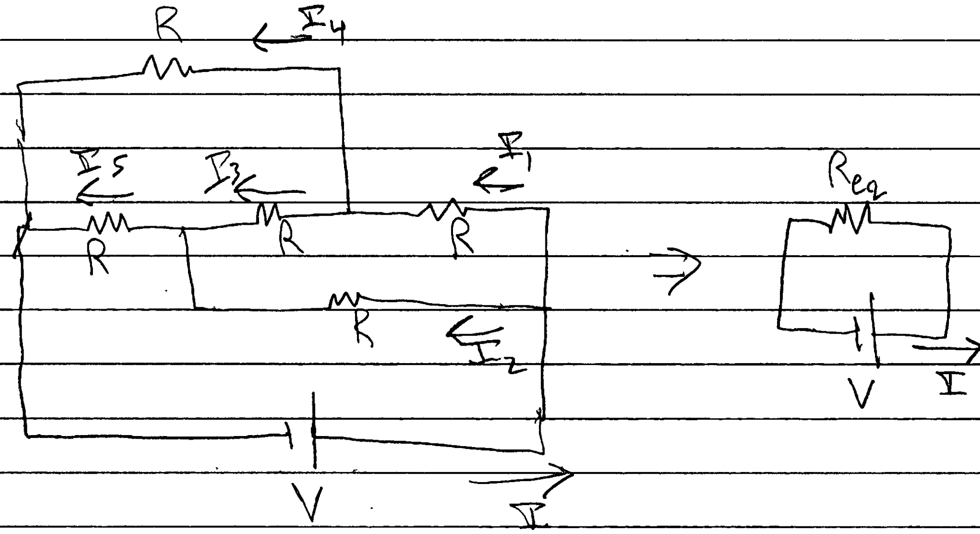
Week 11, Session 2 Solutions

Problem 1.1:



Using Kirchhoff's Laws, we will write down several relations, where we are hoping to solve for I:

Current		Voltage	
$I = I_1 + I_2$	(1)	$V = (I_2 + I_5)R$	(5)
$I_1 = I_3 + I_4$	(2)	$V = (I_1 + I_4)R$	(6)
$I_3 + I_2 = I_5$	(3)	$V = (I_1 + I_3 + I_5)R$	(7)
$I = I_4 + I_5$	(4)	$(I_1 + I_3)R = I_2 R$	(8)
		$(I_3 + I_5)R = I_4 R$	(9)
		$V = I R_{eq}$	(10)

Looking at these equations, we can pick out from adding (5) and (6) that

$$2V = (I_1 + I_2 + I_4 + I_5)R$$

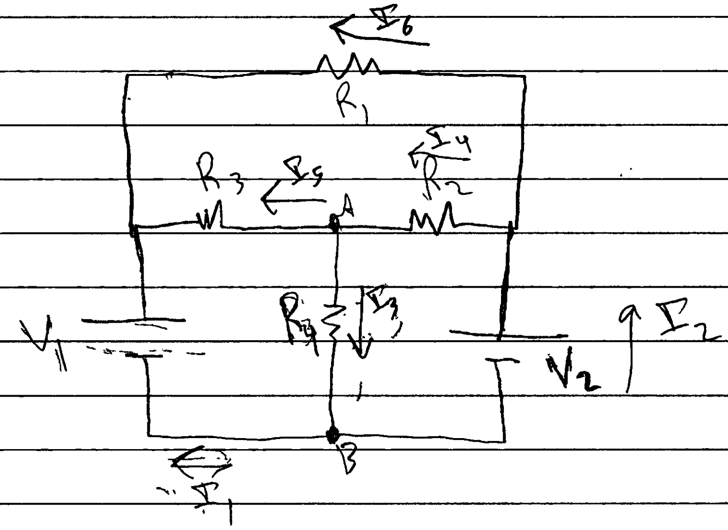
From (1) and (4), this becomes

$$2V = 2IR \Rightarrow I = \frac{V}{R}$$

Then, from (10),

$$V = \frac{V}{R} R_{eq} \Rightarrow R_{eq} = R$$

Problem 1.2:



From the voltage rule,

$$V_2 = I_3 R_4 + I_4 R_2 \Rightarrow I_4 = \frac{V_2 - I_3 R_4}{R_2} = \frac{20-12}{2} = 4\text{ A}$$

Then, at node A,

$$I_4 = I_3 + I_5 \Rightarrow I_5 = I_4 - I_3 = 4 - 3 = 1\text{ A}$$

Over the small loop with V_1,

$$V_1 + I_5 R_3 = I_3 R_4 \Rightarrow V_1 = (12-3) = 9\text{ V}$$

Now, through the outer loop,

$$V_2 - V_1 = I_6 R_1 \Rightarrow I_6 = \frac{V_2 - V_1}{R_1} = \frac{20-9}{1} = 11\text{ A}$$

Now,

$$I_2 = I_4 + I_6 = 15\text{ A}$$

So

$$I_3 = I_1 + I_2 \Rightarrow I_1 = I_3 - I_2 = -12\text{ A}$$

This just means the direction of I_1 is opposite what we drew.

(a) $I_3 = 3\text{ A}$, $I_4 = 4\text{ A}$, $I_5 = 1\text{ A}$, $I_6 = 11\text{ A}$

(b) $V_1 = 9\text{ V}$

(c) $P_2 = I_2 V_2 = (15)(20) = 300\text{ W}$

$P_1 = I_1 V_1 = (-12)(9) = -108\text{ W}$

Problem 1.3:

(a) $V_1 = Q_0/C_1$, so immediately after the switch is closed,

$$I = V/R = \frac{Q_0}{C_1 R}$$

(b) We will need $V_1 = V_2$, so

$$Q_1/C_1 = Q_2/C_2$$

and

$$Q_1 + Q_2 = Q_0 \Rightarrow Q_0 - Q_2 = Q_2 \frac{C_1}{C_2}$$
$$\Rightarrow Q_2 = Q_0/(1 + C_1/C_2)$$
$$\Rightarrow Q_1 = Q_0/(1 + C_2/C_1)$$

(c) Our loop has the total voltage drop

$$Q_1(t)/C_1 + \frac{dQ_1(t)}{dt} R = Q_2(t)/C_2$$
$$\Rightarrow \frac{-dQ_2}{dt} R = \frac{Q_2}{C_2} - \frac{(Q_0 - Q_2)}{C_1}$$
$$\Rightarrow -C_2 R \frac{dQ_2}{dt} = Q_2(1 + C_2/C_1) - C_2 Q_0/C_1$$
$$\Rightarrow \frac{-dQ_2}{dt} + Q_2 \frac{1}{RC_2}\left(1 + \frac{C_2}{C_1}\right) = -\frac{Q_0}{C_1 R}$$

This is of the form

$$\dot{y} + ay = b$$

which is rewritten to have the solution

$$\frac{d}{dt}(y e^{at}) = b e^{at} \Rightarrow y e^{at} = \frac{b}{a} e^{at} + C \Rightarrow y = \frac{b}{a} + C e^{-at}$$

Therefore,

$$Q_2(t) = \frac{C_2}{C_1 + C_2} Q_0 + C e^{-\frac{C_1 + C_2}{RC_1 C_2} t}$$

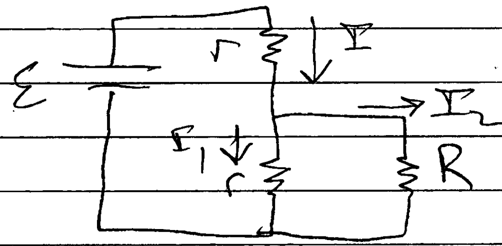
At $t = 0$, $Q_2 = 0$, so $C = -\frac{C_2}{C_1 + C_2} Q_0$

$$Q_2(t) = \frac{C_2}{C_1 + C_2} Q_0 \left(1 - e^{-\frac{C_1 + C_2}{RC_1 C_2} t}\right)$$

(d) C_2 changes by a factor of ϵ?

$$Q_2(t) = \frac{\epsilon C_2}{C_1 + \epsilon C_2} Q_0 \left(1 - e^{-\frac{C_1 + \epsilon C_2}{R_1 \epsilon C_1 C_2} t}\right)$$

Problem 1.4:



We have

$$I = I_1 + I_2 \qquad (1)$$

$$\mathcal{E} = (I + I_1) r \qquad (2)$$

$$I_1 r = I_2 R \qquad (3)$$

From (1) and (3),

$$I = I_1\left(1 + \frac{r}{R}\right) \Rightarrow I_1 = I\,\frac{R}{r+R}$$

Then, from (3),

$$I_2 = I_1 \frac{r}{R} = I \frac{r}{r+R}$$

So the power through this resistor is

$$P_0 = I_2^2 R$$

$$= I^2 \frac{R r^2}{(r+R)^2}$$

Now, if we want this power to stay constant even if R varies slightly, we write

$$P = I^2 r^2 \frac{R + \Delta R}{(r+R+\Delta R)^2}$$

$$= I^2 r^2 \left[\frac{R+\Delta R}{(r+R)^2}\left(1 + \frac{\Delta R}{r+R}\right)^{-2}\right]$$

$$= I^2 \frac{r^2 R}{(r+R)^2}\left[\left(1 + \frac{\Delta R}{R}\right)\left(1 + \frac{\Delta R}{r+R}\right)^{-2}\right]$$

$$\approx P_0 \left[\left(1 + \frac{\Delta R}{R}\right)\left(1 - \frac{2\Delta R}{r+R}\right)\right]$$

$$= P_0 \left(1 + \Delta R\left(\frac{1}{R} - \frac{2}{r+R}\right) - \Delta R^2 \frac{2}{R(r+R)}\right)$$

We ignore second order small terms and to have the ΔR coefficient go to zero,

$$\frac{1}{R} = \frac{2}{r+R} \Rightarrow r + R = 2R \Rightarrow r = R$$