STT 861 Compendium

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Definitions

Def A **random experiment** is an action which will result in one of the many possible outcomes.

Def A **sample space** is the collection of all possible outcomes of a random experiment. We shall denote by **S**.

Def An **Event** is a subset of sample space **S** for which we can define probability.

Def Suppose A and B are two sets. $A \subset B$ (A is a **subset** of B) if $x \in A$ implies $x \in B$. If $A \subset B$ and $B \subset A$ then $A = B$.

Def A set is called an empty set (or **null set**) if it contains no elements.
Notation: $\{\emptyset\}$
Convention: $\emptyset \subset A$, for any set A
Corrolary: $\forall A$, $\emptyset \subset A \subset \mathbf{S}$

Def **Complement** A^c is the set such that $x \in A^c \Rightarrow x \notin A$.
In other words, $A^c = \{x : x \notin A\}$
Notation: A^c or A' or $\overline{A}$

Def **Intersection** A, B are two events.
$A \cap B = \{x : x \in A \text{ and } x \in B\}$

Def **Union** A, B are two events.
$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}$

Def A and B are **disjoint** if $A \cap B = \emptyset$

Properties of set theory
Commutative
$A \cup B = B \cup A$ and $A \cap B = B \cap A$
Associative
$(A \cup B) \cup C = A \cup (B \cup C) =: A \cup B \cup C$
$(A \cap B) \cap C = A \cap (B \cap C) =: A \cap B \cap C$
Distributive
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
De Morgan's Law
$(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$

Def **Set Difference**
$A \setminus B = \{x : x \in A, \text{ but } x \notin B\}$

Def **Symmetric Difference**
$A \triangle B = \{x : x \in A \setminus B, \text{ or } x \in B \setminus A\}$

Def A set A is **finite** if there exists a 1-1 fn $A \mapsto \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$

Def A set A is **finite** if there exists a 1-1 fn $A \mapsto \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$

Def A set A is **countably infinite** if there exists a one-to-one function from $A \mapsto \mathbb{N}$.

Def A set is called **coutable** if it is either finite or countably infinite.

Def $\mathscr{A}$ is a collection of subsets of $\mathbf{S}[\neq \emptyset]$ satisfying:
i) $\mathbf{S} \in \mathscr{A}$
ii) if $A \subset \mathscr{A}$, then $A^c \in \mathscr{A}$
iii) if $A_1, A_2, ... \in \mathscr{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathscr{A}$
We call $\mathscr{A}$ a σ - algebra (or σ - field)
Any domain should be a σ - field

Def $(\mathbf{S}, \mathscr{A}, P)$ is a (probability) measure space

Def Given sample space $\mathbf{S}(\neq \emptyset)$, and the measurable space $(\mathbf{S}, \mathscr{A})$
A function $P : \mathscr{A} \mapsto \mathbb{R}$ is called probability if it satisfies:

a. $P(A) \geq 0$ for any $A \in mathscrA$

b. $P(\mathbf{S}) = 1$

c. if $A_1, A_2 ...$ are disjoint sets from $\mathscr{A}$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Def A collection of sets $\{E_1, E_2, ...\}$ is called a **partition** of event A if:
i) $E_i \cap E_j = \emptyset$, $\forall\, i \neq j$ *(pairwise disjoint)*
ii) $\cup_{i=1}^{\infty} E_i = A$ *(exhaustive)*

Def A sequence of events $\{A_1, A_2 ...\}$ is **increasing to event** A if:
$A_1 \subset A_2 \subset ...$
and $A = \cup_{n=1}^{\infty} A_n$
Notation: $A_n \uparrow A$

Def Similarly, $B_n \downarrow B$ if $B_1 \supset B_2 \supset ...$ and $B = \cap_{n=1}^{\infty} B_n$

Def Counting Methods

	WOR	**WR**
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\frac{n!}{r!(n-r)!} = \binom{n}{r}$	$\binom{n+r-1}{r}$

Def **Conditional Probability** - if two events A, B with $P(B) >= 0$, then the conditional probability of $A|B$ is $P(A|B) = \frac{P(A \cap B)}{P(B)}$
If $P(B) = 0$, $P(A|B)$ is undefined
$P(A \cap B) = P(A)P(A|B) = P(B)P(B|A)$

Def **Multiplication Rule -** If E_1, E_2, ... E_n are events,
$P(E_1 \cap E_2 \cap ... \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)...\ P(E_n|E_1 \cap\ ... \cap E_{n-1})$

Def **Bayes' Rule**
If $E_1, E_2 ... E_n$ is partition of S then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$$

Def Two events A, B are **Independent** if $P(A \cap B) = P(A)P(B)$
This is equivalent to $P(A|B) = P(A)$ and $P(B) = P(B|A)$

Def A collection of events $A_1, A_2, ..., A_n$ are **mutually independent** if for any subcollection $A_{i1}, ..., A_{ik}$, we have:

$$P(\cap_{j=1}^{k} A_{i_j} = \prod_{j=1}^{k} P(A_{i_j})$$

Results

a. $(A^c)^c = A$

b. $\mathbf{S}^c = \emptyset$

c. $\emptyset^c = \mathbf{S}$

d. if $A \subset B$, then $B^c \subset A^c$

Remark Proving Distributive Property:
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

> ***Proof.*** Prove left direction:
> $x \in (A \cup B) \cap C \Rightarrow x \in A \cup B$ and $x \in C$
> $\Rightarrow (x \in A$ or $x \in B)$ and $x \in C$
> $\Rightarrow (x \in A$ and $x \in C)$ or $(x \in A$ and $x \in C)$ or both
> $\Rightarrow x \in A \cap C$ or $x \in B \cap C$ or both
> $\Rightarrow x \in (A \cap C) \cup (B \cap C)$
> Right direction is reverse of above.

a. $A \cap B \subset A$ and $A \cap B \subset B$

b. $A \cap A = A$

c. if $A \subset B$ then $A \cap B = A$

d. $A \subset A \cup B$ and $B \subset A \cup B$

e. $A \cup A = A$

f. if $A \subset B$ then $A \cup B = B$

g. $A \cup A^c = \mathbf{S}$ and $A \cap A^c = \emptyset$

h. if $A \subset C$ and $B \subset C$ then $A \cap B \subset A \cup B \subset C$

i. $\emptyset$ is disjoint to all events

j. if $a \cap B = \emptyset$ then $A \subset B^c$ and $B \subset A^c$

k. if $A \subset B$, then $B^c \subset A^c$

l. $A \setminus B = A \cap B^c$ and $B \setminus A = A^c \cap B$

m. $A \triangle B = (A \cup B) \setminus (A \cap B)$

Remark Consider $\mathbf{S} = \{H, T\}$ and $\mathscr{P}(\mathbf{S}) = \{\emptyset, \{H\}, \{T\}, \mathbf{S}\}$
If $\mathbf{S}$ is countable, we can take $\mathscr{P}(\mathbf{S})$ as the domain for probability function P.
However, if $\mathbf{S}$ is uncountable, then $\mathbf{S}$ is too large, and it is not possible to define a function for $\mathscr{P}(\mathbf{S})$

Remark **Desired Properties of P(.)**

a. $P(\emptyset) = 0$

b. If A and B are disjoint then $P(A \cup B) = P(A) + P(B)$

c. $P(A^c) = 1 - P(A)$

d. If $A \subset B$ then $P(A) \leq P(B)$

e. $P(A) \leq 1$

Remark Partition $\{E_1, ..., E_n\}$ is a finite partition

Remark For $\mathbf{S}$, $\{A, A^c\}$ is a partition

Remark If $E_n : n \geq 1$ is a partition of A, then $P(A) = \sum_{i=1}^{\infty} P(E_i)$

a. Suppose A and B are two events. Then $\{A \cap B, A^c \cap B\}$ is a partition of B. Also, $P(A^c \cap B) = P(B) - P(A \cap B)$
   Proof. $(A \cap B) \cap (A^c \cap B) = (A \cap A^c) \cap (B \cap B) = 0$ *(pairwise disjoint)*
   $(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = B$ *(exhaustive)*

b. A and B are two events. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   (This is the generalized version of b))

c. if $\{C_1, C_2, ...\}$ is a partition of $\mathbf{S}$, then
   $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$

d. **Boole's Inequality**
   $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$

e. **Bonferroni's Inequality**
   $P(\cap_{i=1}^{\infty} A_i) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

Remark Proving Bonferroni from Boole:
$P(\cup_{i=1}^{\infty} A_i^c) \leq \sum_{i=1}^{\infty} P(A_i^c)$
$1 - P(\cup_{i=1}^{\infty} A_i^c) \ \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$
$= P[(\cup_{i=1}^{\infty} A_i^c)^c] \ \geq ...$
$= P[\cap_{i=1}^{\infty} (A_i^c)^c] \ \geq ...$
$= P(\cap_{i=1}^{\infty} A_i) \ \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

a. If $A_n \uparrow A$, then $P(A) = \lim_{n \to \infty} P(A_n)$

b. If $B_n \downarrow B$ then $P(B) = \lim_{n \to \infty} P(B_n)$

Remark Suppose for some set $A, P(A) = 1$ Does this imply $A = S$?
Does $P(B) = 0$ imply $B \neq \emptyset$?
Does $P(A \cap B) = 0$ imply A, B are disjoint?
Not necessarily for all above!

a. $P(A^c|B) = 1 - P(A|B)$

b. If $A \subseteq B$ the $P(A|B) = \frac{P(A)}{P(B)}$ and $P(B|A) = 1$

c. If $E_1, E_2 ... E_n$ is partition of S then
   $P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$

Remark If A, B are disjoint, then A, B are independent if either $P(A) = 0$, $P(B) = 0$ or both.

Remark If A and B are independent events, then the following pairs are also independent:

a. A and B^c

b. A^c and B

c. A^c and B^c