Week 4, Meeting 2 Solutions

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1 Cycle Through the Cycles

This problem will develop a useful reference: a list of all quantities associated with thermodynamic processes of ideal gases. Suppose that there are N molecules of an ideal gas with d degrees of freedom (use $\gamma = \frac{d+2}{d}$ where it is more convenient). Suppose the gas starts at (P_0, V_0). Then $T_0 = P_0V_0/(Nk)$. Complete the following table and *draw each process on a P-V diagram.*

Table 1: This table is also available in the workbook on pg. 153.

Quantity	**Isochoric**	**Isovolumetric**	**Isothermal**	**Adiabatic**
P_f	P_0	P_f (given)	$P_0\frac{V_0}{V_f}$	$P_0\left(\frac{T_f}{T_0}\right)^{\frac{\gamma}{\gamma-1}}$
V_f	V_f (given)	V_0	V_f (given)	$V_0\left(\frac{T_0}{T_f}\right)^{\frac{1}{\gamma-1}}$
T_f	$T_0\frac{V_f}{V_0}$	$T_0\frac{P_f}{P_0}$	T_0	T_f (given)
ΔE	$\frac{d}{2}Nk_BT_0\left(\frac{V_f}{V_0}-1\right)$	$\frac{d}{2}Nk_BT_0\left(\frac{P_f}{P_0}-1\right)$	0 J	$\frac{d}{2}Nk_B(T_f-T_0)$
Q	$Nk_BT_0\left(\frac{V_f}{V_0}-1\right)\left(\frac{d}{2}-1\right)$	$\frac{d}{2}Nk_BT_0\left(\frac{P_f}{P_0}-1\right)$	$Nk_BT_0ln\left(\frac{V_f}{V_0}\right)$	0
W	$P_0(V_f-V_0)$	0 J	$Nk_BT_0ln\left(\frac{V_f}{V_0}\right)$	$-\frac{d}{2}Nk_B(T_f-T_0)$
ΔS			$Nk_Bln\left(\frac{V_f}{V_0}\right)$	0

2 Problems

2.1 Heat from the Ocean

(a) Maximum efficiency is through Carnot's theroem combined with the efficiency of a Carnot cycle to get

$$e = 1 - \frac{T_L}{T_H} \tag{1}$$

So in this case evaluating yields

$$\begin{aligned} e &= 1 - (273.15 + 4K)/(273.15 + 22K) \\ &\approx 0.061 \end{aligned}$$

Thus, the maximum theoretical efficiency would be about 6.1%.

(b) The efficiency can also be written as

$$e = \frac{W}{Q_H} \tag{2}$$

Producing GW of power means 1 GJ of work is being done every second since $[W] = \frac{[J]}{[s]}$. Inverting the above equation, $Q_H = \frac{W}{e}$, so the heat per second is $\frac{dQ_H}{dt} = \frac{P}{e}$. The heat input is related to the volume by

$$\begin{aligned} Q &= mc\Delta T \\ &= \rho c V \Delta T \\ V &= \frac{Q}{\rho c \Delta T} \\ \frac{dV}{dt} &= \frac{P}{e\rho c \Delta T} \end{aligned}$$

Plugging in numbers, the volume processed in one second is

$$\begin{aligned} \frac{VolumeProcessed}{1s} &= \frac{1 \times 10^9 J/s}{(0.061)(1000kg/m^3)(480J/kg * K)(22 - 4K)} \\ &\approx 218m^3 \\ &= 2.18 \times 10^5 L \end{aligned}$$

2.2 Challenge: Adiabatic Atmosphere

From the Ideal Gas Law, $PV = nRT = NkT$. Notice that $\rho = \frac{Nm}{V}$, so

$$P = \rho \frac{kT}{m} \tag{3}$$

Differentiating the given equation of $P\rho^{-\gamma} = constant$, we have

$$\begin{aligned}
\frac{dP}{dh}\rho^{-\gamma} - \gamma\rho^{-\gamma-1}\frac{d\rho}{dh}P &= 0 \\
\frac{dP}{dh} - \gamma\rho^{-1}\frac{d\rho}{dh}P &= 0 \\
\frac{d\rho}{dh} &= \frac{\rho}{P}\frac{1}{\gamma}\frac{dP}{dh}
\end{aligned}$$

From Eq. 3, $\frac{\rho}{P} = \frac{m}{kT}$, so

$$\frac{d\rho}{dh} = \frac{m}{kT}\frac{1}{\gamma}\frac{dP}{dh} \tag{4}$$

Now, differentiating Eq. 3,

$$\frac{dP}{dh} = \frac{kT}{m}\frac{d\rho}{dh} + \frac{\rho k}{m}\frac{dT}{dh} \tag{5}$$

Plutting into Eq. 4, we have

$$\begin{aligned}
\frac{dP}{dh} &= \frac{kT}{m}\left(\frac{m}{kT}\frac{1}{\gamma}\frac{dP}{dh}\right) + \frac{\rho k}{m}\frac{dT}{dh} \\
\frac{\rho k}{m}\frac{dT}{dh} &= \frac{dP}{dh}\left(1 - \frac{1}{\gamma}\right)
\end{aligned}$$

Now, for any substance, the hydrodynamic condition posits that $\frac{dP}{dh} = -\rho g$ (e.g., think of pressure in a lake). Plugging this into the above equation,

$$\begin{aligned}
\frac{dT}{dh} &= \frac{m}{\rho k}(-\rho g)\left(\frac{\gamma - 1}{\gamma}\right) \\
&= -\frac{mg}{k}\frac{\gamma - 1}{\gamma}
\end{aligned}$$

If we assume the atmosphere is monoatomic, $\gamma = \frac{5}{3}$, so $\frac{\gamma-1}{\gamma} = \frac{2}{5}$ and we have

$$\frac{dT}{dh} = -2mg/5k \tag{6}$$