Week 8, Session 1 Solutions

1.1

The volume charge density for a uniform charge distribution is

$$\rho = Q/V$$
$$= Q/(\tfrac{4}{3}\pi R^3)$$
$$= 3Q/4\pi R^3$$

By symmetry, the electric field will point radially outwards. For $r < R$,

$$Q_{enc} = \int dQ$$
$$= \int \rho \, dV$$
$$= \rho \cdot \tfrac{4}{3}\pi r^3$$
$$= Q \cdot (r/R)^3$$

Then, by Gauss's Law with our Gaussian surface as a sphere of radius r,

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}(r/R)^3 \Rightarrow E = \frac{Qr}{4\pi\varepsilon_0 R^3}$$

For $r \geq R$, $Q_{enc} = Q$ so

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Thus,

$$\vec{E}(\vec{r}) = \begin{cases} kQr/R^3 \ \hat{r}, & r < R \\ kQ/r^2 \ \hat{r}, & r \geq R \end{cases}$$

1.2

We have that

$$Q = \int \rho \, dV$$
$$= \int_0^{R/2} \frac{AR}{r^2} 4\pi r^2 dr + \int_{R/2}^{R} \frac{A}{r} 4\pi r^2 dr$$
$$= 4\pi A R \, r\Big|_0^{R/2} + 2\pi A \, r^2\Big|_{R/2}^{R}$$
$$= \cancel{4}\pi A R^2 + 2\pi A \left(\tfrac{3}{4} R^2\right)$$
$$= \frac{7}{2} \pi A R^2$$

Hence,

$$A = \frac{2Q}{7\pi R^2}$$

For the electric field, we use a Gaussian surface of a sphere of radius r. For $r < R/2$,

$$Q_{enc} = 4\pi A R r$$
$$= \frac{8Q}{7} \frac{r}{R}$$

By rotational symmetry, $\hat{E} = \hat{r}$ and now

$$E(4\pi r^2) = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E = \frac{8kQ}{7rR}$$

For $R/2 \le r < R$,

$$Q_{enc} = \frac{8Q}{7} \frac{R/2}{R} + 2\pi A \left(r^2 - R^2/4\right)$$
$$= \frac{4Q}{7} + \frac{4Q}{7R^2}\left(r^2 - R^2/4\right)$$
$$= \frac{4Q}{7}\left(\left(\frac{r}{R}\right)^2 + 3/4\right)$$

Then,

$$E = \frac{4kQ}{7r^2}\left(\frac{r^2}{R^2} + 3/4\right)$$

For $r \ge R$,

$$E = \frac{kQ}{r^2}$$

Hence,

$$\vec{E}(r) = \hat{r} \begin{cases} 8kQ/7rR, & 0 < r < R/2 \\ 4kQ\left(\frac{r^2}{R^2} + \frac{3}{4}\right)/7r^2, & R/2 \le r < R \\ kQ/r^2, & R \le r \end{cases}$$

1.3

Generalizing 1.1 and 1.2, for $r<R$,

$$Q_{enc} = \int_0^r \rho(\vec{r}') 4\pi r'^2 dr'$$

$$E = kQ_{enc}/r^2$$

For $r \geq R$,

$$Q_{enc} = \int_0^R \rho(r') 4\pi r'^2 dr'$$

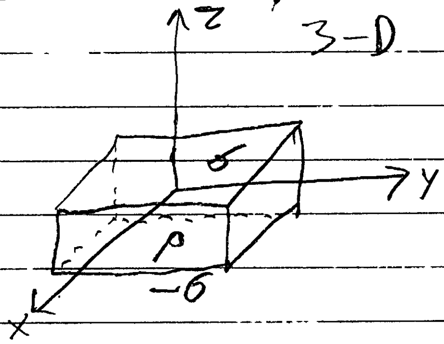
$$E = k\, Q_{enc}/r^2$$

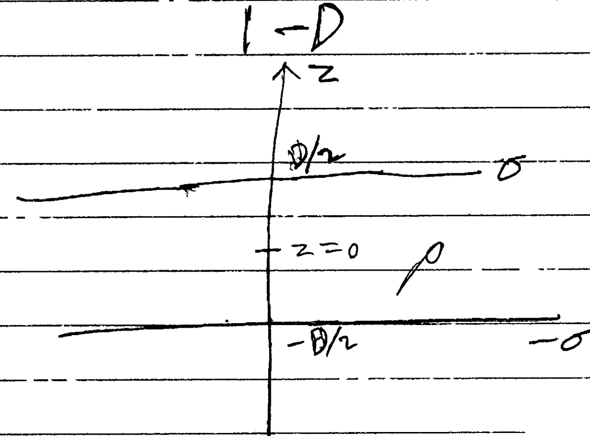
Hence,

$$\vec{E}(r) = \hat{r} \begin{cases} 4\pi k \int_0^r \rho(r') r'^2 dr' / r^2, & 0 \leq r < R \\ 4\pi k \int_0^R \rho(r') r'^2 dr' / r^2, & R \leq r \end{cases}$$

1.4

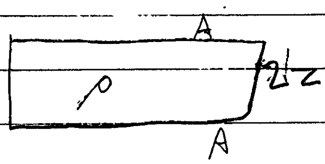
We will put this on the z-axis. By symmetry, our field can only depend on z and will be in the $\hat{z}$ direction.





We will always use Gaussian pillboxes of area A. For $-D/2 < z < D/2$,

$$E(2A) = \frac{\rho(2Az)}{\varepsilon_0}$$



$$\Rightarrow \vec{E} = \frac{\rho z}{\varepsilon_0}\hat{z}$$

This is only the contribution from the slab. The sheets contribute an additional $-\frac{\sigma}{\varepsilon_0}\hat{z}$, so

$$\vec{E} = \frac{\rho z - \sigma}{\varepsilon_0}\hat{z}$$

For $|z| > D/2$, the sheets contribute no field, so

$$\vec{E}(2A) = \frac{\rho(AD)}{\varepsilon_0} \Rightarrow E = \frac{\rho D}{2\varepsilon_0}$$

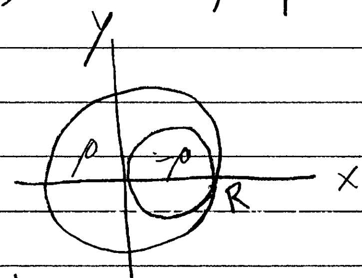
Hence,

$$\vec{E} = \hat{z}\begin{cases} (\rho z - \sigma)/\varepsilon_0, & |z| < D/2 \\ \rho D/2\varepsilon_0, & |z| > D/2 \end{cases}$$

1.5

This is equivalent to having the full sphere of radius R and charge density ρ superposed with a sphere of radius $R/2$ and charge density $-\rho$ where the cavity would have been.

(a) The large sphere contributes no field here.



The small sphere contributes

$$\vec{E} = \frac{kQ_{small}}{(R/2)^2}(-\hat{x})$$

where

$$Q_{small} = (-\rho)\left(\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\right)$$
$$= -\rho\pi R^3/6$$

Hence,

$$\vec{E} = \frac{k\rho\pi R^3}{6R^2}\cdot 4\,\hat{x}$$
$$= 2k\rho\pi R/3\,\hat{x}$$

(b) Each sphere contributes here, so

$$\vec{E} = \frac{kQ_{large}}{R^2}(-\hat{x}) + \frac{kQ_{small}}{(3R/2)^2}(-\hat{x})$$
$$= k(4\pi R^3\rho/3)/R^2(-\hat{x}) + k(-\rho\pi R^3/6)/(9R^2/4)\,(-\hat{x})$$
$$= k\pi\rho R\hat{x}\left(\frac{2}{27} - \frac{4}{3}\right)$$
$$= -34k\pi\rho R/27\,\hat{x}$$

(c) The large sphere contributes, but with only part of its charge, and the small sphere does not. Hence,

$$Q_{enc} = \rho\left(\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\right)$$
$$= \rho\pi R^3/6$$

and

$$\vec{E} = \frac{kQ_{enc}}{(R/2)^2}(\hat{x})$$
$$= 2k\rho\pi R/3\,\hat{x}$$

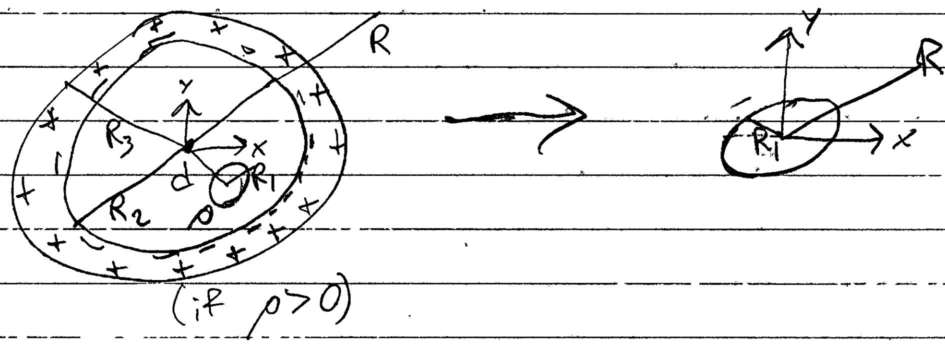
1.6

The charge on the inner surface of the conductor rearranges itself such that the field inside the conductor is zero. The charge on the outer surface will then be uniform since there is no electric field so they want to be as far from each other as possible. Hence, the field will look like that of the cylinder place at the origin for $R > R_3$:

$$E(2\pi R L) = \frac{\rho(\pi R_1^2 L)}{\varepsilon_0}$$

$$\Rightarrow E = \rho R_1^2 / 2\varepsilon_0 R$$

where I have used rotational and translational symmetry to claim $\vec{E} = E(R)\hat{R}$;



Hence,

$$\vec{E}(R) = \rho R_1^2 / 2\varepsilon_0 R \quad \hat{R}$$