

Probability Distributions and Gases

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1 Ideal Gas Law

1.1 Pressured balloons

The lake adds a pressure ρgh where h is the height beneath the surface. So letting D be the depth of the lake,

$$\begin{aligned}PV &= nRT \\P' &= nRT'/V' \\P_{\text{balloon}} + P_{\text{depth}} &= nRT'/V' \\nRT/V + \rho gh &= nRT'/V' \\\rho gh &= nR(T'/V' - T/V) \\h &= \frac{nR}{\rho g}(T'/V' - T/V) \\&= \frac{PV}{\rho gT}(T'/V' - T/V) \\&= \frac{P}{\rho g} \left(\frac{VT'}{TV'} - 1 \right)\end{aligned}$$

1.2 Pressure gauges

Have fun!

1.3 Estimations with the ideal gas law

In a standard room, $P \approx 1\text{atm}$, $T \approx 300\text{K}$, $R \approx 8\text{Jmol}^{-1}\text{K}^{-1}$, $V \approx (5\text{m})^2(4\text{m}) = 100\text{m}^3$, and $1\text{atm} \approx 10^5\text{Nm}^{-2}$. So putting all this together,

$$n = \frac{PV}{RT} \approx \frac{(10^5\text{Nm}^{-2})(100\text{m}^3)}{(8\text{Nm}^{-1}\text{mol}^{-1}\text{K}^{-1})(300\text{K})} \approx \frac{100}{25} \times 10^3\text{mol} = 4000\text{moles} \quad (1)$$

Multiplying by Avogadro's number (6.022×10^{23}), we get

$$N \approx 2.5 \times 10^{26} \quad (2)$$

1.4 Partitioned boxes

- (a) The gas expands into the remainder of the box that used to be a vacuum. So $V/2 \rightarrow V$ and by the Ideal Gas Law,

$$P_0 \frac{V}{2} = nRT_0$$
$$P_f V = nRT_f$$

The temperature remains unaltered, however, because nothing has been done to increase or decrease the kinetic energy of the molecules, so $T_f = T_0$ and therefore by the two equations above, $P_f = P_0/2$.

- (b) This is trickier because along the way, something has to hold the wall in place, which does negative work on the gas (you can also think about the gas doing work on the wall to expand and push it outwards), so the temperature no longer remains constant. Hence, T decreases, V increases ($V/2 \rightarrow V$ as before) and P decreases (by more than half now).

2 Probability Distributions

2.1 Statistics on distributions

- (a) We simply integrate the distribution over all possible values of v and enforce that it must equal one (to be a valid probability distribution):

$$\int_{-\sqrt{a/b}}^{\sqrt{a/b}} f(v) dv = 1 \implies \left[Cav - \frac{Cb}{3} v^3 \right]_{-\sqrt{a/b}}^{\sqrt{a/b}} = 1 \quad (3)$$

Plugging in the numbers, we have

$$Ca(2\sqrt{a/b}) - \frac{Cb}{3} \frac{a}{b} (2\sqrt{a/b}) = 1$$

$$C = \frac{3}{4a} \sqrt{\frac{b}{a}}$$

- (b) To compute average velocity, we integrate the distribution weighted by the velocity over the full range, using the value for C that we found in part (a):

$$\begin{aligned} \bar{v} &= \int_{-\sqrt{a/b}}^{\sqrt{a/b}} v f(v) dv \\ &= \int_{-\sqrt{a/b}}^{\sqrt{a/b}} (Cav - Cbv^3) dv \\ &= \left[\frac{Ca}{2} v^2 - \frac{Cb}{4} v^4 \right]_{-\sqrt{a/b}}^{\sqrt{a/b}} \\ &= \frac{Ca}{2} \left(\frac{a}{b} - \frac{a}{b} \right) - \frac{Cb}{4} \left(\left(\frac{a}{b} \right)^2 - \left(\frac{a}{b} \right)^2 \right) \\ &= 0 \end{aligned}$$

Equivalently, we could have immediately concluded $\bar{v} = 0$ m/s by simply saying integrating an odd function over a domain symmetric about the origin always yields zero.

- (c) We proceed exactly as in part (b), except with v^2 as our weight on the

probability distribution:

$$\begin{aligned}
\bar{v}^2 &= \int_{-\sqrt{a/b}}^{\sqrt{a/b}} v^2 f(v) dv \\
&= \int_{-\sqrt{a/b}}^{\sqrt{a/b}} (Cav^2 - Cbv^4) dv \\
&= \left[\frac{Ca}{3} v^3 - \frac{Cb}{5} v^5 \right]_{-\sqrt{a/b}}^{\sqrt{a/b}} \\
&= C \left[\frac{a}{3} \sqrt{\frac{a}{b}} \frac{2a}{b} - \frac{b}{5} \sqrt{\frac{a}{b}} \frac{2a^2}{b^2} \right] \\
&= \left(\frac{3}{4a} \sqrt{\frac{b}{a}} \right) \left(\frac{2a^2}{b} \sqrt{\frac{a}{b}} \right) \left(\frac{1}{3} - \frac{1}{5} \right) \\
&= \frac{3a}{2b} \frac{2}{15} \\
&= \frac{a}{5b}
\end{aligned}$$

- (d) Quite simply, the number distribution is the probability distribution of the velocities multiplied by the number of particles:

$$N(v) = Nf(v) \quad (4)$$

2.2 Concrete numbers

- (a) Proceeding formulaicly:

$$\begin{aligned}
\bar{v} &= \frac{2 * 10 + 4 * 12 + 2 * 14 + 1 * 15 + 1 * 17}{10} \\
&= 12.8m/s \\
\sqrt{\bar{v}^2} &= \sqrt{\frac{2 * 10^2 + 4 * 12^2 + 2 * 14^2 + 1 * 15^2 + 1 * 17^2}{10}} \\
&= 12.96m/s \\
&\approx 13.0m/s
\end{aligned}$$

- (b) The average kinetic energy is related to both temperature and the average squared speed velocity via

$$\begin{aligned}
\bar{K} &= \frac{1}{2} m \bar{v}^2 = \frac{3}{2} k_B T \\
T &= \frac{1}{3} \frac{m}{k_B} \bar{v}^2 \\
&\approx \frac{13^2}{3} \frac{m}{k_B} \times \frac{m^2}{s^2}
\end{aligned}$$

Note that I explicitly keep the units of $m^2 s^{-2}$ since otherwise they would be lost with the 13^2 being put into v^2 .

2.3 Setting up expressions

- (a) Let $f(v)$ be the usual Maxwell distribution. Then,

$$0.1 = \left(\int_{\alpha}^{\infty} f(v) dv \right) / N \quad (5)$$

- (b) This is given by

$$p_E = \frac{\int_{\alpha}^{\infty} v^2 f(v) dv}{\int_0^{\infty} v^2 f(v) dv} \quad (6)$$

- (c) $N \rightarrow 0.9N$ and $E \rightarrow 0.72E$, so with $T_0 = \frac{2}{3k_B} \frac{E}{N}$ and $T_1 = \frac{2}{3k_B} \frac{0.72E}{0.9N}$, we have

$$\begin{aligned} T_1 &= \frac{0.72}{0.9} T_0 \\ &= \frac{8}{10} T_0 \end{aligned}$$

- (d) We want $\left(\frac{8}{10}\right)^n T_0 < \frac{1}{2} T_0$, where n is the number of evaporative cooling cycles:

$$\begin{aligned} \left(\frac{8}{10}\right)^n &< \frac{1}{2} \\ n \log\left(\frac{8}{10}\right) &< \log\left(\frac{1}{2}\right) \\ -n \log\left(\frac{10}{8}\right) &< -\log(2) \\ n &> \log(2) / \log\left(\frac{10}{8}\right) \\ n &> 3.1 \end{aligned}$$

Hence, we must go through at least four cycles.

2.4 Puddles!

Puddles evaporate because the molecules move with a distribution of velocities and the ones with the top velocities could have enough to escape the surface tension of the puddle and go into the atmosphere. In a sealed jar, the water cannot evaporate because those fast molecules that escape will “bounce” off the walls and go right back in.

2.5 Building intuition

As N increases (keeping T constant), the distribution just gets vertically stretched since it is a number distribution, so this really isn't terribly insightful. As T increases (keeping N constant), the distribution gets a smaller peak but the tail increases, so you get on average faster molecules as the tail with the high speed ones becomes more probable/populated.