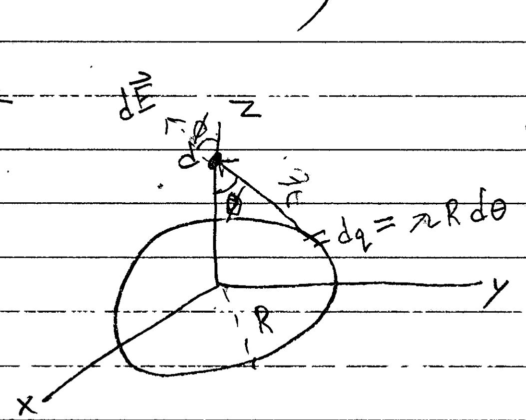
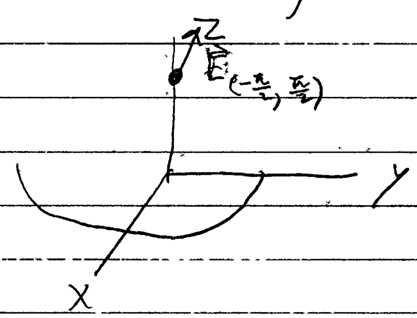
Week 7, Session 2 Solutions

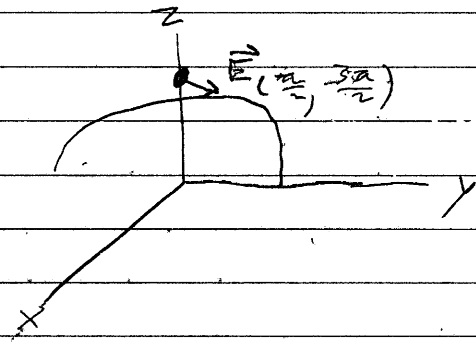
2.1



$r^2 = |\vec{r}|^2 = R^2 + d^2$

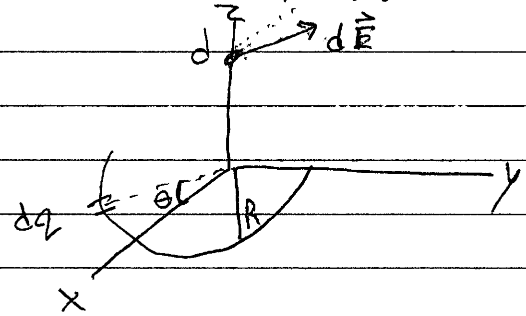
(a) For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\lambda(\theta) > 0$ whereas for $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, $\lambda(\theta) < 0$, so the contributions will look something like this:





Hence, the direction of the electric field will be along $-\hat{x}$, since the $\hat{y}$ components cancel for each semicircle separately, and when combined the $\hat{z}$ components will cancel.

(b) We can calculate the $\hat{x}$ component of one of the semicircles and multiply by 2:



$$\vec{E}_{one} = -\hat{x}\int \frac{dq}{4\pi\varepsilon_0 r^2}\sin\phi\cos\theta$$

$$= \frac{-\hat{x}\, R^2 \lambda_0}{4\pi\varepsilon_0 (R^2+d^2)^{3/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{R^2 \lambda_0 (-\hat{x})}{4\pi\varepsilon_0 (R^2+d^2)^{3/2}} \cdot \left(\frac{1}{2}\pi\right)$$

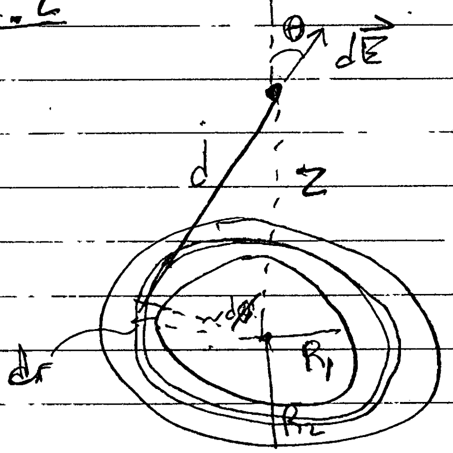
$$= \frac{R^2 \lambda_0}{8\varepsilon_0 (R^2+d^2)^{3/2}} (-\hat{x})$$

$$\vec{E}_{total} = \frac{R^2 \lambda_0}{4\varepsilon_0 (R^2+d^2)^{3/2}} (-\hat{x})$$

(c) This new ring will produce the same field relative to it, except in the $+\hat{x}$ direction. Then, for some point z on the symmetry axis,

$$\vec{E}_{net} = \frac{R^2 \lambda_0}{4\epsilon_0}\left[1/(R^2+(z-2R)^2)^{3/2} - 1/(R^2+z^2)^{3/2}\right]\hat{x}$$

2.2



$\sin\theta = \frac{r}{d}$ $\cos\theta = \frac{z}{d}$

$\sigma(r) = \frac{\beta}{r}$, $\beta > 0$

$dQ = \sigma(r)\, r\, d\phi\, dr$

$d = \sqrt{r^2 + z^2}$

$\hat{d} = \cos\phi \sin\theta\, \hat{x} + \sin\phi \sin\theta\, \hat{y} + \cos\theta\, \hat{z}$

Applying our general formula,

$$\vec{E} = \int d\vec{E}$$
$$= \int_{\text{charge}} \frac{dQ}{4\pi\varepsilon_0 d^2} \hat{d}$$
$$= \int_{R_1}^{R_2} \int_0^{2\pi} \frac{(\beta/r)\, r\, d\phi\, dr}{4\pi\varepsilon_0 (r^2+z^2)} \left[\cos\phi \frac{r}{\sqrt{r^2+z^2}} \hat{x} + \sin\phi \frac{r}{\sqrt{r^2+z^2}} \hat{y} + \frac{z}{\sqrt{r^2+z^2}} \hat{z}\right]$$

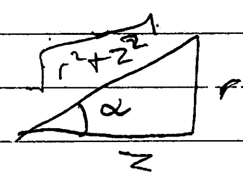
The integral of $\cos\phi$ and $\sin\phi$ over $\theta \in (0, 2\pi)$ are both zero, so we are only left with the $\hat{z}$ component:

$$\vec{E} = \frac{\beta \hat{z}}{2\varepsilon_0} \int_{R_1}^{R_2} \frac{z}{(r^2+z^2)^{3/2}} dr$$

We will substitute $\tan\alpha = \frac{r}{z} \Rightarrow z\sec^2\alpha\, d\alpha = dr$:

$$\vec{E} = \frac{\beta}{2\varepsilon_0 z^2} \hat{z} \int_{r=R_1}^{r=R_2} \frac{1}{(1+\tan^2\alpha)^{3/2}} z\sec^2\alpha\, d\alpha$$

$$= \frac{\beta}{2\varepsilon_0 z} \hat{z} \int_{r=R_1}^{r=R_2} \cos\alpha\, d\alpha$$

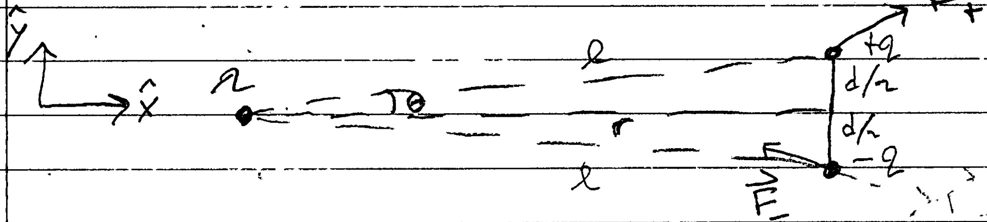


$$= \frac{\beta}{2\varepsilon_0 z} \hat{z} \sin\alpha \Big|_{r=R_1}^{r=R_2}$$

$$= \frac{\beta}{2\varepsilon_0 z} \hat{z} \left(\frac{R_2}{\sqrt{z^2+R_2^2}} - \frac{R_1}{\sqrt{z^2+R_1^2}} \right)$$

2.3

90°:



$$\vec{F}_{net} = 2\,\frac{q\lambda}{2\pi\varepsilon_0 l}\sin\theta\,\hat{y}$$

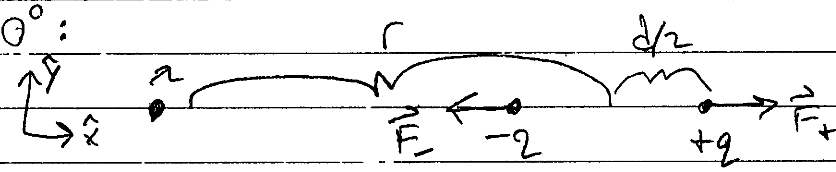
$$= \frac{q\lambda}{\pi\varepsilon_0}\,\frac{d}{2(d^2/4+r^2)}\,\hat{y}$$

$$= \frac{q\lambda}{2\pi\varepsilon_0 r^2}\,d\left(1+\left(\frac{d}{2r}\right)^2\right)^{-1}\hat{y}$$

$$\approx \frac{q\lambda}{2\pi\varepsilon_0 r^2}\,d\left(1-\frac{d^2}{4r^2}\right)\hat{y}$$

$$\approx \frac{q\lambda d}{2\pi\varepsilon_0 r^2}$$

0°:



$$\vec{F}_{net} = \frac{q\lambda}{2\pi\varepsilon_0}\left[\frac{1}{r+d/2}-\frac{1}{r-d/2}\right]\hat{x}$$

$$= \frac{q\lambda}{2\pi\varepsilon_0 r}\left[\left(1+\frac{d}{2r}\right)^{-1}-\left(1-\frac{d}{2r}\right)^{-1}\right]\hat{x}$$

$$\approx \frac{q\lambda}{2\pi\varepsilon_0 r}\left[1-\frac{d}{2r}-1-\frac{d}{2r}\right]\hat{x}$$

$$= \frac{-q\lambda d}{2\pi\varepsilon_0 r^2}\,\hat{x}$$