Week 9, Session 2 Solutions

Problem 1.1:

The total induced charge on the surface of cavity 1 will be

$$Q_{1,\text{induced}} = -Q_1$$

in order to cancel out the electric field from Q_1 inside the conductor (it will not be evenly distributed). Similarly

$$Q_{2,\text{induced}} = -Q_2$$

Since the conductor is uncharged, it must have no net charge, so the outer surface has total induced charge

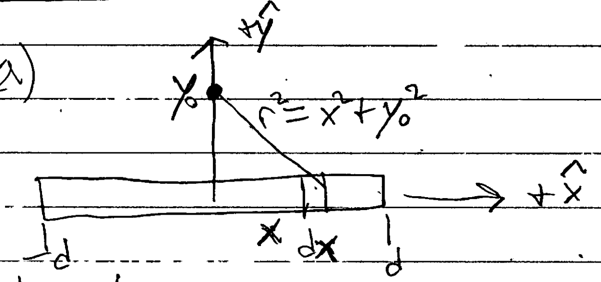
$$Q_{\text{outer, induced}} = Q_1 + Q_2$$

This will be evenly distributed on the surface. So outside for $r > R$, the field looks like that of a point charge $Q_1 + Q_2$ at the origin to get

$$\vec{E}_{out} = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 r^2}\hat{r}$$

Problem 1.2:

(a)



$$dq = \lambda\, dx = a x\, dx$$

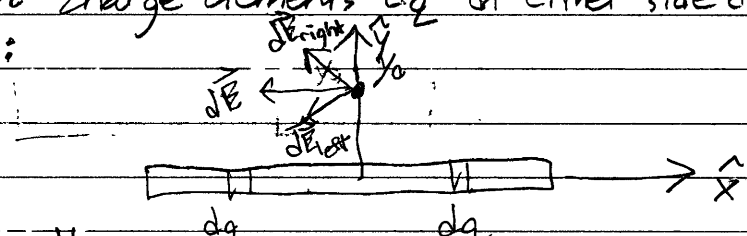
We have that

$$V = \int \frac{dq}{4\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\int_{-d}^{d} \frac{a x\, dx}{\sqrt{x^2+y_0^2}}$$

Let $u = x^2 + y_0^2$, $du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$ to get

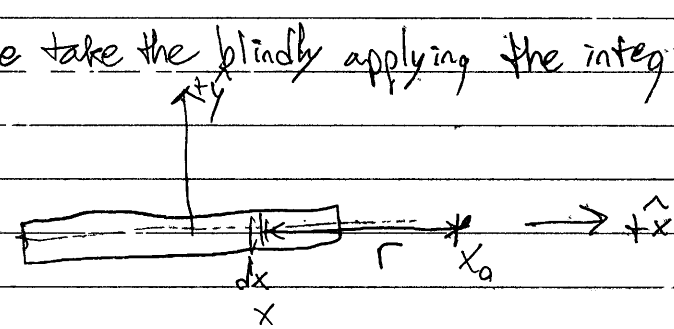
$$V = \frac{1}{4\pi\varepsilon_0}\int_{x=-d}^{x=d} \frac{a}{\sqrt{u}}\,\frac{du}{2} = \frac{a}{8\pi\varepsilon_0}\, 2\sqrt{u}\,\Big|_{x=-d}^{x=d} = \frac{a}{4\pi\varepsilon_0}\left(\sqrt{d^2+y_0^2} - \sqrt{(-d)^2+y_0^2}\right) = 0\ \text{V}$$

Now, I really only did that for giggles and to show it can be done blindly applying the integral. All we had to do was look at the electric field direction from two charge elements dq on either side of the y-axis:



The "left" dq is negative if $a > 0$ while the "right" is positive, so the electric field will be in the $-\hat{x}$ direction. So integrating in from infinity along the y-axis, $\vec{E}\cdot d\vec{\ell} = 0$ so $V = 0$ V.

(b) We take the blindly applying the integral approach:



We have that

$$V = \int \frac{dq}{4\pi\varepsilon_0 r}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-d}^{d} \frac{a \cancel{x}\, dx}{(x_0 - x)}$$

Use the substitution $u = x_0 - x$, $x = x_0 - u$; $du = -dx$ to get

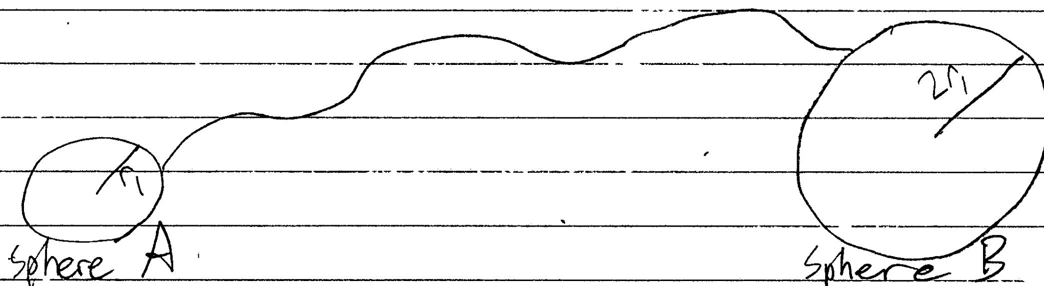
$$V = \frac{a}{4\pi\varepsilon_0}\int_{x_0+d}^{x_0-d} \frac{(x_0-u)(-du)}{u}$$

$$= \frac{a}{4\pi\varepsilon_0}\int_{x_0-d}^{x_0+d} \left(\frac{x_0}{u} - 1\right) du$$

$$= \frac{a}{4\pi\varepsilon_0}\left[x_0 \ln|u| - u\right]_{x_0-d}^{x_0+d}$$

$$= \frac{a}{4\pi\varepsilon_0}\left[x_0 \ln\left|\frac{x_0+d}{x_0-d}\right| - 2d\right]$$

Problem 1.3:



The wire connection means both conductors are equipotentials with each other. Ignoring electric fields from the wire and treating the spheres as isolated, individually, since the wire is long, whatever charge builds up on each surface will be uniformly distributed. Outside the spheres, these look like point charges at the center of the respective spheres.
So we know

$$V_A = V_B \qquad \Rightarrow \frac{Q_A}{4\pi\varepsilon_0 (r_1)} = \frac{Q_B}{4\pi\varepsilon_0 (2r_1)}$$
$$\Rightarrow Q_B = 2Q_A$$

Now, if we compare the electric fields near the surfaces,

$$|\vec{E}_A|_{surface} = \frac{Q_A}{4\pi\varepsilon_0 (r_1)^2}$$
$$|\vec{E}_B|_{surface} = \frac{Q_B}{4\pi\varepsilon_0 (2r_1)^2}$$
$$= \frac{2Q_A}{4(4\pi\varepsilon_0 r_1^2)}$$
$$= \frac{1}{2}\frac{Q_A}{4\pi\varepsilon_0 r_1^2}$$
$$= \frac{1}{2}|E_A|_{surface}$$

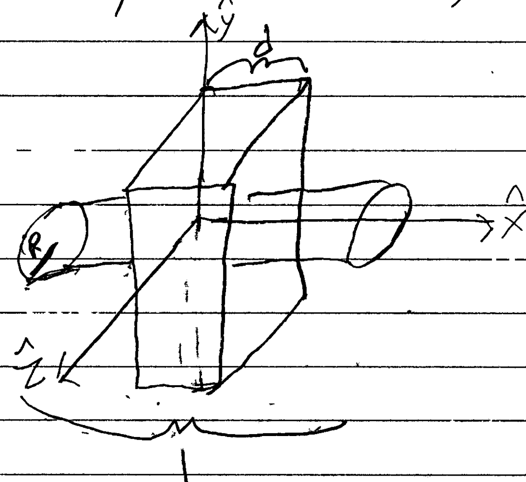
Thus, the faint glow will appear first around the smaller sphere with radius r_1 since the electric field is stronger near its surface than near the surface of the larger sphere and ionisation only occurs in strong electric fields.

Problem 1.4:

we have translational symmetry along the y and z axes along with rotational symmetry by π rotations about the x axis since cosine is an even function. From these, we conclude

$$\vec{E} = E(x)\hat{x}$$

so we have reduced the complexity significantly. Now, using a Gaussian cylinder of radius R and height L with its axis along the $\hat{x}$ direction, we have the picture:



We can find the total flux as

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{a}$$
$$= \int_{\text{left cap}} \vec{E}\cdot d\vec{a} + \int_{\text{sides}} \vec{E}\cdot d\vec{a} + \int_{\text{right cap}} \vec{E}\cdot d\vec{a}$$

Now, if $\vec{E} = \vec{0}$ for $x \leq 0$ and $x \geq d$, $\vec{E} = \vec{0}$ in the "left cap" and "right cap" integrals. Along the sides, $d\vec{a}$ is in the yz-plane and we found earlier that $\vec{E} = E(x)\hat{x}$, so $\vec{E}\cdot d\vec{a} = 0$ since $\vec{E}$ and $d\vec{a}$ are perpendicular. Then,

$$\Phi_E = 0 \ \frac{N}{C}m^2$$

By Gauss' Law,

$$\Phi_E = \frac{Q_{enc}}{\epsilon_0}$$

so we must have $Q_{enc} = 0$ C. In our cylinder,

$$Q_{enc} = \int_{\substack{\text{charge}\\ \text{enclosed}}} \rho\, dV = \pi R^2 \int_0^d \rho_0 \cos\left(\frac{n\pi x}{d}\right)dx,$$

Use the substitution $u = \frac{n\pi x}{d}$, $dx = \frac{d}{n\pi}\,du$ to get

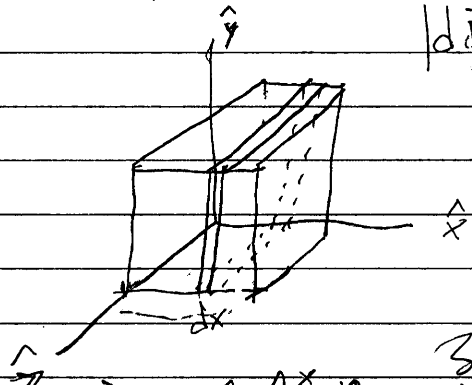
$$Q_{enc} = \frac{\pi R^2 \rho_0 d}{n\pi}\int_0^{n\pi} \cos u \, du$$
$$= \frac{R^2 \rho_0 d}{n} \sin u \Big|_0^{n\pi}$$
$$= \frac{R^2 \rho_0 d}{n} \sin n\pi$$

For this to be null, we must have $n = 1, 2, \ldots$ or $n \in \mathbb{N}$. (Note we do not include 0 since there we would have $\rho(x) = \rho_0$ and there would be no way to have $\vec{E} = \vec{0}$).

Hence, $n = 1, 2, \ldots$ or $n \in \mathbb{N}$ for short.

(b) We start at $x = 0$ where we will arbitrarily define $V(0) = 0$. Then, we can find the electric field at an arbitrary x with $0 \le x \le d$ by breaking up our slab into infinite sheets:

$$|d\vec{E}| = \frac{|\sigma|}{2\varepsilon_0} = \frac{|\rho|\,dx}{2\varepsilon_0}$$



Now, the sheets to the left contribute a $d\vec{E}$ in $+\hat{x}$ direction while those to the right contribut a $d\vec{E}$ in the $-\hat{x}$ direction.

So overall

$$\vec{E}(x) = \hat{x}\int_0^x \frac{\rho_0}{2\varepsilon_0}\cos\left(\frac{n\pi x'}{d}\right)dx' - \hat{x}\int_x^d \frac{\rho_0}{2\varepsilon_0}\cos\left(\frac{n\pi x'}{d}\right)dx'$$
$$= \hat{x}\frac{\rho_0}{2\varepsilon_0}\frac{d}{n\pi}\left[\sin\frac{n\pi x'}{d}\Big|_0^x - \sin\frac{n\pi x'}{d}\Big|_x^d\right]$$
$$= \hat{x}\frac{\rho_0}{2\varepsilon_0}\frac{d}{n\pi}\left(\sin\frac{n\pi x}{d} - 0 - \left(0 - \sin\frac{n\pi x}{d}\right)\right)$$
$$= \hat{x}\frac{\rho_0 d}{n\pi\varepsilon_0}\sin\frac{n\pi x}{d}$$

Then, to find potential we integrate along the slab:

$$V(x) = \int_0^x E(x')\,dx'$$
$$= \frac{\rho_0 d}{n\pi\varepsilon_0}\int_0^x \sin\frac{n\pi x'}{d}\,dx'$$
$$= \frac{\rho_0 d}{n\pi\varepsilon_0}\frac{d}{n\pi}\left[-\cos\frac{n\pi x'}{d}\right]_0^x$$
$$= \frac{\rho_0 d^2}{n^2\pi^2\varepsilon_0}\left(1 - \cos\frac{n\pi x}{d}\right)$$