Probability Distributions and Gases

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August 29, 2017

1 Probability Distributions

1.1 Statistics on distributions

(a) We simply integrate the distribution over all possible values of v and enforce that it must equal one (to be a valid probability distribution):

$$\int_{-\sqrt{a/b}}^{\sqrt{a/b}} f(v)dv = 1 \implies \left[Cav - \frac{Cb}{3}v^3 \right]_{-\sqrt{a/b}}^{\sqrt{a/b}} = 1 \tag{1}$$

Plugging in the numbers, we have

$$Ca(2\sqrt{a/b}) - \frac{Cb}{3}\frac{a}{b}(2\sqrt{a/b}) = 1$$

$$C = \frac{3}{4a}\sqrt{\frac{b}{a}}$$

(b) To compute average velocity, we integrate the distribution weighted by the velocity over the full range, using the value for C that we cound in part (a):

$$\bar{v} = \int_{-\sqrt{a/b}}^{\sqrt{a/b}} v f(v)dv$$

$$= \int_{-\sqrt{a/b}}^{\sqrt{a/b}} (Cav - Cbv^3)dv$$

$$= \left[\frac{Ca}{2}v^2 - \frac{Cb}{4}v^4 \right]_{-\sqrt{a/b}}^{\sqrt{a/b}}$$

$$= \frac{Ca}{2}\left(\frac{a}{b} - \frac{a}{b} \right) - \frac{Cb}{4}\left(\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^2 \right)$$

$$= 0$$

Equivalently, we could have immediately concluded $\bar{v} = 0$ m/s by simply saying integrating an odd function over a domain symmetric about the origin always yields zero.

(c) We proceed exactly as in part (b), except with v^2 as our weight on the probability distribution:

$$\bar{v^2} = \int_{-\sqrt{a/b}}^{\sqrt{a/b}} v^2 f(v)dv$$

$$= \int_{-\sqrt{a/b}}^{\sqrt{a/b}} (Cav^2 - Cbv^4)dv$$

$$= \left[\frac{Ca}{3}v^3 - \frac{Cb}{5}v^5\right]\Bigg|_{-\sqrt{a/b}}^{\sqrt{a/b}}$$

$$= C\left[\frac{a}{3}\sqrt{\frac{a}{b}}\frac{2a}{b} - \frac{b}{5}\sqrt{\frac{a}{b}}\frac{2a^2}{b^2}\right]$$

$$= \left(\frac{3}{4a}\sqrt{\frac{b}{a}}\right)\left(\frac{2a^2}{b}\sqrt{\frac{a}{b}}\right)\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{3a}{2b}\frac{2}{15}$$

$$= \frac{a}{5b}$$

(d) Quite simply, the number distribution is the probability distribution of the velocities multiplied by the number of particles:

$$N(v) = Nf(v) \tag{2}$$

1.2 Concrete numbers

(a) Proceeding formulaicly:

$$\bar{v} = \frac{2*10 + 4*12 + 2*14 + 1*15 + 1*17}{10}$$

$$= 12.8m/s$$

$$\sqrt{\bar{v^2}} = \sqrt{\frac{2*10^2 + 4*12^2 + 2*14^2 + 1*15^2 + 1*17^2}{10}}$$

$$= 12.96m/s$$

$$\approx 13.0m/s$$

(b) The average kinetic energy is related to both temperature and the average squared speed velocity via

$$\bar{K} = \frac{1}{2}m\bar{v^2} = \frac{3}{2}k_B T$$

$$T = \frac{1}{3}\frac{m}{k_B}\bar{v^2}$$

$$\approx \frac{13^2}{3}\frac{m}{k_B} \times \frac{m^2}{s^2}$$

Note that I explicitly keep the units of $m^2 s^{-2}$ since otherwise they would be lost with the 13^2 being put into $\bar{v^2}$.

1.3 Setting up expressions

(a) Let $f(v)$ be the usual Maxwell distribution. Then,

$$0.1 = \left(\int_\alpha^\infty f(v) dv \right) / N \qquad (3)$$

(b) This is given by

$$p_E = \frac{\int_\alpha^\infty v^2 f(v) dv}{\int_0^\infty v^2 f(v) dv} \qquad (4)$$

(c) $N \to 0.9N$ and $E \to 0.72E$, so with $T_0 = \frac{2}{3k_B} \frac{E}{N}$ and $T_1 = \frac{2}{3k_B} \frac{0.72E}{0.9N}$, we have

$$T_1 = \frac{0.72}{0.9} T_0$$
$$= \frac{8}{10} T_0$$

(d) We want $\left(\frac{8}{10} \right)^n T_0 < \frac{1}{2} T_0$, where n is the number of evaporative cooling cycles:

$$\left(\frac{8}{10} \right)^n < \frac{1}{2}$$
$$n \log \left(\frac{8}{10} \right) < \log \left(\frac{1}{2} \right)$$
$$-n \log \left(\frac{10}{8} \right) < -\log(2)$$
$$n > \log(2)/\log \left(\frac{10}{8} \right)$$
$$n > 3.1$$

Hence, we must go through at least four cycles.

1.4 Puddles!

Puddles evaporate because the molecules move with a distribution of velocities and the ones with the top velocities could have enough to escape the surface tension of the puddle and go into the atmosphere. In a sealed jar, the water cannot evaporate because those fast molecules that espace will "bounce" off the walls and go right back in.

1.5 Building intuition

As N increases (keeping T constant), the distribution just gets vertically stretched since it is a number distribution, so this really isn't terribly insightful. As T increases (keeping N constant), the distribution gets a smaller peak but the tail increases, so you get on average faster molecules as the tail with the high speed ones becomes more probable/populated.