Math Review Solutions

GSI: Caleb Eades

August 28, 2017

1 Dimensional Analysis

1.1 Checking the answer, part 1

Yes you should be concerned, because lg has units of Newtons (units of force), not Joules (units of energy).

1.2 Checking the answer, part 2

When solving of the force acting in the y-direction of a block falling down a ramp, I got $F_y = mgysin(\alpha) + \mu mgcos(y\alpha)$. What is everything wrong with this equation?

- mgy is units of energy or work (Joules), not of force (Newtons).
- α must have units of radians based on the first term, so then in the second term, you cannot take a cosine of $y\alpha$ because that has units of meters.
- It is just outright the wrong answer for force acting in the y-direction for a block falling down a ramp, but for the purposes here, we were concerned mostly with the units issues.

1.3 Inferring units

Suppose we had an equation for a distance D that relates to the acceleration a and velocity v by

$$D = \frac{5v}{3n^2} + \frac{a}{wv}e^{ha}, \tag{1}$$

where n, w and h are all constants. What can we infer about the units of these constants?

From the first term, we can infer that n has units of seconds to the negative one-half power, or $s^{-1/2}$. From the second term, the argument of the exponential must be unitless, so h has units of inverse acceleration or s^2m^{-1}. From the coefficient of the exponential, w has units of s^{-1}.

2 Taylor Series

2.1 Finding Maclaurin Series

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... \tag{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{3}$$

$$\begin{aligned}
(e^x)^2 &= e^{2x} \\
&= 1 + (2x) + \frac{(2x)^2}{2!} + ... \\
&= 1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + ...
\end{aligned}$$

We cannot find a Maclaurin series for $ln(x)$ because the derivatives are not well defined at $x = 0$.

2.2 Approximating using Taylor Series

$$\begin{aligned}
1/2.0001^2 &= (2.0001)^{-2} \\
&= (2 + 0.0001)^{-2} \\
&= 2^{-2}(1 + 0.00005)^{-2} \\
&\approx \frac{1}{4}(1 - 0.0001) \\
&= \frac{1}{4}(0.9999) \\
&= 0.249975
\end{aligned}$$

2.3 Special relativity

For small v,

$$\gamma = (1 - v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \tag{4}$$

This is a second-order correction for momentum:

$$p \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) mv \approx mv \tag{5}$$

Whereas for energy mc^2 is a constant so we must keep the second-order correction. Ignoring the constant since we are primarily concerned with energy differences (we can define the 'zero' anywhere), we get:

$$\begin{aligned} E &\approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) mc^2 \\ &= mc^2 + \frac{1}{2}mv^2 \\ &\to \frac{1}{2}mv^2 \end{aligned}$$

For a car traveling at 60 mph,

$$\begin{aligned} v &= 60\frac{miles}{1hr} \times \frac{1hr}{3600s} \times \frac{5280ft}{1mile} \times \frac{12in}{1ft} \times \frac{2.54cm}{1in} \times \frac{1m}{100cm} \\ &\approx 26.82\frac{m}{s} \end{aligned}$$

Then,

$$\begin{aligned} (p_{rel} - p_{cla})/p_{cla} &= \gamma - 1 \\ &= \sqrt{\frac{1}{1 - (26.82m/s)^2/(3 \times 10^8 m/s)^2}} - 1 \\ &\approx 4 \times 10^{-15} \end{aligned}$$

or more simply using the binomial approximation,

$$\gamma - 1 \approx 1 - \frac{v^2}{c^2} \approx 4 \times 10^{-15} \tag{6}$$

Bonus thought: note that the binomial approximation is really accurate here. This is for two reasons, one being $v \ll c$ and the other being it is a second-order approximation in $\left(\frac{v}{c}\right)$, making it more precise than most first-order binomial approximations.

2.4 A harder approximation

$$\begin{aligned} GM(R+h)^{-2} &= \frac{GM}{R^2}(1 + h/R)^{-2} \\ &= g(1 + (-2)(h/R) + (-2)(-3)(h/R)^2/(2!) + ...) \\ &\approx g(1 - 2h/R) \end{aligned}$$

On top of Everest,

$$\frac{-2gh/R}{g} = -2(8.5/6400) \approx -0.003 \tag{7}$$

At the edge of the thermosphere,

$$\begin{aligned} \frac{GM}{(R+h)^2} &= \frac{(6.674 \times 10^{-11} m^3 kg^{-1} s^{-2})(5.972 \times 10^{24} kg)}{(7000 \times 10^3 m)^2} \\ &\approx 8.134 ms^{-2} \end{aligned}$$

A cone-percent interval means we are looking for something in the window $(0.99a, 1.01a) = (8.053, 8.215)ms^{-2}$. The Taylor Series approximations in successive orders are:

$$
\begin{aligned}
P_0 &= g \approx 9.731ms^{-2} \\
P_1 &= g(1 - 2h/R) \approx 7.906ms^{-2} \\
P_2 &= g(1 - 2h/R + 3(h/R)^2) \approx 8.162ms^{-2}
\end{aligned}
$$

Hence we only need a second-order Taylor series approximation to get within one percent (and it is significantly better than one percent at that).