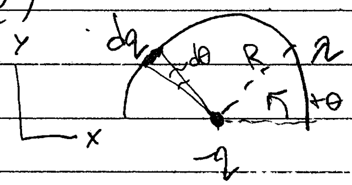
Week 7, Session 1 Solutions

1.1

(a) $Q = \lambda \cdot \frac{1}{2}(2\pi R)$
$= \pi \lambda R$

(b)



$$\vec{F} = \int d\vec{F} = \int \frac{(-q)\,dQ}{4\pi R^2}\hat{r}$$
$$= \frac{-q}{4\pi R^2}\int \lambda R\, d\theta\, \hat{r}$$
$$= \frac{-q\lambda}{4\pi R}\int_0^{\pi} d\theta\, \hat{r}$$

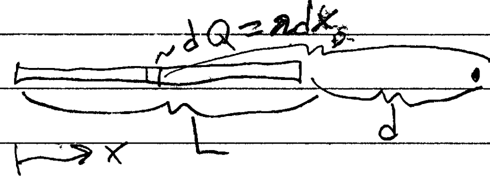
(c) The force will point in the $\hat{y}$ direction by symmetry. It will be in the $+\hat{y}$ direction if $\lambda > 0$ and $-\hat{y}$ if $\lambda < 0$.

(d) Since we only want the $\hat{y}$ component, we can reshape our integral:

$$\vec{F} = \int d\vec{F} = \hat{y}\,\frac{-q\lambda}{4\pi R}\int_0^{\pi} \sin\theta\, d\theta$$
$$= \frac{-q\lambda}{4\pi R}\hat{y}\left[-\cos\theta\right]_0^{\pi}$$
$$= \frac{q\lambda}{2\pi R}\hat{y}$$

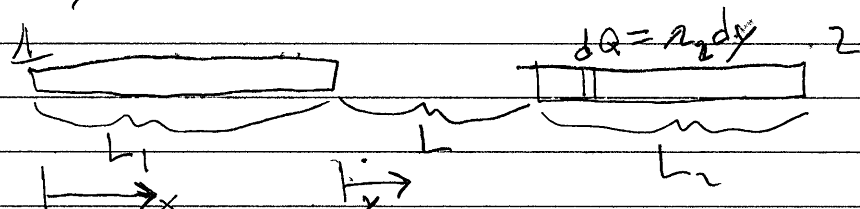
1.2

(a) We first calculate the field from a rod of length L and charge density λ on its axis:



$$\vec{E} = \int d\vec{E}$$
$$= \int \frac{dQ}{4\pi\varepsilon_0 D^2}\hat{x}$$
$$= \frac{\lambda \hat{x}}{4\pi\varepsilon_0}\int_0^L dx/(d+L-x)^2$$
$$= \frac{-\lambda}{4\pi\varepsilon_0}\hat{x}\,(d+L-x)^{-1}\Big|_0^L$$
$$= \frac{-\lambda\hat{x}}{4\pi\varepsilon_0}\left(\frac{1}{d} - \frac{1}{d+L}\right)$$
$$= \frac{-\lambda L}{4\pi\varepsilon_0 d(d+L)}\hat{x}$$

Now, we use this result in finding the force exerted on rod 2 by rod 1:



From above, the field from the first rod is
$$\vec{E}_1 = \frac{k\lambda_1 L_1}{y(y+L_1)}\hat{x}$$
We can take this over the second rod to calculate the total force on it:
$$\vec{F} = \int d\vec{F}$$
$$= \int \vec{E}_1 dQ$$
$$= k\lambda_1\lambda_2 L_1 \int_L^{L+L_2} \frac{dy}{y(y+L_1)}\hat{x}$$
To calculate this integral, we use partial fractions:
$$\frac{A}{y} + \frac{B}{y+L_1} = \frac{1}{y(y+L_1)} \Rightarrow A(y+L_1) + By = 1$$
$$\Rightarrow A+B = 0 \quad \text{and} \quad A = \frac{1}{L_1}$$
$$\Rightarrow B = -\frac{1}{L_1}$$
Then,
$$\vec{F} = k\lambda_1\lambda_2\hat{x}\int_L^{L+L_2}\left(\frac{1}{y} + \frac{1}{y+L_1}\right)dy$$

$$= k\lambda_1\lambda_2\left[\ln\left(\frac{y}{y+L_1}\right)\right]\Big|_L^{L+L_2}\hat{x}$$

$$= k\lambda_1\lambda_2 \ln\left(\frac{L+L_2}{L+L_1+L_2}\cdot\frac{L+L_1}{L}\right)\hat{x}$$

$$= k\lambda_1\lambda_2 \ln\left[\frac{(L+L_1)(L+L_2)}{L(L+L_1+L_2)}\right]\hat{x}$$

(b) When $L \gg L_1$ and $L \gg L_2$, we can rewrite

$$\vec{F} = k\lambda_1\lambda_2 \ln\left[\frac{(1+L_1/L)(1+L_2/L)}{1+(L_1+L_2)/L}\right]\hat{x}$$

$$= k\lambda_1\lambda_2\left[\ln(1+L_1/L)+\ln(1+L_2/L)-\ln(1+(L_1+L_2)/L)\right]\hat{x}$$

$$\approx k\lambda_1\lambda_2\left[\left(L_1/L - L_1^2/2L^2\right)+\left(L_2/L - L_2^2/2L^2\right)-\left(\frac{L_1+L_2}{L}+\frac{(L_1+L_2)^2}{2L^2}\right)\right]\hat{x}$$

$$= k\lambda_1\lambda_2 \frac{2L_1L_2}{L^2}\hat{x}$$

$$= k\lambda_1 L_1 \lambda_2 L_2 / L^2\,\hat{x}$$

$$= k Q_1 Q_2 / L^2\,\hat{x}$$

where we have defined $Q_1 \equiv \lambda_1 L_1$ and $Q_2 \equiv \lambda_2 L_2$.

1.3

The torque on a dipole is given by

$$\vec{\tau} = \vec{p} \times \vec{E}$$

such that the equilibrium position is when $\vec{p}$ and $\vec{E}$ are parallel ($\vec{p} \cdot \vec{E} = 0$). This is when $\sin\theta = 0$ when we use the cross product definition of the torque as

$$|\vec{\tau}| = |\vec{p}||\vec{E}| \sin\theta$$

with θ the angle between $\vec{p}$ and $\vec{E}$. By the right-hand-rule, this torque opposes an increase in θ, so we write

$$\tau = -pE\sin\theta$$

For small oscillations, $\sin\theta \approx \theta$. From the rotational kinematics, $\tau = I\alpha = I\ddot{\theta}$, so we have

$$I\ddot{\theta} \approx -pE\theta \Rightarrow \ddot{\theta} = -\frac{pE}{I}\theta$$

$$= -\omega^2\theta$$

where we have defined $\omega^2 \equiv \frac{pE}{I}$.

Hence the angular frequency of small oscillations is $\omega = \sqrt{pE/I}$.

1.4

This field $\vec{E}$ must point from one plate to another by translational and rotational symmetry. So $\vec{E} = E\hat{x}$.

This means for our drawing the right plate is negatively charged and the left positively. The acceleration of the electron is then

$$\vec{a}_e = \frac{\vec{F}}{m}$$
$$= \frac{-eE}{m}\hat{x}$$

The acceleration of the proton is

$$\vec{a}_p = \frac{eE}{M}\hat{x}$$

Since we are in one-dimension for the motion of interest, we drop vector signs and write

$$a_e = \frac{-eE}{m}$$
$$a_p = \frac{eE}{M}$$

By kinematics, the distance covered by each of this is

$$x_e = \frac{1}{2} a_e t^2$$
$$x_p = \frac{1}{2} a_p t^2$$

Since the times of motion will be equal when then meet, we can divide to get

$$x_e/x_p = a_e/a_p$$
$$= -M/m$$

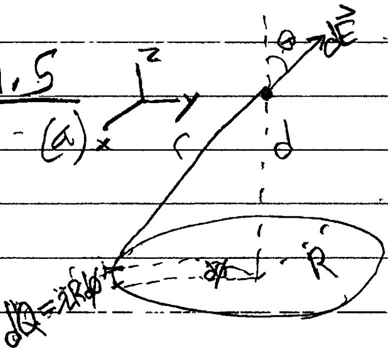
We must have $x_p - x_e = D$, where subtraction comes from a_e and hence x_e being negative. ~~x_p~~ will be x_{meet} if we define $x=0$ at the left, positively charged plate. So

$$x_p - \left(-\frac{M}{m}x_p\right) = D \Rightarrow x_p\left(1+\frac{M}{m}\right) = D$$
$$\Rightarrow x_{meet} = D/\left(\frac{m+M}{m}\right)$$
$$= \frac{m}{m+M}D$$

This result does not depending on the strength of the field since the relative accelerations are based on the charges and masses, not field strength in a uniform field. That is, time is irrelevant since we only want where they meet.

1.5

(a)

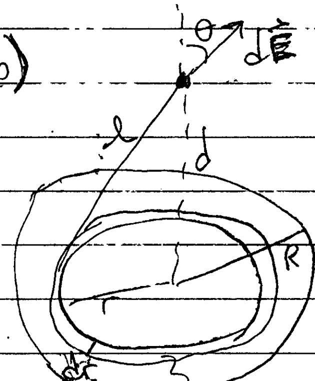


By symmetry, the field will be in the $\hat{z}$ direction, $\hat{z}$ if $\lambda > 0$ and $-\hat{z}$ if $\lambda < 0$. Now,
$r^2 = d^2 + R^2$

Then,

$$\vec{E} = \int d\vec{E}$$
$$= \int_0^{2\pi} \frac{\lambda R\, d\phi}{4\pi\varepsilon_0 r^2} \cos\theta\, \hat{z}$$
$$= \int_0^{2\pi} \frac{\lambda R d}{4\pi\varepsilon_0 r^3}\, d\phi\, \hat{z}$$
$$= \frac{\lambda R d}{2\varepsilon_0 (R^2+d^2)^{3/2}}\, \hat{z}$$

(b)



$dQ = 2\pi r\, dr\, \sigma$
$\equiv 2\pi r \lambda$

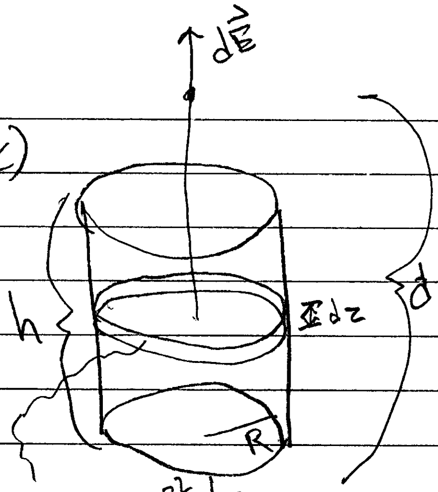
For each ring of radius dr in the disk, the total charge in it is $dQ = 2\pi r\, dr\, \sigma$ whereas in part (a) the ring had total charge $\lambda 2\pi R$. So here, we again take an integral as

$$\vec{E} = \int d\vec{E}$$
$$= \int_{\text{rings}} \frac{(2\pi r \lambda) d}{4\pi\varepsilon_0 (r^2+d^2)^{3/2}}\, \hat{z}$$
$$= \int_0^R \frac{\sigma 2\pi r\, dr\, d}{4\pi\varepsilon_0 (r^2+d^2)^{3/2}}\, \hat{z}$$
$$= \frac{\sigma d}{2\varepsilon_0}\hat{z} \int_0^R \frac{r}{(r^2+d^2)^{3/2}}\, dr$$

We integrate with the substitution $u = r^2 + d^2$, $\frac{du}{2} = r\,dr$:

$$\vec{E} = \frac{\sigma d}{2\varepsilon_0}\hat{z} \int_{d^2}^{R^2+d^2} \frac{1}{2u^{3/2}}\, du$$
$$= \frac{\sigma d}{4\varepsilon_0}\hat{z} \cdot u^{-1/2} \cdot -2 \Big|_{d^2}^{R^2+d^2}$$
$$= \frac{-\sigma d \hat{z}}{2\varepsilon_0}\left(\frac{1}{\sqrt{d^2+R^2}} - \frac{1}{d}\right)$$
$$= \hat{z}\, \frac{\sigma d}{2\varepsilon_0}\left(1/d - 1/\sqrt{d^2+R^2}\right)$$
$$= \hat{z}\, \frac{\sigma}{2\varepsilon_0}\left(1 - d/\sqrt{d^2+R^2}\right)$$

(c)



We proceed similarly to part (b), noting here that each little cylinder has charge

$$dQ = \rho\pi R^2 dz$$

whereas for (b) the total charge of the disk was $\pi R^2\sigma$, so we use (b) in

$$dQ = \rho\pi R^2 dz$$

$$\vec{E} = \int d\vec{E}$$
$$= \hat{z}\int_{\text{cylinders}} \frac{\sigma(\pi R^2)\, d}{2\varepsilon_0(\pi R^2)}\left(1 - d/\sqrt{d^2+R^2}\right)$$
$$= \hat{z}\int_0^h \frac{\rho\pi R^2\, dz}{2\varepsilon_0\pi R^2}\left(1 - d/\sqrt{d^2+R^2}\right)$$

We have to set $d \to d - z$ here since d in part (b) represented the perpendicular distance between the ring and point. Then,

$$\vec{E} = \hat{z}\frac{\rho}{2\varepsilon_0}\int_0^h dz\left(1 - (d-z)/\sqrt{(d-z)^2+R^2}\right)$$
$$= \hat{z}\frac{\rho h}{2\varepsilon_0} + \hat{z}\frac{\rho}{2\varepsilon_0}\int_0^h \frac{z-d}{\sqrt{(z-d)^2+R^2}}\, dz$$

We again use a substitution, $u = (z-d)^2 + R^2$, $\frac{du}{2} = (z-d)\, dz$:

$$\vec{E} = \hat{z}\frac{\rho h}{2\varepsilon_0} + \hat{z}\frac{\rho}{2\varepsilon_0}\int_{d^2+R^2}^{(d-h)^2+R^2} \frac{du}{2u^{1/2}}$$
$$= \hat{z}\frac{\rho h}{2\varepsilon_0} + \hat{z}\frac{\rho}{2\varepsilon_0}\cdot u^{1/2}\Big|_{d^2+R^2}^{(d-h)^2+R^2}$$
$$= \hat{z}\frac{\rho}{2\varepsilon_0}\left[h + \sqrt{(d-h)^2+R^2} - \sqrt{d^2+R^2}\right]$$