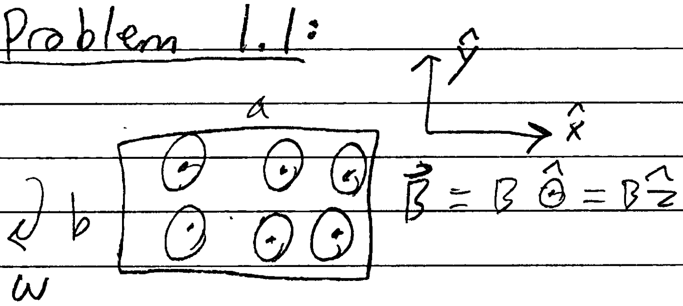
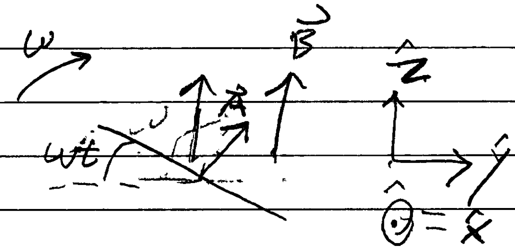
Week 14, Session 1 Solutions

Problem 1.1:





The EMF induced will have a magnitude

$$\mathcal{E} = \left| -\frac{d\Phi_B}{dt} \right|$$
$$= \left| \frac{d\Phi_B}{dt} \right|$$

The area of the loop is ab and there are N turns.
The magnetic flux if the area of the loop is parallel to the field at time $t=0$ is

$$\Phi_B = \vec{B}\cdot\vec{A}$$
$$= NabB\cos\omega t$$

So

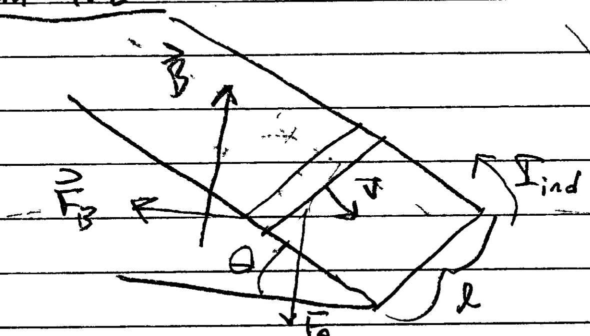
$$\frac{d\Phi_B}{dt} = -\omega NabB\sin\omega t$$

and

$$\mathcal{E} = \omega NabB\sin\omega t$$
$$= \mathcal{E}_0 \sin\omega t$$

if we let $\mathcal{E}_0 = \omega NabB$

Problem 1.2:



(a) The area between the sliding rod and the bottom of the conducting rails is shrinking at a rate

$$\frac{dA}{dt} = -v\ell$$

So the induced emf in the loop is given by

$$\begin{aligned}\mathcal{E} &= -\frac{d\Phi_B}{dt} \\ &= -B\frac{dA}{dt}\cos\theta \\ &= Bv\ell\cos\theta\end{aligned}$$

This will produce a current

$$\begin{aligned}I &= \frac{\mathcal{E}}{R} \\ &= \frac{1}{R}Bv\ell\cos\theta\end{aligned}$$

Hence there will be a magnetic force on the rod as shown. So a terminal velocity is reached when

$$|F_{B,\text{along rails}}| = |F_{g,\text{along rails}}|$$

$$F_B\cos\theta = F_g\sin\theta \Rightarrow I\ell B = mg\tan\theta$$

$$\Rightarrow \frac{1}{R}B^2\ell^2 v\cos\theta = mg\tan\theta$$

$$\Rightarrow v = \frac{Rmg}{B^2\ell^2}\tan\theta\sec\theta$$

This is the terminal velocity.

(b) The internal energy of the bar increases according to

$$\begin{aligned}P &= I^2R \\ &= \frac{1}{R}B^2v^2\ell^2\cos^2\theta\end{aligned}$$

In steady state, $v = v_T$ from (a) and

$$P = \frac{R\,m^2g^2}{B^2\ell^2}\tan^2\theta$$

Now, the rate at which the rod is losing gravitational energy is given by

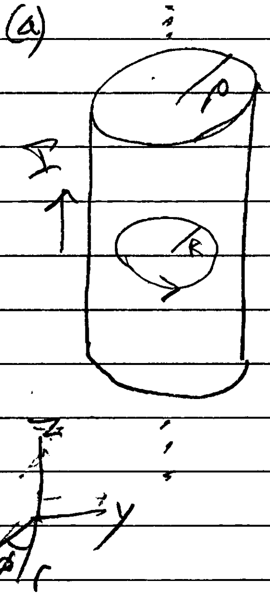
$$\frac{dU}{dt} = mg\frac{dh}{dt}$$
$$= mg\, v_T \sin\theta$$
$$= \frac{R m^2 g^2}{B^2 \ell^2} \tan^2\theta$$

Hence, we see the rate at which the internal energy of the rod is increasing is equal to the rate at which the rod is losing gravitational potential energy.

(c) If $\vec{B}$ were direction down instead of up, the induced current would switch direction, but everything else would stay the same.

Problem 2.1:

(a)



Inside the wire, we have by Amperès Law

$$\oint_R \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{j}\cdot d\vec{a}$$

$$\Rightarrow B(2\pi R) = \mu_0 m \int_0^R r^2\, 2\pi r\, dr$$

$$\Rightarrow B = \frac{\mu_0 m}{R}\int_0^R r^3 dr$$

$$= \frac{\mu_0 m}{4} R^3$$

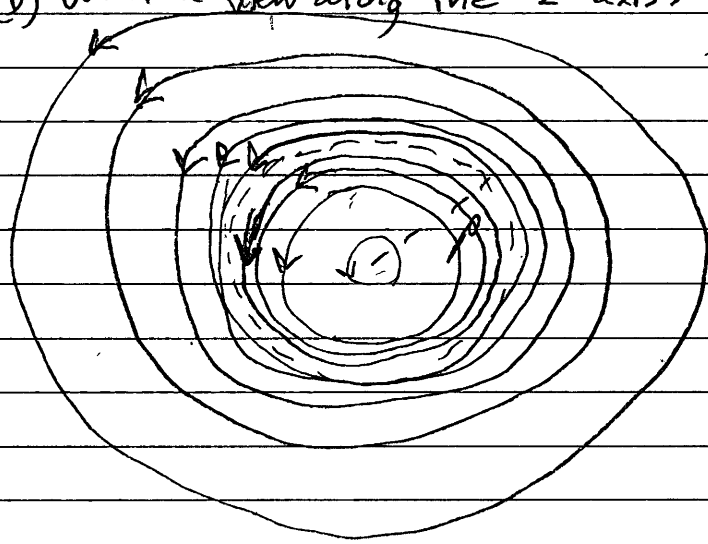
Outside the wire, the $\int \vec{j}\cdot d\vec{a}$ is from 0 to ρ and we instead get

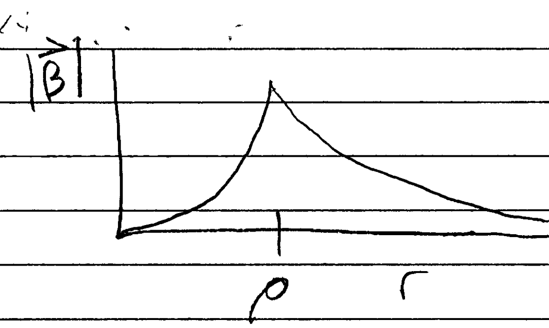
$$B = \frac{\mu_0 m}{4R}\rho^4$$

Hence,

$$\vec{B}(r) = \begin{cases} \frac{\mu_0 m}{4} r^3\, \hat{\phi}, & r<\rho \\ \frac{\mu_0 m}{4}\rho^4 \frac{1}{r}\, \hat{\phi}, & r>\rho \end{cases}$$

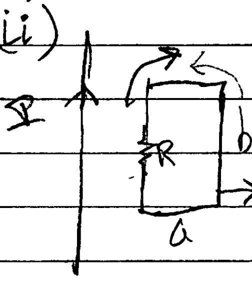
(b) With a view along the z-axis:





(c) (i) $I_{induced} = 0$ in this case as neither $\vec{B}$ nor $\vec{A}$ changes

(ii)



The area of the loop stays constant at ab while the magnetic flux changes:

$$\Phi_B = \int \vec{B}\cdot d\vec{a}$$

$$= \int_d^{d+a} \frac{\mu_0 m}{4}\rho^4 \frac{1}{r}\, b\,dr$$

$$= \frac{\mu_0 m}{4}\rho^4\, b \ln\left(\frac{d+a}{d}\right)$$

Hence,

$$I_{induced} = \frac{\varepsilon}{R}$$
$$= \frac{1}{R}\left(-\frac{d\Phi_B}{dt}\right)$$
$$= \frac{\mu_0 m b}{4R} \rho^4 \frac{-d}{d+a}\left(\frac{-a}{d^2} V\right)$$
$$= \frac{\mu_0 m a b V}{4 R d (d+a)}$$

This current will be clockwise as indicated since the magnetic flux through the loop is decreasing as the loop moves away (decreasing into the page).

Problem 2.2:

We will look at each loop separately.

Left loop:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$
$$= -wvB$$

This will induce a clockwise current since the magnetic flux is increasing out of the page. The total resistance is given by

$$R = \frac{\rho}{A}(2x+w)$$

So the induced current is

$$I = \frac{\mathcal{E}}{R}$$
$$= \frac{AwvB}{\rho(2x+w)}$$

again in the clockwise direction. This will produce a magnetic force to the left with magnitude

$$F_l = IwB$$
$$= \frac{Avw^2B^2}{\rho(2x+w)}$$

Right loop:

The flux is now decreasing out of the page, so the induced current is counterclockwise with again

$$\mathcal{E} = wvB$$
$$R = \frac{\rho}{A}(2(\ell-x)+w)$$
$$I = \frac{\mathcal{E}}{R}$$
$$= \frac{AwvB}{\rho(2\ell-2x+w)}$$

again in the counterclockwise direction. This will produce a magnetic force to the left with magnitude

$$F = IwB$$
$$= \frac{Avw^2B^2}{\rho(2\ell-2x+w)}$$

So the total force that must be placed on the rod when it is at the position shown so that the velocity of the rod is constant is

$$F_{total} = \frac{Avw^2B^2}{\rho}\left[\frac{1}{2x+w} + \frac{1}{2\ell-2x+w}\right]$$

to the right.