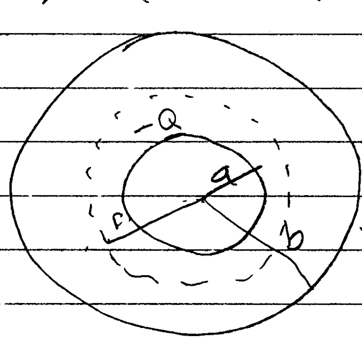
Week 10, Session 1 Solutions

Problem 1.1:

Imagine a charge $+Q$ distributed uniformly over the outer surface and a charge $-Q$ distributed uniformly over the inner surface.



By Gauss' Law with a Gaussian cylinder, the electric field everywhere between the inner and outer surfaces is given by

$$E(2\pi r \ell) = \frac{-Q}{\varepsilon_0}$$

$$\Rightarrow E(r) = \frac{-Q}{2\pi\varepsilon_0 r \ell}$$

(where I have used symmetry to know $\vec{E} = E(r)\hat{r}$ and evaluated $\int \vec{E}\cdot d\vec{a}$ over the three faces of the Gaussian cylinder.) Then,

$$V(b) - V(a) = -\int_a^b \frac{-Q}{2\pi\varepsilon_0 r \ell}\,dr$$

$$= \frac{Q}{2\pi\varepsilon_0 \ell}\ln\frac{b}{a}$$

Calling $V = V(b) - V(a)$, we know for a capacitor,

$$Q = CV \Rightarrow C = \frac{Q}{V}$$

$$= \frac{2\pi\varepsilon_0 \ell}{\ln(b/a)}$$

Now, for the energy stored,

$$U = \tfrac{1}{2}CV^2$$

$$= \frac{1}{2}\left(\frac{2\pi\varepsilon_0\ell}{\ln(b/a)}\right)\left(\frac{Q}{2\pi\varepsilon_0\ell}\ln\frac{b}{a}\right)^2$$

$$= \frac{Q^2}{4\pi\varepsilon_0\ell}\ln\frac{b}{a}$$

and, equivalently,

$$U = \int \mathcal{E}\,dV$$

$$= \frac{\varepsilon_0}{2}\int_a^b (E(r))^2\, 2\pi r \ell\, dr$$

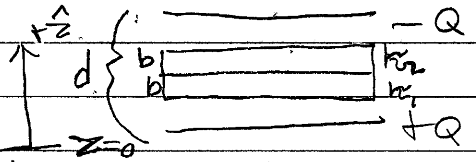
$$= \varepsilon_0 \pi \ell \int_a^b \frac{Q^2}{4\pi^2\varepsilon_0^2 r^2 \ell^2}\, r\, dr$$

$$= \frac{Q^2}{4\pi\varepsilon_0\ell}\int_a^b \frac{1}{r}\,dr$$

$$= \frac{Q^2}{4\pi\varepsilon_0\ell}\ln\frac{b}{a}$$

Problem 1.2:

1. In both the dielectric-free regions,



$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$$
$$= \frac{Q}{A\epsilon_0}\hat{z}$$

With the dielectric, there is an induced electric field opposing the parallel-plate one. The overall electric field is reduced to

$$\vec{E}_2 = \frac{E}{\kappa_2}$$
$$= \frac{Q}{A\kappa_2\epsilon_0}\hat{z}$$
$$\vec{E}_1 = \vec{E}/\kappa_1$$
$$= \frac{Q}{A\kappa_1\epsilon_0}\hat{z}$$

2. $\vec{E}$ is constant in each of the four regions, so the oveall potential difference between the plates is

$$V(+Q \text{ plate}) - V(-Q \text{ plate}) = -\int_{top}^{bottom} \vec{E}\cdot d\vec{l}$$
$$\Rightarrow V = \int_0^d E\,dz$$
$$= \frac{Q}{A\epsilon_0}(d-2b) + \frac{Qb}{A\epsilon_0}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)$$
$$= \frac{Q}{A\epsilon_0}\left[d + b\left(\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2} - 2\right)\right]$$

3. The capacitance is related to charge and potential difference via

$$Q = CV \Rightarrow C = \frac{Q}{V}$$
$$= A\epsilon_0 \Big/ \left[d + b\left(\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2} - 2\right)\right]$$

Problem 2.1:

By rotational symmetry, we know that $\vec{E} = E(r)\,\hat{r}$.

1. If $\vec{E} = E_0 \hat{r}$, then since $\vec{E} = -\vec{\nabla} V$,

$$\frac{dV}{dr} = -E_0$$

In words, the derivative of potential along the radial axis must be constant. Now, in a circuit, the current will be some constant value I (at least in circuits that only have batteries and resistors), so by $V = IR$,

$$\frac{dV}{dr} = I\frac{dR}{dr} \Rightarrow \frac{dR}{dr} = -\frac{E_0}{I}$$

Now, $R = \rho\frac{\ell}{A}$, so $dR = \rho(r)\frac{dr}{4\pi r^2}$, so

$$\frac{\rho(r)}{r^2} = -\frac{4\pi E_0}{I}$$

That is, $\rho(r)/r^2$ must be a constant. This implies we must have $s = 2$.

2. The total voltage drop across this spherical resistor must be the battery voltage, so

$$V = \int I\,dR$$
$$= I\frac{\rho_0}{a^2}\int_a^b r^2\,dr$$
$$= I\frac{\rho_0}{a^2}\left(b^3/3 - a^3/3\right) \Rightarrow I = \frac{3Va^2}{\rho_0(b^3 - a^3)}$$

Problem 2.2:

For each wire,

$$R_i = \rho_i \frac{\ell_i}{A}$$

So the total resistance if we combine in series is

$$R = \frac{1}{A}(\rho_1 \ell_1 + \rho_2 \ell_2)$$

Resistivity ρ varies with temperature according to

$$\rho_1 \to \rho_1(1 + \alpha_1 T)$$

$$\rho_2 \to \rho_2(1 + \alpha_2 T)$$

So then

$$R = \frac{1}{A}\left(\rho_1 \ell_1 (1 + \alpha_1 T) + \rho_2 \ell_2 (1 + \alpha_2 T)\right)$$

$$= \frac{1}{A}(\rho_1 \ell_1 + \rho_2 \ell_2) + \frac{T}{A}(\rho_1 \ell_1 \alpha_1 + \rho_2 \ell_2 \alpha_2)$$

To be independent of temperature, we need the T term to vanish, so

$$\rho_1 \ell_1 \alpha_1 = -\rho_2 \ell_2 \alpha_2 \Rightarrow \ell_2 = -\ell_1 \frac{\rho_1 \alpha_1}{\rho_2 \alpha_2}$$

So if we choose $\ell_1 = 1\,\text{cm}$, then

$$\ell_2 = (-1\,\text{cm}) \frac{\rho_1 \alpha_1}{\rho_2 \alpha_2}$$

[Note this ℓ_2 is positive since $\alpha_1 > 0$ and $\alpha_2 < 0$.]

Problem 2.3:

The resistance is normally given by

$$R = \rho \frac{l}{A}$$

where A and ρ do not vary over the length l of the resistor through which current flows. In general, then, we write

$$dR = \rho(l) \frac{dl}{A(l)}$$

where l is measured along the direction current flows through the resistor. Here, $l = r$ as the radial distance from the center of the resistor and

$$\rho(r) = \rho$$

differential is dr

$$A(r) = 2\pi r l$$

where l is the length of the cylinder in this formulation of A (different from the previous l's that were just distances over which current flows through the resistor).

Our integral becomes

$$R = \int_{\substack{\text{distance} \\ \text{current} \\ \text{flows through}}} dR$$

$$= \int_{r_1}^{r_2} \frac{\rho}{2\pi r l} dr$$

$$= \frac{\rho}{2\pi l} \int_{r_1}^{r_2} \frac{1}{r} dr$$

$$= \frac{\rho}{2\pi l} \ln \frac{r_2}{r_1}$$

Problem 2.4:

(a) When the batteries are connected in series, their voltages add and we essentially have 3 V. This voltage must drop across the resistance of the bulb so

$$V = IR \Rightarrow R = \frac{V}{I}$$
$$= \frac{3\ V}{380\ mA}$$
$$\approx 7.89\ \Omega$$

The power dissipated can be calculated using a couple formulations but we will use

$$P = IV$$
$$= (380\ mA)(3\ V)$$
$$= 1.14\ W$$

(b) The power would increase by a factor of two since $P = IV$ so if $V \to 2V$, $P \to 2P$. You probably shouldn't try this because the filament might burn out fairly quickly as most lightbulbs have a certain power rating.