Week 1, Meeting 1 Solutions (Taylor Series)

8/24 & 8/25

2.1

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$(e^x)^2 = e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$
$$= 1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \cdots$$

Can't find a Maclaurin series for $\ln(x)$ because the derivatives are not well defined at $x=0$.

2.2

$$1/2.0001^2 = (2.0001)^{-2} = (2+0.0001)^{-2}$$
$$\approx 2^{-2}\cancel{(1-2)(0.0001)}(1+0.00005)^{-2}$$
$$\approx \cancel{0.2498}\ \tfrac{1}{4}(1-0.0001) = \tfrac{1}{4}(0.9999)$$
$$= 0.249975$$

2.3

For small v,
$$\gamma = (1 - v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$$
This is a second-order correction for momentum
$$p \approx \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)mv \approx mv$$
Whereas for energy mc^2 is a constant so we must keep the second-order correction. Ignoring the constant since we are concerned with energy differences

primarily (we can define the "zero" anywhere), we get

$$E \approx \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)mc^2 = mc^2 + \frac{1}{2}mv^2$$
$$\rightarrow \frac{1}{2}mv^2$$

For a car traveling at 60 mph,

$$v = 60\ \frac{\text{miles}}{1\text{hr}} \cdot \frac{1\text{ hr}}{3600\text{ s}} \cdot \frac{5280\text{ ft}}{1\text{ mile}} \cdot \frac{12\text{ in}}{1\text{ ft}} \cdot \frac{2.54\text{ cm}}{1\text{ in}} \cdot \frac{1\text{ m}}{10^2\text{ cm}}$$
$$\approx 26.82\ \frac{m}{s}$$

Then,

$$(p_{rel} - p_{cla})/p_{cla} = \gamma - 1 = \frac{1}{\sqrt{1-(26.82)^2/(3\times10^8)^2}} - 1$$
$$\approx 4\times10^{-15}$$

Or using a binomial approximation,

$$\gamma - 1 \approx \frac{1}{2}\frac{v^2}{c^2} \approx 4\times10^{-15}$$

Bonus thought: note that the binomial approximation is really accurate here. This is for two reasons, one being $v \ll c$ and the other being it is a second-order approximation in $\left(\frac{v}{c}\right)$, making it more precise than most first-order binomial approximations.

2.4

$$GM(R+h)^{-2} = \frac{GM}{R^2}\left(1+\frac{h}{R}\right)^{-2}$$
$$= g\left(1+(-2)\left(\frac{h}{R}\right)+\frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2+\dots\right)$$
$$\approx g\left(1-2\frac{h}{R}\right)$$

On top of Everest,

$$\frac{-2g\frac{h}{R}}{g} = -2\left(\frac{8.5}{6400}\right) \approx -0.003$$

At the edge of the thermosphere,

$$\frac{GM}{(R+h)^2} = \frac{(6.674 \times 10^{-11}\ m^3/kgs^2)(5.972 \times 10^{24}\ kg)}{(7000 \times 10^3\ m)^2}$$

$$\approx 8.134\ \frac{m}{s^2}$$

A one-percent interval means we are looking for something in the window $(0.99a, 1.01a)$
$= (8.053, 8.215)\ m/s^2$

The Taylor Series approximations:

$$P_0 = g = 9.731\ \frac{m}{s^2} \text{ (using R value)}$$

$$P_1 = g\left(1 - 2\frac{h}{R}\right) \approx 7.906\ \frac{m}{s^2}$$

$$P_2 = g\left(1 - 2\frac{h}{R} + 3\left(\frac{h}{R}\right)^2\right) \approx 8.162\ m/s^2$$

Hence, we only need a second-order Taylor series approximation to get within one percent.

2.5

Problems: 1) Can jump discontinuities

2) Ignores higher-order terms $\Rightarrow$ "bad" approximation depending on the function and domain

3) Does poorly with rapidly changing/high frequency functions if the step size is too big

4) Much more...

Might fail: See (1) and (3) in the problems

Improvements: i) Include higher-order terms

ii) Smaller or adaptive step sizes

iii) Vet the functions it will be used on