More Gas Math, Solutions

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1 More Statistics

1.1 Continuous vs Discrete

(a) First, we solve for A:

$$\begin{aligned}\int_0^\infty f(v)dv &= \int_0^1 f(v)dv \\ &= A\int_0^1 sin(\pi v)|_0^1 \\ &= \frac{-A}{\pi}cos(\pi v)|_0^1 \\ &= \frac{2A}{\pi}\end{aligned}$$

We mush have $\int_0^\infty f(v)dv = 1$, so $A = \frac{\pi}{2}$. Now, solving for B:

$$\begin{aligned}P(v=0)+P(v=1/4)+P(v=1/2)+P(v=3/4)+P(v=1) &= 1 \\ 0 + B/\sqrt{2} + B + B/\sqrt{2} + 0 &= 1 \\ B(1+\sqrt{2}) &= 1 \\ B &= \frac{1}{1+\sqrt{2}}\end{aligned}$$

For the quesiton of units, A has units of seconds/meter, B is unitless, and π has units of seconds/meter. If this last one confuses you, you could equivalently have a "ghost" one in there with units of seconds/meter and let π be unitless.

(b) For each distribution, we calculate $< v >= \bar{v}$, v_{rms}, σ, and v_p. Starting with A:

$$
\begin{aligned}
< v > &= \int_0^\infty v f(v) dv \\
&= \int_0^1 \frac{\pi}{2} v sin(\pi v) dv \\
&= \frac{\pi}{2}\left[\frac{-v}{\pi} cos(\pi v)|_0^1 + \int_0^1 \frac{1}{\pi} cos(\pi v) dv\right] \\
&= \frac{1}{2}\left[-v cos(\pi v) + \frac{1}{\pi} sin(\pi v)|0^1\right] \\
&= \frac{1}{2} m/s
\end{aligned}
$$

$$
\begin{aligned}
< v^2 > &= \int_0^\infty v^2 f(v) dv \\
&= \int_0^1 \frac{\pi}{2} v^2 sin(\pi v) dv \\
&= \frac{\pi}{2}\left[\frac{-v^2}{\pi} cos(\pi v)|_0^1 + \int_0^1 \frac{2}{\pi} v cos(\pi v) dv\right] \\
&= \frac{-v^2}{2} cos(\pi v)|_0^1 + \frac{v}{\pi} sin(\pi v) + \frac{1}{\pi^2} cos(\pi v)|_0^1 \\
&= \frac{1}{2} - \frac{2}{\pi^2} \\
&= \frac{\pi^2 - 4}{2\pi^2} \\
&\approx 0.30 m^2 s^{-2}
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= \sqrt{< v^2 > - < v >^2} \\
&= \sqrt{\frac{1}{2} - \frac{2}{\pi^2} - \frac{1}{4}} \\
&= \sqrt{\frac{1}{4} - \frac{2}{\pi^2}} \\
&= \frac{1}{2\pi}\sqrt{\pi^2 - 8} \\
&\approx 0.22 m/s
\end{aligned}
$$

Lastly, taking derivatives and finding the local maximum between 0 and 1:

$$
f'(v) = \frac{\pi^2}{2} cos(\pi v) \implies v_p = \frac{1}{2} m/s \tag{1}
$$

where this can be verified with a second derivative test if you want. Now, for B, we do the same thing in the discrete case:

$$\begin{aligned}< v >= \frac{1}{1+\sqrt{2}}\left(\frac{1}{\sqrt{2}}\frac{1}{4}+\frac{1}{\sqrt{2}}\frac{3}{4}+\frac{1}{2}\right) \\ &= \frac{1}{1+\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right) \\ &= \frac{1}{1+\sqrt{2}}\frac{1+\sqrt{2}}{2} \\ &= \frac{1}{2}m/s\end{aligned}$$

$$\begin{aligned}< v^2 >= \frac{1}{1+\sqrt{2}}\left(\frac{1}{\sqrt{2}}\frac{1}{16}+\frac{1}{\sqrt{2}}\frac{9}{16}+\frac{1}{4}\right) \\ &= \frac{1}{1+\sqrt{2}}\frac{5+2\sqrt{2}}{8\sqrt{2}} \\ &\approx 0.29m^2s^{-2}\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{< v^2 > - < v >^2} \\ &\approx 0.19m/s\end{aligned}$$

Lastly, $v_p = \frac{1}{2}$ m/s since it is simply the speed with the highest probability.

(c) A:

$$\begin{aligned}< v^2 > &= 3\frac{k}{m}T \\ T &= \frac{m}{3k} < v^2 > \\ &\approx \frac{1}{10}mk(m^2s^{-2})\end{aligned}$$

B:

$$T = \frac{m}{3k} < v^2 > \approx 0.096\frac{m}{k}(m^2s^{-2}) \tag{2}$$

(d) Even though the discrete distribution was taken from samples of the continuous one, the statistics differe slightly. Nonetheless, for only five samples, the difference are fairly small: $\sigma_A \approx 0.22$ m/s whereas $\sigma_B \approx 0.19$ m/s. Hence, B has a tighter distribution, which makes sense since the tails of A are ignored with the resampling.

In general, you have to be careful with how you resample a continuous distribution to get a discrete spectrum that is more computable (in "real-life" cases where the continuous distribution does not follow a pretty function or is unknown entirely).

1.2 Fun with Maxwell

(1) Method 1 (following pgs. 477-78 of Giancoli): With wall collisions, $\Delta(mv) = 2mv_x$ on the x-direction walls. In a box of dimensions L, the time between collisions in this direction is $\Delta t = 2L/v_x$, so

$$F = \frac{\Delta(mv)}{\Delta t} = \frac{mv_x^2}{L} \tag{3}$$

for the wall's force exerted on a molecule in a collision (averaged). Summing over all the molecules,

$$F_{total} = \frac{m}{L}\left(v_{x1}^2 + \cdot + v_{xN}^2\right) \tag{4}$$

Mulitplying by N/N and noting that $\bar{v_x^2} = \frac{v_{x1}^2 + \cdot + v_{xN}^2}{N}$, we have

$$F = \frac{m}{L} N \bar{v_x^2} \tag{5}$$

Now, $\bar{v^2} = 3\bar{v_x^2}$, so with $P = \frac{F}{A} = \frac{F}{L^2}$, we have

$$P = \frac{1}{3}\frac{Nm\bar{v^2}}{L^3} = \frac{1}{3}\frac{Nm\bar{v^2}}{V} \tag{6}$$

Rearranging, $PV = \frac{2}{3}N\left(\frac{1}{2}m\bar{v^2}\right)$. From the Ideal Gas Law, $PV = NkT$, so

$$\frac{2}{3}E = kT \implies E = < \frac{1}{2}m\bar{v^2} > = \frac{3}{2}kT \tag{7}$$

(2) Method 2 (from the Maxwell distribution): Using $f(v)$ from the Maxwell distribution and $< v^2 > = \int_0^\infty v^2 f(v)dv$, we have

$$< v^2 > = \int_0^\infty A v^4 e^{-Bv^2} dv \tag{8}$$

where $A = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}$ and $B = \frac{1}{2}\frac{m}{kT}$. Observe that

$$v^4 e^{-Bv^2} = \frac{d^2}{dB^2}\left(e^{-Bv^2}\right) \tag{9}$$

So we can use this to rewrite (and using the formula for the integral of a Gaussian on the second step)

$$\begin{aligned} < v^2 > &= A\frac{d^2}{dB^2}\int_0^\infty e^{-Bv^2} dv \\ &= A\frac{d^2}{dB^2}\left(\frac{1}{2}\sqrt{\frac{\pi}{B}}\right) \\ &= \frac{1}{2}\sqrt{\pi}\frac{-1}{2}\frac{-3}{2}B^{-5/2} \\ &= \frac{3\sqrt{\pi}}{8}AB^{-5/2} \end{aligned}$$

Plugging in the constants again,

$$\begin{aligned}
< v^2 > &= \frac{3\sqrt{\pi}}{8} \times 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \times \left(\frac{1}{2}\frac{m}{kT}\right)^{-5/2} \\
&= 3\left(\frac{m}{kT}\right)^{3/2}\left(\frac{kT}{m}\right)^{5/2} \\
&= 3\frac{kT}{m}
\end{aligned}$$

Hence,

$$< \frac{1}{2}mv^2 >= \frac{3}{2}kT \tag{10}$$