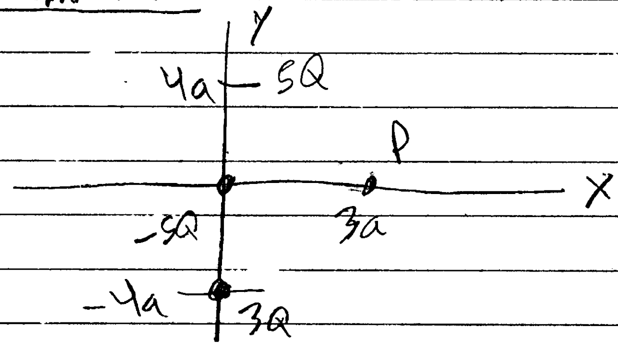
# Week 9, Session 1 Solutions

## Problem 1.1:

(a)




(b) We sum the pairwise energies from $U = q\Delta V$ to get

$$U_{total} = \frac{Q^2}{4\pi\varepsilon_0 a}\left[\frac{-25}{4} + \frac{15}{8} - \frac{15}{4}\right]$$
$$= \frac{-65Q^2}{32\pi\varepsilon_0 a}$$

(c) We have

$$V(P) = V(3a, 0, 0)$$
$$= \frac{Q}{4\pi\varepsilon_0 a}\left[\frac{5}{5} - \frac{5}{3} + \frac{3}{5}\right]$$
$$= \frac{-1Q}{60\pi\varepsilon_0 a}$$

(d) Note that in general,

$$V(x, y, z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{5}{(x^2+(y-4a)^2+z^2)^{1/2}} - \frac{5}{(x^2+y^2+z^2)^{1/2}} + \frac{3}{(x^2+(y+4a)^2+z^2)^{1/2}}\right]$$

Then,

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{5}{(x^2+(y-4a)^2+z^2)^{3/2}} - \frac{5}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{(x^2+(y+4a)^2+z^2)^{3/2}}\right](x\hat{x} + z\hat{z})$$
$$+ \frac{Q}{4\pi\varepsilon_0}\left[\frac{5(y-4a)}{(x^2+(y-4a)^2+z^2)^{3/2}} - \frac{5y}{(x^2+y^2+z^2)^{3/2}} + \frac{3(y+4a)}{(x^2+(y+4a)^2+z^2)^{3/2}}\right]\hat{y}$$

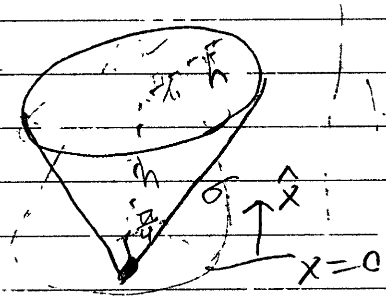
So

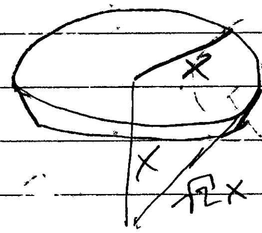
$$\vec{E}(P) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{5}{125a^3} - \frac{5}{27a^3} + \frac{3}{125a^3}\right](3a\hat{x})$$
$$+ \frac{Q}{4\pi\varepsilon_0}\left[\frac{-20a}{125a^3} + \frac{12a}{125a^3}\right]\hat{y}$$
$$= \frac{Q}{500\pi\varepsilon_0 a^2}\left(\left[24 - 125\cdot\frac{15}{27}\right]\hat{x} - 8\hat{y}\right)$$

(e) $\vec{F} = Q\vec{E}(P) = \frac{Q^2}{500\pi\varepsilon_0 a^2}\left(\left[24 - 125\cdot\frac{15}{27}\right]\hat{x} - 8\hat{y}\right)$

(f) $W = U = QV = \frac{-Q^2}{60\pi\varepsilon_0 a}$

Problem 1.2:




$d\ell = \sqrt{2}\,dx$

$dq = \sigma\, 2\pi x \sqrt{2}\,dx$

We have that

$$V = \int \frac{dq}{4\pi\varepsilon_0 r}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_0^h \frac{\sigma\, 2\pi x \sqrt{2}\, dx}{\sqrt{2}\, x}$$

$$= \frac{\sigma}{2\varepsilon_0} \int_0^h dx$$

$$= \frac{\sigma h}{2\varepsilon_0}$$

## Problem 1.3:

We seek the potential function for each:

(a) From the x-component,

$$\frac{\partial V}{\partial x} = -E_x = -kyx \Rightarrow V = -\frac{k}{2}yx^2 + f(y,z)$$

From the y-component,

$$\frac{\partial V}{\partial y} = -E_y = -2yz \Rightarrow V = -y^2 z + g(x,z)$$

Already, we have an irreconciable difference between these two potentials. That is, we cannot find $f(y,z)$ and $g(x,z)$ such that

$$-\frac{k}{2}yx^2 + f(y,z) = -y^2 z + g(x,z)$$

Hence, this is an impossible electric field.

[Note: $f(y,z)$ just means some function of only y and/or z such that $\frac{\partial f(y,z)}{\partial x} = 0$.]

(b) From the x-component,

$$\frac{\partial V}{\partial x} = -E_x = -ky^2 \Rightarrow V = -kxy^2 + f(y,z)$$

From the y-component,

$$\frac{\partial V}{\partial y} = -E_y = -2kxy - kz^2 \Rightarrow V = -kxy^2 - kyz^2 + g(x,z)$$

From the z-component,

$$\frac{\partial V}{\partial z} = -E_z = -2kyz \Rightarrow V = -kyz^2 + h(x,y)$$

If we set

$$f(y,z) = -kyz^2$$
$$g(x,z) = 0$$
$$h(x,y) = -kxy^2$$

Then we get

$$V = -k(xy^2 + yz^2)$$

Now, to check our answer,

$$\vec{E} = -\nabla V$$
$$= -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$
$$= k\left[y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}\right]$$

as expected/desired.

Problem 1.4:

Solving for the potential directly involves a nasty integral, so we will first solve for the electric field using Gauss' Law:

By rotational symmetry, $\vec{E} = E(r)\hat{r}$.

$r < R$:

$$q_{enc} = q\,\frac{r^3}{R^3}$$
$$A = 4\pi r^2$$
$$E = \frac{q_{enc}}{A\varepsilon_0} = \frac{qr}{4\pi\varepsilon_0 R^3}$$

$r \geq R$:

$$q_{enc} = q$$
$$A = 4\pi r^2$$
$$E = \frac{q_{enc}}{A\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 r^2}$$

Now, we solve for the potential using $\Delta V = -\int \vec{E}\cdot d\vec{l}$ and setting $V(\infty) = 0$ V.

$r \geq R$:

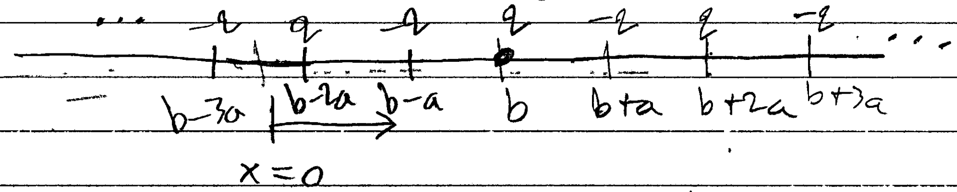
$$V(r) - \cancel{V(\infty)}^{\,0} = -\int_\infty^r \frac{q}{4\pi\varepsilon_0 r'^2}\,dr'$$
$$= \frac{q}{4\pi\varepsilon_0}\int_r^\infty \frac{1}{r'^2}\,dr'$$
$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r'}\right|_r^\infty$$
$$= \frac{q}{4\pi\varepsilon_0 r}$$

$r < R$:

$$V(r) - V(R) = -\int_R^r \frac{qr'}{4\pi\varepsilon_0 R^3}\,dr'$$
$$\Rightarrow V(r) = V(R) + \frac{q}{4\pi\varepsilon_0 R^3}\int_r^R r'\,dr'$$
$$= \frac{q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R^3}\left[r'^2/2\right|_r^R$$
$$= \frac{q}{4\pi\varepsilon_0 R} + \frac{q}{8\pi\varepsilon_0 R} - \frac{qr^2}{8\pi\varepsilon_0 R^3}$$
$$= \frac{q}{8\pi\varepsilon_0 R}\left(3 - r^2/R^2\right)$$

Problem 1.5:

Let us consider the potential of one charge pairwise with every other charge. If this one is at some position ~~x=~~b and has charge +q, the setup is as follows:




We then have that the potential for this charge is

$$V = \frac{q}{4\pi\varepsilon_0}\left[-\frac{q}{a} - \frac{q}{a} + \frac{q}{2a} + \frac{q}{2a} - \frac{q}{3a} - \frac{q}{3a} + \dots\right]$$

$$= \frac{-2q}{4\pi\varepsilon_0 a}\left[1 - \frac{1}{2} + \frac{1}{3} - \dots\right]$$

Recall that the Taylor series expansion for $f(x) = \ln(1+x)$ about $x=0$ is found via the general Taylor expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Now, to find the coefficients,

$$f(0) = \ln(1+0) = \ln(1) = 0$$

$$f'(x) = (1+x)^{-1} \Rightarrow f'(0) = (1+0)^{-1} = 1$$

$$f''(x) = -(1+x)^{-2} \Rightarrow f''(0) = -(1+0)^{-2} = -1$$

$$f^{(3)}(x) = (-1)(-2)(1+x)^{-3} \Rightarrow f^{(3)}(0) = (-1)(-2)(1+0)^{-3} = 2$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n} \Rightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Then, indexing from 1 since for $n=0$, $f(0)=0$,

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(n-1)!}{n!} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

Observe that

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

So since $f(1) = \ln(1+1) = \ln(2)$,

$$V = \frac{-2q}{4\pi\varepsilon_0}\ln(2)$$

This is the potential felt by one charge, so the work required to bring it in is

$$W = qV$$
$$= q\,\frac{q}{4\pi\varepsilon_0 a}\,2\ln(2)$$

Hence, $\alpha = 2\ln(2)$.