

Logique et représentation des connaissances

Spécialités DAC et Android

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Informations Générales

- **Cours:** amphithéâtre B1, mercredi, 13H45-15H45
- **Premier cours:** 20 septembre 2017
- **Dernier cours:** 13 décembre 2017
- **Examens répartis:**
 - **Examen réparti n°1:** 15 novembre 2017
 - **Examen réparti n°2:** 17 janvier 2018
- **Barème:**
 - **Examen réparti n°1:** 0,45
 - **Examen réparti n°2:** 0,45
 - **Contrôle continu:** 0,1



Informations Générales (*suite*)

- **TD/TME**
 - **Lundi après-midi (G1 et G2),**
 - **mardi après-midi (G3),**
 - **mardi soir (TD G5)**
 - **jeudi matin (G4)**
 - **mercredi matin (TME G5)**
- **Moodle:** <https://moodle-sciences.upmc.fr/moodle/>



Organisation TD/TME

39	lun, 25 sep	mar, 26 sep	mer, 27 sep	jeu, 28 sep	ven, 29 sep
Toute la journée	TD M1 semaine 1 Cours M1 semaine 2				
9:00	8:30 - 10:30 4I801-TD1-MLBDA		8:30 - 10:30 4I801-TD2-MLBDA	8:30 - 10:30 4I801-TD3-MLBDA	8:30 - 10:30 4I800-TD4-LRC
10:00	=		=	=	=
11:00	10:45 - 12:45 4I801-TME1-MLBDA		10:45 - 12:45 4I801-TME2-MLBDA	10:45 - 12:45 4I800-TME5-LRC	10:45 - 12:45 4I801-TME3-MLBDA
12:00	=		=	=	=
13:00					
14:00	13:45 - 15:45 4I800-TD1-LRC	13:45 - 15:45 4I800-TD2-LRC	13:45 - 15:45 4I800-TD3-LRC	13:45 - 15:45 4I800-Cours-LRC	13:45 - 15:45 4I801-TME4-MLBDA
15:00	=	=	=	=	=
16:00	16:00 - 18:00 4I800-TME2-LRC	16:00 - 18:00 4I800-TME1-LRC	16:00 - 18:00 4I800-TME3-LRC	16:00 - 18:00 4I801-Cours-MLBDA	
17:00	=	=	=	=	
18:00			18:00 - 20:00 4I800-TD5-LRC		
19:00			=		
20:00					

Cours LRC: logique(s) et représentation(s) des connaissances

1. Introduction à la logique des propositions et des prédictats du premier ordre.
Règles de résolution.
Méthode des tableaux.

Logiques modales

2. Introduction aux logiques modales
3. Logiques épistémique
4. Connaissances communes, connaissances partagées

Représentation des connaissances

5. Représentations sémantiques, graphes conceptuels.
6. Logiques de description (*syntaxe et sémantique*).
7. Logiques de description: raisonnement automatique (*méthode des tableaux et subsomption structurelle*)

Représentations du temps

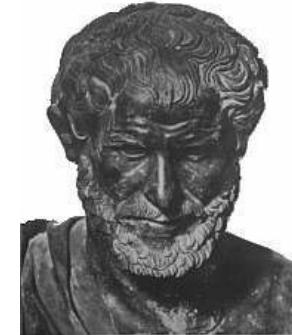
8. Intervalles d'Allen
9. Automates temporisés
10. Réseaux de Petri





History of Logic

(-500 – ...)



1. Describing the correct way of reasoning – **Philosophy** (*Aristotle, ...*)
2. Mathematizing Logic – **Philosophy** (*Leibniz, Boole, ...*)
3. Mechanizing Mathematics – **Mathematics** (*Hilbert, Herbrand, Tarski...*)
4. Computerizing the Proof – **Mathematics, Logic, AI & Computer Science** (...)



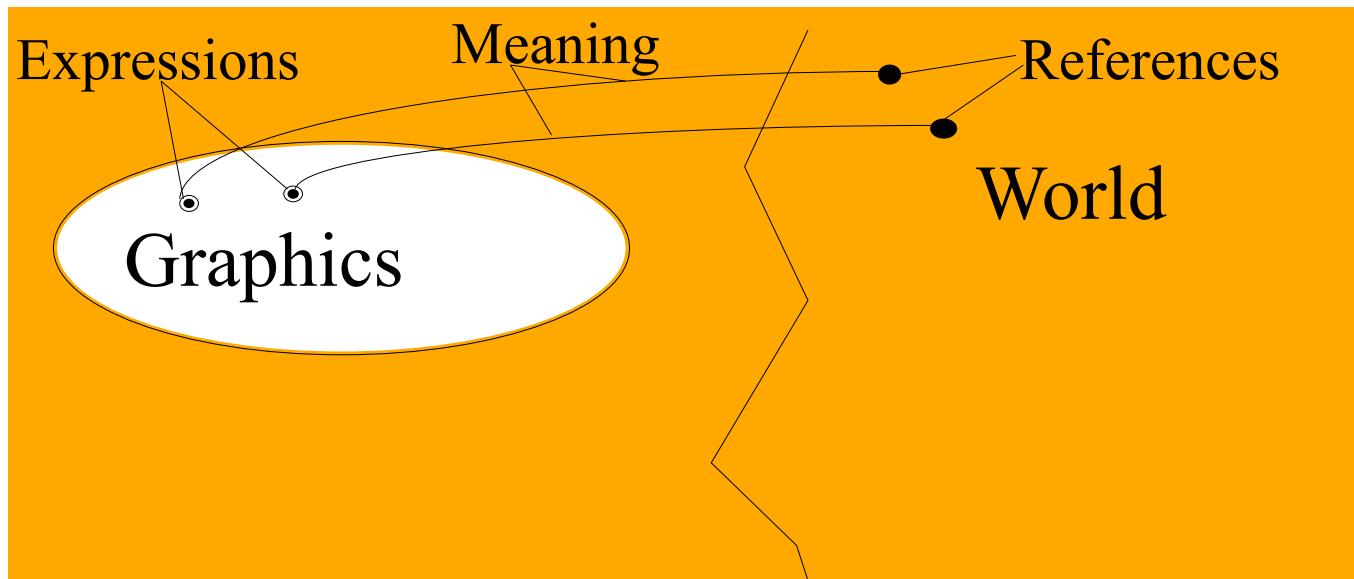
Représentation en logique

- Signe, sens, sémantique et vérité
- Proposition
- Le langage de la logique propositionnelle
- Limites de la logique propositionnelle
- La logique des prédictats
- Sémantique de la logique des prédictats



Sign

- Expression, Meaning, References

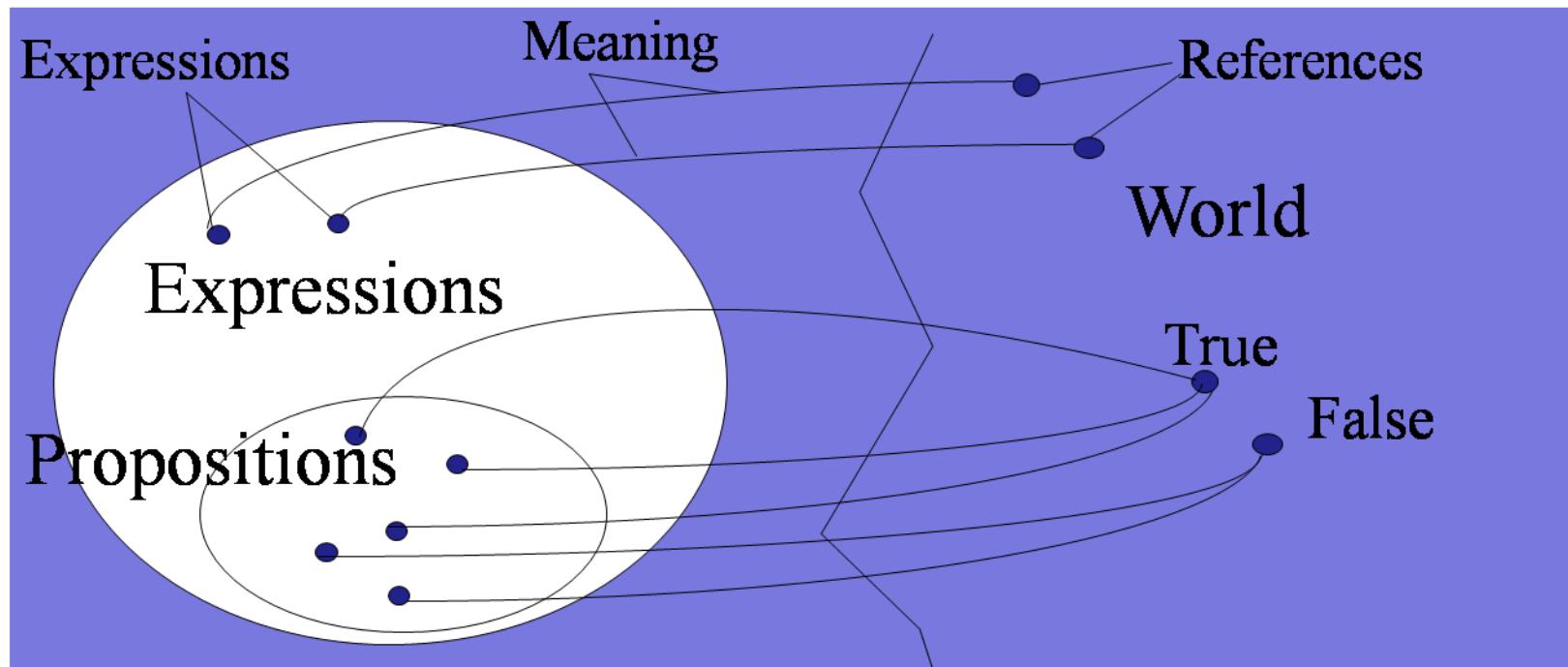


- Sign: graphic with meaning (i.e. intended to express)
 - Examples: “horse”, “reality”, “John”, “!”, “)”, “A”, “E”



Proposition

- Meaning = truth value $\in \{\text{true}, \text{false}\}$
 - “Earth is blue”
 - “The green square is behind the red triangle”
 - “A book was published”



The Propositional Language

- Atom: elementary propositions
 - “It was raining”
 - “The cube is green”
- Connectors
 - Binary: « \wedge », « \vee », « \rightarrow »
 - Unary: « \neg »
- Composed Propositions
 - If A and B are propositions, « $\neg A$ », « $A \wedge B$ », « $A \vee B$ », « $A \rightarrow B$ » are propositions
- Examples:
 - $F_1 = ((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
 - $F_2 = ((p \wedge q \wedge \neg r) \vee ((\neg p \vee q) \rightarrow (\neg p \vee r)))$



Propositional Logic: semantic

- Truth Tables
- Satisfiability, validity, unsatisfiability
- Logical Consequence
- Limitation of Truth Tables



Truth Table

- Every atom a has a *truth value*:
true (v) or false (f)
- How to compute the truth value of a composed proposition?

« $A \wedge B$ »		
$A \backslash B$	v	f
v	v	f
f	f	f

« $A \vee B$ »		
$A \backslash B$	v	f
v	v	v
f	v	f

« $A \rightarrow B$ »		« $\neg A$ »	
$A \backslash B$	v	f	A
v	v	f	
f	v	v	



Example

A	B	C	$(B \rightarrow C)$	$(A \rightarrow (B \rightarrow C))$
v	v	v	v	v
v	v	f	f	f
v	f	v	v	v
v	f	f	v	v
f	v	v	v	v
f	v	f	f	v
f	f	v	v	v
f	f	f	v	v

- Each line of the truth table is called an **interpretation**



- A formula A is said to be *valid* if it is true on all the lines of the truth table, i.e. in all the interpretations
 - Example:
 $(A \rightarrow (B \rightarrow A))$

Validity

A	B	$(B \rightarrow A)$	$(A \rightarrow (B \rightarrow A))$
v	v	v	v
v	f	v	v
f	v	f	v
f	f	v	v



Validity: an other example

- Another example:

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

A	B	C	$(B \rightarrow C)$	$(A \rightarrow (B \rightarrow C))$	$A \rightarrow B$	$A \rightarrow C$	$((A \rightarrow B) \rightarrow (A \rightarrow C))$	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
v	v	v	v	v	v	v	v	v
v	v	f	f	f	v	f	f	v
v	f	v	v	v	f	v	v	v
v	f	f	v	v	f	f	v	v
f	v	v	v	v	v	v	v	v
f	v	f	f	v	v	v	v	v
f	f	v	v	v	v	v	v	v
f	f	f	v	v	v	v	v	v



- A formula A is said to be *satisfiable* if it is **true** on at least one line of its truth table, i.e. in at least on interpretation
 - Example:
 $(A \rightarrow (B \rightarrow \neg A))$

Satisfiability

A	B	$(B \rightarrow \neg A)$	$(A \rightarrow (B \rightarrow \neg A))$
v	v	f	f
v	f	v	v
f	v	v	v
f	f	v	v



I Satisfiability – Unsatisfiability – Validity

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- A formula is said to be
 - “**satisfiable**” if and only if it is true on at least one line of the truth table
 - “**valid**” if and only if it is true on all lines of the truth table
 - “**unsatisfiable**” if and only if it is false on all lines of the truth table, i.e. if it is never true
- **Remarks :**
 - F is *unsatisfiable* iff $\neg F$ is *valid*
 - F is *unsatisfiable* iff F is not *satisfiable*



Logical consequence

Semantic consequence

Entailment

- A is a **semantic consequence** of B if A is true on all the lines of the truth table where B is true
 - Example: $(A \rightarrow B)$ is a semantic consequence of B.

Notation

$\models A$ means A is a valid formula

$B \models A$ means A is a semantic consequence of B



Deduction Theorem

Deduction Theorem: B is a semantic consequence of A ($A \models B$)
if and only if $A \rightarrow B$ is valid ($\models A \rightarrow B$)

Equivalent definitions:

- $A \models$ if and only if $\models (\neg A)$
- $A \models B$ if and only if $\models A \rightarrow B$
- $\models A$ if and only if $(\neg A) \models \neg A$



A Formal System for the Propositional Logic

- *Recall:* a formal system is composed of
 - A formal language
 - A set of axioms
 - A set of inference rules
- Formal system for the propositional logic:
 - **Formal language:** set A of atoms, one unary connector, « \neg », one binary connector, « \rightarrow »
 - **Axioms:** formulae which are obtained by replacing A, B et C by any propositional logic formula in the following axiom schemas:
 - SA1:** $(A \rightarrow (B \rightarrow A))$
 - SA2:** $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
 - SA3:** $((\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A))$
 - **Inference rule:** « *Modus Ponens* » $A, (A \rightarrow B) \vdash B$

Theorems

Definition: any formula which is derived from the axioms by iteratively applying inference rules is a *theorem*.

Notation: $\vdash A$ means A is a theorem

Example: $\vdash (A \rightarrow A)$

Proof:

- 1- $(A \rightarrow ((A \rightarrow A) \rightarrow A))$ SA1
- 2- $((A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)))$ SA2
- 3- $((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ Modus Ponens 2, 1
- 4- $(A \rightarrow (A \rightarrow A))$ SA1
- 5- **$(A \rightarrow A)$** Modus Ponens 4, 3



Demonstration

Definition: a *proof* of a theorem A is a finite sequence of formulae F_0, F_1, \dots, F_n such that

- $F_n = A$
- $\forall i \in [0, n]$ F_i is either an axiom, or obtained by applying the *modus ponens* to the two formulae F_j and F_k where j and $k < i$



Deduction theorem

Theorem: $A_1, \dots, A_{n-1} \vdash (A_n \rightarrow B)$ ssi $A_1, \dots, A_{n-1}, A_n \vdash B$

Proof (direction 1):

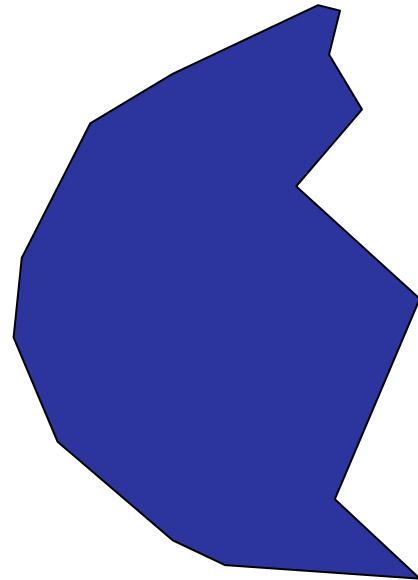
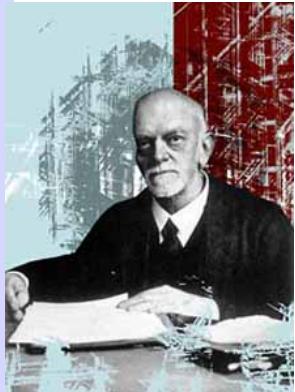
1. $(A_n \rightarrow B)$ Hypothesis 1
2. A_n Hypothesis 2
3. B *Modus Ponens* 1, 2

Proof (direction 2): four possibilities has to be investigated

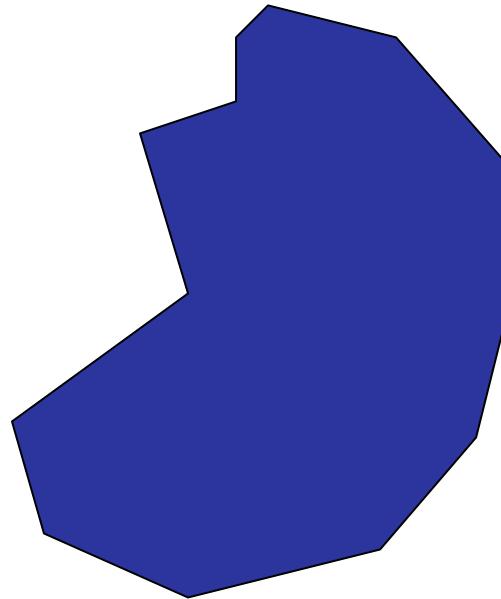
1. B is an axiom
2. B is one of the hypotheses A_1, \dots, A_{n-1}
3. B is the hypothesis A_n
4. B is obtained by applying the *modus ponens* to $(G \rightarrow B)$ and G (*proof by induction on the size of the demonstration*)



Notion of symbolic system



Symbolic system



Mathematical objects

Consistency: each description of the symbolic system corresponds to an object in the reality, i.e. $\forall A \text{ if } \vdash A \text{ then } \models A$

Completeness: each object of the reality can be described in the symbolic system $\forall A \text{ if } \models A \text{ then } \vdash A$



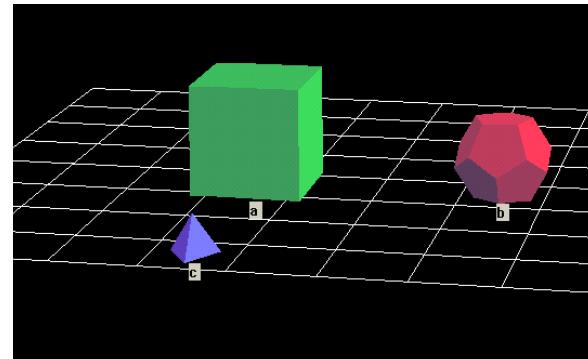
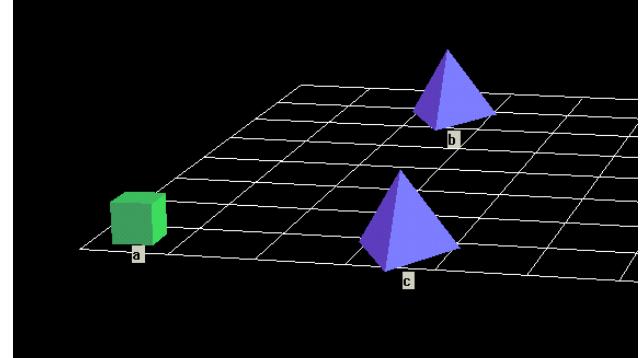
Propositional Logic

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How to translate those sentences in propositional logic:

1. There is an object right to a green cube
2. Every successor of an even number is odd
3. 72 is an even number
4. The successor of 72 is odd



Introduction of propositional functions (predicates):

even(X): « X is even » is a proposition for all values of X.



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Representation in Predicate Logic

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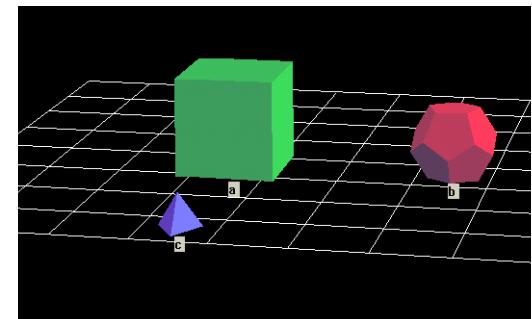
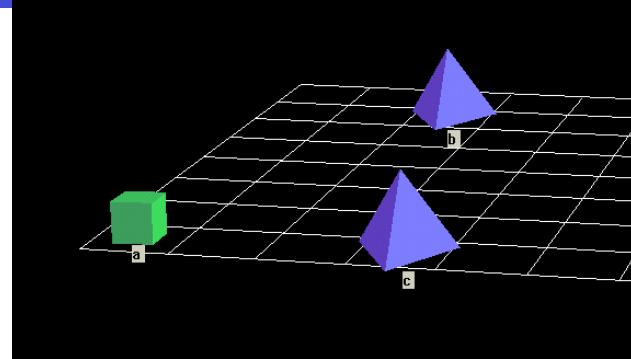
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Notion of propositional function:

1. X is even: **even (X)**
2. 72 is an even number **even (72)**
3. The successor of 72 is odd
odd (successor (72))
4. Every successor of an even number is odd
 $\forall x \text{ (even (x)} \rightarrow \text{odd (successor (x)))}$
5. There is an object to the right of a green cube
 $\exists x \exists y \text{ (right_of (x, y) \& cube (y) \& green (y))}$



Predicate Logic Language

- Terms (\mathcal{T}):
 - Being given $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_n \cup \dots$ a set of *function symbols* [\mathcal{F}_i corresponds to functions of arity i]
 - Being given \mathcal{V} a set of *variable symbols*
 1. Each element of \mathcal{V} is a term
 2. For any n-tuple of terms (t_1, t_2, \dots, t_n) , for any function f_n of \mathcal{F}_n , $f_n(t_1, t_2, \dots, t_n)$ is a term



Predicate Logic Language

- **Atom (\mathcal{A}):**
 - Being given $Q = Q_0 \cup Q_1 \cup Q_2 \cup \dots \cup Q_n \cup \dots$ a set of *predicate symbols* [Q_i corresponds to arity i predicates]

For any n-tuplet of terms (t_1, t_2, \dots, t_n) ,

For any predicate q_n of Q_n ,

$q_n(t_1, t_2, \dots, t_n)$ is an **atom**

Remark: a predicate is a propositional function, i.e. It is a function of which values belong to {true, false}



Examples

- **Terms:**
 - successor(73), plus(72, 1), multiply(72, 2), plus(X, 1).
 - father(Jean), father(X)...
- **Atoms:**
 - even(72), odd(73), odd(successor(X)).
 - smoke(goat, cigar)
 - father(Jean, Peter)...



Formulae

- Quantifier:
 - If $x \in \mathcal{V}$ (x is a variable) and F a formula, then $\forall x F$ et $\exists x F$ are also formulae
 - ◆ $\forall x F$ means “for all x , F ”
 - ◆ $\exists x F$ means “there exists x , F ”
- Formulae:
 - Atoms are formulae
 - If F and G are formulae, « $\neg F$ », « $F \wedge G$ », « $F \vee G$ », « $F \rightarrow G$ » are formulae
 - If $x \in \mathcal{V}$, $\forall x F$ and $\exists x F$ are also formulae



Free – bounded variable

- An **occurrence** of a variable is said to be **free** if it does not appear under the scope of a quantifier (\forall or \exists), else it is called **bounded**

Example: in the following formula

$$\forall x \exists y (\text{mother}(x, y) \wedge \text{married}(x, z))$$

x and **y** are **bounded**

z is **free**.

Remark: a variable may have both free and bounded occurrences in the same formula

$$\forall x (\text{married}(x, y) \wedge \exists y \text{mother}(x, y))$$



Interpretation

- Propositional Logic:
 - Truth values attributed to the different atomic propositions (*line of the truth table*)
- First Order Logic (i.e. predicate logic):
 - A non empty domain \mathcal{D}
 - An attribution of “value” to each symbol:
 - To each n-ary function f_n of \mathcal{F}_n , a function from \mathcal{D}^n to \mathcal{D} , which is denoted $i[f_n]$
 - To each n-ary predicate symbol p_n of \mathcal{P}_n , a function from \mathcal{D}^n to $\{v, f\}$, which is denoted $i[p_n]$



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1st order logic semantics

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- Being given a domain \mathcal{D}
- We call “interpretation” a function i which attributes:
 - A n-ary function f_n of \mathcal{F}_n , a function from \mathcal{D}^n to \mathcal{D} , which is denoted $i[f_n]$
 - To each n-ary predicate symbol p_n of \mathcal{P}_n , a function from \mathcal{D}^n to $\{v, f\}$, which is denoted $i[p_n]$
- A formula is “**valid**” if it is true in all the interpretations of all domains
- A formula is “**satisfiable**” if it is true for at least one interpretation of one domain.
- A formula is “**unsatisfiable**” if it is never true in any domain, i.e. if it is false in all interpretations of all domains.



Examples

- Validity of $\neg\exists x P(x) \rightarrow \forall x(\neg P(x))$
- Unsatisfiability of $\forall x P(x) \wedge \exists y(\neg P(y))$
- Satisfiability of $\forall x(\neg \text{mâle}(x) \rightarrow \text{femelle}(x))$
- Invalidity of $\forall x(\neg \text{mâle}(x) \rightarrow \text{femelle}(x))$
- Validity of $\forall x((\neg \text{mâle}(x) \rightarrow \text{femelle}(x)) \vee (\neg \text{mâle}(x) \wedge \neg \text{femelle}(x)))$



Model

Being given a domain \mathcal{D} and an interpretation i

- We call “model” a couple $\mathcal{M} = \langle \mathcal{D}, i \rangle$
- We call “valuation” a function $v: \mathcal{V} \rightarrow \mathcal{D}$
- $I_{mv}(F)$, the truth value of the formula F is defined as follows:
 - $I_{mv}(P(t_1, t_2, \dots t_n)) = t$ if and only if $i(P(t_1, t_2, \dots t_n)) = t$
 - $I_{mv}(\neg F) = \neg i(F)$
 - $I_{mv}(F \wedge G) = I_{mv}(F) . I_{mv}(G)$
 - $I_{mv}(F \vee G) = I_{mv}(F) + I_{mv}(G)$
 - $I_{mv}(F \rightarrow G) = I_{mv}(\neg F \vee G)$



Example 1

$$F = \exists y P(x, y)$$

Domain: N

Meaning of the predicate: $i(P(n, m)) = t$ iff $n \leq m$

Valuation:

– $v(x) = 5$ and $v(y) = 3$

► We are seeking if for a valuation v' such that

– $v'(x) = 5$

– $v'(y) = p \in N$

– $5 \leq p$

For instance, $p = 10$

$I_{m_{v'}}(P(x, y)) = t$ therefore $I_{m_{v'}}(\exists y P(x, y)) = t$



Example 2

$$F = \forall x \exists y P(x, y)$$

Domain: N

Meaning of the predicate: $i(P(n, m)) = t$ iff $n \leq m$

Valuation:

- $v(x) = 5$ and $v(y) = 3$

➤ *We are seeking if for all valuation v_1 such that*

- $v_1(x) = p \in N$ and $v_1(y) = 3$

$$I_{m \textcolor{red}{v} 1}(\exists y P(x, y)) = t$$

➤ *For all valuation v_1 we are seeking for the existence of a valuation v_2 such that $v_1(x) = v_2(x) = p \in N$ and $v_2(y) = q \in N$, $I_{m \textcolor{red}{v} 2}(P(x, y)) = t$*



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- Being given a domain \mathcal{D} and an interpretation i a model \mathcal{M} is a tuple $\langle \mathcal{D}, i \rangle$ and a valuation v if a function $v: \mathcal{V} \rightarrow \mathcal{D}$
- A formula is “**valid**” if it is true in all the interpretations of all domains,
i.e. if it is true for any models \mathcal{M} and any valuation v
- A formula is “**satisfiable**” if it is true for at least one interpretation of one domain.
i.e. if there exists a model \mathcal{M} such that it is true
- A formula is “**unsatisfiable**” if it is never true in any domain, i.e. if it is false in all interpretations i of all domains \mathcal{D} . i.e. if it is false in all models \mathcal{M}



clause

- **Definition:**

- A **literal** is either an atom or its negation:

Example: $\neg \text{even}(72)$, $\text{odd}(72)$, $\text{successor}(72, 73)$

- A **clause** is a disjunction of literals

Example: $\neg \text{even}(X) \vee \text{odd}(\text{successor}(X))$

Remark: a clause is a logical entailment (implication)

because $(\neg A \vee B)$ is equivalent to $(A \supset B)$

Example: $\text{even}(X) \supset \text{odd}(\text{successor}(X))$



Clausal Form

Theorem: any closed formula F can be transformed into a logically equivalent conjunction of clauses

Example:

$$\forall x \text{ square}(x) \equiv \exists y \text{ multiply}(y, y, x)$$

Can be transformed as a conjunction of two clauses:

$$\neg \text{square}(x) \vee \text{multiply}(r(x), r(x), x) \quad \text{et}$$

$$\text{square}(x) \vee \neg \text{multiply}(r(x), r(x), x)$$



Automatic Theorem Proving

Resolution in Propositional Logic
Tableau Method



Resolution Rule in Propositional Logic

- Been given two clauses C_1 and C_2
- Been given an atomic proposition such that $A \in C_1$ and $\neg A \in C_2$
- The resolution of C_1 and C_2 by A and $\neg A$ is:
$$\begin{aligned} C &= \text{res}(C_1, C_2; A, \neg A) \\ &= [C_1 - \{A\}] \vee [C_2 - \{\neg A\}] \end{aligned}$$

Example: if $C_1 = \neg \text{man} \vee \text{mortal}$,

$C_2 = \neg \text{socrate} \vee \text{man}$, $C_3 = \text{socrate}$ and

$C_4 = \neg \text{mortal}$

$C = \text{res}(C_3, C_2; \text{socrate}, \neg \text{socrate}) = \text{man}$

$C' = \text{res}(C_1, C_4; \text{mortal}, \neg \text{mortal}) = \neg \text{man}$



Resolution (*example*)

$C_1 = \neg \text{man} \vee \text{mortal} = \text{man} \supset \text{mortal}$

$C_2 = \neg \text{socrate} \vee \text{man} = \text{socrate} \supset \text{man}$

$C_3 = \text{socrate}, C_4 = \neg \text{mortal},$

$C = \text{res}(C_3, C_2; \text{socrate}, \neg \text{socrate}) = \text{man}$

$C' = \text{res}(C_1, C_4; \text{mortal}, \neg \text{mortal}) = \neg \text{man}$

- The resolution is more general than the *Modus Ponens* ($A, A \supset B / B$) and the *Modus Tolens* ($\neg B, A \supset B / \neg A$)



Generality of the Resolution

- To prove $S \vdash C$, it is sufficient to prove that $S \cup \{\neg C\}$ is contradictory, i.e. that
$$S \cup \{\neg C\} \models \square$$
 - denotes the empty clause, i.e. the false
- **Theorem:** $S \vdash C$ if and only if it is possible to derive the empty clause by iterative application of the resolution rule on $S \cup \{\neg C\}$



Example

- Consider $S = \{C_1, C_2, C_3\}$

$C_1 = \neg \text{man} \vee \text{mortal} = \text{man} \supset \text{mortal}$

$C_2 = \neg \text{socrate} \vee \text{man} = \text{socrate} \supset \text{man}$

$C_3 = \text{socrate}, C_4 = \neg \text{mortal},$

- To prove $S \models \text{mortel}$, it is sufficient to derive the empty clause, i.e. \square , from $S \cup \{C_4\}$

$C_5: \neg \text{man} \quad \text{res}(C_1, C_4; \text{mortal}, \neg \text{mortal})$

$C_6: \neg \text{socrate} \quad \text{res}(C_2, C_5; \text{man}, \neg \text{man})$

$C_5: \square \quad \text{res}(C_3, C_6; \text{socrate}, \neg \text{socrate})$

QED



Tableau Method in Propositional Logic

Step 1: normalization – transformation into **NNF** –
Negative Normal Form

The negations occurs only before atomic propositions

Push negations inwards the formulas using de Morgan laws and the suppression of the double negation

$$\neg\neg\varphi \equiv \varphi$$

$$\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$$

$$\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$$



Tableau Method in Propositional Logic (following)

Step 2: build a tableau

Root: the formula under *NNF – Negative Normal Form*

Build successors of T using two rules R_\wedge et R_\vee .

We stop when we are unable to apply any rule

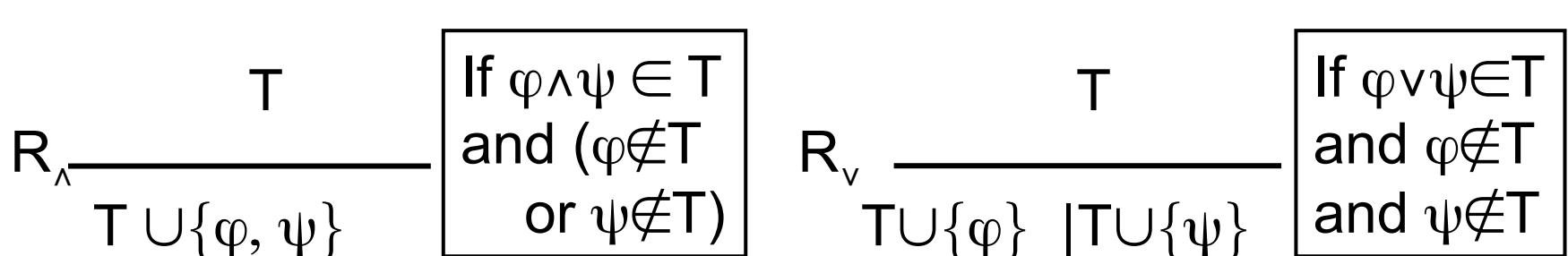


Tableau Method

Definitions:

- **Contradictory tableau**: it contains simultaneously p and $\neg p$ (it is said that it contains a *clash*)
- **Complete tableau**: no rule can be applied
- A tableau is said to be ***closed*** if it contains a clash, ***open*** else

On applies systematically the rules on all the tableau.

The answer is said to be “satisfiable” if one of the generated tableau is open, “unsatisfiable” else.

$$R_\wedge \frac{T}{T \cup \{\varphi, \psi\}} \quad \boxed{\text{If } \varphi \wedge \psi \in T \text{ and } (\varphi \notin T \text{ or } \psi \notin T)}$$

$$R_\vee \frac{T}{T \cup \{\varphi\} \mid T \cup \{\psi\}} \quad \boxed{\text{If } \varphi \vee \psi \in T \text{ and } \varphi \notin T \text{ and } \psi \notin T}$$



Properties

1. There is finite sequence of applications of rules
$$\mathcal{S}_0 = T \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_2 \rightarrow \dots$$
2. If \mathcal{S}' is obtained from a finite set \mathcal{S} of tableau by applying transformation rules, then \mathcal{S}' is consistent iff \mathcal{S} is.
3. All clash tableau are inconsistent
4. All complete and open tableau is consistent



Third step

- T being a complete tableau

The model $M[T]$ that satisfies φ is built as follows:

If p , a proposition, belongs to T , then p is true in $M[T]$,

else p is false



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Example

$$\varphi = \neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

φ rewrite: $\varphi = (p \vee \neg q) \wedge (\neg r \vee q)$

$$(p \vee \neg q) \wedge (\neg r \vee q)$$

 R_\wedge

$$\begin{aligned} & (p \vee \neg q) \wedge (\neg r \vee q) \\ & (\neg r \vee q) \\ & (p \vee \neg q) \end{aligned}$$

Clash**Complete**

$$\begin{aligned} & (p \vee \neg q) \wedge (\neg r \vee q) \\ & (\neg r \vee q) \\ & (p \vee \neg q) \\ & \neg q \\ & q \end{aligned}$$

$$\begin{aligned} & (p \vee \neg q) \wedge (\neg r \vee q) \\ & (\neg r \vee q) \\ & (p \vee \neg q) \\ & \neg q \end{aligned}$$

 R_v R_v

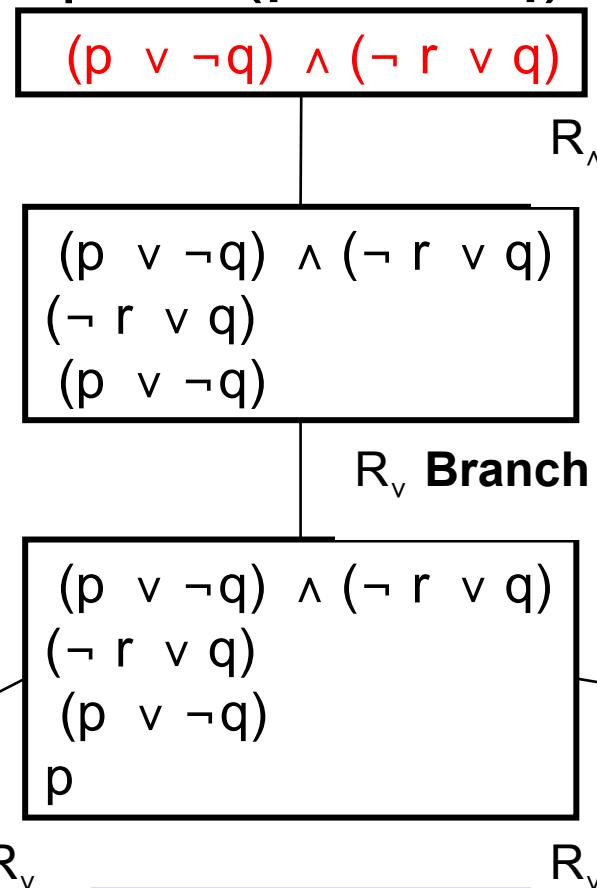
$$\begin{aligned} & (p \vee \neg q) \wedge (\neg r \vee q) \\ & (\neg r \vee q) \\ & (p \vee \neg q) \\ & \neg q \\ & \neg r \end{aligned}$$



Example - follows

$$\varphi = \neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

- Réécriture: $\varphi = (p \vee \neg q) \wedge (\neg r \vee q)$



Complete

Complete



Conclusion

$$\varphi = \neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

φ owns three models:

1. $\neg q, \neg r$
2. $p, \neg r$
3. p, q



Another application of the tableau method

$$F = \neg(v \wedge \neg\neg v) \wedge (\neg\neg v \vee v) \wedge (\neg\neg v \vee \neg(p \vee a))$$

Transformation – NNF:

$$F = (\neg v \vee \neg \neg \neg v) \wedge (\neg \neg v \vee v) \wedge \\ (\neg \neg v \vee (\neg p \wedge \neg a))$$

$$F = (\neg v \vee \neg \neg \neg v) \wedge (\neg \neg v \vee v) \wedge \\ (\neg \neg v \vee \neg p) \wedge (\neg \neg v \vee \neg a)$$



Tableau Method

$$(\neg v \vee \neg n_v) \wedge (n_v \vee v) \wedge (n_v \vee \neg p) \wedge (n_v \vee \neg a)$$
$$3 \times R_\wedge$$
$$(\neg v \vee \neg \neg n_v) \wedge (n_v \vee v) \wedge (n_v \vee \neg p) \wedge (n_v \vee \neg a)$$
$$(\neg v \vee \neg n_v)$$
$$(n_v \vee v)$$
$$(n_v \vee \neg p)$$
$$(n_v \vee \neg a)$$
$$R_v \text{ Branch 1}$$
$$\dots (\neg v \vee \neg \neg n_v)$$
$$(n_v \vee v)$$
$$(n_v \vee \neg p)$$
$$(n_v \vee \neg a)$$
$$n_v$$
$$R_v \text{ Branch 2}$$
$$\dots (\neg v \vee \neg \neg n_v)$$
$$(n_v \vee v)$$
$$(n_v \vee \neg p)$$
$$(n_v \vee \neg a)$$
$$\neg a$$


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Tableau Method – *left part*

Model 1

$n_v, \neg v$

... ($\neg v \vee \neg n_v$)
 $(n_v \vee v)$
 $(n_v \vee \neg p)$
 $(n_v \vee \neg a)$
 n_v

R_v Branch 1

... ($\neg v \vee \neg n_v$)
 $(n_v \vee v)$
 $(n_v \vee \neg p)$
 $(n_v \vee \neg a)$
 n_v
 $\neg v$

Complete

R_v Branch 2

... ($\neg v \vee \neg n_v$)
 $(n_v \vee v)$
 $(n_v \vee \neg p)$
 $(n_v \vee \neg a)$
 n_v
 $\neg n_v$

Clash



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Tableau Method – right part

Model 2

 $n_v, \neg v, \neg a$

... ($\neg v \vee \neg n_v$)
 $(n_v \vee v)$
 $(n_v \vee \neg p)$
 $(n_v \vee \neg a)$
 $\neg a$

 R_v Branch 1

... ($\neg v \vee \neg n_v$)
 $(n_v \vee v)$
 $(n_v \vee \neg p)$
 $(n_v \vee \neg a)$
 $\neg a$
 $\neg v$

 R_v Branch 2

... ($\neg v \vee \neg n_v$)
 $(n_v \vee v)$
 $(n_v \vee \neg p)$
 $(n_v \vee \neg a)$
 $\neg a$
 $\neg n_v$

 R_v Branch 1 R_v Branch 2 R_v Branch 1 R_v Branch 2

Complete

... $\neg a$
 $\neg v$
 v

Clash

... $\neg a$
 $\neg v$
 n_v

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... $\neg a$
 $\neg n_v$
 v

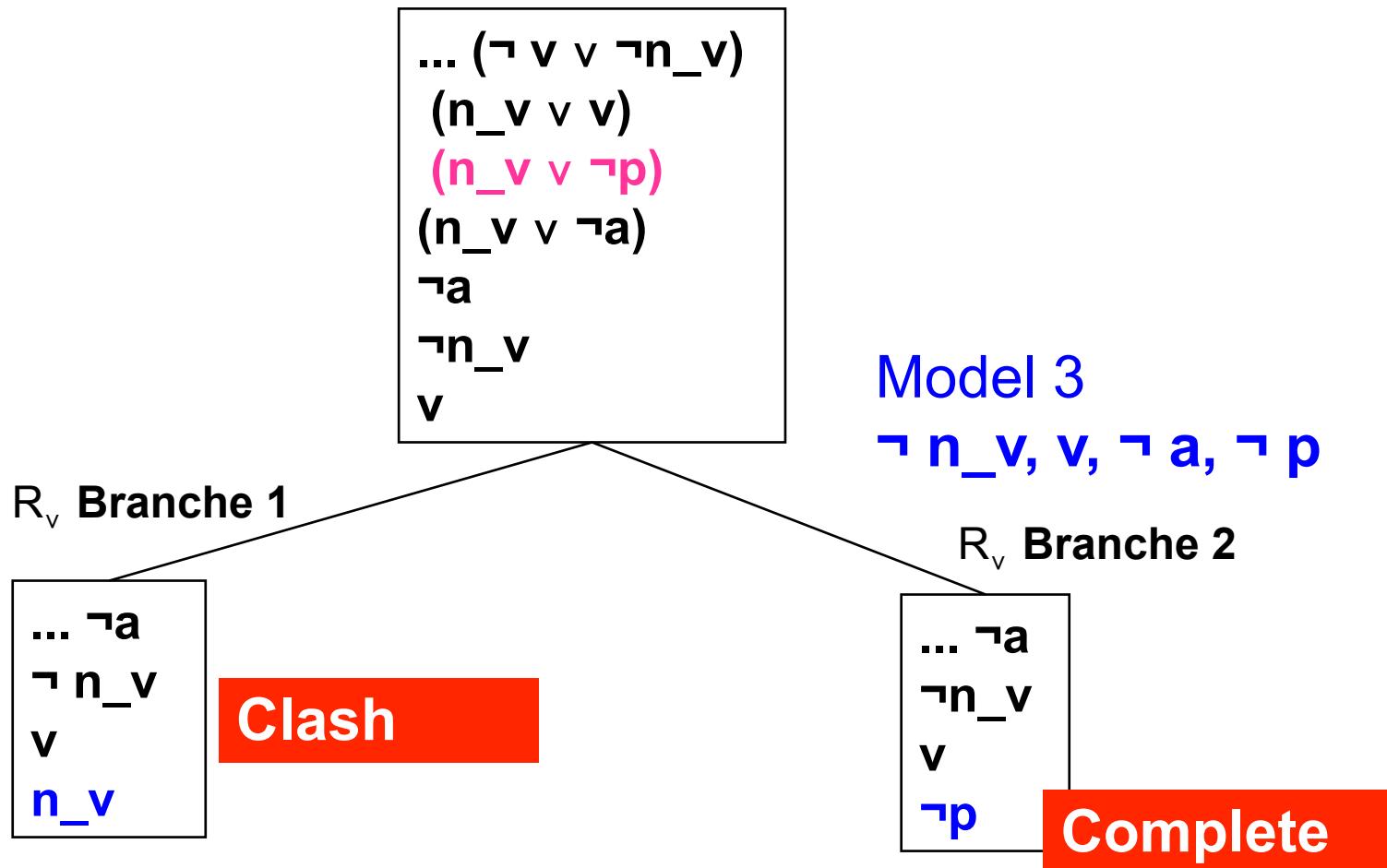
Marie

clash

... $\neg a$
 $\neg n_v$
 n_v

Marie

Following – *right*



Recap

- We obtain three models for F:

$$F = \neg(v \wedge \neg\neg v) \wedge (\neg\neg v \vee v) \wedge \\ (\neg\neg v \vee \neg(p \vee a))$$

Model 1: $\underline{\neg\neg v}$, $\neg v$

Model 2: $\underline{\neg\neg v}$, $\neg v$, $\neg a$

Model 3: $\neg \underline{\neg\neg v}$, v , $\neg a$, $\neg p$



Tableau method in propositional logic

Generalization

Goal: search a model for a set of formulas \mathcal{F}

- **Initialisation:** create a non marked root node initialized with \mathcal{F}
- **Iterative decomposition:** choose iteratively all the non treated nodes (set of formulae with \mathcal{F}) and mark them once treated
 - If the node contained complementary literals, mark the node as closed (clash)
 - Else, if all the formulae associated to the node are propositional variables (i.e. atom or atom negation), mark it as opened
 - Else, choose a non propositional formula F belonging to the current node
 - If F is α -type: create one new node non marked with the set $\mathcal{F} - \{F\} \cup \{\alpha_1, \alpha_2\}$
 - If F is β -type: create two new non marked nodes with the sets $\mathcal{F} - \{F\} \cup \{\beta_1\}$ and $\mathcal{F} - \{F\} \cup \{\beta_2\}$



Decomposition rules

Formula α	α_1	α_2
$\neg\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$

Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$





$$\varphi = \neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

Example

Formula α	α_1	α_2
$\neg\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$

$$\neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

Type α

$$\begin{array}{l} \neg(\neg p \wedge q) \\ \neg(r \wedge \neg q) \end{array}$$

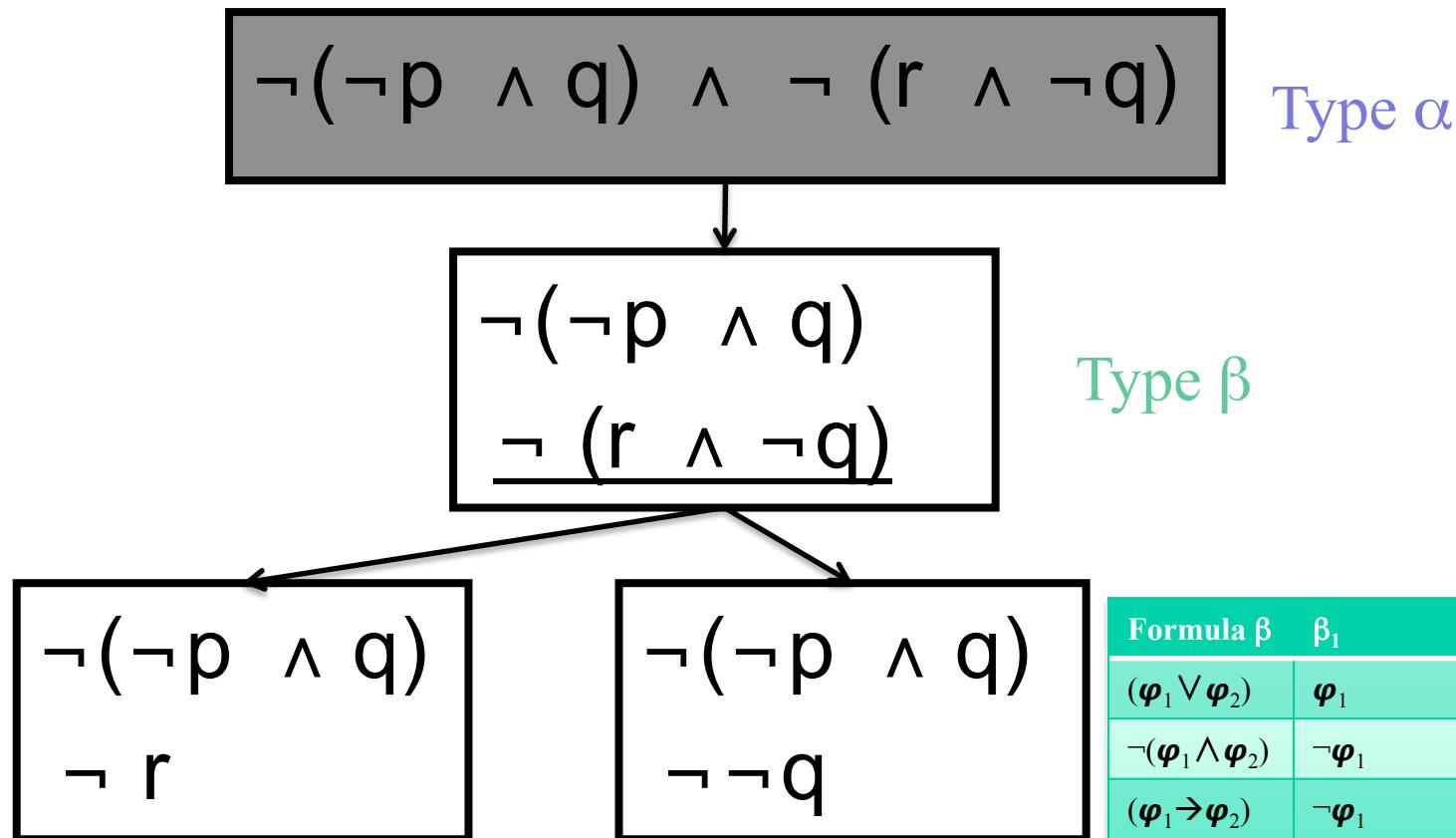
Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$



$$\varphi = \neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

Example

Formula α	α_1	α_2
$\neg\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$



Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$

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$$\neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

$$\neg(\neg p \wedge q) \\ \underline{\neg(r \wedge \neg q)}$$

treated

$$\neg(\neg p \wedge q) \\ \neg r$$

treated

$$\neg(\neg p \wedge q) \\ \neg \neg q$$

treated

$$\neg \neg p \\ r$$

treated

$$(\neg p \wedge q) \\ \neg q \\ q$$

Clash

Formula α	α_1	α_2
$\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$

Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$

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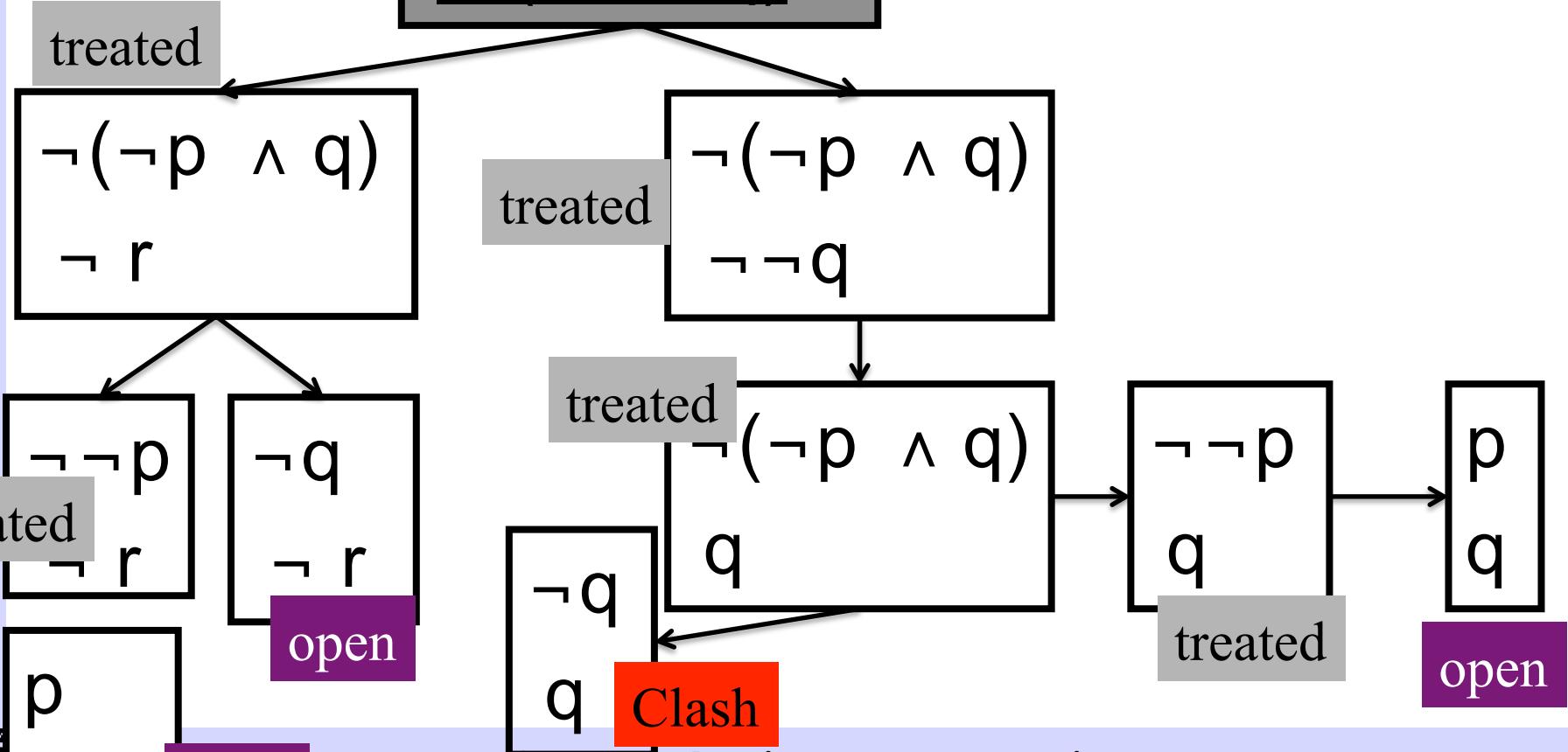
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$$\neg(\neg p \wedge q) \wedge \neg(r \wedge \neg q)$$

$$\neg(\neg p \wedge q) \\ \underline{\neg(r \wedge \neg q)}$$

3 models:

1. p, q
2. $\neg q, \neg r$
3. $p, \neg r$



Another example

$$\neg(v \wedge \neg v) \wedge (\neg v \vee v) \wedge (\neg v \vee \neg(p \vee a))$$


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$$\neg(v \wedge \neg v) \wedge (\neg v \vee v) \wedge (\neg v \vee \neg(p \vee a))$$

treated

treated

$$\begin{aligned} &\neg(v \wedge \neg v) \\ &(\neg v \vee v) \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

treated

$$\begin{aligned} &\neg v \\ &(\neg v \vee v) \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

treated

$$\begin{aligned} &\neg \neg v \\ &(v \vee v) \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

$$\begin{aligned} &\neg v \\ &\neg v \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

$$\begin{aligned} &\neg v \\ &v \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

$$\begin{aligned} &\neg \neg v \\ &v \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

Formula α	α_1	α_2
$\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$

Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$

$$\begin{aligned} &\neg \neg v \\ &\neg v \\ &(\neg v \vee \neg(p \vee a)) \end{aligned}$$

Clash



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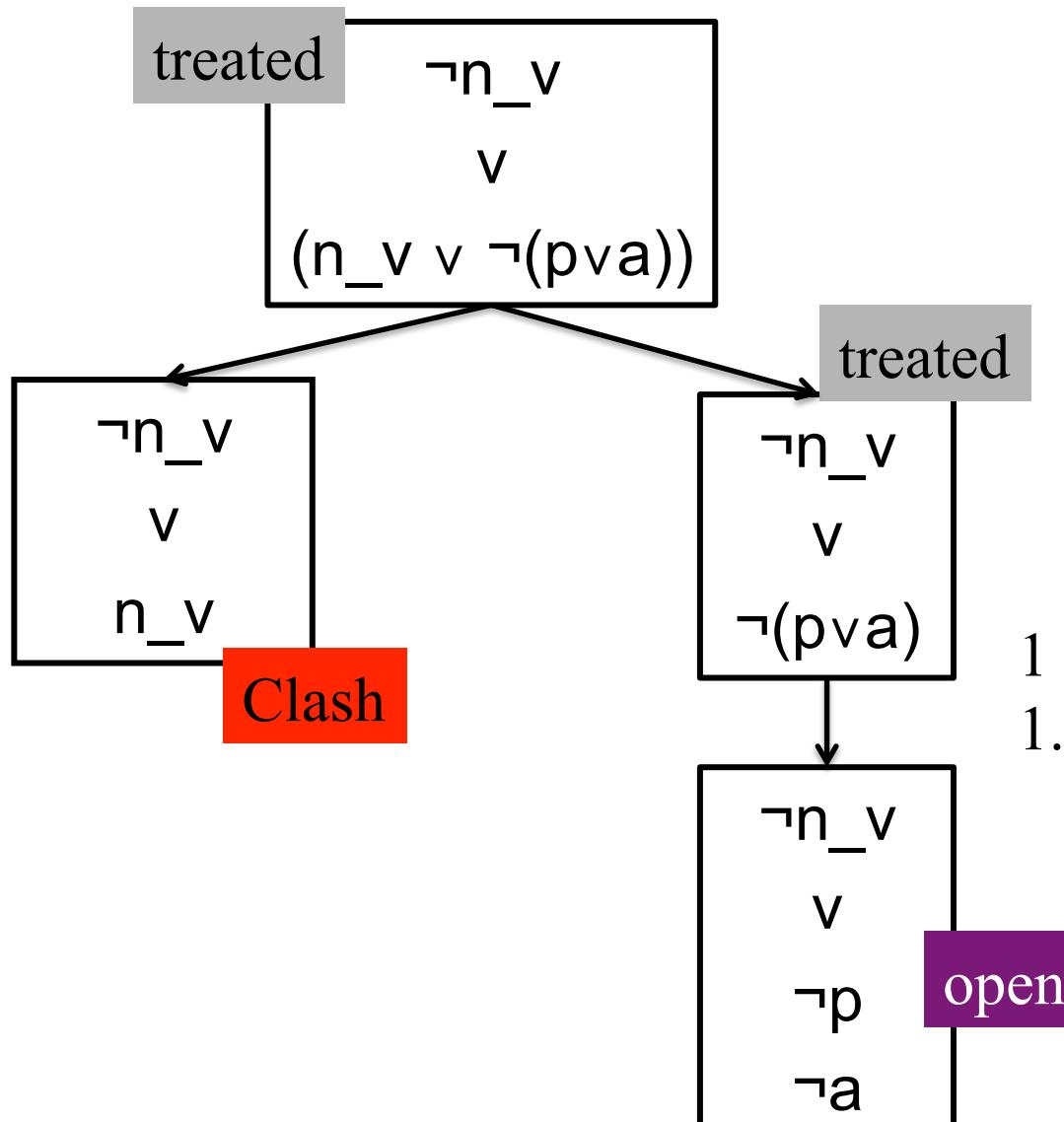
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Right node



Formula α	α_1	α_2
$\neg\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$

Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$

1 model:

1. $\neg n_v, v, \neg p, \neg a$



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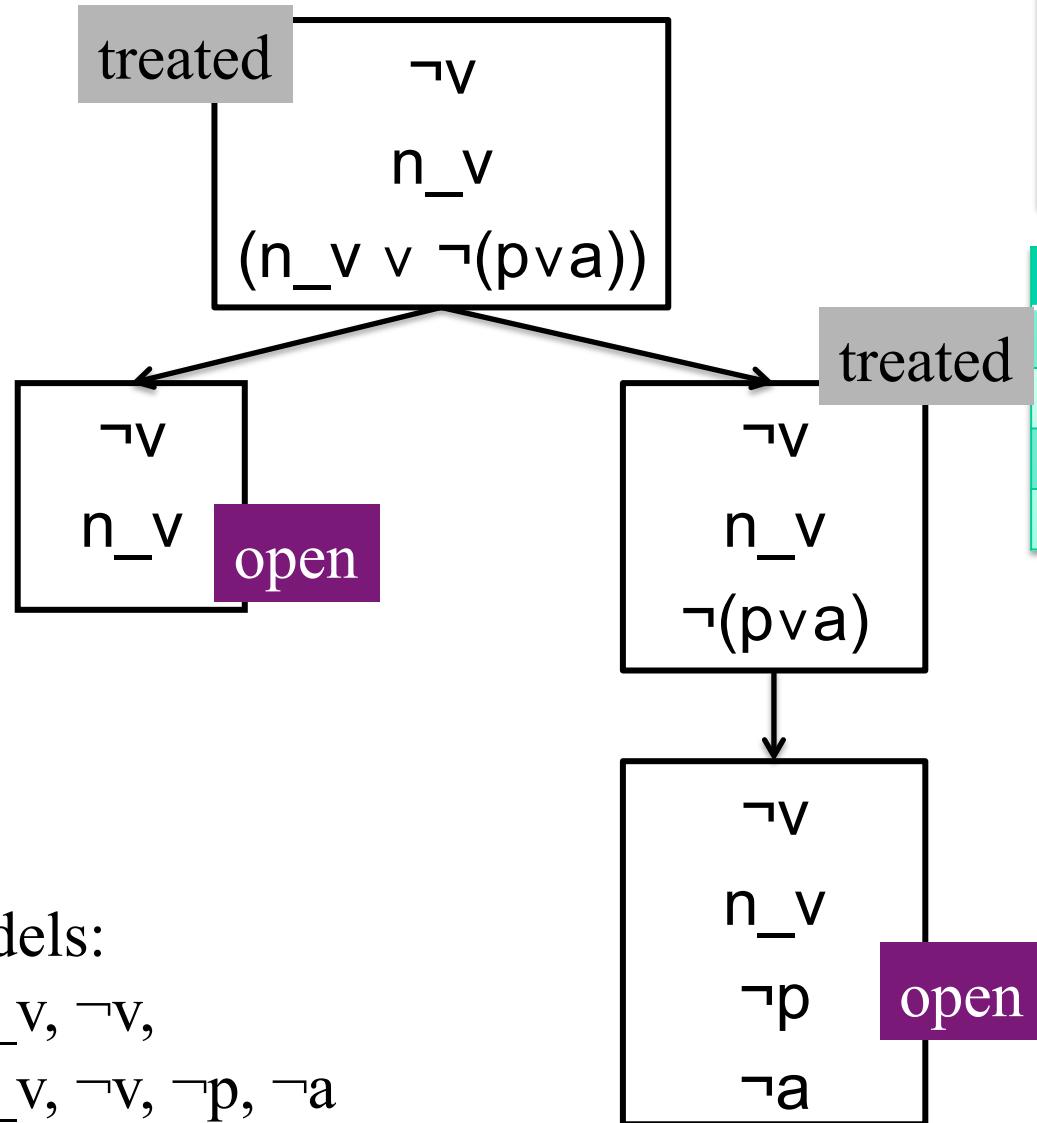
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Left node



Formula α	α_1	α_2
$\neg\neg\varphi$	φ	
$\varphi_1 \wedge \varphi_2$	φ_1	φ_2
$\neg(\varphi_1 \vee \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \rightarrow \varphi_2)$	φ_1	$\neg\varphi_2$
$\varphi_1 \leftrightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$

Formula β	β_1	β_2
$(\varphi_1 \vee \varphi_2)$	φ_1	φ_2
$\neg(\varphi_1 \wedge \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$(\varphi_1 \rightarrow \varphi_2)$	$\neg\varphi_1$	$\neg\varphi_2$
$\neg(\varphi_1 \leftrightarrow \varphi_2)$	$\neg(\varphi_1 \rightarrow \varphi_2)$	$\neg(\varphi_2 \rightarrow \varphi_1)$

2 models:

1. $n_v, \neg v,$
2. $n_v, \neg v, \neg p, \neg a$

