{dy} g_i^{-1}(y) \right| & y \in \mathscr{Y} \\ 0 & otherwise \end{cases}$$