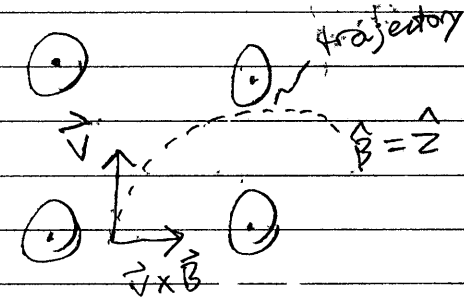
Week 12, Session 1 Solutions

Problem 1.1:

When an ion is accelerated through a potential difference, we know it gains energy $q\Delta V$, so in this case all that energy is going to kinetic energy as the charge is losing potential energy. So as it enters the entrance slit, it has speed given by

$$\frac{1}{2}mv^2 = qV_0 \Rightarrow v = \sqrt{\frac{2qV_0}{m}}$$

Now, in the field, we have the picture to the right, with the magnetic force providing a central force for circular motion. Hence



$$F_c = qvB$$
$$= ma_c$$
$$= m\frac{v^2}{(x/2)}$$

Rearranging,

$$xqB = 2mv \Rightarrow m = \frac{xqB}{2v}$$
$$= \frac{xqB}{2}\left(\frac{m}{2qV_0}\right)^{1/2}$$
$$\Rightarrow m^{1/2} = \frac{xB}{2}\left(\frac{q}{2V_0}\right)^{1/2}$$
$$\Rightarrow m = \frac{B^2 q}{8V_0}x^2$$

Problem 1.2:

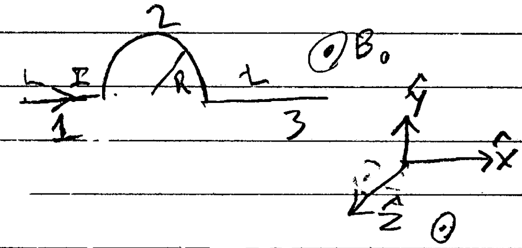
In general, the magnetic force on a wire is given by

$$\vec{F} = I\vec{L} \times \vec{B}$$

or in differential form,

$$d\vec{F} = I\, d\vec{l} \times \vec{B}$$

So we will split our wire into three segments and compute the net magnetic force on each of the three segments.



Segment 1:

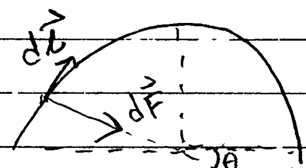
$$\vec{F} = I(L\hat{x}) \times (B_0\hat{z})$$
$$= ILB_0(-\hat{y})$$

Segment 3:

Ditto: $\vec{F} = ILB_0(-\hat{y})$

Segment 2:

$\theta \in (0, \pi)$



By symmetry, the $\hat{x}$ components will cancel, so we will only take the $\hat{y}$ component and

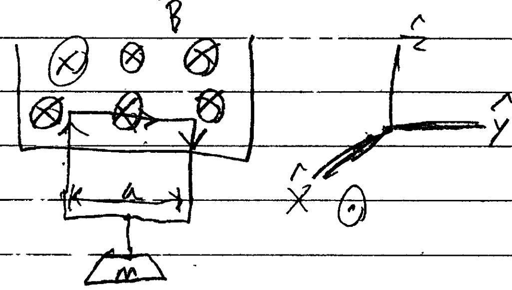
$$\vec{F} = \int d\vec{F}$$
$$= \int I\, d\vec{l} \times \vec{B}$$
$$= \int_0^{\pi} I B_0 R\, d\theta\, (-\sin\theta\, \hat{y})$$
$$= -IB_0R\,\hat{y} \int_0^{\pi} \sin\theta\, d\theta$$
$$= IB_0R\,\hat{y}\, [\cos\theta]_0^{\pi}$$
$$= 2IB_0R\,(-\hat{y})$$

So overall, the net magnetic force on the whole wire is

$$\vec{F}_{net} = -2IB_0(L+R)\,\hat{y}$$

Problem 1.3:

(a) The magnetic force on the wire will be in the $\hat{z}$ direction with



$$\vec{F}_M = (I a \hat{y}) \times (-B_0 \hat{x})$$
$$= I a B_0 \hat{z}$$

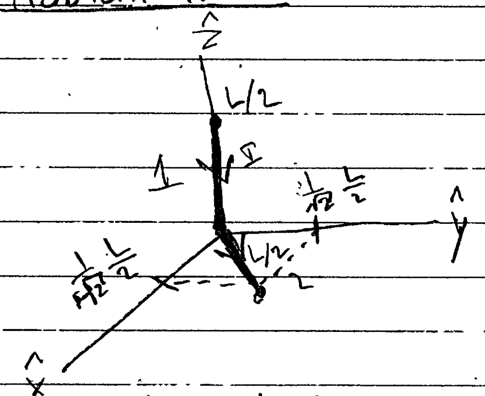
If this is to balance the gravitational force $\vec{F}_g = -mg\,\hat{z}$, we need

$$I a B_0 = mg \Rightarrow I = \frac{mg}{aB_0}$$

(b) When we are increasing the current, we must supply power to the circuit in some form or another (i.e. hooking up a battery). This is doing the work in lifting the weight, which could be crudely thought of as us pushing the electrons.

[Note in part (a) that our $\vec{F}_m$ calculation only included the current going through the top, horizontal wire as the forces on the vertical ones cancel each other and the bottom wire is not in the magnetic field.]

Problem 1.4:



We will calculate the force on each of the two segments separately.

Segment 1:

Along the z-axis, $x = y = 0$, so $\vec{B} = B_0 z^2 \hat{z}$. Then,

$$\begin{aligned}
\vec{F} &= \int d\vec{F} \\
&= \int I\, d\vec{l} \times \vec{B} \\
&= I \int (-dz\, \hat{z}) \times (B_0 z^2 \hat{z}) \\
&= I \int 0 \\
&= 0
\end{aligned}$$

Segment 2:

In the xy plane, $z = 0$, so $\vec{B} = B_0 \left(\frac{x}{L}\hat{x} + \sin\left(\frac{4\sqrt{2}}{L}\pi y\right)\hat{y}\right)$.

In this case,

$$\begin{aligned}
\vec{F} &= I \int d\vec{l} \times \vec{B} \\
&= I \int_0^{L/2} \left(\frac{\sqrt{2}}{2} dl\, \hat{x} + \frac{\sqrt{2}}{2} dl\, \hat{y}\right) \times \left(B_0 \left(\frac{x}{L}\hat{x} + \sin\left(\frac{4\sqrt{2}}{L}\pi y\right)\hat{y}\right)\right)
\end{aligned}$$

Along the integration path, $x = y = \frac{\sqrt{2}}{2} l$, so

$$\begin{aligned}
\vec{F} &= \frac{I B_0}{\sqrt{2}} \int_0^{L/2} dl\, (\hat{x} + \hat{y}) \times \left(\frac{\sqrt{2}}{2}\frac{l}{L}\hat{x} + \sin\left(\frac{4\pi}{L} l\right)\hat{y}\right) \\
&= \frac{I B_0}{2} \int_0^{L/2} \left(-\frac{l}{L}\hat{z} + \sin\left(\frac{4\pi}{L} l\right)\hat{z}\right) dl \\
&= \frac{I B_0}{2} \hat{z} \int_0^{L/2} \left(\sin \frac{4\pi}{L} l - \frac{l}{L}\right) dl \\
&= \frac{I B_0}{2} \hat{z} \left(\left[\frac{-L}{4\pi} \cos \frac{4\pi}{L} l\right]_0^{L/2} - \left[\frac{l^2}{2L}\right]_0^{L/2}\right) \\
&= \frac{I B_0}{2} \hat{z} \left[-\frac{L^2}{8L}\right] \\
&= -\frac{I B_0 L}{8} \hat{z}
\end{aligned}$$

Hence, the net force on the wire is $\vec{F}_{net} = -\frac{I B_0 L}{8}\hat{z}$.

[Note we could have accomplished the sine integral for segment 2 by antisymmetry of sine over one period.]