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Let E and F be Banach spaces with norms

$$f(x+y)=f(x)+f(y)$$

$$f:E\longrightarrow$$

FUNCTIONAL EQUATIONS, DIFFERENCE INEQUALITIES AND ULAM STABILITY NOTIONS (F.U.N.)

$$x, y \in E$$

JOHN MICHAEL RASSIAS
EDITOR

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JOHN MICHAEL RASSIAS
EDITOR

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PREFACE

1. Ulam's Biography:

S. M. ULAM was born in Lwow, Poland on April 3, 1909 and died in Santa Fe, U.S.A. on May 13, 1984. He graduated with a doctorate in pure mathematics from the Polytechnic Institute at Lwow in 1933. Ulam worked at: The Institute for Advanced Study, Princeton (1936), Harvard University (1939-40), University of Wisconsin (1941-43), Los Alamos Scientific Laboratory (1943-65), University of Colorado (1965-76), and University of Florida (1974-). He was a member of the American Academy of Arts and Sciences and the National Academy of Sciences. He made fundamental contributions in mathematics, physics, biology, computer science, and the design of nuclear weapons. His early mathematical work was in set theory, topology, group theory, and measure theory. While still a schoolboy in Lwow, Ulam signed his notebook "S. Ulam, astronomer, physicist and mathematician". As Ulam notes, "the aesthetic appeal of pure mathematics lies not merely in the rigorous logic of the proofs and theorems, but also in the poetic elegance and economy in articulating each step in a mathematical presentation." Ulam worked with Stefan Banach, Kazimir Kuratowski, Karol Borsuk, Stanislaw Mazur, Hugo Steinhaus, John von Neumann, Garrett Birkhoff, Cornelius Everett, Dan Mauldin, D. H. Hyers, Mark Kac, P. R. Stein, Enrico Fermi, John Pasta, Richard Feynman, Ernest Lawrence, J. Robert Oppenheimer, Teller, and many other people of applied and exact sciences. Ulam was invited to Los Alamos by his friend John von Neumann, one of the most influential mathematicians of the twentieth century. Ulam's most remarkable achievement at Los Alamos was his contribution to the postwar development of the thermonuclear or hydrogen (H-) bomb in which nuclear energy is released when two hydrogen or deuterium nuclei fuse together. One of Ulam's early insights was to use the fast computers at Los Alamos to solve a wide variety of problems in a statistical manner using random numbers. This method has become appropriately known as the Monte Carlo method. One example that may have biological relevance is the subfield of cellular automata founded by Ulam and von Neumann. Finally Ulam had a unique ability to raise important unsolved problems. One of these problems was solved by the editor of this F.U.N. volume (*J. Approx. Th.*, Vol. 57, 268-273, 1989).

2. Ulam’s volume F. U. N. :

Functional Equations and Difference Inequalities and Ulam Stability Notions, is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the first centennial birthday anniversary of Stanislaw Marcin ULAM.

This collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical and physical sciences. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.

Ulam's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.

The central scope of this commemorating 100birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.

We have recently observed a more rapid development in the areas of research of Ulam worldwide.

This F.U.N. volume is suitable for graduate students and researchers interested in functional equations, and differential equations and would make an ideal supplementary reading or independent study research text.

This item will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.

This Ulam's volume (F.U.N.) consists of research papers containing various parts of contemporary pure and applied mathematics with emphasis to Ulam's mathematics.

It contains various parts of *Functional Equations and Difference Inequalities as well as related topics in Mathematical Analysis,* namely:

Ulam’s stability of a class of linear Cauchy functional equations;

Sequential antagonistic games with an auxiliary initial phase; Some stability results for equations and inequalities connected with the exponential function; On a problem of John M. Rassias concerning the stability in Ulam sense of Euler-Lagrange equation; Hyers-Ulam-Aoki-Rassias stability and Ulam-Gavruta-Rassias stability of quadratic homomorphisms and quadratic derivations on Banach algebras; Fundamental solutions for the generalized elliptic Gellerstedt equation; Pointwise superstability and superstability of the Jordan equation; A problem with non-local conditions on the line of degeneracy and parallel characteristics for a mixed type equation with singular coefficient; On the stability of an additive functional inequality in normed modules; Cubic derivations and quartic derivations on Banach modules; Tetrahedron isometry Ulam stability problem; Hyers-Ulam stability of Cauchy type additive functional equations; Solution and Ulam stability of a mixed type cubic and additive functional equation; Stability of mappings approximately preserving orthogonality and related topics; The Frank problem for second order nonlinear equations of mixed type with non-smooth degenerate curve.

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Chapter 1

ULAM'S STABILITY OF A CLASS OF LINEAR CAUCHY FUNCTIONAL EQUATIONS

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Abstract

In this work, we describe the solution of (1.1) connected with additive functions and we study the Ulam's problem of this equation. Some applications deal with new equations of type linear Cauchy in Banach spaces, are given.

2000 Mathematics Subject Classifications: 39B32, 39B42, 39B72.

Key words: Functional equation, Ulam problem, stability.

1. Introduction

Let E and F be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$ respectively. A mapping $f : E \longrightarrow F$ is called, additive function, if it satisfies the Cauchy functional equation $f(x + y) = f(x) + f(y)$ for all $x, y \in E$.

In 1940, S. M. Ulam (see [22]) raised the question concerning the stability of group homomorphisms: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?". The first answer to Ulam's question, concerning the Cauchy equation, was given by D. H. Hyers [8]. Thus we speak about the Hyers–Ulam stability. This terminology is also applied to the case of other functional equations. Th. M. Rassias [19] generalized the theorem of

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Hyers for approximately linear mappings [19]. The stability phenomena that was proved by Th. M. Rassias [19] is called the Hyers–Ulam–Rassias stability. The modified Ulam's stability problem with the generalization control function was proved by P. Găvruta [6] in the following way

Theorem 1.1. *Let E be a vector space, E be a Banach space and let $\varphi : E\times E \longrightarrow [0,+\infty[$ be a function satisfying*

$$\psi(x,y) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \varphi(2^k x, 2^k y) < +\infty$$

for all $x, y \in E$. If a function $f : E \longrightarrow F$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| < \varphi(x,y)$$

for all $x, y \in E$. Then there exists a unique additive function $T : E \longrightarrow F$ which satisfies $\|f(x) - T(y)\| \leq \psi(x,y)$ for all $x \in E$.

J. M. Rassias [14]–[18] solved the Ulam's problem for different mappings, in the following way

Theorem 1.2. *Let X be a real normed linear space and let Y be a real normed linear space. Assume in addition that if $f : X \longrightarrow Y$ is a mapping for which there exist constant $\delta > 0$ and $p,\ q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \longrightarrow Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\delta}{\|2^r - 1\|} \|x\|^r$$

for all $x \in X$.

As the words "differing slightly" and "be close" in Ulam's problem have various meanings, different kinds of stability can be dealt with. It may happen that all approximate solution are in fact exact solutions. Then we speak about superstability. To get acquainted with the theory of the stability of functional equation we refer to papers [8]–[15].

In this paper, we introduce the new general linear Cauchy type functional equation of the form

$$\sum_{j=1}^{M} \left(\sum_{i=1}^{n} x_i + a_j \right) = M \sum_{i=1}^{n} f(x_i) \ \text{ for all } \ x_1, \ldots, x_n \in E \tag{1.1}$$

and for any arbitrary fixed elements $a_1, \ldots, a_M$ in E. When $a_j = 0$ for all $j \in \{1, \ldots, M\}$, the equation (1.1) reduces to the equation

$$f\left(\sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_i) \ \text{ for all } \ x_1, \ldots, x_n \in E. \tag{1.2}$$

The Ulam's problem of the question (1.2) was studied by J. M. Rassias in [18 Theorem 2]. If $n = 2$ and $M = 2$, the equation (1.1) becomes

$$f(x+y+a)+f(x+y+b)=2f(x)+2f(y). \tag{1.3}$$

The Ulam's problem of this equation was studied by authors in [1]. Following this investigation, we study here the Ulam's problem of the functional equation (1.1) and a bouquet of special cases, namely

$$f(x+y+z+a)+f(x+y+z+b)=2f(x)+2f(y)+2f(z) \tag{1.4}$$

and

$$f(x+y+a)+f(x+y+b)+f(x+y+c)=3f(x)+3f(y), \tag{1.5}$$

for all $x, y, z \in E$ and any arbitrary $a, b, c \in E$. This paper is organized as follows: in the first section, after this introduction we gave the general solution of (1.1). In the second section we investigate the Ulam's problem for the general linear Cauchy equation (1.1). In Corrollaries (5.5) and (5.6) we deduce the Ulam's stability for equation (1.4) and (1.5).

2. Solution of (1.1)

In this section we give the general solution of the functional equation (1.1).

Theorem 2.1. *Let M, N be integers, $M > 0$ and $N > 1$. Let E and F be vectors spaces. A function $f : E \longrightarrow F$ satisfies the functional equation (1.1) if and only if there exists an additive function $g : E \longrightarrow F$ such that*

$$f(x)=\sum_{j=1}^{M}\frac{g((N-1)x+a_j)}{M(N-1)} \quad \textit{for all} \quad x\in E.$$

Proof. If f is a solution of (1.1), then by substituting $x_1, \ldots, x_{N-1}$ by 0 and x_N by $x_1 + \cdots + x_N$ in (1.1) we have

$$\sum_{j=1}^{M} f\left(\sum_{i=1}^{N} x_i + a_j\right) = M(N-1)f(0)+Mf\left(\sum_{i=1}^{N} x_i\right) \quad \text{for all} \quad x_1,\ldots,x_N \in E. \tag{2.6}$$

By taking $x_1 = \cdots = x_N = 0$ in (1.1) we get

$$\sum_{j=1}^{M}(f-f(0))(a_j) = M(N-1)f(0).$$

From (1.1) and (2.1) we deduce that

$$\sum_{i=1}^{N} f(x_i) = (N-1)f(0) + f\left(\sum_{i=1}^{N} x_i\right)$$

so

$$\sum_{i=1}^{N}(f-f(0))(x_i)=(f-f(0))\left(\sum_{i=1}^{N}x_i\right).$$

We pose $g=f-f(0)$, then

$$f(x)=g(x)+\frac{\sum\limits_{j=1}^{M}g(a_j)}{M(N-1)}=\frac{\sum\limits_{j=1}^{M}g((N-1)x+a_j)}{M(N-1)}.$$

Conversely, let $g:E\longrightarrow F$ be an additive function. It's elementary to verify that $f(x)=\frac{\sum\limits_{j=1}^{M}g((N-1)x+a_j)}{M(N-1)}$ is a solution of (1.1). This ends the proof. □

3. Ulam's Stability for the Functional Equation (1.1)

In this section we establish the Ulam's stability for equation (1.1).

Theorem 3.1. *Let E be a vector space, F a Banach space and $\delta>0$. Suppose that the function $f:E\longrightarrow F$ satisfies the inequality*

$$\left\|\sum_{j=1}^{M}f\left(\sum_{i=1}^{N}x_i+a_j\right)-M\sum_{i=1}^{N}f(x_i)\right\|<\delta \text{ for all } x_1,\dots,x_N\in E. \tag{3.7}$$

Then, there exists a unique function $S:E\longrightarrow F$ solution of (1.1) such that

$$\|f(x)-S(x)\|\le\frac{\delta}{M(N-1)} \text{ for all } x\in E.$$

Proof. Assume that $f:E\longrightarrow F$ satisfies the inequality (3.1), we use induction on n to prove that the sequence functions $f_0(x)=f(x)$ and $f_n(x)=\sum\limits_{j=1}^{M}f_{n-1}(Nx+a_j)$ for all $x\in E$, $n\in\mathbb{N}^*$ satisfy the following statements

$$\|f_n(x)-MNf_{n-1}(x)\|<M^{n-1}\delta \text{ for all } x\in E,\ \ n\in\mathbb{N}^*, \tag{3.8}$$

$$\|f_n(x)-M^nN^nf_0(x)\|<M^{n-1}\frac{N^n-1}{N-1}\delta \text{ for all } x\in E,\ \ n\in\mathbb{N}^*, \tag{3.9}$$

$$\left\|\sum_{j=1}^{M}f_n\left(\sum_{i=1}^{N}x_i+a_j\right)-M\sum_{i=1}^{N}f_n(x_i)\right\|<M^n\delta \text{ for all } x_1,\dots,x_N\in E,\ \ n\in\mathbb{N}^*. \tag{3.10}$$

By taking $x_1=\dots=x_N=x$ in (3.1), we get that

$$\|f_1(x)-MNf_0(x)\|<\delta.$$

We have for all $x_1, \ldots, x_n \in E$ and $n \in \mathbb{N}^*$

$$\begin{aligned}
&\left\| \sum_{j=1}^{M} f_1 \left(\sum_{i=1}^{N} x_i + a_j \right) - M \sum_{i=1}^{N} f_1(x_i) \right\| \\
&\qquad = \left\| \sum_{j=1}^{M} \sum_{l=1}^{M} f_0 \left(N \sum_{i=1}^{N} x_i + N a_j + a_l \right) - \sum_{j=1}^{M} M \sum_{i=1}^{N} f_0(N x_i + a_j) \right\| \\
&\qquad\quad \leq \sum_{j=1}^{M} \left\| \sum_{l=1}^{M} f_0 \left(\sum_{i=1}^{N} (N x_i + a_j) + a_l \right) - M \sum_{i=1}^{N} f_0(N x_i + a_j) \right\| < M\delta.
\end{aligned}$$

Consequently, the assertions (3.2), (3.3) and (3.4) are trues for $n = 1$. Assuming that the assertions are trues for all integers i, $1 \leq i \leq n$. It follows from the induction assumption that

$$\begin{aligned}
\|f_{n+1}(x) - MN f_n(x)\| &= \left\| \sum_{j=1}^{M} f_n(Nx + a_j) - MN \sum_{j=1}^{M} f_{n-1}(Nx + a_j) \right\| \\
&\leq \sum_{j=1}^{M} \|f_n(Nx + a_j) - MN f_{n-1}(Nx + a_j)\| < M(M^{n-1}\delta) = M^n \delta
\end{aligned}$$

for all $x \in E$. We have for all $x \in E$ that

$$\begin{aligned}
&\left\| f_{n+1}(x) - M^{n+1} N^{n+1} f_0(x) \right\| = \left\| \sum_{i=0}^{n} M^i N^i f_{n+1-i}(x) - M^{i+1} N^{i+1} f_{n-i}(x) \right\| \\
&\leq \sum_{i=0}^{n} M^i N^i \|f_{n+1-i}(x) - MN f_{n-i}(x)\| \leq \sum_{i=0}^{n} M^i N^i M^{n-i} \delta = M^n \frac{N^{n+1} - 1}{N - 1} \delta.
\end{aligned}$$

Now, for all x_i, $1 \leq i \leq n$, we get

$$\begin{aligned}
&\left\| \sum_{j=1}^{M} f_{n+1} \left(\sum_{i=1}^{N} x_i + a_j \right) - M \sum_{i=1}^{N} f_{n+1}(x_i) \right\| \\
&\qquad = \left\| \sum_{j=1}^{M} \sum_{l=1}^{M} f_n \left(N \sum_{i=1}^{N} x_i + N a_j + a_l \right) - \sum_{j=1}^{M} M \sum_{i=1}^{N} f_n(N x_i + a_j) \right\| \\
&\leq \sum_{j=1}^{M} \left\| \sum_{l=1}^{M} f_n \left(\sum_{i=1}^{N} (N x_i + a_j) + a_l \right) - M \sum_{i=1}^{N} f_n(N x_i + a_j) \right\| < M(M^n \delta) = M^{n+1} \delta,
\end{aligned} \tag{3.11}$$

which gives the sought results. From (3.2), it follows that the sequence functions $\left(\frac{f_n(x)}{M^n N^n} \right)_n$ is a Cauchy sequence. Since F is complete, the above sequence has a limit in F. We define $S : E \longrightarrow F$ by

$$S(x) = \lim_{n \to +\infty} \frac{f_n(x)}{M^n N^n}\,.$$

From (3.3), it follows that

$$\|f(x) - S(x)\| \le \frac{\delta}{M(N-1)} \text{ for all } x \in E.$$

From (3.4), we deduce that $S : E \longrightarrow F$ satisfies the functional equation (1.1).

In the next we will show the uniqueness of S. Let $S' : E \longrightarrow F$ be another solution of the functional equation (1.1) which satisfies

$$\|f(x) - S'(x)\| \le \frac{\delta}{M(N-1)} \text{ for all } x \in E.$$

We will prove by induction that

$$\left\|f_n(x) - M^n N^n S'(x)\right\| \le \frac{M^{n-1}}{N-1}\,\delta \text{ for all } x \in E. \tag{3.12}$$

Since S' satisfies (1.1), we get

$$\begin{aligned}\left\|f_1(x) - MNS'(x)\right\| &= \left\|\sum_{j=1}^{M} f_0(Nx+a_j) - \sum_{j=1}^{M} S'(Nx+a_j)\right\| \\ &\le \sum_{j=1}^{M} \left\|f(Nx+a_j) - S'(Nx+a_j)\right\| \le \frac{M\delta}{M(N-1)} = \frac{\delta}{N-1}.\end{aligned}$$

Assuming that (3.6) is true for all integers i, $1 \le i \le n$, hence we have

$$\begin{aligned}\left\|f_{n+1}(x) - M^{n+1}N^{n+1}S'(x)\right\| &= \left\|\sum_{j=1}^{M} f_n(Nx+a_j) - M^n N^n \sum_{j=1}^{M} S'(Nx+a_j)\right\| \\ &\le \sum_{j=1}^{M} \left\|f_n(Nx+a_j) - M^n N^n S'(Nx+a_j)\right\| \le \frac{M^n}{N-1}\,\delta.\end{aligned}$$

By letting $n \longrightarrow +\infty$ we get from inequality

$$\left\|\frac{f_n(x)}{M^n N^n} - S'(x)\right\| \le \frac{\delta}{M(N-1)N^n}$$

that $S = S'$. This ends the proof of theorem (3.1). □

Theorem 3.2. *Let E be a vector space, let F be a Banach space and let $\varphi : E^N \longrightarrow [0, +\infty[$ be a function. We define $\varphi_0 = \varphi$ and*

$$\varphi_n(x_1, \dots, x_N) = \sum_{j=1}^{M} \varphi_{n-1}(Nx_1+a_j, \dots, Nx_N+a_j) \text{ for all } x_1, \dots, x_N \in E, \quad n \in \mathbb{N}^*. \tag{3.13}$$

Suppose that

$$\widetilde{\varphi}(x_1,\ldots,x_N)=\sum_{n=0}^{+\infty}\frac{1}{M^{n+1}N^{n+1}}\,\varphi_n(x_1,\ldots,x_N)<+\infty \ \textit{ for all } \ x_1,\ldots,x_N\in E. \tag{3.14}$$

Assume that a function $f:E\longrightarrow F$ satisfies the inequality

$$\|\sum_{j=1}^{M}f(\sum_{i=1}^{N}x_i+a_j)-M\sum_{i=1}^{N}f(x_i)\|<\varphi(x_1,\ldots,x_N)\ for\ all\ x_1,\ldots,x_N\in E. \tag{3.15}$$

Then, there exists a unique function $S:E\longrightarrow F$ solution of (1.1) such that

$$\|f(x)-S(x)\|\leq\widetilde{\varphi}(x,\ldots,x)\ \textit{ for all } \ x\in E. \tag{3.16}$$

Proof. We use induction on n to prove that the sequence functions $f_0(x)=f(x)$ and $f_n(x)=\sum_{j=1}^{M}f_{n-1}(Nx+a_j)$ for all $x\in E$, $n\in\mathbb{N}^*$, satisfy the following statements

$$\|f_n(x)-MNf_{n-1}(x)\|<\varphi_{n-1}(x,\ldots,x)\ \text{ for all } \ x\in E,\ \ n\in\mathbb{N}^*, \tag{3.17}$$

$$\|f_n(x)-M^nN^nf_0(x)\|<\sum_{i=0}^{n-1}M^iN^i\varphi_{n-1-i}(x,\ldots,x)\ \text{ for all } \ x\in E,\ \ n\in\mathbb{N}^*, \tag{3.18}$$

$$\left\|\sum_{j=1}^{M}f_n\left(\sum_{i=1}^{N}x_i+a_j\right)-M\sum_{i=1}^{N}f_n(x_i)\right\|<\varphi_n(x_1,\ldots,x_N)\ \text{ for all } \ x_1,\ldots,x_N\in E,\ \ n\in\mathbb{N}. \tag{3.19}$$

By using the definition of f_n and the inequality (3.9) we have

$$\|f_1(x)-MNf_0(x)\|=\left\|\sum_{j=1}^{M}f(Nx+a_j)-MNf(x)\right\|<\varphi(x,\ldots,x).$$

We have from (3.5)

$$\begin{aligned}&\left\|\sum_{j=1}^{M}f_1\left(\sum_{i=1}^{N}x_i+a_j\right)-M\sum_{i=1}^{N}f_1(x_i)\right\|\\&\qquad\leq\sum_{j=1}^{M}\left\|\sum_{l=1}^{M}f_0\left(\sum_{i=1}^{N}(Nx_i+a_j)+a_l\right)-M\sum_{i=1}^{N}f_0(Nx_i+a_j)\right\|\\&\qquad\qquad<\sum_{j=1}^{M}\varphi(Nx_1+a_j,\ldots,Nx_n+a_j)=\varphi_1(x_1,\ldots,x_N).\end{aligned}$$

The assertions (3.11), (3.12) and (3.13) are now trues for $n = 1$. Assuming that the assertions are trues for all integers i, $1 \leq i \leq n$. It follows from (3.9) and the induction assumption that

$$\begin{aligned} \|f_{n+1}(x) - MNf_n(x)\| &= \left\| \sum_{j=1}^{M} f_n(Nx + a_j) - MN \sum_{j=1}^{M} f_{n-1}(Nx + a_j) \right\| \\ &\leq \sum_{j=1}^{M} \|f_n(Nx + a_j) - MNf_{n-1}(Nx + a_j)\| \\ &< \sum_{j=1}^{M} \varphi_{n-1}(Nx + a_j, \ldots, Nx + a_j) = \varphi_n(x, \ldots, x). \end{aligned}$$

We have from (3.5)

$$\begin{aligned} &\left\| \sum_{j=1}^{M} f_{n+1}\left(\sum_{i=1}^{N} x_i + a_j \right) - M \sum_{i=1}^{N} f_{n+1}(x_i) \right\| \\ &\leq \sum_{j=1}^{M} \left\| \sum_{l=1}^{M} f_n\left(\sum_{i=1}^{N} (Nx_i + a_j) + a_l \right) - M \sum_{l=1}^{M} f_n(Nx_i + a_l) \right\| \\ &< \sum_{j=1}^{M} \varphi_n(Nx_1 + a_j, \ldots, Nx_N + a_j) = \varphi_{n+1}(x_1, \ldots, x_N). \end{aligned}$$

Also we have

$$\begin{aligned} \left\| f_{n+1}(x) - M^{n+1}N^{n+1}f_0(x) \right\| &= \left\| \sum_{i=0}^{n} M^i N^i \left[f_{n+1-i}(x) - MNf_{n-i}(x) \right] \right\| \\ &< \sum_{i=0}^{n} M^i N^i \varphi_{n-i}(x, \ldots, x). \end{aligned}$$

This gives the sought results. It follows from (3.12) that the sequence $\left(\frac{f_n(x)}{M^n N^n} \right)_n$ is a Cauchy sequence. Since F is complete, the sequence has a limit in F. We define $S : E \longrightarrow F$, by

$$S(x) = \lim_{n \to +\infty} \frac{f_n(x)}{M^n N^n} .$$

It follows from (3.12) that

$$\|f(x) - S(x)\| \leq \widetilde{\varphi}(x, \ldots, x) \text{ for all } x \in E.$$

From (3.13), it follows that the function $S : E \longrightarrow F$ satisfies the functional equation (1.1).

In the next we will show the uniqueness of S. Let $S' : E \longrightarrow F$ be another solution of the functional equation (1.1) which satisfies

$$\|f(x) - S'(x)\| \leq \widetilde{\varphi}(x, \ldots, x) \text{ for all } x \in E.$$

We will prove by induction that

$$\left\| f_n(x) - M^n N^n S'(x) \right\| < M^{n-1} N^{n-1} \sum_{i=n}^{+\infty} \frac{1}{M^i N^i} \varphi_i(x, \ldots, x) \ \text{ for all } \ x \in E, \quad n \in \mathbb{N}^*. \tag{3.20}$$

By using (3.5), (3.7), (3.8) and (3.10), we get

$$\begin{aligned}
\left\| f_1(x) - MNS'(x) \right\| &= \left\| \sum_{j=1}^{M} f_0(Nx + a_j) - \sum_{j=1}^{M} S'(Nx + a_j) \right\| \\
&\le \sum_{j=1}^{M} \left\| f_0(Nx + a_j) - S'(Nx + a_j) \right\| \le \sum_{j=1}^{M} \widetilde{\varphi}(Nx + a_j, \ldots, Nx + a_j) \\
&= \sum_{n=0}^{+\infty} \sum_{j=1}^{M} \frac{1}{M^{n+1} N^{n+1}} \varphi_n(Nx + a_j, \ldots, Nx + a_j) \\
&= \sum_{n=0}^{+\infty} \frac{1}{M^{n+1} N^{n+1}} \varphi_{n+1}(x, \ldots, x) = \sum_{i=1}^{+\infty} \frac{1}{M^i N^i} \varphi_i(x, \ldots, x).
\end{aligned}$$

Assuming that the assertion (3.14) is true for all integers i, $1 \le i \le n$, hence we have

$$\begin{aligned}
\left\| f_{n+1}(x) - M^{n+1} N^{n+1} S'(x) \right\| &= \left\| \sum_{j=1}^{M} f_n(Nx + a_j) - M^n N^n \sum_{j=1}^{M} S'(Nx + a_j) \right\| \\
&\le \sum_{j=1}^{M} \left\| f_n(Nx + a_j) - M^n N^n S'(Nx + a_j) \right\| \\
&< \sum_{j=1}^{M} M^{n-1} N^{n-1} \sum_{i=n}^{+\infty} \frac{1}{M^i N^i} \varphi_i(Nx + a_j, \ldots, Nx + a_j) \\
&= M^{n-1} N^{n-1} \sum_{i=n}^{+\infty} \frac{1}{M^i N^i} \sum_{j=1}^{M} \varphi_i(Nx + a_j, \ldots, Nx + a_j) \\
&= M^{n-1} N^{n-1} \sum_{i=n}^{+\infty} \frac{1}{M^i N^i} \varphi_{i+1}(x, \ldots, x) \\
&= M^n N^n \sum_{i=n+1}^{+\infty} \frac{1}{M^i N^i} \varphi_i(x, \ldots, x).
\end{aligned}$$

This gives (3.14). Consequently, by letting $n \longrightarrow +\infty$ we obtain that $S = S'$. This ends the proof of theorem (3.2). $\square$

Corollary 3.3. *Let $\delta > 0$ and $p < 1$. Let $f : E \longrightarrow F$ be a function from a normed vector* *space E into a Banach space F, which satisfies*

$$\left\|\sum_{j=1}^{M} f(\sum_{i=1}^{N} x_i + a_j) - M\sum_{i=1}^{N} f(x_i)\right\| < \delta \sum_{i=1}^{N} \|x_i\|^p \text{ for all } x_1, \ldots, x_N \in E, \quad N \geq 2. \tag{3.21}$$

Then, there exists a unique mapping $S : E \longrightarrow F$ solution of (1.1) such that

$$\|f(x) - S(x)\| \leq \frac{\delta\|x\|^p}{M} + \delta\, \frac{N^p}{M(N-N^p)} \left(\|x\| + \frac{a}{N-1}\right)^p \text{ for all } x \in E, \tag{3.22}$$

where $a = \max(\|a_1\|, \ldots, \|a_M\|)$.

Proof. We pose $\varphi(x_1, \ldots, x_N) = \delta \sum\limits_{i=1}^{N} \|x_i\|^p$, $x_1, \ldots, x_N \in E$. By induction we will prove that

$$\varphi_n(x_1, \ldots, x_N) \leq M^n N^{np} \delta \sum_{i=1}^{N} \left(\|x_i\| + a\sum_{k=1}^{n} \frac{1}{N^k}\right)^p \text{ for all } x_1, \ldots, x_N \in E, \quad n \in \mathbb{N}^*. \tag{3.23}$$

For $n = 1$ we have

$$\varphi_1(x_1, \ldots, x_N) = \sum_{j=1}^{M} \delta \sum_{i=1}^{N} \|Nx_i + a_j\|^p \leq MN^p\delta \sum_{i=1}^{N} \left(\|x_i\| + \frac{a}{N}\right)^p.$$

Assuming that (3.17) is true for all integers i, $1 \leq i \leq n$, hence we have

$$\begin{aligned}
\varphi_{n+1}(x_1, \ldots, x_N) &= \sum_{j=1}^{M} \varphi_n(Nx_1 + a_j, \ldots, Nx_N + a_j) \\
&\leq M^n N^{np} \delta \sum_{j=1}^{M}\sum_{i=1}^{N} \left(\|Nx_i + a_j\| + a\sum_{k=1}^{n} \frac{1}{N^k}\right)^p \\
&\leq M^n N^{np} \delta \sum_{j=1}^{M}\sum_{i=1}^{N} \left(\|Nx_i\| + \|a_j\| + a\sum_{k=1}^{n} \frac{1}{N^k}\right)^p \\
&\leq M^n N^{(n+1)p} \delta \sum_{j=1}^{M}\sum_{i=1}^{N} \left(\|x_i\| + \frac{a}{N} + a\sum_{k=1}^{n} \frac{1}{N^{k+1}}\right)^p \\
&\leq M^{n+1} N^{(n+1)p} \delta \sum_{i=1}^{N} \left(\|x_i\| + a\sum_{k=1}^{n+1} \frac{1}{N^k}\right)^p
\end{aligned}$$

for all $x_1, \ldots, x_N \in E$, $n \in \mathbb{N}^*$. So that gives

$$\widetilde{\varphi}(x_1, \ldots, x_N) = \frac{1}{MN} \sum_{n=0}^{+\infty} \frac{1}{M^n N^n}\, \varphi_n(x_1, \ldots, x_N)$$

$$\leq \frac{\delta}{MN}\sum_{i=1}^{N}\|x_i\|^p + \frac{\delta}{MN}\sum_{n=1}^{+\infty}(N^{p-1})^n\sum_{i=1}^{N}\left(\|x_i\| + a\sum_{k=1}^{n}\frac{1}{N^k}\right)^p$$

$$\leq \frac{\delta}{MN}\sum_{i=1}^{N}\|x_i\|^p + \frac{\delta}{M}\,\frac{N^{p-1}}{N-N^p}\sum_{i=1}^{N}\left(\|x_i\| + \frac{a}{N-1}\right)^p \quad \text{for all} \ \ x_1,\ldots,x_N \in E.$$

From Theorem 3.2 we deduce that there exists a unique function $S : E \longrightarrow F$ solution of (1.1) such that

$$\|f(x) - S(x)\| \leq \widetilde{\varphi}(x,\ldots,x) \leq \frac{\delta\|x\|^p}{M} + \frac{N^p\delta}{M(N-N^p)}\left(\|x\| + \frac{a}{N-1}\right)^p$$

for all $x \in E$. □

Corollary 3.4. *Let $\delta > 0$ and $p < 1$. Suppose that $f : E \longrightarrow F$ be a function from a normed vector space E into a Banach space F such that*

$$\left\|f\left(\sum_{i=1}^{N}x_i\right) - \sum_{i=1}^{N}f(x_i)\right\| < \delta\sum_{i=1}^{N}\|x_i\|^p \ \ \textit{for all} \ \ x_1,\ldots,x_N \in E, \quad N \geq 2.$$

Then, there exists a unique additive mapping $S : E \longrightarrow F$ which satisfies

$$\|f(x) - S(x)\| \leq \frac{\delta\|x\|^p}{1-N^{p-1}} \ \ \textit{for all} \ \ \in E.$$

Corollary 3.5. *Let $f : E \longrightarrow F$ be a function from a normed vector space E into a Banach space F such that*

$$\left\|\sum_{j=1}^{M}f\left(\sum_{i=1}^{N}x_i + a_j\right) - M\sum_{i=1}^{N}f(x_i)\right\| < \alpha + \delta\prod_{i=1}^{N}\|x_i\|^{p_i} \ \ \textit{for all} \ \ x_1,\ldots,x_N \in E, \tag{3.24}$$

where α, δ are a positive numbers and $p_1,\ldots,p_N \in \mathbb{R}$ are such that $p = \sum\limits_{i=1}^{N}p_i < 1$. Then, there exists a unique function $S : E \longrightarrow F$ solution of (1.1) such that

$$\|f(x)-S(x)\| \leq \frac{\alpha}{M(N-1)} + \frac{\delta\|x\|^p}{MN} + \frac{\delta N^{p-1}}{M(N-N^p)}\left(\|x\|+\frac{a}{N-1}\right)^p \ \ \textit{for all} \ \ x \in E, \tag{3.25}$$

where $a = \max(\|a_1\|,\ldots,\|a_M\|)$.

Proof. We pose $\varphi(x_1,\ldots,x_N) = \alpha + \delta\prod\limits_{i=1}^{N}\|x_i\|^{p_i}, x_1,\ldots,x_N \in E$. By induction we will prove that

$$\begin{aligned}\varphi_n(x_1,\ldots,x_N) &\leq M^n\alpha \\ &+ M^nN^{np}\delta\prod_{i=1}^{N}\left(\|x_i\| + a\sum_{k=1}^{n}\frac{1}{N^k}\right)^{p_i} \quad \text{for all} \ \ x_1,\ldots,x_N \in E, \ \ n \in \mathbb{N}^*.\end{aligned} \tag{3.26}$$

For $n = 1$ we have

$$\varphi_1(x_1,\dots,x_N) = M\alpha + \sum_{j=1}^{M}\delta\prod_{i=1}^{N}\|Nx_i + a_j\|^{p_i} \le M\alpha + MN^p\delta\prod_{i=1}^{N}\left(\|x_i\| + \frac{a}{N}\right)^{p_i}.$$

Assuming that (3.20) is true for all integers i, $1 \le i \le n$, hence we have

$$\begin{aligned}
\varphi_{n+1}(x_1,\dots,x_N) &= \sum_{j=1}^{M}\varphi_n(Nx_1 + a_j,\dots,Nx_N + a_j)\\
&\le \alpha M^{n+1} + M^nN^{np}\delta\sum_{j=1}^{M}\prod_{i=1}^{N}\left(\|Nx_i + a_j\| + a\sum_{k=1}^{n}\frac{1}{N^k}\right)^{p_i}\\
&\le \alpha M^{n+1} + M^nN^{np}\delta\sum_{j=1}^{M}\prod_{i=1}^{N}\left(\|Nx_i\| + \|a_j\| + a\sum_{k=1}^{n}\frac{1}{N^k}\right)^{p_i}\\
&\le \alpha M^{n+1} + M^nN^{(n+1)p}\delta\sum_{j=1}^{M}\prod_{i=1}^{N}\left(\|x_i\| + \frac{a}{N} + a\sum_{k=1}^{n}\frac{1}{N^{k+1}}\right)^{p_i}\\
&\le \alpha M^{n+1} + M^{n+1}N^{(n+1)p}\delta\prod_{i=1}^{N}\left(\|x_i\| + a\sum_{k=1}^{n+1}\frac{1}{N^k}\right)^{p_i}
\end{aligned}$$

for all $x_1,\dots,x_N \in E$, $n \in \mathbb{N}^*$. So that gives

$$\begin{aligned}
&\widetilde{\varphi}(x_1,\dots,x_N) = \frac{1}{MN}\sum_{n=0}^{+\infty}\frac{1}{M^nN^n}\varphi_n(x_1,\dots,x_N)\\
&\le \frac{\alpha}{M(N-1)} + \frac{\delta}{MN}\prod_{i=1}^{N}\|x_i\|^{p_i} + \frac{1}{MN}\sum_{n=1}^{+\infty}(N^{p-1})^n\delta\prod_{i=1}^{N}\left(\|x_i\| + a\sum_{k=1}^{n}\frac{1}{N^k}\right)^{p_i}\\
&\le \frac{\alpha}{M(N-1)} + \frac{\delta}{MN}\prod_{i=1}^{N}\|x\|^{p_i} + \frac{\delta N^{p-1}}{M(N-N^p)}\prod_{i=1}^{N}\left(\|x_i\| + \frac{a}{N-1}\right)^{p_i}\\
&\qquad\qquad\qquad\qquad\qquad\text{for all }\ x_1,\dots,x_N \in E.
\end{aligned}$$

From Theorem 3.2, we deduce that there exists a unique function $S : E \longrightarrow F$ solution of (1.1) such that

$$\|f(x)-S(x)\| \le \widetilde{\varphi}(x,\dots,x) \le \frac{\alpha}{M(N-1)}+\frac{\delta}{MN}\|x\|^p+\frac{\delta N^{p-1}}{M(N-N^p)}\left(\|x\| + \frac{a}{N-1}\right)^p$$

for all $x \in E$. In the case where $p = p_1 + p_2 < 1$, we get the following corollary which completes Theorem 1.2. $\square$

Corollary 3.6. *Let $f : E \longrightarrow F$ be a function from a normed vector space E into a Banach space F such that*

$$\left\|f\left(\sum_{i=1}^{N}x_i\right) - \sum_{i=1}^{N}f(x_i)\right\| < \alpha + \delta\prod_{i=1}^{N}\|x_i\|^{p_i}\ \ \textit{for all}\ \ x_1,\dots,x_N \in E, \tag{3.27}$$

where α, δ *are a positive numbers and* $p_1, \ldots, p_N \in \mathbb{R}$ *are such that* $p = \sum_{i=1}^{N} p_i < 1$. *Then, there exists a unique additive mapping* $S : E \longrightarrow F$ *such that*

$$\|f(x) - S(x)\| \leq \frac{\alpha}{M(N-1)} + \frac{\delta\|x\|^p}{(N-N^p)} \quad \textit{for all} \quad x \in E. \tag{3.28}$$

Corollary 3.7. *Let* $\delta > 0$, *let* E *be a vector space and* F *a Banach space. Assume that a function* $f : E \longrightarrow F$ *satisfies the functional inequality*

$$\Big\|f(x+y+z+a)+f(x+y+z+b)-2f(x)-2f(y)-2f(z)\Big\|<\delta \quad \textit{for all} \quad x,y,z\in E, \tag{3.29}$$

where a *and* b *are two arbitrary elements of* E. *Then, there exists a unique function* $S : E \longrightarrow F$ *solution of the functional equation*

$$f(x+y+z+a)+f(x+y+z+b) = 2f(x)+2f(y)+2f(z) \quad \textit{for all} \quad x,y,z \in E, \tag{3.30}$$

such that $\|f(x) - S(x)\| \leq \frac{\delta}{4}$ *for all* $x \in E$.

Corollary 3.8. *Let* $\delta > 0$, *let* E *be a vector space and* F *a Banach space. Assume that a function* $f : E \longrightarrow F$ *satisfies the functional inequality*

$$\Big\|f(x+y+a)+f(x+y+b)+f(x+y+c)-3f(x)-3f(y)\Big\| < \delta \quad \textit{for all} \quad x,y,z \in E, \tag{3.31}$$

where a, b *and* c *are three arbitrary elements of* E. *Then, there exists a unique function* $S : E \longrightarrow F$ *solution of the functional equation*

$$f(x+y+a)+f(x+y+b)+f(x+y+c) = 3f(x)+3f(y) \quad \textit{for all} \quad x,y,z \in E, \tag{3.32}$$

such that $\|f(x) - S(x)\| \leq \frac{\delta}{3}$ *for all* $x \in E$.

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Chapter 2

SEQUENTIAL ANTAGONISTIC GAMES WITH AN AUXILIARY INITIAL PHASE*

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Abstract

We use random walk processes and tools of fluctuation theory to analyze stochastic games of two players running two phases. The casualties between the players are modeled by antagonistic random walks. The game is observed by a point process forming a bivariate random walk process that runs within a fixed rectangle R1 and registers the moves of the game at the observation epochs. Phase 1 of the game ends when the process crosses R1 while being contained within a larger rectangle R2. That is, Phase 1 ends when a players sustains serious but restricted damages. Phase 2 begins thereafter and ends with the ruin of one of the players, which occurs when the random walk process leaves the area of a rectangle R3, containing R2. Unlike the assumptions made in our recent work, the observation process is no longer independent of the antagonistic walks, which suggests an analytical treatment of yet another auxiliary phase (called initial phase). Three functionals are evaluated on each of the three phases that are merged together to form the entire game utilizing not only boundary values but also some key past parameters. The results give predictions of the exit times from phases 1 and 2 in the form of analytically tractable functionals.

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Key words: Antagonistic stochastic games, sequential games, fluctuation theory, marked point processes, Poisson process, ruin time, exit time.

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1. Introduction

The Game. The game presented in this article models a long term conflict between players A and B that exchange hostile actions at random times and exert damages of random magnitudes. Each player can sustain a limited amount of damage and once this threshold is reached or exceeded, the player is ruined. A simple game would be over when one of the players is ruined. Of course, any real conflict continues until the loser is completely destroyed or becomes aware of his defeat and his official surrender takes place. In the game presented here, the conflict continues over several phases.

The Terminology. Cooperation between players is often assumed to occur even if a game is, loosely speaking, noncooperative. However, there is little agreement in the literature on whether the terms "noncooperative" and "antagonistic" are completely synonymous but we have used these terms interchangably in some of our past work. In the game presented here, we assume that the game is totally *antagonistic* and there is no cooperation whatsoever between the players. We also assume that the end of the entire game or *exit* time, and the ruin time of a player are the same. We would not have a problem letting the final phase of the game go past the exit time until the defeated player surrenders, but doing so would not yield any interesting analytical results and only make the final formulas more complex.

Another feature of the game presented in this paper is its *stochasticity*. That is, the game is modeled by stochastic processes and unlike most traditional game-theoretical work, this game does not offer an optimal strategy for winning or reaching equilibrium. In our analysis, we strive to predict major events of the game (such as defeat of a losing player), the status of casualties upon the end of the game, and to impose some reasonable control.

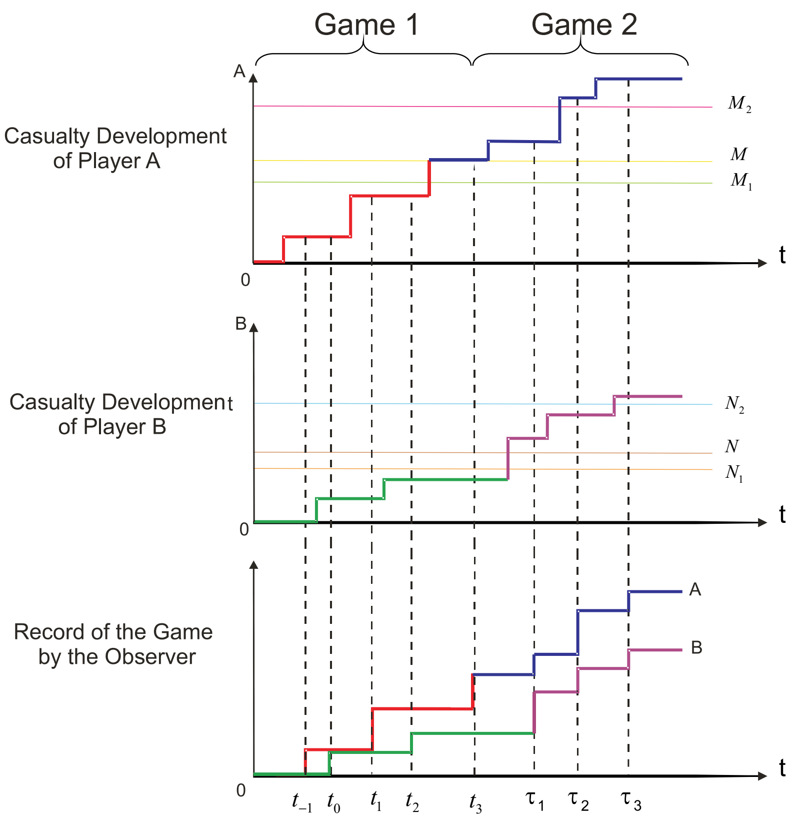
The game presented here runs *sequentially*. That is, the game consists of two separate phases (or games) where the first phase has limited damage inflicted to the players allowing them to continue to the second phase which runs differently.

Modeling. The antagonistic game in this paper is a model of a conflict but in turn, the game is modeled by a multivariate random walk or multivariate marked point process. The conflict starts with a hostile action by one of the players followed by hostile responses from the second player. At some time, the first phase of the conflict ends when one of the players sustains a certain amount of damage. At this time however, no player is ruined and the game simply morphs into a second phase which is more intense. At this point and over the entire second phase of the game, the thresholds are higher than in the first phase.

To make the model more realistic, we allow the status of the conflict to be updated only at certain epochs of time (an *observation process*) which causes certain delays. The status update information can be arbitrarily crude or fine and thus, we have a full control over it. The observation process is loosely specified and very general thereby leaving enough space to manage it.

The Initial Phase. In our past and recent work [5], [6], [8] on antagonistic games, we always assumed that the hostile actions between players A and B start at some point of time $\min\{r_1, w_1\}$ (where $r_1 \geq 0$ is the time of the first attack on player A and $w_1 \geq 0$ is the time of the first attack on player B) and that the game will be observed by a third-party stochastic process beginning with a random time t_0 (or t_1), a nonnegative r.v. (random variable). Under these assumptions, the initial observation time, say t_0, can begin in any interval: $[0, \min\{r_1, w_1\})$, or $[\min\{r_1, w_1\}, \max\{r_1, w_1\})$, or $[\max\{r_1, w_1\}, \infty)$. This type of uncertainty was motivated by analytical complexities where our prior investigation showed that specifying the initial observation time would form a peliminary phase attached to the forthcoming phases using a "doubly-formed" boundary condition. However, in this paper we assume the initial observation t_0 starts after $\max\{r_1, w_1\}$, i.e. at random epoch $t_0 = \max\{r_1, w_1\} + \Delta_0$, where Δ_0 is an independent r.v.. The evolution of the game that goes on beyond t_0 gets scrutinized all the way from 0 to t_0 to make an attachment that is analytically compatible. With this assumption, some preliminary analysis was required that is separate from the rest of the game.

Attaching the Phases. The game studied here has three separate phases. The initial phase begins with time zero and the inception of the conflict at $\min\{r_1, w_1\}$, and lasts until t_0, when the first instant of the observation actually takes place. Phase 1 (Game 1) occurs while the actions of the players and the damage is relatively limited. Phase 2 (Game 2) occurs when more intense actions are taking place and lasts until one of the players is completely ruined. For an illustration, see Figure 1 below:



$$\tau_{-1} = t_{\rho-1} = t_2 \qquad \tau_\mu = \tau_2$$

$$\tau_0 = t_\rho = t_3 \qquad \tau_\nu = \tau_3$$

Figure 1.

It may appear that to merge all three phases (being distinct in nature but linked through input parameters), we need the exit times from each phase and the values of the damages at the exits. However, it turns out that not only the exit times and exit values are required, but also, information from the preceding observation in order to attach the phases. The first effort to accomplish such a merge was made in our recent work [8]. In this paper we include an initial phase with outputs that impact the entire game.

A Formal Description. Let $(\Omega, \mathcal{F}(\Omega), \mathfrak{F}_t, P)$ be a filtered probability space and let $\mathcal{F}_{\mathcal{A}^1}, \mathcal{F}_{\mathcal{B}^1}, \mathcal{F}_S \subseteq \mathcal{F}(\Omega)$ be independent σ-subalgebras. We suppose that

$$\mathcal{A}^1 := \sum_{j\geq 1} d_j \varepsilon_{r_j} \quad \text{and} \quad \mathcal{B}^1 := \sum_{k\geq 1} z_k \varepsilon_{w_k} \tag{1.1}$$

are $\mathcal{F}_{\mathcal{A}^1}$-measurable and $\mathcal{F}_{\mathcal{B}^1}$-measurable marked Poisson random measures (ε_a is a point mass at a), with respective intensities λ_A and λ_B and position independent marking. The random measures are specified by their transforms

$$Ee^{-u\mathcal{A}^1(\cdot)} = e^{\lambda_A|\cdot|[h_A(u)-1]}, \;\; h_A(u) = Ee^{-ud_1}, \;\; Re(u) \geq 0, \tag{1.2}$$

$$Ee^{-u\mathcal{B}^1(\cdot)} = e^{\lambda_B|\cdot|[h_B(u)-1]}, \;\; h_B(u) = Ee^{-uz_1}, \;\; Re(u) \geq 0, \tag{1.3}$$

where $|\cdot|$ is the Borel–Lebesgue measure and d_j and z_k are nonnegative r.v.'s.

Phase 1 (or Game 1) starts with hostile actions initiated by one of the players A or B at r_1 or w_1, and with respective strikes of magnitudes d_1 and z_1 respectively. The players can exchange with several more strikes before the initial information is picked up by an observer at time t_0. We therefore assume that

$$t_0 \geq \max\{r_1, w_1\}. \tag{1.4}$$

The initial observation time t_0 will be formalized below. All forthcoming observations will be rendered in accordance with a point process

$$\begin{gathered} T_0 = \sum_{i\geq 0} \varepsilon_{t_i} = \varepsilon_{t_0} + S, \;\; \text{with} \;\; S = \sum_{i\geq 1} \varepsilon_{t_i}, \\ 0 < t_0 < t_1 < \cdots < t_n < \cdots \;\; (t_n \to \infty, \;\; \text{with} \;\; n \to \infty), \end{gathered} \tag{1.5}$$

and its extension

$$T := \varepsilon_{t_{-1}} + T_0, \;\; \text{with} \;\; t_{-1} := \min\{r_1, w_1\}, \tag{1.6}$$

such that the tail $S = \sum_{i\geq 1} \varepsilon_{t_i}$ of T_0 is $\mathcal{F}_S$-measurable. The increments $\Delta_1 := t_1 - t_0, \Delta_2 := t_2 - t_1, \Delta_3 := t_3 - t_2, \ldots$ are all independent and identically distributed, and all belong to the equivalence class $[\Delta]$ of r.v.'s with the common transform

$$\delta(\theta) := Ee^{-\theta\Delta}. \tag{1.7}$$

Define the initial observation as

$$t_0 = \max\{r_1, w_1\} + \Delta_0, \tag{1.8}$$

where $\Delta_0 \in [\Delta]$ and Δ_0 are independent from the rest of the Δ's. t_0 is included in T_0 of equation (1.5) and because it contains some of the $\mathcal{A}^1$ and $\mathcal{B}^1$, T_0 is not $\mathcal{F}_S$-measurable. However, T_0 is a delayed renewal process.

We assign t_{-1} to the genuine start of the game at time $\min\{r_1, w_1\}$ mentioned in (1.6). That is,

$$t_{-1} = \min\{r_1, w_1\}. \tag{1.9}$$

Now, since t_{-1} and $t_0 - t_{-1}$ are dependent (through r_1 and w_1), the extended process T of (1.6) is not a renewal process, and not even a delayed renewal, as it was in [5], [6], [8].

It should be clear that t_0 depends upon r_1 and w_1 and thus on $\mathcal{A}^1$ and $\mathcal{B}^1$, which makes T_0 depend on the named σ-algebras.

Define the continuous time parameter process

$$(\alpha(t), \beta(t)) := \mathcal{A}^1 \otimes \mathcal{B}^1([0,t]), \;\; t \geq 0, \tag{1.10}$$

to be adapted to the filtration $(\mathfrak{F}_t)_{t\geq 0}$. Also introduce its embedding over T_0 :

$$(\alpha_j, \beta_j) := (\alpha(t_j), \beta(t_j)) = \mathcal{A}^1 \otimes \mathcal{B}^1([0,t_j]), \;\; j = 0, 1, \ldots, \tag{1.11}$$

which form observations of $\mathcal{A}^1 \otimes \mathcal{B}^1$ over T_0, with respective increments

$$(\xi_j, \eta_j) := \mathcal{A}^1 \otimes \mathcal{B}^1((t_{j-1}, t_j]), \;\; j = 1, \ldots. \tag{1.12}$$

In addition, let

$$(\xi_0, \eta_0) := \mathcal{A}^1 \otimes \mathcal{B}^1((\max\{r_1, w_1\}, t_0]) \tag{1.13}$$

to be used later on.

Introduce the embedded bivariate marked point process

$$\mathcal{A}_{T_0} \otimes \mathcal{B}_{T_0} := (\alpha_0, \beta_0)\varepsilon_{t_0} + \sum_{j\geq 1} (\xi_j, \eta_j)\varepsilon_{t_j}, \tag{1.14}$$

where the marginal marked point processes

$$\mathcal{A}_{T_0} = \alpha_0 \varepsilon_{t_0} + \sum_{i\geq 1} \xi_i \varepsilon_{t_i} \;\text{ and }\; \mathcal{B}_{T_0} = \beta_0 \varepsilon_{t_0} + \sum_{i\geq 1} \eta_i \varepsilon_{t_i} \tag{1.15}$$

are with position dependent marking and with ξ_j and η_j being dependent. For the forthcoming sections we introduce the Laplace–Stieltjes transform

$$g(u, v, \theta) := Ee^{-u\xi_j - v\eta_j - \theta\Delta_j}, \;\; Re(u) \geq 0, \;\; Re(v) \geq 0, \;\; Re(\theta) \geq 0, \;\; j \geq 0, \tag{1.16}$$

which will be evaluated as the following:

$$\begin{aligned}
E\left[e^{-u\xi_j - v\eta_j - \theta\Delta_j}\right] &= E\left[e^{-\theta\Delta_j} E\left[e^{-u\xi_j - v\eta_j} \middle| \Delta_j\right]\right] \\
&= E\left[e^{-\theta\Delta_j} \cdot E\left[e^{-u\mathcal{A}^1((t_{j-1}, t_j])} \middle| \Delta_j\right] \cdot E\left[e^{-v\mathcal{B}^1((t_{j-1}, t_j])} \middle| \Delta_j\right]\right] \\
&= E\left[e^{-\theta\Delta_j} \cdot e^{\lambda_A \Delta_j (h_A(u) - 1)} \cdot e^{\lambda_B \Delta_j (h_B(v) - 1)}\right] \\
&= E\left[e^{-\left\{\theta + \lambda_A(1 - h_A(u)) + \lambda_B(1 - h_B(v))\right\}\Delta_j}\right] \\
&= \delta(\theta^*), \;\; j = 1, 2, \ldots,
\end{aligned} \tag{1.17}$$

$$\theta^* := \theta + \lambda_A(1 - h_A(u)) + \lambda_B(1 - h_B(v)), \tag{1.18}$$

where the functional δ has been defined in (1.7).

In Section 2 we will refine the embedded bivariate random walk process of (1.14) to include the information about t_{-1} and the corresponding values of the process at t_{-1}, in notation $\mathcal{A}_T^1 \otimes \mathcal{B}_T^1$. In the upcoming sections, we further extend $\mathcal{A}_T^1 \otimes \mathcal{B}_T^1$ to games 1 and 2. Game 1 (Sections 3 and 4) will continue until one of the players sustains damages exceeding either two fixed thresholds, M_1 and N_1, respectively, but limited to two larger thersholds M $(> M_1)$ and N $(> N_1)$. In other words, the random walk will end game 1 when it enters the area R2$-$R1, where R1$= [0, M_1] \times [0, N_1]$ and R2$= [0, M] \times [0, N]$. After game 1 ends, the extension of $\mathcal{A}_T^1 \otimes \mathcal{B}_T^1$ drifts in R2$-$R1 initiating game 2 (Section 5), which terminates when the damage to one of the players exceeds M_2 $(> M)$ or N_2 $(> N)$. The random walk may drift further in the area R3$-$R2, where R3$= [0, M_2] \times [0, N_2]$, but once it leaves R3$-$R2, the entire game is over.

Related Literature. The modeling and analysis of the game can be classified in two different ways. Topically, our model belongs to the game-theoretical literature [1]–[3], [5]–[12], [14]–[16],[18]–[23], and more particularly to sequential games [3], [7], [8], [10], [11], [14], [18]–[20], [23]. Furthermore, the game falls into the subcategory of stochastic games [1], [5], [6], [8], [15], [16]. It also overlaps with the area of stochastic hybrids [1], [4], [5], [6], [8], of which [1] and [5] are true hybrid stochastic games. The antagonistic nature of our modeling suggests yet another category of games, which are purely antagonistic or noncooperative, and are widely used in economics with highly competitive parties [2], [5], [6], [8], [9], [12], [16], [18], [20], [21] and warfare [5], [6], [8], [22]. Methodologically, the paper falls into the area of fluctuations of random walk processes [5], [6], [8], [17]. The literature on this topic is very rich and we cite only a few pertinent articles.

The Layout of the Paper. The present article generalizes our past and recent work on hybrid and sequential antagonistic games [5], [6], [8] in which the initial observation epoch could take place at any time independently of the inception of the conflict. The latter would make it possible for the initial observation epoch, and even some of its following ones to take place before the conflict begins. This assumption offered tame analytics but turned out to be less realistic. In this paper, we form a strict chronology of the events so that the first observation does not take place before two sides exchange with mutual hostilities. The resulting dependence between all processes (which was not assumed in [5], [6], [8]) yielded analytical challenges that gave rise to this article. We therefore divided the whole game into three separate phases, of which the first phase takes place in the interval $[0, t_0]$ (t_0 =max$\{r_1, w_1\} + \Delta_0)$ and is referred to as the initial phase. The details of the initial phase are developed in Section 2 along with all other formalities of the game. Game 1, which continues from t_0 until one of the players ends up suffering some moderate and limited losses, is treated in Section 3 and results in an explicit functional of the end of game 1 (total exit time), the damages to both players, and other important reference points. In Section 4 we impose restrictions on how much damage each player can sustain and further modify the "truncated" functional obtained in Section 4. This completes the first phase of the conflict. In Section 5, we work on game 2, which begins on the heels of game 1, but under different conditions and under the control of different processes. At the end of Section 5 we calculate the functional that includes only the paths of the game when player A is defeated. All results are given in analytically tractable forms.

2. The Initial Phase of the Game

Extended Game 1 will include the recording of the conflict between players A and B known to an observer upon process T (informally, $\{t_{-1}, t_0, t_1, \ldots\}$) from its inception upon t_{-1} followed by the initial observation at time t_0. Extended Game 1 is defined below. The actual start of the game at t_{-1} is unknown to the observer, as this moment takes place prior to t_0. From the construction of the extended game, the point process T is obviously "doubly delayed" (in light of its attachment t_{-1}). The information on t_{-1} will be presented in the upcoming sections.

The initial phase of the game is specified as follows. Define the respective damages to the players at t_{-1} as

$$(\xi_{-1}, \eta_{-1}) := (\alpha_{-1}, \beta_{-1}) := (\alpha(t_{-1}), \quad \beta(t_{-1})) = (d_1 \mathbf{1}_{\{r_1 \le w_1\}}, z_1 \mathbf{1}_{\{r_1 \ge w_1\}}). \tag{2.1}$$

Therefore, the embedded process $\sum_{k \ge -1} \varepsilon_{t_k}(\alpha_k, \beta_k)$ obeys the extended initial conditions

$$\mathcal{A}^1_{t_{-1}} \otimes \mathcal{B}^1_{t_{-1}} = (\alpha_{-1}, \beta_{-1}) = (d_1, 0), \text{ on trace } \sigma\text{-algebra } \mathcal{F}(\Omega) \cap \{r_1 < w_1\}, \tag{2.2}$$

$$\mathcal{A}^1_{t_{-1}} \otimes \mathcal{B}^1_{t_{-1}} = (\alpha_{-1}, \beta_{-1}) = (0, z_1), \text{ on } \mathcal{F}(\Omega) \cap \{r_1 > w_1\}, \tag{2.3}$$

$$\mathcal{A}^1_{t_{-1}} \otimes \mathcal{B}^1_{t_{-1}} = (\alpha_{-1}, \beta_{-1}) = (d_1, z_1), \text{ on } \mathcal{F}(\Omega) \cap \{r_1 = w_1\}. \tag{2.4}$$

Observe that none of the relations below is correct:

$$\alpha_0 = \xi_{-1} + \xi_0 \text{ and } \beta_0 = \eta_{-1} + \eta_0.$$

The extended form of game 1 is formally defined as the bivariate marked point process

$$\mathcal{A}^1_T \otimes \mathcal{B}^1_T := (\xi_{-1}, \eta_{-1})\varepsilon_{t_{-1}} + (\alpha_0 - \xi_{-1}, \beta_0 - \eta_{-1})\varepsilon_{t_0} + \sum_{j \ge 1} (\xi_j, \eta_j)\varepsilon_{t_j} \tag{2.5}$$

which is embedded over T.

Because r_1 and w_1 are continuous r.v.'s, $\{r_1 = w_1\}$ is a P-null set. Hence, the associated trace σ-algebra $\mathcal{F}(\Omega) \cap \{r_1 = w_1\}$ contains only a.s. negligible paths of game $\mathcal{A}^1_T \otimes \mathcal{B}^1_T$, which will have no impact on the upcoming functionals.

As we will see it in the next section, game 1 will require knowledge of $\mathcal{A}^1_T \otimes \mathcal{B}^1_T$ at t_{-1} and t_0. Consequently, we begin to work on the functional

$$\phi_0 := \phi_0(a_0, b_0, \vartheta_0, u_0, v_0, \theta_0) = E\left[e^{-a_0\alpha_{-1} - u_0\alpha_0 - b_0\beta_{-1} - v_0\beta_0 - \vartheta_0 t_{-1} - \theta_0 t_0}\right] \tag{2.6}$$

that describes what we call, the initial phase of the game.

Theorem 1. *The functional ϕ_0 of the initial phase of the game satisfies the following formula:*

$$\phi_0 = \frac{\lambda_A \lambda_B \delta(\theta^*)}{\vartheta_0 + \theta_0 + \lambda_A + \lambda_B} \left(\frac{1}{\theta_A + \lambda_B} h_A(a_0 + u_0) h_B(v_0) + \frac{1}{\theta_B + \lambda_A} h_A(u_0) h_B(b_0 + v_0) \right), \tag{2.7}$$

where θ^* is defined in (1.18) *and*

$$\theta_A := \theta_0 - \lambda_A(h_A(u_0) - 1), \tag{2.8}$$

$$\theta_B := \theta_0 - \lambda_B(h_B(v_0) - 1), \tag{2.9}$$

$$\delta(\theta) := E\left[e^{-\theta\Delta_0}\right]. \tag{2.10}$$

Proof. Recall that $t_{-1} = \min\{r_1, w_1\}$ and $t_0 = \max\{r_1, w_1\} + \Delta_0$. Then, the functional ϕ_0 may be rewritten as

$$\begin{aligned}\phi_0 = E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\right]\\ = E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1<w_1\}}\right]\\ + E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1>w_1\}}\right].\end{aligned} \tag{2.11}$$

Since $\{r_1 = w_1\}$ is a P-null set,

$$E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1=w_1\}}\right] = 0.$$

Case 1. $\boldsymbol{r_1 < w_1}$. This corresponds to

$$t_{-1} = r_1 \text{ and } t_0 = w_1 + \Delta_0. \tag{2.12}$$

Let

$$\xi' = \mathcal{A}^1((r_1, w_1]). \tag{2.13}$$

To keep up with the abundance of notation, the initial phase is depicted in Figure 2:

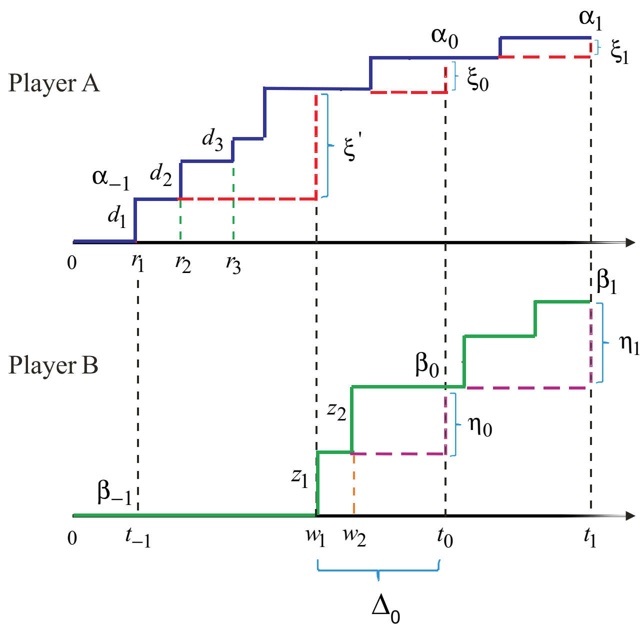


Figure 2.

The first term of ϕ_0 in (2.11) can be evaluated as follows:

$$\begin{aligned}&E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1<w_1\}}\right]\\ &= E\left[e^{-a_0 d_1-u_0(d_1+\xi'+\xi_0)-b_0\cdot 0-v_0(0+z_1+\eta_0)-\vartheta_0 r_1-\theta_0(r_1+(w_1-r_1)+\Delta_0)}\mathbf{1}_{\{r_1<w_1\}}\right]\end{aligned}$$

(due to assumed independent marking and independence of Poisson processes $\mathcal{A}^1$ and $\mathcal{B}^1$)

$$= Ee^{-(a_0+u_0)d_1}Ee^{-v_0 z_1}E\left[e^{-u_0\xi'-(\vartheta_0+\theta_0)r_1-\theta_0(w_1-r_1)}\mathbf{1}_{\{r_1<w_1\}}\right]E\left[e^{-u_0\xi_0-v_0\eta_0-\theta_0\Delta_0}\right].$$

Thus, we have

$$\begin{aligned}
&E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1<w_1\}}\right] \\
&= h_A(a_0+u_0)h_B(v_0)E\left[e^{-u_0\xi'-(\vartheta_0+\theta_0)r_1-\theta_0(w_1-r_1)}\mathbf{1}_{\{r_1<w_1\}}\right]E\left[e^{-u_0\xi_0-v_0\eta_0-\theta_0\Delta_0}\right]
\end{aligned} \tag{2.14}$$

With notation (2.13), the third factor of (2.14) reads

$$\begin{aligned}
&E\left[e^{-u_0\xi'-(\vartheta_0+\theta_0)r_1-\theta_0(w_1-r_1)}\mathbf{1}_{\{r_1<w_1\}}\right] \\
&= E\left[e^{-(\vartheta_0+\theta_0)r_1-\theta_0(w_1-r_1)}\mathbf{1}_{\{r_1<w_1\}}E\left[e^{-u_0\xi'}|r_1,w_1\right]\right] \\
&= E\left[e^{-(\vartheta_0+\theta_0)r_1-\theta_0(w_1-r_1)}\mathbf{1}_{\{r_1<w_1\}}\cdot e^{\lambda_A(w_1-r_1)(h_A(u_0)-1)}\right] \\
&= E\left[e^{-[\vartheta_0+\lambda_A(h_A(u_0)-1)]r_1-[\theta_0-\lambda_A(h_A(u_0)-1)]w_1}\mathbf{1}_{\{r_1<w_1\}}\right] \\
&= E\left[e^{-\vartheta_A r_1-\theta_A w_1}\mathbf{1}_{\{r_1<w_1\}}\right],
\end{aligned} \tag{2.15}$$

where $\vartheta_A=\vartheta_0+\lambda_A(h_A(u_0)-1)$ and $\theta_A=\theta_0-\lambda_A(h_A(u_0)-1)$. Because r_1 and w_1 are independent and exponentially distributed,

$$\begin{aligned}
E\left[e^{-\vartheta_A r_1-\theta_A w_1}\mathbf{1}_{\{r_1<w_1\}}\right] &= \int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-\vartheta_A x-\theta_A y}\lambda_A e^{-\lambda_A x}\lambda_B e^{-\lambda_B y}\,dy\,dx \\
&= \int_{x=0}^{\infty}\lambda_A e^{-(\vartheta_A+\lambda_A)x}\left[\frac{-\lambda_B}{\theta_A+\lambda_B}e^{-(\theta_A+\lambda_B)y}\right]_{y=x}^{\infty}dx \\
&= \int_{x=0}^{\infty}\lambda_A e^{-(\vartheta_A+\lambda_A)x}\frac{\lambda_B}{\theta_A+\lambda_B}e^{-(\theta_A+\lambda_B)x}\,dx \\
&= \frac{\lambda_A\lambda_B}{\theta_A+\lambda_B}\int_{x=0}^{\infty}e^{-(\vartheta_A+\theta_A+\lambda_A+\lambda_B)x}\,dx \\
&= \frac{\lambda_A\lambda_B}{(\theta_A+\lambda_B)(\vartheta_A+\theta_A+\lambda_A+\lambda_B)} = \frac{1}{\theta_A+\lambda_B}\cdot\frac{\lambda_A\lambda_B}{\vartheta_0+\theta_0+\lambda_A+\lambda_B}.
\end{aligned} \tag{2.16}$$

Since Δ_0 belongs to the equivalence class $[\Delta]$ with the common transform $\delta(\theta)$, the last factor of (2.14) is

$$E\left[e^{-u_0\xi_0-v_0\eta_0-\theta_0\Delta_0}\right]=\delta(\theta^*), \tag{2.17}$$

as per (1.17) with $\theta^*:=\theta_0+\lambda_A(1-h_A(u_0))+\lambda_B(1-h_B(v_0))$ defined so in (1.18).

In summary,

$$\begin{aligned}
&E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1<w_1\}}\right] \\
&\qquad = \frac{1}{\theta_A+\lambda_B}\cdot\frac{\lambda_A\lambda_B\delta(\theta^*)}{\vartheta_0+\theta_0+\lambda_A+\lambda_B}\,h_A(a_0+u_0)h_B(v_0).
\end{aligned} \tag{2.18}$$

Case 2. $\boldsymbol{r_1>w_1}$ corresponds to

$$t_{-1}=w_1 \text{ and } t_0=r_1+\Delta_0. \tag{2.19}$$

Denote

$$\eta' = \mathcal{B}^1((t_{-1}, r_1]). \tag{2.20}$$

Then by interchanging the roles of r_1 and w_1 we have from (2.18),

$$E\left[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}\mathbf{1}_{\{r_1>w_1\}}\right]$$
$$= \frac{1}{\theta_B+\lambda_A}\cdot\frac{\lambda_A\lambda_B\delta(\theta^*)}{\vartheta_0+\theta_0+\lambda_A+\lambda_B}\,h_A(u_0)h_B(b_0+v_0). \tag{2.21}$$

Summing up (2.18) and (2.21) yields (2.7). □

3. The Development of Game 1 after t_0

After passing the initial phase, game 1 continues with its status registered at epochs T. Game 1 ends when at least one of the players sustains damages in excess of thresholds M_1 or N_1. To further formalize game 1 past t_0 we introduce the following random exit indices

$$\nu_1 := \inf\{j \geq 0 :\ \alpha_j = \alpha_0 + \xi_1 + \cdots + \xi_j > M_1\}, \tag{3.1}$$
$$\nu_2 := \inf\{k \geq 0 :\ \beta_k = \beta_0 + \eta_1 + \cdots + \eta_k > N_1\}. \tag{3.2}$$

Game 1 is thus assumed to be over at t_ρ (first passage time or exit from game 1) where

$$\rho := \min\{\nu_1, \nu_2\}. \tag{3.3}$$

Note that while one of the players will be defeated, it will not be explicitly revealed which of the two is defeated without going over particular paths of the game. The latter is not our objective during this phase however, because neither player will be ruined as per Section 4.

Definition 1. The Terminated Game 1 is the random measure

$$[\mathcal{A}^1 \otimes \mathcal{B}^1]_\rho = (\xi_{-1}, \eta_{-1})\varepsilon_{t_{-1}} + (\alpha_0 - \xi_{-1}, \beta_0 - \eta_{-1})\varepsilon_{t_0} + \sum_{j=1}^{\rho}(\xi_j, \eta_j)\varepsilon_{t_j} \tag{3.4}$$

on $(\Omega, \mathcal{F}(\Omega), (\mathfrak{F}_t)_{t=0}^{t_\rho}, P)$ (with the first two terms incorporating the pair of the initial conditions), where t_ρ is the end of game 1. Here the process is adapted to the t_ρ-head of filtration $(\mathfrak{F}_t)_{t\geq 0}$.

For this phase of the game, we consider

$$\phi_\rho := \phi_\rho(a_1, b_1, \vartheta_1, u_1, v_1, \theta_1) = E\left[e^{-a_1\alpha_{\rho-1}-u_1\alpha_\rho-b_1\beta_{\rho-1}-v_1\beta_\rho-\vartheta_1 t_{\rho-1}-\theta_1 t_\rho}\right]. \tag{3.5}$$

To evaluate this functional we introduce the Laplace–Carson transform

$$\mathcal{L}_{p_1q_1}(\cdot)(x_1, y_1) := x_1y_1\int_{p_1=0}^{\infty}\int_{q_1=0}^{\infty} e^{-x_1p_1-y_1q_1}(\cdot)\,d(p_1, q_1),\quad Re(x_1) > 0,\quad Re(y_1) > 0, \tag{3.6}$$

with the inverse

$$\mathcal{L}^{-1}_{x_1y_1}(\cdot)(p_1,q_1)=\mathfrak{L}^{-1}(\cdot\frac{1}{x_1y_1})(p_1,q_1), \tag{3.7}$$

where $\mathfrak{L}^{-1}$ is the inverse of the bivariate Laplace transform.

Theorem 2 (below) establishes an explicit formula for ϕ_ρ. We use the following abbreviations based on (1.16):

$$g := g(a_1+u_1+x_1, b_1+v_1+y_1, \vartheta_1+\theta_1), \tag{3.8}$$
$$G := g(u_1+x_1, v_1+y_1, \theta_1), \tag{3.9}$$
$$G^1 := g(u_1, v_1+y_1, \theta_1), \tag{3.10}$$
$$G^2 := g(u_1+x_1, v_1, \theta_1), \tag{3.11}$$
$$G^{12} := g(u_1, v_1, \theta_1), \tag{3.12}$$
$$\Phi_0^* := \phi_0(0,0,0,a_1+u_1+x_1, b_1+v_1+y_1, \vartheta_1+\theta_1), \tag{3.13}$$
$$\Phi_0 := \phi_0(a_1, b_1, \vartheta_1, u_1+x_1, v_1+y_1, \theta_1), \tag{3.14}$$
$$\Phi_0^1 := \phi_0(a_1+x_1, b_1, \vartheta_1, u_1, v_1+y_1, \theta_1), \tag{3.15}$$
$$\Phi_0^2 := \phi_0(a_1, b_1+y_1, \vartheta_1, u_1+x_1, v_1, \theta_1), \tag{3.16}$$
$$\Phi_0^{12} := \phi_0(a_1+x_1, b_1+y_1, \vartheta_1, u_1, v_1, \theta_1). \tag{3.17}$$

Theorem 2. *The functional ϕ_ρ of game 1 satisfies the following formula:*

$$\phi_\rho=\mathcal{L}^{-1}_{x_1y_1}\left(\Phi_0^{12}-\Phi_0+\frac{\Phi_0^*}{1-g}(G^{12}-G)\right)(M_1,N_1), \tag{3.18}$$

provided that $Re(a_1+u_1+x_1)>0,\quad Re(b_1+v_1+y_1)>0,\quad Re(\vartheta_1+\theta_1)>0,$ (3.18a)

with any two of the three strict inequalities relaxed with $\geq$.

Proof. We first extend the random indices ν_1 and ν_2 to the families of indices

$$\Big\{\nu_1(p_1):=\inf\big\{j\geq 0:\ \alpha_j=\alpha_0+\xi_1+\cdots+\xi_j>p_1\big\}, p_1\geq 0\Big\} \tag{3.19}$$

and

$$\Big\{\nu_2(q_1):=\inf\big\{k\geq 0:\ \beta_k=\beta_0+\eta_1+\cdots+\eta_k>q_1\big\}, q_1\geq 0\Big\}. \tag{3.20}$$

The parametric analog of ρ is then

$$\Big\{\rho(p_1,q_1):=\min\big\{\nu_1(p_1),\nu_2(q_1)\big\}, p_1\geq 0, q_1\geq 0\Big\}. \tag{3.21}$$

Next, introduce the following parametric families of measurable sets:

$$H_{1,2}=\big\{\nu_1(p_1)<\nu_2(q_1)\big\},\quad H_{12}=\big\{\nu_1(p_1)=\nu_2(q_1)\big\},\quad H_{2,1}=\big\{\nu_1(p_1)>\nu_2(q_1)\big\}. \tag{3.22}$$

The corresponding parametric extension of the primary functional ϕ_ρ can be decomposed in accordance with (3.22) as follows:

$$\begin{aligned}
\phi_{\rho(p_1,q_1)} :&= \phi_{\rho(p_1,q_1)}(a_1,b_1,\vartheta_1,u_1,v_1,\theta_1)\\
&= E\Big[e^{-a_1\alpha_{\rho(p_1,q_1)-1}-u_1\alpha_{\rho(p_1,q_1)}-b_1\beta_{\rho(p_1,q_1)-1}-v_1\beta_{\rho(p_1,q_1)}-\vartheta_1 t_{\rho(p_1,q_1)-1}-\theta_1 t_{\rho(p_1,q_1)}}\Big]\\
&= E\Big[e^{-a_1\alpha_{\rho(p_1,q_1)-1}-u_1\alpha_{\rho(p_1,q_1)}-b_1\beta_{\rho(p_1,q_1)-1}-v_1\beta_{\rho(p_1,q_1)}-\vartheta_1 t_{\rho(p_1,q_1)-1}-\theta_1 t_{\rho(p_1,q_1)}}\mathbf{1}_{H_{1,2}}\Big]\\
&\quad+ E\Big[e^{-a_1\alpha_{\rho(p_1,q_1)-1}-u_1\alpha_{\rho(p_1,q_1)}-b_1\beta_{\rho(p_1,q_1)-1}-v_1\beta_{\rho(p_1,q_1)}-\vartheta_1 t_{\rho(p_1,q_1)-1}-\theta_1 t_{\rho(p_1,q_1)}}\mathbf{1}_{H_{12}}\Big]\\
&\quad+ E\Big[e^{-a_1\alpha_{\rho(p_1,q_1)-1}-u_1\alpha_{\rho(p_1,q_1)}-b_1\beta_{\rho(p_1,q_1)-1}-v_1\beta_{\rho(p_1,q_1)}-\vartheta_1 t_{\rho(p_1,q_1)-1}-\theta_1 t_{\rho(p_1,q_1)}}\mathbf{1}_{H_{2,1}}\Big],
\end{aligned} \tag{3.23}$$

or in notation, =F$_{1,2} + F_{12} + F_{2,1}$,

Below we will be concerned with transformations of $F_{1,2}$, F_{12} and $F_{2,1}$ under the operator $\mathcal{L}_{p_1q_1}$ to be applied to $\phi_{\rho(p_1,q_1)}$.

Case 1. $\boldsymbol{\nu_1(p_1) < \nu_2(q_1)}$. This will follow the paths of game 1 on the trace σ-algebra $\mathcal{F}(\Omega)\cap\{\nu_1(p_1)<\nu_2(q_1)\}$ and yield $\rho(p_1,q_1)=\nu_1(p_1)$:

$$F_{1,2} = \sum_{j\ge 0}\sum_{k>j} E\Big[e^{-a_1\alpha_{j-1}-u_1\alpha_j-b_1\beta_{j-1}-v_1\beta_j-\vartheta_1 t_{j-1}-\theta_1 t_j}\mathbf{1}_{\{\nu_1(p_1)=j,\nu_2(q_1)=k\}}\Big]. \tag{3.24}$$

By Fubini's theorem,

$$\begin{aligned}
&\mathcal{L}_{p_1q_1}(F_{1,2})(x_1,y_1)\\
&=\sum_{j\ge0}\sum_{k>j}E\Big[e^{-a_1\alpha_{j-1}-u_1\alpha_j-b_1\beta_{j-1}-v_1\beta_j-\vartheta_1 t_{j-1}-\theta_1 t_j}\big(e^{-x_1\alpha_{j-1}}-e^{-x_1\alpha_j}\big)\big(e^{-y_1\beta_{k-1}}-e^{-y_1\beta_k}\big)\Big].
\end{aligned} \tag{3.25}$$

Case: $j=0$. This case will include the entire information on the initial phase observed at t_0 and prior to t_0, including t_{-1}. In a few lines below, we are going to implement the result of Theorem 1 and utilize all necessary versions of the functional ϕ_0 :

$$\begin{aligned}
&\sum_{k>0}E\Big[e^{-a_1\alpha_{-1}-u_1\alpha_0-b_1\beta_{-1}-v_1\beta_0-\vartheta_1 t_{-1}-\theta_1 t_0}\big(e^{-x_1\alpha_{-1}}-e^{-x_1\alpha_0}\big)\big(e^{-y_1\beta_{k-1}}-e^{-y_1\beta_k}\big)\Big]\\
&\qquad=\sum_{k>0}E\Big[e^{-a_1\alpha_{-1}-u_1\alpha_0-b_1\beta_{-1}-v_1\beta_0-\vartheta_1 t_{-1}-\theta_1 t_0}\big(e^{-x_1\alpha_{-1}}-e^{-x_1\alpha_0}\big)\\
&\qquad\qquad\times e^{-y_1\beta_0}e^{-y_1(\eta_1+\,\dots\,+\eta_{k-1})}\big(1-e^{-y_1\eta_k}\big)\Big]\\
&\qquad=\Big\{E\big[e^{-(a_1+x_1)\alpha_{-1}-u_1\alpha_0-b_1\beta_{-1}-(v_1+y_1)\beta_0-\vartheta_1 t_{-1}-\theta_1 t_0}\big]\\
&\qquad\qquad-E\big[e^{-a_1\alpha_{-1}-(u_1+x_1)\alpha_0-b_1\beta_{-1}-(v_1+y_1)\beta_0-\vartheta_1 t_{-1}-\theta_1 t_0}\big]\Big\}\\
&\qquad\qquad\times\sum_{k>0}E\big[e^{-y_1(\eta_1+\cdots+\eta_{k-1})}\big(1-e^{-y_1\eta_k}\big)\big]
\end{aligned}$$

$$= \Big\{ \phi_0(a_1 + x_1, b_1, \vartheta_1, u_1, v_1 + y_1, \theta_1) - \phi_0(a_1, b_1, \vartheta_1, u_1 + x_1, v_1 + y_1, \theta_1) \Big\}$$
$$\times \sum_{k>0} \big[g(0, y_1, 0)\big]^{k-1} \big(1 - g(0, y_1, 0)\big)$$
$$= \Phi_0^1 - \Phi_0, \tag{3.26}$$

where the summation over $k > 0$ converges to 1 as per Lemma 1 of [5]: the associated convergence of $\sum\limits_{k>0} [g(0, y_1, 0)]^{k-1}$ is guaranteed provided that $Re(y_1) > 0$. The last line in (3.26) is due to notation (3.14)–(3.15).

Case: $j > 0$. This case also contains parts of functional ϕ_0 in the information related to the reference point t_0.

Transformation (3.25) for this case is

$$\sum_{j>0} \sum_{k>j} E\Big[e^{-a_1\alpha_{j-1} - u_1\alpha_j - b_1\beta_{j-1} - v_1\beta_j - \vartheta_1 t_{j-1} - \theta_1 t_j} \big(e^{-x_1\alpha_{j-1}} - e^{-x_1\alpha_j}\big)\big(e^{-y_1\beta_{k-1}} - e^{-y_1\beta_k}\big)\Big]$$
$$= \sum_{j>0} \sum_{k>j} \Big\{ E\big[e^{-(a_1+u_1+x_1)\alpha_{j-1} - (b_1+v_1+y_1)\beta_{j-1} - (\vartheta_1+\theta_1)t_{j-1}}\big]$$
$$\times E\big[e^{-u_1\xi_j}(1 - e^{-x_1\xi_j}) e^{-(v_1+y_1)\eta_j - \theta_1\Delta_j}\big] E\big[e^{-y_1(\eta_{j+1}+\cdots+\eta_{k-1})}(1 - e^{-y_1\eta_k})\big]\Big\}$$
$$= \sum_{j>0} \Big\{ E\big[e^{-(a_1+u_1+x_1)\alpha_0 - (b_1+v_1+y_1)\beta_0 - (\vartheta_1+\theta_1)t_0}\big]$$
$$\times E\Big[e^{-(a_1+u_1+x_1)(\xi_1+\cdots+\xi_{j-1}) - (b_1+v_1+y_1)(\eta_1+\cdots+\eta_{j-1}) - (\vartheta_1+\theta_1)(\Delta_1+\cdots+\Delta_{j-1})}\Big]$$
$$\times E\big[e^{-u_1\xi_j}(1 - e^{-x_1\xi_j}) e^{-(v_1+y_1)\eta_j - \theta_1\Delta_j}\big] \sum_{k>j} E\big[e^{-y_1(\eta_{j+1}+\cdots+\eta_{k-1})}(1 - e^{-y_1\eta_k})\big]\Big\}, \tag{3.27}$$

where the third factor can be written as

$$E\big[e^{-u_1\xi_j - (v_1+y_1)\eta_j - \theta_1\Delta_j}\big] - E\big[e^{-(u_1+x_1)\xi_j - (v_1+y_1)\eta_j - \theta_1\Delta_j}\big] = G^1 - G$$

(as per notation (3.9)–(3.10)) and the summation over $k > j$ converges to 1, for $Re(y_1) > 0$, as per Lemma 1 of [5]. Then, after some algebra in (3.27) and the use of notation (3.8)–(3.10) and (3.13), we arrive at

$$\phi_0(0, 0, 0, a_1 + u_1 + x_1, b_1 + v_1 + y_1, \vartheta_1 + \theta_1) \cdot \sum_{j>0} g^{j-1} \cdot (G^1 - G)$$
$$= \Phi_0^* \cdot \sum_{j>0} g^{j-1} \cdot (G^1 - G) = \frac{\Phi_0^*}{1 - g}(G^1 - G), \tag{3.28}$$

with the convergence of $\sum\limits_{j>0} g^{j-1}$ under the condition that the parameters of g satisfy:

$$Re(a_1 + u_1 + x_1) > 0, \quad Re(b_1 + v_1 + y_1) > 0, \quad Re(\vartheta_1 + \theta_1) > 0,$$

with any two of the three strict inequalities relaxed with $\geq$.

With the cases $j = 0$ and $j > 0$ combined together, we will arrive at

$$\mathcal{L}_{p_1q_1}(F_{1,2})(x_1, y_1) = (\Phi_0^1 - \Phi_0) + \frac{\Phi_0^*}{1 - g}(G^1 - G). \tag{3.29}$$

Case 2. $\boldsymbol{\nu_1(p_1) > \nu_2(q_1)}$. This will follow the paths of game 1 on the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\nu_1(p_1) > \nu_2(q_1)\}$ and yielding $\rho(p_1, q_1) = \nu_2(q_1)$.

With the roles of x_1 and y_1 interchanged, we find that

$$\begin{aligned}
&\mathcal{L}_{p_1q_1}(F_{2,1})(x_1, y_1) \\
&= \Big\{\phi_0(a_1, b_1 + y_1, \vartheta_1, u_1 + x_1, v_1, \theta_1) - \phi_0(a_1, b_1, \vartheta_1, u_1 + x_1, v_1 + y_1, \theta_1)\Big\} \\
&\quad +\phi_0(0, 0, 0, a_1 + u_1 + x_1, b_1 + v_1 + y_1, \vartheta_1 + \theta_1) \cdot \sum_{j>0} g^{j-1} \cdot (G^2 - G) \\
&= (\Phi_0^2 - \Phi_0) + \frac{\Phi_0^*}{1-g}(G^2 - G). \qquad (3.30)
\end{aligned}$$

Case 3. $\boldsymbol{\nu_1(p_1) = \nu_2(q_1)}$. This implies

$$\rho(p_1, q_1) = \nu_1(p_1) = \nu_2(q_1).$$

The corresponding transformation is

$$\begin{aligned}
&\mathcal{L}_{p_1q_1}(F_{12})(x_1, y_1) \\
&=\sum_{j\geq 0} E\Big[e^{-a_1\alpha_{j-1}-u_1\alpha_j-b_1\beta_{j-1}-v_1\beta_j-\vartheta_1 t_{j-1}-\theta_1 t_j}(e^{-x_1\alpha_{j-1}}-e^{-x_1\alpha_j})(e^{-y_1\beta_{j-1}}-e^{-y_1\beta_j})\Big]. \qquad (3.31)
\end{aligned}$$

Case: $j = 0$.

$$(e^{-x_1\alpha_{j-1}} - e^{-x_1\alpha_j})(e^{-y_1\beta_{j-1}} - e^{-y_1\beta_j})$$

for $j = 0$ give

$$= e^{-x_1\alpha_{-1}-y_1\beta_{-1}} - e^{-x_1\alpha_{-1}-y_1\beta_0} - e^{-x_1\alpha_0-y_1\beta_{-1}} + e^{-x_1\alpha_0-y_1\beta_0} \qquad (3.32)$$

and thus the transformation (3.31) can be written as

$$\begin{aligned}
&\phi_0(a_1 + x_1, b_1 + y_1, \vartheta_1, u_1, v_1, \theta_1) - \phi_0(a_1 + x_1, b_1, \vartheta_1, u_1, v_1 + y_1, \theta_1) \\
&-\phi_0(a_1, b_1 + y_1, \vartheta_1, u_1 + x_1, v_1, \theta_1) + \phi_0(a_1, b_1, \vartheta_1, u_1 + x_1, v_1 + y_1, \theta_1) \\
&= \Phi_0^{12} - \Phi_0^1 - \Phi_0^2 + \Phi_0 \qquad (3.33)
\end{aligned}$$

Case: $j > 0$.
Transformation (3.31) reads

$$\begin{aligned}
&\sum_{j>0} E\big[e^{-a_1\alpha_{j-1}-u_1\alpha_j-b_1\beta_{j-1}-v_1\beta_j-\vartheta_1 t_{j-1}-\theta_1 t_j}(e^{-x_1\alpha_{j-1}} - e^{-x_1\alpha_j})(e^{-y_1\beta_{j-1}} - e^{-y_1\beta_j})\big] \\
&= \sum_{j>0} E\big[e^{-(a_1+u_1+x_1)\alpha_{j-1}-(b_1+v_1+y_1)\beta_{j-1}-(\vartheta_1+\theta_1)t_{j-1}}\big] \\
&\quad \times E\big[e^{-u_1\xi_j-v_1\eta_j-\theta_1\Delta_j}(1 - e^{-x_1\xi_j})(1 - e^{-y_1\eta_j})\big]
\end{aligned}$$

$$= \varPhi_0^* \cdot \sum_{j>0} g^{j-1} \cdot (G^{12} - G^1 - G^2 + G) = \frac{\varPhi_0^*}{1-g}(G^{12} - G^1 - G^2 + G). \tag{3.34}$$

Thus,

$$\mathcal{L}_{p_1 q_1}(F_{12})(x_1, y_1) = (\varPhi_0^{12} - \varPhi_0^1 - \varPhi_0^2 + \varPhi_0) + \frac{\varPhi_0^*}{1-g}(G^{12} - G^1 - G^2 + G). \tag{3.35}$$

Finally after simple algebra, the sum all of three cases is

$$\begin{aligned}\mathcal{L}_{p_1 q_1}(\phi_{\rho(p_1,q_1)})(x_1, y_1) &= \mathcal{L}_{p_1 q_1}(F_{1,2})(x_1, y_1) + \mathcal{L}_{p_1 q_1}(F_{12})(x_1, y_1) + \mathcal{L}_{p_1 q_1}(F_{2,1})(x_1, y_1) \\ &= \varPhi_0^{12} - \varPhi_0 + \frac{\varPhi_0^*}{1-g}(G^{12} - G). \qquad \square\end{aligned}$$

4. The Restricted Random Walk

In this section we form a bridge from the first phase to the second phase (game 2). Because at the end of game 1, each player is supposed to have sustained only limited damage not in excess of M or N, and because the winner of game 1 is not specified, we need to reduce the damages to their maximal values of M or N in the event excesses take place. A similar procedure was rendered in [8]. Let us define

$$\widehat{\alpha}_\rho = \min\{\alpha_\rho, M\} \tag{4.1}$$

and

$$\widehat{\beta}_\rho = \min\{\beta_\rho, N\}. \tag{4.2}$$

The corresponding functional to be worked on is

$$\widehat{\phi}_\rho := \widehat{\phi}_\rho(a_2, b_2, \vartheta_2, u_2, v_2, \theta_2) = E\left[e^{-a_2\alpha_{\rho-1} - u_2\widehat{\alpha}_\rho - b_2\beta_{\rho-1} - v_2\widehat{\beta}_\rho - \vartheta_2 t_{\rho-1} - \theta_2 t_\rho}\right]. \tag{4.3}$$

Theorem 3 (below) is similar to that of [8] (applied to a different functional) but for the sake of consistency we give a proof.

Theorem 3. *The functional $\widehat{\phi}_\rho$ of the tandem game upon the beginning of phase 2 satisfies the following formula:*

$$\begin{aligned}\widehat{\phi}_\rho = {} & \mathcal{L}^{-1}_{x_2 y_2}\big[\phi_\rho(a_2, b_2, \vartheta_2, u_2 + x_2, v_2 + y_2, \theta_2) - e^{-u_2 M}\phi_\rho(a_2, b_2, \vartheta_2, x_2, v_2 + y_2, \theta_2) \\ & - e^{-v_2 N}\phi_\rho(a_2, b_2, \vartheta_2, u_2 + x_2, y_2, \theta_2) + e^{-u_2 M - v_2 N}\phi_\rho(a_2, b_2, \vartheta_2, x_2, y_2, \theta_2)\big](M, N) \\ & + e^{-v_2 N}\mathcal{L}^{-1}_{x_2}\big[\phi_\rho(a_2, b_2, \vartheta_2, u_2 + x_2, 0, \theta_2) - e^{-u_2 M}\phi_\rho(a_2, b_2, \vartheta_2, x_2, 0, \theta_2)\big](M) \\ & + e^{-u_2 M}\mathcal{L}^{-1}_{y_2}\big[\phi_\rho(a_2, b_2, \vartheta_2, 0, v_2 + y_2, \theta_2) - e^{-v_2 N}\phi_\rho(a_2, b_2, \vartheta_2, 0, y_2, \theta_2)\big](N) \\ & + e^{-u_2 M - v_2 N}\phi_\rho(a_2, b_2, \vartheta_2, 0, 0, \theta_2), \quad Re(x_2) > 0, \quad Re(y_2) > 0.\end{aligned} \tag{4.4}$$

Here $\mathcal{L}^{-1}_{xy}$ is the inverse of the Laplace–Carson transform introduced in the earlier sections.

Proof. Let

$$\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)=E\left[e^{-a_2\alpha_{\rho-1}-u_2\widehat{\alpha}_\rho-b_2\beta_{\rho-1}-v_2\beta_\rho-\vartheta_2 t_{\rho-1}-\theta_2 t_\rho}\right], \tag{4.5}$$

which is the "truncated" functional $\phi_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)$ only w.r.t. the first component α_ρ (but not β_ρ). That is, in the event the total damage to player A upon exit from game 1 exceeds M (while surely crossing M_1 which is greater than M), it will be reduced to M, because of our assumption on the maximum casualty to player A. Analogously, we introduce the truncated functional w.r.t. the second component β_ρ:

$$\widehat{\phi}_2(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)=E\left[e^{-a_2\alpha_{\rho-1}-u_2\alpha_\rho-b_2\beta_{\rho-1}-v_2\widehat{\beta}_\rho-\vartheta_2 t_{\rho-1}-\theta_2 t_\rho}\right], \tag{4.6}$$

which will represent the joint functional of the damages to players A and B and exit time from game 1, with restricted casulaties to player B, but not to player A.

Due to Theorem 2 [3], we have

$$\begin{aligned} T_1\phi_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2) &:= \widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)\\ =\mathcal{L}^{-1}_{x_2}\left[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,v_2,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,v_2,\theta_2)\right](M)&\\ +e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,0,v_2,\theta_2),& \end{aligned} \tag{4.7}$$

expressed through operator T_1 acting on variable u w.r.t. a fixed parameter M. Define operator T_2 which is similar to T_1, only acting on variable v w.r.t. another fixed parameter N:

$$\begin{aligned} T_2\phi_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2) &:= \widehat{\phi}_2(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)\\ =\mathcal{L}^{-1}_{y_2}\left[\phi_\rho(a_2,b_2,\vartheta_2,u_2,v_2+y_2,\theta_2)-e^{-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,u_2,y_2,\theta_2)\right](N)&\\ +e^{-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,u_2,0,\theta_2).& \end{aligned} \tag{4.8}$$

Thus, merging operators T_1 and T_2 makes

$$\begin{aligned} \widehat{\phi}_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2) &= T_2\circ T_1\phi_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)\\ =\mathcal{L}^{-1}_{y_2}\left[\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,v_2+y_2,\theta_2)-e^{-v_2N}\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,y_2,\theta_2)\right](N)&\\ +e^{-v_2N}\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,0,\theta_2).& \end{aligned} \tag{4.9}$$

Then, in light of (4.7), the first term of (4.9) can be rewritten as

$$\begin{aligned} &\mathcal{L}^{-1}_{y_2}\Big[\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,v_2+y_2,\theta_2)-e^{-v_2N}\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,y_2,\theta_2)\Big](N)\\ &=\Big\{\mathcal{L}^{-1}_{x_2y_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,v_2+y_2,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,v_2+y_2,\theta_2)\Big](M,N)\\ &\qquad+e^{-u_2M}\mathcal{L}^{-1}_{y_2}\left[\phi_\rho(a_2,b_2,\vartheta_2,0,v_2+y_2,\theta_2)\right](N)\Big\}\\ &\quad-e^{-v_2N}\Big\{\mathcal{L}^{-1}_{x_2y_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,y_2,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,y_2,\theta_2)\Big](M,N)\\ &\qquad+e^{-u_2M}\mathcal{L}^{-1}_{y_2}\left[\phi_\rho(a_2,b_2,\vartheta_2,0,y_2,\theta_2)\right](N)\Big\}\\ &=\mathcal{L}^{-1}_{x_2y_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,v_2+y_2,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,v_2+y_2,\theta_2)\\ &\quad-e^{-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,y_2,\theta_2)+e^{-u_2M-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,x_2,y_2,\theta_2)\Big](M,N) \end{aligned}$$

$$+e^{-u_2M}\mathcal{L}^{-1}_{y_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,0,v_2+y_2,\theta_2)-e^{-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,0,y_2,\theta_2)\Big](N) \tag{4.10}$$

and the second term

$$\begin{aligned}
&e^{-v_2N}\widehat{\phi}_1(a_2,b_2,\vartheta_2,u_2,0,\theta_2)\\
&= e^{-v_2N}\Big\{\mathcal{L}^{-1}_{x_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,0,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,0,\theta_2)\Big](M)\\
&\qquad +e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,0,0,\theta_2)\Big\}\\
&= e^{-v_2N}\mathcal{L}^{-1}_{x_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,0,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,0,\theta_2)\Big](M)\\
&\qquad +e^{-u_2M-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,0,0,\theta_2).
\end{aligned} \tag{4.11}$$

Hence,

$$\begin{aligned}
&T_2\circ T_1\phi_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)\\
&= \mathcal{L}^{-1}_{x_2y_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,v_2+y_2,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,v_2+y_2,\theta_2)\\
&\quad -e^{-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,y_2,\theta_2)+e^{-u_2M-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,x_2,y_2,\theta_2)\Big](M,N)\\
&\quad +e^{-v_2N}\mathcal{L}^{-1}_{x_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,u_2+x_2,0,\theta_2)-e^{-u_2M}\phi_\rho(a_2,b_2,\vartheta_2,x_2,0,\theta_2)\Big](M)\\
&\quad +e^{-u_2M}\mathcal{L}^{-1}_{y_2}\Big[\phi_\rho(a_2,b_2,\vartheta_2,0,v_2+y_2,\theta_2)-e^{-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,0,y_2,\theta_2)\Big](N)\\
&\quad +e^{-u_2M-v_2N}\phi_\rho(a_2,b_2,\vartheta_2,0,0,\theta_2),
\end{aligned} \tag{4.12}$$

which is a closed form of $\widehat{\phi}_\rho(a_2,b_2,\vartheta_2,u_2,v_2,\theta_2)$. □

5. The Final Phase

During the final phase of the game, the conflict may intensify. To realistically model this part of the conflict we assume the presence of different parameters beginning t_ρ, which is the exit time from phase 1. Perhaps there is a time of truce between the players lasting from t_ρ and ending at some epoch when game 2 starts, but this would be analytically insignificant and we bypass this time as nonexistent. We therefore allow game 2 to develop under different parameters, but inheriting the values of the process at t_ρ and $t_{\rho-1}$. As we will see, merging the two phases will automatically require us to bypass this time on attaching the initial phase to phase 1.

Now, the rest of the procedure is very similar to the development in our last paper [8], but to make this paper self-contained we include some details.

We assume that the processes describing the casualties to player A and B, as well as observation process, will be different.

We start again with some independent σ-algebras $\mathcal{F}_A,\mathcal{F}_B,\mathcal{F}_\mathcal{T}\subseteq\mathcal{F}(\Omega)$. We also assume that they are independent from the previously introduced σ-subalgebras $\mathcal{F}_{A^1}$, $\mathcal{F}_{B^1}$, $\mathcal{F}_T$.

We need two auxiliary marked Poisson random measures

$$\mathcal{P}_A:=\sum_{j\ge1}\pi_j^A\varepsilon_{\varphi_j}\ \text{ and }\ \mathcal{P}_B:=\sum_{k\ge1}\pi_k^B\varepsilon_{\zeta_k} \tag{5.1}$$

with respective intensities Λ_A and Λ_B and position independent marking, so that $\mathcal{P}_A$ and $\mathcal{P}_B$ are $\mathcal{F}_A$- and $\mathcal{F}_B$-measurable, respectively. Now based on (5.1), we form the damage processes:

$$\mathcal{A} := \sum_{j\geq 0} \pi_j^A \varepsilon_{\delta_j} \text{ and } \mathcal{B} := \sum_{k\geq 0} \pi_k^B \varepsilon_{\varsigma_k}, \tag{5.2}$$

where

$$\delta_0 = t_\rho, \quad \delta_j = t_\rho + \varphi_j, \quad j = 1, 2, \ldots, \quad \pi_0^A = \widehat{\alpha}_\rho, \tag{5.3}$$

$$\varsigma_0 = t_\rho, \quad \varsigma_k = t_\rho + \zeta_k, \quad k = 1, 2, \ldots, \quad \pi_0^B = \widehat{\beta}_\rho. \tag{5.4}$$

To attach game 2 to game 1, we use the delayed components of $\mathcal{A}$ and $\mathcal{B}$ in (5.3-5.4). While the increments of the associated point processes $\sum_{j\geq 0} \varepsilon_{\delta_j}$ and $\sum_{k\geq 0} \varepsilon_{\varsigma_k}$ are independent and exponentially distributed with respective parameters Λ_A and Λ_B as per (5.1)–(5.4), the associated marks (being a.s. nonnegative) counting from the first one are iid with the transforms

$$H_A(u) = Ee^{-u\pi_1^A}, \quad Re(u) \geq 0, \tag{5.5}$$

$$H_B(u) = Ee^{-u\pi_1^B}, \quad Re(u) \geq 0. \tag{5.6}$$

The start of game 2 will be at points δ_0 and ς_0 (actually, at $\min\{\delta_0, \varsigma_0\}$) and their initial positions π_0^A and π_0^B of processes (5.2). They are to be known from the functional of

$$\widehat{\phi}_\rho(a_2, b_2, \vartheta_2, u_2, v_2, \theta_2) = E\left[e^{-a_2\alpha_{\rho-1} - u_2\widehat{\alpha}_\rho - b_2\beta_{\rho-1} - v_2\widehat{\beta}_\rho - \vartheta_2 t_{\rho-1} - \theta_2 t_\rho}\right]$$

derived in Theorem 3. According to (5.3) and (5.4), the corresponding marginal transformations are

$$Ee^{-\theta\delta_0} = Ee^{-\theta\varsigma_0} = \widehat{\phi}_\rho(0, 0, 0, 0, 0, \theta), \tag{5.7}$$

$$Ee^{-u\widehat{\alpha}_\rho} = \widehat{\phi}_\rho(0, 0, 0, u, 0, 0), \quad Re(u) \geq 0, \tag{5.8}$$

$$Ee^{-v\widehat{\beta}_\rho} = \widehat{\phi}_\rho(0, 0, 0, 0, v, 0), \quad Re(v) \geq 0. \tag{5.9}$$

Notice that these two delayed Poisson marked processes describe the conflict between players A and B in the second phase, and they start in accordance with the truncated terminal conditions of the two-variate random walk process from the previous phase.

Furthermore, as in game 1, we introduce an $\mathcal{F}_{\mathcal{T}}$-measurable point process

$$\mathcal{T} := \sum_{i\geq 0} \varepsilon_{\tau_i}, \quad \tau_0 := t_\rho, \tag{5.10}$$

which is a delayed renewal process aimed to oversee the second phase of the game.

If

$$(A(t), B(t)) := \mathcal{A} \otimes \mathcal{B}([0, t]), \quad t \geq 0, \tag{5.11}$$

then

$$(A_j, B_j) := (A(\tau_j), B(\tau_j)) = \mathcal{A} \otimes \mathcal{B}([0, \tau_j]), \quad j = 0, 1, \ldots, \tag{5.12}$$

form observations of $\mathcal{A} \otimes \mathcal{B}$ embedded upon $\mathcal{T}$, with respective increments

$$(X_j, Y_j) = \mathcal{A} \otimes \mathcal{B}((\tau_{j-1}, \tau_j]), \ \ j = 1, \ldots, \tag{5.13}$$

$$X_0 = A_0, \quad Y_0 = B_0, \quad \tau_{-1} := t_{\rho-1}, \quad A_{-1} := \alpha_{\rho-1}, \quad B_{-1} := \beta_{\rho-1}. \tag{5.14}$$

Note that due to the formation of sequential games and the upcoming analysis, not only do the initial values of game 2 absorb the terminal values of game 1, but they will also need "pre-exit" values of game 1, which we then include in (5.14).

Obviously, the bivariate marked point measure

$$\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}} := \sum_{j \geq 0} (X_j, Y_j) \varepsilon_{\tau_j}, \tag{5.15}$$

with marginal measures

$$\mathcal{A}_{\mathcal{T}} = \sum_{i \geq 0} X_i \varepsilon_{\tau_i} \ \text{ and } \ \mathcal{B}_{\mathcal{T}} = \sum_{i \geq 0} Y_i \varepsilon_{\tau_i}, \tag{5.16}$$

is with position dependent marking thus making X_j and Y_j dependent. With the notation

$$\Delta_j := \tau_j - \tau_{j-1}, \ \ j = 1, 2, \ldots, \quad \Delta_0 := \tau_0, \tag{5.17}$$

we can evaluate the functional

$$\gamma(u, v, \theta) = E e^{-uX_j - vY_j - \theta \Delta_j} \tag{5.18}$$

using straightforward probabilistic arguments (cf. [5], formula (3.19)),

$$\gamma(u, v, \theta) = \widehat{\gamma}\left\{\theta + \Lambda_A(1 - H_A(u)) + \Lambda_B(1 - H_B(v))\right\}, \ \ j = 1, 2, \ldots, \tag{5.19}$$

where

$$\widehat{\gamma}(\theta) = E e^{-\theta \Delta_j} \tag{5.20}$$

is the marginal Laplace–Stieltjes transform of $\Delta_1, \Delta_2, \ldots$.

Now we introduce the exit indices of game 2:

$$\mu = \inf\left\{m \geq 0 : \ A_0 + X_1 + \cdots + X_m = A_m > M_2; \ A_0 = \widehat{\alpha}_\rho\right\}, \tag{5.21}$$

$$\nu = \inf\left\{n \geq 0 : \ B_0 + Y_1 + \cdots + Y_n = B_n > N_2; \ B_0 = \widehat{\beta}_\rho\right\}. \tag{5.22}$$

Since in game 2, and thus the game as the whole, we are interested in the paths that lead to the defeat of player A, the main functional of the game will be

$$\Phi_{\mu\nu} := \Phi_{\mu\nu}(a, a', b, b', h, h') = E\left[e^{-aA_{\mu-1} - a'A_\mu - bB_{\mu-1} - b'B_\mu - h\tau_{\mu-1} - h'\tau_\mu} \mathbf{1}_{\{\mu < \nu\}}\right] \tag{5.23}$$

Definition 2. Game 2 will be defined as the random measure

$$[\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}}]_\mu = [\mathcal{A}^1 \otimes \mathcal{B}^1]_\rho + \sum_{j=0}^{\mu} (X_j, Y_j) \varepsilon_{\tau_j}$$

on $(\Omega, \mathcal{F}(\Omega), (\mathfrak{F}_t)_{t=0}^{\tau_\mu}, P)$, with the history $[\mathcal{A}^1 \otimes \mathcal{B}^1]_\rho$ of the conflict and the pair of initial conditions:

$$(A_{-1} = A(\tau_{-1}) = \alpha_{\rho-1}, \quad B_{-1} = B(\tau_{-1}) = \beta_{\rho-1}), \tag{5.24}$$

$$(A_0 = A(\tau_0) = \widehat{\alpha}_\rho, \quad B_0 = B(\tau_0) = \widehat{\beta}_\rho). \tag{5.25}$$

The moment of time τ_μ is called the end of game 2 and this will also be the end of the entire game. Game 2 will be observed along the paths from the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$, which all terminate at τ_μ. The game process will be adapted to the head-filtration $(\mathfrak{F}_t)_{t=0}^{\tau_\mu}$.

Below Theorem 4 establishes an explicit formula for $\Phi_{\mu\nu}$. We use the following abbreviations based on (5.18) with all involved variables being fixed:

$$\gamma := \gamma(a + a' + c, b + b' + s, h + h'), \tag{5.26}$$

$$\Gamma := \gamma(a' + c, b' + s, h'), \tag{5.27}$$

$$\Gamma^1 := \gamma(a', b' + s, h'). \tag{5.28}$$

Theorem 4. *The tandem game of two players A and B is with the both players initiating the conflict and with player A losing the entire game; the functional $\Phi_{\mu\nu}$ describing the game satisfies the following formula:*

$$\begin{aligned}
&\Phi_{\mu\nu}(a, a', b, b', h, h') \\
&= \mathcal{L}_{cs}^{-1}\Big[\widehat{\phi}_\rho(a + c, b, h, a', b' + s, h') - \widehat{\phi}_\rho(a, b, h, a' + c, b' + s, h') \\
&\quad + \widehat{\phi}_\rho(0, 0, 0, a + a' + c, b + b' + s, h + h') \cdot \frac{1}{1-\gamma}\left(\Gamma^1 - \Gamma\right)\Big](M_2, N_2)
\end{aligned}$$

under the conditions that

$$Re(c) > 0, \quad Re(s) > 0, \quad Re(a + a' + c) > 0, \quad Re(b + b' + s) > 0, \quad e(h + h') > 0, \tag{5.29}$$

with any two strict inequalities from the latter three being replaced with $\geq$.

The proof of Theorem 4 is identical to that of Theorem 3 of [8] and thus will be omitted.

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Chapter 3

SOME STABILITY RESULTS FOR EQUATIONS AND INEQUALITIES CONNECTED WITH THE EXPONENTIAL FUNCTION

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Abstract

We generalize some earlier results connected with the Hyers-Ulam stability of functional equations and inequalities related to the exponential function. In our first theorem we prove the stability of equation $f = f'$ in reflexive normed spaces. Further, we apply this result jointly with some other facts to prove the stability of several related functional equations.

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1. Introduction

Throughout the paper I stands for an open real interval, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N} = \{1, 2, \ldots\}$.

C. Alsina and J. L. Garcia-Roig [1] investigated the following functional inequalities:

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(x) + f(y)}{2}, \quad x, y \in \mathbb{R}, \quad x \neq y, \tag{1}$$

and

$$0 \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(x) + f(y)}{2}, \quad x, y \in \mathbb{R}, \quad x \neq y. \tag{2}$$

They have proved that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies (1) if and only if there exists a nonincreasing function $d\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in \mathbb{R}$ [1, Theorem 1].

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Further, $f\colon \mathbb{R} \to \mathbb{R}$ is a solution of (2) if and only if there exists a continuous nonincreasing function $d\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in \mathbb{R}$ and $d(x+t) \geq e^{-t}d(x)$ for $x \in \mathbb{R}$ and $t > 0$ [1, Theorem 2]. Moreover, their results remain true (with the same proofs) if the real line $\mathbb{R}$ is replaced by an open interval I.

C. Alsina and R. Ger [2] and later W. Fechner [3] have dealt with the following functional inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(y)-f(x)}{y-x}, \quad x, y \in I, \quad x < y. \tag{3}$$

In particular, it has been proved that if a function $f\colon I \to \mathbb{R}$ satisfies (3) jointly with the following condition:

$$\limsup_{h \to 0+} f(x+h) \geq f(x), \quad x \in I. \tag{4}$$

then there exists a nondecreasing nonnegative function $i\colon I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for $x \in I$ [3, Theorem 2].

In [4] the following inequality is examined:

$$6\,\frac{f(y)-f(x)}{y-x} \leq 4f\left(\frac{x+y}{2}\right) + f(x) + f(y), \quad x, y \in I, \quad x \neq y, \tag{5}$$

as well as a more general functional inequality:

$$\frac{f(y)-f(x)}{y-x} \leq N\left(g(M_1(x,y)), g(M_2(x,y))\right), \quad x, y \in I, \quad x \neq y, \tag{6}$$

where M_1, M_2 and N stand for arbitrary means and N is continuous. It was proved that solutions of (5) which satisfy (4) are of the form $f(x) = i(x)e^x$ for $x \in I$, where i is a nondecreasing function. Further, continuous solutions of (6) satisfy inequality

$$f(y) - f(x) \leq \int_x^y g(t)\,dt, \quad x, y \in I, \quad x \leq y. \tag{7}$$

Remark 1. The representation of function $f\colon I \to \mathbb{R}$ in the form $f(x) = d(x)e^x$ or $f(x) = i(x)e^x$ with d and i being arbitrary nonincreasing and a nondecreasing function, respectively, is equivalent to the validity of the following respective estimations:

$$f(y) \geq f(x) \cdot e^{y-x}, \quad x, y \in I, \quad x < y;$$
$$f(y) \leq f(x) \cdot e^{y-x}, \quad x, y \in I, \quad x < y.$$

It seems that C. Alsina and R. Ger [2] were the first authors who investigated the Hyers–Ulam stability of differential equations. In particular, they have shown [2, Theorem 1 and Remark] that the equation $f' = f$ is stable; more precisely, they have proved that given an $\varepsilon \geq 0$ if $f\colon I \to \mathbb{R}$ is a differentiable mapping such that

$$|f'(x) - f(x)| \leq \varepsilon, \quad x \in I,$$

then there exists a constant $c_0 \in \mathbb{R}$ such that

$$|f(x) - c_0 \cdot e^x| \le 3\varepsilon, \;\; x \in I.$$

The purpose of the present paper is to investigate the Hyers–Ulam stability of the above-mentioned functional inequalities and of related functional equations. Moreover, we generalize the above-mentioned result of C. Alsina and R. Ger from [2] for mappings with values in a reflexive normed linear space.

2. Results

Our first result is the following theorem.

Theorem 1. *Given an $\varepsilon \ge 0$ and a reflexive real normed linear space $(X, \|\cdot\|)$ assume that $f\colon (0,+\infty) \to X$ is a differentiable mapping such that*

$$\|f'(x) - f(x)\| \le \varepsilon, \;\; x \in (0,+\infty).$$

Then there exists a vector $c_0 \in X$ such that

$$\|f(x) - e^x c_0\| \le 3\varepsilon, \;\; x \in (0,+\infty). \tag{8}$$

Proof. For every member x^* of the closed unit ball S^* in the dual space X^* one has

$$\left|x^*(f'(x) - f(x))\right| \le \varepsilon, \;\; x \in (0,+\infty),$$

stating that the map $x^* \circ f$ satisfies the assumptions of the above-mentioned result of C. Alsina and R. Ger from [2]. Therefore, there exists a real constant $c = c(x^*)$ such that

$$\left|(x^* \circ f)(x) - c(x^*)e^x\right| \le 3\varepsilon, \;\; x \in (0,+\infty). \tag{9}$$

Clearly, we may assume that the assignment

$$S^* \ni x^* \mapsto c(x^*) \in \mathbb{R}$$

yields a function from S^* into $\mathbb{R}$. Moreover, for every other element $y^* \in S^*$ such that $\|x^* + y^*\| \le 1$ one has also

$$\left|(y^* \circ f)(x) - c(y^*)e^x\right| \le 3\varepsilon, \;\; x \in (0,+\infty),$$

and

$$\left|((x^* + y^*) \circ f)(x) - c(x^* + y^*)e^x\right| \le 3\varepsilon, \;\; x \in (0,+\infty).$$

Consequently,

$$\left|[c(x^* + y^*) - c(x^*) - c(y^*)]e^x\right| \le 9\varepsilon, \;\; x \in (0,+\infty),$$

whence, by passing here to the infinity with x, we deduce that

$$x^*, y^*, x^* + y^* \in S^* \;\text{ implies }\; c(x^* + y^*) = c(x^*) + c(y^*). \tag{10}$$

Likewise, if $\alpha \in [-1,1]$ and $x^* \in S^*$ we get

$$\left|\left[((\alpha x^*)\circ f)(x) - c(\alpha x^*)\right]e^x\right| \leq 3\varepsilon, \quad x \in (0,+\infty),$$

as well as

$$\left|\left[((\alpha x^*)\circ f)(x) - \alpha c(x^*)\right]e^x\right| \leq 3|\alpha|\varepsilon, \quad x \in (0,+\infty),$$

whence

$$\left|\left[c(\alpha x^*) - \alpha c(x^*)\right]e^x\right| \leq 3(1+|\alpha|)\varepsilon, \quad x \in (0,+\infty),$$

which forces the equality

$$c(\alpha x^*) = \alpha c(x^*) \tag{11}$$

for every $\alpha \in [-1,1]$ and every $x^* \in S^*$.

Now, we may extend c onto the whole dual space X^* by putting

$$c(x^*) := nc\left(\frac{1}{n}\,x^*\right), \quad x^* \in X^*,$$

where $n \in \mathbb{N}$ is large enough to have $\frac{1}{n}\|x^*\| \leq 1$. To see that such an extension is well defined (does not depend on the choice of n) we apply a standard reasoning based on (10) and (11). For, observe first that (10) implies (induction)

$$nc\left(\frac{1}{n}\,y^*\right) = c(y^*) \ \text{ provided that } \ y^* \in S^*.$$

Therefore, fixing arbitrarily a member x^* of the dual space X^* and taking $n, m \in \mathbb{N}$ large enough to have $\frac{1}{n}x^*, \frac{1}{m}x^*$ in S^* we infer that

$$nc\left(\frac{1}{nm}\,x^*\right) = c\left(\frac{1}{m}\,x^*\right)$$

as well as

$$mc\left(\frac{1}{nm}\,x^*\right) = c\left(\frac{1}{n}\,x^*\right).$$

Consequently,

$$\frac{1}{n}\,c\left(\frac{1}{m}\,x^*\right) = c\left(\frac{1}{nm}\,x^*\right) = \frac{1}{m}c\left(\frac{1}{n}\,x^*\right),$$

whence

$$mc\left(\frac{1}{m}\,x^*\right) = nc\left(\frac{1}{n}\,x^*\right).$$

Now, a simple calculation shows that $c\colon X^* \to \mathbb{R}$ is both additive and homogeneous, i.e. linear. On the other hand, in virtue of (9) applied for $x = 1$, we get

$$\begin{aligned}\sup_{\|x^*\|\leq 1} |c(x^*)| &\leq \sup_{\|x^*\|\leq 1} \frac{1}{e}\,\left(3\varepsilon + \left|(x^*\circ f)(1)\right|\right)\\ &\leq \frac{3}{e}\,\varepsilon + \sup_{\|x^*\|\leq 1}\frac{1}{e}\left|(x^*\circ f)(1)\right| \leq \frac{3}{e}\,\varepsilon + \frac{1}{e}\,\|f(1)\| < +\infty\end{aligned}$$

stating that c is continuous. Thus

$$c \in X^{**}$$

and, due to the reflexivity of X, there exists a $c_0 \in X$ such that

$$c(x^*) = x^*(c_0)$$

for all $x^* \in X^*$. Consequently, relation (9) assumes the form

$$\left|x^*(f(x) - e^x c_0)\right| \leq 3\varepsilon, \quad x \in (0, +\infty), \quad x^* \in S^*,$$

which immediately implies the assertion claimed. □

Remark 2. Generalizations to semireflexive locally convex linear topological spaces in the spirit of L. Székelyhidi (see [8]) as well as to sequentially complete locally convex linear topological spaces (see Z. Gajda [5] and [6]) are also possible. We have omitted the details of such approach to keep greater readibility of the statements.

In what follows we will deal with the stability of functional inequalities (1) and (3) and we use the results obtained to prove stability of the corresponding functional equations. Then, we derive similar results for (5). Finally, we will focus on the stability of (6) and the related equality.

Remark 3. Using a result of J. Ger [7, Theorem 1] concerning the Sahoo–Riedel equations on an interval one can easily check that the only solution of each of the following functional equations

$$\frac{f(x) + f(y)}{2} = \frac{f(x) - f(y)}{x - y}\,, \quad x, y \in I, \quad x < y;$$

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) - f(y)}{x - y}\,, \quad x, y \in I, \quad x < y; \tag{12}$$

and

$$f(x) + f(y) + 4f\left(\frac{x + y}{2}\right) = 6\frac{f(x) - f(y)}{x - y}\,, \quad x, y \in I, \quad x < y, \tag{13}$$

is the zero function $f = 0$. Therefore, in order to prove that these equations are stable in the sense of Hyers–Ulam, we need to show that each solution of the corresponding “approximate equation” is in a sense close to the zero function.

Proposition 1. *Assume that $\varepsilon \geq 0$ and $f_0 \colon I \to \mathbb{R}$ satisfies*

$$\frac{f_0(y) - f_0(x)}{y - x} \leq \frac{f_0(x) + f_0(y)}{2} + \varepsilon, \quad x, y \in \mathbb{R}, \quad x \neq y.$$

Then $f_0(x) = f(x) - \varepsilon$ for $x \in I$, where f is a solution of (1).

Proof. It is enough to put $f(x) := f_0(x) + \varepsilon$ for $x \in I$ and check that f satisfies (1). □

Now, we are going to discuss the following stability problem:

$$\left\| \frac{1}{y-x}\left[f(y)-f(x)\right]-\frac{f(x)+f(y)}{2} \right\| \leq \varepsilon, \quad x,y \in I, \quad x<y, \tag{14}$$

where f maps I into a normed linear space (not necessarily complete). Our result reads as follows.

Theorem 2. *Given an $\varepsilon \geq 0$, let $(X, \|\cdot\|)$ be a real normed linear space and let $f\colon I \to X$ satisfy* (14). *Then the estimate*

$$\left\| f(y) - e^{y-x} f(x) \right\| \leq [e^{y-x}-1]\varepsilon, \quad x,y \in I, \quad x<y, \tag{15}$$

holds true.

Proof. First, we derive (15) from Proposition 1 and C. Alsina and J. L. Garcia–Roig [1, Theorem 1] in a special case, where $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$. In this situation (14) is equivalent to the following system of two functional inequalities:

$$\begin{aligned} \frac{f(x)+f(y)}{2} &\leq \frac{f(y)-f(x)}{y-x}+\varepsilon, \quad x,y \in I, \quad x<y; \\ \frac{f(x)+f(y)}{2} &\geq \frac{f(y)-f(x)}{y-x}-\varepsilon, \quad x,y \in I, \quad x<y. \end{aligned}$$

Now, apply Proposition 1 jointly with [1, Theorem 1] and Remark 1 twice, for f and for $-f$, to get

$$\begin{gathered} f(y)+\varepsilon \leq [f(x)+\varepsilon]e^{y-x}, \quad x,y \in I, \quad x<y; \\ -f(y)+\varepsilon \leq [-f(x)+\varepsilon]e^{y-x}, \quad x,y \in I, \quad x<y, \end{gathered}$$

which leads to

$$\left| f(y) - e^{y-x} f(x) \right| \leq [e^{y-x}-1]\varepsilon, \quad x,y \in I, \quad x<y.$$

Now, the general case can be derived from the real one. Indeed, if f satisfies (14), then for each linear and continuous functional x^* the map $x^* \circ f$ satisfies the real version of this inequality and (15) follows from the Hahn–Banach Theorem. □

In a similar way we may proceed with the inequality (3). In order to make [3, Theorem 2] applicable we will be assuming that the functions in question are continuous (condition (4) needs to be satisfied for both mappings f and $-f$).

Proposition 2. *Given an $\varepsilon \geq 0$, let $f_0\colon I \to \mathbb{R}$ satisfy*

$$f_0\left(\frac{x+y}{2}\right) \leq \frac{f_0(y)-f_0(x)}{y-x}+\varepsilon, \quad x,y \in I, \quad x<y.$$

Then $f_0(x) = f(x)+\varepsilon$ for $x \in I$, where f is a solution of (3).

Theorem 3. *Given an* $\varepsilon \geq 0$, *let* $(X, \|\cdot\|)$ *be a real normed linear space and let* $f \colon I \to X$ *be a continuous solution to the functional inequality*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\,[f(y) - f(x)] \right\| \leq \varepsilon, \quad x, y \in I, \quad x < y. \tag{16}$$

Then the estimate (15) *holds true.*

Proof. It suffices to apply Proposition 2, then [3, Theorem 2], Remark 1 and the Hahn–Banach Theorem. □

One may obtain analogous results for equation (13). It suffices to apply [4, Theorem 7] and repeat the reasoning used previously.

Proposition 3. *Given an* $\varepsilon \geq 0$, *let* $f_0 \colon I \to \mathbb{R}$ *satisfy*

$$6\,\frac{f_0(y) - f_0(x)}{y-x} \leq 4 f_0\left(\frac{x+y}{2}\right) + f_0(x) + f_0(y) + \varepsilon,$$

for each $x, y \in I$ *such that* $x < y$. *Then* $f_0(x) = f(x) - \frac{1}{6}\varepsilon$ *for* $x \in I$, *where* f *is a solution of* (5).

Theorem 4. *Given an* $\varepsilon \geq 0$, *let* $(X, \|\cdot\|)$ *be a real normed linear space and let*$f \colon I \to X$ *be continuous and satisfies*

$$\left\| \frac{2}{3}\, f\left(\frac{x+y}{2}\right) + \frac{f(x) + f(y)}{6} - \frac{1}{y-x}\,[f(y) - f(x)] \right\| \leq \varepsilon, \tag{17}$$

for each $x, y \in I$ *such that* $x < y$. *Then the estimate* (15) *holds true.*

Remark 4. In view of Remark 3, it is reasonable to expect that each continuous solution of (14), (16) and (17) has to be in a sense close to the zero function. That is really the case. We will derive this fact from the previous theorems. For (16) let $f \colon I \to X$ satisfy assumptions of Theorem 3. Fix $x \in I$ and $h > 0$ such that $x + 2h \in I$. By Theorem 3 the estimate (15) is valid whence

$$\|f(x+h) - e^h f(x)\| \leq [e^h - 1]\varepsilon, \tag{18}$$

$$\|f(x+2h) - e^{2h} f(x)\| \leq [e^{2h} - 1]\varepsilon. \tag{19}$$

On the other hand, (16) applied for $y = x + 2h$ implies that

$$\left\| f(x+h) - \frac{1}{2h}\, f(x+2h) + \frac{1}{2h}\, f(x) \right\| \leq \varepsilon.$$

Let us rewrite these three estimates in the following form:

$$\|2hf(x+h) - 2he^h f(x)\| \leq [2he^h - 2h]\varepsilon,$$

$$\| - f(x+2h) + e^{2h} f(x)\| \leq [e^{2h} - 1]\varepsilon,$$

$$\| - f(x) - 2hf(x+h) + f(x+2h)\| \leq 2h\varepsilon.$$

Adding these three inequalities side-by-side we finally arrive at

$$\|f(x)\| \le \frac{e^{2h} + 2he^h - 1}{e^{2h} - 2he^h - 1} \cdot \varepsilon.$$

Note that the right-hand side of this estimation does not depend upon x. Therefore, we may conclude that in the class of continuous mappings the functional equation (12) is stable in the sense of Hyers-Ulam. Moreover, if the interval I is unbounded from the right, then we may pass with h to $+\infty$, to obtain the estimation

$$\|f(x)\| \le \varepsilon, \quad x \in I. \tag{20}$$

An analogous reasoning can be applied for (14) (without assuming the continuity of f). Indeed, with the aid of this estimation for $y = x + 2h$ we get

$$\left\| f(x) + f(x+2h) - \frac{1}{h}\left[f(x+2h) - f(x)\right] \right\| \le 2\varepsilon,$$

which jointly with (19) (which is valid by Theorem 2) gives us

$$\|f(x)\| \le \left| \frac{(h-1)(e^{2h}-1) + 2h}{(h-1)e^{2h} + h + 1} \right| \cdot \varepsilon, \quad x \in I, \quad h > 0, \quad x + h \in I,$$

stating that (14) is stable. Again, if additionally I is unbounded from the right, then by letting h tend to $+\infty$ we can see that (20) holds.

Finally, (17) applied for $y = x + 2h$ implies that

$$\left\| 4hf(x+h) + (h+3)f(x) + (h-3)f(x+2h) \right\| \le 6h\varepsilon,$$

which jointly with (18) and (19) (which is valid by Theorem 4) leads to the estimation

$$\|f(x)\| \le \left| \frac{he^{2h} + 4he^h + h - 1}{(h-3)e^{2h} + 4he^h + h + 3} \right| \cdot \varepsilon, \quad x \in I, \quad h > 0, \quad x + h \in I,$$

stating that (14) is stable in the sense of Hyers–Ulam. Again, if additionally I is unbounded from the right, then (20) holds.

Now, we will discuss the stability of a more general equality:

$$\frac{f(y) - f(x)}{y - x} = N\left(g(M_1(x,y)), g(M_2(x,y))\right), \quad x, y \in I, \quad x \neq y, \tag{21}$$

under an additional assumption that there exist an injective function $\varphi\colon I \to \mathbb{R}$ such that

$$N(u+h, v+h) = N(u,v) + \varphi(h), \tag{22}$$

for each $u, v \in I$ and $h \in \mathbb{R}$ such that $u + h, v + h \in I$.

Proposition 4. *Given an $\varepsilon \ge 0$, let M_1, M_2 and N be arbitrary means and assume that N satisfies* (22) *with an injective mapping $\varphi\colon I \to \mathbb{R}$ and $f_0\colon I \to \mathbb{R}$ and $g_0\colon I \to \mathbb{R}$ satisfy*

$$\frac{f_0(y) - f_0(x)}{y - x} \le N\left(g_0(M_1(x,y)), g_0(M_2(x,y))\right) + \varepsilon,$$

for each $x, y \in I$ such that $x < y$. Then $f_0 = f$ and $g_0(x) = g(x) - \varphi^{-1}(\varepsilon)$ for $x \in I$,where (f, g) is a solution of (6).

By a suitable modification of some previously used arguments, we obtain the following result, which states that in the stability inequality the general mean appearing in the right-hand side of (21) can be replaced by the integral mean.

Theorem 5. *Given an* $\varepsilon \geq 0$*, let* M_1*,* M_2 *and* N *be arbitrary means and assume that* N *is continuous and satisfies* (22) *with an injective mapping* $\varphi\colon I \to \mathbb{R}$ *and* $f\colon I \to \mathbb{R}$ *and* $g\colon I \to \mathbb{R}$ *satisfy*

$$\left|\frac{f(y)-f(x)}{y-x} - N\left(g(M_1(x,y)), g(M_2(x,y))\right)\right| \leq \varepsilon,$$

for each $x, y \in I$ *such that* $x < y$*. Then*

$$\left|\frac{f(y)-f(x)}{y-x} - \frac{1}{y-x}\int\limits_x^y g(t)\,dt\right| \leq \varphi^{-1}(\varepsilon), \quad x, y \in I.$$

We terminate the paper with a stability result for (7).

Proposition 5. *Assume that* $\varepsilon \geq 0$ *and* $f_0\colon I \to \mathbb{R}$ *is a continuous function satisfying*

$$\frac{1}{y-x}\int\limits_x^y f_0(t)\,dt \leq \frac{f_0(y)-f_0(x)}{y-x} + \varepsilon, \quad x, y \in I, \quad x < y.$$

Then $f_0(x) = f(x) + \varepsilon$ *for* $x \in I$*, where* f *is a solution of* (7)*.*

Proof. Put $f(x) := f_0(x) - \varepsilon$ for $x \in I$ and check that f satisfies (7). □

Corollary 1. *Given an* $\varepsilon \geq 0$ *and a reflexive real normed linear space* $(X, \|\cdot\|)$ *assume that* $f\colon (0, +\infty) \to X$ *is a differentiable mapping such that*

$$\left\|\frac{f(y)-f(x)}{y-x} - \frac{1}{y-x}\int\limits_x^y f(t)\,dt\right\| \leq \varepsilon, \quad x, y \in I, \quad x < y.$$

Then there exists a vector $c_0 \in X$ *such that the estimation* (8) *holds true.*

Proof. Clearly, a differentiable function is Pettis (Bochner) integrable. Consequently, the integral mean

$$\frac{1}{y-x}\int\limits_x^y f(t)\,dt$$

will tend to the value $f(x)$ whenever y tends to x. Therefore, it suffices to pass $y \to x$ and to apply Theorem 1. □

Corollary 2. *Given an* $\varepsilon \geq 0$*, let* M_1*,* M_2 *and* N *be arbitrary means and assume that* N *is continuous and satisfies* (22) *with an injective mapping* $\varphi\colon I \to \mathbb{R}$ *whereas* $f\colon I \to \mathbb{R}$ *satisfies*

$$\left|\frac{f(y)-f(x)}{y-x} - N\left(f(M_1(x,y)), f(M_2(x,y))\right)\right| \leq \varepsilon,$$

for each $x, y \in I$ such that $x < y$. Then, there exists a constant $c \in \mathbb{R}$ such that

$$|f(x) - ce^x| \leq 3\varphi^{-1}(\varepsilon), \quad x \in I.$$

Added in proof. On January 20, 2009 some of the above results were presented at the Seminar on Functional Equations held at the Silesian University of Katowice (Poland). During the discussion Peter Volkmann [9] remarked that the re exivity assumption in Theorem 1 (and a fortiori in Corollary 1) may be dropped.

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Chapter 4

ON A PROBLEM OF JOHN M. RASSIAS CONCERNING THE STABILITY IN ULAM SENSE OF EULER–LAGRANGE EQUATION

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Abstract

In this paper we solve a problem posed by John M. Rassias in 1992, concerning the stability of Euler-Lagrange equation in the Ulam sense.

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Key words: Ulam stability, Euler–Lagrange mapping.

1. Introduction

The study of stability problems for functional equations originated from a talk of S. Ulam in 1940 (see [19]) when he proposed the following problem:

Let G be a group endowed with a metric d. Given $\varepsilon > 0$, does there exist a $k > 0$ such that for every function $f : G \to G$ satisfying the inequality

$$d(f(xy), f(x)f(y)) < \varepsilon, \ \ \forall\, x, y \in G,$$

there exists an automorphism a of G with

$$d(f(x), a(x)) < k\varepsilon, \ \ \forall\, x \in G \ \ ?$$

For results concerning this area see the papers [1]–[18]. In the following theorem John M. Rassias [12] proved the stability of Euler–Lagrange equation in the Ulam sense:

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Theorem 1.1. *Let X be a normed linear space, Y be a Banach space, and $f : X \to Y$. If there exists $a \geq 0$, $b \geq 0$ such that $a + b < 2$, and $c_2 \geq 0$ such that:*

$$\|f(x+y) + f(x-y) - 2 \cdot [f(x) + f(y)]\| \leq c_2 \cdot \|x\|^a \cdot \|y\|^b$$

for all $x, y \in X$, there exists a unique non-linear mapping $N : X \to Y$ such that:

$$\|f(x) - N(x)\| \leq c \cdot \|x\|^{a+b}$$

and

$$N(x+y) + N(x-y) = 2 \cdot [N(x) + N(y)]$$

for all $x, y \in X$, where $c = c_2/(4 - 2^{a+b})$

In the same paper he puts the following question: "What is the situation in the above theorem in the case $a + b = 2$?" In the present paper we give an answer to this problem.

A similar result for Cauchy functional equation was obtained in 1999 by the second author of this paper[4].

2. The Result

Theorem 2.1. *Let be $0 < a < 2$. Then, there exists a mapping $f : \mathbb{R} \to \mathbb{R}$ so that:*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq k|x|^a|y|^{2-a}, \tag{2.1}$$

where k does not depend on x, y, for all $x, y \in \mathbb{R}$ and for any quadratic mapping $Q : \mathbb{R} \to \mathbb{R}$ and every $\alpha \in \mathbb{R}$ we have:

$$\sup_{x \neq 0} \frac{|f(x) - Q(x)|}{|x|^\alpha} = \infty \tag{2.2}$$

Proof. We take the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \ln|x|, & \text{if } \ x \neq 0 \\ 0, & \text{if } \ x = 0 \end{cases}$$

Step (I). We verify (2.1) for all $x \geq y > 0$. For $x = y > 0$:

$$\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right| = 4x^2 \ln 2x - 4x^2 \ln x = (4 \ln 2)x^2.$$

If $x > y > 0$ we have:

$$\begin{aligned}
&\frac{f(x+y) + f(x-y) - 2f(x) - 2f(y)}{x^a y^{2-a}} \\
&\quad = \frac{(x+y)^2 \ln(x+y) + (x-y)^2 \ln(x-y) - 2x^2 \ln x - 2y^2 \ln y}{x^a y^{2-a}} \\
&\quad = \frac{(x^2+y^2) \ln(x^2-y^2) + 2xy[\ln(x+y) - \ln(x-y)] - x^2 \ln x^2 - y^2 \ln y^2}{x^a y^{2-a}}
\end{aligned}$$

$$= \frac{x^2 \ln \frac{x^2-y^2}{x^2} + y^2 \ln \frac{x^2-y^2}{y^2} + 2xy \ln \frac{x+y}{x-y}}{x^a y^{2-a}}$$
$$= \left(\frac{x}{y}\right)^{2-a} \ln\left(1-\frac{y^2}{x^2}\right) + \left(\frac{y}{x}\right)^a \ln\left(\frac{x^2}{y^2}-1\right) + 2\left(\frac{x}{y}\right)^{1-a} \ln \frac{\frac{x}{y}+1}{\frac{x}{y}-1}$$
$$= t^{2-a} \ln\left(1-\frac{1}{t^2}\right) + \frac{1}{t^a} \ln\left(t^2-1\right) + 2t^{1-a} \ln \frac{t+1}{t-1},$$

where $t := \frac{x}{y} > 1$.

We prove that the function $F_a : (1,\infty) \to \mathbb{R}$,

$$F_a(t) = t^{2-a} \ln\left(1-\frac{1}{t^2}\right) + \frac{1}{t^a} \ln\left(t^2-1\right) + 2t^{1-a} \ln \frac{t+1}{t-1}$$

is bounded.

We prove that

$$\lim_{t\to\infty} F_a(t) = 0. \tag{2.3}$$

First term:

$$\lim_{t\to\infty} t^{2-a} \ln\left(1-\frac{1}{t^2}\right) \underset{\frac{1}{t}=u}{=} \lim_{\substack{u\to 0\\ u>0}} u^{a-2} \ln(1-u^2)$$
$$= \lim_{\substack{u\to 0\\ u>0}} \frac{\ln(1-u^2)}{u^{2-a}}$$
$$= \lim_{\substack{u\to 0\\ u>0}} \frac{\frac{-2u}{1-u^2}}{(2-a)u^{1-a}}$$
$$= -\frac{2}{2-a} \lim_{\substack{u\to 0\\ u>0}} \frac{u^a}{1-u^2}.$$

Second term:

$$\lim_{t\to\infty} \frac{\ln(t^2-1)}{t^a} = \lim_{t\to\infty} \frac{\frac{2t}{t^2-1}}{at^{a-1}}$$
$$= \frac{2}{a} \lim_{t\to\infty} \frac{t^{2-a}}{t^2-1}$$
$$= 0,$$

since $2-a<2$.

The last term:

$$\lim_{t\to\infty} t^{1-a} \ln \frac{t+1}{t-1} = \lim_{t\to\infty} \frac{1}{t^a} \ln\left(\frac{t+1}{t-1}\right)^t = 0\cdot 2 = 0$$

We prove that:

$$\lim_{\substack{t\to 1\\ t>1}} F_a(t) \tag{2.4}$$

exists and it is finite. We write:

$$\begin{aligned}F_a(t) &= t^{2-a}\ln(t-1) + \frac{1}{t^a}\ln(t-1) - 2t^{1-a}\ln(t-1)\\ &\quad + t^{2-a}\ln(t+1) + \frac{1}{t^a}\ln(t+1) + 2t^{1-a}\ln(t+1)\\ &\quad - 2t^{2-a}\ln t\end{aligned}$$

hence:

$$F_a(t) = \frac{(t-1)^2}{t^a} ln(t-1) + \frac{(t+1)^2\ln(t+1)}{t^a} - 2t^{2-a}\ln t.$$

We have

$$\lim_{\substack{t\to 1\\ t>1}} F_a(t) = 4\ln 2$$

Since F_a is continuous on $(1,\infty)$, it follows that it is a bounded function.

Step (II). We prove (2.1) for all $x, y \in \mathbb{R}$ using the following lemma:

Lemma 2.2. *Let be $f : \mathbb{R} \to \mathbb{R}$ an even function so that $f(0) = 0$ and for every $0 < a < 2$ there exists $c = c(a) > 0$ so that:*

$$\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right| \le cx^a y^{2-a} \quad \textit{for all} \quad x \ge y > 0 \tag{2.5}$$

Then, for every $0 < a < 2$ there is $k = k(a)$ such that:

$$\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right| \le k|x|^a|y|^{2-a} \quad \textit{for all} \quad x, y \in \mathbb{R}. \tag{2.6}$$

Proof of Lemma. Case 1) $0 < x < y$. Then:

$$\begin{aligned}&\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right|\\ &\qquad = \left|f(x+y) + f(y-x) - 2f(y) - 2f(x)\right|\\ &\qquad \le c(2-a)y^{2-a}x^a.\end{aligned}$$

If $k = max\{c(a), c(2-a)\}$ it follows

$$\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right| \le kx^a y^{2-a}, \quad x, y > 0$$

Case 2) $x = 0$ or $y = 0$. Clear.
Case 3) $x > 0$, $y < 0$. If $u = -y > 0$, then

$$\begin{aligned}&\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right|\\ &\qquad = \left|f(x-u) + f(x+u) - 2f(x) - 2f(u)\right|\\ &\qquad \le kx^a u^{2-a} = kx^a|y|^{2-a}\end{aligned}$$

Case 4) $x < 0$, $y > 0$. Put $v = -x$

$$\begin{aligned}&\left|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right|\\ &\qquad = \left|f(-v+y) + f(-v-y) - 2f(-v) - 2f(y)\right|\end{aligned}$$

$$\begin{aligned} &= \left|f(v+y)+f(y-v)-2f(v)-2f(y)\right| \\ &\leq kv^a y^{2-a}. \end{aligned}$$

Case 5) $x<0$, $y<0$. If $u=-y, v=-x$ we have:

$$\begin{aligned} &\left|f(x+y)+f(x-y)-2f(x)-2f(y)\right| \\ &\qquad = \left|f(u+v)+f(v-u)-2f(v)-2f(u)\right| \\ &\qquad \leq kv^a u^{2-a} = k|x|^a|y|^{2-a}. \end{aligned}$$

Step (III). We prove (2.2). First we prove that if $Q:\mathbb{R}\to\mathbb{R}$ is a quadratic mapping then

$$Q(2^n)=4^nQ(1), \quad n\in\mathbb{Z}.$$

From $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ for $x=y$ we obtain

$$Q(2x)+Q(0)=4Q(x)$$

For $x=0$ it follows $2Q(0)=4Q(0)\Longrightarrow Q(0)=0$.

Hence $Q(2x)=4Q(x)$. We obtain: $Q(2^nx)=4^nQ(x), n\in\mathbb{N}$.

That implies

$$Q(2^n)=4^nQ(1), \quad n\in\mathbb{N}$$

and for $x=\frac{1}{2^n}$ we have:

$$Q(1)=4^nQ(2^{-n})\Longleftrightarrow Q(2^{-n})=4^{-n}Q(1).$$

For $\alpha\leq 2$:

$$\begin{aligned} \sup_{x\neq 0}\frac{|f(x)-Q(x)|}{|x|^\alpha} &\geq \sup_{n\in\mathbb{N}}\frac{|f(2^n)-Q(2^n)|}{2^{n\alpha}} \\ &= \sup_{n\in\mathbb{N}}\frac{|4^n n\ln 2-4^nQ(1)|}{2^{n\alpha}} \\ &= \sup_{n\in\mathbb{N}} 2^{n(2-\alpha)}\left|n\ln 2-Q(1)\right| = \infty. \end{aligned}$$

For $\alpha>2$:

$$\begin{aligned} \sup_{x\neq 0}\frac{|f(x)-Q(x)|}{|x|^\alpha} &\geq \sup_{n\in\mathbb{N}}\frac{|f(2^{-n})-Q(2^{-n})|}{2^{-n\alpha}} \\ &= \sup_{n\in\mathbb{N}}\frac{|4^{-n} n\ln 2-4^{-n}Q(1)|}{2^{-n\alpha}} \\ &= \sup_{n\in\mathbb{N}} 2^{n(\alpha-2)}\left|n\ln 2-Q(1)\right| = \infty. \end{aligned}$$

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Chapter 5

HYERS–ULAM–AOKI–RASSIAS SSTABILITY AND ULAM–GAVRUTA–RASSIAS STABILITY OF QUADRATIC HOMOMORPHISMS AND QUADRATIC DERIVATIONS ON BANACH ALGEBRAS

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Abstract

In this paper, we establish the Hyers–Ulam–Aoki–Rassias stability and Ulam–Gavruta–Rassias stability of the quadratic homomorphisms and quadratic derivations on Banach algebras.

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Key words: Hyers–Ulam–Aoki–Rassias stability, Ulam–Gavruta–Rassias stability-Homomorphism- Derivation- Quadratic function.

1. Introduction

Throughout this paper we suppose that A is a Banach algebra and X is a Banach A-module. Quadratic functional equation was used characterize inner product spaces [2,3,18]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

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is related to symmetric bi-additive function [2], [18]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [2], [18]). The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}\,(f(x + y) - f(x - y)). \tag{1.2}$$

A mapping $f : A \longrightarrow A$ is called a quadratic homomorphism if f is a quadratic function satisfies $f(ab) = f(a)f(b)$ for all $a, b \in A$. For instance, let A be commutative, then the mapping $f : A \longrightarrow A$ defined by $f(a) = a^2 (a \in A)$, is a quadratic homomorphism.

A mapping $f : A \longrightarrow X$ is called a quadratic derivation if f is a quadratic function satisfies $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. The following example is a slight modification of an example due to [11].

Example. Let $\mathcal{A}$ be a Banach algebra. Then we take

$$\mathcal{T} = \begin{bmatrix} 0 & \mathcal{A} & \mathcal{A} \\ 0 & 0 & \mathcal{A} \\ 0 & 0 & 0 \end{bmatrix},$$

$\mathcal{T}$ is a Banach algebra equipped with the usual matrix-like operations and the following norm:

$$\left\| \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right\| = \|a\| + \|b\| + \|c\| \quad (a, b, c \in \mathcal{A}).$$

It is known that

$$\mathcal{T}^* = \begin{bmatrix} 0 & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & \mathcal{A}^* \\ 0 & 0 & 0 \end{bmatrix},$$

is the dual of $\mathcal{T}$ under the following norm

$$\left\| \begin{bmatrix} 0 & f & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \right\| = \max\big\{\|f\|, \|g\|, \|h\|\big\} \;\; (f, g, h \in \mathcal{A}^*).$$

Let the left module action of $\mathcal{T}$ on $\mathcal{T}^*$ be trivial and let the right module action of $\mathcal{T}$ on $\mathcal{T}^*$ is defined as follows.

$$\left\langle \begin{bmatrix} 0 & f & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \right\rangle = f(ax) + g(by) + h(cz),$$

for all $f, g, h \in \mathcal{A}^*$, $a, b, c, x, y, z \in \mathcal{A}$. Then $\mathcal{T}^*$ is a Banach $\mathcal{T}$-module. Let $\begin{bmatrix} 0 & k & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*$. We define $D : \mathcal{T} \longrightarrow \mathcal{T}^*$ by

$$D\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & k & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (a, b, c \in \mathcal{A}).$$

Then we can see that D is a quadratic derivation from $\mathcal{T}$ into $\mathcal{T}^*$ [11].

It is easy to see that a quadratic homomorphism (derivation) is a linear homomorphism (derivation) if and only if it is zero function.

The stability problem of functional equations originated from a question of Ulam [67] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, dose there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition dose there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [19] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in t for each fixed $x \in E$, then T is linear. In 1950, T. Aoki [3] was the second author to treat this problem for additive mappings.

Finally in 1978, Th. M. Rassias [61] proved the following Theorem.

Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \epsilon\big(\|x\|^p + \|y\|^p\big) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\,\|x\|^p, \tag{1.4}$$

for all $x \in E$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$, and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is linear.

In 1991, Z. Gajda [14] answered the question for the case $p > 1$, which was rased by Rassias. On the other hand J. M. Rassias [53]–[55], generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem:

Theorem 1.2. *Let $\Theta \geq 0$ and let $p_1, p_2 \in \mathbb{R}$ with $p = p_1 + p_2 \neq 1$. Suppose $f : E \longrightarrow E'$ is a mapping from a norm space E into a Banach space E' such that the inequality*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \epsilon\|x\|^{p_1}\|y\|^{p_2},$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p} \|x\|^p,$$

for all $x \in E$. If in addition for every $x \in E$, $f(tx)$ is continuous, then T is linear.

Following the techniques of the proof of the corollary of D. H. Hyers [19] it is observed that D. H. Hyers introduced (in 1941) the following Hyers continuity condition: about the continuity of the mapping $f(tx)$ in real t for each fixed x, and then he proved homogenouity of degree one and therefore the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers–Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of T. Aoki [3], Th. M. Rassias [61] and J. M. Rassias [53]–[55], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by T. Aoki, Th. M. Rassias and J. M. Rassias is called Hyers–Ulam–Aoki–Rassias stability for the sum and Ulam–Gavruta–Rassias stability for the product of powers of norms. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [12], [13] (see also [1], [4], [6], [15], [16], [18]–[22] and [56]–[64]). A Ulam–Gavruta–Rassias stability Theorem for quadratic functional equation (1.1) was proved by J. M. Rassias [53]–[55] (see also [10]). During the 34th International Symposium on Functional Equations, Gy. Maksa [27] posed the problem concerning the Hyers–Ulam stability of the functional equation

$$f(xy) = f(x)y + xf(y), \tag{1.5}$$

on the interval $(0, 1]$ and J. Tabor gave an answer to the equation of Maksa in [66]. On the other hand, Zs. P$\acute{a}$les [47] remarked that the functional equation (1.1) for real-valued functions has a superstability on the interval $[1, \infty)$. In 1997, C. Borelli [7] demonstrated the Hyers–Ulam stability of functional equation (1.5) on restricted domain of $\mathbb{R}$. Jung and Park [23] have solved the functional equation $f(x+y+xy) = f(x)+f(y)+f(x)y+xf(y)$ motivated by the equation (1.5), and then investigated the Hyers–Ulam–Aoki–Rassias stability problem on the interval $(-1, 0]$ and the superstability on $[0, \infty)$ (see also [16], [20]–[22]).

Hyers–Ulam–Aoki–Rassias stability and problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \longrightarrow B$, were A is normed space and B Banach space(see [65]). Cholewa [9] noticed that the Theorem of Skof is still true if relevant domain A is replaced an abelian group. In the paper [10], Czerwik proved the Hyers–Ulam–Aoki–Rassias stability of the equation (1.3). Grabiec [17] has generalized these result mentioned above (see [2], [10], [24], [25], [46] and [48]–[51]). For the stability of linear homomorphisms and derivations we refer the reader to [4], [5], [8], [26] and [28]–[45]. In section two we investigate the situation that the generalized Hyers–Ulam–Aoki–Rassias stability and Ulam–Gavruta–Rassias stability for quadratic homomorphisms on Banach algebras. In section three we study the generalized Hyers–Ulam–Aoki–Rassias stability and Ulam-Gavruta-Rassias stability of quadratic derivations from a Banach algebras into its Banach modules.

2. Quadratic Homomorphisms

In this section we study the stability of quadratic homomorphisms on Banach algebras.

Theorem 2.1. *Let A, B be Banach algebras. Suppose functions $\psi, Q : A \times A \to [0, \infty)$ satisfying*

$$\tilde{Q}(x) := \sum_{i=0}^{\infty} \frac{1}{4^i} Q(2^i x, 2^i x) < \infty, \tag{2.6}$$

$$\lim \frac{1}{16^i} \psi(2^i x, 2^i y) = 0, \tag{2.7}$$

and

$$\lim Q(2^i x, 2^i y) = 0 \tag{2.8}$$

for all $x, y \in A$. If $f : A \to B$ is a mapping such that

$$\big\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \big\| \le Q(x, y), \tag{2.9}$$

and that

$$\| f(xy) - f(x)f(y) \| \le \psi(x, y), \tag{2.10}$$

for all $x, y \in A$, then there exists a unique quadratic homomorphism $H : A \to B$ such that

$$\| f(x) - H(x) \| \le \frac{1}{4} \tilde{Q}(x) \tag{2.11}$$

for all $x \in A$.

Proof. Letting y=x in (2.4), we get

$$\left\| \frac{1}{4} f(2x) - f(x) \right\| \le \frac{1}{4} Q(x, x). \tag{2.12}$$

Replace x by $2x$ in (2.7) and result divide by 4 to obtain

$$\left\| \frac{1}{4^2} f(4x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{4^2} Q(2x, 2x). \tag{2.13}$$

Now, combine (2.7) and (2.8) by use of the triangle inequality to get

$$\left\| \frac{1}{4^2} f(4x) - f(x) \right\| \le \frac{1}{4} Q(x, x) + \frac{1}{4^2} Q(2x, 2x).$$

Now, proceed in this way to prove by induction that

$$\left\| \frac{1}{4^n} f(2^n x) - f(x) \right\| \le \sum_{i=0}^{n-1} \frac{1}{4^{i+1}} Q(2^i x, 2^i x). \tag{2.14}$$

In order to show that functions $h_n(x) = \frac{1}{4^n} f(2^n x)$ form a convergent sequence, we used Cauchy convergence criterion. Indeed, replace x by $2^m x$ in (2.9) and result divide by 4^m, where m is an arbitrary positive integer. We find that,

$$\left\| \frac{1}{4^{n+m}} f(2^{n+m}x) - \frac{1}{4^m} f(2^m x) \right\| \leq \frac{1}{4} \sum_{i=m}^{m+n-1} \frac{1}{4^i} Q(2^i x, 2^i x). \tag{2.15}$$

By (2.1) and since B is complete, it follows that $\lim_{n\to\infty} H_n(x)$ exists for all $x \in A$.

Let m=0 and $n \to \infty$ in (2.10), we have

$$\|H(x) - f(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{4^i} Q(2^i x, 2^i x) = \frac{1}{4} \tilde{Q}(x)$$

such that H defined by $H : A \to X$, $H(x) = \lim \frac{1}{4^n} f(2^n x)$. On the other hand, by using (2.1), for all $x, y \in A$, we have

$$\begin{aligned}
&\|H(x+y) + H(x-y) - 2H(x) - 2H(y)\| \\
&\qquad = \lim \frac{1}{4^n} \|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\| \\
&\qquad\qquad \leq \lim \frac{1}{4^n} Q(2^n x, 2^n y) = 0.
\end{aligned}$$

This means that H is quadratic. Using (2.2) to obtain

$$\begin{aligned}
\|H(xy) - H(x)H(y)\| &= \left\| \lim \frac{1}{4^n} f(2^n xy) - \lim \frac{1}{4^n} f(2^n x). \lim \frac{1}{4^n} f(2^n y) \right\| \\
&= \left\| \lim \frac{1}{16^n} f(2^n x 2^n y) - \lim \frac{1}{16^n} f(2^n x) f(2^n y) \right\| \\
&\leq \lim \frac{1}{16^n} \psi(2^n x, 2^n y) = 0.
\end{aligned}$$

Hence, $H(xy) = H(x)H(y)$. □

Now, suppose there is another such function $\acute{H} : A \to B$ satisfies $\acute{H}(x+y) + \acute{H}(x-y) = 2\acute{H}(x) + 2\acute{H}(y)$ and $\|\acute{H}(x) - f(x)\| \leq \frac{1}{4}\tilde{Q}(x)$. Then for all $x \in A$, we have

$$\begin{aligned}
\|H(x) - \acute{H}(x)\| &= \frac{1}{4^n} \|H(2^n x) - \acute{H}(2^n x)\| \\
&\leq \frac{1}{4^n} \Big(\|H(2^n x) - f(2^n x)\| + \|\acute{H}(2^n x) - f(2^n x)\| \Big) \\
&\leq \frac{2}{4^{n+1}} \tilde{Q}(2^n x) = \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^{i+n}} Q(2^{i+n} x, 2^{i+n} x) \\
&= \frac{1}{2} \sum_{i=n}^{\infty} \frac{1}{4^i} Q(2^i x, 2^i x).
\end{aligned}$$

By $n \to \infty$ we get, $H(x) = \acute{H}(x)$.

Corollary 2.2. *Let $P < 2$, $\Theta > 0$, and let A, B be Banach algebras. Suppose mapping $\psi : A \times A \to [0, \infty)$ satisfies*

$$\lim \frac{1}{16^i} \psi(2^i x, 2^i y) = 0,$$

for all $x, y \in A$, moreover, suppose mapping $f : A \to B$ satisfies

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \leq \Theta(\|x\|^p + \|y\|^p),$$

and

$$\|f(xy) + f(x)f(y)\| \leq \psi(x, y),$$

for all $x, y \in A$. Then there exists a unique quadratic homomorphism $H : A \to B$ such that for all $x \in A$,

$$\|H(x) - f(x)\| \leq \frac{\Theta\|x\|^p}{2} \cdots \frac{1}{1 - 2^{p-2}}\,.$$

Proof. It follows from above Theorem by taking $Q(x, y) := \theta(\|x\|^p + \|y\|^p)$. □

In the following Corollary, we show that the superstability for the inequality (2.4) is valid when f is a quadratic function.

Corollary 2.3. *Let let A and B be Banach algebras. Suppose a mapping $\psi : A \times A \to [0, \infty)$ satisfies*

$$\lim \frac{1}{16^i} \psi(2^i x, 2^i y) = 0,$$

moreover, suppose a quadratic mapping $f : A \to B$ satisfies

$$\|f(xy) + f(x)f(y)\| \leq \psi(x, y),$$

for all $x, y \in A$. Then f is a homomorphism.

Proof. It follows from above Theorem by taking $Q(x, y) := 0$. □

Theorem 2.4. *Let A, B be Banach algebras. Suppose functions $\psi, Q : A \times A \to [0, \infty)$ satisfying*

$$\tilde{Q}(x) := \sum_{i=1}^{\infty} 4^i Q\left(\frac{x}{2^i}, \frac{x}{2^i}\right) < \infty, \tag{2.16}$$

$$\lim_{i\to\infty} 16^i \psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0, \tag{2.17}$$

and

$$\lim_{i\to\infty} Q\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0, \tag{2.18}$$

for all $x, y \in A$. Moreover, if $f : A \to B$ is a mapping such that

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \leq Q(x, y), \tag{2.19}$$

and that

$$\|f(xy) - f(x)f(y)\| \leq \psi(x, y), \tag{2.20}$$

for all $x, y \in A$, then there exists a unique quadratic homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{4}\,\tilde{Q}(x), \tag{2.21}$$

for all $x \in A$.

Proof. Letting y=x in (2.14) we get

$$\|f(2x) - 4f(x)\| \leq Q(x, x), \tag{2.22}$$

replace x by $\frac{x}{2}$ in (2.17), to obtain

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq Q\left(\frac{x}{2}, \frac{x}{2}\right). \tag{2.23}$$

Replacing x by $\frac{x}{2}$ in (2.18) and result divide by $\frac{1}{4}$, we lead to

$$\left\|4F\left(\frac{x}{2}\right) - 16f\left(\frac{x}{4}\right)\right\| \leq 4Q\left(\frac{x}{4}, \frac{x}{4}\right). \tag{2.24}$$

Combine (2.18) and (2.19), to get

$$\left\|f(x) - 16f\left(\frac{x}{4}\right)\right\| \leq Q\left(\frac{x}{2}, \frac{x}{2}\right) + 4Q\left(\frac{x}{2^2}, \frac{x}{2^2}\right). \tag{2.25}$$

Now, Proceed in this way to prove by induction that,

$$\left\|H^n f\left(\frac{x}{2^2}\right) - f(x)\right\| \leq \sum_{i=1}^{n} 4^{i-1} Q\left(\frac{x}{2^i}, \frac{x}{2^i}\right). \tag{2.26}$$

In order to show that functions $H_n(x) = 4^n f\left(\frac{x}{2^n}\right)$ form a convergent sequence, we used Cauchy convergence criterion. Indeed, replace x by $\frac{x}{2^m}$ in (2.21) and result divide by $\frac{1}{4^m}$, where m is an arbitrary positive integer. we lead to

$$\left\|4^{n+m} f\left(\frac{x}{2^n 2^m}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{i=1+m}^{m+n} 4^{i-1} Q\left(\frac{x}{2^i}, \frac{x}{2^i}\right). \tag{2.27}$$

Since B is complete, then by $\lim_{n\to\infty} H_n(x)$ exists for all $x \in A$. Let m=0 and $n \to \infty$ in (2.22), we have, $\|H(x) - f(x)\| \leq \frac{1}{4}\,\tilde{Q}(x)$, which $H : A \to B$ defined by, $H(x) = \lim 4^n f\left(\frac{x}{2^n}\right)$.

Now, for all $x, y \in A$, it follows that

$$\begin{aligned}
&\left\|H(x+y) + H(x-y) - 2H(x) - 2H(y)\right\| \\
&\qquad = \lim 4^n \left\|f(2^{-n}(x+y)) + f(2^{-n}(x-y)) - 2f(2^{-n}x) - 2f(2^{-n}y)\right\| \\
&\qquad\qquad \leq \lim 4^n Q(2^{-n}x, 2^{-n}y) = 0.
\end{aligned}$$

Thus

$$H(x+y) + H(x-y) = 2H(x) + 2H(y).$$

So we obtain

$$\begin{aligned}\|H(xy)-H(x)H(y)\| &= \left\|\lim 4^n f(2^{-n}xy) - \lim 4^n f(2^{-n}x)\lim 4^n f(2^{-n}y)\right\| \\ &\leq \lim 16^n \psi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0.\end{aligned}$$

This means that, $H(xy)=H(x)H(y)$.

Now, suppose there is another quadratic homomorphism $\acute{H} : A \to B$ satisfies (2.16). Then we have

$$\begin{aligned}\|H(x)-\acute{H}(x)\| &= 4^n\left\|H\left(\frac{x}{2^n}\right)-\acute{H}\left(\frac{x}{2^n}\right)\right\| \\ &\leq 4^n\left(\left\|H\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|+\left\|\acute{H}\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\leq 4^n\left(\frac{1}{4}\tilde{Q}\left(\frac{x}{2^n}\right)+\frac{1}{4}\tilde{Q}\left(\frac{x}{2^n}\right)\right) \\ &= \frac{1}{2}\sum 4^{i+n}Q\left(\frac{x}{2^{i+n}},\frac{x}{2^{i+n}}\right)=\frac{1}{2}\sum_{i=1+n}^{\infty}4^iQ\left(\frac{x}{2^i},\frac{x}{2^i}\right).\end{aligned}$$

By (2.11) it follows that $H(x)=\acute{H}(x)$, for all $x\in A$. □

Corollary 2.5. *Let P, Θ be positive real numbers such that $p>2$, and let A,B be Banach algebras. Suppose function $\psi : A\times A\to[0,\infty)$ satisfies*

$$\lim 16^i\psi\left(\frac{x}{2^i},\frac{y}{2^i}\right)=0,$$

for all $x,y\in A$. Moreover, if $f : A\to B$ is a mapping such that

$$\left\|f(x+y)+f(x-y)-2f(x)-2f(y)\right\|\leq\Theta\big(\|x\|^p+\|y\|^p\big),$$

and

$$\left\|f(xy)-f(x)f(y)\right\|\leq\psi(x,y),$$

then there exists a unique quadratic homomorphism $H : A\to B$ such that

$$\|H(x)-f(x)\|\leq\Theta\frac{1}{2}\,\|x\|^p.\frac{1}{2^{P-2}-1}\,,$$

for all $x\in A$.

3. Quadratic Derivations

In this section we establish the stability of quadratic derivations.

Theorem 3.1. *Let A be a Banach algebra and X be a Banach A-Module. Suppose maps $\psi,Q : A\times A\to[0,\infty)$ satisfying*

$$\tilde{Q}(x):=\sum\frac{1}{4^i}\,Q(2^ix,2^ix)<\infty, \tag{3.28}$$

$$\lim\frac{1}{8^i}\psi(2^ix,2^iy)=0, \tag{3.29}$$

and

$$\lim Q(2^i x, 2^i y) = 0. \tag{3.30}$$

Moreover, if $f : A \to X$ is a mapping such that for all $x, y \in A$,

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \leq Q(x,y), \tag{3.31}$$

and that

$$\big\|2^{-n} f(4^n xy) + 2^n x f(2^n y) - 2^n f(2^n x).y\big\| \leq \psi(2^n x, 2^n y), \tag{3.32}$$

for all $n \in \mathbb{N}$, then there exists a unique quadratic derivation $D : A \to X$ such that for all $x \in A$

$$\|f(x) - D(x)\| \leq \frac{1}{4}\tilde{Q}(x). \tag{3.33}$$

Proof. By Theorem 2.1, the limit $D(x) := \lim \frac{1}{4^n} f(2^n x)$ exists for every $x \in A$. Now for all $x, y \in A$, we have

$$\begin{aligned}
&\big\|D(x+y) + D(x-y) - 2D(x) - 2D(y)\big\| \\
&= \left\|\lim \frac{1}{4^n} f(2^n(x+y)) + \lim \frac{1}{4^n} f(2^n(x-y)) - \lim \frac{1}{4^n} 2f(2^n x) - \lim \frac{1}{4^n} 2f(2^n y)\right\| \\
&\leq \lim \frac{1}{4^n} Q(2^n x, 2^n y) = 0.
\end{aligned}$$

Therefore $D(x+y) + D(x-y) = 2D(x) + 2D(y)$ and D is quadratic.

On the other hand, we have

$$\begin{aligned}
&\big\|D(xy) - xD(y) - D(x)y\big\| \\
&= \left\|\frac{1}{4^n} D(2^n xy) - xD(y) - D(x)y\right\| \left\|\lim \frac{1}{16^n} f(4^n xy) - x \lim \frac{1}{4^n} f(2^n y) - \lim \frac{1}{4^n} f(2^n x).y\right\| \\
&= \left\|\lim \frac{1}{16^n} f(2^n x.2^n y) - \lim \frac{1}{8^n} (2^n x).f(2^n y) - \lim \frac{1}{8^n} f(2^n x).(2^n y)\right\| \\
&= \lim \frac{1}{8^n} \left\|2^{-n} f(2^n x.2^n y) - (2^n x).f(2^n y) - f(2^n x).(2^n y)\right\| \\
&\leq \lim \frac{1}{8^n} \psi(2^n x, 2^n y) = 0.
\end{aligned}$$

Thus $D(xy) = xD(y) + D(x)y$.

Now, suppose there is another such function $\acute{D} : A \to X$ with $\acute{D}(x+y) + \acute{D}(x-y) = 2\acute{D}(x) + 2\acute{D}(y)$ and $\|\acute{D}(x) - f(x)\| \leq \frac{1}{4}\tilde{Q}(x)$ for all $x \in A$. So, for all $x \in A$ we have

$$\begin{aligned}
\big\|D(x) - \acute{D}(x)\big\| &= \frac{1}{4^n} \big\|D(2^n x) - \acute{D}(2^n x)\big\| \\
&= \frac{1}{4^n} \Big(\big\|D(2^n x) - f(2^n x)\big\| + \big\|\acute{D}(2^n x) - f(2^n x)\big\|\Big) \\
&\leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^{i+n}} Q(2^{i+n} x, 2^{i+n} x) = \frac{1}{2} \sum_{i=n}^{\infty} \frac{1}{4^i} Q(2^i x, 2^i x).
\end{aligned}$$

Using (3.1) and taking $n \to \infty$ we get, $D(x) = \acute{D}(x)$. □

Corollary 3.2. *Let $P < 2$ and $\Theta > 0$, and let A be a Banach algebra, X be a Banach A-module. Suppose function $\psi : A \times A \to [0, \infty)$ satisfies*

$$\lim \frac{1}{8^i}\, \psi(2^i x, 2^i y) = 0,$$

moreover, if $f : A \to X$ is a mapping such that,

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \le \Theta\big(\|x\|^P + \|y\|^P\big)$$

and

$$\big\|2^{-n} f(4^n xy) + 2^n x f(2^n y) - 2^n f(2^n x).y\big\| \le \psi(2^n x, 2^n y),$$

then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \le \frac{\theta\|x\|^P}{2}.\frac{1}{1-2^{P-2}}\,.$$

Theorem 3.3. *Let A be a Banach algebra and X be a Banach A-module. Suppose functions $Q, \psi : A \times A \to [0, \infty)$ satisfying*

$$\tilde{Q}(x) := \sum_{i=1}^{\infty} 4^i Q\left(\frac{x}{2^i}, \frac{x}{2^i}\right) < \infty, \tag{3.34}$$

$$\lim_{i\to\infty} 8^i \psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0, \tag{3.35}$$

and

$$\lim_{i\to\infty} Q\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0. \tag{3.36}$$

Moreover, suppose the mapping $f : A \to X$ satisfies

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \le Q(x,y), \tag{3.37}$$

and

$$\big\|2^n f(2^{-n}x.2^{-n}y) - 2^{-n} x f(2^{-n}y) - f(2^{-n}x).(2^{-n}y)\big\| \le \psi(x,y), \tag{3.38}$$

for all $x, y \in A$, and for all $n \in \mathbb{N}$. Then there exists a unique quadratic derivation $D : A \to X$ such that for all $x \in A$,

$$\|D(x) - f(x)\| \le \frac{1}{4}\,\tilde{Q}(x). \tag{3.39}$$

Proof. By Theorem 2.3, the limit $D(x) := \lim 4^n f\big(\frac{x}{2^n}\big)$ exists for all $x \in A$. Now, by (3.7), for all $x, y \in A$, we have

$$\begin{aligned}
&\big\|D(x+y) + D(x-y) - 2D(x) - 2D(y)\big\| \\
&= \Big\|\lim 4^n f(2^{-n}(x+y)) + \lim 4^n f(2^{-n}(x-y)) - 2\lim 4^n f(2^{-n}x) - 2\lim 4^n f(2^{-n}y)\Big\| \\
&\le \lim 4^n q(2^{-n}x, 2^{-n}y) = 0.
\end{aligned}$$

Therefore $D(x+y)+D(x-y)=2D(x)+2D(y)$. This means that D is quadratic. On the other hand by (3.8) and (3.11), for all $x,y\in A$, we have

$$\begin{aligned}
\big\|D(xy)-xD(y)-D(x)y\big\| &= \big\|4^nD(2^{-n}xy)-xD(y)-D(x)y\big\|\\
&= \Big\|\lim 16^nf(2^{-n}x2^{-n}y)-x\lim 4^nf(2^{-n}y)-\lim 4^nf(2^{-n}x).y\Big\|\\
&= \Big\|\lim 16^nf(2^{-n}x.2^{-n}y)-(2^{-n}x)\lim 8^nf(2^{-n}y)-\lim 8^nf(2^{-n}x).(2^{-n}y)\Big\|\\
&= \lim 8^n\Big\|2^nf(2^{-n}x.2^{-n}y)-2^{-n}xf(2^{-n}y)-f(2^{-n}x).(2^{-n}y)\Big\|\\
&\le \lim 8^n\psi(2^{-n}x,2^{-n}y)=0.
\end{aligned}$$

On the other word, D is multiplicative derivation. Now, suppose there is another such function $\acute{D}:A\to X$, with $\acute{D}(x+y)+\acute{D}(x-y)=2\acute{D}(x)+2\acute{D}(y)$ and $\|\acute{D}(x)-f(x)\|\le\frac{1}{4}\tilde{Q}(x)$. It follows that

$$\begin{aligned}
\|D(x)-\acute{D}(x)\| &= 4^n\big\|D(2^{-n}x)-\acute{D}(2^{-n}x)\big\|\\
&= 4^n\Big(\big\|D(2^{-n}x)-f(2^{-n}x)\big\|+\big\|\acute{D}(2^{-n}x)-f(2^{-n}x)\big\|\Big)\\
&\le 4^n\left(\frac{1}{4}\tilde{Q}(2^{-n}x)+\frac{1}{4}\tilde{Q}(2^{-n}x)\right)=\frac{1}{2}\sum_{i=1+n}^{\infty}4^iQ(2^{-i}x,2^{-i}x).
\end{aligned}$$

By (3.7), we get $D(x)=\acute{D}(x)$ for all $x\in A$ by taking $n\longrightarrow\infty$. □

Corollary 3.4. *Let $P>2$ and $\theta>0$, and let A be a Banach algebra, X be a Banach A-module. Suppose mapping $\psi:A\times A\longrightarrow[0,\infty)$ satisfies*

$$\lim_{i\to\infty}8^i\psi\left(\frac{x}{2^i},\frac{y}{2^i}\right)=0,$$

moreover, if $f:A\to X$ is mapping such that,

$$\big\|f(x+y)+f(x-y)-2f(x)-2f(y)\big\|\le\theta\big(\|x\|^P+\|y\|^P\big),$$

and that

$$\big\|2^nf(2^{-n}x.2^{-n}y)-2^{-n}xf(2^{-n}y)-f(2^{-n}x).(2^{-n}y)\big\|\le\psi(x,y), \tag{3.11}$$

for all $x,y\in X$ and for all $n\in\mathbb{N}$. Then there exists a unique quadratic derivation $D:A\to X$ such that for all $x\in A$,

$$\|D(x)-f(x)\|\le\frac{\theta}{2}\,\|x\|^P\frac{1}{2^{P-2}-1}\,.$$

Proof. The proof follows from above Theorem by taking $Q(x,y):=\theta(\|x\|^P+\|y\|^P)$. □

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Chapter 6

FUNDAMENTAL SOLUTIONS FOR THE GENERALIZED ELLIPTIC GELLERSTEDT EQUATION

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Abstract

In 2002, J. M. Rassias "Uniqueness of quasi-regular solutions for a bi-parabolic elliptic bi-hyperbolic Tricomi problem", *Complex Variables*, **47(8)** (2002), 707-718) imposed and investigated the bi-parabolic elliptic bi-hyperbolic mixed type partial differential equation of second order. In this paper fundamental solutions for the generalized and degenerated elliptic Gellerstedt equation are constructed in the first quadrant via Appell hypergeometric functions of two variables. Besides employing the formula expansion of Appell hypergeometric functions, we prove that the four constructed fundamental solutions possess a certain logarithmic singularity. Classical references in this field of mixed type partial differential equations are given by: J. M. Rassias (Lecture Notes on Mixed Type Partial Differential Equations, *World Scientific*, 1990, pp. 1-144) and M. M. Smirnov (Equations of Mixed Type, Translations of Mathematical Monographies, 51, *American Mathematical Society, Providence, R. I.*, 1978, pp. 1-232).

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1. Introduction

The theory of the degenerated equations of elliptic type is one of the most important topics of the modern theory of partial differential equations. Degenerated elliptic equations appear in the solution of many important applied problems. Determination of fundamental solutions of partial differential equations is one of the core targets. Both the theory of potentials and solution of boundary value problems are established by means of fundamental solutions. Well-known fundamental solutions exist today only for several degenerated elliptic partial differential equations.

Example. Let us consider the classical degenerated elliptic Gellerstedt equation [1, 5]

$$y^m u_{xx} + u_{yy} = 0,\ m = const > 0, \tag{1.1}$$

\in the field of $\mathbb{R}_+^1 = \{(x,y) : -\infty < x < \infty, y > 0\}$. Regarding this equation, we now know pertinent fundamental solutions expressed via hypergeometric Gauss functions ([4], p. 41, (2.7)-(2.8), see also [5]):

$$q_1(x,y;x_0,y_0) = k_1 \left(r_1^2\right)^{-\beta} F(\beta,\beta;2\beta;1-\sigma), \tag{1.2}$$

$$q_2(x,y;x_0,y_0) = k_2 \left(r_1^2\right)^{-\beta} (1-\sigma)^{1-2\beta} F(1-\beta,1-\beta;2-2\beta;1-\sigma), \tag{1.3}$$

where

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = (x-x_0)^2 + \frac{4}{(m+2)^2}\left(\frac{2}{m+2} y^{\frac{m+2}{2}} \mp \frac{2}{m+2} y_0^{\frac{m+2}{2}}\right)^2,\ \sigma = \frac{r^2}{r_1^2}, \tag{1.4}$$

$$\beta = \frac{m}{2(m+2)},\ k_1 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2\beta} \frac{\Gamma^2(\beta)}{\Gamma(2\beta)},\ k_2 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta} \frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)}, \tag{1.5}$$

and hypergeometric Gauss function $F(a;b;c;x)$ [6], [7], as well as

$$F(a,b;c;x) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i\, i!} x^i, \tag{1.6}$$

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt,\ \operatorname{Re} c > \operatorname{Re} b > 0, \tag{1.7}$$

$\left(\lambda\right)_{\mu}=\Gamma\left(\lambda+\mu\right)/\Gamma\left(\lambda\right)$ is a symbol of Pochhammer (or the *shifted* factorial).

In 2002, J. M. Rassias in [8]-[9] imposed the *bi-parabolic elliptic bi-hyperbolic* mixed type partial differential equation of second order

$$K_{1}\left(y\right)\left[M_{2}\left(x\right)u_{x}\right]_{x}+M_{1}\left(x\right)\left[K_{2}\left(y\right)u_{y}\right]_{y}+r\left(x,y\right)u=f\left(x,y\right). \tag{1.8}$$

which is parabolic on both lines $x=0, y=0$, elliptic in the first quadrant $x>0, y>0$ and hyperbolic in both quadrants $x<0, y>0$; $x>0, y<0$ and established the proof of quasi-regular solutions of the Tricomi problem (or Problem T) associated to this equation (1.8).

If we suppose equation (1.8) in the region $R_{+}^{2}=\left\{\left(x,y\right):x>0,y>0\right\}$, such that

$$M_{1}\left(x\right)=x^{n_{1}},\ M_{2}\left(x\right)=x^{n_{2}},K_{1}\left(y\right)=y^{m_{1}},K_{2}\left(y\right)=y^{m_{2}},r\left(x,y\right)=f\left(x,y\right)=0,$$

then the following generalized elliptic Gellerstedt equation

$$L\left(u\right)\equiv y^{m_{1}}\left[x^{n_{2}}u_{x}\right]_{x}+x^{n_{1}}\left[y^{m_{2}}u_{y}\right]_{y}=0, \tag{1.9}$$

holds with conditions

$$n_{1}+n_{2}>0,\ \ 0\le n_{2}<1,\ \ m_{1}+m_{2}>0,\ \ 0\le m_{2}<1,\ \ n_{1},n_{2},m_{1},m_{2}\in\mathbb{R}. \tag{1.10}$$

In this paper we explicitly construct four fundamental solutions for the afore-mentioned generalized elliptic Gellerstedt equation (1.9) in the field of $\mathbb{R}_{+}^{2}=\left\{\left(x,y\right):x>0,y>0\right\}$, associated to hypergeometric Appell functions in two variables. Furthermore and by means of expansions for hypergeometric function, we prove that constructed fundamental solutions possess a logarithmic singularity as $r\to 0$. Classical references in this field of mixed type partial differential equations are by: J. M. Rassias (Lecture Notes on Mixed Type Partial Differential Equations, *World Scientific*, 1990, pp. 1-144) and M. M. Smirnov (Equations of Mixed Type, Translations of Mathematical Monographies, 51, *American Mathematical Society, Providence, R. I.*, 1978, pp. 1-232).

2. Green's Formulas

Let us consider equation (1.9) in a bounded domain $D\subset\mathbb{R}_{+}^{2}=\left\{\left(x,y\right):x>0,y>0\right\}$ with smooth $\Gamma=\partial D$ its boundary. We can obviously prove the following differential identity

$$uL(v)-vL(u)=\frac{\partial}{\partial x}\left[x^{n_2}y^{m_1}\left(uv_x-vu_x\right)\right]+\frac{\partial}{\partial y}\left[x^{n_1}y^{m_2}\left(uv_y-vu_y\right)\right]. \tag{2.1}$$

Integrating this identity, and applying Green's formula, we obtain the integral identity

$$\iint\limits_D \left[uL(v)-vL(u)\right]dxdy=\int\limits_\Gamma(-x^{n_1}y^{m_2})\left(uv_y-vu_y\right)dx+x^{n_2}y^{m_1}\left(uv_x-vu_x\right)dy, \tag{2.2}$$

where $\Gamma=\partial D$ is the contour of domain D.

We note that formula (2.2) holds under the following three assumptions:

i). Functions $u(x,y)$, $v(x,y)$ and their partial derivatives of the first order are continuous in the closure $\bar{D}=D\cup\partial D=D\cup\Gamma$;

ii). Partial derivatives of the second order are continuous in the interior of D; and

iii). Integrals involving $L(u)$ and $L(v)$ exist.

Let $u(x,y)$ and $v(x,y)$ be solutions of the equation (1.9). From formula (2.2), we have

$$\int\limits_\Gamma \left(uA_s[v]-vA_s[u]\right)ds=0, \tag{2.3}$$

where s the arc length along the smooth boundary $\Gamma=\partial D$ and

$$A_s[\ \ ]=x^{n_2}y^{m_1}\frac{dy}{ds}\frac{\partial}{\partial x}-x^{n_1}y^{m_2}\frac{dx}{ds}\frac{\partial}{\partial y}. \tag{2.4}$$

Besides from the following differential identity

$$uL(u)\equiv\left(x^{n_2}y^{m_1}uu_x\right)_x+\left(x^{n_1}y^{m_2}uu_y\right)_y-x^{n_2}y^{m_1}u_x^2-x^{n_1}y^{m_2}u_y^2=0, \tag{2.5}$$

we establish the Green's integral identity

$$\iint\limits_D \left[x^{n_2}y^{m_1}u_x^2+x^{n_1}y^{m_2}u_y^2\right]dxdy=\int\limits_\Gamma uA_s[u]ds, \tag{2.6}$$

where $u(x,y)$ a solution of equation (1.9). From formula (2.3) and letting $v=1$, we obtain

$$\int\limits_\Gamma A_s[u]ds=0. \tag{2.7}$$

This relation (2.7) is a compatibility condition for the solution of the exterior Neumann problem for (1.9).

3. Fundamental Solutions

A solution of equation (1.9) is established in $\mathbb{R}^2_+ = \{(x,y) : x > 0, y > 0\}$, in the form of

$$u = P(x, y; x_0, y_0)\,\omega(\xi, \eta), \tag{3.1}$$

where

$$\left.\begin{matrix} r^2 \\ r_1^2 \\ r_2^2 \end{matrix}\right\} = \left(\frac{1}{q}x^q \begin{smallmatrix} - \\ + \\ - \end{smallmatrix} \frac{1}{q}x_0^q\right)^2 + \left(\frac{1}{p}y^p \begin{smallmatrix} - \\ - \\ + \end{smallmatrix} \frac{1}{p}y_0^p\right)^2, \quad P(x, y; x_0, y_0) = \left(r^2\right)^{-\alpha-\beta}, \tag{3.2}$$

$$\xi = \frac{r^2 - r_1^2}{r^2},\ \eta = \frac{r^2 - r_2^2}{r^2},\ q = \frac{n_1 - n_2 + 2}{2},\ p = \frac{m_1 - m_2 + 2}{2}, \tag{3.3}$$

$$\alpha = \frac{n_1 + n_2}{2(n_1 - n_2 + 2)}, \beta = \frac{m_1 + m_2}{2(m_1 - m_2 + 2)}, 0 < 2\alpha, 2\beta < 1. \tag{3.4}$$

In fact, substituting (3.1) into (1.9), we get

$$A_1\omega_{\xi\xi} + A_2\omega_{\xi\eta} + A_3\omega_{\eta\eta} + B_1\omega_\xi + B_2\omega_\eta + C\omega = 0, \tag{3.5}$$

where

$$\begin{aligned}
A_1 &= P\left[y^{m_1}x^{n_2}\xi^2{}_x + x^{n_1}y^{m_2}\xi^2{}_y\right],\\
A_2 &= 2P\left[y^{m_1}x^{n_2}\xi_x\eta_x + x^{n_1}y^{m_2}\xi_y\eta_y\right],\\
A_3 &= P\left[y^{m_1}x^{n_2}\eta^2{}_x + x^{n_1}y^{m_2}\eta^2{}_y\right],\\
B_1 &= 2\left(y^{m_1}x^{n_2}P_x\xi_x + x^{n_1}x^{m_2}P_y\xi_y\right) + P\left(y^{m_1}x^{n_2}\xi_{xx} + x^{n_1}x^{m_2}\xi_{yy}\right)\\
&\qquad + P\left(n_2y^{m_1}x^{n_2-1}\xi_x + m_2x^{n_1}y^{m_2-1}\xi_y\right),\\
B_2 &= 2\left(y^{m_1}x^{n_2}P_x\eta_x + x^{n_1}x^{m_2}P_y\eta_y\right) + P\left(y^{m_1}x^{n_2}\eta_{xx} + x^{n_1}x^{m_2}\eta_{yy}\right)\\
&\qquad + P\left(n_2y^{m_1}x^{n_2-1}\eta_x + m_2x^{n_1}y^{m_2-1}\eta`_y\right),\\
C &= y^{m_1}x^{n_2}P_{xx} + x^{n_1}y^{m_2}P_{yy} + n_2y^{m_1}x^{n_2-1}P_x + m_2x^{n_1}y^{m_2-1}P_y.
\end{aligned}$$

Therefore we determine

$$A_1=-4P\frac{x^{n_1}y^{m_1}}{r^2}x^{-q}x_0^q\xi\left(1-\xi\right), \qquad (3.6)$$

$$A_2=4P\frac{x^{n_1}y^{m_1}}{r^2}y^{-p}y_0^p\xi\eta+4P\frac{x^{n_1}y^{m_1}}{r^2}x^{-q}x_0^q\xi\eta, \tag{3.7}$$

$$A_3=-4P\frac{x^{n_1}y^{m_1}}{r^2}y^{-p}y_0^p\,\eta\left(1-\eta\right), \qquad (3.8)$$

$$B_1=-4P\frac{x^{n_1}y^{m_1}}{r^2}x^{-q}x_0^q\left[2\alpha-\left(1+2\alpha+\beta\right)\xi\right]+4P\frac{x^{n_1}y^{m_1}}{r^2}y^{-p}y_0^p\beta\xi, \tag{3.9}$$

$$B_2=4P\frac{x^{n_1}y^{m_1}}{r^2}x^{-q}x_0^q\alpha\eta-4P\frac{x^{n_1}y^{m_1}}{r^2}y^{-p}y_0^p\left[2\beta-\left(1+\alpha+2\beta\right)\eta\right], \tag{3.10}$$

$$C=4P\frac{x^{n_1}y^{m_1}}{r^2}\left(\alpha+\beta\right)\alpha x^{-q}x_0^q+4P\frac{x^{n_1}y^{m_1}}{r^2}\left(\alpha+\beta\right)\beta y^{-p}y_0^p. \tag{3.11}$$

Employing substitutions (3.6)-(3.11) in the equation (3.5), we determine the following differential system of hypergeometric Appell functions ([6], p.44, (F_2)), such that:

$$\begin{cases}\xi\left(1-\xi\right)\omega_{\xi\xi}-\xi\eta\omega_{\xi\eta}+\left[2\alpha-\left(1+2\alpha+\beta\right)\xi\right]\omega_\xi-\alpha\eta\omega_\eta-\left(\alpha+\beta\right)\alpha\omega=0,\\ \eta\left(1-\eta\right)\omega_{\eta\eta}-\xi\eta\omega_{\xi\eta}+\left[2\beta-\left(1+\alpha+2\beta\right)\eta\right]\omega_\eta-\beta\xi\omega_\xi-\left(\alpha+\beta\right)\beta\omega=0.\end{cases} \tag{3.12}$$

This system pertinent to the above equation (3.5) has the solutions ([6], p. 50, (11)):

$$\omega_1\left(\xi,\eta\right)=F_2\left(\alpha+\beta;\alpha,\beta;2\alpha,2\beta;\xi,\eta\right), \tag{3.13}$$

$$\omega_2\left(\xi,\eta\right)=\xi^{1-2\alpha}F_2\left(1-\alpha+\beta;1-\alpha,\beta;2-2\alpha,2\beta;\xi,\eta\right), \tag{3.14}$$

$$\omega_3\left(\xi,\eta\right)=\eta^{1-2\beta}F_2\left(1+\alpha-\beta;\alpha,1-\beta;2\alpha,2-2\beta;\xi,\eta\right), \tag{3.15}$$

$$\omega_4\left(\xi,\eta\right)=\xi^{1-2\alpha}\eta^{1-2\beta}F_2\left(2-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta\right), \tag{3.16}$$

where $F_2\left(a;b,b';c,c';x,y\right)$ hypergeometric Appell function ([6], p. 14 (12)):

$$F_2\left(a;b_1,b_2;c_1,c_2;x,y\right)=\sum_{i,j=0}^{\infty}\frac{\left(a\right)_{i+j}\left(b_1\right)_i\left(b_2\right)_j}{\left(c_1\right)_i\left(c_2\right)_j\,i!\,j!}x^iy^j, \tag{3.17}$$

having the following integral representation ([6], p. 28 (2)):

$$F_2\left(a;b_1,b_2;c_1,c_2;x,y\right)=\frac{\Gamma\left(c_1\right)\Gamma\left(c_2\right)}{\Gamma\left(b_1\right)\Gamma\left(b_2\right)\Gamma\left(c_1-b_1\right)\Gamma\left(c_2-b_2\right)}$$
$$\cdot\int_0^1\int_0^1 t_1^{b_1-1}t_2^{b_2-1}\left(1-t_1\right)^{c_1-b_1-1}\left(1-t_2\right)^{c_2-b_2-1}\left(1-xt_1-yt_2\right)^{-a}dt_1dt_2, \tag{3.18}$$

$$\operatorname{Re}c_1>\operatorname{Re}b_1>0,\ \operatorname{Re}c_2>\operatorname{Re}b_2>0.$$

Therefore by substituting solutions (3.13) - (3.16) into the solution (3.1), we find four solutions of the generalized elliptic Gellerstedt equation (1.9):

$$q_1\left(x,y;x_0,y_0\right)=l_1\left(r^2\right)^{-\alpha-\beta}F_2\left(\alpha+\beta;\alpha,\beta;2\alpha,2\beta;\xi,\eta\right), \tag{3.19}$$

$$l_1=\frac{1}{4\pi}\left(\frac{4}{n_1-n_2+2}\right)^{2\alpha}\left(\frac{4}{m_1-m_2+2}\right)^{2\beta}\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(2\alpha)\Gamma(2\beta)}; \tag{3.20}$$

$$q_2\left(x,y;x_0,y_0\right)=l_2\left(r^2\right)^{\alpha-\beta-1}x^{1-n_2}x_0^{1-n_2}F_2\left(1-\alpha+\beta;1-\alpha,\beta;2-2\alpha,2\beta;\xi,\eta\right), \tag{3.21}$$

$$l_2=\frac{1}{4\pi}\left(\frac{4}{n_1-n_2+2}\right)^{2-2\alpha}\left(\frac{4}{m_1-m_2+2}\right)^{2\beta}\frac{\Gamma(1-\alpha)\Gamma(\beta)\Gamma(1-\alpha+\beta)}{\Gamma(2-2\alpha)\Gamma(2\beta)}; \tag{3.22}$$

$$q_3\left(x,y;x_0,y_0\right)=l_3\left(r^2\right)^{-\alpha+\beta-1}y^{1-m_2}y_0^{1-m_2}F_2\left(1+\alpha-\beta;\alpha,1-\beta;2\alpha,2-2\beta;\xi,\eta\right), \tag{3.23}$$

$$l_3=\frac{1}{4\pi}\left(\frac{4}{n_1-n_2+2}\right)^{2\alpha}\left(\frac{4}{m_1-m_2+2}\right)^{2-2\beta}\frac{\Gamma(\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\beta)}{\Gamma(2\alpha)\Gamma(2-2\beta)}; \tag{3.24}$$

$$q_4\left(x,y;x_0,y_0\right)$$
$$=l_4\left(r^2\right)^{\alpha+\beta-2}x^{1-n_2}y^{1-m_2}x_0^{1-n_2}y_0^{1-m_2}F_2\left(2-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta\right), \tag{3.25}$$

$$l_4=\frac{1}{4\pi}\left(\frac{4}{n_1-n_2+2}\right)^{2-2\alpha}\left(\frac{4}{m_1-m_2+2}\right)^{2-2\beta}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(2-\alpha-\beta)}{\Gamma(2-2\alpha)\Gamma(2-2\beta)}. \tag{3.26}$$

It is easy to note that the constructed functions possesses the following eight properties:

$$x^{n_2}\frac{\partial}{\partial x}q_1\left(x,y;x_0,y_0\right)\Big|_{x=0}=0,\ y^{m_2}\frac{\partial}{\partial y}q_1\left(x,y;x_0,y_0\right)\Big|_{y=0}=0, \tag{3.27}$$

$$q_2\left(x,y;x_0,y_0\right)\big|_{x=0}=0,\quad y^{m_2}\frac{\partial}{\partial y}q_2\left(x,y;x_0,y_0\right)\Big|_{y=0}=0, \tag{3.28}$$

$$x^{n_2}\frac{\partial}{\partial x}q_3\left(x,y;x_0,y_0\right)\Big|_{x=0}=0,\quad q_3\left(x,y;x_0,y_0\right)\big|_{y=0}=0, \tag{3.29}$$

$$q_4\left(x,y;x_0,y_0\right)\big|_{x=0}=0,\quad q_4\left(x,y;x_0,y_0\right)\big|_{y=0}=0. \tag{3.30}$$

Applying the formula of differentiation ([6], p. 19, (20)), from (3.19) we get

$$\frac{\partial^{i+j}}{\partial x^i\partial y^j}F_2\left(a;b_1,b_2;c_1,c_2;x,y\right)=\frac{(a)_{i+j}(b_1)_i(b_2)_j}{(c_1)_i(c_2)_j}F_2\left(a+i+j;b_1+i,b_2+j;c_1+i,c_2+j;x,y\right),$$

and

$$\begin{aligned}
&x^{n_2}\frac{\partial}{\partial x}q_1\left(x,y;x_0,y_0\right)=\\
&-2l_1(\alpha+\beta)\left(r^2\right)^{-\alpha-\beta-1}x^{\frac{n_1+n_2}{2}}\left(\frac{1}{q}x^q-\frac{1}{q}x_0^q\right)F_2\left(\alpha+\beta;\alpha,\beta;2\alpha,2\beta;\xi,\eta\right)\\
&-2l_1(\alpha+\beta)\left(r^2\right)^{-\alpha-\beta-1}x^{\frac{n_1+n_2}{2}}\frac{1}{q}x_0^qF_2\left(1+\alpha+\beta;1+\alpha,\beta;1+2\alpha,2\beta;\xi,\eta\right)\\
&-2l_1(\alpha+\beta)\left(r^2\right)^{-\alpha-\beta-1}x^{\frac{n_1+n_2}{2}}\left(\frac{1}{q}x^q-\frac{1}{q}x_0^q\right)\left[\frac{\alpha}{2\alpha}\xi F_2\left(1+\alpha+\beta;1+\alpha,\beta;1+2\alpha,2\beta;\xi,\eta\right)\right.\\
&\left.+\frac{\beta}{2\beta}\eta F_2\left(1+\alpha+\beta;\alpha,1+\beta;2\alpha,1+2\beta;\xi,\eta\right)\right].
\end{aligned}\tag{3.31}$$

By virtue of an adjacent relation for hypergeometric Appell functions ([6], p. 21), one finds

$$\frac{b_1}{c_1}xF_2\left(1+a;1+b_1,b_2;1+c_1,c_2;x,y\right)+\frac{b_2}{c_2}yF_2\left(1+a;b_1,1+b_2;c_1,1+c_2;x,y\right)$$
$$=F_2\left(1+a;b_1,b_2;c_1,c_2;x,y\right)-F_2\left(a;b_1,b_2;c_1,c_2;x,y\right),$$

and from (3.31), we establish

$$x^{n_2}\frac{\partial}{\partial x}q_1\left(x,y;x_0,y_0\right)=$$
$$-2l_1\left(\alpha+\beta\right)\left(r^2\right)^{-\alpha-\beta-1}x^{\frac{n_1+n_2}{2}}\frac{1}{q}x_0^qF_2\left(1+\alpha+\beta;1+\alpha,\beta;1+2\alpha,2\beta;\xi,\eta\right) \quad (3.32)$$
$$-2l_1\left(\alpha+\beta\right)\left(r^2\right)^{-\alpha-\beta-1}x^{\frac{n_1+n_2}{2}}\left(\frac{1}{q}x^q-\frac{1}{q}x_0^q\right)F_2\left(1+\alpha+\beta;\alpha,\beta;2\alpha,2\beta;\xi,\eta\right)$$

Considering conditions (1.10) from (3.32), one finds (3.27). Properties (3.28)-(3.30) are similarly proved. We shall note that properties (3.27)-(3.30) will be used for the solution of boundary value problems associated to the afore-mentioned generalized elliptic Gellerstedt equation (1.9).

4. Logarithmic Singularities of Fundamental Solutions

We claim that the constructed solutions of the equation (1.9), have logarithmic singularities as $r\to 0$. In fact, we first determine a fundamental solution $q_1\left(x,y;x_0,y_0\right)$:

By virtue of a well-known expansion ([10], p. 253 (26)), we find

$$F_2\left(a;b_1,b_2;c_1,c_2;x,y\right)$$
$$=\sum_{i=0}^{\infty}\frac{(a)_i(b_1)_i(b_2)_i}{(c_1)_i(c_2)_i\,i!}x^iy^iF\left(a+i,b_1+i;c_1+i;x\right)F\left(a+i,b_2+i;c_2+i;y\right), \quad (4.1)$$

and thus for function $q_1\left(x,y;x_0,y_0\right)$, we have

$$q_1\left(x,y;x_0,y_0\right)=l_1\left(r^2\right)^{-\alpha-\beta}\sum_{i=0}^{\infty}\frac{(\alpha+\beta)_i(\alpha)_i(\beta)_i}{(2\alpha)_i(2\beta)_i\,i!}\left(\frac{r^2-r_1^2}{r^2}\right)^i\left(\frac{r^2-r_2^2}{r^2}\right)^i$$
$$\times F\left(\alpha+\beta+i,\alpha+i;2\alpha+i;\frac{r^2-r_1^2}{r^2}\right)F\left(\alpha+\beta+i,\beta+i;2\beta+i;\frac{r^2-r_2^2}{r^2}\right), \quad (4.2)$$

Using the formula $F\left(a,b;c;x\right)=\left(1-x\right)^{-b}F\left(c-a,b;c;x/\left(x-1\right)\right)$ and (4.2), we get

$$q_1\left(x,y;x_0,y_0\right)=l_1\left(r_1^2\right)^{-\alpha}\left(r_2^2\right)^{-\beta}\sum_{i=0}^{\infty}\frac{(\alpha+\beta)_i(\alpha)_i(\beta)_i}{(2\alpha)_i(2\beta)_i\,i!}\left(\frac{r_1^2-r^2}{r_1^2}\right)^i\left(\frac{r_2^2-r^2}{r_2^2}\right)^i$$
$$\times F\left(\alpha-\beta,\alpha+i;2\alpha+i;\frac{r_1^2-r^2}{r_1^2}\right)F\left(\beta-\alpha,\beta+i;2\beta+i;\frac{r_1^2-r^2}{r_1^2}\right). \tag{4.3}$$

Therefore [7], we establish as $r\to 0$, the following essential relations:

$$F\left(a,b;c;1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c\neq 0,-1,-2,...,\mathrm{Re}\left(c-a-b\right)>0, \tag{4.4}$$

$$F\left(\alpha-\beta,\alpha+i;2\alpha+i;1-\frac{r^2}{r_1^2}\right)\to F\left(\alpha-\beta,\alpha+i;2\alpha+i;1\right)=\frac{\Gamma(2\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\alpha)}\frac{(2\alpha)_i}{(\alpha+\beta)_i},$$
$$F\left(\beta-\alpha,\beta+i;2\beta+i;1-\frac{r^2}{r_2^2}\right)\to F\left(\beta-\alpha,\beta+i;2\beta+i;\right)=\frac{\Gamma(2\beta)\Gamma(\alpha)}{\Gamma(\alpha+\beta)\Gamma(\beta)}\frac{(2\beta)_i}{(\alpha+\beta)_i}. \tag{4.5}$$

Hence, as $r\to 0$, from expansion (4.3) we obtain

$$q_1\left(x,y;x_0,y_0\right)\to l_1\left(r_1^2\right)^{-\alpha}\left(r_2^2\right)^{-\beta}\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma^2(\alpha+\beta)}F\left(\alpha,\beta;\alpha+\beta;\frac{r^4}{r_1^2r_2^2}-\frac{r^2}{r_1^2}-\frac{r^2}{r_2^2}+1\right). \tag{4.6}$$

Similarly, we get

$$F\left(a,b;a+b;z\right)=-\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}F\left(a,b;1;1-z\right)\ln\left(1-z\right)$$
$$+\frac{\Gamma(a+b)}{\Gamma^2(a)\Gamma^2(b)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)\Gamma(b+j)}{(j!)^2}\left[2\psi(1+j)-\psi(a+j)-\psi(b+j)\right](1-z)^j, \tag{4.7}$$
$$-\pi<\arg\left(1-z\right)<\pi,\ a,b\neq 0,-1,-2,...$$

via the logarithmic derivative $\psi(z)$ of $\Gamma(z)$ [7]:

$$\psi(z)=\ln z-\sum_{n=0}^{\infty}\left[\frac{1}{n+z}-\ln\left(1+\frac{1}{n+z}\right)\right],\ z>0;$$
$$\psi(z)=\int_0^{\infty}e^{-t}\ln t\,dt+\int_0^1\frac{1-t^{z-1}}{1-t}dt,\ \mathrm{Re}\,z>0$$

Thus from relation (4.6) as $r \to 0$, we establish

$$q_1(x,y;x_0,y_0) = l_1\left(r_1^2\right)^{-\alpha}\left(r_2^2\right)^{-\beta}\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}$$
$$\times\left\{-F\left[\alpha,\beta;1;r^2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{r^2}{r_1^2r_2^2}\right)\right]\left\{\ln r^2+\ln\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{r^2}{r_1^2r_2^2}\right)\right\}\right. \tag{4.8}$$
$$\left.+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+j)\Gamma(\beta+j)}{(j!)^2}h_j\left[r^2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{r^2}{r_1^2r_2^2}\right)\right]^j\right\}+O(1),$$
$$h_j = \left[2\psi(1+j)-\psi(\alpha+j)-\psi(\beta+j)\right].$$

Formula (4.8) proves that the solution $q_1(x,y;x_0,y_0)$ has a logarithmic singularity as $r \to 0$. Hence $q_1(x,y;x_0,y_0)$ is the first fundamental solution of the above-mentioned equation (1.9). Similarly, we determine the other three fundamental solutions $q_2(x,y;x_0,y_0)$, $q_3(x,y;x_0,y_0)$, $q_4(x,y;x_0,y_0)$ of the generalized and degenerated Elliptic Gellerstedt equation (1.9).

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Chapter 7

POINTWISE SUPERSTABILITY AND SUPERSTABILITY OF THE JORDAN EQUATION

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Abstract

In this paper, ε-approximate Jordan mappings and strong ε-approximate Jordan mappings are introduced, $(\mathscr{A},\mathscr{B})$-pointwise superstability and $(\mathscr{A},\mathscr{B})$-superstability of the Jordan equation are defined. It is proved that if $\mathscr{A}$ and $\mathscr{B}$ are normed algebras such that the norm of $\mathscr{B}$ is multiplicative, then the Jordan equation is both $(\mathscr{A},\mathscr{B})$-pointwise superstable and $(\mathscr{A},\mathscr{B})$-superstable.

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1. Introduction

In 1940 S. M. Ulam [32] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following celebrated Ulam question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric ρ. Given an $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all x, y in G, then a homomorphism $g : G \to G'$ exists with $\rho(f(x), g(x)) < \varepsilon$ for all x in G?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an approximate homomorphism.

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In 1941 D. H. Hyers [10] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies Hyers inequality

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \varepsilon \tag{1.1}$$

for all x, y in E. It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{1.2}$$

exists for all $x \in E$ and that L is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \varepsilon$ for all x in E, where ε is a positive constant. No continuity conditions are required for this pioneering Hyers–Ulam stability result. However, if one applies the Hyers continuity condition that $t \mapsto f(tx)$ is continuous in the real variable t for each fixed $x \in E$, then he obtains that L is real linear, and if f is continuous at a single point of E, then L is also continuous.

Besides D. H. Hyers [10] studied the problem of knowing if in the case that a map f is "near" to hold the Cauchy additive functional equation, then there exists another map L acting in the same space "near" to f and satisfying this equation. Here "near" means that f and L are close in the sense of a metric structure inside the considered functional space.

In 1982–1994, a generalization of Hyers stability result was proved by J. M. Rassias ([21]–[24], [26]). This author assumed the following generalized condition (or weaker inequality or Cauchy–Gavruta–Rassias inequality)

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \theta \|x\|^p \|y\|^q \tag{1.3}$$

for all x, y in E, controlled by (or involving) a product of different powers of norms, where $\theta \geq 0$ and real p, q are real numbers such that $r = p + q \neq 1$, and retained the condition of continuity of $f(tx)$ in t for a any fixed x in E. Besides J. M. Rassias investigated that it is possible to replace ε in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in these results, the approach to the existence question was to prove asymptotic type formulas of the form:

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), \quad \text{or} \quad L(x) = \lim_{n\to\infty} 2^{n} f(2^{-n} x). \tag{1.4}$$

J. M. Rassias stability Theorem ([21]–[23], [26]) states: Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and satisfies the inequality (1.3) for all x, y in X. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}\,\|x\|^r \tag{1.5}$$

for all $x \in X$. If, in addition, the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for a any fixed x in X, then the L is a $\mathbb{R}$-linear mapping. In 1999, P. Gavruta [7] gave a nice counterexample to the Ulam–Gavruta–Rassias stability of this theorem in the singular case: $r = 1$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting research results concerning the above Ulam stability problem ([1], [2], [4]–[9], [12], [21]–[24], [26], [29]). A large list of references can be found therein. Interesting pertinent counterexamples can be found in ([5], [7], [15]). Euler–Lagrange (E–L) mappings and Euler–Lagrange–Rassias (E–L–R) mappings were introduced and named by J. M. Rassias [25] (in 1992) and via ([27], [28]). Already several specialists on the Ulam stability problem have employed these new functional mappings ([11], [15], [17]–[20], [30]). These E–L–R quadratic functional equations (and mappings) are important in analysis, because they marry functional equations (and mappings) with the probability theory and mathematical statistics (stochastic analysis) through the weighted quadratic means and quadratic mean equations. Besides the Ulam–Gavruta–Rassias (UGR) stability was introduced by J. M. Rassias ([21]–[23], [26]) (in 1982–1989, 1994) for linear mappings via the above J. M. Rassias stability theorem, where the initial fixed bound proposed by D. H. Hyers [10] was changed in the sequel by another condition involving a product of different powers of norms. Already several authors have used this new stability ([18], [30]). Also the Ulam–Aoki–Rassias (UAR) stability or equivalently Hyers–Ulam–Rassias (HUR) or Hyers–Ulam–Aoki–Rassias (HUAR) stability was introduced by T. Aoki [1] (in 1950) for additive mappings and by Th. M. Rassias [29] (in 1978) for linear mappings, where the initial Hyers fixed bound [10] was changed by a condition involving a sum of powers of norms.

Young Whan Lee, Gwang Hui Kim in [33] discussed the superstability of Jordan functional on a normed algebra. In this paper, we will discuss the pointwise superstability and superstability of the Jordan equation. We first recall the definitions of ε-homomorphism, Jordan mapping, Jordan equation, ε-approximate Jordan mapping and strong ε-approximate Jordan mapping.

Definition 1.1. Let $\mathscr{A}, \mathscr{B}$ be normed algebras, and $f : \mathscr{A} \to \mathscr{B}$ be a linear mapping, if f satisfies the inequality

$$\big\|f(ab) - f(a)f(b)\big\| \leq \varepsilon\|a\|\,\|b\|$$

for all $a, b \in \mathscr{A}$, then f is called an *ε-homomorphism.*

Definition 1.2. Let $\mathscr{A}, \mathscr{B}$ be Banach algebras, $\phi : \mathscr{A} \to \mathscr{B}$ be a linear mapping. Then the equation

$$\phi(a^2) - (\phi(a))^2 = 0 \ \ (\forall\, a \in \mathscr{A}) \tag{1.1}$$

is said to be the *Jordan equation.*

Moreover, a solution of the Jordan equation (1.1) is called a Jordan mapping. Especially, a Jordan mapping from $\mathscr{A}$ into $\mathbb{C}$ is called a Jordan functional on $\mathscr{A}$.

Definition 1.3. Let $\mathscr{A}, \mathscr{B}$ be normed algebras, $f : \mathscr{A} \to \mathscr{B}$ be a linear mapping satisfying the inequality

$$\big\|f(a^2) - (f(a))^2\big\| \leq \varepsilon\|a\|^2$$

for all $a \in \mathscr{A}$. Then f is called an *ε-approximate Jordan mapping*. Especially, if $\mathscr{B}$ is the complex field $\mathbb{C}$, then f is called an ε-approximate Jordan functional on $\mathscr{A}$.

Definition 1.4. Let $\mathscr{A}, \mathscr{B}$ be normed algebras, $f : \mathscr{A} \to \mathscr{B}$ be a linear mapping satisfying the inequality

$$\left\|f(a^2)-(f(a))^2\right\| \le \varepsilon$$

for all $a \in \mathscr{A}$. Then f is called a *strong ε-approximate Jordan mapping*. Especially, if $\mathscr{B}$ is the complex field $\mathbb{C}$, then f is called a *strong ε-approximate Jordan functional* on $\mathscr{A}$.

2. The Pointwise Superstability and Superstability of the Jordan Equation

In this section, we will study the pointwise superstability and superstability of the Jordan equation.

Let $f : \mathscr{A} \to \mathscr{B}$ be an ε-approximate Jordan mapping, define

$$D_f = \left\{a \in \mathscr{A} : \ f(a^2) = (f(a))^2\right\},$$
$$E_f^\varepsilon = \left\{a \in \mathscr{A} : \ \|f(a)\| \le C(\varepsilon)\|a\|\right\},$$

where

$$C(\varepsilon) = \frac{1+\sqrt{1+4\varepsilon}}{2}.$$

Definition 2.1. If every ε-approximate Jordan mapping $f : \mathscr{A} \to \mathscr{B}$ satisfies

$$\mathscr{A} = D_f \cup E_f^\varepsilon,$$

then the Jordan equation (1.1) is called *$(\mathscr{A}, \mathscr{B})$-pointwise superstable.*

Definition 2.2. If a mapping $f : \mathscr{A} \to \mathscr{B}$ satisfies

$$\inf_{\|a\|=1} \|f(a)\| > C(\varepsilon),$$

then it is said to be *ε-lower bounded.*

Definition 2.3. If every ε-lower bounded ε-approximate Jordan mapping $f : \mathscr{A} \to \mathscr{B}$ is a Jordan mapping, then (1.1) is called *$(\mathscr{A}, \mathscr{B})$-superstable.*

Theorem 2.1. *Let $\mathscr{A}$ be an algebra, $\mathscr{B}$ be a commutative normed algebra with the multiplicative norm, and $f : \mathscr{A} \to \mathscr{B}$ be a strong ε-approximate Jordan mapping. Then for each $a \in \mathscr{A} \setminus \ker f$, $f(a^2) = (f(a))^2$. In particular, if f is also an injection, then f is a Jordan mapping on $\mathscr{A}$.*

Proof. Since f is a strong ε-approximate Jordan mapping, for every $x, y \in \mathscr{A} \setminus \ker f$, we have $\|f((x+y)^2) - (f(x+y))^2\| \le \varepsilon$. Thus,

$$\begin{aligned}
\varepsilon &\ge \left\|f((x+y)^2) - (f(x+y))^2\right\| \\
&= \left\|f(x^2+xy+yx+y^2) - (f(x))^2 - (f(y))^2 - 2f(x)f(y)\right\| \\
&= \left\|f(x^2) - (f(x))^2 + f(y^2) - (f(y))^2 + f(xy+yx) - 2f(x)f(y)\right\| \\
&\ge \left\|f(xy+yx) - 2f(x)f(y)\right\| - \left\|f(x^2) - (f(x))^2\right\| - \left\|f(y^2) - (f(y))^2\right\| \\
&\ge \left\|f(xy+yx) - 2f(x)f(y)\right\| - 2\varepsilon.
\end{aligned}$$

Therefore, $\|f(xy+yx)-2f(x)f(y)\|\le 3\varepsilon$. This proves that

$$\big\|f(xy)-f(x)f(y)\big\|\le \frac{3\varepsilon}{2}\,,\ \ \forall\, x\in\{y\}'. \tag{2.1}$$

Clearly, $C(\varepsilon)^2-C(\varepsilon)=\varepsilon$ and $C(\varepsilon)>1$. Let $a\in \mathscr{A}\setminus \ker f$, for $f(a)\neq 0$, we may assume that $\|f(a)\|>C(\varepsilon)$ because $\|f(ta)\|=\|tf(a)\|>C(\varepsilon)$ for some $t\in\mathbb{R}$ and $f((ta)^2)=(f(ta))^2$ implies $f(a^2)=(f(a))^2$. So there exists a $p>0$ such that $\|f(a)\|=C(\varepsilon)+p$. Then

$$\begin{aligned}
\|f(a^2)\| &= \big\|f(a^2)-(f(a))^2+(f(a))^2\big\|\\
&\ge \|(f(a))^2\|-\big\|f(a^2)-(f(a))^2\big\|\\
&= \|f(a)\|^2-\|f(a^2)-(f(a))^2\|\\
&\ge (C(\varepsilon)+p)^2-\varepsilon\\
&= 2pC(\varepsilon)+p^2+C(\varepsilon)\\
&> 2pC(\varepsilon)+C(\varepsilon)\\
&> C(\varepsilon)+2p.
\end{aligned}$$

This proves that when $n=1$, inequality

$$\|f(a^{2^n})\|>C(\varepsilon)+2np \tag{2.2}$$

holds. Suppose that when $n=k$, (2.2) holds, that is $\|f(a^{2^k})\|>C(\varepsilon)+2kp$, then when $n=k+1$, we have

$$\begin{aligned}
\|f(a^{2^{k+1}})\| &= \|f((a^{2^k})^2)\|\\
&= \big\|f((a^{2^k})^2)-(f(a^{2^k}))^2+(f(a^{2^k}))^2\big\|\\
&\ge \|(f(a^{2^k}))^2\|-\big\|f((a^{2^k})^2)-(f(a^{2^k}))^2\big\|\\
&\ge (C(\varepsilon)+2kp)^2-\varepsilon\\
&= C(\varepsilon)+4kpC(\varepsilon)+4k^2p^2\\
&> C(\varepsilon)+4kpC(\varepsilon)\\
&> C(\varepsilon)+4kp\\
&\ge C(\varepsilon)+(2k+2)p\\
&= C(\varepsilon)+2(k+1)p.
\end{aligned}$$

So by induction, (2.2) holds for all $n=1,2,3,\ldots$. For all mutually commutative elements $x,y,z\in\mathscr{A}\setminus\ker f$, we see from (2.1) that

$$\big\|f(xyz)-f(xy)f(z)\big\|\le\frac{3\varepsilon}{2}\,,\quad \big\|f(xyz)-f(x)f(yz)\big\|\le\frac{3\varepsilon}{2}\,.$$

Hence

$$\begin{aligned}
\big\|f(xy)f(z)-f(x)f(yz)\big\| &= \big\|f(xy)f(z)-f(xyz)+f(xyz)-f(x)f(yz)\big\|\\
&\le \|f(xy)f(z)-f(xyz)\|+\|f(xyz)-f(x)f(yz)\|\\
&\le 3\varepsilon.
\end{aligned}$$

So

$$\begin{aligned}\big\|f(xy)f(z)-f(x)f(y)f(z)\big\| &= \big\|f(xy)f(z)-f(x)f(yz)+f(x)f(yz)-f(x)f(y)f(z)\big\|\\ &\le \|f(xy)f(z)-f(x)f(yz)\|+\|f(x)\|\cdot\|f(yz)-f(y)f(z)\|\\ &\le 3\varepsilon+\|f(x)\|\cdot\frac{3\varepsilon}{2}\,.\end{aligned}$$

In particular, letting $x=y=a, z=a^{2^n}$, by the above inequality and (2.2), we have

$$\begin{aligned}\big\|f(a^2)-(f(a))^2\big\| &= \frac{\|f(a^2)f(a^{2^n})-(f(a))^2f(a^{2^n})\|}{\|f(a^{2^n})\|}\\ &\le \frac{3\varepsilon+\|f(a)\|\cdot\frac{3\varepsilon}{2}}{\|f(a^{2^n})\|}\\ &< \frac{3\varepsilon+\|f(a)\|\cdot\frac{3\varepsilon}{2}}{C(\varepsilon)+2np}\\ &\to 0\ \ (n\to\infty).\end{aligned}$$

Therefore $f(a^2)=(f(a))^2$. This completes the proof. □

Lemma 2.1. *Let $\mathscr{A}$ be a normed algebra, $\mathscr{B}$ be a normed algebra with the multiplicative norm, and $f:\mathscr{A}\to\mathscr{B}$ be an ε-approximate Jordan mapping. If $a\in\mathscr{A}$ with $\|f(a)\|>C(\varepsilon)\|a\|$, then there exists a $p>0$ such that*

$$\|f(a^{2^n})\|\ge(C(\varepsilon)+2np)\|a\|^{2^n}\ \ (n=1,2,\ldots). \tag{2.3}$$

Proof. Clearly, $a\neq0$. Putting $p=\frac{\|f(a)\|}{\|a\|}-C(\varepsilon)$, we see that $p>0$ and $\|f(a)\|=(C(\varepsilon)+p)\|a\|$. Thus

$$\begin{aligned}\|f(a^2)\| &= \big\|f(a^2)-(f(a))^2+(f(a))^2\big\|\\ &\ge \|(f(a))^2\|-\|f(a^2)-(f(a))^2\|\\ &= \|f(a)\|^2-\|f(a^2)-(f(a))^2\|\\ &\ge \big[(C(\varepsilon)+p)^2-\varepsilon\big]\|a\|^2\\ &= \big[2pC(\varepsilon)+p^2+C(\varepsilon)\big]\|a\|^2\\ &> \big[2pC(\varepsilon)+C(\varepsilon)\big]\|a\|^2\\ &> \big[C(\varepsilon)+2p\big]\|a\|^2.\end{aligned}$$

This proves that when $n=1$, inequality

$$\|f(a^{2^n})\|\ge(C(\varepsilon)+2np)\|a\|^{2^n}$$

holds. Suppose that when $n=k$, (2.3) holds, that is $\|f(a^{2^k})\|\ge[C(\varepsilon)+2kp]\|a\|^{2^k}$, then when $n=k+1$, we have

$$\begin{aligned}\|f(a^{2^{k+1}})\| &= \big\|f((a^{2^k})^2)\big\|\\ &= \big\|f((a^{2^k})^2)-(f(a^{2^k}))^2+(f(a^{2^k}))^2\big\|\end{aligned}$$

$$\begin{aligned}
&\geq \|(f(a^{2^k}))^2\| - \left\|f((a^{2^k})^2) - (f(a^{2^k}))^2\right\| \\
&\geq \left[(C(\varepsilon) + 2kp)^2 - \varepsilon\right] \|a\|^{2^{k+1}} \\
&= \left[C(\varepsilon) + 4kpC(\varepsilon) + 4k^2p^2\right] \|a\|^{2^{k+1}} \\
&> \left[C(\varepsilon) + 4kpC(\varepsilon)\right] \|a\|^{2^{k+1}} \\
&> \left[C(\varepsilon) + 4kp\right] \|a\|^{2^{k+1}} \\
&\geq \left[C(\varepsilon) + (2k+2)p\right] \|a\|^{2^{k+1}} \\
&= \left[C(\varepsilon) + 2(k+1)p\right] \|a\|^{2^{k+1}}.
\end{aligned}$$

Therefore, (2.3) holds for all $n = 1, 2, 3, \ldots$. □

Theorem 2.2. *Let $\mathscr{A}, \mathscr{B}$ be normed algebras and $\mathscr{B}$ with the multiplicative norm, $f : \mathscr{A} \to \mathscr{B}$ be an ε-approximate Jordan mapping. If $\|f(a)\| > C(\varepsilon)\|a\|$ for some $a \in \mathscr{A}$, then $f(a^k) = (f(a))^k (k = 1, 2, \ldots)$.*

Proof. Clearly, $a \neq 0$. Let $c = \frac{a}{\|a\|}$. Since $\|f(a)\| > C(\varepsilon)\|a\|$, by Lemma 2.1, there exists a constant number $p > 0$ such that

$$\|f(c^{2^n})\| \geq C(\varepsilon) + 2np \quad (n = 1, 2, \ldots). \tag{2.4}$$

For all $m, n \in \mathbb{N}^+$, we get from the definition of ε-approximate Jordan mapping that

$$\begin{aligned}
&\left\|f(c^n c^m) - f(c^n)f(c^m)\right\| \\
&= \left\|\frac{1}{2}\left[f((c^n+c^m)^2) - f((c^n)^2) - f((c^m)^2)\right] - \frac{1}{2}\left[(f(c^n+c^m))^2 - (f(c^n))^2 - (f(c^m))^2\right]\right\| \\
&= \frac{1}{2}\left\|\left[f((c^n+c^m)^2) - (f(c^n+c^m))^2\right] - \left[f((c^n)^2) - (f(c^n))^2\right] - \left[f((c^m)^2) - (f(c^m))^2\right]\right\| \\
&\leq \frac{1}{2}\left(\left\|f((c^n+c^m)^2) - (f(c^n+c^m))^2\right\| + \left\|f((c^n)^2) - (f(c^n))^2\right\| + \left\|f((c^m)^2) - (f(c^m))^2\right\|\right) \\
&\leq \frac{\varepsilon}{2}\left(\|c^n + c^m\|^2 + \|c^n\|^2 + \|c^m\|^2\right) \\
&\leq 3\varepsilon.
\end{aligned}$$

Therefore

$$\left\|f(c^n c^m) - f(c^n)f(c^m)\right\| \leq 3\varepsilon. \tag{2.5}$$

We get from (2.4) and (2.5) that

$$\begin{aligned}
\|f(c^k) - (f(c))^k\| &= \frac{1}{\|f(c^{2^n})\|}\left\|\left[f(c^k)f(c^{2^n}) - f(c^k \cdot c^{2^n})\right]\right. \\
&\quad \left. - \sum_{i=0}^{k-1}\left[(f(c))^{k-i} f(c^i c^{2^n}) - (f(c))^{k-i-1} \cdot f(c^{i+1} \cdot c^{2^n})\right]\right\| \\
&\leq \frac{1}{\|f(c^{2^n})\|}\left(\left\|f(c^k)f(c^{2^n}) - f(c^k \cdot c^{2^n})\right\|\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{i=0}^{k-1} \left\| (f(c))^{k-i} f(c^i \cdot c^{2^n}) - (f(c))^{k-i-1} \cdot f(c^{i+1} \cdot c^{2^n}) \right\| \Bigg) \\
&= \frac{1}{\|f(c^{2^n})\|} \Bigg(\|f(c^k) f(c^{2^n}) - f(c^k \cdot c^{2^n})\| \\
&\qquad + \sum_{i=0}^{k-1} \|f(c)\|^{k-i-1} \cdot \left\| f(c) \cdot f(c^i \cdot c^{2^n}) - f(c^{i+1} \cdot c^{2^n}) \right\| \Bigg) \\
&\le \frac{1}{\|f(c^{2^n})\|} \left\{ 3\varepsilon + 3\varepsilon \sum_{i=0}^{k-1} \|f(c)\|^{k-i-1} \right\} \\
&\le \frac{1}{C(\varepsilon) + 2np} \left\{ 3\varepsilon + 3\varepsilon \sum_{i=0}^{k-1} \|f(c)\|^{k-i-1} \right\} \\
&\to 0 \ \ (n \to \infty).
\end{aligned}
$$

Thus $f(c^k) = (f(c))^k$, that is $f(a^k) = (f(a))^k$. This completes the proof. □

As an application of Theorem 2.2., we can get the following corollary.

Corollary 2.1. *Let $\mathscr{A}, \mathscr{B}$ be normed algebras and $\mathscr{B}$ have multiplicative norm. Then the Jordan equation (1.1) is $(\mathscr{A}, \mathscr{B})$-pointwise superstable.*

Theorem 2.3. *Let $\mathscr{A}, \mathscr{B}$ be normed algebras and Jordan equation (1.1) be $(\mathscr{A}, \mathscr{B})$-pointwise superstable. Then (1.1) is $(\mathscr{A}, \mathscr{B})$-superstable.*

Proof. Let $f : \mathscr{A} \to \mathscr{B}$ be an ε-approximate Jordan mapping and ε-lower bounded. For all $a \in \mathscr{A} \setminus \{0\}$, we have $\|f(a)\| > C(\varepsilon)\|a\|$. Since the Jordan equation (1.1) is $(\mathscr{A}, \mathscr{B})$-pointwise superstable, $f(a^2) = (f(a))^2$. f is therefore a Jordan mapping. This shows that the Jordan equation (1.1) is $(\mathscr{A}, \mathscr{B})$-superstable. This completes the proof. □

Use Corollary 2.1 and Theorem 2.3, we obtain the following corollary.

Corollary 2.2. *Let $\mathscr{A}, \mathscr{B}$ be normed algebras, and $\mathscr{B}$ have multiplicative norm. Then the Jordan equation (1.1) is $(\mathscr{A}, \mathscr{B})$-superstable.*

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Chapter 8

A PROBLEM WITH NON-LOCAL CONDITIONS ON THE LINE OF DEGENERACY AND PARALLEL CHARACTERISTICS FOR A MIXED TYPE EQUATION WITH SINGULAR COEFFICIENT

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Abstract

In the paper, we consider the problem for a mixed type equation with singular coefficient with non-local conditions on the line of degeneracy and parallel characteristics. We prove the uniqueness of solution for the problem by an analogue of the Bitsadze extremume principle. And we use the method of integral equations to prove the existence of solution.

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Key words and phrases. Non-local condition, mixed type equation, parallel characteristics, singular coefficient, extremum principle.

Introduction

Some boundary value problems for partial differential equations were investigated in (Polosin, 1996; Rassias, 1999; Protter, 1951; Holmgren, 1926). In (Wen, Chen, Cheng, 2007), the general Tricomi-Rassias problem was investigated for generalized Chaplygin equation. In the paper, the representation of solution of the general Tricomi-Rassias problem is given for the first time, as well as the uniqueness and the existence of solution for the problem is proved by a new method. In (Rassias, 2002), the general Tricomi problem was investigated for the new bi-parabolic elliptic bi-hyperbolic equation.

Unlike known problems for mixed type equations, in the present paper we investigate new non-local problem with the Bitsadze-Samarskii condition on parallel characteristics and an analogue of the Frankl condition on a segment of the degeneracy line for a class of mixed type equations.

Statement of the Problem

Consider the equation

$$signy\,|\,y\,|^m\,u_{xx}+u_{yy}-(m/2y)u_y=0\,, \tag{1}$$

where $m=const>0,$ in a finite one-connected domain Ω of the plane of independent variables $x,\ \ y$ bounded at $y>0$ by the normal curve $\sigma_0^2: x^2+4(m+2)^{-2}y^{m+2}=1,$ with ends in points $A(-1,0),\ \ B(1,0),$ and at $y<0$ – by characteristics AC and BC of the equation (1).

Denote the parts of Ω lying in the half-planes $y>0$ and $y<0$ as Ω^+ and Ω^-, respectively. Let C_0 and C_1 be points of intersection of characteristics AC and BC with the characteristic starting from the point $E(c,0),$ respectively, here $c\in J=(-1,1)$ is an interval of the axis $y=0$.

Let $p(x)\in C^2[-1,c]$ be a diffeomorphism from the set of points of the segment $[-1,c]$ into the set of points of the segment $[c,1],$ and also $p'(x)<0,\quad p(-1)=1,$ $p(c)=c.$ As an example of such function, one can mention a linear function $p(x)=\delta-kx$ where $k=(1-c)/(1+c),\ \ \delta=2c/(1+c).$

Problem GF. To find the function $u(x,y)\in C\left(\overline{\Omega}\right)$ in Ω satisfying the following conditions:

1. $u(x,y)\in C^2(\Omega^+)$ and satisfies (1) in this domain;

2. $u(x,y)$ is a generalized solution of the class R_1 (Smirnov, 1985) ($\tau'(x),\ \ \nu(x)\in H,$ definition for $\tau(x)$ and $\nu(x)$ is given below) in the domain $\Omega^-\setminus\left(EC_0\cup EC_1\right);$

3. On the degeneracy interval, the following conjugation condition

$$\lim_{y\to-0}\left(-y\right)^{-m/2}\frac{\partial u}{\partial y}=\lim_{y\to+0}y^{-m/2}\frac{\partial u}{\partial y},\quad x\in J\setminus\{c\} \tag{2}$$

hold, moreover these limits can have at $x = \pm 1,\ \ x = c$ singularities of the order less than unit;

4.

$$u(x, y)|_{\sigma_0} = \varphi(x), \quad -1 \le x \le 1, \tag{3}$$

$$u[\theta(x)] - u[\theta^*(p(x))] = \psi(x), \qquad -1 \le x \le c, \tag{4}$$

$$u(x,0) - u(p(x),0) = f(x), \qquad -1 \le x \le c\,, \tag{5}$$

where

$$\theta(x_0) = (x_0 - 1)/2 - i[(m+2)(1+x_0)/4]^{2/(m+2)},$$

$$\theta^*(p(x_0)) = (c + p(x_0))/2 - i[(m+2)/(p(x_0) - c)/4]^{2/(m+2)},$$

i.e. $\theta(x_0)$ ($\theta^*(p(x_0))$) is the affix of the point of intersection of the characteristic AC_0 (EC_1) with the characteristic starting from the point $(x_0, 0),\ \ x_0 \in [-1, c]$ $((p(x_0), 0),$ $p(x_0) \in [c,1]$), given functions $f(x), \psi(x) \in C[-1,c] \cap C^{1,\alpha}(-1,c),\ \ \varphi(x) \in C^{0,\alpha}[-1,1],$ and also $\varphi(x) = (1 - x^2)\tilde{\varphi}(x)$, where $\tilde{\varphi}(x) \in C^{0,\alpha}[-1,1]$, moreover, by virtue of the coordination condition, we have $f(-1) = \varphi(-1) - \varphi(1) = 0$ from (5) at $x = -1$.

Conditions (4) and (5) are analogues of the Bitsadze-Samarskii (Bitsadze, Samarskii, 1969) and Frankl (Morawetz, 1954; Frankl, 1956) conditions, respectively.

Uniqueness of the Solution of Problem GF

The following assertion takes place.

Theorem 1. *Problem* GF *with homogeneous boundary conditions* ($\varphi(x) \equiv \psi(x) \equiv f(x) \equiv 0$) *has only the trivial solution.*

Proof. By the d'Alembert's formula (Bitsadze, 1981) giving the solution of the modified Cauchy problem in Ω^- with data

$$\tau(x) = u(x,0), \quad x \in \overline{J}; \qquad \nu(x) = \lim_{y \to -0} (-y)^{-m/2} \frac{\partial u}{\partial y}, \quad x \in J,$$

we have from the boundary condition (4) the following:

$$\tau'(x) - \nu(x) - \tau'(p(x))p'(x) + \nu(p(x))p'(x) = 2\psi'(x), \quad x \in [-1, c]. \tag{6}$$

Rewrite (5) in the form

$$\tau(x)-\tau(p(x))=f(x),\quad x\in[-1,c]. \tag{7}$$

Transform (6) with the help of (7) to the form

$$\nu(x)-\nu(p(x))p'(x)=f'(x)-2\psi'(x),\quad x\in[-1,c]. \tag{8}$$

Let's show that if $\varphi(x)\equiv 0$, $\psi(x)\equiv 0$, $f(x)\equiv 0$, then the solution of problem GF in the domain $\overline{\Omega}$ equals identically to zero. Suppose the opposite, let $u(x,y)\neq 0$ in the domain $\overline{\Omega^+}$, hence it has the positive maximum and negative minimum in $\overline{\Omega^+}$. By the Hopf's principle (Bitsadze, 1981), the function $u(x,y)$ to be found does not attain its positive maximum in interior points of Ω^+. According to the corresponding homogeneous condition (3) we have the same in σ_0.

Let $u(x,y)$ attain its positive maximum in an interior point of the segment $AB\setminus\{E\}$. Then by virtue of the corresponding homogeneous condition (5), this extremum is attained in two points: $(x_0,0)$ and $(p(x_0),0)$. Therefore we have in these points $\nu(x_0)<0$, $\nu(p(x_0))<0$, (Volkodavov, 1970), what follows $\nu(x_0)-\nu(p(x_0))p'(x_0)<0$. But this contradicts to the corresponding homogeneous relation (8) at $x=x_0$, hence there is no positive maximum on $AB\setminus\{E\}$.

Thus, the function $u(x,y)$ to be found attains its positive maximum in $\overline{\Omega^+}$ in the point $E(c,0)$. Analogously, one can prove that $u(x,y)$ attains its negative minimum in $\overline{\Omega^+}$ also in $E(c,0)$. Obtained contradiction shows that $u(x,y)\equiv 0$ in $\overline{\Omega^+}$, hence $u(x,y)\equiv 0$ in $\overline{\Omega}$. □

Existence of the Solution of Problem GF

Theorem 2. *Problem* GF *is uniquely solvable.*

Proof. The following formula (Mirsaburov, Eshankulov, 2003):

$$u(x,y)=\frac{1}{2\pi}\int_{-1}^{1}\nu(t)\left\{\ln\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]-\right.$$
$$\left.-\ln\left[(1-xt)^2+\frac{4t^2}{(m+2)^2}y^{m+2}\right]\right\}dt+$$

$$+\frac{(m+2)(1-R^2)}{4\pi}\int_{-1}^{1}\varphi(\xi)[\eta(\xi)]^{-(m+2)/2}(r^{-2}+r_1^{-2})d\xi,$$

where

$$R^2=x^2+4(m+2)^{-2}y^{m+2},\ \ \xi^2+4(m+2)^{-2}\eta^{m+2}=1,$$

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\}=(\xi-x)^2+\frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}}\pm\eta^{\frac{m+2}{2}}\right)^2,$$

gives in Ω^+ the solution of the modified problem N:

$$u(x,y)|_{\sigma_0}=\varphi(x),\quad x\in\overline{J};\qquad \lim_{y\to+0}y^{-m/2}\frac{\partial u}{\partial y}=\nu(x),\quad x\in J,$$

for the equation (1). By this formula,

$$\tau'(x)=-\frac{1}{\pi}\int_{-1}^{1}\nu(t)\left(\frac{1}{t-x}-\frac{t}{1-xt}\right)dt+F(x),\quad x\in[-1,1],\tag{9}$$

where

$$F(x)=\frac{m+2}{2\pi}\frac{d}{dx}\left[(1-x^2)\int_{-1}^{1}\varphi(\xi)[\eta(\xi)]^{-(m+2)/2}[1-2x\xi+x^2]^{-1}d\xi\right],$$

$$F(x)\in C\left(\overline{J}\right)\cap C^1(J).$$

By virtue of (9), we obtain from the condition (7) that

$$\int_{-1}^{1}\nu(t)\left(\frac{1}{t-x}-\frac{t}{1-xt}\right)dt-\int_{-1}^{1}\nu(t)\left(\frac{1}{t-p(x)}-\frac{t}{1-p(x)t}\right)p'(x)dt=$$

$$=F_0(x),\qquad x\in(-1,c),\tag{10}$$

where

$$F_0(x)=\pi(F(x)-F(p(x))p'(x)-f'(x))\in C^{0,\alpha}(-1,c).$$

Now decomposing each integral on the left side of (10) onto two ones with the intervals $(-1,c)$ and $(c,1)$, substituting $t=p(s)$ into integrals with limits $(c,1)$, taking into account (8), we have

$$\int_{-1}^{c}\nu(t)\left(\frac{1}{t-x}-\frac{t}{1-xt}\right)dt-\int_{-1}^{c}\nu(s)\left(\frac{1}{p(s)-x}-\frac{p(s)}{1-xp(s)}\right)ds-$$

$$-\int_{-1}^{c}\nu(t)\left(\frac{1}{t-p(x)}-\frac{t}{1-p(x)t}\right)p'(x)dt+$$

$$+\int_{-1}^{c}\nu(s)\left(\frac{1}{p(s)-p(x)}-\frac{p(s)}{1-p(x)p(s)}\right)p'(x)ds=F_1(x),x\in(-1,c)\,, \tag{11}$$

where

$$F_1(x)=F_0(x)+\int_{-1}^{c}(2\psi'(s)-f'(s))\left(\frac{1}{p(s)-x}-\frac{p(s)}{1-xp(s)}\right)ds-$$

$$-\int_{-1}^{c}(2\psi'(s)-f'(s))\left(\frac{1}{p(s)-p(x)}-\frac{p(s)}{1-p(x)p(s)}\right)p'(x)ds\in C^{0,\alpha}(-1,c).$$

Further we investigate problem GF in two cases: $c=0$ and $c\neq 0$.

1. The case of $c=0$, $p(x)=-x$. In this case, (11) has the form

$$4\int_{-1}^{0}\nu(t)\left(\frac{1}{t^2-x^2}-\frac{1}{1-x^2t^2}\right)tdt=F_1(x),\quad x\in(-1,0).$$

Substituting $\xi=t^2$, $t=-\sqrt{\xi}$; $y=x^2$, $x=-\sqrt{y}$, we obtain the following first order singular integral Tricomi equation:

$$\int_{0}^{1}\rho(\xi)\left(\frac{1}{\xi-y}-\frac{1}{1-y\xi}\right)d\xi=F_2(y),\quad y\in(0,1), \tag{12}$$

where

$$\rho(\xi)=\nu\left(-\sqrt{\xi}\right),\ F_2(y)=-\frac{1}{2}F_1\left(-\sqrt{y}\right).$$

We'll look up the solution $\rho(y)$ of the equation (12) in the Hölder class of functions $H(0,1)$ bounded at the point $y=1$ and unbounded at the point $y=0$, i.e. in the class $h(1)$. Applying the Carleman method (Smirnov, 1985), we obtain the inversion formula for the singular integral equation (12):

$$\rho(y)=-\frac{1}{\pi^2}\int_{0}^{1}\sqrt{\frac{\xi}{y}\frac{1+y}{1+\xi}}\left(\frac{1}{\xi-y}-\frac{1}{1-y\xi}\right)F_2(\xi)d\xi.$$

2. The case of $c\neq 0$, $p(x)=\delta-kx$ where $\delta=2c/(1+c)$, $k=(1-c)/(1+c)$. In this case, introduce the function

$$c(x,t)=\frac{p'(x)p(t)}{1-p(x)p(t)}-\frac{t}{1-xt}\in C\left([-1,c]\times[-1,c]\right)$$

except for the point $(-1,-1)$ where the function is bounded. In this case, we transform (11) into the form:

$$\int_{-1}^{c}\left(\frac{1}{t-x}-\frac{t}{1-xt}\right)\nu(t)dt=\frac{1}{2}\int_{-1}^{c}\left(\frac{1}{p(t)-x}+\frac{p'(x)}{t-p(x)}\right)\nu(t)dt+$$
$$+R[\nu]+\frac{1}{2}F_1(x),\qquad x\in(-1,c)\,, \tag{13}$$

where

$$R[\nu]=-\frac{1}{2}\int_{-1}^{c}\left(\frac{p(t)}{1-xp(t)}+\frac{tp'(x)}{1-p(x)t}-c(x,t)\right)\nu(t)dt \tag{14}$$

is a regular operator. The integrand expression $\frac{1}{p(t)-x}+\frac{p'(x)}{t-p(x)}$ of the right side of (13) have at $t=c,\ \ x=c$ the first order singularity by virtue of the equality $p(c)=c$, therefore this summand is selected separately (Polosin, 1996; Mirsaburov, 2001).

Rewrite (13) in the following form:

$$\int_{-1}^{c}\left(\frac{1}{t-x}-\frac{t}{1-xt}\right)\nu(t)dt=F_3(x),\quad x\in(-1,c)\,, \tag{15}$$

where

$$F_3(x)=\frac{1}{2}\int_{-1}^{c}\left(\frac{1}{p(t)-x}+\frac{p'(x)}{t-p(x)}\right)\nu(t)dt+R[\nu]+\frac{1}{2}F_1(x). \tag{16}$$

Solving (15) by the Carleman method, we have

$$\nu(x)=-\frac{1}{\pi^2}\int_{-1}^{c}\sqrt{\frac{(c-t)(1-ct)}{(c-x)(1-cx)}}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)F_3(t)dt. \tag{17}$$

Transform (17) with regard to (16) to the form

$$\nu(x)=-\frac{1}{2\pi^2}\int_{-1}^{c}\nu(s)ds\int_{-1}^{c}\sqrt{\frac{c-t}{c-x}}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\times$$
$$\times\left(\frac{1}{p(s)-t}-\frac{p'(t)}{s-p(t)}\right)dt+R_1[\nu]+F_4(x),\quad x\in(-1,c)\,, \tag{18}$$

where

$$R_1[\nu]=-\frac{1}{2\pi^2}\int_{-1}^{c}\nu(s)ds\int_{-1}^{c}\left(\sqrt{\frac{1-ct}{1-cx}}-1\right)\sqrt{\frac{c-t}{c-x}}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\times$$
$$\times\left(\frac{1}{p(s)-t}-\frac{p'(t)}{s-p(t)}\right)dt-\frac{1}{\pi^2}\int_{-1}^{c}\sqrt{\frac{(c-t)(1-ct)}{(c-x)(1-cx)}}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)R[\nu]dt,$$

$$F_4(x)=-\frac{1}{2\pi^2}\int_{-1}^{c}\sqrt{\frac{(c-t)(1-ct)}{(c-x)(1-cx)}}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)F_1(t)dt\in C^{0,\alpha}(-1,c).$$

Calculating the interior integral of the right side in (18), we have

$$\nu(x)=-\frac{1}{2\pi^2}\int_{-1}^{c}\frac{\nu(s)ds}{\sqrt{c-x}}\left\{\frac{1}{\delta-ks-x}\left[-2(1+c)^{1/2}F\left(1,-\frac{1}{2},\frac{1}{2};\frac{c-x}{1+c}\right)+\right.\right.$$
$$+\frac{2}{3}\frac{(1+c)^{3/2}}{1+\delta-ks}F\left(1,1,\frac{5}{2};\frac{1+c}{1+\delta-ks}\right)\bigg]-\frac{k}{s-\delta+kx}[-2(1+c)^{1/2}\times$$
$$\times F\left(1,-\frac{1}{2},\frac{1}{2};\frac{c-x}{1+c}\right)+\frac{2k(1+c)^{3/2}}{3(1-s)}F\left(1,1,\frac{5}{2};\frac{k(1+c)}{1-s}\right)\bigg]+$$
$$+\frac{1}{1-\delta x+kxs}\left[\frac{2x}{3}\frac{(1+c)^{3/2}}{1-xc}F\left(\frac{3}{2},1,\frac{5}{2};-\frac{x(1+c)}{1-xs}\right)-\frac{2}{3}\frac{(1+c)^{3/2}}{1+\delta-ks}\times\right.$$
$$\times F\left(1,1,\frac{5}{2};\frac{1+c}{1+\delta-ks}\right)\bigg]+\frac{k}{k+sx-\delta x}\left[\frac{2x(1+c)^{3/2}}{3(1-xc)}F\left(\frac{3}{2},1,\frac{5}{2};\frac{-x(1+c)}{1-xc}\right)-\right.$$
$$\left.\left.-\frac{2k}{3(1-s)}F\left(1,1,\frac{5}{2};\frac{k(1+c)}{1-s}\right)\right]\right\}+R_1[\nu]+F_4(x),\quad x\in(-1,c). \tag{19}$$

Introduce the notations:

$$K_0(x)=-2(1+c)^{1/2}F\left(1,-\frac{1}{2},\frac{1}{2};\frac{c-x}{1+c}\right)+2(1+c)^{1/2}, \tag{20}$$

$$L(s)=\frac{2}{3}\frac{(1+c)^{3/2}}{1+\delta-ks}F\left(1,1,\frac{5}{2};\frac{1+c}{1+\delta-ks}\right)-2(1+c)^{1/2}+\pi k^{1/2}(c-s)^{1/2}, \tag{21}$$

$$M(s)=\frac{2k(1+c)^{3/2}}{3(1-s)}F\left(1,1,\frac{5}{2};\frac{k(1+c)}{1-s}\right)-2(1+c)^{1/2}+\pi k^{-1/2}(c-s)^{1/2}. \tag{22}$$

It is easy to verify that functions $K_0(x),\ L(s),\ M(s)$ are infinitesimals of the order less than 1/2 at the point c, i.e.

$$K_0(x)=o\left((c-x)^{1/2}\right), L(s)=o\left((c-s)^{1/2}\right), M(s)=o\left((c-s)^{1/2}\right). \tag{23}$$

We prove only the last relation in (23). Applying the L'Hospital rule for calculating limits, we have

$$\lim_{s\to c}\frac{M(s)}{(c-s)^{\frac{1}{2}}}=\lim_{s\to c}\frac{M'(s)}{-\frac{1}{2}(c-s)^{-\frac{1}{2}}}=$$
$$-2\lim_{s\to c}\left\{\frac{2}{3}(1+c)^{\frac{1}{2}}\frac{d}{ds}\left[\frac{k(1+c)}{1-s}F\left(1,1,\frac{5}{2};\frac{k(1+c)}{1-s}\right)\right]-\right.$$
$$\left.-\frac{\pi k^{-\frac{1}{2}}}{2}(c-s)^{-\frac{1}{2}}\right\}(c-s)^{\frac{1}{2}}.$$

Applying the well-known formulas for the Gauss hypergeometric functions (Smirnov, 1985)

$$\frac{d}{dx}x^aF(a,b,c;x)=ax^{a-1}F(a+1,b,c;x),$$

$$F(a,b,c;x)=(1-x)^{c-a-b}F(c-a,c-b,c;x),$$

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \Gamma^2\left(\frac{1}{2}\right)=\pi,$$

it is not difficult to calculate that

$$\lim_{s\to c}\frac{M(s)}{(c-s)^{\frac{1}{2}}}=-2\left(\frac{k^{-\frac{1}{2}}\pi}{2}-\frac{k^{-\frac{1}{2}}\pi}{2}\right)=0.$$

Hence, $M(s)=o\left((c-s)^{\frac{1}{2}}\right)$. Other two relations in (23) can be proved analogously.

Taking into consideration (23) and notations (20)-(22), we transform the equation (19) to the form

$$\nu(x)=\frac{k^{1/2}}{2\pi}\int_{-1}^{c}\sqrt{\frac{c-s}{c-x}}\left(\frac{1}{\delta-ks-x}-\frac{1}{s-\delta+kx}\right)\nu(s)ds+$$
$$+R_2[\nu]+F_4(x),\qquad x\in(-1,c), \tag{24}$$

where

$$R_2[\nu] = R_1[\nu] - \frac{1}{2\pi^2}\int_{-1}^{c}\frac{\nu(s)ds}{\sqrt{c-x}}\left\{\frac{K_0(x)+L(s)}{\delta-ks-x} - \frac{k(K_0(x)+M(s))}{s-\delta+kx} + \right.$$

$$+\frac{1}{1-\delta x+kxs}\left[\frac{2x(1+c)^{3/2}}{3(1-xc)}F\left(\frac{3}{2},1,\frac{5}{2};-\frac{x(1+c)}{1-xc}\right) - \frac{2}{3}\frac{(1+c)^{3/2}}{1+\delta-ks}\times\right.$$

$$\left.\times F\left(1,1,\frac{5}{2};\frac{1+c}{1+\delta-ks}\right)\right] + \frac{k}{k+sx-\delta x}\left[\frac{2x(1+c)^{3/2}}{3(1-xc)}\times\right.$$

$$\left.\left.\times F\left(\frac{3}{2},1,\frac{5}{2};\frac{-x(1+c)}{1-xc}\right) - \frac{2k}{3(1-s)}F\left(1,1,\frac{5}{2};\frac{k(1+c)}{1-s}\right)\right]\right\} \tag{25}$$

is a regular operator.

Rewrite (24) in the form

$$\nu(x) = \frac{k^{1/2}}{2\pi}\int_{-1}^{c}\sqrt{\frac{c-s}{c-x}}\frac{1}{c-s}\left(\frac{1}{k+(c-x)/(c-s)} + \right.$$

$$\left. + \frac{1}{1+k(c-x)/(c-s)}\right)\nu(s)ds + R_2[\nu] + F_4(x). \tag{26}$$

If we substitute in (26) $s = c-(1+c)e^{-t}$, $x = c-(1+c)e^{-y}$ (Polosin, 1996) and introduce the new unknown function $\rho(y) = \nu[c-(1+c)e^{-y}]e^{-y}$, then (26) has the form

$$\rho(y) = \int_{0}^{\infty} K(y-t)\rho(t)dt + R_3[\rho] + F_5(y), \tag{27}$$

where

$$K(x) = \frac{k^{1/2}}{2\pi}\left[\frac{1}{ke^{x/2}+e^{-x/2}} + \frac{1}{e^{x/2}+ke^{-x/2}}\right],$$

$$R_3[\rho] = R_2[\rho e^{y}]e^{-y}, \quad F_5(y) = F_4[c-(1+c)e^{-y}]e^{-y},$$

$R_3[\rho]$ is a regular operator.

The equation (27) is the Winner-Hopf equation (Gakhov, Cherskii, 1978). The index of (27) will be the index of the expression $1-K^{\wedge}(x)$ where

$$K^{\wedge}(x) = \int_{-\infty}^{\infty} e^{ixt}K(t)dt.$$

Calculating the Fourier integral with the help of the residues theory, we obtain

$$K^{\wedge}(x) = \frac{k^{1/2}}{2\pi}\left(\frac{\pi e^{-ixlnk}}{\sqrt{k}ch\pi x} + \frac{\pi e^{ixlnk}}{\sqrt{k}ch\pi x}\right) = \frac{cos(xlnk)}{ch\pi x} \leq 1.$$

Thus, $1 - K^{\wedge}(x) \geq 0,$ hence , $Ind(1 - K^{\wedge}(x)) = 0,$ i.e. changing of the argument of $1 - K^{\wedge}(x)$ on the real axis expressed in terms of total turns equals to zero (Gakhov, Cherskii, 1978). This and the uniqueness of the solution of problem GF yield uniquely solvability of (27). Therefore problem GF is uniquely solvable.

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Chapter 9

ON THE STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY IN NORMED MODULES

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To the memory of Professor Stanislaw Marcin Ulam
on the occasion of his 100-th birthday anniversary

Abstract

In this paper, we investigate the following additive functional inequality $\|f(x)+f(y)+f(z)+f(w)\| \le \|f(x)+f(y+z+w)\|$ in normed modules over a C^*-algebra. This is applied to understand homomorphisms in C^*-algebras.

Moreover, we prove the generalized Hyers–Ulam stability of the following functional inequality $\|f(x)+f(y)+f(z)+f(w)\| \le \|f(x)+f(y+z+w)\|+\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$ in real Banach spaces, where θ, p are positive real numbers with $p \neq 1$.

Using fixed point methods, we prove the generalized Hyers–Ulam stability of the previous functional inequality in real Banach spaces.

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Key words: Functional equation, fixed point, generalized Hyers–Ulam stability, functional inequality, linear mapping in normed modules over a C^*-algebra.

1. Introduction

The stability problem of functional equations originated from a question of Ulam (33) concerning the stability of group homomorphisms. Hyers (13) gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki (1) for additive mappings and by Th. M. Rassias (28) for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias (28) has provided a lot

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of influence in the development of what we call *generalized Hyers–Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta (10) by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. In 1982, J. M. Rassias (24) followed the innovative approach of the Th. M. Rassias' theorem (28) in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

Theorem 1.1. (J. M. Rassias: (24)–(27)). *Let X be a real normed linear space and Y a real Banach space. Assume that $f : X \to Y$ is a mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the functional inequality*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see (6), (9), (14), (21), (22), (29)–(31)).

We recall a fundamental result in fixed point theory.

Theorem 1.2 ((2),(7))**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \forall\, n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \big\{y \in X \mid d(J^{n_0} x, y) < \infty\big\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th. M. Rassias (15) were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see (4), (17), (18), (19), (23)).

Gilányi (11) showed that if f satisfies the functional inequality

$$\big\|2f(x) + 2f(y) - f(x-y)\big\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also (32). Fechner (8) and Gilányi (12) proved the generalized Hyers–Ulam stability of the functional inequality (1.1). Park, Cho and Han (20) investigated the functional inequality

$$\big\|f(x)+f(y)+f(z)\big\| \le \|f(x+y+z)\| \tag{1.2}$$

in Banach spaces, and proved the generalized Hyers–Ulam stability of the functional inequality (1.2) in Banach spaces.

In this paper, we investigate a module linear mapping associated with the functional inequality

$$\big\|f(x)+f(y)+f(z)+f(w)\big\| \le \|f(x)+f(y+z+w)\|. \tag{1.3}$$

This is applied to understand homomorphisms in C^*-algebras. Moreover, we prove the generalized Hyers–Ulam stability of the functional inequality

$$\begin{aligned}&\big\|f(x)+f(y)+f(z)+f(w)\big\|\\ &\le \|f(x)+f(y+z+w)\| + \theta\big(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p\big)\end{aligned} \tag{1.4}$$

in real Banach spaces. Using fixed point methods, we prove the generalized Hyers–Ulam stability of the functional inequality (1.4) in real Banach spaces.

2. Functional Inequalities in Normed Modules over a C^*-Algebra

Throughout this section, let A be a unital C^*-algebra with unitary group $U(A)$ and unit e, and let B be a C^*-algebra. Assume that X is a normed A-module with norm $\|\cdot\|$ and that Y is a normed A-module with norm $\|\cdot\|$.

In this section, we investigate an A-linear mapping associated with the functional inequality (1.3).

Theorem 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\big\|f(x)+f(y)+f(z)+uf(w)\big\| \le \big\|f(x)+f(y+z+uw)\big\| \tag{2.5}$$

for all $x,y,z,w \in X$ and all $u \in U(A)$. Then the mapping $f : X \to Y$ is A-linear.

Proof. Letting $x=y=z=w=0$ and $u=e\in U(A)$ in (2.1), we get

$$\|4f(0)\| \le \|2f(0)\|.$$

So $f(0)=0$.

Letting $x=w=0$ in (2.1), we get

$$\|f(y)+f(z)\| \le \|f(y+z)\| \tag{2.6}$$

for all $y,z\in X$.

Replacing y and z by x and $y+z+w$ in (2.2), respectively, we get

$$\big\|f(x)+f(y+z+w)\big\| \le \|f(x+y+z+w)\|$$

for all $x, y, z, w \in X$. So

$$\big\|f(x) + f(y) + f(z) + f(w)\big\| \le \|f(x + y + z + w)\| \tag{2.7}$$

for all $x, y, z, w \in X$.

Letting $z = w = 0$ and $y = -x$ in (2.3), we get

$$\|f(x) + f(-x)\| \le \|f(0)\| = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ and $w = 0$ in (2.3), we get

$$\big\|f(x) + f(y) - f(x + y)\big\| = \big\|f(x) + f(y) + f(-x - y)\big\| \le \|f(0)\| = 0$$

for all $x, y \in X$. Thus

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

Letting $z = -uw$ and $x = y = 0$ in (2.1), we get

$$\| - f(uw) + uf(w)\| = \|f(-uw) + uf(w)\| \le \|2f(0)\| = 0$$

for all $w \in X$ and all $u \in U(A)$. Thus

$$f(uw) = uf(w) \tag{2.8}$$

for all $u \in U(A)$ and all $w \in X$.

Now let $a \in A$ $(a \neq 0)$ and M an integer greater than $4|a|$. Then $|\frac{a}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 of (16), there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3\,\frac{a}{M} = u_1 + u_2 + u_3$. So by (2.4)

$$\begin{aligned}
f(ax) &= f\left(\frac{M}{3} \cdot 3\frac{a}{M}\,x\right) = M \cdot f\left(\frac{1}{3} \cdot 3\frac{a}{M}\,x\right) = \frac{M}{3}\,f\left(3\frac{a}{M}\,x\right) = \frac{M}{3}\,h(u_1x + u_2x + u_3x) \\
&= \frac{M}{3}\,\big(f(u_1x) + f(u_2x) + f(u_3x)\big) = \frac{M}{3}\,(u_1 + u_2 + u_3)f(x) = \frac{M}{3} \cdot 3\frac{a}{M}\,f(x) \\
&= af(x)
\end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-linear, as desired. $\square$

Corollary 2.2. *Let $f : A \to B$ be a multiplicative mapping such that*

$$\big\|f(x) + f(y) + f(z) + \mu f(w)\big\| \le \big\|f(x) + f(y + z + \mu w)\big\| \tag{2.9}$$

for all $x, y, z, w \in A$ and all $\mu \in \mathbb{T} := \big\{\lambda \in \mathbb{C} \mid |\lambda| = 1\big\}$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. By Theorem 2.1, the multiplicative mapping $f : A \to B$ is $\mathbb{C}$-linear, since C^*-algebras are normed modules over $\mathbb{C}$. So the multiplicative mapping $f : A \to B$ is a C^*-algebra homomorphism, as desired. $\square$

3. Generalized Hyers–Ulam Stability of Functional Inequalities

Throughout this section, assume that X is a real normed linear space and that Y is a real Banach space.

In this section, we prove the generalized Hyers–Ulam stability of the functional inequality (1.4) in real Banach spaces.

Theorem 3.1. *Assume that $f : X \to Y$ is an odd mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R}$ such that $p \neq 1$ and $f : X \to Y$ satisfies the functional inequality*

$$\begin{aligned} &\big\|f(x) + f(y) + f(z) + f(w)\big\| \\ &\leq \|f(x) + f(y+z+w)\| + \theta\big(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\big) \end{aligned} \tag{3.10}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{2^p + 2}{|2^p - 2|}\,\theta\|x\|^p \tag{3.11}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is an $\mathbb{R}$-linear mapping.

Proof. Since f is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$.

Letting $x = 0$, $z = y$ and $w = -2y$ in (3.1), we get

$$\|2f(y) - f(2y)\| \leq (2 + 2^p)\theta\|y\|^p \tag{3.12}$$

for all $y \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2 + 2^p}{2^p}\,\theta\|x\|^p$$

for all $x \in X$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq \frac{2 + 2^p}{2^p}\sum_{j=l}^{m-1}\frac{2^j}{2^{pj}}\,\theta\|x\|^p \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

Assume that $p > 1$. It follows from (3.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

Letting $l = 0$ and $m \to \infty$ in (3.4), we get

$$\|f(x) - A(x)\| \leq \frac{2^p + 2}{2^p - 2}\,\theta\|x\|^p$$

for all $x \in X$.

It follows from (3.1) that

$$\begin{aligned}&\left\|2^k f\left(\frac{x}{2^k}\right)+2^k f\left(\frac{y}{2^k}\right)+2^k f\left(\frac{z}{2^k}\right)+2^k f\left(\frac{w}{2^k}\right)\right\|\\&\quad\le\left\|2^k f\left(\frac{x}{2^k}\right)+2^k f\left(\frac{y+z+w}{2^k}\right)\right\|+\frac{2^k\theta}{2^{pk}}\left(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p\right)\end{aligned}\tag{3.14}$$

for all $x, y, z, w \in X$. Letting $k \to \infty$ in (3.5), we get

$$\big\|A(x)+A(y)+A(z)+A(w)\big\| \le \|A(x)+A(y+z+w)\| \tag{3.15}$$

for all $x, y, z, w \in X$. It is easy to show that $A : X \to Y$ is odd. Letting $w = -y - z$ and $x = 0$ in (3.6), we get $A(y + z) = A(y) + A(z)$ for all $y, z \in X$. So there exists a Cauchy additive mapping $A : X \to Y$ satisfying (3.2) for the case $p > 1$.

Now, let $T : X \to Y$ be another Cauchy additive mapping satisfying (3.2). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= 2^q\left\|A\left(\frac{x}{2^q}\right)-T\left(\frac{x}{2^q}\right)\right\|\\&\le 2^q\left(\left\|L\left(\frac{x}{2^q}\right)-f\left(\frac{x}{2^q}\right)\right\|+\left\|T\left(\frac{x}{2^q}\right)-f\left(\frac{x}{2^q}\right)\right\|\right)\\&\le \frac{2^p+2}{2^p-2}\cdot\frac{2\cdot 2^q}{2^{pq}}\,\theta\|x\|^p,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

Assume that $p < 1$. It follows from (3.3) that

$$\left\|f(x)-\frac{1}{2}\,f(2x)\right\| \le \frac{2^p+2}{2}\,\theta\|x\|^p$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}\,f(2^l x)-\frac{1}{2^m}\,f(2^m x)\right\| \le \frac{2^p+2}{2}\sum_{j=l}^{m-1}\frac{2^{pj}}{2^j}\,\theta\|x\|^p \tag{3.16}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

It follows from (3.7) that the sequence $\left\{\frac{1}{2^k}f(2^k x)\right\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{2^k}\,f(2^k x)\right\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty}\frac{1}{2^k}\,f\left(2^k x\right)$$

for all $x \in X$.

Letting $l = 0$ and $m \to \infty$ in (3.7), we get

$$\|f(x)-A(x)\| \le \frac{2^p+2}{2-2^p}\theta\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the above proof. So there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \le \frac{2^p + 2}{|2^p - 2|}\,\theta\|x\|^p \tag{3.17}$$

for all $x \in X$.

Assume that $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. By the same reasoning as in the proof of Theorem 1.1, one can prove that A is an $\mathbb{R}$-linear mapping. □

Using fixed point methods, we prove the generalized Hyers–Ulam stability of the functional inequality (1.4) in Banach spaces.

Theorem 3.2. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^4 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,y,z,w) \le \frac{1}{2}\,L\varphi(2x,2y,2z,2w)$ for all $x,y,z,w \in X$, and*

$$\big\|f(x) + f(y) + f(z) + f(w)\big\| \le \|f(x) + f(y+z+w)\| + \varphi(x,y,z,w) \tag{3.18}$$

for all $x,y,z,w \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \le \frac{L}{2-2L}\,\varphi(0,x,x,-2x) \tag{3.19}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\Big\{K \in \mathbb{R}_+ :\ \|g(x) - h(x)\| \le K\varphi(0,x,x,-2x),\ \forall\, x \in X\Big\}.$$

It is easy to show that (S,d) is complete. (See the proof of Theorem 2.5 of (3).)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of (2) that

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g,h \in S$.

Since $f : X \to Y$ is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. Letting $z = y = x$ and $w = -2x$ in (3.9), we get

$$\|2f(x) - f(2x)\| = \|2f(x) + f(-2x)\| \le \varphi(0,x,x,-2x) \tag{3.20}$$

for all $x \in X$.

It follows from (3.11) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \varphi\left(0, \frac{x}{2}, \frac{x}{2}, -x\right) \le \frac{L}{2}\,\varphi(0, x, x, -2x)$$

for all $x \in X$. Hence $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}\,A(x) \tag{3.21}$$

for all $x \in X$. Then $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : \ d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.12) such that there exists a $K \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le K\varphi(0, x, x, -2x)$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x) \tag{3.22}$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2 - 2L}\,.$$

This implies that the inequality (3.10) holds.

It follows from (3.9) and (3.13) that

$$\left\|A(x) + A(y) + A(z) + A(w)\right\| \le \|A(x) + A(y + z + w)\|$$

for all $x, y, z, w \in X$. By Theorem 2.1, the mapping $A : X \to Y$ is a Cauchy additive mapping.

Therefore, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (3.11), as desired. $\square$

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\begin{aligned} &\left\|f(x) + f(y) + f(z) + f(w)\right\| \\ &\le \|f(x) + f(y + z + w)\| + \theta\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right) \end{aligned} \tag{3.23}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r + 2}{2^r - 2}\,\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y,z,w) := \theta\big(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\big)$$

for all $x, y, z, w \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. $\square$

Remark 3.4. Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^4 \to [0,\infty)$ satisfying (3.9). By a similar method to the proof of Theorem 3.2, one can show that if there exists an $L < 1$ such that $\varphi(x,y,z,w) \le 2L\varphi\big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\big)$ for all $x, y, z, w \in X$, then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \le \frac{1}{2-2L}\,\varphi(0,x,x,-2x)$$

for all $x \in X$.

For the case $0 < r < 1$, one can obtain a similar result to Corollary 3.3: Let $0 < r < 1$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be an odd mapping satisfying (3.14). Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \le \frac{2+2^r}{2-2^r}\,\theta\|x\|^r$$

for all $x \in X$.

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Chapter 10

CUBIC DERIVATIONS AND QUARTIC DERIVATIONS ON BANACH MODULES

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To the memory of Professor Stanislaw Marcin Ulam
on the occasion of his 100-th birthday anniversary

Abstract

In this paper, we define a cubic derivation and a quartic derivation on a Banach module over a normed algebra and prove the generalized Hyers–Ulam stability of the cubic derivation and the quartic derivation on a Banach module over a normed algebra.

2000 Mathematics Subject Classifications: Primary 39B82, 39B72.

Key words: Cubic functional equation, quartic functional equation, cubic derivation, quartic derivation, generalized Hyers–Ulam stability.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam (23) concerning the stability of group homomorphisms. Hyers (5) gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki (1) for additive mappings and by Th. M. Rassias (15) for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias (15) has provided a lot of influence in the development of what we call *generalized Hyers–Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by

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Găvruta (4) by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

A square norm on an inner product space satisfies the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof (22) for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa (2) noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In (3), Czerwik proved the generalized Hyers–Ulam stability of the quadratic functional equation. Several functional equations have been investigated in (6), (7), (9) and (11)–(21).

Jun and Kim (8) introduced the following functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.1}$$

and they established the general solution and the generalized Hyers–Ulam stability problem for the functional equation (1.1).

It is easy to see that the function $f(x) = cx^3$ is a solution of the above functional equation (1.1). Thus, it is natural that (1.1) is called a *cubic functional equation* and every solution of the cubic functional equation (1.1) is said to be a *cubic mapping*.

In (10), S. Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = cx^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Throughout this paper, we suppose that A is a normed algebra and that X is a Banach A-module.

Definition 1.1. A cubic mapping $f : A \to X$ is called a *cubic derivation* if f satisfies $f(xy) = x^3 f(y) + f(x)y^3$ for all $x, y \in A$.

Definition 1.2. A quartic mapping $f : A \to X$ is called a *quartic derivation* if f satisfies $f(xy) = x^4 f(y) + f(x)y^4$ for all $x, y \in A$.

Example 1.3. Assume that A is a commutative normed algebra. Let $\omega \in X$ be fixed.

(i) Define $f : A \to X$ by $f(x) := x^3\omega - \omega x^3$ for all $x \in A$. It is easy to show that $f : A \to X$ is a cubic derivation.

(ii) Define $f : A \to X$ by $f(x) := x^4\omega - \omega x^4$ for all $x \in A$. It is easy to show that $f : A \to X$ is a quartic derivation.

In this paper, we prove the generalized Hyers-Ulam stability of the cubic derivation and of the quartic derivation on a Banach module over a normed algebra.

2. On the Stability of Cubic Derivations on Banach Modules

In this section, we prove the generalized Hyers–Ulam stability of the cubic derivation on a Banach module over a normed algebra.

Theorem 2.1. *Suppose that a function $\psi : A \times A \to [0, \infty)$ satisfies*

$$\widetilde{\psi}(x,y) := \sum_{i=0}^{\infty} \frac{1}{8^i}\, \psi(2^i x, 2^i y) < \infty \tag{2.3}$$

for all $x, y \in A$. If $f : A \to X$ is a mapping such that

$$\big\| f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) \big\| \le \psi(x,y), \tag{2.4}$$

$$\big\| f(xy) - x^3 f(y) - f(x)y^3 \big\| \le \psi(x,y) \tag{2.5}$$

for all $x, y \in A$, then there exists a unique cubic derivation $D : A \to X$ such that

$$\|f(x) - D(x)\| \le \frac{1}{16}\, \widetilde{\psi}(x,0) \tag{2.6}$$

for all $x \in A$

Proof. Putting $y = 0$ in (2.2), we get

$$\|2f(2x) - 16f(x)\| \le \psi(x,0) \tag{2.7}$$

for all $x \in A$. So

$$\left\| f(x) - \frac{1}{8}\, f(2x) \right\| \le \frac{1}{16}\, \psi(x,0)$$

for all $x \in A$. Hence

$$\left\| \frac{1}{8^n}\, f(2^n x) - \frac{1}{8^m}\, f(2^m x) \right\| \le \frac{1}{16} \sum_{k=n}^{m-1} \frac{1}{8^k}\, \psi(2^k x, 0) \tag{2.8}$$

for all nonnegative integers n, m with $n < m$. Thus $\left\{ \frac{1}{8^n}\, f(2^n x) \right\}$ is a Cauchy sequence in X. Since X is complete, there exists a mapping $D : A \to X$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$$

for all $x \in A$. Letting $n = 0$ and $m \to \infty$ in (2.6), we get the inequality (2.4).

It follows from (2.2) that

$$\begin{aligned} &\big\| D(2x+y) + D(2x-y) - 2D(x+y) - 2D(x-y) - 12D(x) \big\| \\ &= \lim_{n\to\infty} \frac{1}{8^n} \big\| f(2^n(2x+y)) + f(2^n(2x-y)) - 2f(2^n(x+y)) - 2f(2^n(x-y)) - 12f(2^n x) \big\| \\ &\le \lim_{n\to\infty} \frac{1}{8^n}\, \psi(2^n x, 2^n y) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$D(2x+y)+D(2x-y)-2D(x+y)-2D(x-y)-12D(x)=0$$

for all $x, y \in A$. Hence $D : A \to X$ is a cubic mapping.

Let $C : A \to X$ be another cubic mapping satisfying (2.4). Then

$$\begin{aligned}\|D(x)-C(x)\| &= \frac{1}{8^n}\left\|D(2^n x)-C(2^n x)\right\| \\ &\le \frac{1}{8^n}\Big(\|f(2^n x)-D(2^n x)\|+\|f(2^n x)-C(2^n x)\|\Big) \\ &\le \frac{1}{8^{n+1}}\,\widetilde{\psi}(2^n x, 0),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we have $D(x) = C(x)$ for all $x \in A$. This proves the uniqueness of D.

On the other hand, it follows from (2.3) that

$$\begin{aligned}&\left\|D(xy)-x^3D(y)-D(x)y^3\right\| \\ &\quad= \left\|\frac{1}{64^n}\left(D(4^n xy)-8^n x^3 D(2^n y)-8^n D(2^n x)y^3\right)\right\| \\ &\quad= \left\|\lim_{n\to\infty}\frac{1}{64^n}\,f(4^n xy)-x^3\lim_{n\to\infty}\frac{1}{8^n}f(2^n y)-\lim_{n\to\infty}\frac{1}{8^n}\,f(2^n x)y^3\right\| \\ &\quad\le \lim_{n\to\infty}\frac{1}{64^n}\,\psi(2^n x, 2^n y) \\ &\quad\le \lim_{n\to\infty}\frac{1}{8^n}\,\psi(2^n x, 2^n y)=0\end{aligned}$$

for all $x, y \in A$. Thus

$$D(xy)=x^3D(y)+D(x)y^3$$

for all $x, y \in A$, as desired. □

Corollary 2.2. *Let $p < 3$ and θ be positive real numbers. If $f : A \to X$ is a mapping such that*

$$\left\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\right\| \le \theta\big(\|x\|^p+\|y\|^p\big), \tag{2.9}$$

$$\left\|f(xy)-x^3f(y)-f(x)y^3\right\| \le \theta\big(\|x\|^p+\|y\|^p\big) \tag{2.10}$$

for all $x, y \in A$, then there exists a unique cubic derivation $D : A \to X$ such that

$$\|D(x)-f(x)\| \le \frac{\theta}{16-2^{p+1}}\,\|x\|^p$$

for all $x \in A$.

Proof. Define $\psi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 2.1 to get the desired result. □

Theorem 2.3. *Suppose that a function $\psi : A \times A \to [0, \infty)$ satisfies*

$$\sum_{i=1}^{\infty} 64^i \psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.11}$$

for all $x, y \in A$. If $f : A \to X$ is a mapping satisfying (2.2) *and* (2.3)*, then there exists a unique cubic derivation $D : A \to X$ such that*

$$\|f(x) - D(x)\| \le \frac{1}{16}\, \widetilde{\psi}(x, 0) \tag{2.12}$$

for all $x \in A$, where $\widetilde{\psi}(x, y) := \sum_{i=1}^{\infty} 8^i \psi\big(\frac{x}{2^i}, \frac{y}{2^i}\big)$ for all $x, y \in A$.

Proof. It follows from (2.5) that

$$\left\|f(x) - 8f\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\,\psi\left(\frac{x}{2}, 0\right)$$

for all $x \in A$. So

$$\left\|8^n f\left(\frac{x}{2^n}\right) - 8^m f\left(\frac{x}{2^m}\right)\right\| \le \frac{1}{16} \sum_{k=n+1}^{m} 8^k \psi\left(\frac{x}{2^k}, 0\right) \tag{2.13}$$

for all nonnegative integers n, m with $n < m$. Thus $\left\{8^n f\big(\frac{x}{2^n}\big)\right\}$ is a Cauchy sequence in X. Since X is complete, there exists a mapping $D : A \to X$ defined by

$$D(x) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Letting $n = 0$ and $m \to \infty$ in (2.11), we get the inequality (2.10).

It follows from (2.2) that

$$\begin{aligned}
&\big\|D(2x + y) + D(2x - y) - 2D(x + y) - 2D(x - y) - 12D(x)\big\| \\
&\quad = \lim_{n\to\infty} 8^n \left\|f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right) - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right)\right\| \\
&\quad \le \lim_{n\to\infty} 8^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} 64^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$D(2x + y) + D(2x - y) - 2D(x + y) - 2D(x - y) - 12D(x) = 0$$

for all $x, y \in A$. Hence $D : A \to X$ is a cubic mapping.

On the other hand, it follows from (2.3) and (2.9) that

$$\begin{aligned}
&\big\|D(xy) - x^3 D(y) - D(x) y^3\big\| \\
&\quad = \left\|64^n \left(D\left(\frac{xy}{4^n}\right) - \frac{1}{8^n}\, x^3 D\left(\frac{y}{2^n}\right) - \frac{1}{8^n} D\left(\frac{x}{2^n}\right) y^3\right)\right\| \\
&\quad = \left\|\lim_{n\to\infty} 64^n f\left(\frac{xy}{4^n}\right) - x^3 \lim_{n\to\infty} 8^n f\left(\frac{y}{2^n}\right) - \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) y^3\right\| \\
&\quad \le \lim_{n\to\infty} 64^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$. Thus

$$D(xy) = x^3 D(y) + D(x) y^3$$

for all $x, y \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $p > 6$ and θ be positive real numbers. If $f : A \to X$ is a mapping satisfying* (2.7) *and* (2.8)*, then there exists a unique cubic derivation $D : A \to X$ such that*

$$\|D(x) - f(x)\| \le \frac{\theta}{2^{p+1} - 16} \|x\|^p$$

for all $x \in A$.

Proof. Define $\psi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 2.3 to get the desired result. □

Definition 2.5. Let A, B be algebras. A cubic mapping $f : A \to B$ is called a *cubic homomorphism* if f satisfies $f(xy) = f(x)f(y)$ for all $x, y \in A$.

Example 2.6. Assume that A is an algebra and that B is a commutative algebra. Let $f : A \to B$ be a homomorphism and $F(x) := f(x)^3$ for all $x \in A$. It is easy to show that $F : A \to B$ is a cubic homomorphism.

Remark 2.7. By the same methods as in the proofs of the results in this section, one can prove the generalized Hyers–Ulam stability of cubic homomorphisms in Banach algebras.

3. On the Stability of Quartic Derivations on Banach Modules

In this section, we prove the generalized Hyers–Ulam stability of the quartic derivation on a Banach module over a normed algebra.

Theorem 3.1. *Suppose that a function $\psi : A \times A \to [0, \infty)$ satisfies*

$$\widetilde{\psi}(x, y) := \sum_{i=0}^{\infty} \frac{1}{16^i} \psi(2^i x, 2^i y) < \infty \tag{3.14}$$

for all $x, y \in A$. If $f : A \to X$ is a mapping such that

$$\left\| f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) - 6f(y) \right\| \le \psi(x, y), \tag{3.15}$$

$$\left\| f(xy) - x^4 f(x) - f(x) y^4 \right\| \le \psi(x, y) \tag{3.16}$$

for all $x, y \in A$, then there exists a unique quartic derivation $D : A \to X$ such that

$$\|f(x) - D(x)\| \le \frac{1}{32} \widetilde{\psi}(x, 0) \tag{3.17}$$

for all $x \in A$

Proof. Putting $y = 0$ in (3.2), we get

$$\|2f(2x) - 32f(x)\| \leq \psi(x, 0) \tag{3.18}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{16}\, f(2x)\right\| \leq \frac{1}{32}\,\psi(x, 0)$$

for all $x \in A$. Hence

$$\left\|\frac{1}{16^n}\, f(2^n x) - \frac{1}{16^m}\, f(2^m x)\right\| \leq \frac{1}{32}\sum_{k=n}^{m-1} \frac{1}{16^k}\,\psi(2^k x, 0) \tag{3.19}$$

for all nonnegative integers n, m with $n < m$. Thus $\left\{\frac{1}{16^n} f(2^n x)\right\}$ is a Cauchy sequence in X. Since X is complete, there exists a mapping $D : A \to X$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{16^n}\, f(2^n x)$$

for all $x \in A$. Letting $n = 0$ and $m \to \infty$ in (3.6), we get the inequality (3.4).

It follows from (3.2) that

$$\begin{aligned}
&\big\|D(2x + y) + D(2x - y) - 4D(x + y) - 4D(x - y) - 24D(x) - 6D(y)\big\| \\
&\qquad = \lim_{n\to\infty} \frac{1}{16^n}\,\big\|f(2^n(2x + y)) + f(2^n(2x - y)) - 4f(2^n(x + y)) \\
&\qquad\qquad - 4f(2^n(x - y)) - 24f(2^n x) - 6f(2^n y)\big\| \\
&\qquad \leq \lim_{n\to\infty} \frac{1}{16^n}\,\psi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$D(2x + y) + D(2x - y) - 4D(x + y) - 4D(x - y) - 24D(x) - 6D(y) = 0$$

for all $x, y \in A$. Hence $D : A \to X$ is a quartic mapping.

Let $Q : A \to X$ be another quartic mapping satisfying (3.4). Then

$$\begin{aligned}
\|D(x) - Q(x)\| &= \frac{1}{16^n}\,\|D(2^n x) - Q(2^n x)\| \\
&\leq \frac{1}{16^n}\,\Big(\|f(2^n x) - D(2^n x)\| + \|f(2^n x) - Q(2^n x)\|\Big) \\
&\leq \frac{1}{16^{n+1}}\,\widetilde{\psi}(2^n x, 0),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we have $D(x) = Q(x)$ for all $x \in A$. This proves the uniqueness of D.

On the other hand, it follows from (3.3) that

$$\begin{aligned}
&\left\|D(xy) - x^4D(y) - D(x)y^4\right\| \\
&= \left\|\frac{1}{256^n}\left(D(4^nxy) - 16^nx^4D(2^ny) - 16^nD(2^nx)y^4\right)\right\| \\
&= \left\|\lim_{n\to\infty}\frac{1}{256^n}f(4^nxy) - x^4\lim_{n\to\infty}\frac{1}{16^n}f(2^ny) - \lim_{n\to\infty}\frac{1}{16^n}f(2^nx)y^4\right\| \\
&\le \lim_{n\to\infty}\frac{1}{256^n}\psi(2^nx, 2^ny) \\
&\le \lim_{n\to\infty}\frac{1}{16^n}\psi(2^nx, 2^ny) = 0
\end{aligned}$$

for all $x, y \in A$. Thus

$$D(xy) = x^4D(y) + D(x)y^4$$

for all $x, y \in A$, as desired. □

Corollary 3.2. *Let $p < 4$ and θ be positive real numbers. If $f : A \to X$ is a mapping such that*

$$\left\|f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)-6f(y)\right\| \le \theta\left(\|x\|^p+\|y\|^p\right), \tag{3.20}$$

$$\left\|f(xy) - x^4f(y) - f(x)y^4\right\| \le \theta\left(\|x\|^p + \|y\|^p\right) \tag{3.21}$$

for all $x, y \in A$, then there exists a unique quartic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \le \frac{\theta}{32 - 2^{p+1}}\|x\|^p$$

for all $x \in A$.

Proof. Define $\psi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 3.1 to get the desired result. □

Theorem 3.3. *Suppose that a function $\psi : A \times A \to [0, \infty)$ satisfies*

$$\sum_{i=1}^{\infty} 256^i\psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{3.22}$$

for all $x, y \in A$. If $f : A \to X$ is a mapping satisfying (3.2) *and* (3.3)*, then there exists a unique quartic derivation $D : A \to X$ such that*

$$\|f(x) - D(x)\| \le \frac{1}{32}\widetilde{\psi}(x, 0) \tag{3.23}$$

for all $x \in A$, where $\widetilde{\psi}(x, y) := \sum_{i=1}^{\infty} 16^i\psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right)$ for all $x, y \in A$.

Proof. It follows from (3.5) that

$$\left\|f(x) - 16f\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\,\psi\left(\frac{x}{2}, 0\right)$$

for all $x \in A$. So

$$\left\|16^n f\left(\frac{x}{2^n}\right) - 16^m f\left(\frac{x}{2^m}\right)\right\| \le \frac{1}{32}\sum_{k=n+1}^{m} 16^k \psi\left(\frac{x}{2^k}, 0\right) \tag{3.24}$$

for all nonnegative integers n, m with $n < m$. Thus $\left\{16^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in X. Since X is complete, there exists a mapping $D : A \to X$ defined by

$$D(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Letting $n = 0$ and $m \to \infty$ in (3.11), we get the inequality (3.10).

It follows from (3.2) that

$$\begin{aligned}
\big\|D(2x+y) &+ D(2x-y) - 4D(x+y) - 4D(x-y) - 24D(x) - 6D(y)\big\| \\
&= \lim_{n\to\infty} 16^n \left\|f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 4f\left(\frac{x+y}{2^n}\right)\right. \\
&\qquad \left. - 4f\left(\frac{x-y}{2^n}\right) - 24f\left(\frac{x}{2^n}\right) - 6f\left(\frac{y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 16^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} 256^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$D(2x+y) + D(2x-y) - 4D(x+y) - 4D(x-y) - 24D(x) - 6D(y) = 0$$

for all $x, y \in A$. Hence $D : A \to X$ is a quartic mapping.

On the other hand, it follows from (3.3) and (3.9) that

$$\begin{aligned}
&\|D(xy) - x^4 D(y) - D(x)y^4\| \\
&\quad = \left\|256^n \left(D\left(\frac{xy}{4^n}\right) - \frac{1}{16^n}\, x^4 D\left(\frac{y}{2^n}\right) - \frac{1}{16^n}\, D\left(\frac{x}{2^n}\right) y^4\right)\right\| \\
&\quad = \left\|\lim_{n\to\infty} 256^n f\left(\frac{xy}{4^n}\right) - x^4 \lim_{n\to\infty} 16^n f\left(\frac{y}{2^n}\right) - \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) y^4\right\| \\
&\quad \le \lim_{n\to\infty} 256^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$. Thus

$$D(xy) = x^4 D(y) + D(x)y^4$$

for all $x, y \in A$, as desired. □

Corollary 3.4. *Let* $p > 8$ *and* θ *be positive real numbers. If* $f : A \to X$ *is a mapping satisfying* (3.7) *and* (3.8)*, then there exists a unique quartic derivation* $D : A \to X$ *such that*

$$\|D(x) - f(x)\| \le \frac{\theta}{2^{p+1} - 32}\,\|x\|^p$$

for all $x \in A$.

Proof. Define $\psi(x,y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 3.3 to get the desired result. $\square$

Definition 3.5. Let A, B be algebras. A quartic mapping $f : A \to B$ is called a *quartic homomorphism* if f satisfies $f(xy) = f(x)f(y)$ for all $x, y \in A$.

Example 3.6. Assume that A is an algebra and that B is a commutative algebra. Let $f : A \to B$ be a homomorphism and $F(x) := f(x)^4$ for all $x \in A$. It is easy to show that $F : A \to B$ is a quartic homomorphism.

Remark 3.7. By the same methods as in the proofs of the results in this section, one can prove the generalized Hyers–Ulam stability of quartic homomorphisms in Banach algebras.

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Chapter 11

TETRAHEDRON ISOMETRY ULAM STABILITY PROBLEM

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Abstract

In this paper we investigate the tetrahedron isometry Ulam stability problem.

1. Introduction

In 1940 S. M. Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Ulam stability problem. *We are given a group G and a metric group* G' *with metric* $\rho(.,.)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta > 0$ *such that if* $f : G \to G'$ *satisfies* $\rho(f(xy), f(x)f(y)) < \delta$ *for all* x, y *in G, then a homomorphism* $h : G \to G'$ *exists with* $\rho(f(x), h(x)) < \varepsilon$ *for all* $x \in G$ *?*

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism.*

In 1941 D. H. Hyers (*on the stability of the linear functional equation,* Proc. Nat. Acad. Sci. USA **27**(1941), 222-224) considered the case of approximately additive mappings $f : E \to E'$ where *E* and E' are Banach spaces and f satisfies *Hyers inequality*

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$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all x, y *in* $E.$ It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L: E \to E'$ is the unique additive mapping satisfying

$$\| f(x) - L(x) \| \leq \varepsilon \text{ .}$$

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable *t* for each fixed *x* ,then L is linear ,and if *f* is continuous at a single point of E then $L: E \to E'$ is also continuous.

A generalization of this result was proved via the following theorems. In the first two theorems 1-2, we assumed the following *weaker condition* (or *weaker inequality*)

$$\| f(x+y) - [f(x) + f(y)] \| \leq \theta \|x\|^p \|y\|^q \text{ for all } x, y \text{ in } E,$$

involving a product of different powers of norms, where $\theta \geq 0$ and real *p, q* such that $\rho = p + q \neq 1$, and retained the condition of continuity of $f(tx)$ in *t* for fixed *x*. Besides through the last two theorems 1.3-1.4, we investigated that it is possible to replace ε in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in this article, the approach to the existence question was to prove *asymptotic type formulas* of the form $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, or $L(x) = \lim_{n\to\infty} 2^{n} f(2^{-n} x)$.

However, in 2002 we (J. Ind. Math. Soc. **69** (2002), 155-160) considered and investigated quadratic equations involving <u>*a product of powers of norms*</u> in which an approximate quadratic mapping *degenerates* to a *genuine* quadratic mapping. Analogous results could be investigated with additive type equations involving a product of powers of norms.

In 1982, J. M. Rassias ("*on approximation of approximately linear mappings by linear mappings*", J. Funct. Anal. 46 (1), 5-9) provided a generalization of Hyers' stability Theorem which allows the *Cauchy difference to be unbounded,* as follows:

Theorem 1.1 ([1], [2], [5]). *Let* $f: E \to E'$ *be a mapping from a normed vector space E into a Banach space* E' *subject to the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon \|x\|^p \|y\|^p$$

for all $x, y \in E$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $0 \leq p < 1/2$. *Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\| f(x) - L(x) \| \le \frac{\varepsilon}{2 - 2^{2p}} \|x\|^{2p}$$

for all $x \in E$. *If* $p < 0$ *then inequality* (1.3) *holds for* $x, y \ne 0$ *and* (1.4) *for* $x \ne 0$. *If* $p > 1/2$ *then inequality* (1.3) *holds for all* $x, y \in E$ *and the limit*

$$A(x) = \lim_{n\to\infty} 2^n \left(\frac{x}{2^n} \right)$$

exists for all $x \in E$ *and* $A : E \to E'$ *is the unique additive mapping which satisfies*

$$\| f(x) - A(x) \| \le \frac{\varepsilon}{2^{2p} - 2} \|x\|^{2p}$$

for all $x \in E$. *If in addition* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then L is* $\mathbb{R}$ *- linear mapping.*

The case $p = 1/2$ in inequality (1.3) is singular. A counter-example has been given by P. Gavruta ("*An answer to a question of John M. Rassias concerning the stability of Cauchy equation*", in: Advances in Equations and Inequalities, in: *Hadronic Math. Ser.*, 1999, 67-71). Our above-mentioned stability is called *Ulam - Gavruta - Rassias stability*.

Theorem 1.2 ([3]). *Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that* $f : X \to Y$ *is an approximately additive mapping for which there exist constants* $\theta \ge 0$ *and* $p, q \in \mathbb{R}$ *such that* $\rho = p + q \ne 1$ *and f satisfies inequality*

$$\| f(x+y) - [f(x) + f(y)] \| \le \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *satisfying*

$$\| f(x) - L(x) \| \le \frac{\theta}{|2^\rho - 2|} \|x\|^\rho$$

for all $x \in X$. *If in addition* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in R$ *for each fixed* $x \in X$, *then L is* $\mathbb{R}$ *- linear mapping.*

Theorem 1.3 ([4]). *Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that* $f : X \to Y$ *is an approximately additive mapping for which there exists a constant* $\theta \geq 0$ *such that f satisfies inequality*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \leq \theta K(x_1, x_2, ..., x_n) \qquad (*)$$

for all $(x_1, x_2, ..., x_n) \in X^n$ *and* $K : X^n \to \mathbb{R}^+ \cup \{0\}$ *is a non-negative real-valued function such that*

$$R_n = R_n(x) = \sum_{j=0}^{\infty} n^{-j} K(n^j x, n^j x, ..., n^j x) \; (< \infty)$$

is a non-negative function of x, and the condition

$$\lim_{m \to \infty} n^{-m} K(n^m x_1, n^m x_2, ..., n^m x_n) = 0$$

holds. Then there exists a unique additive mapping $L_n : X \to Y$ *satisfying*

$$\left\| f(x) - L_n(x) \right\| \leq \frac{\theta}{n} R_n(x)$$

for all $x \in X$. *If in addition* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* L_n *is an* $\mathbb{R}$ *linear mapping.*

Replacing $x_i = x$ *for* $i = 1, 2, ..., n$ in (*), we obtain

$$\left\| f(x) - n^{-\nu} f(n^{\nu} x) \right\| \leq \frac{\theta}{n} \sum_{j=0}^{\nu-1} n^{-j} K(n^j x, n^j x, ..., n^j x) = \theta \sum_{j=0}^{\nu-1} n^{-(j+1)} K(n^j x, n^j x, ..., n^j x),$$

and

$$L_n(x) = \lim_{\nu \to \infty} n^{-\nu} f(n^{\nu} x).$$

Analogous stability results we get if we substitute $x_i = \frac{x}{n}$ *for* $i = 1, 2, ..., n$ in (*).

Theorem 1.4 ([4]). *Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that* $f : X \to Y$ *is an approximately additive mapping such that f satisfies inequality*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \leq N(x_1, x_2, ..., x_n) \qquad (**)$$

for all $(x_1, x_2, ..., x_n) \in X^n$ *and* $N : X^n \to R^+ \cup \{0\}$ *is a non-negative real-valued function such that* $N(x, x, \ldots, x)$ *is bounded on the unit ball of X, and*

$$N(tx_1, tx_2, ..., tx_n) \le k(t) N(x_1, x_2, ..., x_n)$$

for all $t \ge 0$ *,where* $k(t) < \infty$ *and*

$$R_n^0 = R_n^0(x) = \sum_{j=0}^{\infty} n^{-j} k(n^j) < \infty \ .$$

If in addition $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in R$ *for each fixed* $x \in X$ *and* $f : X \to Y$ *is bounded on some ball of X, then there exists a unique* $\mathbb{R}$ *- linear mapping* $L_n : X \to Y$ *satisfying*

$$\| f(x) - L_n(x) \| \le MN(x, x, ..., x),$$

for all $x \in X$ *, where*

$$M = \sum_{m=0}^{\infty} n^{-(m+1)} k(n^m).$$

Replacing $x_i = x$ *for* $i = 1, 2, ..., n$ in (**), we obtain the results of this theorem.

Analogous stability results we get if we substitute $x_i = \dfrac{x}{n}$ *for* $i = 1, 2, ..., n$ in (**).

In 2007, S. Xiang, M. J. Rassias and we [15] investigated the Aleksandrov and triangle perimeter isometry Ulam stability problem.

In this paper we study *the tetrahedron edge perimeter isometry Ulam stability problem on bounded domains.*

2. Tetrahedron Perimeter Isometry Stability

Let X and Y be real Banach spaces. A mapping $I : X \to Y$ introduced by John Michael Rassias, is called a tetrahedron edge perimeter isometry if I satisfies the tetrahedron edge perimeter identity

$$\begin{aligned} &\|I(x) - I(y)\| + \|I(y) - I(z)\| + \|I(z) - I(x)\| + \|I(x)\| + \|I(y)\| + \|I(z)\| \\ &= \|x - y\| + \|y - z\| + \|z - x\| + \|x\| + \|y\| + \|z\| \end{aligned} \tag{*}$$

for all $x, y, z \in X$.

In this section, we establish isometry stability results pertinent to the famous Ulam stability problem and the tetrahedron edge perimeter mapping $T : X \to Y$,

$$T_i\left(x, y, z\right) = T\left(x, y, z\right) = \|x - y\| + \|y - z\| + \|z - x\| + \|x\| + \|y\| + \|z\|$$

with respect to a tetrahedron $ABCD$ of vertices $A(0), B(x), C(y), D(z)$, and the corresponding mapping

$$T_f\left(x, y, z\right) = T\left(f\left(x\right), f\left(y\right), f\left(z\right)\right) = \left\| f\left(x\right) - f\left(y\right)\right\| + \left\| f\left(y\right) - f\left(z\right)\right\| + \left\| f\left(z\right) - f\left(x\right)\right\| + \left\| f\left(x\right)\right\| + \left\| f\left(y\right)\right\| + \left\| f\left(z\right)\right\|$$

as well as the difference operator D_f such that

$$D_f\left(x, y, z\right) = T_f\left(x, y, z\right) - T_i\left(x, y, z\right) = \left\| f\left(x\right) - f\left(y\right)\right\| + \left\| f\left(y\right) - f\left(z\right)\right\| + \left\| f\left(z\right) - f\left(x\right)\right\| + \left\| f\left(x\right)\right\| + \left\| f\left(y\right)\right\| + \left\| f\left(z\right)\right\| - \left[\|x - y\| + \|y - z\| + \|z - x\| + \|x\| + \|y\| + \|z\|\right]$$

in the *bounded* ball $B = \left\{x \in X : \|x\| \le r\right\}$ $\left(0 < r \le 1\right)$ of a real Hilbert space X associated with an inner product $\langle .\,, .\rangle$, where the norm $\|.\|$ is given by the formula $\|x\|^2 = \langle x\,, x\rangle$.

Theorem 2.1. *If a mapping* $f : X \to Y$ *satisfies the following tetrahedron edge perimeter inequality*

$$\left|D_f\left(x, y, z\right)\right| \le \vartheta\left[\|x - y\|^p + \|y - z\|^p\right], \tag{2.1}$$

for all $x, y, z \in B\left(\subseteq X\right)$ *and some* $\vartheta \ge 0$, *and* $p > 1$, *then there exists a unique linear tetrahedron edge perimeter* $I : X \to Y$, *such that the following inequality*

$$\left\| f\left(x\right) - I\left(x\right)\right\| \le \theta\left[\frac{2^{\frac{p-1}{2}}}{2^{\frac{p-1}{2}} - 1}\right]\|x\|^{\frac{1+p}{2}} \tag{2.2}$$

holds for all $x \in B$, *where*

$$\theta = \sqrt{2}\sqrt{\vartheta}\sqrt{1+r^{p-1}} \quad \left(\leq 2\sqrt{\vartheta} \;\; if \;\; 0 \leq \vartheta \leq 1\right) \tag{2.3}$$

and $I(x) = \lim_{n\to\infty} 2^n f\left(2^{-n}x\right)$ *for all* $x \in B$.

Proof. Let $x, y, z \in B$ .Substituting $x = y = z = 0$ in (2.1), one obtains $f(0) = 0$. Setting $z = y = x$ in (2.1), we get $\|f(x)\| = \|x\|$ for all $x \in B$. Thus replacing $(x, y, z) = \left(x, \frac{x}{2}, 0\right)$ in (2.1) and then employing $\|f(x)\| = \|x\|$ and the standard triangle inequality, we obtain

$$\left|\left\|f(x) - f\left(\frac{1}{2}x\right)\right\| - \frac{1}{2}\|x\|\right| \leq (2\vartheta)2^{-p}\|x\|^p$$

for all $x \in B$, or equivalently inequality

$$\left\|f(x) - f\left(\frac{1}{2}x\right)\right\| \leq \frac{1}{2}\|x\| + (2\vartheta)2^{-p}\|x\|^p, \tag{2.4}$$

for all $x \in B$. Thus

$$\begin{aligned}\left(\frac{1}{2}\|x\| + 2^{-p}(2\vartheta)\|x\|^p\right)^2 &\geq \left\|f(x) - f\left(\frac{1}{2}x\right)\right\|^2 = \|f(x)\|^2 + \left\|f\left(\frac{1}{2}x\right)\right\|^2 \\ &-2\left\langle f(x), f\left(\frac{1}{2}x\right)\right\rangle\end{aligned}, \tag{2.5}$$

for all $x \in B$. Employing $\|f(x)\| = \|x\|$ and (2.5) and the quadratic identity

$$\begin{aligned}\frac{1}{2}\left\|f(x) - 2f\left(\frac{1}{2}x\right)\right\|^2 &= \frac{1}{2}\|f(x)\|^2 + 2\left\|f\left(\frac{1}{2}x\right)\right\|^2 - 2\left\langle f(x), f\left(\frac{1}{2}x\right)\right\rangle \\ &= -\frac{1}{2}\|f(x)\|^2 + \left\|f\left(\frac{1}{2}x\right)\right\|^2 + \left\|f(x) - f\left(\frac{1}{2}x\right)\right\|^2,\end{aligned} \tag{2.6}$$

for all $x \in B$, we get the inequality

$$\frac{1}{2}\left\| f\left(x\right)-2f\left(\frac{1}{2}x\right)\right\|^2 = \left(-\frac{1}{2}+\frac{1}{4}\right)\|x\|^2 + \left\| f\left(x\right)- f\left(\frac{1}{2}x\right)\right\|^2$$

$$\leq -\frac{1}{4}\|x\|^2 + \left(\frac{1}{2}\|x\| + 2^{-p}\left(2\vartheta\right)\|x\|^p\right)^2 = 2^{-p}\left(2\vartheta\right)\|x\|^{p+1} + 2^{-2p}\left(2\vartheta\right)^2\|x\|^{2p},$$

or

$$\left\| f\left(x\right)-2f\left(\frac{1}{2}x\right)\right\|^2 \leq 2^{1-p}\left(2\vartheta\right)\|x\|^{p+1} + 2^{1-2p}\left(2\vartheta\right)^2\|x\|^{2p}$$

$$\left\| f\left(x\right)-2f\left(\frac{1}{2}x\right)\right\|^2 \leq 2\vartheta\left(1+\vartheta r^{p-1}\right)\|x\|^{1+p},$$

or *the fundamental inequality*

$$\left\| f\left(x\right)-2f\left(\frac{1}{2}x\right)\right\| \leq \sqrt{2}\sqrt{\vartheta}\sqrt{1+\vartheta r^{p-1}}\,\|x\|^{\frac{1+p}{2}} = \theta\|x\|^{\frac{1+p}{2}}, \tag{2.7}$$

for all $x\in B$, where $\theta=\sqrt{2}\sqrt{\vartheta}\sqrt{1+r^{p-1}}$ $\left(\leq 2\sqrt{\vartheta}\ \ if\ \ 0\leq\vartheta\leq 1\right)$, because

$$2^{1-2p}\|x\|^{2p} = \frac{1}{2}2^{2(1-p)}\|x\|^{(1+p)+(p-1)} = \frac{1}{2}\left(\frac{1}{4}\right)^{p-1}\|x\|^{p-1}\|x\|^{1+p}$$
$$< \frac{1}{2}(1)^{p-1}r^{p-1}\|x\|^{1+p} = \frac{1}{2}r^{p-1}\|x\|^{1+p}.$$

Therefore, by (or without) induction on n, we obtain *the general inequality*

$$\left\| f\left(x\right)-2^n f\left(2^{-n}x\right)\right\| \leq \theta\left[\sum_{j=0}^{n-1} 2^{j\left(\frac{1-p}{2}\right)}\right]\|x\|^{\frac{1+p}{2}} = \theta\left[\frac{1-2^{n\left(\frac{1-p}{2}\right)}}{1-2^{\frac{1-p}{2}}}\right]\|x\|^{\frac{1+p}{2}}, \tag{2.8}$$

for all $x\in B$, on every natural number n, and some $\vartheta\geq 0$, and $p>1$.

From (2.8), it is clear that the sequence $\left\{I_n\left(x\right)\right\}$, with $I_n\left(x\right)=2^n f\left(2^{-n}x\right)$, is a Cauchy sequence, because X is a complete space and $p>1$. Therefore the limit

$$I\left(x\right)=\lim_{n\to\infty}I_n\left(x\right)=\lim_{n\to\infty}2^n f\left(2^{-n}x\right),$$

exists and satisfies (*) for all $x\in B$, yielding the existence of a tetrahedron edge perimeter isometry $I:X\to X$.

The proof for the linearity and uniqueness of the mapping $I:X\to X$ follows standard techniques ([1]-[6]).

Corollary 2.2. *If a mapping $f : X \to Y$ satisfies the following tetrahedron edge perimeter inequality*

$$\left|D_f\left(x,y,z\right)\right| \le \left(2\vartheta\right)\left[\left\|x-y\right\|^p \left\|y-z\right\|^q\right], \tag{2.9}$$

for all $x,y,z \in B\left(\subseteq X\right)$ and some $\vartheta \ge 0$, and $\rho = p+q > 1$, then there exists a unique linear tetrahedron edge perimeter $I : X \to Y$, such that the following inequality

$$\left\|f\left(x\right)-I\left(x\right)\right\| \le \theta\left[\frac{2^{\frac{\rho-1}{2}}}{2^{\frac{\rho-1}{2}}-1}\right]\left\|x\right\|^{\frac{1+\rho}{2}}, \tag{2.10}$$

holds for all $x \in B$, where $\theta = \sqrt{2}\sqrt{\vartheta}\sqrt{1+r^{\rho-1}}$ $\left(\le 2\sqrt{\vartheta} \text{ if } 0 \le \vartheta \le 1\right)$, and $I\left(x\right) = \lim_{n\to\infty} 2^n f\left(2^{-n}x\right)$ for all $x \in B$.

Proof. Let $x,y,z \in B$ .Substituting $x = y = z = 0$ in (2.9), one obtains $f\left(0\right) = 0$. Setting $z = y = x$ in (2.9), we get $\left\|f\left(x\right)\right\| = \left\|x\right\|$ for all $x \in B$. Thus replacing $\left(x,y,z\right) = \left(x,\frac{x}{2},0\right)$ in (2.1) and then employing $\left\|f\left(x\right)\right\| = \left\|x\right\|$ and the standard triangle inequality, we obtain

$$\left|\left\|f\left(x\right)-f\left(\frac{1}{2}x\right)\right\| - \frac{1}{2}\left\|x\right\|\right| \le \left(2\vartheta\right)2^{-\rho}\left\|x\right\|^\rho$$

for all $x \in B$, or equivalently inequality

$$\left\|f\left(x\right)-f\left(\frac{1}{2}x\right)\right\| \le \frac{1}{2}\left\|x\right\| + \left(2\vartheta\right)2^{-\rho}\left\|x\right\|^\rho$$

for all $x \in B$.

The rest of the proof is omitted as analogous to that of the above Theorem 2.1.

Corollary 2.3. *If a mapping $f : X \to Y$ satisfies the following tetrahedron edge perimeter inequality*

$$\left|D_f\left(x,y,z\right)\right| \le \left(\frac{2}{3}\vartheta\right)\left[\left\|x-y\right\|^{\frac{p}{2}}\left\|y-z\right\|^{\frac{p}{2}} + \left(\left\|x-y\right\|^p + \left\|y-z\right\|^p\right)\right], \tag{2.11}$$

for all $x, y, z \in B\left(\subseteq X\right)$ *and some* $\vartheta \geq 0$, *and* $p > 1$, *then there exists a unique linear tetrahedron edge perimeter* $I : X \to Y$ *, such that the following inequality*

$$\left\| f\left(x\right) - I\left(x\right)\right\| \leq \theta \left[\frac{2^{\frac{p-1}{2}}}{2^{\frac{p-1}{2}} - 1}\right] \left\|x\right\|^{\frac{1+p}{2}}, \tag{2.12}$$

holds for all $x \in B$*,where* $\theta = \sqrt{2}\sqrt{\vartheta}\sqrt{1+r^{p-1}}$ $\left(\leq 2\sqrt{\vartheta}\ \textit{if}\ \ 0 \leq \vartheta \leq 1\right)$, *and*

$$I\left(x\right) = \lim_{n\to\infty} 2^n f\left(2^{-n}x\right)$$

for all $x \in B$.

Proof. Let $x, y, z \in B$ .Substituting $x = y = z = 0$ in (2.11), one obtains $f\left(0\right) = 0$. Setting $z = y = x$ in (2.11), we get $\left\|f\left(x\right)\right\| = \left\|x\right\|$ for all $x \in B$. Replacing $\left(x, y, z\right) = \left(x, \frac{x}{2}, 0\right)$ in (2.11) and then employing $\left\|f\left(x\right)\right\| = \left\|x\right\|$ and the standard triangle inequality, we obtain

$$\left|\left\| f\left(x\right) - f\left(\frac{1}{2}x\right)\right\| - \frac{1}{2}\left\|x\right\|\right| \leq \left(2\vartheta\right)2^{-p}\left\|x\right\|^p$$

for all $x \in B$, or equivalently inequality

$$\left\| f\left(x\right) - f\left(\frac{1}{2}x\right)\right\| \leq \frac{1}{2}\left\|x\right\| + \left(2\vartheta\right)2^{-p}\left\|x\right\|^p$$

for all $x \in B$.

The rest of the proof is omitted as analogous to that of the above Theorem 2.1.

Note 2.4. *The "product-sum" of powers of norms*

$$\left\|x-y\right\|^{\frac{p}{2}}\left\|y-z\right\|^{\frac{p}{2}} + \left(\left\|x-y\right\|^p + \left\|y-z\right\|^p\right)$$

in (2.11) was introduced by J. M. Rassias and several specialists have already employed it.

Open Ulam Isometry Stability Problem 2.5.

To investigate the tetrahedron edge perimeter isometry stability <u>*on unbounded domains.*</u>

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Chapter 12

HYERS–ULAM STABILITY OF CAUCHY TYPE ADDITIVE FUNCTIONAL EQUATIONS

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Abstract

In 1940 (and 1964) S. M. Ulam proposed the well-known Ulam stability problem. In 1941 D. H. Hyers solved the Hyers–Ulam problem for linear mappings. In this paper we introduce a Cauchy type additive functional equation and investigate the Hyers-Ulam stability of this equation.

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Key words: Hyers–Ulam stability, Cauchy type additive functional equation.

1. Introduction

In 1940 (and 1964) Stanislaw M. Ulam (5) proposed the following stability problem, well-known as *Ulam stability problem*:

"When is true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In particular he stated the stability question:

"Let G_1 be a group and G_2 a metric group with the metric $\rho(.,.)$. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941 D. H. Hyers (1) solved this problem for linear mappings as follows:

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Theorem 1.1. *If a mapping $f : R \to R$ satisfies the approximately additive inequality*

$$\big|f(x+y) - f(x) - f(y)\big| \leq \delta, \tag{1.1}$$

for some fixed $\delta > 0$ and all $x, y \in R$, where R is the set of real numbers, then there exists a unique additive mapping $A : R \to R$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), \tag{1.2}$$

and inequality

$$|f(x) - A(x)| \leq \delta \tag{1.3}$$

for some fixed $\delta > 0$ and all $x \in R$. If, moreover, $f(tx)$ is continuous in t for each fixed $x \in R$, then $A(tx) = tA(x)$ for all $t, x \in R$.

$A : R \to R$ is a unique linear additive mapping satisfying equation

$$A(x+y) = A(x) + A(y). \tag{1.4}$$

In this paper we introduce a Cauchy type additive functional equation and investigate the Hyers–Ulam stability of this equation.

2. Cauchy Type Additive Functional Equations

Definition 2.1. A mapping $f : R \to R$ is called approximately Cauchy type additive, if the approximately Cauchy additive functional inequality

$$\big|f(x+y) + f(x-y) + f(y-x) - f(x) - f(y)\big| \leq \varepsilon \tag{2.5}$$

holds for every $x, y \in R$ with $\varepsilon \geq 0$.

Theorem 2.2. *Assume that $f : R \to R$ is an approximately Cauchy type additive mapping satisfying (2.5). Define, $f_n(x) = 2^{-n} f(2^n x)$.*

Then, there exists a unique Cauchy type additive mapping $A : R \to R$ such that

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{2.6}$$

for all $x \in R$ and $n \in N = \{1, 2, \ldots\}$, which is the set of natural numbers and

$$|f(x) - A(x)| \leq 3\varepsilon \tag{2.7}$$

for some fixed $\varepsilon > 0$ and all $x \in R$. If, moreover, $f(tx)$ is continuous in t for each fixed $x \in R$, then $A(tx) = tA(x)$ for all $t, x \in R$.

$A : R \to R$ is a unique linear Cauchy type additive mapping satisfying equation

$$A(x+y) + A(x-y) + A(y-x) = A(x) + A(y). \tag{2.8}$$

Proof. Step 1. By substituting $x = y = 0$ and $x = y$ in (2.5), respectively, we can observe that

$$|f(0)| \leq \varepsilon \tag{2.9}$$

and

$$\left|f(x) - 2^{-1}f(2x)\right| \leq \frac{3}{2}\varepsilon. \tag{2.10}$$

Hence, for $n \in N - \{0\}$

$$\begin{aligned}\left|f(x) - 2^{-n}f(2^n x)\right| &\leq \left|f(x) - 2^{-1}f(2x)\right| + \left|2^{-1}f(2x) - 2^{-2}f(2^2 x)\right| + \cdots \\ &\quad + \left|2^{-(n-1)}f(2^{n-1}x) - 2^{-n}f(2^n x)\right| \\ &\leq \frac{3}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\varepsilon \\ &= 3(1 - 2^{-n})\varepsilon. \end{aligned} \tag{2.11}$$

Step 2. Following, we need to show that if there is a sequence $\{f_n\} : f_n(x) = 2^{-n}f(2^n x)$, then $\{f_n\}$ converges.

For every $m > n > 0$, we can obtain

$$\begin{aligned}\left|f_m(x) - f_n(x)\right| &= \left|2^{-m}f(2^m x) - 2^{-n}f(2^n x)\right| \\ &= 2^{-n}\left|2^{-(m-n)}f(2^m x) - f(2^n x)\right| \\ &\leq 2^{-n}3(1 - 2^{-(m-n)})\varepsilon \\ &< \frac{3\varepsilon}{2^n} \to 0,\end{aligned}$$

for $n \to \infty$. Since R is *complete* we can conclude that $\{f_n\}$ is convergent. Thus, there is a well-defined $A : R \to R$ such that $A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$.

Step 3. Observe that

$$|f(x) - f_n(x)| = \left|f(x) - 2^{-n}f(2^n x)\right| \leq 3(1 - 2^{-n})\varepsilon,$$

from which by letting $n \to \infty$ we obtain

$$|f(x) - A(x)| \leq 3\varepsilon.$$

Step 4. By letting $x \to 2^n x$ and $y \to 2^n y$, from (2.5), we have:

$$\left|f(2^n(x+y)) + f(2^n(x-y)) + f(2^n(y-x)) - f(2^n x) - f(2^n y)\right| \leq \varepsilon.$$

Next, by multiplying with 2^{-n} and by letting $n \to \infty$, we can conclude that truly exists an $A : R \to R$ such that: $A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$ satisfies the *Cauchy-type additivity property*

$$A(x+y) + A(x-y) + A(y-x) = A(x) + A(y). \tag{2.12}$$

Step 5. We need to prove that A is *unique.*

Observe, from (2.12), that

$$A(0) = 0 \quad \text{and} \quad A(2x) = 2A(x).$$

Therefore, by *induction hypothesis* we can show that

$$A(2^n x) = 2A(2^{n-1}x) = 2^n A(x)$$

or equivalently

$$A(x) = 2^{-n} A(2^n x).$$

Assume, now, the existence of $A' : R \to R$, such that $A'(x) = 2^{-n}A'(2^n x)$. With the aid of the triangular inequality,

$$\begin{aligned}|A(x) - A'(x)| &\leq \left|2^{-n}A(2^n x) - 2^{-n}f(2^n x)\right| + \left|2^{-n}f(2^n x) - 2^{-n}A'(2^n x)\right|\\ &\leq 2^{-n}3\varepsilon + 2^{-n}3\varepsilon\\ &\to 0,\end{aligned}$$

as $n \to \infty$. Thus, the *uniqueness* of A is proved and the stability of *Cauchy-type additive mapping* $A : R \to R$ is established.

Step 6. To complete the proof of Theorem 2.2, we only need to examine whether $A : R \to R$ is a *linear Cauchy-type mapping*. To be more precise, we need to show that:

(1) $A(x + y) + A(x - y) + A(y - x) = A(x) + A(y)$, and

(2) $A(rx) = rA(x), \quad \forall r \in R.$

Recall that we have shown already that **(1)** holds.

Therefore, we only need to show that **(2)** is valid $\forall r \in R$.

For that we will study four cases.

Case 1: Let $r = k \in N = \{0, 1, 2, \ldots\}$.

For $k = 0$, from **(2)**, we have $A(0) = 0$. This is verified if we substitute $x = y = 0$ in (2.12).

Assume, that $A\big((k-1)x\big) = (k-1)A(x)$ is true $\forall k$.

Then, we need to prove that $A(kx) = kA(x)$.

Note that for $x = x$, and $y = 0$ from (2.12), we can easily obtain $A(-x) = (-1)A(x)$.

Let $x = x$ and $y = (k-1)x$ in (2.12). Then,

$$A(kx) + A(-(k-2)x) + A((k-2)x) = A(x) + A((k-1)x),$$

or

$$A(kx) = kA(x), \ \ \forall\, k \in N = \{0, 1, 2, \ldots\}.$$

Case 2: Let $r = k \in Z$.

We only need to observe that A is *odd.* Since, we have already proved that **(2)** is valid $\forall k \in N = \{0, 1, 2, ...\}$ we can then conclude that

$$A(kx) = kA(x), \ \ \forall\, k \in Z.$$

Case 3: Let $r = \frac{k}{l} \in Q$, for $k \in Z$, $l \in Z - \{0\}$.

Then, $A(x) = A\big(l\frac{1}{l}\,x\big) = lA\big(\frac{1}{l}\,x\big)$, for $l \in Z - \{0\}$. Hence, $A\big(\frac{1}{l}\,x\big) = \frac{1}{l}\,A(x)$.

Besides, for $k \in Z$, $A\big(\frac{k}{l}x\big) = A\big(k\frac{1}{l}\,x\big) = kA\big(\frac{1}{l}\,x\big)$, from *Case 2.*

Thus, $A\big(\frac{k}{l}\,x\big) = \frac{k}{l}\,A(x)$, or $A(rx) = rA(x)$ for $r \in Q$.

Case 4: Let $r \in R$, where $r = q_n$: rational numbers.

Since R is a *complete* space, every sequence $\{q_n\}$ converges in R, i.e. $\lim_{n\to\infty} q_n = q \in R$.

Recall that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ and $f(tx)$ is continuous in t for each fixed x in R. Therefore, $A(tx)$ is continuous in t for each fixed x in R. Besides,

$$\lim_{n\to\infty} A(q_n x) = A\big(\lim_{n\to\infty} q_n x\big) = A(qx) \tag{2.13}$$

and

$$\lim_{n\to\infty} A(q_n x) = \lim_{n\to\infty} q_n A(x) = qA(x). \tag{2.14}$$

From (2.13) and (2.14) *Case 4.* is now proved, which completes *Step 6.* and thus the proof of our Theorem 2.2 .

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Chapter 13

SOLUTION AND ULAM STABILITY OF A MIXED TYPE CUBIC AND ADDITIVE FUNCTIONAL EQUATION

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Abstract

In this paper, the authors investigate the general solution and Ulam stability of mixed type cubic and additive functional equation of the form

$$\begin{aligned}&3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)\\&\quad+4\left[f(x)+f(y)+f(z)\right]=4\left[f(x+y)+f(x+z)+f(y+z)\right]\end{aligned} \tag{*}$$

introduced by the first author. We also investigate the first author's stability of the equation (*) controlled by a mixed type product – sum of powers of norms .

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1. Introduction

In 1940, S. M. Ulam [55] raised the following question concerning the stability of group homomorphisms: "Under what conditions does there is an additive mapping near an approximately additive mapping between a group and a metric group ? "

In 1941, D. H. Hyers [15] answered the stability problem of Ulam under the assumption that the groups are Banach spaces. In 1950, Aoki [3] generalized the Hyers theorem for additive mappings. In 1978, Th. M. Rassias [47] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become bounded. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors [5], [8]-[10], [12]-[14], [59], [30], [44], [46], [49], [54].

In 1982, J. M. Rassias [39] gave a further generalization of the result of D. H. Hyers and proved theorems using weaker conditions controlled by a product of different powers of norms. Very recently, J. M. Rassias introduced the mixed type product sum of powers of norms [51]. The investigation of stability of functional equations involving a mixed type product - sum of powers of norms is known as Ulam-Gavruta-Rassias stability.

Theorem 1.1 ([39]). Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\| f(x+y) - f(x) - f(y)\| \le \epsilon \|x\|^p \|y\|^p$$

for all $x, y \in E$, *where* ϵ *and* p *are constants with* $\epsilon > 0$ *and* $0 \le p < \frac{1}{2}$. *Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f\left(2^n x\right)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\| f(x) - L(x)\| \le \frac{\epsilon}{2-2^{2p}} \|x\|^{2p}$$

for all $x \in E$.

The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability by M. A. Sibaha et al., [53], Nakmahachalasint [32,33], Ravi and Arunkumar [50], Ravi and Senthil Kumar [52]. Besides, J. M. Rassias [42] also introduced and investigated the Euler-Lagrange type quadratic mappings, called Euler-Lagrange-Rassias quadratic mappings.

The functional equation

$$f\left(x+y\right) + f\left(x-y\right) = 2f\left(x\right) + 2f\left(y\right) \tag{1.1}$$

is said to be quadratic functional equation because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.1). Quadratic functional equations were used to characterize inner product spaces [1], [2], [19]. A square norm on an inner product space satisfies the important parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

It is well known that a function f is a solution of (1.1) if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x,x)$ for all x [1, 26]. The bi-additive function B is given by

$$B(x,y) = \frac{1}{4}\left[f(x+y) + f(x-y)\right]. \tag{1.2}$$

Functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

is called *cubic functional equation*, because the cubic function $f(x) = cx^3$ is a solution of the equation (1.3). The general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3) was discussed by K. W. Jun and H. M. Kim [20]. They proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

The general solution and the generalized Hyers-Ulam-Rassias stability of the cubic functional equation

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x) \tag{1.4}$$

on abelain groups was investigated by K. H. Park and Y. S. Jung [36].

K. W. Jun and H. M. Kim [22] introduced the following generalized quadratic and additive type functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j) \tag{1.5}$$

in the class of function between real vector spaces. For $n = 3$, Pl. Kannappan proved that a function f satisfies the functional equation (1.5) if and only if there exists a symmetric bi-additive function A and additive function B such that $f(x)=B(x,x)+A(x)$ for all x [26]. The Hyers-Ulam stability for the equation $n=3$ was proved by S. M. Jung [24]. The Hyers-Ulam-Rassias stability for the equation $n=4$ was also investigated by I. S. Chang et al [7].

The general solution and the generalized Hyers-Ulam stability for the ***quadratic and additive type functional equation***

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y) \tag{1.6}$$

for any positive integer a with $a \neq -1,0,1$ was discussed by K. W. Jun and H. M. Kim [21].

The general solution and the generalized Hyers-Ulam stability for mixed type of ***cubic and quadratic functional equation*** of the form

$$6f(x+y)-6f(x-y)+4f(3y)=3f(x+y)-3f(x-2y)+9f(2x) \tag{1.7}$$

was investigated by I. S. Chang and Y. S. Jung [6].

In [34], W. G. Park and J. H. Bae considered the following ***quartic functional equation***

$$6f(x+y)-6f(x-y)+4f(3y)=3f(x+y)-3f(x-2y)+9f(2x) \tag{1.8}$$

and proved that a function f between real vector spaces X and Y is a solution of (1.8) if and only if there exists a unique symmetric multi–additive function $Q: X\times X\times X\times X \to Y$ such that $f(x)=Q(x,x,x,x)$ for all $x\in X$.

Stability of a ***functional equation deriving from cubic and quartic functions*** of the form

$$\begin{aligned}4\left[f(3x+y)+f(3x-y)\right]&=-12\left[f(x+y)+f(x-y)\right]\\&+12\left[f(2x+y)+f(2x-y)\right]-8f(y)-192f(x)+f(2y)+30f(2x)\end{aligned} \tag{1.9}$$

and its general solution and the generalized Hyers-Ulam-Rassias stability was discussed by M. Eshaghi Gordji et al., [11].

In this paper, we discuss the general solution and Ulam stability of mixed type cubic and additive functional equation and investigate the J. M. Rassias stability of Mixed Type product – sum of powers of norms for the equation

$$\begin{aligned}&3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)\\&+4\left[f(x)+f(y)+f(z)\right]=4\left[f(x+y)+f(x+z)+f(y+z)\right].\end{aligned} \tag{1.10}$$

It is easy to see that the function $f(x) = ax^3 + bx$ is a solution of the functional equation (1.10) where a, b are real constants.

2. General Solution of the Functional Equation (1.10)

Through out this section, we assume that A and B are two real vector spaces.

Lemma 2.1. *Let* $f : A \to B$ *be a function satisfying the functional equation* (1.10) *then* f *is an odd function.*

Proof. Letting (x, y, z) be $(0,0,0)$ in (1.10), we get

$$f(0) = 0. \tag{2.1}$$

Replacing (x, y, z) by $(x,0,0)$ in (1.10), we obtain

$$f(-x) = -f(x) \tag{2.2}$$

for all $x \in A$. Hence f is an odd function.

Lemma 2.2. *Let* $f : A \to B$ *be a function satisfying the functional equation* (1.10) *then* f *is a cubic function.*

Proof. Replacing (x, y, z) by $(x-y, y-z, z-x)$ and using (2.1), (2.2) in (1.10), we obtain

$$\begin{aligned} f\left(2(x-y)\right) + f\left(2(y-z)\right) + f\left(2(z-x)\right) \\ = 8\left[f(x-y) + f(y-z) + f(z-x)\right] \end{aligned} \tag{2.3}$$

for all $x, y, z \in A$. Setting $(x-y, y-z, z-x)$ by (u, v, w) in (2.3), we arrive

$$f(2u) + f(2v) + f(2w) = 8\left[f(u) + f(v) + f(w)\right] \tag{2.4}$$

for all $u, v, w \in A$. Replacing (u, v, w) by (u, u, u) in (2.4), we obtain

$$f(2u) = 8f(u) \tag{2.5}$$

for all $u \in A$. Again replacing (x, y, z) by (x, x, x) in (1.10), we get

$$f(3x) = 4f(2x) - 5f(x) \tag{2.6}$$

for all $x \in A$. Now with the help of (2.5), we arrive $f(3x) = 27f(x)$ for all $x \in A$. In general for any positive integer n, we obtain $f(nx) = n^3 f(x)$ for all $x \in A$. Therefore f is a cubic function.

Lemma 2.3. *Let* $f : A \to B$ *be a function satisfying the functional equation* (1.10) *then* f *is an additive function.*

Proof. Replacing z by $x+y$ in (1.10), we get

$$3f(2(x+y)) + f(2y) + f(2x) + 4f(x) + 4f(y) = 4f(2x+y) + 4f(x+2y) \tag{2.7}$$

for all $x, y \in A$. Again replacing z by $-x-y$ in (1.10), we obtain

$$f(2x) + f(2y) + 8f(x+y) = 8f(x) + 8f(y) + f(2(x+y)) \tag{2.8}$$

for all $x, y \in A$. Multiplying (2.8) by 3 and using (2.7), we get

$$f(2x) + f(2y) + 6f(x+y) = 5f(x) + 5f(y) + f(2x+y) + f(x+2y) \tag{2.9}$$

for all $x, y \in A$. Subtracting (2.9) from (2.8), we have

$$f(2x+y) + f(x+2y) = 3f(x) + 3f(y) + f(2(x+y)) - 2f(x+y) \tag{2.10}$$

for all $x, y \in A$. Substituting y by $-y$ in (2.10), we arrive

$$f(2x-y) + f(x-2y) = 3f(x) - 3f(y) + f(2(x-y)) - 2f(x-y) \tag{2.11}$$

for all $x, y \in A$. Adding (2.10) and (2.11), we get

$$\begin{aligned} &f(2x+y) + f(x+2y) + f(2x-y) + f(x-2y) \\ &\qquad = 6f(x) + f(2(x+y)) - 2f(x+y) + f(2(x-y)) - 2f(x-y) \end{aligned} \tag{2.12}$$

for all $x, y \in A$. Setting $(x+y, x-y)$ by (u, v) in (2.12), we arrive

$$\begin{aligned} f(u+x)+f(u+y)+f(x+v)+f(x-v) \\ =6\left(\frac{u+v}{2}\right)+f(2u)+f(2v)-2f(u)-2f(v) \end{aligned} \tag{2.13}$$

for all $x,y,u,v \in A$. Replacing (x,y,u,v) by (z,z,z,z) in (2.13), we obtain

$$f(2z)=2f(z) \tag{2.14}$$

for all $z \in A$. Again replacing (x,y,z) by (x,x,x) in (1.10), we get

$$f(3x)=4f(2x)-5f(x) \tag{2.15}$$

for all $x \in A$. With the help of (2.14), we arrive $f(3x)=3f(x)$ for all $x \in A$. In general for any positive integer n, we obtain $f(nx)=n\,f(x)$ for all $x \in A$. Therefore f is an additive function.

Theorem 2.4. *A mapping $f: A \to B$ is a function satisfying the functional equation* (1.10) *for all $x,y,z \in A$, if and only if there exists two mappings $T: A\times A\times A \to B$ and $R: A\times A\times A \to B$ such that $f(x)=T(x,x,x)+R(x)$ for all $x \in A$, where T is symmetric for each fixed one variable and is additive for fixed two variables and R is additive.*

Proof. Replacing z by $-z$ in (1.10), we get

$$\begin{aligned} 3f(x+y-z)+f(-x+y-z)+f(x-y-z)+f(x+y+z) \\ +4\left[f(x)+f(y)-f(z)\right]=4\left[f(x+y)+f(x-z)+f(y-z)\right] \end{aligned} \tag{2.16}$$

for all $x,y,z \in A$. Adding (1.10) and (2.16) and dividing by 4, we arrive

$$\begin{aligned} f(x+y+z)+f(x+y-z)+2f(x)+2f(y) \\ =2f(x+y)+f(x+z)+f(y+z)+f(x-z)+f(y-z) \end{aligned} \tag{2.17}$$

for all $x,y,z \in A$. This equation (2.17) was already dealt by H.M. Kim [28]. By Theorem 2.1 [23], the proof is completed.

3. Generalized Ulam Stability of the Functional Equation (1.10)

For convenience, we define,

$$D f(x,y,z) = 3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)$$
$$+4\left[f(x)+f(y)+f(z)\right]-4\left[f(x+y)+f(x+z)+f(y+z)\right]$$

for all $x, y, z \in A$.

Theorem 3.1. *Let* A *be a real vector space and* B *be a Banach space. Let* $\alpha : A\times A\times A \to [0,\infty)$ *be a function such that*

$$\sum_{i=0}^{\infty} \frac{\alpha\left(3^i x, 3^i y, 3^i z\right)}{27^i} \text{ converges and } \lim_{n\to\infty} \frac{\alpha\left(3^n x, 3^n y, 3^n z\right)}{27^n} = 0 \tag{3.1}$$

for all $x, y, z \in A$. *If* $f_c : A \to B$ *is a cubic function satisfying*

$$\left\| D f_c(x,y,z)\right\| \le \alpha(x,y,z) \tag{3.2}$$

for all $x, y, z \in A$, *then there exists a unique cubic function* $T : A \to B$ *satisfying* (1.10) *and*

$$\left\| f_c(x) - T(x)\right\| \le \frac{1}{81}\sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{27^k} \tag{3.3}$$

for all $x \in A$. *Function* $T(x)$ *is defined by*

$$T(x) = \lim_{n\to\infty} \frac{f_c\left(3^n x\right)}{27^n} \tag{3.4}$$

for all $x \in A$.

Proof. Replacing (x, y, z) by (x, x, x) in (3.2) and using (2.5), we get

$$\left\| \frac{f_c(3x)}{27} - f_c(x)\right\| \le \frac{1}{81}\alpha(x,x,x) \tag{3.5}$$

for all $x \in A$. Replacing x by $3x$ and divided by 27 in (3.5) and adding the resultant inequality with (3.5), we obtain

$$\left\| \frac{f_c\left(3^2 x\right)}{27^2} - f_c(x) \right\| \leq \frac{1}{81}\left[\alpha(x,x,x) + \frac{\alpha(3x,3x,3x)}{27} \right] \tag{3.6}$$

for all $x \in A$. In general for any positive integer n, we have

$$\begin{aligned} \left\| \frac{f_c\left(3^n x\right)}{27^n} - f_c(x) \right\| &\leq \frac{1}{81}\sum_{k=0}^{n-1} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{27^k} \\ &\leq \frac{1}{81}\sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{27^k} \end{aligned} \tag{3.7}$$

for all $x \in A$. We have to show that sequence $\left\{ \frac{f_c\left(3^n x\right)}{27^n} \right\}$ converges for all $x \in A$.

Replacing x by $3^m x$ and dividing by 27^m in (3.7) for any $n, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f_c\left(3^n 3^m x\right)}{27^m 27^n} - \frac{f_c\left(3^m x\right)}{27^m} \right\| &\leq \frac{1}{27^m} \left\| \frac{f_c\left(3^n 3^m x\right)}{27^n} - f_c\left(3^m x\right) \right\| \\ &\leq \frac{1}{27^m}\frac{1}{81}\sum_{k=0}^{n-1} \frac{\alpha\left(3^{k+m} x, 3^{k+m} x, 3^{k+m} x\right)}{27^k} \\ &\leq \frac{1}{81}\sum_{k=0}^{\infty} \frac{\alpha\left(3^{k+m} x, 3^{k+m} x, 3^{k+m} x\right)}{27^{k+m}} \end{aligned} \tag{3.8}$$

for all $x \in A$. By condition (3.1) the right hand side of (3.8) converges to 0 as $n \to \infty$. Thus the sequence $\left\{ \frac{f_c\left(3^n x\right)}{27^n} \right\}$ is a Cauchy sequence. Due to completeness of the Banach space B, there exists a mapping $T : A \to B$ such that

$$T(x) = \lim_{n\to\infty} \frac{f_c\left(3^n x\right)}{27^n}, \quad \text{for all } x \in A.$$

Letting $n \to \infty$ in (3.7), we obtain (3.3). To show that T satisfies (1.10), we are setting (x, y, z) by $\left(3^n x, 3^n y, 3^n z\right)$ in (3.2) and dividing by 27^n, we get

$$\frac{1}{27^n}\Big\| 3f_c\left(3^n(x+y+z)\right)+f_c\left(3^n(-x+y+z)\right)+f_c\left(3^n(x-y+z)\right)+f_c\left(3^n(x+y-z)\right)$$
$$+4\left[f_c\left(3^n x\right)+f_c\left(3^n y\right)+f_c\left(3^n z\right)\right]-4\left[f_c\left(3^n(x+y)\right)+f_c\left(3^n(x+z)\right)+f_c\left(3^n(y+z)\right)\right]\Big\|$$
$$\leq \frac{\alpha\left(3^n x,3^n y,3^n z\right)}{27^n}.$$

for all $x,y,z\in A$. Taking limit $n\to\infty$ and using the definition of $T(x)$ in the above inequality, it becomes

$$3T(x+y+z)+T(-x+y+z)+T(x-y+z)+T(x+y-z)$$
$$+4\left[T(x)+T(y)+T(z)\right]=4\left[T(x+y)+T(x+z)+T(y+z)\right]$$

for all $x,y,z\in A$. Therefore T satisfies (1.10). To prove uniqueness of T, suppose that there exists another cubic mapping $U:A\to B$ satisfying (3.3) and (3.4). Therefore

$$\left\|T(x)-U(x)\right\|\leq\frac{1}{27^n}\left\{\left\|T(x)-f(x)\right\|+\left\|f(x)-U(x)\right\|\right\}$$
$$\leq\frac{1}{27^n}\,\frac{2}{81}\sum_{k=0}^{\infty}\frac{\alpha\left(3^{k+n}x,3^{k+n}x,3^{k+n}x\right)}{27^k}$$
$$\leq\frac{2}{81}\sum_{k=0}^{\infty}\frac{\alpha\left(3^{k+n}x,3^{k+n}x,3^{k+n}x\right)}{27^{k+n}}$$

for all $x\in A$. By condition (3.1), the right hand side goes to 0 as $n\to\infty$ and it follows that $T(x)=U(x)$ for all $x\in A$. Hence T is unique. Hence the proof is complete.

Theorem 3.2. *Let A be a real vector space and B be a Banach space. Let $\alpha:A\times A\times A\to[0,\infty)$ be a function such that*

$$\sum_{i=1}^{\infty}27^i\alpha\left(\frac{x}{3^i},\frac{y}{3^i},\frac{z}{3^i}\right)\text{ converges and }\lim_{n\to\infty}27^n\alpha\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right)=0 \tag{3.9}$$

for all $x,y,z\in A$. If $f_c:A\to B$ is a cubic function satisfying (3.2) *for all $x,y,z\in A$. Then there exists a unique cubic function $T:A\to B$ which satisfies* (1.10) *and*

$$\left\| f_c(x) - T(x) \right\| \le \frac{1}{81} \sum_{k=1}^{\infty} 27^k \alpha\left(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k} \right) \tag{3.10}$$

for all $x \in A$. *The function* $T(x)$ *is defined by*

$$T(x) = \lim_{n\to\infty} 27^n f_c\left(\frac{x}{3^n} \right) \tag{3.11}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (3.5) and proceeding the same way as that of Theorem 3.1 the proof is complete.

Corollary 3.3. *Let* A *be a real normed space and* B *be a Banach space. If* $f_c : A \to B$ *is a cubic function satisfying the functional inequality*

$$\left\| D\, f_c(x, y, z) \right\| \le \varepsilon \left\{ \|x\|^p + \|y\|^p + \|z\|^p \right\} \tag{3.12}$$

with $p < 3$ (*or*) $p > 3$, *for some* $\varepsilon > 0$ *and for all* $x, y, z \in A$. *Then there exists a unique cubic function* $T : A \to B$ *which satisfies* (1.10) and

$$\left\| f_c(x) - T(x) \right\| \le \begin{cases} \dfrac{\varepsilon \|x\|^p}{27 - 3^p} & for \ \ p < 3, \\ \dfrac{\varepsilon \|x\|^p}{3^p - 27} & for \ \ p > 3 \end{cases} \tag{3.13}$$

for all $x \in A$.

Proof. If we choose $\alpha(x, y, z) = \varepsilon \left\{ \|x\|^p + \|y\|^p + \|z\|^p \right\}$ for all $x, y, z \in A$. Then by Theorem 3.1, we arrive

$$\left\| f_c(x) - T(x) \right\| \le \frac{\varepsilon \|x\|^p}{27 - 3^p}, \text{ for all } x \in A \text{ and } p < 3.$$

Using Theorem 3.2, we arrive

$$\left\| f_c(x) - T(x) \right\| \le \frac{\varepsilon \|x\|^p}{3^p - 27}, \text{ for all } x \in A \text{ and } p > 3.$$

Corollary 3.4. *Let* A *be a real normed space and* B *be a Banach space. If a cubic function* $f_c : A \to B$ *satisfies the functional inequality*

$$\left\| D\, f_c\left(x, y, z\right)\right\| \leq \varepsilon \tag{3.14}$$

for some $\varepsilon > 0$ *and for all* $x, y, z \in A$*. Then there exists a unique cubic function* $T : A \to B$ *which satisfies* (1.10) *and*

$$\left\| f_c\left(x\right) - T\left(x\right)\right\| \leq \frac{\varepsilon}{78} \tag{3.15}$$

for all $x \in A$.

Proof. If we choose $\alpha\left(x, y, z\right) = \varepsilon$ for all $x, y, z \in A$. Then by Theorem 3.1 it follows that

$$\left\| f_c\left(x\right) - T\left(x\right)\right\| \leq \frac{\varepsilon}{78}, \text{ for all } x \in A.$$

Corollary 3.5. *If* $f_c : A \to B$ *is a cubic function from a normed vector space* A *into a Banach space* B *satisfies*

$$\left\| D\, f_c\left(x, y, z\right)\right\| \leq \varepsilon \left\|x\right\|^p \left\|y\right\|^p \left\|z\right\|^p \tag{3.16}$$

for all $x, y, z \in A$*, where* ε and p *are constants with* $p < 1$ *(or)* $p > 1$*, then there exists a unique cubic function* $T : A \to B$ *which satisfies* (1.10) and

$$\left\| f_c\left(x\right) - T\left(x\right)\right\| \leq \begin{cases} \dfrac{\varepsilon \left\|x\right\|^{3p}}{81 - 3^{3p+1}} & \text{for } p < 1, \\ \dfrac{\varepsilon \left\|x\right\|^{3p}}{3^{3p+1} - 81} & \text{for } p > 1 \end{cases} \tag{3.17}$$

for all $x \in A$.

Proof. If we choose $\alpha\left(x, y, z\right) = \varepsilon \left\|x\right\|^p \left\|y\right\|^p \left\|z\right\|^p$ for all $x, y, z \in A$. Then by Theorem 3.1, we arrive

$$\left\| f_c\left(x\right) - T\left(x\right)\right\| \leq \frac{\varepsilon \left\|x\right\|^{3p}}{81 - 3^{3p+1}}, \text{ for all } x \in A \text{ and } p < 1.$$

Using Theorem 3.2, we arrive

$$\left\| f_c(x) - T(x) \right\| \le \frac{\varepsilon \|x\|^{3p}}{3^{3p+1} - 81}, \text{ for all } x \in A \text{ and } p > 1.$$

Theorem 3.6. *Let* A *be a real vector space and* B *be a Banach space. Let* $\alpha : A \times A \times A \to [0, \infty)$ *be a function such that*

$$\sum_{i=0}^{\infty} \frac{\alpha\left(3^i x, 3^i y, 3^i z\right)}{3^i} \text{ converges and } \lim_{n\to\infty} \frac{\alpha\left(3^n x, 3^n y, 3^n z\right)}{3^n} = 0 \tag{3.18}$$

for all $x, y, z \in A$. *If* $f_a : A \to B$ *is a additive function satisfying*

$$\left\| D\, f_a(x, y, z) \right\| \le \alpha(x, y, z) \tag{3.19}$$

for all $x, y, z \in A$. *Then there exists a unique additive function* $R : A \to B$ *which satisfies* (1.10) *and*

$$\left\| f_a(x) - R(x) \right\| \le \frac{1}{9} \sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{3^k} \tag{3.20}$$

for all $x \in A$. *The function* $R(x)$ *is defined by*

$$R(x) = \lim_{n\to\infty} \frac{f_a\left(3^n x\right)}{3^n} \tag{3.21}$$

for all $x \in A$.

Proof. Replacing (x, y, z) by (x, x, x) in (3.19) and using (2.14), we get

$$\left\| \frac{f_a(3x)}{3} - f_a(x) \right\| \le \frac{1}{9} \alpha(x, x, x) \tag{3.22}$$

for all $x \in A$. Replacing x by $3x$ and dividing by 3 in (3.22) and adding the resultant inequality with (3.22), we obtain

$$\left\| \frac{f_a\left(3^2 x\right)}{3^2} - f_a(x) \right\| \le \frac{1}{9} \left[\alpha(x, x, x) + \frac{\alpha(3x, 3x, 3x)}{3} \right] \tag{3.23}$$

for all $x \in A$. In general for any positive integer n, we have

$$\left\| \frac{f_a\left(3^n x\right)}{3^n} - f_a\left(x\right) \right\| \leq \frac{1}{9} \sum_{k=0}^{n-1} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{3^k}$$

$$\leq \frac{1}{9} \sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{3^k} \tag{3.24}$$

for all $x \in A$. We have to show that the sequence $\left\{ \frac{f_a\left(3^n x\right)}{3^n} \right\}$ converges for all $x \in A$.

Replacing x by $3^m x$ and divide by 3^m in (3.24) for any $n,\ m > 0$, we obtain

$$\left\| \frac{f_a\left(3^n 3^m x\right)}{3^m 3^n} - \frac{f_a\left(3^m x\right)}{3^m} \right\| \leq \frac{1}{3^m} \left\| \frac{f_a\left(3^n 3^m x\right)}{3^n} - f_a\left(3^m x\right) \right\|$$

$$\leq \frac{1}{3^m} \frac{1}{9} \sum_{k=0}^{n-1} \frac{\alpha\left(3^{k+m} x, 3^{k+m} x, 3^{k+m} x\right)}{3^k}$$

$$\leq \frac{1}{9} \sum_{k=0}^{\infty} \frac{\alpha\left(3^{k+m} x, 3^{k+m} x, 3^{k+m} x\right)}{3^{k+m}} \tag{3.25}$$

for all $x \in A$. By condition (3.18) the right hand side of (3.25) converges to zero as $n \to \infty$. Thus the sequence $\left\{ \frac{f_a\left(3^n x\right)}{3^n} \right\}$ is a Cauchy sequence. Due to completeness of the Banach space B, there exists a mapping $R : A \to B$ such that $R\left(x\right) = \lim_{n\to\infty} \frac{f_a\left(3^n x\right)}{3^n}$, for all $x \in A$.

Letting $n \to \infty$ in (3.24), we establish (3.20). To show that R satisfies (1.10) and is unique, the proof will be similar to that of Theorem 3.1.

Theorem 3.7. *Let A be a real vector space and B be a Banach space. Let $\alpha : A \times A \times A \to [0, \infty)$ be a function such that*

$$\sum_{i=1}^{\infty} 3^i \alpha\left(\frac{x}{3^i}, \frac{y}{3^i}, \frac{z}{3^i}\right) \text{ converges and } \lim_{n\to\infty} 3^n \alpha\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0 \tag{3.26}$$

for all $x, y, z \in A$. *If* $f_a : A \to B$ *is a additive function satisfying* (3.19) *for all* $x, y, z \in A$. *Then there exists a unique additive function* $R : A \to B$ *which satisfies* (1.10) *and*

$$\left\| f_a(x) - R(x) \right\| \leq \frac{1}{9} \sum_{k=1}^{\infty} 3^k \alpha\left(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k}\right) \tag{3.27}$$

for all $x \in A$. *The function* $R(x)$ *is defined by*

$$R(x) = \lim_{n \to \infty} 3^n f_a\left(\frac{x}{3^n}\right) \tag{3.28}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (3.22) and proceeding the same way as that of Theorem 3.6 the proof is complete.

Corollary 3.8. *Let* A *be a real normed space and* B *be a Banach space. If* $f_a : A \to B$ *is a additive function satisfying the functional inequality*

$$\left\| D\, f_a(x, y, z) \right\| \leq \varepsilon \left\{ \|x\|^p + \|y\|^p + \|z\|^p \right\} \tag{3.29}$$

with $p < 1$ *and* $p > 1$, *for some* $\varepsilon > 0$ *and for all* $x, y, z \in A$. *Then there exists a unique additive function* $R : A \to B$ *which satisfies* (1.10) *and*

$$\left\| f_a(x) - R(x) \right\| \leq \begin{cases} \dfrac{\varepsilon \|x\|^p}{3 - 3^p} & \text{for } \; p < 1, \\[2ex] \dfrac{\varepsilon \|x\|^p}{3^p - 3} & \text{for } \; p > 1 \end{cases} \tag{3.30}$$

for all $x \in A$.

Proof. If we choose $\alpha(x, y, z) = \varepsilon \left\{ \|x\|^p + \|y\|^p + \|z\|^p \right\}$ for all $x, y, z \in A$. Then by Theorem 3.6, we arrive

$$\left\| f_a(x) - R(x) \right\| \leq \frac{\varepsilon \|x\|^p}{3 - 3^p}, \text{ for all } x \in A \text{ and } p < 1.$$

Using Theorem 3.7, we arrive

$$\left\| f_a(x) - R(x) \right\| \le \frac{\varepsilon \left\| x \right\|^p}{3^p - 3}, \text{ for all } x \in A \text{ and } p > 1.$$

Corollary 3.9. *Let A be a real normed space and B be a Banach space. If a additive function $f_a : A \to B$ satisfies the functional inequality*

$$\left\| D\, f_a(x, y, z) \right\| \le \varepsilon \tag{3.31}$$

for some $\varepsilon > 0$ and for all $x, y, z \in A$. Then there exists a unique additive function $R : A \to B$ which satisfies (1.10) *and*

$$\left\| f_a(x) - R(x) \right\| \le \frac{\varepsilon}{6} \tag{3.32}$$

for all $x \in A$.

Proof. If we choose $\alpha(x, y, z) = \varepsilon$ for all $x, y, z \in A$. Then by Theorem 3.6 it follows that

$$\left\| f_a(x) - R(x) \right\| \le \frac{\varepsilon}{6}, \text{ for all } x \in A.$$

Corollary 3.10. *If $f_a : A \to B$ is a additive function from a normed vector space A into a Banach space B satisfies*

$$\left\| D\, f_a(x, y, z) \right\| \le \varepsilon \left\| x \right\|^p \left\| y \right\|^p \left\| z \right\|^p \tag{3.33}$$

for all $x, y, z \in A$ where ε and p are constants with $p < \frac{1}{3}$ (or) $p > \frac{1}{3}$, then there exists a unique additive function $R : A \to B$ which satisfies (1.10) *and*

$$\left\| f_a(x) - R(x) \right\| \le \begin{cases} \dfrac{\varepsilon \left\| x \right\|^{3p}}{9 - 3^{3p+1}} & \text{for } p < \dfrac{1}{3}, \\[2ex] \dfrac{\varepsilon \left\| x \right\|^{3p}}{3^{3p+1} - 9} & \text{for } p > \dfrac{1}{3} \end{cases} \tag{3.34}$$

for all $x \in A$.

Proof. If we choose $\alpha(x, y, z) = \varepsilon \left\| x \right\|^p \left\| y \right\|^p \left\| z \right\|^p$ for all $x, y, z \in A$. Then by Theorem 3.6, we arrive

$$\left\| f_a(x) - R(x) \right\| \le \frac{\varepsilon \|x\|^{3p}}{9 - 3^{3p+1}}, \text{ for all } x \in A \text{ and } p < \frac{1}{3}.$$

Using Theorem 3.7, we arrive

$$\left\| f_a(x) - R(x) \right\| \le \frac{\varepsilon \|x\|^{3p}}{3^{3p+1} - 9}, \text{ for all } x \in A \text{ and } p > \frac{1}{3}.$$

Theorem 3.11. *Let* A *be a real vector space and* B *be a Banach space. Let* $\alpha : A \times A \times A \to [0,\infty)$ *be a function satisfying* (3.1), (3.9), (3.18), (3.26) *for all* $x, y, z \in A$. *If* $f : A \to B$ *be a function satisfying*

$$\left\| D f(x,y,z) \right\| \le \alpha(x,y,z) \tag{3.35}$$

for all $x, y, z \in A$. *Then there exists a unique cubic function* $T : A \to B$ *and a unique additive function* $R : A \to B$ *which satisfies* (1.10) *and*

$$\left\| f(x) - T(x) - R(x) \right\| \le \frac{1}{81} \sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{27^k} + \frac{1}{9} \sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{3^k} \tag{3.36}$$

and

$$\left\| f(x) - T(x) - R(x) \right\| \le \frac{1}{81} \sum_{k=1}^{\infty} 27^k \alpha\left(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k}\right) + \frac{1}{9} \sum_{k=1}^{\infty} 3^k \alpha\left(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k}\right) \tag{3.37}$$

for all $x \in A$.

Proof. Let $f(x) = f_c(x) + f_a(x)$. Then

$$\begin{aligned} \left\| f(x) - T(x) - R(x) \right\| &= \left\| f_c(x) + f_a(x) - T(x) - R(x) \right\| \\ &\le \left\| f_c(x) - T(x) \right\| + \left\| f_a(x) - R(x) \right\| \end{aligned} \tag{3.38}$$

Using Theorems 3.1, 3.6 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \frac{1}{81} \sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{27^k} + \frac{1}{9} \sum_{k=0}^{\infty} \frac{\alpha\left(3^k x, 3^k x, 3^k x\right)}{3^k}$$

for all $x \in A$.

Again using Theorems 3.2, 3.7 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \frac{1}{81}\sum_{k=1}^{\infty} 27^k \alpha\left(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k}\right) + \frac{1}{9}\sum_{k=1}^{\infty} 3^k \alpha\left(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k}\right)$$

for all $x \in A$.

Corollary 3.12. *Let A be a real normed space and B be a Banach space. Let $f : A \to B$ be function satisfying the functional inequality*

$$\left\| D f(x, y, z) \right\| \le \varepsilon \left\{ \|x\|^p + \|y\|^p + \|z\|^p \right\} \tag{3.39}$$

with $p < 3$ (or) $p > 1$ for some $\varepsilon > 0$ and for all $x, y, z \in A$. Then there exists a unique cubic function $T : A \to B$ and a unique additive function $R : A \to B$ which satisfies (1.10) *and*

$$\left\| f(x) - T(x) - R(x) \right\| \le \begin{cases} \varepsilon \|x\|^p \left\{ \dfrac{1}{27 - 3^p} + \dfrac{1}{3 - 3^p} \right\} & \textit{for} \quad p < 3, \\[2ex] \varepsilon \|x\|^p \left\{ \dfrac{1}{3^p - 27} + \dfrac{1}{3^p - 3} \right\} & \textit{for} \quad p > 1 \end{cases} \tag{3.40}$$

for all $x \in A$.

Proof. Using Corollaries 3.3, 3.8 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \varepsilon \|x\|^p \left\{ \frac{1}{27 - 3^p} + \frac{1}{3 - 3^p} \right\}, \text{ for all } x \in A \text{ and } p < 3.$$

Again using Corollaries 3.3, 3.8 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \varepsilon \|x\|^p \left\{ \frac{1}{3^p - 27} + \frac{1}{3^p - 3} \right\}, \text{ for all } x \in A \text{ and } p > 1.$$

Corollary 3.13. *Let A be a real normed space and B be a Banach space. If a function $f : A \to B$ satisfies the functional inequality*

$$\left\| D f(x, y, z) \right\| \le \varepsilon \tag{3.41}$$

for some $\varepsilon > 0$ and for all $x, y, z \in A$. Then there exists a unique cubic function $T : A \to B$ and a unique additive function $R : A \to B$ which satisfies (1.10) *and*

$$\left\| f(x) - T(x) - R(x) \right\| \le \frac{7\varepsilon}{39} \tag{3.42}$$

for all $x \in A$.

Proof. Using Corollaries 3.4 and 3.9 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \frac{7\varepsilon}{39}, \text{ for all } x \in A.$$

Corollary 3.14. If $f : A \to B$ be a function from a normed vector space A into a Banach space B satisfies

$$\left\| D f(x, y, z) \right\| \le \varepsilon \|x\|^p \|y\|^p \|z\|^p \tag{3.43}$$

for all $x, y, z \in A$, *where* ε and p *are constants, with* $p < 1$ *(or)* $p > \frac{1}{3}$ *then there exists a unique cubic function* $T : A \to B$ *and a unique additive function* $R : A \to B$ *which satisfies* (1.10) *and*

$$\left\| f(x) - T(x) - R(x) \right\| \le \begin{cases} \varepsilon \|x\|^{3p} \left\{ \dfrac{1}{81 - 3^{3p+1}} + \dfrac{1}{9 - 3^{3p+1}} \right\} & \text{for } p < 1, \\ \varepsilon \|x\|^{3p} \left\{ \dfrac{1}{3^{3p+1} - 81} + \dfrac{1}{3^{3p+1} - 9} \right\} & \text{for } p > \dfrac{1}{3} \end{cases} \tag{3.44}$$

for all $x \in A$.

Proof. Using Corollaries 3.5, 3.10 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \varepsilon \|x\|^{3p} \left\{ \frac{1}{81 - 3^{3p+1}} + \frac{1}{9 - 3^{3p+1}} \right\},$$

for all $x \in A$ and $p < 1$.

Again using Corollaries 3.5, 3.10 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le \varepsilon \|x\|^{3p} \left\{ \frac{1}{3^{3p+1} - 81} + \frac{1}{3^{3p+1} - 9} \right\},$$

for all $x \in A$ and $p > \frac{1}{3}$.

4. Mixed Type Product – Sum Stability of Functional Equation (1.10)

Through out this section, we assume that A be a normed space and B be a Banach space respectively.

Theorem 4.1. *If* $f_c : A \to B$ *is a cubic function satisfying*

$$\left\| D\, f_c\left(x,y,z\right)\right\| \le \varepsilon\ \left\{\|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right)\right\} \tag{4.1}$$

for all $x,y,z \in A$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $p < 1$. *Then there exists a unique cubic function* $T : A \to B$ *defined by* (3.4) *satisfies* (1.10) *and*

$$\left\| f_c\left(x\right) - T\left(x\right)\right\| \le \frac{4\varepsilon}{81 - 3^{3p+1}}\|x\|^{3p} \tag{4.2}$$

for all $x \in A$.

Proof. Setting $\left(x,y,z\right)$ by $\left(0,0,0\right)$ in (4.1), we get $f\left(0\right) = 0$. Replacing $\left(x,y,z\right)$ by $\left(x,x,x\right)$ in (4.1) and using (2.5), we get

$$\left\| \frac{f_c\left(3x\right)}{27} - f_c\left(x\right)\right\| \le \frac{4\varepsilon}{81}\|x\|^{3p} \tag{4.3}$$

for all $x \in A$. Replacing x by $3x$ and divided by 27 in (4.3) and adding the resultant inequality with (4.3), we obtain

$$\left\| \frac{f_c\left(3^2 x\right)}{27^2} - f_c\left(x\right)\right\| \le \frac{4\varepsilon}{81}\left[1 + \frac{3^{3p}}{27}\right]\|x\|^{3p} \tag{4.4}$$

for all $x \in A$. In general for any positive integer n , we have

$$\begin{aligned}\left\| \frac{f_c\left(3^n x\right)}{27^n} - f_c\left(x\right)\right\| &\le \frac{4\varepsilon}{81}\sum_{k=0}^{n-1}\left(\frac{3^{3p}}{27}\right)^k \|x\|^{3p} \\ &\le \frac{4\varepsilon}{81}\sum_{k=0}^{\infty}\left(\frac{3^{3p}}{27}\right)^k \|x\|^{3p}\end{aligned} \tag{4.5}$$

for all $x \in A$. We have to show that the sequence $\left\{ \frac{f_c\left(3^n x\right)}{27^n} \right\}$ is a Cauchy sequence, replacing x by $3^m x$ and divide by 27^m in (4.5) for any $n, m > 0$, we obtain

$$\begin{aligned}
\left\| \frac{f_c\left(3^n 3^m x\right)}{27^m 27^n} - \frac{f_c\left(3^m x\right)}{27^m} \right\| &\leq \frac{1}{27^m} \left\| \frac{f_c\left(3^n 3^m x\right)}{27^n} - f_c\left(3^m x\right) \right\| \\
&\leq \frac{1}{27^m} \frac{4\varepsilon}{81} \sum_{k=0}^{n-1} \left(\frac{3^{3p}}{27} \right)^k \left\| 3^m x \right\|^{3p} \\
&\leq \frac{4\varepsilon}{81} \sum_{k=0}^{n-1} \left(\frac{3^{3p}}{27} \right)^{k+m} \left\| x \right\|^{3p} \\
&\leq \frac{4\varepsilon}{81} \sum_{k=0}^{\infty} \frac{1}{3^{3(1-p)(k+m)}} \left\| x \right\|^{3p}
\end{aligned} \tag{4.6}$$

for all $x \in A$. As $p < 1$, the right hand side of (4.6) tends to 0 as $m \to \infty$. Thus the sequence $\left\{ \frac{f_c\left(3^n x\right)}{27^n} \right\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $T : A \to B$ such that

$$T(x) = \lim_{n \to \infty} \frac{f_c\left(3^n x\right)}{27^n}, \quad \text{for all } x \in A.$$

Letting $n \to \infty$ in (4.5), we arrive (4.2). To show T satisfies (1.10) and it is unique the proof is similar to that of Theorem 3.1.

Theorem 4.2. *If* $f_c : A \to B$ *is a cubic function satisfying* (4.1) *for all* $x, y, z \in A$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $p > 1$. *Then there exists a unique cubic function* $T : A \to B$ *defined by* (3.11) *satisfies* (1.10) *and*

$$\left\| f_c(x) - T(x) \right\| \leq \frac{4\varepsilon}{3^{3p+1} - 81} \left\| x \right\|^{3p} \tag{4.7}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (4.3) and proceeding the same way as that of Theorem 4.1 the proof is complete.

Theorem 4.3. *If* $f_a : A \to B$ *is a additive function satisfying*

$$\left\| D\, f_a(x,y,z) \right\| \le \varepsilon \left\{ \|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p} \right) \right\} \tag{4.8}$$

for all $x, y, z \in A$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $p < \frac{1}{3}$. *Then there exists a unique additive function* $R : A \to B$ *defined by* (3.21) *satisfies* (1.10) *and*

$$\| f_a(x) - R(x) \| \le \frac{4\varepsilon}{9 - 3^{3p+1}} \|x\|^{3p} \tag{4.9}$$

for all $x \in A$.

Proof. Replacing (x, y, z) by (x, x, x) in (4.8) and using (2.14), we get

$$\left\| \frac{f_a(3x)}{3} - f_a(x) \right\| \le \frac{4\varepsilon}{9} \|x\|^{3p} \tag{4.10}$$

for all $x \in A$. Replacing x by $3x$ and divided by 3 in (4.10) and adding the resultant inequality with (4.10), we obtain

$$\left\| \frac{f_a(3^2 x)}{3^2} - f_a(x) \right\| \le \frac{4\varepsilon}{9} \left[1 + \frac{3^{3p}}{3} \right] \|x\|^{3p} \tag{4.11}$$

for all $x \in A$. In general for any positive integer n , we have

$$\begin{aligned} \left\| \frac{f_a(3^n x)}{3^n} - f_a(x) \right\| &\le \frac{4\varepsilon}{9} \sum_{k=0}^{n-1} \left(\frac{3^{3p}}{3} \right)^k \|x\|^{3p} \\ &\le \frac{4\varepsilon}{9} \sum_{k=0}^{\infty} \left(\frac{3^{3p}}{3} \right)^k \|x\|^{3p} \end{aligned} \tag{4.12}$$

for all $x \in A$. We have to show that the sequence $\left\{ \frac{f_a(3^n x)}{3^n} \right\}$ is a Cauchy sequence, replacing x by $3^m x$ and divide by 3^m in (4.12) for any $n, m > 0$, we obtain

$$\left\| \frac{f_a\left(3^n 3^m x\right)}{3^m 3^n} - \frac{f_a\left(3^m x\right)}{3^m} \right\| \leq \frac{1}{3^m} \left\| \frac{f_a\left(3^n 3^m x\right)}{3^n} - f_a\left(3^m x\right) \right\|$$

$$\leq \frac{4\varepsilon}{9} \sum_{k=0}^{n-1} \left(\frac{3^{3p}}{3} \right)^{k+m} \|x\|^{3p}$$

$$\leq \frac{4\varepsilon}{9} \sum_{k=0}^{\infty} \frac{1}{3^{(1-3p)(k+m)}} \|x\|^{3p} \tag{4.13}$$

for all $x \in A$. As $p < \frac{1}{3}$, the right hand side of (4.13) tends to 0 as $m \to \infty$.

Thus the sequence $\left\{ \frac{f_a\left(3^n x\right)}{3^n} \right\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $R : A \to B$ such that

$$R(x) = \lim_{n \to \infty} \frac{f_a\left(3^n x\right)}{3^n}, \text{ for all } x \in A.$$

Letting $n \to \infty$ in (4.12), we arrive (4.9). To show R satisfies (1.10) and it is unique the proof is similar to that of Theorem 3.6.

Theorem 4.4. *If* $f_a : A \to B$ *is a additive function satisfying* (4.8) *for all* $x, y, z \in A$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $p > \frac{1}{3}$. *Then there exists a unique additive function* $R : A \to B$ *defined by* (3.28) *satisfies* (1.10) *and*

$$\| f_a(x) - R(x) \| \leq \frac{4\varepsilon}{3^{3p+1} - 9} \|x\|^{3p} \tag{4.14}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (4.10) and proceeding the same way as that of Theorem 4.3 the proof is complete.

Theorem 4.5. *If* $f : A \to B$ *be a function satisfying*

$$\|D f(x, y, z)\| \leq \varepsilon \left\{ \|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p} \right) \right\} \tag{4.15}$$

for all $x, y, z \in A$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $p < 1$ *(or)* $p > \frac{1}{3}$.
Then there exists a unique cubic function $T : A \to B$ and a unique additive function $R : A \to B$ which satisfies (1.10) *and*

$$\left\| f(x) - T(x) - R(x) \right\| \le 4\varepsilon \|x\|^{3p} \left\{ \frac{1}{81 - 3^{3p+1}} + \frac{1}{9 - 3^{3p+1}} \right\}, \qquad \textit{for } p < 1 \quad (4.16)$$

for all $x \in A$. *Also*

$$\left\| f(x) - T(x) - R(x) \right\| \le 4\varepsilon \|x\|^{3p} \left\{ \frac{1}{3^{3p+1} - 81} + \frac{1}{3^{3p+1} - 9} \right\}, \qquad \textit{for } p > \frac{1}{3} \quad (4.17)$$

for all $x \in A$.

Proof. Using Theorems 4.1, 4.3 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le 4\varepsilon \|x\|^{3p} \left\{ \frac{1}{81 - 3^{3p+1}} + \frac{1}{9 - 3^{3p+1}} \right\},$$

for all $x \in A$ and $p < 1$.
Again using Theorems 4.2, 4.4 and (3.38), we arrive

$$\left\| f(x) - T(x) - R(x) \right\| \le 4\varepsilon \|x\|^{3p} \left\{ \frac{1}{3^{3p+1} - 81} + \frac{1}{3^{3p+1} - 9} \right\},$$

for all $x \in A$ and $p > \frac{1}{3}$.

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Chapter 14

STABILITY OF MAPPINGS APPROXIMATELY PRESERVING ORTHOGONALITY AND RELATED TOPICS

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Abstract

Speaking of the *stability* we follow the question of S. Ulam [21, p.63]: "when is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In this chapter we study stability of mappings between Hilbert spaces which nearly preserve orthogonality, inner product or its absolute value. In the first section we present some results on linear approximately orthogonality preserving mappings. In the second section we study orthogonality equation and in the last one we present some stability results on Wigner equation.

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1. Mappings Approximately Preserving Orthogonality

Let $\mathcal{H}$ and $\mathcal{K}$ be real or complex Hilbert spaces with an inner product denoted by $\langle \cdot, \cdot \rangle$. As usual, vectors x and y are said to be orthogonal, $x \perp y$, if $\langle x, y \rangle = 0$. A mapping $T : \mathcal{H} \to \mathcal{K}$ is called *orthogonality preserving*, if it preserves orthogonality, that is

$$x \perp y \Rightarrow Tx \perp Ty, \quad x, y \in \mathcal{H}.$$

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It is known that a linear mapping T is orthogonality preserving if and only if $T = \gamma U$, where U is an isometry and $\gamma \geq 0$, see [4]. Let us say that for a given $\varepsilon \in [0,1)$ vectors $x, y \in \mathcal{H}$ are approximately orthogonal or ε-orthogonal, denoted by $x \perp^{\varepsilon} y$, if

$$|\langle x, y\rangle| \leq \varepsilon \|x\| \, \|y\|.$$

Thus one can consider the class of *approximately orthogonality preserving* mappings as all those satisfying the condition

$$x \perp y \Rightarrow Tx \perp^{\varepsilon} Ty, \quad x, y \in \mathcal{H}.$$

Hence, the natural stability question is whether an approximately orthogonality preserving linear mapping $T : \mathcal{H} \to \mathcal{K}$ must be close to a linear orthogonality preserving mapping. More precisely: if $T : \mathcal{H} \to \mathcal{K}$ is a linear approximately orthogonality preserving mapping, is there a linear orthogonality preserving mapping $V : \mathcal{H} \to \mathcal{K}$ and $\delta(\varepsilon) > 0$, such that

$$\|T - V\| \leq \delta(\varepsilon) \min\left\{\|T\|, \|V\|\right\} \text{ and } \delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Let us fix some notation. The Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and we write $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H}, \mathcal{H})$. The spectrum of an operator T is denoted by $\sigma(T)$.

In the next lemma we show that linear approximately orthogonality preserving mappings are automatically bounded and “almost” multiples of isometries.

Lemma 1.1 ([4], [20]). *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ a nonzero linear approximately orthogonality preserving mapping, $\varepsilon \in [0,1)$. Then*

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \, \|T\| \, \|x\| \leq \|Tx\| \leq \|T\| \, \|x\|, \quad x \in \mathcal{H}.$$

Proof. We show that for unit vectors u and v

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \, \|Tv\| \leq \|Tu\| \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \, \|Tv\|. \tag{1}$$

If u and v are linearly dependent, then (1) is satisfied. Hence, we may assume that u and v are linearly independent unit vectors. Choose $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $\langle u, \lambda v\rangle \in \mathbb{R}$. (If $\mathcal{H}$ is real one can take $\lambda = 1$). Then $u + \lambda v \perp u - \lambda v$ and hence $Tu + \lambda Tv \perp^{\varepsilon} Tu - \lambda Tv$. Therefore,

$$\left|\left\langle Tu + \lambda Tv, Tu - \lambda Tv\right\rangle\right| \leq \varepsilon \|Tu + \lambda Tv\| \, \|Tu - \lambda Tv\|.$$

This is equivalent to

$$\left(\|Tu\|^2 - \|Tv\|^2\right)^2 + (2\mathrm{Im}\langle Tu, \lambda Tv\rangle)^2 \leq \varepsilon^2 \left[\left(\|Tu\|^2 + \|Tv\|^2\right)^2 - (2\mathrm{Re}\langle Tu, \lambda Tv\rangle)^2\right],$$

hence

$$\left| \, \|Tu\|^2 - \|Tv\|^2\right| \leq \varepsilon \left(\|Tu\|^2 + \|Tv\|^2\right),$$

which gives (1). Now it follows from (1) that T is bounded and then that

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \, \|T\| \leq \|Tu\| \leq \|T\|.$$

Thus for $u = x/\|x\|$ one obtains

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\, \|T\|\, \|x\| \le \|Tx\| \le \|T\|\, \|x\|, \quad x \in \mathcal{H}, \tag{2}$$

and the proof is completed. □

Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and let $T = U|T|$ be its polar decomposition, where $|T| \in \mathcal{B}(\mathcal{H})$ is a positive square root of T^*T and $U \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a partial isometry. Usually the polar decomposition is stated for the operators in $\mathcal{B}(\mathcal{H})$, see [18, p.96], however for the operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ the proof is the same with obvious modifications.

Lemma 1.2 ([20], Lemma 2.1)**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $T \in \mathcal{B}(\mathcal{H}, K)$ and $T = U|T|$ its polar decomposition. If for $m, M > 0$*

$$m\|x\| \le \|Tx\| \le M\|x\|, \quad x \in \mathcal{H}, \tag{3}$$

then U is an isometry and

$$\|T - U\| \le \max\big\{|M-1|, |m-1|\big\}.$$

Proof. Assume first that $\mathcal{H}$ is a complex Hilbert space. From (3) it follows that T is injective which shows that U is an isometry. Thus a positive operator $|T|$ satisfies (3) as well, hence $|T|$ is invertible. The inequalities

$$\frac{1}{M}\, \|x\| \le \big\|\, |T|^{-1}x\big\| \le \frac{1}{m}\, \|x\|$$

and (3) imply that $\sigma(|T|) \subseteq [m, M]$. Thus $\sigma(|T| - 1) \subseteq [m-1, M-1]$ and, since the norm of a selfadjoint operator equals its spectral radius, it follows that $\||T| - 1\| \le \max\{|M-1|, |m-1|\}$. Therefore

$$\|T - U\| = \|U(|T| - 1)\| = \big\|\, |T| - 1\big\| \le \max\big\{|M-1|, |m-1|\big\}.$$

If $\mathcal{H}$ is a real Hilbert space, let $\mathcal{H}_C$ be its complexification, see [11, p. 150] for the details, and $|T|_C : \mathcal{H}_C \to \mathcal{H}_C$ a positive linear operator defined by

$$|T|_C(x, y) = (|T|x, |T|y).$$

Then it is easy to check that $|T|_C$ also satisfies (3), and by the first part of the proof, it follows that

$$\||T|_C - 1\| \le \max\big\{|M-1|, |m-1|\big\}.$$

The proof is completed using the fact that U is an isometry and that the latter inequality implies $\||T| - 1\| \le \max\big\{|M-1|, |m-1|\big\}$. □

Remark 1.1. Let $f : \mathcal{H} \to \mathcal{K}$ be a nonlinear mapping such that $m\|x\| \le \|f(x)\| \le M\|x\|$ with m and M both close to 1. Suppose that we can approximate f by a linear mapping S, i.e., $\|f(x) - Sx\| \le \varepsilon\|x\|$ for all $x \in \mathcal{H}$ and for some $\varepsilon \in [0, m)$. Then we can approximate

f by a linear isometry U from the polar decomposition of S. Indeed, from our assumptions on f and S we obtain

$$(m-\varepsilon)\|x\| \le \|Sx\| \le (M+\varepsilon)\|x\|.$$

By the previous lemma it follows that $\|S-U\| \le \max\left\{|M+\varepsilon-1|, |m-\varepsilon-1|\right\}$. Hence U is close to S and then U is also close to f.

Now we can prove stability of linear approximately orthogonality preserving mappings.

Theorem 1.3 ([20], Theorem 2.3)**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $T : \mathcal{H} \to \mathcal{K}$ a nonzero linear approximately orthogonality preserving mapping, $\varepsilon \in [0,1)$, and $T = U|T|$ its polar decomposition. Then U is an isometry and*

$$\|T - \|T\|U\| \le \left(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right)\|T\|. \tag{4}$$

Proof. Combine Lemma 1.1 and Lemma 1.2 with $m = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$, $M = 1$ and $\frac{T}{\|T\|}$. □

Remark 1.2. The estimate in the previous theorem is sharp. An example of such a mapping is $T = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) : \mathbb{R} \oplus \mathbb{R}^{n-1} \to \mathbb{R} \oplus \mathbb{R}^{n-1}$, $n \ge 2$, where $a = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$ for some $\varepsilon \in [0,1)$ and 1 in the lower right corner is the identity on $\mathbb{R}^{n-1}$. However, the approximating isometries are not unique, see [20] for the details.

Remark 1.3. There is also a variant of Theorem 1.3 for mappings between Hilbert C^*-modules, see [12, Theorem 4.4]. Let $\mathcal{A}$ be a C^*-algebra of compact operators on $\mathcal{H}$ and let $\mathcal{V}$ and $\mathcal{W}$ be Hilbert $\mathcal{A}$-modules. Let $T : \mathcal{V} \to \mathcal{W}$ be an $\mathcal{A}$-linear approximately orthogonality preserving mapping with some $\varepsilon \in [0,1)$. Then there is an $\mathcal{A}$-linear isometry $U : \mathcal{V} \to \mathcal{W}$ such that (4) holds.

Example 1.4. *Nonlinear orthogonality preserving or approximately orthogonality preserving mappings need not be approximated by linear mappings at all. Indeed, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$f(x_1,x_2) = \begin{cases} (x_1,x_2), & x_1x_2 \ne 0 \\ (1,1), & x_1 \ne 0, x_2 = 0 \\ (-1,1), & x_1 = 0, x_2 \ne 0 \\ (0,0), & x_1 = x_2 = 0 \end{cases}$$

Then f is a nonlinear orthogonality preserving, hence also approximately orthogonality preserving mapping. Let us suppose that we can find a linear operator A such that $\|f(x) - Ax\| \le \|x\|$ for all $x \in \mathbb{R}^2$. This implies, since $f(\pm 1, 0) = (1,1)$, that

$$\|(1,1) - A(1,0)\| \le 1 \ \text{ and } \ \|(1,1) + A(1,0)\| \le 1.$$

By the parallelogram identity it follows that $2 + \|A(1,0)\|^2 \le 1$, a contradiction.

Question 1. Is the Theorem 1.3 true also for unitary (pre-Hilbert) spaces? Namely, in Lemma 1.2 we used the concept of the polar decomposition for which the spaces need to be complete.

Question 2. Can we approximate a nonlinear approximately orthogonality preserving mapping by a nonlinear orthogonality preserving mapping?

1.1. Orthogonality in Normed Spaces

The notion of orthogonality in an arbitrary normed space may be introduced in various ways. One of the possibilities is the following definition introduced by Birkhoff [1], see also James [13]. Let $\mathcal{X}$ be a real or complex normed space; then for $x, y \in \mathcal{X}$

$$x \perp_B y \iff \|x + \lambda y\| \geq \|x\| \text{ for all scalars } \lambda.$$

We call the relation $\perp_B$ a *Birkhoff–James orthogonality*. It is easily seen that, for inner-product spaces, this last definition is equivalent to the usual definition of orthogonality. In general normed spaces Birkhoff-James orthogonality is neither symmetric nor additive, but it is always homogeneous. It is clear from the definition that scalar multiples of linear isometries preserve orthogonality. Converse was proved by Koldobsky [15] for real spaces and later by Blanco and Turnšek in general.

Theorem 1.5 ([2], Theorem 3.1)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is orthogonality preserving if and only if it is a scalar multiple of a linear isometry.*

Our aim is to define an approximate Birkhoff orthogonality generalizing the $\perp^\varepsilon$ one for inner product spaces; we follow the approach of Chmieliński [5].

Let $\mathcal{X}$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The norm in $\mathcal{X}$ need not come from an inner product. However, see Lumer [16] and Giles [10], there exists a mapping $[\cdot, \cdot] : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ with the following properties:

(s1) $[\lambda x + \mu y, z] = \lambda[x, z] + \mu[y, z]$, $x, y, z \in \mathcal{X}$, $\lambda, \mu \in \mathbb{K}$;

(s2) $[x, \lambda y] = \overline{\lambda}[x, y]$, $x, y \in \mathcal{X}$, $\lambda \in \mathbb{K}$;

(s3) $[x, x] = \|x\|^2$, $x \in \mathcal{X}$;

(s4) $|[x, y]| \leq \|x\|\|y\|$, $x, y \in \mathcal{X}$.

A mapping satisfying (s1)-(s4) is called a *semi-inner product* (s.i.p.). Note that there may exist infinitely many different semi-inner products in $\mathcal{X}$. There is a unique s.i.p. in $\mathcal{X}$ if and only if $\mathcal{X}$ is *smooth* (i.e., there is a unique supporting hyperplane at each point of the unit sphere or, equivalently, the norm is *Gâteaux differentiable*). Recall that the norm is Gâteaux differentiable at $x \neq 0$ if the limit

$$\lim_{t \to 0,\, t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $y \in \mathcal{X}$. Recall also that in this case this last limit is equal to the real part of $f_x(y)$, where f_x is a *support functional* at x, i.e., a norm one linear functional such that $f_x(x) = \|x\|$. If we define $[y, x] = \|x\| f_x(y)$, then $[\cdot, \cdot]$ is a unique s.i.p. Now in smooth spaces, by an analogy with inner product spaces, we define *semi-orthogonality* and *approximate semi-orthogonality*:

$$x \perp_s y \iff [y, x] = 0;$$
$$x \perp_s^\varepsilon y \iff |[y, x]| \leq \varepsilon \|x\|\|y\|,$$

for some $\varepsilon \in [0,1)$.

Our next aim is to express the above approximate semi-orthogonality relation without s.i.p., thus involving only norm, and take it as a definition of approximate Birkhoff orthogonality in general normed space $\mathcal{X}$.

Proposition 1.6 ([5], Proposition 3.1, 3.2)**.** *Let $\mathcal{X}$ be a smooth normed space and let $\varepsilon \in [0,1)$. Then $|[y,x]| \leq \varepsilon\|x\|\|y\|$ if and only if $\|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|$ for all scalars λ.*

Proof. ($\Rightarrow$): Recall that $[y,x] = \|x\| f_x(y)$, where f_x is a support functional at x. Then $|f_x(y)| \leq \varepsilon\|y\|$, and for some $\theta \in [0,1]$ and for some $\varphi \in [-\pi,\pi]$ we have

$$f_x(y) = \theta\varepsilon\|y\|e^{i\varphi}.$$

For arbitrary $\lambda \in \mathbb{K}$ we have

$$\begin{aligned}\|x+\lambda y\| &\geq |f_x(x+\lambda y)| = \left|\|x\| + \lambda\theta\varepsilon\|y\|e^{i\varphi}\right| \\ &= \left|\|x\| + \theta\varepsilon\|y\|\big(\mathrm{Re}(\lambda e^{i\varphi}) + i\mathrm{Im}(\lambda e^{i\varphi})\big)\right|,\end{aligned}$$

hence

$$\begin{aligned}\|x+\lambda y\|^2 &\geq \big(\|x\| + \theta\varepsilon\|y\|\mathrm{Re}(\lambda e^{i\varphi})\big)^2 + \big(\theta\varepsilon\|y\|(\mathrm{Im}(\lambda e^{i\varphi})\big)^2 \\ &\geq \|x\|^2 + 2\theta\varepsilon\|x\|\|y\|(\mathrm{Re}(\lambda e^{i\varphi}) \\ &\geq \|x\|^2 - 2\theta\varepsilon\|x\|\|y\||\lambda| \\ &\geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|.\end{aligned}$$

($\Leftarrow$): Let $\gamma = \mathrm{Arg}(f_x(y))$. Then $|f_x(y)| = e^{-i\gamma}f_x(y) = f_x(e^{-i\gamma}y) = \mathrm{Re}(f_x(e^{-i\gamma}y))$. From $\|x+\lambda e^{-i\gamma}y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|$ it follows that

$$\frac{\|x+\lambda e^{-i\gamma}y\| - \|x\|}{|\lambda|} \geq \frac{-2\varepsilon\|x\|\|y\|}{\|x+\lambda e^{-i\gamma}y\| + \|x\|}.$$

Taking the right and left limits as $\lambda \to 0$ we get $|f_x(y)| \leq \varepsilon\|y\|$ and the proof is completed. □

Taking into account the last proposition we define the approximate Birkhoff orthogonality on any normed space.

Definition 1 ([5])**.** Let $\mathcal{X}$ be a normed space and let $\varepsilon \in [0,1)$ be given. We say that x is ε-Birkhoff orthogonal to y, $x \perp_B^\varepsilon y$, if $\|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|$ for all scalars λ.

Question 3. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let $T : \mathcal{X} \to \mathcal{Y}$ be a linear mapping which approximately preserves Birkhoff orthogonality. Is then T close to some multiple of an isometry?

2. Stability of the Orthogonality Equation

Since orthogonality preserving mappings can be far from being linear or continuous, see Example 1.4, we will now impose a stronger condition. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A mapping $f : \mathcal{H} \to \mathcal{K}$ is called *inner product preserving* if it is a solution of the *orthogonality equation*:

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \quad \text{for } x, y \in \mathcal{H}. \tag{O}$$

It is easy to see that f satisfies (O) if and only if it is a linear isometry. We say that a mapping $f : \mathcal{H} \to \mathcal{K}$ approximately preserves the inner product if it satisfies

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \varphi(x, y) \tag{AO}$$

for some appropriate control function φ and all $x, y \in \mathcal{H}$.

Theorem 2.1 ([8], Theorem 4.1)**.** *If $f : \mathcal{H} \to \mathcal{K}$ satisfies (AO) with a function $\varphi : \mathcal{H}\times\mathcal{H} \to [0,\infty)$ such that $\lim_{m+n\to\infty} c^{m+n}\varphi(c^{-m}x, c^{-n}y) = 0$ for all $x, y \in \mathcal{H}$ and for some $1 \ne c > 0$, then there exists a unique mapping $U : \mathcal{H} \to \mathcal{K}$ satisfying the orthogonality equation (O) and such that*

$$\|f(x) - Ux\| \le \sqrt{\varphi(x, x)} \ \text{ for all } \ x \in \mathcal{H}.$$

Proof. We give just the idea of the proof which goes back to Hyers. One defines $f_n(x) = c^n f(c^{-n}x)$ and shows that the sequence is Cauchy, hence convergent. The limit Ux satisfies the requirements of the theorem. □

Let $\varphi(x, y) = \varepsilon\|x\|^p\|y\|^p$ with $p \in \mathbb{R} \setminus \{1\}$. Then φ satisfies the conditions of the above theorem and we get the following result.

Corollary 2.2 ([3], Theorem 2)**.** *Let $\varepsilon > 0$ and $p \in \mathbb{R} \setminus \{1\}$ be fixed. Then, for a mapping $f : \mathcal{H} \to \mathcal{K}$ satisfying*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \le \varepsilon\|x\|^p\|y\|^p, \quad x, y \in \mathcal{H}_p, \tag{5}$$

where $\mathcal{H}_p = \mathcal{H}$ for $p \ge 0$ and $\mathcal{H}_p = \mathcal{H} \setminus \{0\}$ for $p < 0$, there exists a unique mapping $U : \mathcal{H} \to \mathcal{K}$ satisfying (O) and such that

$$\|f(x) - Ux\| \le \sqrt{\varepsilon}\|x\|^p.$$

The constant $\sqrt{\varepsilon}$ which appears in the assertion of the previous corollary is the best possible. To see it let us consider the example.

Example 2.3 ([8], Example 2.4)**.** *Let $f : l^2 \to l^2$ be a mapping defined by $f(x) = (\sqrt{\varepsilon}\|x\|^p, x)$. Then f satisfies* (5) *and $Ux = (0, x)$ is a solution of (O) such that $\|f(x) - Ux\| = \sqrt{\varepsilon}\|x\|^p$. That the estimate is indeed sharp follows from the uniqueness.*

Remark 2.1. Results of Theorem 2.1 and Corollary 2.2 can be generalised to the setting of Hilbert C^*-modules, see [9].

2.1. The case $p = 1$

The case $p = 1$ seems to be a singular one. So, let us see what we can say about mappings $f : \mathcal{H} \to \mathcal{K}$ satisfying the following condition:

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \varepsilon\|x\|\,\|y\|, \quad x, y \in \mathcal{H}, \quad \varepsilon \in [0, 1). \tag{6}$$

From (6) it follows that $\sqrt{1-\varepsilon}\|x\| \le \|f(x)\| \le \sqrt{1+\varepsilon}\|x\|$. But we can even assume that f preserves the norm.

Proposition 2.4 ([3], Proposition 1)**.** *Let $f : \mathcal{H} \to \mathcal{K}$ satisfies* (6)*. Define*

$$g(x) = \begin{cases} \dfrac{f(x)\|x\|}{\|f(x)\|} & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$$

Then:

(i) $|\langle g(x), g(y)\rangle - \langle x, y\rangle| \le 2\varepsilon\|x\|\,\|y\|$,

(ii) $\|g(x)\| = \|x\|$,

(iii) $\|f(x) - g(x)\| \le (1 - \sqrt{1-\varepsilon})\|x\|$.

Let $\mathcal{H}$ be an n-dimensional Hilbert space and $g : \mathcal{H} \to \mathcal{K}$ a mapping satisfying (6) for some $\varepsilon \in [0, 1)$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis in $\mathcal{H}$ and for $x \in \mathcal{H}$, $x = \sum_{i=1}^{n} \alpha_i e_i$, define a linear operator $S : \mathcal{H} \to \mathcal{K}$ by $Sx = \sum_{i=1}^{n} \alpha_i g(e_i)$.

Proposition 2.5 ([20], Proposition 2.5)**.** *If $g : \mathcal{H} \to \mathcal{K}$ satisfies* (6) *and $\|g(x)\| = \|x\|$, $x \in \mathcal{H}$, then*

$$\|g(x) - Sx\| \le \sqrt{\varepsilon(n + 2\sqrt{n} - 1)}\,\|x\|.$$

Proof. Let $x = \sum_{i=1}^{n} \alpha_i e_i$ and denote

$$\overline{\alpha_i}\langle g(x), g(e_i)\rangle - \overline{\alpha_i}\langle x, e_i\rangle = \lambda_i.$$

Thus

$$\overline{\alpha_i}\langle g(x), g(e_i)\rangle = |\alpha_i|^2 + \lambda_i,$$

where $|\lambda_i| \le \varepsilon|\alpha_i|\,\|x\|$ because of (6). Note also that $|\langle g(e_i), g(e_j)\rangle| \le \varepsilon$ for $i \neq j$. Then

$$\begin{aligned}
\|g(x) - Sx\|^2 &= \big\langle g(x) - Sx, g(x) - Sx\big\rangle \\
&= \|x\|^2 - 2\mathrm{Re}\Big(\sum_{i=1}^{n} \overline{\alpha_i}\langle g(x), g(e_i)\rangle\Big) + \sum_{i,j=1}^{n} \alpha_i\overline{\alpha_j}\langle g(e_i), g(e_j)\rangle \\
&= \|x\|^2 - 2\sum_{i=1}^{n}|\alpha_i|^2 - 2\mathrm{Re}\Big(\sum_{i=1}^{n}\lambda_i\Big) + \sum_{i=1}^{n}|\alpha_i|^2 + \sum_{i\neq j, i,j=1}^{n} \alpha_i\overline{\alpha_j}\langle g(e_i), g(e_j)\rangle \\
&\le 2\varepsilon\|x\|\sum_{i=1}^{n}|\alpha_i| + \varepsilon\left(\Big(\sum_{i=1}^{n}|\alpha_i|\Big)^2 - \sum_{i=1}^{n}|\alpha_i|^2\right) \\
&\le 2\varepsilon\|x\|\,\|x\|\sqrt{n} + \varepsilon\left(\|x\|^2 n - \|x\|^2\right) = \varepsilon\left(n + 2\sqrt{n} - 1\right)\|x\|^2,
\end{aligned}$$

and the result follows. □

Now we can prove stability of the orthogonality equation for the finite-dimensional domain.

Theorem 2.6. *Let $\mathcal{H}$ be an n-dimensional Hilbert space and $f : \mathcal{H} \to \mathcal{K}$ a mapping satisfying* (6)*. Then there exists a linear isometry $U : \mathcal{H} \to \mathcal{K}$ and a continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, given by*

$$\eta(\varepsilon) = \min\Big\{1 - \sqrt{1-\varepsilon} + \sqrt{8\varepsilon(n + 2\sqrt{n} - 1)}, 1 + \sqrt{1+\varepsilon}\Big\},$$

such that

$$\|f(x) - Ux\| \le \eta(\varepsilon)\|x\|, \quad x \in \mathcal{H}.$$

Proof. Since $\|f(x)\| \le \sqrt{1+\varepsilon}\|x\|$, we get

$$\|f(x) - Ux\| \le \|f(x)\| + \|Ux\| \le \big(1 + \sqrt{1+\varepsilon}\big)\|x\|$$

for any isometry U. Assume now that $\varepsilon < \varepsilon_0 = \frac{1}{2(n+2\sqrt{n}-1)}$ and let $\delta = \sqrt{2\varepsilon(n + 2\sqrt{n} - 1)} < 1$. Let the function $g : \mathcal{H} \to \mathcal{K}$ be as in Proposition 2.4. From Proposition 2.5 one obtains a linear operator S such that $\|g(x) - Sx\| \le \delta\|x\|$, hence

$$(1-\delta)\|x\| \le \|Sx\| \le (1+\delta)\|x\|.$$

From Lemma 1.2, see also Remark 1.1, it follows that $\|S - U\| \le \delta$, where U is the isometry from the polar decomposition of S. Thus $\|g(x) - Ux\| \le 2\delta\|x\|$ and

$$\|f(x) - Ux\| \le \big(1 - \sqrt{1-\varepsilon} + 2\delta\big)\,\|x\| \ \text{ for } \ \varepsilon < \varepsilon_0.$$

Since $\delta = 1$ for $\varepsilon = \varepsilon_0$ and since $3 - \sqrt{1-\varepsilon} > 1 + \sqrt{1+\varepsilon}$, the function η is continuous and the proof is completed. $\square$

We can formulate the previous theorem in a more compact form. See [22], [23] for similar results on bounded domains.

Theorem 2.7. *There is a universal constant C with the following property. Let $\mathcal{H}$ be an n-dimensional Hilbert space and $f : \mathcal{H} \to \mathcal{K}$ a mapping satisfying* (6)*. Then there exists a linear isometry $U : \mathcal{H} \to \mathcal{K}$ with*

$$\|f(x) - Ux\| \le C\sqrt{\varepsilon\, n}\,\|x\|$$

for all $x \in \mathcal{H}$.

Remark 2.2. One can show, see [4, Lemma 2], that a mapping f satisfying (6) has to be *quasi linear* in the following sense:

$$\|f(x+y) - f(x) - f(y)\| \le 2\sqrt{\varepsilon}\,(\|x\| + \|y\|) \tag{7}$$

and

$$\|f(\lambda x) - \lambda f(x)\| \le 2\sqrt{\varepsilon}\,|\lambda|\|x\| \tag{8}$$

Remark 2.3. The solution of (6) need not be neither additive nor homogeneous. Indeed, consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $f(x) = (\sqrt{\varepsilon}\|x\|, x)$. However, we can assume with no loss of generality that solutions of (6) are real homogeneous. For this we need the following observations. Define

$$\Omega(x) = \|x\| \left(f\Big(\frac{x}{2\|x\|}\Big) - f\Big(\frac{-x}{2\|x\|}\Big) \right),$$

where f is a solution of (6). Then Ω is real homogeneous and satisfies (6). To see this denote $\lambda = \frac{1}{2\|x\|}$ and $\mu = \frac{1}{2\|y\|}$. Then

$$\begin{aligned}
&\langle \Omega(x), \Omega(y)\rangle - \langle x, y\rangle \\
&\quad = \|x\|\|y\| \Big[\big(\langle f(\lambda x), f(\mu y)\rangle - \langle \lambda x, \mu y\rangle\big) + \big(\langle \lambda x, -\mu y\rangle - \langle f(\lambda x), f(-\mu y)\rangle\big) \\
&\qquad + \big(\langle -\lambda x, \mu y\rangle - \langle f(-\lambda x), f(\mu y)\rangle\big) + \big(\langle f(-\lambda x), f(-\mu y)\rangle - \langle -\lambda x, -\mu y\rangle\big)\Big]
\end{aligned}$$

and by using the triangle inequality it follows that Ω satisfies (6). Furthermore, Ω is close to f. Indeed,

$$\begin{aligned}
\Omega(x) - f(x) &= \|x\|\big(f(\lambda x) - f(-\lambda x)\big) - \|x\|\big(\lambda f(x) + \lambda f(x)\big) \\
&= \|x\|\big(f(\lambda x) - \lambda f(x)\big) - \|x\|\big(f(-\lambda x) - (-\lambda) f(x)\big).
\end{aligned}$$

Using the triangle inequality and (8) it follows that $\|\Omega(x) - f(x)\| \le 2\sqrt{\varepsilon}\,\|x\|$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map which satisfies the condition (6) for some $\varepsilon \in [0, 1)$. Then by Proposition 2.4 and Remark 2.3 we can assume that f preserves the norm, that is $\|f(x)\| = \|x\|$, and that f is homogeneous. Furthermore, if we can approximate f by a linear map, then by Remark 1.1, we can approximate f also with an isometry. By a result of Kalton, [14, Theorem 2.2], we have the following theorem.

Theorem 2.8. *There is a universal constant C with the following property. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map which satisfies the condition* (6). *Then there is an isometry $U : \mathbb{R}^n \to \mathbb{R}^n$ with*

$$\|f(x) - Ux\| \le C\sqrt{\varepsilon}\,(\log n + 1)\|x\|$$

for all $x \in \mathbb{R}^n$.

Question 4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy (6) for some $\varepsilon \in [0, 1)$. Suppose that f preserves the norm, that is $\|f(x)\| = \|x\|$, and that f is homogeneous. Let $\alpha(f) = \inf_T \sup_{\|x\|=1} \|f(x) - Tx\|$, where the infimum is taken over all linear mappings $T : \mathbb{R}^n \to \mathbb{R}^n$. Let $\alpha(n, \varepsilon) = \sup_f \alpha(f)$, the supremum being taken over all f as above. Then by the Theorem 2.7, $\alpha(n, \varepsilon) \le C\sqrt{\varepsilon\, n}$ ($\alpha(n, \varepsilon) \le C\sqrt{\varepsilon}\,(\log n + 1)$) if f is continuous). Is it true that the "approximation error" $\alpha(n, \varepsilon)$ depends on the dimension? Find its lower bound?

Question 5. Is the orthogonality equation with control function $\varphi(x, y) = \varepsilon\|x\|\|y\|$ stable also in the case of the infinite dimensional domain?

3. Stability of the Wigner Equation

In the Hilbert space formulation of quantum mechanics, which is mainly due to von Neumann, several mathematical objects appear whose physical meaning is connected with the probabilistic aspects of the theory, see [17]. For example $\mathcal{S}(\mathcal{H})$, the set of all positive trace-class operators on $\mathcal{H}$ with trace 1. The elements of $\mathcal{S}(\mathcal{H})$ are called states of the system. The extreme points of $\mathcal{S}(\mathcal{H})$ as a convex set in $\mathcal{B}(\mathcal{H})$ are called pure states. It is easy to see that they are exactly the rank-one projections on $\mathcal{H}$. A rank-one projection can be trivially identified with its range or with any unit vector which spans its range. Hence, one can regard pure states in three different ways: rank-one projections, one-dimensional subspaces, unit vectors (in this latter case the identification is one-to-one only up to multiplication by a scalar of modulus 1). If $P = x \otimes x$ and $Q = y \otimes y$ are pure states, then the *transition probability* between them is defined by tr$(PQ) = |\langle x, y\rangle|^2$, where tr denotes the usual trace functional. In this context it is very important Wigner's theorem which we can formulate in different ways, see[17, p. 12]. The classical formulation, see [19], says:

Theorem 3.1. *If $f : \mathcal{H} \to \mathcal{K}$ satisfies*

$$\big|\langle f(x), f(y)\rangle\big| = |\langle x, y\rangle| \quad \textit{for} \quad x, y \in \mathcal{H}, \tag{W}$$

then f is phase-equivalent to a linear or a conjugate-linear isometry.

Recall that functions $f, g : \mathcal{H} \to \mathcal{K}$ are phase-equivalent if there exists a function $\gamma : \mathcal{H} \to S^1$, where S^1 is the unit circle in the complex plane, such that $g(x) = \gamma(x) f(x)$ for all $x \in \mathcal{H}$. Recall also that conjugate-linear means $f(\lambda x + \mu y) = \overline{\lambda} f(x) + \overline{\mu} f(y)$.

Stability of the Wigner equation (W) is explained in the following analogue of the Theorem 2.1.

Theorem 3.2 ([8], Theorem 2.1)**.** *If $f : \mathcal{H} \to \mathcal{K}$ satisfies*

$$\big||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\big| \le \varphi(x, y) \quad \textit{for } x, y \in \mathcal{H}$$

with a function $\varphi : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ such that $\lim_{m+n\to\infty} c^{m+n}\varphi(c^{-m}x, c^{-n}y) = 0$ for all $x, y \in \mathcal{H}$ and for some $1 \neq c > 0$, then there exists a unique (up to a phase-equivalent function) mapping $U : \mathcal{H} \to \mathcal{K}$ satisfying the Wigner equation (W) and such that

$$\|f(x) - Ux\| \le \sqrt{\varphi(x, x)} \quad \textit{for all} \quad x \in \mathcal{H}.$$

This theorem covers also the case of control function $\varphi(x, y) = \varepsilon\|x\|^p\|y\|^p$ for $p \neq 1$. However the case $p = 1$ is again a singular one. Only some partial results for the dimension of the domain equal to 1 or 2 have been obtained.

Theorem 3.3 ([3], Theorem 3)**.** *If $\dim \mathcal{H} = 1$ and $f : \mathcal{H} \to \mathcal{K}$ satisfies*

$$\big||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\big| \le \varepsilon\|x\|\|y\| \tag{9}$$

with $\varepsilon \in [0, 1)$, then there exists a mapping U satisfying the Wigner equation (W) and such that

$$\|f(x) - Ux\| \le 2\sqrt{\varepsilon}\|x\|, \quad x \in \mathcal{H}.$$

Theorem 3.4 ([6], Corollary 2)**.** *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ if $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies* (9)*, then there exists a mapping $U : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the Wigner equation on $\mathbb{R}^2$ and such that*

$$\|f(x) - Ux\| \leq \delta(\varepsilon)\|x\|, \;\; x \in \mathbb{R}^2$$

for some function $\delta : (0, \varepsilon_0) \to \mathbb{R}_+$ satisfying the condition $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$.

Question 6. Is the Wigner equation stable also for domains of dimension greater than two?

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Chapter 15

THE FRANKL PROBLEM FOR SECOND ORDER NONLINEAR EQUATIONS OF MIXED TYPE WITH NON-SMOOTH DEGENERATE CURVE

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Abstract

In [1]–[6], the authors posed and discussed the Tricomi and Frankl problems of some second order equations of mixed type, but they only consider some special mixed equations. In [3], the authors discussed the uniqueness of solutions of Tricomi problem for some second order mixed equation with nonsmooth degenerate line. The present paper deals with the Tricomi and Frankl problems for second order nonlinear mixed equations with non-smooth degenerate curve, we first give the formulation of the Tricomi problem, and derive some estimates and existence of solutions of the Tricomi problem for the equations with nonsmooth degenerate line. Finally we discuss the Frankl problem for the above mixed equations with non-smooth degenerate curve. Thus the results obtained in this paper generalize the results obtained by the authors of [2]–[6].

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Key words: Tricomi and Frankl problems, nonlinear equations of mixed type, nonsmooth degenerate line.

1. Formulation of Tricomi Problem for Mixed Equations

Let D be a simply connected bounded domain in the complex plane $\mathbf{C}$ with the boundary $\partial D = \Gamma \cup L$, where $\Gamma(\subset \{x > 0, y > 0\}) \in C^2_\mu$ $(0 < \mu < 1)$ is a curve with the end points $z = 1, i$, and $L = L_1 \cup L_2 \cup L_3 \cup L_4$, where L_1, L_2, L_3, L_4 are four characteristics with the slopes $-H_2(x)/H_1(y)$, $H_2(x)/H_1(y)$, $-H_2(x)/H_1(y)$, $H_2(x)/H_1(y)$ passing through the points $z = x + iy = 0, 1, 0, i$ respectively as follows

$$
\begin{aligned}
L_1 &= \left\{ -G_1(y) = -\int_0^y H_1(t)\,dt = G_2(x) = \int_0^x H_2(t)\,dt,\ x \in (0, x_1) \right\}, \\
L_2 &= \left\{ -G_1(y) = -\int_0^y H_1(t)\,dt = \int_x^1 H_2(t)\,dt = G_2(1) - G_2(x),\ x \in (x_1, 1) \right\}, \\
L_3 &= \left\{ G_1(y) = \int_0^y H_1(t)\,dt = -\int_0^x H_2(t)\,dt = -G_2(x),\ y \in (0, y_2) \right\}, \\
L_4 &= \left\{ G_1(1) - G_1(y) = \int_y^1 H_1(t)\,dt = -\int_0^x H_2(t)\,dt = -G_2(x),\ y \in (y_2, 1) \right\}.
\end{aligned} \tag{1.1}
$$

Here $H_1(y) = \sqrt{|K_1(y)|}, H_2(x) = \sqrt{|K_2(x)|}$, $K_1(0) = 0$, $K_2(0) = 0$, $K_1(y) = \mathrm{sgn} y|y|^{m_1} h_1(y)$, $K_2(x) = \mathrm{sgn} x|x|^{m_2} h_2(x)$ are continuous in $\overline{D}$, possess the first order derivative and $yK_1(y) > 0$ on $y \neq 0$, $xK_2(x) > 0$ on $x \neq 0$, $m_1, m_2\,(< \min(1, m_1))$ are positive constants, $h_1(y)$, $h_2(x)$ in $\overline{D}$ are continuously differentiable positive functions, and (x_1, y_1), (x_2, y_2) are the intersection points of L_1, L_2 and L_3, L_4 respectively. There is no harm in assuming that the boundary Γ of the domain D is a smooth curve, which possesses the form $G_2(x) = G_2(1) - G_1(y)$ and $G_1(y) = G_1(1) - G_2(x)$ near the points $z = 1$ and i respectively. Denote $D^+ = D \cap \{x > 0, y > 0\}$, $D^- = D_1^- \cup D_2^-$, $D_1^- = D \cap \{y < 0\}$, $D_2^- = D \cap \{x < 0\}$. In this paper we use the notation of the complex number in D^+ and the hyperbolic number in D^- (see [10]).

Now we introduce the second order nonlinear equation of mixed type with nonsmooth degenerate line

$$
Lu = K_1(y)u_{xx} + K_2(x)u_{yy} + au_x + bu_y + c^* u = -d \ \text{ in } \ \overline{D}, \tag{1.2}
$$

where $c^* = c - |u|^\sigma$, a, b, c, d are real functions of $z(\in \overline{D})$, $u, u_x, u_y\,(\in \mathbf{R})$, σ is a non-negative constant, and suppose that the equation (1.1) satisfies **Condition** C:

1) The coefficients a, b, c, d are measurable in D^+ and continuous in $\overline{D^-}$ for any continuously differentiable function $u(z)$ in $D^* = \overline{D} \backslash Z'$, $Z' = \{0, 1, i\}$, and satisfy

$$
\begin{gathered}
L_\infty[\eta, \overline{D^+}] \le k_0, \ \ \eta = a, b, c, \ \ L_\infty[d, \overline{D^+}] \le k_1, \ \ c \le 0 \ \text{ in } \ D^+, \\
\hat{C}[d, \overline{D^-}] = C[d, \overline{D^-}] + C[d_x, \overline{D^-}] \le k_1, \quad \hat{C}[\eta, \overline{D^-}] \le k_0, \eta = a, b, c, \\
|a|/H_1 = o(1) \ \text{ as } \ y = \mathrm{Im} z (z \in D_1^-) \to 0, \ \ m_1 \ge 2, \\
|b|/H_2 = o(1) \ \text{ as } \ x = \mathrm{Re} z (z \in D_2^-) \to 0, \ \ m_2 \ge 2, \\
|\eta|/H_1 H_2, \ \ |\eta|/H_1^2, \ \ |\eta|/H_2^2 = o(1) \ \text{ as } \ z\,(\in D^-) \to 0, \ \ \eta = a, b, \\
\eta|x|^{-m_2/2}, \eta_x|x|^{-m_2/2-1}, \eta|y|^{-m_1/2}, \eta_y|y|^{-m_1/2-1} = O(1) \ \text{ as } \ z \to 0, \ \ \eta = c, d,
\end{gathered} \tag{1.3}
$$

in which $k_0\,(\ge \max[2\sqrt{h(y)}, 1/\sqrt{h(y)}, 1]$, $k_1\,(\ge \max[6k_0, 1])$ are positive constants.

2) For any continuously differentiable functions $u_1(z), u_2(z)$ in D^*, $F(z,u,u_z) = au_x+bu_y+cu+d$ satisfies the following condition

$$F(z,u_1,u_{1z})-F(z,u_2,u_{2z})=\tilde{a}(u_1-u_2)_x+\tilde{b}(u_1-u_2)_x+\tilde{c}(u_1-u_2) \text{ in } \overline{D},$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ satisfy the conditions as those of a, b, c. Obviously equation (1.1) with the condition $K_2(x)=1$, $a=b=c=d=0$ is the so-called Chaplygin equation.

If $H_1(y)=[|y|^{m_1}h_1]^{1/2}$, $H_2(x)=[|x|^{m_2}h_2(x)]^{1/2}$ as stated before, then we have

$$\begin{aligned} &Y=G_1(y)=\int_0^y H_1(t)\,dt, \quad |Y|\le\frac{k_0}{m_1+2}|y|^{(m_1+2)/2},\\ &X=G_2(x)=\int_0^x H_2(t)dt,\ |X|\le\frac{k_0}{m_2+2}|x|^{(m_2+2)/2} \text{ in } \overline{D^\pm}, \end{aligned} \tag{1.4}$$

and their inverse functions $y=\pm|(G_1)^{-1}(Y)|$, $y=\pm|(G_2)^{-1}(X)|$ satisfy the inequalities

$$\begin{aligned} &|y|=|(G_1)^{-1}(Y)|\le\left(\frac{k_0(m_1+2)}{2}\right)^{2/(m_1+2)}|Y|^{2/(m_1+2)}=J_1|Y|^{2/(m_1+2)},\\ &|x|=|(G_2)^{-1}(X)|\le\left(\frac{k_0(m_2+2)}{2}\right)^{2/(m_2+2)}|X|^{2/(m_2+2)}=J_2|X|^{2/(m_2+2)}. \end{aligned} \tag{1.5}$$

The Tricomi problem for equation (1.2) may be formulated as follows:

Problem T. Find a continuous solution $u(z)$ of (1.1) in $\overline{D}\backslash\{0\}$, where u_x, u_y are continuous in $D^*=\overline{D}\backslash\{1,i,0\}$, and satisfy the boundary conditions

$$u(z)=\phi(z) \text{ on } \Gamma, \quad u(z)=\psi_1(x) \text{ on } L_2, \quad u(z)=\psi_2(y) \text{ on } L_4, \tag{1.6}$$

where $\phi(1)=\psi_1(1), \phi(i)=\psi_2(i)$, and $\phi(z),\psi_1(x),\psi_2(y)$ satisfy the conditions

$$C^2_\alpha[\phi(z),\Gamma]\le k_2,\ \ C^2_\alpha[\psi_1(x),L_2]\le k_2,\ \ C^2_\alpha[\psi_2(y),L_4]\le k_2, \tag{1.7}$$

in which α $(0<\alpha<1)$, k_2 are positive constants.

If the boundary Γ near $z=1,\ i$ possesses the form

$$G_2(x)=G_2(1)-G_1(y), \quad G_1(y)=G_1(1)-G_2(x)$$

respectively, we find the derivative for (1.6) according to the parameter $s=\operatorname{Re}z=x$ on Γ

near $z=1$ and the parameter $s=\operatorname{Im} z=y$ on Γ near $z=i$, and obtain

$$
\begin{gathered}
u_s=u_x+u_y y_x=u_x-H_2(x)u_y/H_1(y)=\phi'(x),\ \text{ i.e.}\\
H_1(y)u_x-H_2(x)u_y=H_1(y)\phi'(x)\ \text{ on }\ \Gamma\ \text{ near }\ z=1,\\
u_s=u_x x_y+u_y=-H_1(y)u_x/H_2(x)+u_y=\phi'(y),\ \text{ i.e.}\\
H_1(y)u_x-H_2(x)u_y=-H_2(x)\phi'(y)\ \text{ on }\ \Gamma\ \text{ near }\ z=i,\\
u_s=u_x+u_y y_x=u_x+H_2(x)u_y/H_1(y)=\psi_1'(x),\ \text{ i.e.}\\
H_1(y)u_x+H_2(x)u_y=H_1(y)\psi_1'(x)\ \text{ on }\ L_2,\\
u_s=u_x x_y+u_y=H_1(y)u_x/H_2(x)+u_y=\psi_2'(y),\ \text{ i.e.}\\
H_1(y)u_x+H_2(x)u_y=H_2(x)\psi_2'(y)\ \text{ on }\ L_4,\\
H_2(x)u_y(x)=2\tilde R_0(x)\ \text{ or }\ H_1(y)u_x(y)=0\ \text{ on }\ L_0',\\
H_1(y)u_x(y)=2\hat R_0(y)\ \text{ or }\ H_2(x)u_y(x)=0\ \text{ on }\ L_0'',
\end{gathered}
\tag{1.8}
$$

where $L_0'=\{0\le x\le 1, y=0\}$, $L_0''=\{x=0, 0\le y\le 1\}$, $L_0=L_0'\cup L_0''$, and $\tilde R_0(x)$, $\hat R_0(y)$ are undetermined real functions. It is clear that the complex form of (1.8) is as follows

$$
\begin{aligned}
\operatorname{Re}\big[\overline{\lambda(z)}(U+iV)\big]&=\operatorname{Re}\big[\overline{\lambda(z)}(H_1(y)u_x-iH_2(x)u_y)\big]/2=R(z)\ \text{ on }\ \Gamma\cup L_0,\\
\operatorname{Re}\big[\overline{\lambda(z)}(U+jV)\big]&=\operatorname{Re}\big[\overline{\lambda(z)}(H_1(y)u_x-jH_2(x)u_y)\big]/2=R(z)\ \text{ on }\ L_2\cup L_4,\\
\operatorname{Im}\big[\overline{\lambda(z)}(U+jV)\big]_{z=z_1}&=\operatorname{Im}\big[\overline{\lambda(z)}(H_1(y)u_x-jH_2(x)u_y)\big]/2|_{z=z_1}=c_1,\\
\operatorname{Im}\big[\overline{\lambda(z)}(U+jV)\big]_{z=z_2}&=\operatorname{Im}\big[\overline{\lambda(z)}(H_1(y)u_x-jH_2(x)u_y)\big]/2|_{z=z_2}=c_2,
\end{aligned}
\tag{1.9}
$$

where j is the hyperbolic unit such that $j^2=1$,

$$
\begin{gathered}
U(z)=\frac{H_1(y)}{2}u_x,\quad V(z)=-\frac{H_2(x)}{2}v_y,\quad d_1=\phi(1)=b_0,\\
c_1=\frac{1}{2\sqrt2}\big[-H_1(y_1)\psi_1'(x_1)\big],\quad c_2=\frac{1}{2\sqrt2}\big[-H_2(x_2)\psi_2'(y_2)\big],
\end{gathered}
$$

and

$$
\lambda(z)=\begin{cases}(1+i)/\sqrt2,\\(1+i)/\sqrt2,\\(1+j)/\sqrt2,\\(1+j)/\sqrt2,\\1\ \text{ or }\ i,\\ i\ \text{ or }\ 1,\end{cases}\qquad
R(z)=\begin{cases}H_1(y)\phi'(x)/2\sqrt2 & \text{on }\ \Gamma\ \text{ at }\ z=1,\\-H_2(x)\phi'(y)/2\sqrt2 & \text{on }\ \Gamma\ \text{ at }\ z=i,\\H_1(y)\psi_1'(x)/2\sqrt2 & \text{on }\ L_2,\\H_2(x)\psi_2'(y)/2\sqrt2 & \text{on }\ L_4,\\0\ \text{ or }\ \tilde R_0(x) & \text{on }\ L_0'\\\hat R_0(y)\ \text{ or }\ 0 & \text{on }\ L_0''.\end{cases}
\tag{1.10}
$$

where $\tilde R_0(x)$, $\hat R_0(y)$ are as stated before. Denoting $t_1=1$, $t_2=i$, $t_3=0$, we have

$$
\begin{aligned}
e^{i\phi_1}&=\frac{\lambda(t_1-0)}{\lambda(t_1+0)}=e^{0\pi i-\pi i/4}=e^{-\pi i/4},\quad \gamma_1=-\frac14-K_1=-\frac14,\quad K_1=0,\\
e^{i\phi_2}&=\frac{\lambda(t_2-0)}{\lambda(t_2+0)}=e^{\pi i/4-\pi i/2}=e^{-\pi i/4},\gamma_2=-\frac14-K_2=-\frac14,\quad K_2=0,\\
e^{i\phi_3}&=\frac{\lambda(t_3-0)}{\lambda(t_3+0)}=e^{\pi i/2-0\pi i}=e^{\pi i/2},\quad \gamma_3=\frac{\pi/2}{\pi}-K_3=\frac12,\quad K_3=0,
\end{aligned}
\tag{1.11}
$$

in which we consider $\mathrm{Re}[W(z)] = H_1(y)u_x/2 = 0$ on L'_0 and $\mathrm{Im}[\bar{i}W(z)] = -H_2(x)u_y/2 = 0$ on L''_0, thus the index of $\lambda(z)$ on $\partial D^+ = \Gamma \cup L_0$ is

$$K = (K_1 + K_2 + K_3)/2 = 0. \tag{1.12}$$

Obviously the Tricomi problem for Chaplygin equation is a special case of Problem T for equation (1.2).

Noting that $\phi(z) \in C^2_\alpha(\Gamma)$, $\psi_1(x) \in C^2(L_2)$, $\psi_2(y) \in C^2_\alpha(L_4)$ $(0 < \alpha < 1)$, we can find two twice continuously differentiable functions $u_0^\pm(z)$ in $\overline{D}^\pm$, for instance, which are the solutions of the Dirichlet problem with the boundary condition on $\Gamma \cup L_2 \cup L_4$ in (1.6) for harmonic equations in $D^\pm$, thus the functions $v(z) = v^\pm(z) = u(z) - u_0^\pm(z)$ in D is the solution of the equation in the form

$$Lv = K_1(y)v_{xx} + K_2(x)v_{yy} + \hat{a}v_x + \hat{b}v_y + \hat{c}^* v = -\hat{d} \ \text{ in } \ D \tag{1.13}$$

satisfying the corresponding boundary conditions

$$\begin{gathered} v(z) = 0 \ \text{ on } \ \Gamma\cup L_2\cup L_4, \ \text{ i.e. } \mathrm{Re}[\overline{\lambda(z)}W(z)] = R(z) \ \text{ on } \ \Gamma\cup L_2\cup L_4, \\ v(1) = b_0, \ \ v(0) = 0, \end{gathered} \tag{1.14}$$

where the coefficients of (1.13) satisfy the conditions similar to Condition C, $W(z) = U + iV = v^+_{\tilde{z}}$ in D^+ and $W(z) = U + jV = v^-_{\tilde{z}}$ in $\overline{D^-}$, hence later on we only discuss the case of $R(z) = 0$ on $\Gamma \cup L_2 \cup L_4$ and $c_1 = c_2 = d_1 = 0$ in (1.14) and the case of index $K = 0$, which is called Problem $\tilde{T}$, the other case can be similarly discussed. From $v(z) = v^\pm(z) = u(z) - u_0^\pm(z)$ in $\overline{D^\pm}$, we have $u(z) = v^\pm(z) + u_0^\pm(z)$ in $\overline{D^\pm}$, and

$$\begin{gathered} v^+(z) = v^-(z) - u_0^+(z) + u_0^-(z) \ \text{ on } \ L_0, \\ u_y = v_y^\pm + u_{0y}^\pm, v_y^+ = v_y^- - u_{0y}^+ + u_{0y}^- = 2\hat{R}_1(x), v_y^- = 2\tilde{R}_1(x) \ \text{ on } \ L'_0, \\ u_x = v_x^\pm + u_{0x}^\pm, v_x^+ = v_x^- - u_{0x}^+ + u_{0x}^- = 2\hat{R}_2(y), v_x^- = 2\tilde{R}_2(y) \ \text{ on } \ L''_0. \end{gathered}$$

2. Representation of Solutions of Tricomi Problem for Mixed Equations

In this section, we first write the complex form of equation (1.2). Denote

$$\begin{gathered} W(z) = U + iV = \frac{1}{2}\left[H_1(y)u_x - iH_2(x)u_y\right] = u_{\tilde{z}} = \frac{H_1(y)H_2(x)}{}2[u_X - iu_Y], \\ H_1(y)H_2(x)W_{\overline{Z}} = \frac{H_1(y)H_2(x)}{2}[W_X + iW_Y] = \frac{1}{2}\left[H_1(y)W_x + iH_2(x)W_y\right] = W_{\overline{\tilde{z}}} \ \text{ in } \ \overline{D^+}, \end{gathered} \tag{2.1}$$

we have

$$\begin{aligned}
&H_1(y)H_2(x)W_{\overline{Z}}=H_1H_2[W_X+iW_Y]/2=H_1H_2\big[(U+iV)_X+i(U+iV)_Y\big]/2\\
&=iH_1H_2\big[(U+V)-i(U-V)\big]_{\mu+i\nu}=iH_1H_2\overline{\big[(U+V)+i(U-V)\big]_{\mu-i\nu}}\\
&=\Big\{\big[iH_2H_{1y}/H_1-a/H_1+H_1H_{2x}/H_2-ib/H_2\big]W\\
&+\big[iH_1H_{2x}/H_2-a/H_1-H_1H_{2x}/H_2+ib/H_2\big]\overline{W}-c^*u-d\Big\}/4\\
&=A_1(z)W+A_2(z)\overline{W}+A_3(z)u+A_4(z)=g(Z),\quad\text{i.e.}\\
&\big[(U+V)+i(U-V)\big]_{\mu-i\nu}\\
&=\Big\{2[H_2H_{1y}/H_1]U+2[H_1H_{2x}/H_2]V-i[au_x+bu_y+c^*u+d]\Big\}/(4H_1H_2)\\
&=i\overline{g(Z)}\ \text{ in }\ D^+_Z,
\end{aligned}\tag{2.2}$$

in which D^+_Z, D^+_τ are the image domains of D^+ with respect to the mapping $Z=Z(z)=X+iY$, $\tau=\mu+i\nu=\tau(z)$ respectively, and

$$\mu=G_2(x)+G_1(y)=X+Y,\ \ \nu=G_2(x)-G_1(y)=X-Y\ \text{ in }\ D^+.\tag{2.3}$$

Similarly introduce the hyperbolic unit j such that $j^2=1$, we can obtain

$$\begin{aligned}
&W(z)=U+jV=\frac{1}{2}\big[H_1(y)u_x-jH_2(x)u_y\big]=\frac{H_1(y)H_2(x)}{2}[u_X-ju_Y]=H_1(y)H_2(x)u_Z,\\
&H_1(y)H_2(x)W_{\overline{Z}}=\frac{H_1(y)H_2(x)}{2}[W_X+jW_Y]=\frac{1}{2}\big[H_1(y)W_x+jH_2(x)W_y\big]=W_{\overline{z}}\ \text{ in }\ \overline{D^-},\\
&-K_1(y)u_{xx}-K_2(x)u_{yy}=H_1(y)\big[H_1(y)u_x-jH_2(x)u_y\big]_x\\
&+jH_2(x)\big[H_1(y)u_x-jH_2(x)u_y\big]_y-jH_2(x)H_{1y}u_x+jH_1(y)H_{2x}u_y\\
&=4H_1(y)H_2(x)W_{\overline{Z}}-j[H_2H_{1y}/H_1]H_1u_x+j[H_1H_{2x}/H_2]H_2u_y=au_x+bu_y+c^*u+d,\ \text{ i.e.}\\
&H_1(y)H_2(x)W_{\overline{Z}}=H_1H_2[W_X+jW_Y]/2=H_1H_2\big\{(U+V)_\mu e_1+(U-V)_\nu e_2\big\}\\
&=\Big\{2j\big[H_2H_{1y}/H_1\big]U+2j\big[H_1H_{2x}/H_2\big]V+au_x+bu_y+c^*u+d\Big\}/4\\
&\Big\{\big[jH_2H_{1y}/H_1+a/H_1\big](W+\overline{W})+\big[H_1H_{2x}/H_2-jb/H_2\big](W-\overline{W})+c^*u+d\Big\}/4\\
&=\Big\{\big[a/H_1+H_1H_{2x}/H_2+H_2H_{1y}/H_1-b/H_2\big](U+V)\\
&+\big[a/H_1-H_1H_{2x}/H_2+H_2H_{1y}/H_1+b/H_2\big](U-V)+c^*u+d\Big\}e_1/4\\
&+\Big\{\big[a/H_1-H_1H_{2x}/H_2-H_2H_{1y}/H_1-b/H_2\big](U+V)\\
&+\big[a/H_1+H_1H_{2x}/H_2-H_2H_{1y}/H_1+b/H_2\big](U-V)+c^*u+d\Big\}e_2/4,\quad\text{i.e.}\\
&\left.\begin{aligned}(U+V)_\mu&=\hat{A}_1(U+V)+\hat{B}_1(U-V)+\hat{C}_1u+\hat{D}_1,\\ (U-V)_\nu&=\hat{A}_2(U+V)+\hat{B}_2(U-V)+\hat{C}_2u+\hat{D}_2,\end{aligned}\right.\ \text{ in }\ D^-_\tau,
\end{aligned}\tag{2.4}$$

in which $e_1=(1+j)/2$, $e_2=(1-j)/2$, D_Z^-, D_τ^- are the image sets of D_1^- with respect to the mapping $Z=Z(z), \tau=\mu+j\nu=\tau(z)$ respectively, and

$$\begin{aligned}
\hat{A}_1&=\frac{1}{4H_1H_2}\left[\frac{a}{H_1}+\frac{H_1H_{2x}}{H_2}+\frac{H_2H_{1y}}{H_1}-\frac{b}{H_2}\right],\ \ \hat{C}_1=\frac{c^*}{4H_1H_2}\,,\\
\hat{B}_1&=\frac{1}{4H_1H_2}\left[\frac{a}{H_1}-\frac{H_1H_{2x}}{H_2}+\frac{H_2H_{1y}}{H_1}+\frac{b}{H_2}\right],\ \ \hat{C}_2=\frac{c^*}{4H_1H_2}\,,\\
\hat{A}_2&=\frac{1}{4H_1H_2}\left[\frac{a}{H_1}-\frac{H_1H_{2x}}{H_2}-\frac{H_2H_{1y}}{H_1}-\frac{b}{H_2}\right],\ \ \hat{D}_1=\frac{d}{4H_1H_2},\\
\hat{B}_2&=\frac{1}{4H_1H_2}\left[\frac{a}{H_1}+\frac{H_1H_{2x}}{H_2}-\frac{H_2H_{1y}}{H_1}+\frac{b}{H_2}\right],\ \ \hat{D}_2=\frac{d}{4H_1H_2}\ \text{ in }\ D^-.
\end{aligned}\tag{2.5}$$

For the domain D_2^-, we can also write the coefficients of equation (2.4) in D_τ^-, where $\tau=\mu+j\nu=G_1(y)+G_2(x)+j[G_1(y)-G_2(x)]$. It is clear that a special case of (2.2), (2.4) is the complex equation

$$W_{\overline{Z}}=0\ \text{ in }\ D_Z^+\cup D_Z^-.\tag{2.6}$$

The boundary value problem for equations (2.2), (2.4) with the boundary condition (1.14) and the relation: the first formula in (2.7) below will be called Problem A. Here we mention that if we denote $\mu=x+G_1(y)$, $\nu=x-G_1(y)$ in D_1^-, and $\mu=G_2(x)+y$, $\nu=G_2(x)-y$ in D_2^-, then the last system in (2.4) is true still.

Now we state and verify the representation of solutions of Problem T for equation (1.2).

Theorem 2.1. *Under Condition C, any solution $u(z)$ of Problem T for equation* (1.2) *in $\overline{D}$ can be expressed as follows*

$$\begin{gathered}
u(z)=u(x)-2\int_0^y\frac{V(z)}{H_2(x)}\,dy=2\mathrm{Re}\int_1^z\left[\frac{\mathrm{Re}w}{H_1(y)}+\binom{i}{-j}\frac{\mathrm{Im}w}{H_2(x)}\right]dz+b_0\ \text{ in }\left(\frac{\overline{D^+}}{\overline{D^-}}\right),\\
w(z)=\Phi(Z)+\Psi(Z)=\hat{\Phi}(Z)+\hat{\Psi}(Z),\quad T(Z)=-\frac{1}{\pi}\iint_{D_t}\frac{f(t)}{t-Z}\,d\sigma_t,\\
\Psi(Z)=T(Z)+\overline{T(\overline{Z})},\quad \hat{\Psi}(Z)=T(Z)-\overline{T(\overline{Z})}\ \text{ in }\ \overline{D_Z^+},\\
w(z)=\phi(z)+\psi(z)=\xi(z)e_1+\eta(z)e_2\ \text{ in }\ \overline{D^-},\\
\eta(z)=-\int_0^\nu\frac{g_2(y)}{4H_1(y)H_2(x)}\,d\nu=\theta(z)+\int_0^y g_2(z)\,dy=\int_{S_2}g_2(y)\,dy+\int_0^y g_2(z)\,dy\\
=\int_{y_0}^{|y|}\hat{g}_2(z)\,dy,\quad z\in s_2,
\end{gathered}\tag{2.7}$$

$$\begin{aligned}
&\xi(z)=\zeta(z)+\int_0^y g_1(z)\,dy,\quad z\in s_1,\\
&g_l(z)=\tilde A_l(U+V)+\tilde B_l(U-V)+2\tilde C_lU+\tilde D_lu+\tilde E_l,\quad l=1,2,\\
&\xi(z)=\zeta(z)+\int_0^x g_1(z)\,dx,\quad z\in s_1,\eta(z)=\theta(z)+\int_0^x g_2(z)\,dx,\quad z\in s_2,\\
&g_l(z)=\hat A_l(U+V)+\hat B_l(U-V)+2\hat C_lV+\hat D_lu+\hat E_l,\quad l=1,2,
\end{aligned}\tag{2.7}$$

in which $Z=X+iY=G_2(x)+iG_1(y)$, $f(Z)=g(Z)/H_1H_2$, $U=H_1u_x/2$, $V=-H_2u_y/2$, $\zeta(z)e_1+\theta(z)e_2$ *is a solution of* (2.6) *in* D_Z^-, s_1, s_2 *are two families of characteristics in* D^-:

$$s_1:\ \frac{dx}{dy}=\frac{H_1(y)}{H_2(x)},\quad s_2:\ \frac{dx}{dy}=-\frac{H_1(y)}{H_2(x)}\tag{2.8}$$

passing through the point $z=x+jy\in D^-$, S_1, S_2 *are the characteristic curves from the points on* L_1, L_2 *to two points on* L_0' *respectively,*

$$\theta(z)=\int_{S_2} g_2(z)\,dy,\quad \eta(z)=-\int_0^\nu\big[g_2(z)/4H_1(y)H_2(x)\big]\,d\nu$$

is the integral along characteristic curve s_1 *from a point* $z_0=x_0+jy_0$ *on* L_2 *to the point* $z=x+jy\in\overline{D_Z^-}$, $\theta(x)=-\zeta(x)$ *on* L_0', *and* $\zeta(z)=-\theta(G_2(x)-G_1(y))$ *on the characteristic curves of* s_1,s_2 *passing through the point* $z=x$ *respectively, and*

$$\begin{aligned}
&w(z)=U(z)+jV(z)=\frac12 H_1u_x-\frac j2 H_2u_y,\\
&\xi(z)=\mathrm{Re}\psi(z)+\mathrm{Im}\psi(z),\quad \eta(z)=\mathrm{Re}\psi(z)-\mathrm{Im}\psi(z),\\
&\tilde A_1=\frac14\left[\frac{h_{1y}}{h_1}+\frac{H_1h_{2x}}{H_2h_2}-\frac{2b}{H_2^2}\right],\quad \tilde B_1=\frac14\left[\frac{h_{1y}}{h_1}-\frac{H_1h_{2x}}{H_2h_2}+\frac{2b}{H_2^2}\right],\\
&\tilde A_2=\frac14\left[\frac{h_{1y}}{h_1}+\frac{H_1h_{2x}}{H_2h_2}+\frac{2b}{H_2^2}\right],\quad \tilde B_2=\frac14\left[\frac{h_{1y}}{h_1}-\frac{H_1h_{2x}}{H_2h_2}-\frac{2b}{H_2^2}\right],\\
&\hat A_1=\frac14\left[\frac{H_2h_{1y}}{H_1h_1}+\frac{h_{2x}}{h_2}+\frac{2a}{H_1^2}\right],\quad \hat B_1=\frac14\left[\frac{H_2h_{1y}}{H_1h_1}-\frac{h_{2x}}{h_2}+\frac{2a}{H_1^2}\right],\\
&\hat A_2=\frac14\left[-\frac{H_2h_{1y}}{H_1h_1}-\frac{h_{2x}}{h_2}+\frac{2a}{H_1^2}\right],\quad \hat B_2=\frac14\left[-\frac{H_2h_{1y}}{H_1h_1}+\frac{h_{2x}}{h_2}+\frac{2a}{H_1^2}\right],\\
&\tilde C_1=\frac{a}{2H_1H_2}+\frac{m_1}{4y},\quad \tilde C_2=-\frac{a}{2H_1H_2}+\frac{m_1}{4y},\quad \tilde D_1=-\tilde D_2=\frac{c^*}{2H_2},\\
&\tilde E_1=-\tilde E_2=\frac{d}{2H_2}\ \text{ in }\ D_1^-,\quad \hat C_1=-\frac{b}{2H_1H_2}+\frac{m_2}{4x},\\
&\hat C_2=-\frac{b}{2H_1H_2}-\frac{m_2}{4x},\quad \hat D_1=\hat D_2=\frac{c^*}{2H_1},\quad \hat E_1=\hat E_2=\frac{d}{2H_1}\ \text{ in }\ D_2^-,
\end{aligned}$$

in which

$$H_1(y) = \left[|y|^{m_1}h_1(y)\right]^{1/2}, \quad H_2(x) = \left[|x|^{m_2}h_1(x)\right]^{1/2},$$

herein $h_1(y)$, $h_2(x)$ are positive continuously differentiable functions.

Proof. Here and later on we only discuss the integrals in $\overline{D_1^-}$, the case in $\overline{D_2^-}$ can be similarly discussed. From (2.4) it is easy to see that equation (1.2) in $\overline{D_1^-}$ can be reduced to the system of integral equations: (2.7). Moreover we can extend the equation (2.4) onto the the symmetrical domain $\hat{D}_Z$ of D_{1Z}^- with respect to the real axis $\mathrm{Im}Z = 0$, namely introduce the function $\hat{W}(Z)$ as follows:

$$\hat{W}(Z) = \begin{cases} W[z(Z)], \\ -\overline{W[z(\overline{Z})]}, \end{cases} \qquad \hat{u}(z) = \begin{cases} u(Z) & \text{in } D_{1Z}^-, \\ -u(\overline{Z}) & \text{in } \hat{D}_Z, \end{cases}$$

and then the equation (2.4) is extended as

$$\hat{W}_{\overline{z}} = \hat{A}_1\hat{W} + \hat{A}_2\overline{\hat{W}} + \hat{A}_3\hat{u} + \hat{A}_4 = \tilde{g}(Z) \text{ in } \overline{D_{1Z}^-} \cup \overline{\hat{D}_Z},$$

where

$$\begin{gathered} \hat{A}_l(Z) = \begin{cases} A_l(Z), \\ \overline{\tilde{A}_l(\overline{Z})}, \end{cases} \quad l=1,2,3, \quad \hat{A}_4(Z) = \begin{cases} A_4(Z), \\ -\overline{A_4(\overline{Z})}, \end{cases} \\ \tilde{g}_l(Z) = \begin{cases} g_l(z) & \text{in } \overline{D_{1Z}^-}, \\ -\overline{g_l(\overline{Z})} & \text{in } \overline{\hat{D}_Z}, \end{cases} \quad l=1,2, \end{gathered} \tag{2.9}$$

here $\tilde{A}_1(\overline{Z}) = A_2(\overline{Z})$, $\tilde{A}_2(\overline{Z}) = A_1(\overline{Z})$, $\tilde{A}_3(\overline{Z}) = A_3(\overline{Z})$, and we mention that in general $\hat{u}(z)$ on L_0' may not be continuous. It is easy to see that the system of integral equations (2.9) can be written in the form

$$\begin{gathered} \eta(z) = \theta(z) + \int_0^{\tilde{y}} g_2(z)\,dy = \int_{y_0}^{\hat{y}} \tilde{g}_2(z)\,dy, \\ \xi(z) = \zeta(z) + \int_0^{y} g_1(z)dy = \int_{y_0}^{\hat{y}} \tilde{g}_1(z)\,dy, \quad \hat{z} = x + j\hat{y} = x + j|y| \text{ in } \tilde{D}_1^-, \end{gathered} \tag{2.10}$$

where $x_0 + jy_0$ is the intersection point of L_2 and the characteristic curve s_1 passing through $z = x + jy \in \overline{D_1^-}$, the function $\zeta(z)$ is determined by $\theta(z)$, i.e. the function $\zeta(z)$ can be defined by $\zeta(z) = -\theta(z) = -\theta(G_2(x) - G_1(y))$, for the extended integral, which can be appropriately defined in D_{1Z}^-, for convenience later on the above form $g_2(z)$ is written still, and the numbers $\hat{y} - y_0$, $\hat{t} - y_0$ will be written by $\tilde{y}$, $\tilde{t}$ respectively. $\square$

3. Existence of Solutions of Tricomi Problem for Mixed Equations

For proving the existence of solutions of Tricomi problem for mixed equations with nonsmooth degenerate line in D, we first give the estimates of the solutions of Problem $\tilde{T}$ for

(1.2) in $\overline{D_Z}=\overline{D_Z^+}$. It is clear that Problem $\tilde{T}$ is equivalent to Problem A for the complex equation

$$\begin{gathered}W_{\overline{Z}}=\frac{1}{H_1H_2}\left[A_1W+A_2\overline{W}+A_3u+A_4\right]\ \text{ in }\ D_Z,\\ A_1=\frac{iH_2H_{1y}}{4H_1}+\frac{H_1H_{2x}}{4H_2}-\frac{a}{4H_1}-\frac{ib}{4H_2},\quad A_3=\frac{-c^*}{4},\\ A_2=\frac{iH_2H_{1y}}{4H_1}-\frac{H_1H_{2x}}{4H_2}-\frac{a}{4H_1}+\frac{ib}{4H_2},\quad A_4=\frac{-d}{4},\end{gathered}\tag{3.1}$$

with the boundary condition

$$\mathrm{Re}\big[\overline{\lambda(z)}W(z)\big]=R(z)\ \text{ on }\ \Gamma\cup L_2\cup L_4,\ \ u(1)=d_1,\ \ u(0)=0,\tag{3.2}$$

and the relation

$$u(z)=u(x)-2\int_0^y\frac{V(z)}{H_2(x)}\,dy=2\mathrm{Re}\int_0^z\left[\frac{\mathrm{Re}W}{H_1(y)}+i\frac{\mathrm{Im}W}{H_2(x)}\right]dz\ \text{ in }\ \overline{D^+}.\tag{3.3}$$

As stated in Section 1, we can assume $R(x)=0$ on $\Gamma\cup L_2\cup L_4$ in (3.2), $d_1=0$, $u(0)=0$, because the index $K=0$ of $\lambda(z)$ on ∂D_Z. In the following we first prove that there exists a solution of Problem A^+ for (3.1), (3.3) with the boundary condition (3.2) on Γ and

$$\mathrm{Re}[-iW(x)]=-\frac{1}{2}\,H_2(x)\hat{R}_1(x)\ \text{ on }\ L_0',\quad \mathrm{Re}[W(iy)]=\frac{1}{2}\,H_1(y)\hat{R}_2(y)\ \text{ on }\ L_0'',$$

and the boundary value problem for (3.1), (3.3) with the boundary condition (3.2) on $L_2\cup L_4$ and

$$\mathrm{Re}[-jW(x)]=\frac{H_2(x)}{2}\,\tilde{R}_1(x)=R(z)\ \text{ on }\ L_0',\quad \mathrm{Re}[W(jy)]=\frac{1}{2}\,H_1(y)\tilde{R}_2(y)=R(z)\ \text{ on }\ L_0''$$

will be called Problem A^-, where $\hat{R}_1(x)$, $\hat{R}_2(y)$, $\tilde{R}_1(x)$, $\tilde{R}_2(y)$ are as stated in (1.14). From the method and result in [8]–[11], we know that Problem A^+ for equation (3.1), (3.3) in D^+ has a solution $W(z)$. Hence in the following we only prove the unique solvability of Problem A^- for (3.1), (3.3) in D^-, which is the Darboux type problem (see [2]).

Theorem 3.1. *If equation* (1.1) *satisfies Condition C, then there exists a solution* $[w(z),u(z)]$ *of Problem* A^- *for* (3.1)–(3.3).

Proof. We can only discuss in D_1^-, because the case in D_2^- can be similarly discussed. By using the method in [10], we may only discuss the problem in $D_*=\overline{D_1^-}\cap\{(0\le)a_0=\delta_0\le x\le b_0=1-\delta_0\,(<1),\ -\delta\le y\le 0\}$, and s_1, s_2 are the characteristics of families in Theorem 2.1 emanating from any two points $(a_0,0)$, $(b_0,0)$ $(0\le a_0<b_0<1)$, where δ, δ_0 are sufficiently small positive numbers. In this case, we can omit the function $K_2(x)$, and may only consider the function $K(y)=K_1(y)=-|y|^m h(y)=-|y|^{m_1}h_1(y)$, where $m=m_1$, $h(y)=h_1(y)$ is a continuously differentiable positive function in $\overline{D_1^-}$. It is clear that for two characteristics s_1, s_2 passing through a point $z=x+jy\in\overline{D_1^-}$

and x_1, x_2 are the intersection points with the axis $y=0$ respectively, for any two points $\tilde{z}_1=\tilde{x}_1+j\tilde{y}\in s_1$, $\tilde{z}_2=\tilde{x}_2+j\tilde{y}\in s_2$, we have

$$\begin{gathered}|\tilde{x}_1-\tilde{x}_2|\le|x_1-x_2|\\ =2\left|\int_0^y\sqrt{-K(t)}\,dt\right|\le\frac{2k_0}{m+2}\,|y|^{1+m/2}\le M|y|^{m/2+1}\ \text{ for }\ -\delta\le y\le 0,\end{gathered}\tag{3.4}$$

where M is a positive constant as stated in (3.6) below, and d is the diameter of D_1^-. From Condition C, we can assume that the coefficients of (2.7) possess continuously differentiable with respect to $x\in L_0'$ and satisfy the conditions

$$\begin{gathered}|\tilde{A}_l|,|\tilde{A}_{lx}|,|\tilde{B}_l|,|\tilde{B}_{lx}|,|\tilde{D}_l|,|\tilde{D}_{lx}|\le k_0\le k_1/6,\\ |\tilde{E}_l|,|\tilde{E}_{lx}|\le k_1/2,2\sqrt{h},1/\sqrt{h},|h_y/h|\le k_0\le k_1/6\ \text{ in }\ \bar{D},\quad l=1,2,\end{gathered}\tag{3.5}$$

and we shall use the constants

$$\begin{gathered}M=4\max[M_1,M_2,M_3],\quad M_1=\max\left[8(k_1d)^2,\frac{M_3}{k_1}\right],\\ M_2=\frac{(2+m)k_0d^2}{\delta^{2+m}}\left[4k_1+\frac{4\varepsilon_0+m}{\delta}\right],\quad M_3=2k_1^2\left[d+\frac{1}{2H(y_1')}\right],\\ \gamma=\max\left[2k_1d\delta^{\beta}+\frac{4\varepsilon(y)+m}{2\beta'}\right]<1,\quad -\delta\le y\le 0,\end{gathered}\tag{3.6}$$

and M_l $(l=1,2,3)$ are positive constants, d is the diameter of D, $\beta'=(1+m/2)(1-3\beta)$, $\varepsilon_0=\max_{\overline{D^-}}\varepsilon(z)$, $1/2H(y_1')\le k_0[(m+2)a_0/k_0]^{-m/(2+m)}$, δ, β are sufficiently small positive constants, and y_1' is an appropriately negative number. We choose $v_0=0$, $\xi_0=0$, $\eta_0=0$ and substitute them into the corresponding positions of v, ξ, η in the right-hand sides of (2.7), and by the successive iteration, we find the sequences of functions $\{v_k\}$, $\{\xi_k\}$, $\{\eta_k\}$, which satisfy the relations

$$\begin{aligned}&v_{k+1}(z)=v_{k+1}(x)-2\int_0^y V_k(z)dy=v_{k+1}(x)+\int_0^y(\eta_k-\xi_k)\,dy,\\ &\xi_{k+1}(z)=\zeta_{k+1}(z)+\int_0^y g_{1k}(z)\,dy=\int_{y_0}^{\hat{y}}g_{lk}\,dy,\\ &\eta_{k+1}(z)=\theta_{k+1}(z)+\int_0^y g_{2k}(z)dy=\int_{y_0}^{\hat{y}}g_{2k}(z)\,dy,\\ &\quad g_{lk}(z)=\tilde{A}_l\xi_k+\tilde{B}_l\eta_k+\tilde{C}_l(\xi_k+\eta_k)+\tilde{D}_lv_k+\tilde{E}_l,\quad l=1,2,\quad k=0,1,2,\ldots,\end{aligned}\tag{3.7}$$

setting $\tilde{g}_{lk+1}(z)=g_{lk+1}(z)-g_{lk}(z)$ $(l=1,2)$ and

$$\begin{gathered}\tilde{y}=\hat{y}-y_1,\quad \tilde{t}=\hat{t}-y_1,\quad \tilde{v}_{k+1}(z)=v_{k+1}(z)-v_k(z),\quad \tilde{\xi}_{k+1}(z)=\xi_{k+1}(z)-\xi_k(z),\\ \tilde{\eta}_{k+1}(z)=\eta_{k+1}(z)-\eta_k(z),\quad \tilde{\zeta}_{k+1}(z)=\zeta_{k+1}(z)-\zeta_k(z),\quad \tilde{\theta}_{k+1}(z)=\theta_{k+1}(z)-\theta_k(z),\end{gathered}$$

where $v(x) = u(x) - u_0(x)$ on L_0' as stated before, and $z_1 = x_1 + jy_1$ is the intersection point of the characteristic curve s_1 and the boundary L_2. Moreover we can prove that $\{\tilde{v}_k\}$, $\{\tilde{\xi}_k\}$ $\{\tilde{\eta}_k\}$ $\{\tilde{\zeta}_k\}$, $\{\tilde{\theta}_k\}$ in D_* satisfy the estimates

$$\begin{gathered}
|\tilde{v}_k(z) - \tilde{v}_k(x)|, |\tilde{\xi}_k(z) - \tilde{\zeta}_k(z)|, |\tilde{\eta}_k(z) - \tilde{\theta}_k(z)| \le M'\gamma^{k-1}|y|^{1-\beta}, \\
|\tilde{\xi}_k(z)|, |\tilde{\eta}_k(z)| \le M(M_2|\tilde{y}|)^{k-1}/(k-1)! \le M'\gamma^{k-1}, \\
\left|\tilde{\xi}_k(z_1) - \tilde{\xi}_k(z_2) - \tilde{\zeta}_k(z_1) - \tilde{\zeta}_k(z_2)\right| \le M(M_2|\tilde{y}|)^{k-1}|x_1 - x_2|^{1-\beta}/(k-1)! \\
\le M'\gamma^{k-1}\left[|x_1 - x_2|^{1-\beta} + |x_1 - x_2|^{\beta}|t|^{\beta'}\right], |\tilde{\eta}_k(z_1) - \tilde{\eta}_k(z_2) - \tilde{\theta}_k(z_1) - \tilde{\theta}_k(z_2)|, \\
|\tilde{v}_k(z_1) - \tilde{v}_k(z_2)|,\ |\tilde{\xi}_k(z_1) - \tilde{\xi}_k(z_2)|,\ |\tilde{\eta}_k(z_1) - \tilde{\eta}_k(z_2)| \le M(M_2|\tilde{t}|)^{k-1} \\
\times|x_1 - x_2|^{1-\beta}/(k-1)! \le M'\gamma^{k-1}\left[|x_1 - x_2|^{1-\beta} + |x_1 - x_2|^{\beta}|t|^{\beta'}\right], \\
\left|\tilde{\xi}_k(z) + \tilde{\eta}_k(z) - \tilde{\zeta}_k(z) - \tilde{\theta}_k(z)\right| \le M'\gamma^{k-1}|x_1 - x_2|^{\beta}|y|^{\beta'}, \\
|\tilde{\xi}_k(z) + \tilde{\eta}_k(z)| \le M(M_2|\tilde{y}|)^{k-1}|x_1 - x_2|^{\beta}|y|^{\beta'}/(k-1)! \\
\le M'\gamma^{k-1}|x_1 - x_2|^{1-\beta},\quad 0 \le |y| \le \delta,
\end{gathered} \tag{3.8}$$

where $z = x + jy$, $z = x + jt$ is the intersection point of s_1, s_2 passing through the points z_1, z_2, $\beta' = (1 + m/2)(1 - 3\beta/2)$, β is a sufficiently small positive constant, such that $(2 + m)\beta < 1$, and M' is a sufficiently large positive constant.

On the basis of the above estimate (3.8), the convergence of two sequences of functions $\{M(M_2|\tilde{y}|)^{k-1}/(k-1)!\}$, $\{M'\gamma^{k-1}|y|^{\beta'}\}$ and the comparison test, we can derive that $\{v_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ in D_* uniformly converge to v_*, ξ_*, η_* satisfying the system of integral equations

$$\begin{aligned}
v_*(z) &= v_*(x) - 2\int_0^y V_*\,dy = u_*(x) + \int_0^y (\eta_* - \xi_*)\,dy, \\
\xi_*(z) &= \zeta_*(z) + \int_0^y \left[\tilde{A}_1\xi_* + \tilde{B}_1\eta_* + \tilde{C}_1(\xi_* + \eta_*) + \tilde{D}_1 v_* + \tilde{E}_1\right] dy, \quad z \in s_1, \\
\eta_*(z) &= \theta_*(z) + \int_0^y \left[\tilde{A}_2\xi_* + \tilde{B}_2\eta_* + \tilde{C}_2(\xi_* + \eta_*) + \tilde{D}_2 v_* + \tilde{E}_2\right] dy, \quad z \in s_2,
\end{aligned}$$

and the function $[W_*(z), v_*(z)] = [(\xi_* + \eta_* + j\xi_* - j\eta_*)/2, v_*(z)]$ is a solution of Problem A^- for equation (3.1). Moreover the function $u(z) = v_*(z) + u_0(z)$ is a solution of Problem T for (1.2) in D^-. The proof is finished. Besides by the similar method, we can prove the uniqueness of solution of Problem A^- in D_1 in (3.1)–(3.3) with $c = c^*$. $\square$

From the above discussion, we obtain the following theorem.

Theorem 3.2. *Let equation* (1.2) *with* $c = c^*$ *satisfy Condition C. Then the above Tricomi problem* (*Problem* T) *for* (1.2) *with* $c = c^*$ *has a unique solution.*

4. The Frankl Problem for Mixed Equations

Now we consider some general domains with non-characteristic boundary and prove the solvability of Frankl problem for equation (1.2).

1) Let D be a simply connected bounded domain D in the complex plane $\mathbf{C}$ with the boundary $\partial D = \Gamma \cup L$, where Γ, L are as stated before. Now, we consider the domain D' with the boundary $\Gamma \cup L_1' \cup L_2' \cup L_3' \cup L_4'$, as stated in Section 1, the curve Γ can be replaced by another smooth curve Γ', because it can be realized through a conformal mapping. The parameter equations of the curves L_1', L_2', L_3', L_4' are as follows:

$$\begin{aligned} &L_1'=\{\gamma_1(s)+y=0,\ 0\le s\le s_1'\}, \quad L_2'=\{x-G(y)=1,\ l_1\le x\le 1\},\\ &L_3'=\{\gamma_2(s)+x=0,\ 0\le s\le s_2'\}, \quad L_4'=\{y-G(x)=1,\ l_2\le y\le 1\}, \end{aligned} \tag{4.1}$$

where $Y = G_1(y) = \int_0^y \sqrt{|K_1(y)|}\, dy$ in $\overline{D_1}$, $X = G_2(x) = \int_0^x \sqrt{|K_2(x)|}\, dx$ in $\overline{D_2}$, $\gamma_k(s)$ on $S_k = \{0 \le s \le s_k'\}$ ($l_1 = G_1[-\gamma_1(s_1')]$, $l_2 = G_2[-\gamma_2(s_2')]$ are continuously differentiable, $\gamma_k(0) = 0$, $\gamma_k(s) > 0$ on $\{0 < s \le s_k'\}$ ($k = 1, 2$), $G_1'(y) = H_1(y)$, $G_2'(x) = H_2(x)$, the slope of the curve $y = -\gamma_1(s)$ at the intersection point z_1^* of L_1' and the characteristic curve of $s_1:\ dy/dx = 1/H_1(y)$ in $x + jy-$plane is not equal to that of the characteristic curve at the point, and the slope of the curve $y = -\gamma_2(s)$ at the intersection point z_2^* of L_3' and the characteristic curve of $s_1:\ dy/dx = 1/H_1(y)$ in $x + jy-$plane is not equal to that of the characteristic curve at the point, $z_1' = l_1 - j\gamma_1(s_1')$, $z_2' = -\gamma_1(s_1') + jl_2$ are the intersection point of L_1', L_2' and L_3', L_4' respectively. Actually we can permit that the curve L_1' with any characteristic curve of $s_1\ :\ dy/dx = 1/H_1(y)$ has at most one intersection point, similarly we can discuss the curve L_3'. From the above conditions, we can determine the $x-$coordinate of L_1' and $y-$coordinate of L_3'. Here we mention that in [2], under the non-characteristic curve $y = -\gamma(x)$ satisfying $0 < \gamma'(x) \le 1$ on $L_0 = \{0 \le x \le 1\}$, A.V. Bitsadze discussed the mixed equation $\operatorname{sgn} y u_{xx} + u_{yy} = 0$ by the method of integral equations, even though the reasoning occupied 26 pages (pp. 379–406, [2]), the Frankl problem had not been completely solved.

We consider the Frankl problem (Problem F') for equation (1.2) in D' with the boundary conditions

$$\begin{gathered} u(z)=\phi(z)\ \text{ on }\ \Gamma, \quad u(z)=\psi(z)\ \text{ on }\ L',\ \text{ i.e. } \operatorname{Re}[\overline{\lambda(z)}w(z)]=R(z),\ \ z\in\Gamma\cup\hat{L}',\\ \operatorname{Im}[\overline{\lambda(z)}w(z)]\big|_{z=z_1'} = c_1, \quad \operatorname{Im}[\overline{\lambda(z)}w(z)]\big|_{z=z_2'} = c_2',\ \ u(t_1) = d_1. \end{gathered} \tag{4.2}$$

Herein denote $\hat{L}' = L_2' \cup L_4'$, $t_1 = 1$, $w(z) = u_{\tilde{z}}$, $\lambda(z) = a(x) + ib(x)$ on Γ, $\lambda(z) = a(x) + jb(x)$ on L', and $\lambda(z)$, $r(z)$, c_l ($l = 1, 2$), d_1 satisfy the conditions

$$\begin{gathered} C_\alpha^1[\lambda(z),\Gamma] \le k_0,\ C_\alpha^1[r(z),\Gamma] \le k_2,\\ C_\alpha^1[\lambda(z),\hat{L}'] \le k_0,\ C_\alpha^1[r(z),\hat{L}'] \le k_2, |c_l|, |d_1| \le k_2, l=1,2,\\ \max_{z\in L_2'} \frac{1}{|a(x)+b(x)|},\ \max_{z\in L_4'} \frac{1}{|a(y)+b(y)|} \le k_0, \end{gathered} \tag{4.3}$$

where $\lambda(z)$, $r(z)$ are as stated in (1.6)–(1.7), and α $(0 < \alpha < 1)$, k_0, k_2 are positive constants.

Setting $Y = G_1(y) = \int\limits_0^y \sqrt{|K_1(t)|}\,dt$, $X = G_2(x) = \int\limits_0^x \sqrt{|K_2(t)|}\,dt$. By the conditions in (4.1), the inverse function $x = \sigma_1(\nu) = (\mu+\nu)/2$ of $\nu = x - G_1(y)$ can be found, i.e. $\mu = 2\sigma_1(\nu) - \nu$, $0 \le \nu \le 1$ and the curve L_1' can be expressed by $\mu = 2\sigma_1(\nu) - \nu = 2\sigma_1(x+\gamma_1(s)) - x - \gamma_1(s)$ on S_1, and $y = \sigma_2(\nu) = (\mu-\nu)/2$ of $\mu = G_2(x) + y$ can be found, i.e. $\mu = 2\sigma_2(\nu) + \nu$, $0 \le \nu \le 1$ and the curve L_3' can be expressed by $\mu = 2\sigma_2(\nu) + \nu = 2\sigma_2(\gamma_2(s) + y) + \gamma_2(s) + y$ on S_2. We make a transformation

$$\begin{aligned}&\tilde\mu = \big[\mu - 2\sigma_1(\nu) + \nu\big]/\big[1 - 2\sigma_1(\nu) + \nu\big],\ \ \tilde\nu = \nu,\ \ 2\sigma_1(\nu) - \nu \le \mu \le 1,\\ &\tilde\mu = \big[\mu - 2\sigma_2(\nu) - \nu\big]/\big[1 - 2\sigma_2(\nu) - \nu\big],\ \ \tilde\nu = \nu,\ \ 2\sigma_2(\nu) + \nu \le \mu \le 1,\end{aligned} \tag{4.4}$$

where μ, ν are real variables, their inverse transformations are

$$\begin{aligned}&\mu = \big[1 - 2\sigma_1(\nu) + \nu\big]\tilde\mu + 2\sigma_1(\nu) - \nu,\ \ \nu = \tilde\nu,\ \ 0 \le \tilde\mu,\ \ \nu \le 1,\\ &\mu = \big[1 - 2\sigma_2(\nu) - \nu\big]\tilde\mu + 2\sigma_2(\nu) + \nu,\ \ \nu = \tilde\nu,\ \ 0 \le \tilde\mu,\ \ \nu \le 1.\end{aligned} \tag{4.5}$$

It is not difficult to see that the transformation in (4.4) maps the set D' onto D. Denote by

$$\begin{gathered}\tilde Z = \tilde x + j\tilde Y = \tilde x + j\tilde G_1(y) = \hat f(x + jY) = \hat f(Z),\\ Z = x + jY = x + jG(y) = \hat f^{-1}(\tilde Z)\end{gathered} \tag{4.6}$$

the above transformation and its inverse transformation respectively, where $\tilde x = [\tilde\mu + \tilde\nu]/2$, $\tilde Y = [\tilde\mu - \tilde\nu]/2$, and by

$$\begin{gathered}\tilde z = \tilde x + j\tilde y = \tilde z(\tilde Z) = \tilde z[\hat f(Z(z))] = f(z),\\ z = x + jy = f^{-1}(\tilde z),\end{gathered} \tag{4.7}$$

the corresponding transformation and its inverse transformation respectively. In this case, the last system of equations in (2.4) can be rewritten as

$$\begin{aligned}&\xi_\mu = A_1\xi + B_1\eta + C_1(\xi+\eta) + Du + E,\\ &\eta_\nu = A_2\xi + B_2\eta + C_2(\xi+\eta) + Du + E,\end{aligned}\quad z \in D'^-. \tag{4.8}$$

Suppose that (1.2) in D' satisfies Condition C, through the transformation (4.4), we obtain $\xi_{\tilde\mu} = [1 - 2\sigma_1(\nu) + \nu]\xi_\mu$, $\eta_{\tilde\nu} = \eta_\nu$ in $D_1' = D' \cap \{y < 0\}$, and $\xi_{\tilde\mu} = [1 - 2\sigma_2(\nu) - \nu]\xi_\mu$ $\eta_{\tilde\nu} = \eta_\nu$ in $D_2' = D' \cap \{x < 0\}$, where $\xi = U + V, \eta = U - V$, and then

$$\begin{aligned}&\xi_{\tilde\mu} = \big[1 - 2\sigma_1(\nu) + \nu\big]\,\big[A_1\xi + B_1\eta + C_1(\xi+\eta) + Du + E\big],\\ &\eta_{\tilde\nu} = A_2\xi + B_2\eta + C_2(\xi+\eta) + Du + E \ \text{ in } \ D_1 = D \cap \{y<0\},\\ &\xi_{\tilde\mu} = \big[1 - 2\sigma_2(\nu) - \nu\big]\,\big[A_1\xi + B_1\eta + C_1(\xi+\eta) + Du + E\big],\\ &\eta_{\tilde\nu} = A_2\xi + B_2\eta + C_2(\xi+\eta) + Du + E \ \text{ in } \ D_2 = D \cap \{x<0\},\end{aligned} \tag{4.9}$$

and through the transformation (4.6), the boundary condition (4.2) is reduced to

$$\begin{aligned}&\mathrm{Re}\big[\overline{\lambda(f^{-1}(\tilde z))}W(f^{-1}(\tilde z))\big] = R(f^{-1}(\tilde z)),\ \ \tilde z \in \Gamma \cup L_2 \cup L_4,\\ &\mathrm{Im}\big[\overline{\lambda(f^{-1}(\tilde z_k))}W(f^{-1}(\tilde z_k)\big] = c_k,\ \ k=1,2,\ \ u(1)=d_1,\end{aligned} \tag{4.10}$$

in which $\tilde{z}_k = f(z'_k)$, $k = 1, 2$. Therefore the boundary value problem (4.8), (4.2) (Problem A') is transformed into the boundary value problem (4.9), (4.10), i.e. the corresponding Problem A in D. On the basis of Theorem 3.2, we see that the boundary value problem (4.9),(4.10) has a solution $w(\tilde{z})$, and

$$u(z)=2R\int_0^z \left[\frac{\mathrm{Re}W}{H_1(y)}+\binom{i}{-j}\frac{\mathrm{Im}W}{H_2(x)}\right]dz+d_1 \ \text{ in } \ \binom{D^+}{D^-}$$

is just a solution of Problem F' for (1.2) in D' with the boundary condition (4.2).

Theorem 4.1. *If equation* (1.2) *in* D' *satisfies Condition* C *in the domain* D' *with the boundary* $\Gamma\cup L'_1\cup L'_2\cup L'_3\cup L'_4$, *where* L'_1, L'_2, L'_3, L'_4 *are as stated in* (4.1), *then Problem* F' *for* (1.2) *with the boundary conditions* (4.2) *has a solution* $u(z)$.

2) Next let the domain D'' be a simply connected domain with the boundary $\Gamma \cup L''_1 \cup L''_2 \cup L''_3 \cup L''_4$, where Γ is as stated before, which can be replaced by another smooth curve Γ'', and similarly to the case 1, the parameter equations of the curves L''_1, L''_2, L''_3, L''_4 are as follows:

$$\begin{aligned}&L''_1=\{\gamma_1(s)+y=0,\ 0\le s\le s'_1\},\quad L''_2=\{\gamma_2(s)+y=0,\ 0\le s\le s'_2\},\\&L''_3=\{\gamma_2(s)+x=0,\ 0\le s\le s'_3\},\quad L'_4=\{\gamma_4(s)+x=0,\ 0\le s\le s'_4\},\end{aligned}\tag{4.11}$$

in which $\gamma_k(0) = 0$, $\gamma_k(s)$ on $S_k = \{0 \le s \le s'_k\}$ $(k = 1, 2, 3, 4)$ are continuously differentiable, $z''_1 = l_1 - j\gamma_1(s'_1)$, $z''_2 = -\gamma_2(s'_2) + jl_2$ are the intersection points of L''_1, L''_2 and L''_3, L''_4 respectively, the slope of curve L''_2 at the intersection point z^*_1 of L''_2 and the characteristic curve of $s_1 :\ dy/dx = -1/H_1(y)$ is not equal to that of the characteristic curve at the point, and the slope of curve L''_4 at the intersection point z^*_2 of L''_4 and the characteristic curve of $s_2 :\ dx/dy = -1/H_2(x)$ is not equal to that of the characteristic curve at the point. The curves L''_1, L''_3 satisfy some conditions as stated below.

The so-called Frankl problem (Problem F'') for equation (1.2) in the domain D'' is to find a solution of (1.2) in D'' satisfying the boundary conditions

$$\begin{aligned}&u(z)=\phi(z)\ \text{ on }\ \Gamma,\quad u(z)=\psi(z)\ \text{ on }\ L'',\ \text{ i.e. }\ \mathrm{Re}[\overline{\lambda(z)}w(z)]=R(z),\ \ z\in\Gamma\cup\hat{L}'',\\&\mathrm{Im}[\overline{\lambda(z)}w(z)]\big|_{z=z''_1}=c_1,\quad \mathrm{Im}[\overline{\lambda(z)}w(z)]\big|_{z=z''_2}=c_2,\ \ u(t_1)=d_1.\end{aligned}\tag{4.12}$$

Herein denote $w(z) = u_{\tilde{z}}$, $\hat{L}'' = L''_2 \cup L''_4$, $\lambda(z) = a(x) + ib(x)$ on Γ, $\lambda(z) = a(x) + jb(x)$ on L'', and $\lambda(z)$, $r(z)$, c_l $(l = 1, 2)$, d_1 satisfy the conditions

$$\begin{gathered}C^1_\alpha[\lambda(z),\Gamma]\le k_0,\ C^1_\alpha[r(z),\Gamma]\le k_2,\\C^1_\alpha[\lambda(z),\hat{L}'']\le k_0,\ C^1_\alpha[r(z),\hat{L}'']\le k_2,|c_l|,|d_1|\le k_2,l=1,2,\\\max_{z\in L''_2}\frac{1}{|a(x)+b(x)|},\ \max_{z\in L''_4}\frac{1}{|a(y)+b(y)|}\le k_0,\end{gathered}\tag{4.13}$$

where $\lambda(z)$, $r(z)$ are as stated in (1.6)–(1.7), and α $(0 < \alpha < 1)$, k_0, k_2 are positive constants.

By the conditions in (4.11), the inverse function $x = \tau_1(\mu) = (\mu + \nu)/2$ of $\mu = x + G_1(y)$ can be found, i.e. $\nu = 2\tau_1(\mu) - \mu$, $0 \le \mu \le 1$, the inverse function $y = \tau_2(\mu) =$

$(\mu-\nu)/2$ of $\mu = G_2(x)+y$ can be found, i.e. $\nu=-2\tau_2(\mu)+\mu$, $0\le\mu\le 1$, and the curve L_2'', L_4'' can be expressed by

$$\begin{aligned}&\nu = 2\tau_1(\mu)-\mu = 2\tau_1(x-\gamma_2(s))-x+\gamma_2(s),\\&\nu=-2\tau_2(\mu)+\mu=-2\tau_2(x-\gamma_4(s))+x-\gamma_4(s).\end{aligned}\tag{4.14}$$

We make a transformation

$$\begin{aligned}&\tilde{\mu}=\mu,\quad \tilde{\nu}=\frac{\nu-2\tau_1(\mu)+\mu}{2\tau_1(\mu)-\mu}+1,\quad 0\le\nu\le 2\tau_1(\mu)-\mu,\\&\tilde{\mu}=\mu,\quad \tilde{\nu}=\frac{\nu+2\tau_2(\mu)-\mu}{-\tau_2(\mu)+\mu}+1,\quad 0\le\nu\le -2\tau_2(\mu)-\mu.\end{aligned}\tag{4.15}$$

It is clear that their inverse transformations are

$$\begin{aligned}&\mu=\tilde{\mu},\quad \nu=(\tilde{\nu}-1)(2\tau_1(\mu)-\mu)+2\tau_1(\mu)-\mu,\quad 0\le\tilde{\mu},\quad \tilde{\nu}\le 1,\\&\mu=\tilde{\mu},\quad \nu=(\tilde{\nu}-1)(-2\tau_2(\mu)+\mu)-2\tau_2(\mu)+\mu,\quad 0\le\tilde{\mu},\quad \tilde{\nu}\le 1,\end{aligned}\tag{4.16}$$

Denote by

$$\begin{gathered}\tilde{Z}=\tilde{x}+j\tilde{Y}=\tilde{x}+j\tilde{G}(y)=\hat{g}(x+jY)=\hat{g}(Z),\\Z=x+jY=x+jG(y)=\hat{g}^{-1}(\tilde{Z}),\end{gathered}$$

the above transformation and its inverse transformation respectively, where $\tilde{x}=[\tilde{\mu}+\tilde{\nu}]/2$, $\tilde{Y}=[\tilde{\mu}-\tilde{\nu}]/2$, and by

$$\begin{gathered}\tilde{z}=\tilde{x}+j\tilde{y}=\tilde{z}(\tilde{Z})=\tilde{z}[\hat{g}(Z(z))]=f(z),\\z=x+jy=g^{-1}(\tilde{z}),\end{gathered}\tag{4.17}$$

the corresponding transformation and its inverse transformation respectively. In this case, the last system of equations in (2.4) can be rewritten as

$$\begin{aligned}&\xi_\mu=A_1\xi+B_1\eta+C_1(\xi+\eta)+Du+E,\\&\eta_\nu=A_2\xi+B_2\eta+C_2(\xi+\eta)+Du+E,\end{aligned}\quad z\in D''^{-}.\tag{4.18}$$

Through the transformation (4.15), we obtain

$$\begin{aligned}&(u+v)_{\tilde{\mu}}=(u+v)_\mu,\quad (u-v)_{\tilde{\nu}}=[2\tau_1(\mu)-\mu](u-v)_\nu\ \text{ in }\ D_1''=D''\cap\{y<0\},\\&(u+v)_{\tilde{\mu}}=(u+v)_\mu,\quad (u-v)_{\tilde{\nu}}=[-2\tau_2(\mu)+\mu](u-v)_\nu\ \text{ in }\ D_2''=D''\cap\{x<0\}.\end{aligned}\tag{4.19}$$

System (4.18) in D''^{-} is reduced to

$$\begin{aligned}&\xi_{\tilde{\mu}}=A_1\xi+B_1\eta+C_1(\xi+\eta)+Du+E,\\&\eta_{\tilde{\nu}}=\big[2\tau_1(\mu)-\mu\big]\big[A_2\xi+B_2\eta+C_2(\xi+\eta)+Du+E\big]\ \text{ in }\ D_1',\\&\xi_{\tilde{\mu}}=A_1\xi+B_1\eta+C_1(\xi+\eta)+Du+E,\\&\eta_{\tilde{\nu}}=\big[-2\tau_2(\mu)+\mu\big]\big[A_2\xi+B_2\eta+C_2(\xi+\eta)+Du+E\big]\ \text{ in }\ D_2'.\end{aligned}\tag{4.20}$$

Moreover, through the transformation (4.17), the boundary condition (4.12) on L_2'', L_4'' is reduced to

$$\begin{aligned} &\mathrm{Re}\big[\overline{\lambda(g^{-1}(\tilde{z}))}W(g^{-1}(\tilde{z}))\big] = R[g^{-1}(\tilde{z})], \quad \tilde{z} \in \Gamma \cup L_2' \cup L_4', \\ &\mathrm{Im}\big[\overline{\lambda(g^{-1}(\tilde{z}_k'))}W(g^{-1}(\tilde{z}_k'))\big] = c_k, \quad k = 1, 2, \quad u(1) = d_1, \end{aligned} \tag{4.21}$$

in which $\tilde{z}_k' = g(z_k'')$, $k = 1, 2$. Therefore the boundary value problem (4.8), (4.12) in D'' is transformed into the boundary value problem (4.20), (4.21), where we require that the boundaries $L_k' = g(L_k'')$ $(k = 1, 3)$ satisfy the similar conditions in (4.1). According to the method in the proof of Theorem 4.1, we can see that the boundary value problem (4.20), (4.21) has a solution $u(\tilde{z})$, and then the corresponding $u = u(z)$ is a solution of Problem F'' of equation (1.2).

Theorem 4.2. *If the mixed equation* (1.2) *satisfies Condition* C *in the domain* D'' *with the boundary* $\Gamma \cup L_1'' \cup L_2'' \cup L_3'' \cup L_4''$, *where* L_1'', L_2'', L_3'', L_4'' *are as stated in* (4.11), *then Problem* F'' *for* (1.2) *in* D'' *has a solution* $u(z)$.

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