

# Week 5, Meeting 1 Solutions

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## 1 Looking Forward to Spring(s)?

### 1.1 Springs and Expansions

Initially,  $(P_1 - P_0)A = 0N$ , so  $P_1 = P_0$  and via the Ideal Gas Law,  $P_1 V_1 = nRT_1$ .

(a) As  $T$  increases,  $P$  and  $V$  will as well, so the spring will start resisting:

$$(P_2 - P_0)A = k(V_2 - V_1)/A$$
$$P_2 V_2 = nRT_2 = 3nRT_1 = 3P_1 V_1$$

Hence  $P_2 = 3P_1 V_1 / V_2$  and

$$V_2 - V_1 = \frac{A^2}{k} \left( \frac{3P_1 V_1}{V_2} - P_1 \right)$$
$$0 = V_2^2 + V_2 \left( \frac{A^2 P_1}{k} - V_1 \right) - \frac{3A^2 P_1 V_1}{k}$$
$$V_2 = \frac{1}{2} \left( V_1 - \frac{A^2 P_1}{k} \pm \sqrt{\left( \frac{A^2 P_1}{k} \right)^2 - 2 \left( \frac{A^2 P_1}{k} \right) (V_1) + V_1^2 + 12 \frac{A^2 P_1 V_1}{k}} \right)$$
$$= \frac{1}{2} \left( V_1 - \frac{A^2 P_1}{k} \pm \sqrt{\left( \frac{A^2 P_1}{k} \right)^2 + 10 \left( \frac{A^2 P_1}{k} \right) (V_1) + V_1^2} \right)$$

Since the radical will be bigger than  $\frac{A^2 P_1}{k}$ , we choose the addition such that

$$V_2 = \frac{1}{2} \left( V_1 - \frac{A^2 P_1}{k} + \sqrt{\left( \frac{A^2 P_1}{k} \right)^2 + 10 \left( \frac{A^2 P_1}{k} \right) (V_1) + V_1^2} \right) \quad (1)$$

(b) Whatever work the gas did is the negative of what the spring did, which

is  $-k\frac{\Delta h^2}{2}$ , so

$$\begin{aligned} W_{gas} &= \frac{k}{2} \left( \frac{V_2}{A} - \frac{V_1}{A} \right)^2 \\ &= \frac{k}{2A^2} (V_2 - V_1)^2 \\ &= \frac{k}{8A^2} \left( \left( \frac{A^2 P_1}{k} \right)^2 - 2 \frac{A^2 P_1}{k} \sqrt{\left( \frac{A^2 P_1}{k} \right)^2 + 10 \frac{A^2 P_1 V_1}{k} + V_1^2} + \left( \frac{A^2 P_1}{k} \right)^2 + 10 \frac{A^2 P_1 V_1}{k} + V_1^2 \right) \end{aligned}$$

(c) From the first law,

$$\begin{aligned} Q &= \Delta E_{int} + W_{gas} \\ &= 3nRT_1 \\ &\quad + \frac{k}{8A^2} \left( \left( \frac{A^2 P_1}{k} \right)^2 - 2 \frac{A^2 P_1}{k} \sqrt{\left( \frac{A^2 P_1}{k} \right)^2 + 10 \frac{A^2 P_1 V_1}{k} + V_1^2} + \left( \frac{A^2 P_1}{k} \right)^2 + 10 \frac{A^2 P_1 V_1}{k} + V_1^2 \right) \end{aligned}$$

(d) Let's define a new variable  $V'$  as

$$V' = \frac{1}{2} \left( V_1 - \frac{A^2 P_1}{k} + \sqrt{\left( \frac{A^2 P_1}{k} \right)^2 + 10 \frac{A^2 P_1 V_1}{k} + V_1^2} \right) \quad (2)$$

Then the answers for each of the previous parts are much simplified. (a)  $V_2 = V'$ . (b)  $W_{gas} = \frac{k}{2A^2} (V' - V_1)^2$ . (c)  $Q = 3nRT_1 + \frac{k}{2A^2} (V' - V_1)^2$ .

## 2 Test-like and other Interesting Problems

### 2.1 Interlude on Entropy

(a) For an estimate, we follow the textbook's approach of calculating  $T_f$ ,  $Q_{iron}$  and  $Q_{water}$ , and finding entropies from the average temperature over the process:

$$\begin{aligned} Q_{iron} &= m_{iron} c_{iron} \Delta T_{iron} \\ &= (10kg)(450Jkg^{-1}K^{-1})(T_f - 1000K) \\ Q_{water} &= (100kg)(4186Jkg^{-1}K^{-1})(T_f - 300K) \end{aligned}$$

Setting  $Q_{water} = -Q_{iron}$  since all the heat lost by one must be gained by the other, we have

$$\begin{aligned} 4500(1000 - T_f) &= 418600(T_f - 300) \\ T_f &= \frac{4500000 + 418600 * 300}{418600 + 4500} \\ &\approx 307.4K \end{aligned}$$

Then,

$$\begin{aligned} Q_{iron} &\approx -3.1 \times 10^6 J \\ Q_{water} &\approx 3.1 \times 10^6 J \end{aligned}$$

Hence,

$$\begin{aligned} \Delta S_{iron} &\approx \frac{Q_{iron}}{T_{iron,avg}} \\ &= \frac{Q_{iron}}{(1000 + T_f)/2} \\ &\approx -4.7 \times 10^3 JK^{-1} \Delta S_{water} \quad \approx \frac{Q_{water}}{T_{water,avg}} \\ &= \frac{Q_{water}}{(300 + T_f)/2} \\ &\approx 1.0 \times 10^4 JK^{-1} \end{aligned}$$

So  $\Delta S_{system} \approx 5.3 \times 10^3 JK^{-1}$ .

(b)  $T_f$  is the same as calculated in part (a), as is  $Q_{iron}$  and  $Q_{water}$ . But now,

$$\begin{aligned} \Delta S_{iron} &= \int_{1000K}^{307.4K} \frac{(4500 JK^{-1})}{T} dT \\ &= (-4500 JK^{-1}) \ln \left( \frac{1000}{307.4} \right) \\ &\approx -5.3 \times 10^3 JK^{-1} \end{aligned}$$

Similarly,

$$\Delta S_{water} = (418600 JK^{-1}) \ln \left( \frac{307.4}{300} \right) \approx 1.0 \times 10^4 JK^{-1}$$

So  $\Delta S_{system} \approx 4.7 \times 10^3 JK^{-1}$ . This is less than part (a) because first-order log approximations are less accurate when there is a significant difference in the numerator and denominator.

## 2.2 Derivations

(a) Let the top isotherm be at temperature  $T_H$  and the lower one at  $T_C$ . We will look at each segment in the cycle.

A to B:  $\Delta E_{int,A \rightarrow B} = 0$ , so

$$\begin{aligned} Q_H &= Q_{A \rightarrow B} = W_{A \rightarrow B} \\ &= \int_{V_A}^{V_B} \frac{nRT_H}{V} dV \\ &= nRT_H \ln \left( \frac{V_B}{V_A} \right) \end{aligned}$$

B to C:  $Q_{B \rightarrow C} = 0$  and  $\Delta E_{int, B \rightarrow C} = \frac{3}{2}nR(T_C - T_H)$ . By the first law,

$$\begin{aligned} W_{B \rightarrow C} &= -\Delta E_{int, B \rightarrow C} \\ &= \frac{3}{2}nR(T_H - T_C) \end{aligned}$$

C to D:  $\Delta E_{int, C \rightarrow D} = 0$ , so  $Q_C = |Q_{C \rightarrow D}|$  and

$$\begin{aligned} Q_{C \rightarrow D} &= W_{C \rightarrow D} \\ &= -nRT_C \ln \left( \frac{V_C}{V_D} \right) \\ Q_C &= nRT_C \ln \left( \frac{V_C}{V_D} \right) \end{aligned}$$

D to A:  $Q_{D \rightarrow A} = 0$  and  $\Delta E_{int, D \rightarrow A} = \frac{3}{2}nR(T_H - T_C)$ . By the first law,

$$\begin{aligned} W_{D \rightarrow A} &= -\Delta E_{int, D \rightarrow A} \\ &= \frac{3}{2}nR(T_C - T_H) \end{aligned}$$

Now, putting all these pieces together and looking at the overall cycle,

$$W_{net} = nRT_H \ln \left( \frac{V_B}{V_A} \right) - nRT_C \ln \left( \frac{V_C}{V_D} \right) \quad (3)$$

The efficiency is then given by

$$\begin{aligned} e &= \frac{W_{net}}{Q_H} \\ &= \frac{nRT_H \ln \left( \frac{V_B}{V_A} \right) - nRT_C \ln \left( \frac{V_C}{V_D} \right)}{nRT_H \ln \left( \frac{V_B}{V_A} \right)} \\ &= 1 - \frac{T_C}{T_H} \ln \left( \frac{V_C}{V_D} \right) \ln \left( \frac{V_B}{V_A} \right) \end{aligned}$$

Along an adiabat,  $PV^\gamma = \text{constant}$ , so  $P_B V_B^\gamma = P_C V_C^\gamma$ . Also, from the Ideal Gas Law,  $P_B V_B = nRT_H$  while  $P_C V_C = nRT_C$ , so

$$\begin{aligned} nRT_H V_B^{\gamma-1} &= nRT_C V_C^{\gamma-1} \\ V_C &= V_B \left( \frac{T_H}{T_C} \right)^{1/(\gamma-1)} \end{aligned}$$

Similarly,

$$V_D = V_A \left( \frac{T_H}{T_C} \right)^{1/(\gamma-1)} \quad (4)$$

Hence,

$$\begin{aligned}
e &= 1 - \frac{T_C}{T_H} \ln \left( \frac{V_B (T_H/T_C)^{1/(\gamma-1)}}{V_A (T_H/T_C)^{1/(\gamma-1)}} \right) / \ln \left( \frac{V_B}{V_A} \right) \\
&= 1 - \frac{T_C}{T_H} \ln \left( \frac{V_B}{V_A} \right) / \ln \left( \frac{V_B}{V_A} \right) \\
&= 1 - \frac{T_C}{T_H}
\end{aligned}$$

(b) We again look at each segment in the cycle.

A to B:

$$\begin{aligned}
Q_{A \rightarrow B} &= 0 \\
\Delta E_{int, A \rightarrow B} &= \frac{3}{2} nR(T_B - T_A) \\
W_{A \rightarrow B} &= -\Delta E_{int, A \rightarrow B} = \frac{3}{2} nR(T_A - T_B)
\end{aligned}$$

B to C:

$$\begin{aligned}
\Delta E_{int, B \rightarrow C} &= \frac{3}{2} nR(T_C - T_B) \\
W_{B \rightarrow C} &= 0 \\
Q_{B \rightarrow C} &= \Delta E_{int, B \rightarrow C} = \frac{3}{2} nR(T_C - T_B)
\end{aligned}$$

C to D:

$$\begin{aligned}
Q_{C \rightarrow D} &= 0 \\
\Delta E_{int, C \rightarrow D} &= \frac{3}{2} nR(T_D - T_C) \\
W_{C \rightarrow D} &= -\Delta E_{int, C \rightarrow D} = \frac{3}{2} nR(T_C - T_D)
\end{aligned}$$

D to A:

$$\begin{aligned}
\Delta E_{int, D \rightarrow A} &= \frac{3}{2} nR(T_A - T_D) \\
W_{D \rightarrow A} &= 0 \\
Q_{D \rightarrow A} &= \Delta E_{int, D \rightarrow A} = \frac{3}{2} nR(T_A - T_D)
\end{aligned}$$

Overall in the cycle then,

$$\begin{aligned}
W_{net} &= \frac{3}{2} nR(T_A + T_C - T_B - T_D) \\
Q_H &= \frac{3}{2} nR(T_A - T_D)
\end{aligned}$$

Again we resort to the  $PV^\gamma = \text{constant}$  relation for adiabats to get

$$\begin{aligned} P_A V_A^\gamma &= P_B V_B^\gamma \\ T_A V_A^{\gamma-1} &= T_B V_B^{\gamma-1} \\ T_B &= T_A \left( \frac{V_A}{V_B} \right)^{\gamma-1} \end{aligned}$$

Similarly,  $T_C = T_D \left( \frac{V_D}{V_C} \right)^{\gamma-1}$ . However, we can observe that  $V_A = V_D$  and  $V_B = V_C$ , so

$$\begin{aligned} e &= 1 - \frac{T_A \left( \frac{V_A}{V_B} \right)^{\gamma-1} - T_D \left( \frac{V_A}{V_B} \right)^{\gamma-1}}{T_A - T_D} \\ &= 1 - \frac{T_A - T_D}{T_A - T_D} \left( \frac{V_A}{V_B} \right)^{\gamma-1} \\ &= 1 - \left( \frac{V_A}{V_B} \right)^{\gamma-1} \end{aligned}$$

### 2.3 Net Efficiency of Two Engines

- (a) We define efficiency as  $e = \frac{W_{net}}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}}$ , so

$$\begin{aligned} W_A &= Q_{in,A} e_A \\ Q_{out,A} &= Q_{in,A} (1 - e_A) \end{aligned}$$

- (b)

$$\begin{aligned} W_B &= Q_{in,A} (1 - e_A) e_B \\ Q_{out,B} &= Q_{in,A} (1 - e_A) (1 - e_B) \end{aligned}$$

- (c)

$$\begin{aligned} W_{total} &= W_A + W_B \\ &= Q_{in,A} (e_A + e_B - e_A e_B) \end{aligned}$$

- (d)

$$\begin{aligned} e_C &= \frac{W_{total}}{Q_{in,A}} \\ &= e_A + e_B - e_A e_B \end{aligned}$$

- (e) Observe that  $\frac{\partial e_C}{\partial e_A} = 1 - e_B$  and  $\frac{\partial e_C}{\partial e_B} = 1 - e_A$ . These have critical points at  $e_B = 1$  and  $e_A = 1$ , respectively. So  $e_C$  is maximized when  $e_A = e_B = 1$ , so  $e_C = 1$ . Hence, for  $e_A < 1$  and  $e_B < 1$ ,  $e_C < 1$ .

(f)  $e_A = 1 - \frac{T_M}{T_H}$  and  $e_B = 1 - \frac{T_C}{T_M}$ , so

$$\begin{aligned}
 e_C &= 1 - \frac{T_M}{T_H} + 1 - \frac{T_C}{T_M} - \left(1 - \frac{T_M}{T_H}\right) \left(1 - \frac{T_C}{T_M}\right) \\
 &= 2 - \frac{T_M}{T_H} - \frac{T_C}{T_M} - 1 + \frac{T_M}{T_H} + \frac{T_C}{T_M} - \frac{T_C}{T_H} \\
 &= 1 - \frac{T_C}{T_H}
 \end{aligned}$$