Finding Similar Sets

Vassilis Christophides
christop@csd.uoc.gr
http://www.csd.uoc.gr/~hy562
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Motivation

- Many Web-mining problems can be expressed as finding "similar" sets:
  - Pages with similar words, e.g., for classification by topic
  - NetFlix users with similar tastes in movies for recommendation systems
    - Dual: movies with similar sets of fans
  - Images of related things
- The best techniques depend on whether you are looking for items that are very similar or only somewhat similar
  - Special cases are easy, e.g., identical documents, or one document contained character-by-character in another
  - General case, where many small pieces of one document appear out of order in another, is very hard

Comparing Documents for Near Duplicates

- Applications: Given a body of documents, find pairs of documents with a lot of text in common, e.g.:
  - Mirror Web sites, or approximate mirrors
    - Application: Don't want to show both in a search
  - Plagiarism, including large quotations
  - Similar news articles at many news sites
    - Application: Cluster articles by "same story"

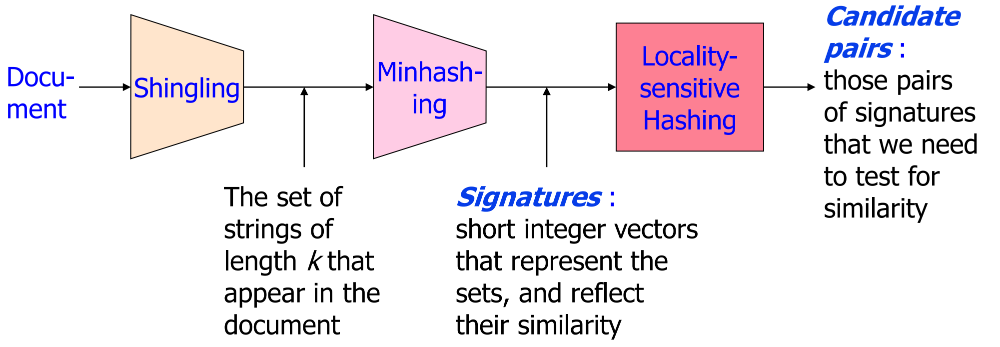
- Simple IR approaches are not suited:
  - Document = set of words appearing in document
  - Document = set of "important" words

  Why? we need to account for ordering of words!

Main Issues

- What is the right representation of the document when we check for similarity?
  - E.g., representing a document as a set of characters will not do (why?)
- When we have billions of documents, keeping the full text in memory is not an option
  - We need to find a shorter representation
- How do we do pairwise comparisons of billions of documents?
  - If exact match was the issue it would be ok, can we replicate this idea?

Three Essential Techniques for Detecting Similar Documents



- Shingling: convert documents, emails, etc., to *sets*
- Minhashing: convert *large sets to short signatures*, while preserving similarity
- Locality-sensitive hashing: focus on *pairs of signatures likely to be similar*

Shingles

- A k -shingle (or k -gram) for a document is a sequence of k characters that appears in the document
  - Represent a document by its set of k-shingles
- Example: k=2; doc= abcab. Set of 2-shingles = {ab, bc, ca}
  - Option: regard shingles as a bag (multiset), and count ab twice
- Working Assumption: Documents that have lots of shingles in common have similar text, even if the text appears in different order
  - What if two documents differ by a word?
    - Affects only k-shingles within distance k from the word
  - What if we reorder paragraphs?
    - Affects only the 2k shingles that cross paragraph boundaries

Shingle Size

- Is k=2 a good choice for size?
- Example: k=2;
  - `doc1 = abcab. 2-shingles = {ab, bc, ca}`
  - `doc2 = cabc. 2-shingles = {ab, bc, ca}`
- Careful decision: you must pick k large enough, or most documents will have most shingles
  - k = 5 is OK for short documents
  - k = 10 is better for long documents

Shingles: Compression Option

- How about space overhead?
  - Each character can be represented as a byte
  - k-shingle requires k bytes
- If k=9, to compare shingles we need to compare 9 bytes
- To improve efficiency, we can compress long shingles:
  - hash them to (say) 4 bytes, and
  - represent a document by the set of hash values of its k-shingles

  `(aaabbbccc)(abcabcabc)` → `h(aaabbbccc)h(abcabcabc)`

  18 bytes → 8 bytes
- Working Assumption: Two documents could (rarely) appear to have shingles in common, when in fact only the hash-values were shared

Thought Question

- Why is it better to hash 9-shingles (say) to 4 bytes than to use 4-shingles?
  - There are many more possible shingles, this reduces the likelihood that documents that share many shingles are not similar

- Hint: How random are the 32-bit sequences that result from 4-shingling?
  - Assuming 20 characters are common in English, there are $(20)^4$ = 160000 4-shingles < 2^{32}
  - Using 9-shingles there are $(20)^9 >> 2^{32}$

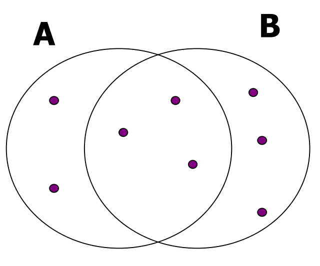
MinHashing

Basic Data Model: Sets

- Many similarity problems can be couched as finding subsets of some universal set that have significant intersection
- Examples include:
  - Documents represented by their sets of shingles (or hashes of those shingles): $C_i=S(D_i)$
  - Similar customers or products
- Equivalently, each document is a 0/1 vector in the space of k-shingles
  - Each unique shingle is a dimension
  - Vectors are very sparse
- Interpret set intersection as bitwise AND, and set union as bitwise OR

Jaccard Similarity of Sets

- The *Jaccard similarity* of two sets is the size of their intersection divided by the size of their union
  - $Sim\ (C_1,\ C_2) = |C_1 \cap C_2| / |C_1 \cup C_2|$
- Example:



3 in intersection
8 in union
Jaccard similarity = 3/8

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Motivation for Min-Hash

- Suppose we need to find near-duplicate documents among $N=1$ million (10^6) documents
- Naïvely, we would have to compute pairwise Jaccard similarities for every pair of docs
  - $N(N-1)/2 \approx 5*10^{11}$ comparisons
  - At $\approx 10^5$ secs/day and 10^6 comparisons/sec, it would take 5 days
- For $N = 10$ million (10^7), it takes more than a year…

From Sets to Boolean Matrices

- Rows = elements (shingles) of the universal set
- Columns = sets (documents)
  - 1 in row *e* and column *S* if and only if *e* is a member of *S*
  - Column similarity is the Jaccard similarity of the sets of their rows with 1
- Typical matrix is sparse (most rows are of type d, see later)
  - Sparse matrices are usually better represented by the list of places where there is a non-zero value
  - But the boolean matrix picture is conceptually useful

Documents

Shingles				
	1	1	1	0
	1	1	0	1
	0	1	0	1
	0	0	0	1
	1	0	0	1
	1	1	1	0
	1	0	1	0

Example: Jaccard Similarity of Columns

	C_1	C_2		
a	0	1		*
b	1	0		*
c	1	1	*	*
d	0	0		
e	1	1	*	*
f	0	1		*

Sim (C_1, C_2) = **2/5 = 0.4**

$d(C_1,C_2)$ =1– (Jaccard similarity) = **0.6**

Outline: Finding Similar Columns

- Naïve approach:
  - ❶ Compute signatures of columns = small summaries of columns
  - ❷ Examine pairs of signatures to find similar columns
    - Requirement: similarities of signatures and columns are related
  - ❸ Optional: check that columns with similar signatures are really similar
- This scheme works but …
  - What if the set of signatures (or k-shingles) is too large to fit in the memory?
  - Or the number of documents are too large?
- Idea: Find a way to hash a document (column) to a single (small size) value! and similar documents to the same value!
  - Warning: These methods can produce *false negatives*, and even *false positives* (if the above optional check is not made)

Signatures

- Key idea: "hash" h(·) each column C to a small signature, such that:
  - ❶ $h(C)$ is small enough that we can fit a signature in main memory for each column
  - ❷ $Sim(C_1, C_2)$ is the same as the "similarity" of $h(C_1)$ and $h(C_2)$
- By hashing columns into buckets we expect that "most" pairs of near duplicate documents hash into the same bucket!
- *Goal: Find a hash function h(·) such that:*
  - If $sim(C_1,C_2)$ is high, then with high probability $h(C_1) = h(C_2)$
  - If $sim(C_1,C_2)$ is low, then with high probability $h(C_1) \neq h(C_2)$
- Clearly, the hash function depends on the similarity metric:
  - Not all similarity metrics have a suitable hash function!
  - There is a suitable hash function for the Jaccard similarity:
    - It is called Min-Hashing!

Minhashing

- History: invented by Andrei Broder in 1997 (AltaVista) to detect near duplicate web pages
- Imagine the rows of the Boolean matrix permuted under random permutation **π**
- Define a "hash" function $h_\pi(C)$ = the index of the **first** (in the permuted order **π**) row in which column C has value **1**:
  - $h_\pi(C) = min_\pi\, \pi(C)$

Min-hashing Example

2nd element of the permutation
is the first to map to a 1 in col 1

Permutations

1	**4**	3
3	**2**	4
7	**1**	7
6	**3**	6
2	**6**	1
5	**7**	2
4	**5**	5

Input matrix

1	**0**	**1**	**0**
1	**0**	**0**	**1**
0	**1**	**0**	**1**
0	**1**	**0**	**1**
0	**1**	**0**	**1**
1	**0**	**1**	**0**
1	**0**	**1**	**0**

Signature matrix *M*

2	**1**	**2**	**1**
2	**1**	**4**	**1**
1	**2**	**1**	**2**

$h_2(3)=4$ (permutation 2, column 3)

4th element of the permutation
is the first to map to a 1 in col 3

Surprising Property

- The probability (over all permutations of the rows) that $h(C_1)=h(C_2)$ is the same as $Sim(C_1, C_2)$:
  - $Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1,C_2)$
- With multiple signatures we get a good approximation
- Use several independent hash functions to create a signature of a column
  - The similarity of signatures is the fraction of the hash functions in which they agree
  - Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures

Why?

0	**0**
0	0
1	**1**
0	0
0	1
1	0

- Let ***X*** be a column (set of shingles), $y \in X$ is a shingle
- **Then:** $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$
  - It is equally likely that any shingle $y \in X$ is mapped to the ***min*** element
- Let y be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$
- **Then either:** $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, **or**
  $\pi(y) = \min(\pi(C_2))$ if $y \in C_2$
- So the prob. that **both** are true is the prob. $y \in C_1 \cap C_2$
- $\Pr[\min(\pi(C_1))=\min(\pi(C_2))]=|C_1 \cap C_2|/|C_1 \cup C_2|= sim(C_1,C_2)$

One of the two cols had to have 1 at position y

Four Types of Rows

- Given columns C_1 and C_2, rows may be classified as:

	C_1	C_2	
a	1	1	*1 in both columns*
b	1	0	*columns are different*
c	0	1	
d	0	0	*0 in both columns*

- Also, *a* = # rows of type *a* , etc.
- The ratio of type a, b, and c that determine the similarity and the probability that $h(C_1) = h(C_2)$
  - Note $Sim(C_1,C_2) = a/(a+b+c)$
  - **Then:** $\Pr[h(C_1)=h(C_2)] = Sim(C_1,C_2)$
- Look down the permuted columns C_1 and C_2 until we see a 1
  - If it's a type-a row, then $h(C_1)=h(C_2)$
  - If a type-b or type-c row, then not

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Min Hashing – Example

Input matrix

1	4	3	1	0	1	0
3	2	4	1	0	0	1
7	1	7	0	1	0	1
6	3	6	0	1	0	1
2	6	1	0	1	0	1
5	7	2	1	0	1	0
4	5	5	1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

Similarities:

	1-3	2-4	1-2	3-4
Col/Col	0.75	0.75	0	0
Sig/Sig	0.67	1.00	0	0

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MinHash – False Positive/Negative

- Instead of comparing sets, we now compare only 1 bit!
- False positive?
  - False positive can be easily dealt with by doing an additional layer of checking (treat minhash as a filtering mechanism)
- False negative?
  - Requiring full match of signature is strict, some similar sets will be lost
- High error rate! Can we do better?

Minhash Signatures

- Pick (say) 100 random permutations of the rows
- Think of *Sig(C)* as a column vector
- Let ***Sig(C)[i] = min(π$_i$(C))***
  according to the *i* th permutation, the number of the first row that has a 1 in column *C*
- Note: The sketch (signature) of column C is small **~400** bytes!
  - We achieved our goal! We "compressed" long bit vectors into short signatures

Implementation Trick

- Permuting rows even once is prohibitive
  - Suppose 1 billion rows
  - Hard to pick a random permutation from 1…billion
    - Sorting would take a long time
    - Representing a random permutation requires 1 billion entries
- A good approximation to permuting rows: pick 100 (?) hash functions h_i
  - Simulate the effect of a random permutation by a random hash function that maps row numbers to as many buckets as there are rows
  - Row hashing: ordering under h_i gives a random row permutation!
- One-pass implementation
  - For each column C and each hash function h_i, keep a "slot" M(i,C) for the min-hash value
  - Intent: M(i,C) will become the smallest value of h_i(r) for which column C has 1 in row r
    - i.e., h_i(r) gives order of rows for i th permutation

Implementation

```
M(i,C) = ∞
for each row r
    for each column c
      if c has 1 in row r // Scan rows looking for 1s
        for each hash function h_i do
          // Suppose row r has 1 in column C
          if h_i(r) is a smaller value than M(i,C) then
              M(i,C):= h_i(r);
```

How to pick a random hash function h(x)?
Universal hashing:

$h_{a,b}(x)=((a \cdot x+b) \bmod p) \bmod N$ where:

a,b … random integers

p … prime number (p > N)

Example

Row	C1	C2
1	1	0
2	0	1
3	1	1
4	1	0
5	0	1

Jaccard=1/5

$h(x) = x \bmod 5$
$g(x) = 2x+1 \bmod 5$

	Sig1	Sig2
$h(1) = 1$	1	∞
$g(1) = 3$	3	∞
$h(2) = 2$	1	2
$g(2) = 0$	3	0
$h(3) = 3$	1	2
$g(3) = 2$	2	0
$h(4) = 4$	1	2
$g(4) = 4$	2	0
$h(5) = 0$	1	0
$g(5) = 1$	2	0

M(i,C)

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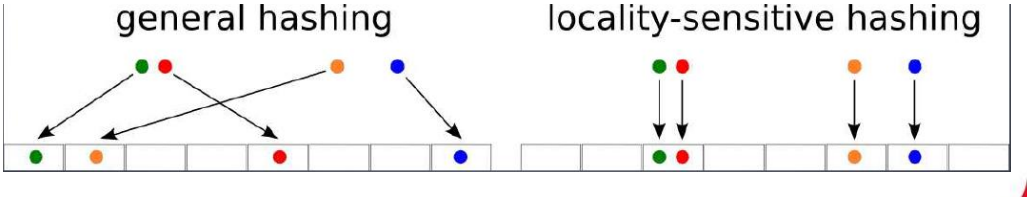
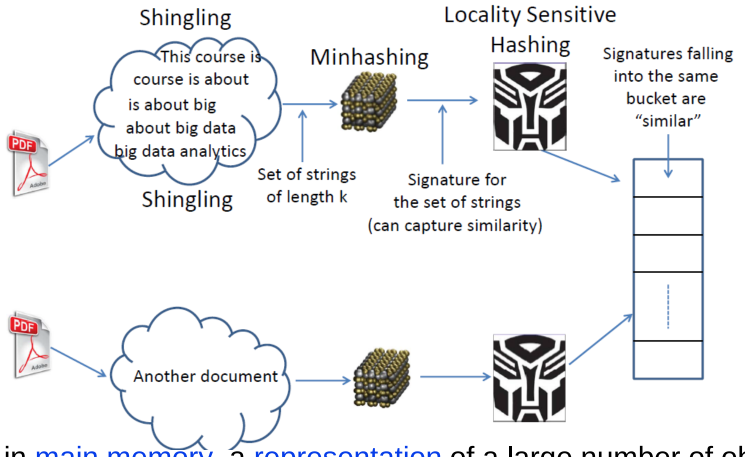
So far …

- Represent a document as a set of hash values (of its k-shingles)
- Transform set of k-shingles to a set of minhash signatures
- Use Jaccard to compare two documents by comparing their signatures
- Is this method (i.e., transforming sets to signature) necessarily “better”??

Locality-Sensitive Hashing



The BIG Picture (All-pair comparison)



- Suppose, in main memory, a representation of a large number of objects
  - May be signatures of documents as in minhashing
- We want to pair-wise compare each for finding those pairs that are sufficiently similar

Finding Similar Pairs

- While the signatures of all columns may fit in main memory, comparing the signatures of all pairs of columns is quadratic in the number of columns
- Naïve solution
  - For each document, compare with the other N-1 documents
    - Takes N-1 comparisons
    - Can be optimized using *filter-and-refine* mechanisms
  - Requires N*(N-1)/2 comparisons
- Example:
  - 10^7 columns implies ~ 10^{14} column-comparisons
  - At 1 μs/comparison 10^8 (~ 3 years!)

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Locality-Sensitive Hashing

- Use a function `f(x,y)` that tells whether or not x and y is a candidate pair: a pair of elements whose similarity must be evaluated

- With only one hash function on one entire column of signature, likely to have many *false negatives*

- Key idea: Apply the hash function on the columns of signature matrix M multiple times, each on a partition of the column
  - Arrange that (only) similar columns are likely to hash (i.e., with high probability) to the same bucket
  - Each pair of columns that hashes at least once into the same bucket is a candidate pair

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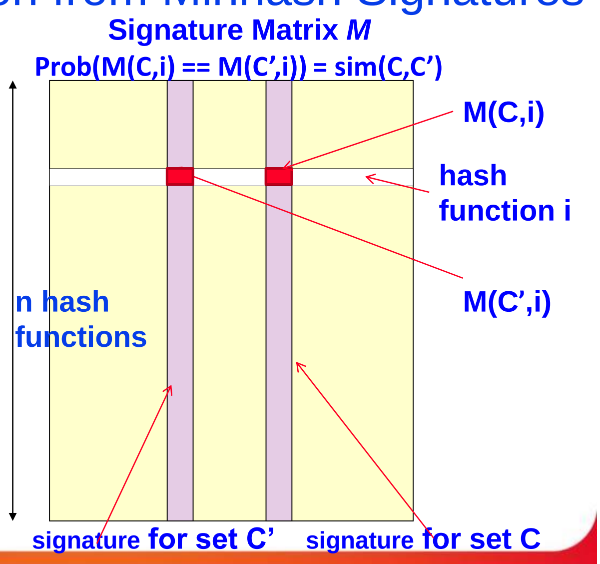
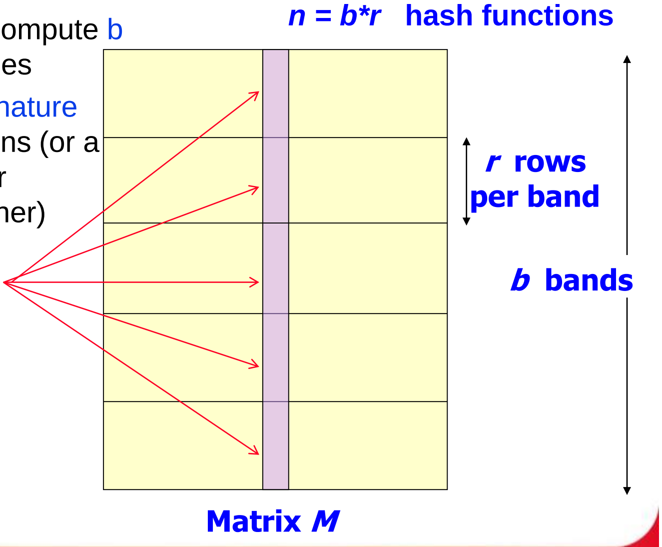
Candidate Generation from Minhash Signatures

- Pick a similarity threshold s, a fraction $0 < s < 1$
- A pair of columns x and y is a candidate pair if their signatures agree in at least fraction s of the rows
  - i.e., `M(i,x) = M(i,y)` for at least fraction s values of i
  - we expect documents x and y to have the same (Jaccard) similarity as their signatures



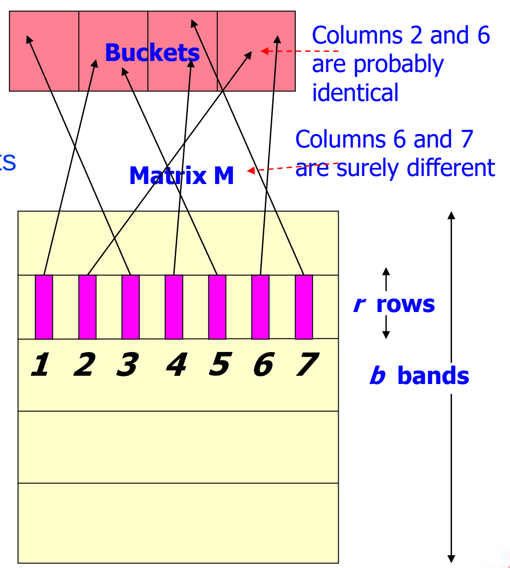
Partition Into Bands

- Divide matrix M into b bands of r rows
  - For each document, compute b sets of r minhash values
  - Each set is a mini-signature with r minhash functions (or a concatenation of the r minhash values together)



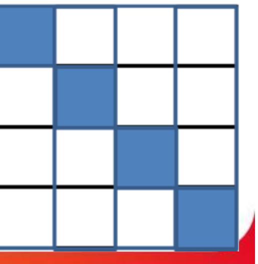
Partition into Bands

- For each band, hash its portion of each column (i.e., the concatenated values) to a hash table with k buckets
  - this has the "same" effect as ensuring all columns have the same values
  - make k as large as possible to minimize collision
- Candidate column pairs are those that hash to the same bucket for ≥ 1 band
- Tune b and r to catch *most similar pairs*, but few *non-similar pairs*



Simplifying Assumption

- There are enough buckets that columns are unlikely to hash to the same bucket unless they are *identical* in a particular band
  - Hereafter, we assume that "same bucket" means "identical in that band"
  - Assumption needed only to simplify analysis, not for correctness of algorithm
- Finding all pairs within a bucket become computationally cheaper!
  - Declare all pairs within a bucket to be "matching" OR
  - Perform pair-wise comparisons for those documents that fall into the same bucket
    - Much smaller than pair-wise over all documents



Example: Effect of Bands

- Suppose 10^5 columns of M (100k docs)
- Signatures of 10^2 integers (rows)
- If each signature is represented as a 4 byte integer value, we need only 10^2*4*10^5 = 40Mb of memory!
- $5*10^9$ pairs of signatures can take a while to compare
- Choose 20 bands of 5 integers/band
- Goal: Find pairs of documents that are *at least s = 0.8 similar*

Suppose C_1, C_2 are 80% Similar

- Find pairs of $\geq s$=0.8 similarity, set b=20, r=5
- Assume: $\mathrm{sim}(C_1, C_2) = 0.8$
  - Since $\mathrm{sim}(C_1, C_2) \geq s$, we want C_1, C_2 to be a candidate pair
  - We want them to hash to at least 1 common bucket (at least one band is identical)

- Probability C_1, C_2 identical in one particular band: $(0.8)^5 = 0.328$
- Probability C_1, C_2 are *not* similar in all of the 20 bands: $(1-0.328)^{20} = 0.00035$
  - i.e., about 1/3000th of the 80%-similar column pairs are false negatives (we miss them)

- We would find 99.965% pairs of truly similar documents

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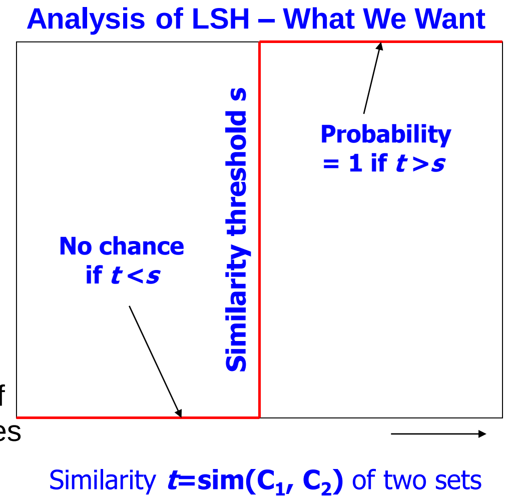
Suppose C_1, C_2 are 30% Similar

- Find pairs of $\geq s$=0.8 similarity, set b=20, r=5
- Assume: $\mathrm{sim}(C_1,C_2) = 0.3$
  - Since $\mathrm{sim}(C_1, C_2) < s$ we want C_1, C_2 to hash to NO common buckets (all bands should be different)

- Probability C_1, C_2 identical in one particular band: $(0.3)^5 = 0.00243$
  - Probability C_1, C_2 identical in at least 1 of 20 bands: $1-(1-0.00243)^{20} = 0.0474$
  - In other words, approximately 4.74% pairs of docs with similarity 0.3 end up becoming candidate pairs
  - They are false positives since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold s

LSH Involves a Tradeoff

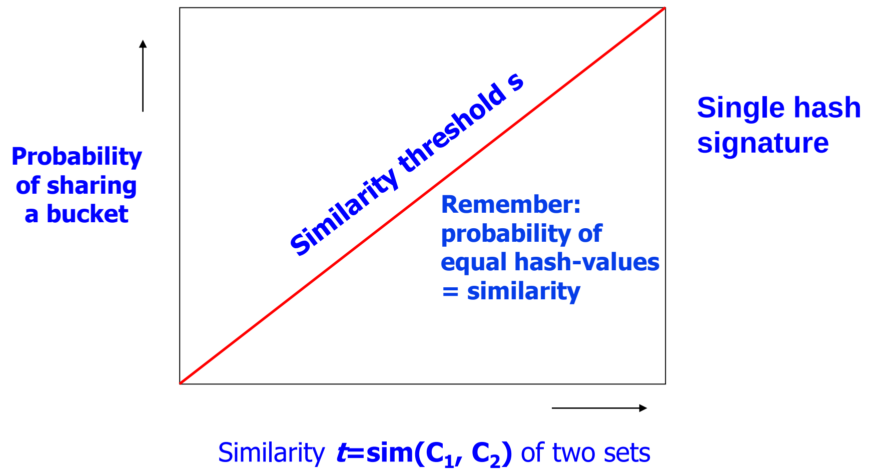
Probability of sharing a bucket



- How to get a step-function?
- Pick:
  - The number of Min-Hashes (rows of *M*)
  - The number of bands *b*, and
  - The number of rows *r* per band

  to balance false positives/negatives
- Example: if we had only 20 bands of 5 rows, the number of false negatives would go down, but the number of false positives would go up

What One Band Gives You

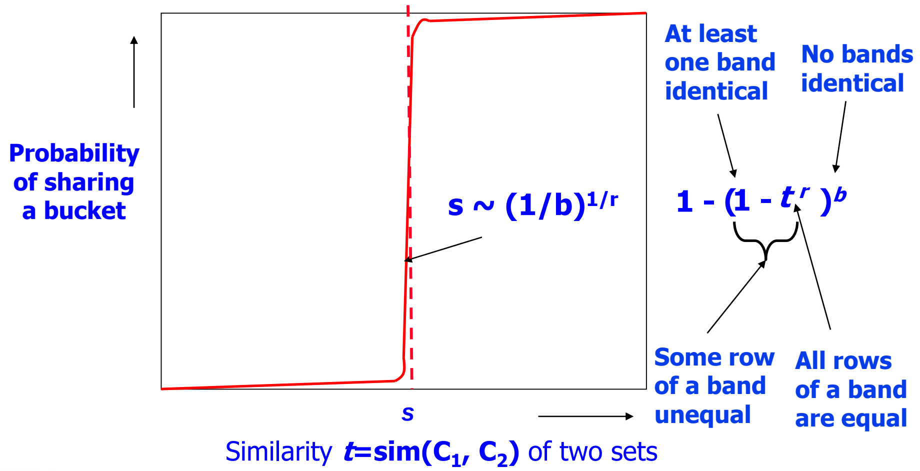


- This is what 1 hash-code gives you

  $\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$

What b Bands of r Rows Gives You

- The S-curve is where the "magic" happens



Example: $b = 20$; $r = 5$

t	$1-(1-t^r)^b$
.2	.006
.3	.047
.4	.186
.5	.470
.6	.802
.7	.975
.8	.9996

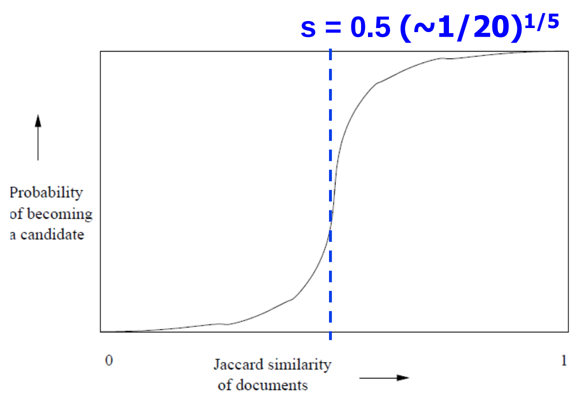
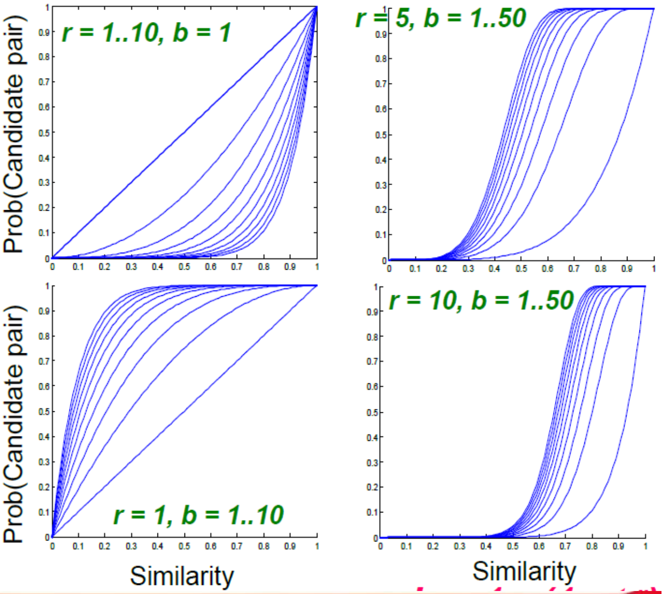


Figure 3.7: The S-curve

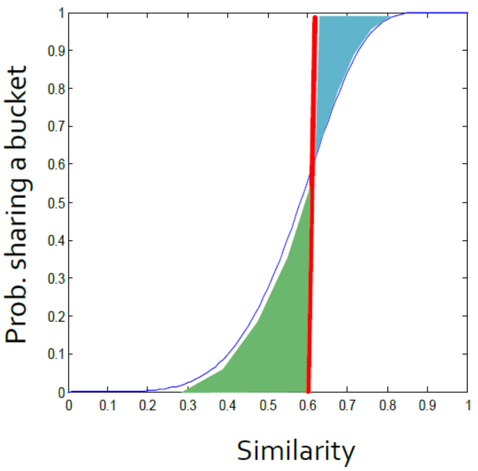
S-curves as a Function of *b* and *r*

- Given a fixed threshold `s`
- We want choose `r` and `b` such that the `Pr(Candidate pair)` has a "step" right around `s`



Picking *r* and *b*: The S-Curve

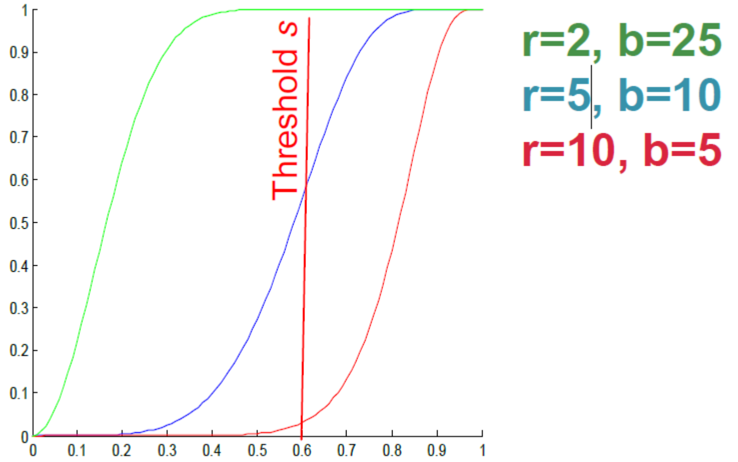
- Picking r and b to get the best S-curve



Blue area: *False Negative* rate
These are pairs with `sim > s` but the X fraction won't share a band and then will never become candidates. This means we will never consider these pairs for (slow/exact) similarity calculation!
Green area: *False Positive* rate
These are pairs with `sim < s` but we will consider them as candidates. This is not too bad, we will consider them for (slow/exact) similarity computation and discard them.

Picking *r* and *b* to Get Desired Performance

- 50 hash-functions (r * b = 50)



Limitations of Minhash

- Minhash is great for near-duplicate detection
  - Set high threshold for Jaccard similarity
- Limitations:
  - Jaccard similarity only
  - Set-based representation, no way to assign weights to features
- Random projections:
  - Works with arbitrary vectors using cosine similarity
  - Same basic idea, but details differ
  - Slower but more accurate: no free lunch!

LSH Generalizations

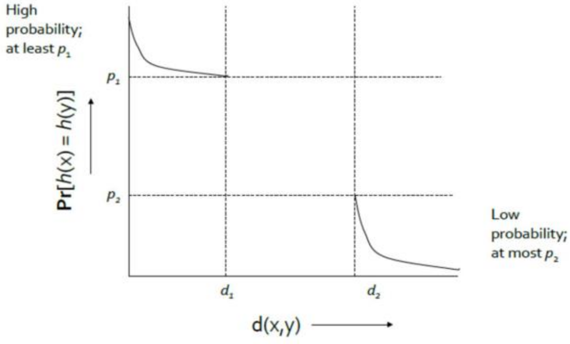
Multiple Hash Functions

- For Min-Hashing signatures, we got a Min-Hash function for each permutation of rows
- So far, we have assumed only one hash function (even applied multiple times)
  - Shorthand: `h(x)=h(y)` implies "h says x and y are equal"
- We could have used a family of hash functions
  - A (large) set of related hash functions generated by some mechanism
  - We should be able to efficiently pick a hash function at random from such a family

Locality-Sensitive (LS) Families

- Consider a space S of points with a distance measure d
- A family **H** of hash functions is said to be (d_1, d_2, p_1, p_2)- sensitive if for any x and y in S:
  - If $d(x,y) \leq d_1$, then prob over all h in **H** that $h(x)=h(y)$ is at least p_1
  - If $d(x,y) \geq d_2$, then prob over all h in **H** that $h(x)=h(y)$ is at most p_2

Small distance, high probability of hashing to the same value



Large distance, low probability of hashing to the same value

Example of LS Family: MinHash

- Let
  - **S** = space of all sets,
  - **d** = Jaccard distance,
  - **H** is family of Min-Hash functions for all permutations of rows
- Minhashing gives a (d_1, d_2, p_1, p_2)-sensitive family for any $d_1 < d_2$
  - E.g., **H** is a (1/3, 2/3, 2/3, 1/3)-sensitive family for S and d
  - If distance ≤ 1/3 (i.e., similarity ≥ 2/3), then probability that minhash values agree is ≥ 2/3
  - This is because for any hash function $h \in H$ $Pr(h(x)=h(y))=1-d(x,y)$
- Simply restates theorem about Min-Hashing in terms of distances rather than similarities!

Example of LS Family: MinHash

- **Claim: Min-hash *H*** is a ($\boxed{1/3}$, 2/3, $\boxed{2/3}$, 1/3)-sensitive family for ***S*** and ***d***

If distance < 1/3
(so similarity ≥ 2/3)

Then probability that Min-Hash values agree ≥ 2/3

- For Jaccard similarity, Min-Hashing gives a $(d_1,d_2,(1-d_1),(1-d_2))$-sensitive family for any $d_1<d_2$

- Theory leaves unknown what happens to pairs that are at distance between d_1 and d_2
  - Consequence: No guarantees about fraction of false positives in that range

Amplifying a LS-family

- Can we reproduce the “S-curve” effect we saw before for any LS family?

- The “bands” technique we learned for signature matrices carries over to this more general setting
  - So we can do LSH with any *(d1, d2, p1, p2)-sensitive* family

- Two constructions:
  - **AND** construction like “rows in a band”
  - **OR** construction like “many bands”

AND Construction of Hash Functions

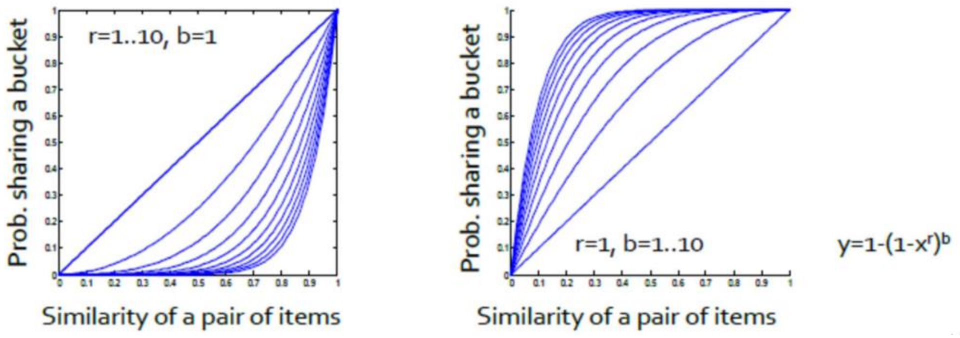
- Given family **H**, construct family **H'** consisting of *r* functions from **H**
- For $h=[h_1,\dots,h_r]$ in **H'**, $h(x)=h(y)$ if and only if $h_i(x)=h_i(y)$ for all i, $1 \le i \le r$
- Note this has the same effect as "r signatures"
  - x and y are considered a candidate pair if every one of the r rows say that x and y are equal
- Theorem: If **H** is (d_1, d_2, p_1, p_2)-sensitive, then **H'** is (d_1, d_2, p_1^r, p_2^r)-sensitive
  - That is, for any p, if p is the probability that a member of **H** will declare (x,y) to be a candidate pair, then the probability that a member of **H'** will so declare is p^r
  - Proof**:** Use the fact that h_i 's are independent

OR Construction of Hash Functions

- Given family **H**, construct family **H'** consisting of *b* functions from **H**
- For $h=[h_1,\dots,h_b]$ in **H'**, $h(x)=h(y)$ if and only if $h_i(x)=h_i(y)$ for at least one i, $1 \le i \le b$
- Mirrors the effect of combining "b bands":
  - x and y become a candidate pair if any set makes them a candidate pair
- Theorem: If **H** is (d_1, d_2, p_1, p_2)-sensitive, then **H'** is $(d_1, d_2, 1-(1-p_1)^b, 1-(1-p_2)^b)$-sensitive
  - That is, for any p, if p is the probability that a member of **H** will declare (x,y) to be a candidate pair, then $(1-p)$ is the probability that it will not declare so
  - $(1-p)^b$ is the probability that none of the family h_1, h_b will declare (x,y) a candidate pair
  - $1-(1-p)^b$ is the probability that at least one h_i will declare (x,y) a candidate pair, and therefore that **H'** will declare (x,y) to be a candidate pair

Effect of AND & OR Constructions

- **AND** makes all probabilities shrink, but by choosing r correctly, we can make the *lower probability approach 0* while the higher does not
- **OR** makes all probabilities grow, but by choosing b correctly, we can make the *upper probability approach 1* while the lower does not



Composing Constructions: AND-OR Composition

- r-way **AND** construction followed by b-way **OR** construction
  - Exactly what we did with minhashing
    - If b bands match in all r values hash to same bucket
    - Columns that are hashed into ≥ 1 common bucket -> candidate

- Take points x and y s.t. `Pr[h(x)=h(y)] = p`
  - **H** will make `(x,y)` a candidate pair with probability p

- Construction makes `(x,y)` a candidate pair with probability $1-(1-p^r)^b$
  - The S-Curve!

Example

- Example: Take **H** and construct **H’** by the **AND** construction with *r* = 4. Then, from **H’**, construct **H’’** by the **OR** construction with *b* = 4
- E.g., transform a (0.2, 0.8, 0.8, 0.2)-sensitive family into a (0.2, 0.8, 0.8785, 0.0064)-sensitive family

p	$1-(1-p^4)^4$
.2	.0064
.3	.0320
.4	.0985
.5	.2275
.6	.4260
.7	.6666
.8	.8785
.9	.9860

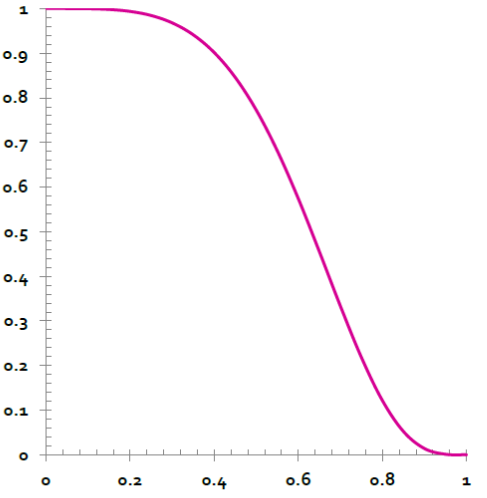
Composing Constructions: OR-AND Composition

- b-way **OR** construction followed by r-way **AND** construction
- Transforms probability p into $(1-(1-p)^b)^r$
  - The same S-curve, mirrored horizontally and vertically

Example

- Example: Take **H** and construct **H'** by the **OR** construction with b = 4. Then, from **H'**, construct **H''** by the **AND** construction with r = 4
- E.g., transform a (0.2, 0.8, 0.8, 0.2)-sensitive family into a (0.2, 0.8, 0.9936, 0.1215)-sensitive family

p	$(1-(1-p)^4)^4$
.1	.0140
.2	.1215
.3	.3334
.4	.5740
.5	.7725
.6	.9015
.7	.9680
.8	.9936

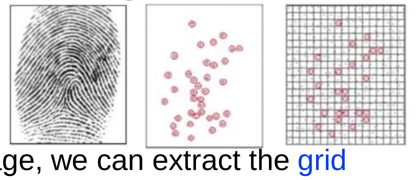


Cascading Constructions

- Example: Apply the (4,4) OR-AND construction followed by the (4,4) AND-OR construction
- Transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.9999996,.0008715)-sensitive family
  - Note this family uses 256 (= 4*4*4*4) of the original hash functions

Applications of LSH

Application 2: A LHS Family for Fingerprint Matching



- Fingerprint can be uniquely defined by its minutiae
- By overlaying a grid on the fingerprint image, we can extract the grid squares where the minutiae are located
- Two fingerprints are similar if the set of grid squares significantly overlap
  - Jaccard distance and minhash can be used, but …
- Let F be a family of functions
  - f ∈ F is defined by, say 3, grid squares such that f returns the same bucket whenever the fingerprint has minutiae in all three grid squares
  - f sends all fingerprints that have minutiae in all three of f's grid points to the same bucket
  - Two fingerprints match if they are in the same bucket

LSH for Fingerprint Matching

- Suppose probability of finding a minutiae in a random grid square of a random finger is 0.2
- And probability of finding one in the same grid square of the same finger (different fingerprint) is 0.8
- Prob two fingerprints from different fingers match=$(0.2)^3$x $(0.2)^3$= 0.000064
- Prob two fingerprints from the same finger match=$(0.2)^3$x $(0.8)^3$= 0.004096
- Use more functions from F!
- Take 1024 functions and do a OR construction
  - Prob putting the fingerprints from the same finger in at least one bucket is $1 – (1-0.004096)^{1024} = 0.985$
  - Prob two fingerprints from different fingers falling into the same bucket is $1 – (1-0.000064)^{1024} = 0.063$
  - We have 1.5% false negatives and 6.3% false positives
- Using AND construction will
  - Greatly reduce the prob of a false positive
  - Small increase in false-negative rate

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