Foundations of Machine Learning
CentraleSupélec — Fall 2017

7. Regularized linear regression

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Practical matters

- **Class representatives** (délégués de classe)

HW04

Assume p random variables $X_1, \ldots, X_p$, conditionally independent given Y. Y is a discrete random variable that can take one of K values $\{y_1, \ldots, y_K\}$, corresponding to K classes.

We suppose that each X_j is distributed normally:

$$p(X_j = u | Y = y_k) = \frac{1}{\sigma_{jk}\sqrt{2\pi}} \exp\left(\frac{-(u - \mu_{jk})^2}{2\sigma_{jk}^2}\right).$$

Let us assume that the variance is independent of the class k and the feature j, i.e. $\sigma_{jk} = \sigma$.

We observe n datapoints $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\}$ and their labels $\{y^1, \ldots, y^n\}$. Derive the maximum likelihood estimator for μ_{jk}.

What is known? What are the parameters to be estimated?

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variable

parameters to be determined

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given

Maximum likelihood estimation

- **Find θ such that $\mathcal{X}$ is the most likely to be drawn.**
- **Likelihood** of θ given the i.i.d. sample $\mathcal{X}$:
$$\ell(\theta|\mathcal{X}) = p(\mathcal{X}|\theta) = p(\boldsymbol{x}^1|\theta)p(\boldsymbol{x}^2|\theta)\ldots p(\boldsymbol{x}^n|\theta)$$
- **Log likelihood:**
$$\mathcal{L}(\theta|\mathcal{X}) = \log \ell(\theta|\mathcal{X}) = \log p(\boldsymbol{x}^1|\theta) + \cdots + \log p(\boldsymbol{x}^n|\theta)$$
- **Maximum likelihood estimation (MLE):**
$$\theta^* = \arg\max_\theta \mathcal{L}(\theta|\mathcal{X})$$

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given

For feature j, class k

Maximum likelihood estimation

$$\boxed{\mu_{jk}^* = \arg\max_{\mu_{jk}} \sum_{i=1}^{n} \log p(X_j = x_j^i | \mu_{jk}, \sigma)}$$ when $Y=y_k$

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I_{ik}

$$= \frac{1}{\sigma_{jk}\sqrt{2\pi}} \exp\left(\frac{-(x_j^i - \mu_{jk})^2}{2\sigma_{jk}^2}\right)$$

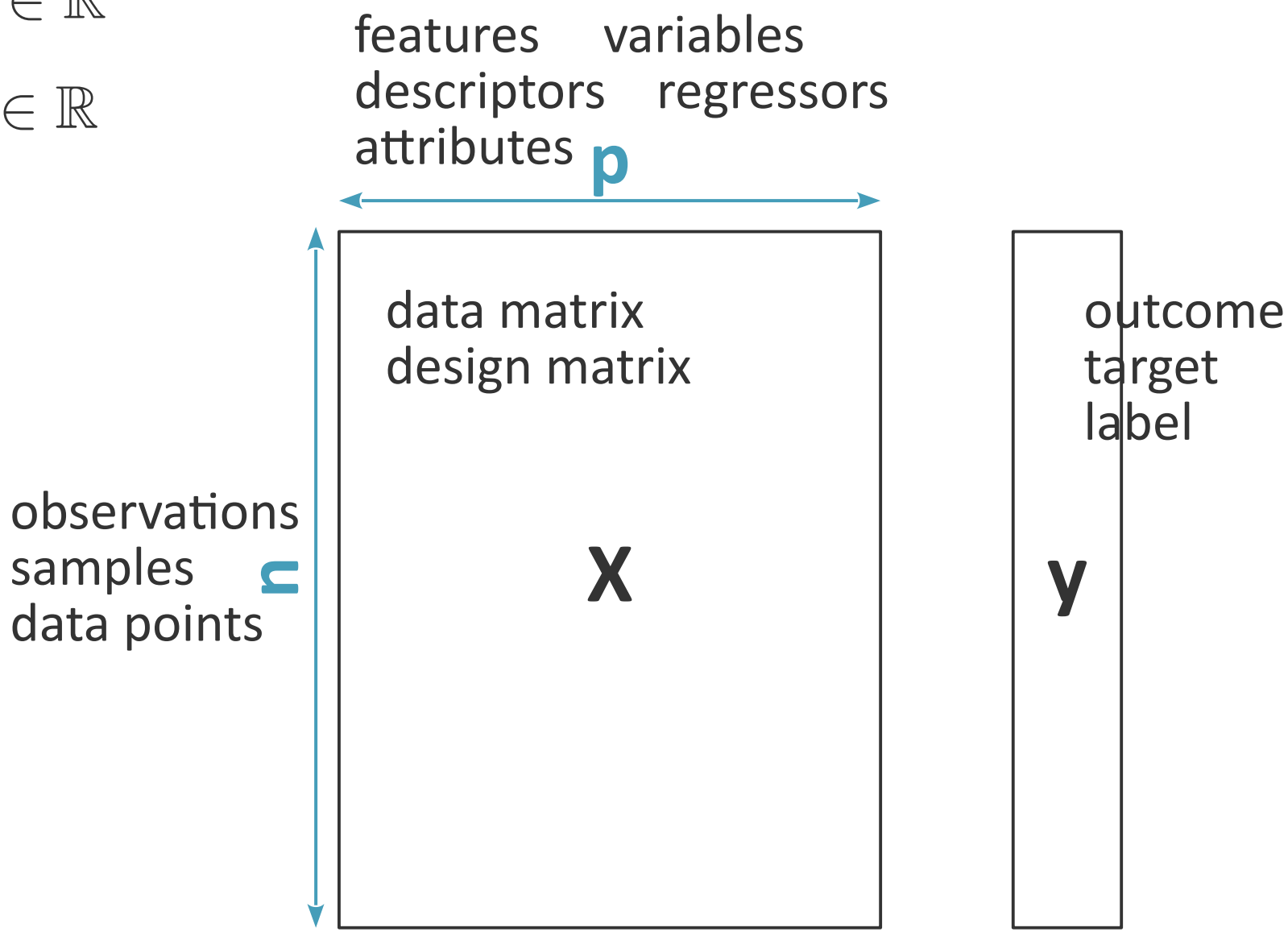
Learning objectives

- Understand **regularization** as a means to control model complexity.
- Define **Lasso**, **ridge regression**, **elastic net.**
- Understand the role of the **l1 and l2 norms** in regularization
- Interpret **solution paths** for Lasso and ridge regression.

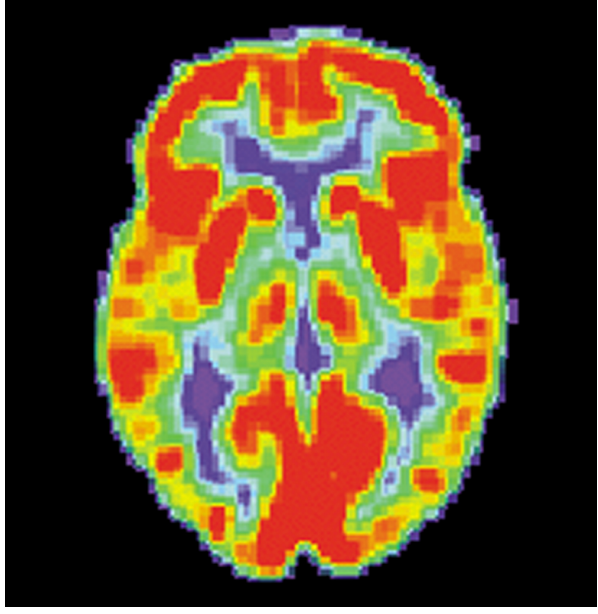
Regression setting

$x_j^i \in \mathbb{R}$

$y^i \in \mathbb{R}$



Large p, small n



- **neuroimaging**

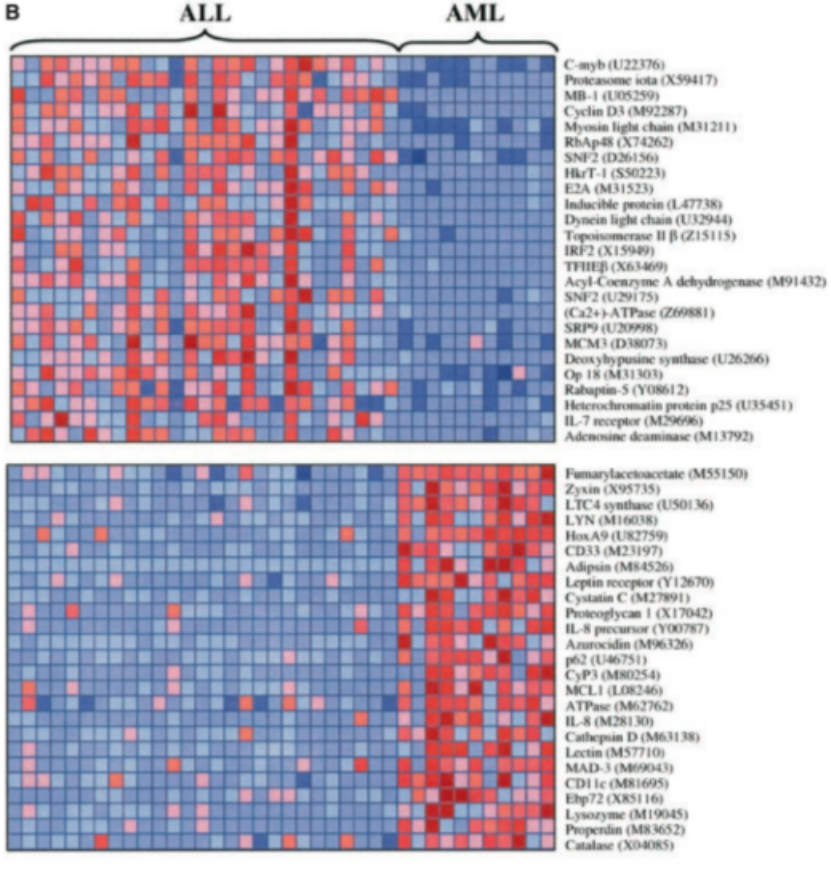
  thousands of brain regions / pixels / voxels

  much fewer patients

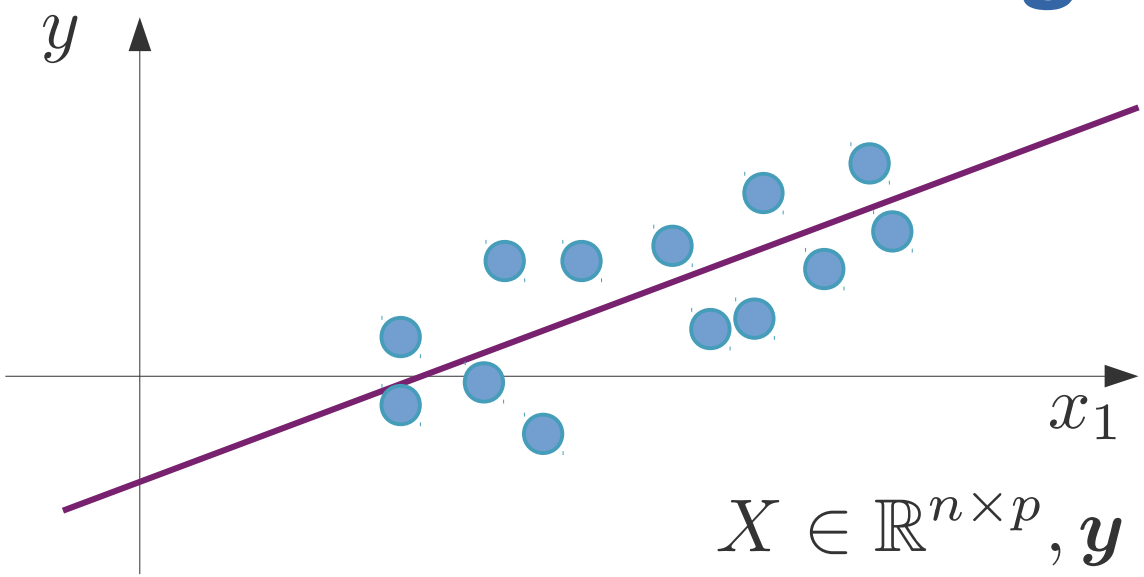
- **genetics and genomics**

  thousands of genes, millions of SNPs..

  usually, at best thousands of patients



Linear regression



$$\boldsymbol{x} \in \mathbb{R}^p$$

$$f(\boldsymbol{x}|\boldsymbol{\beta}) = \sum_{j=1}^{p} \beta_j x_j + \beta_0$$

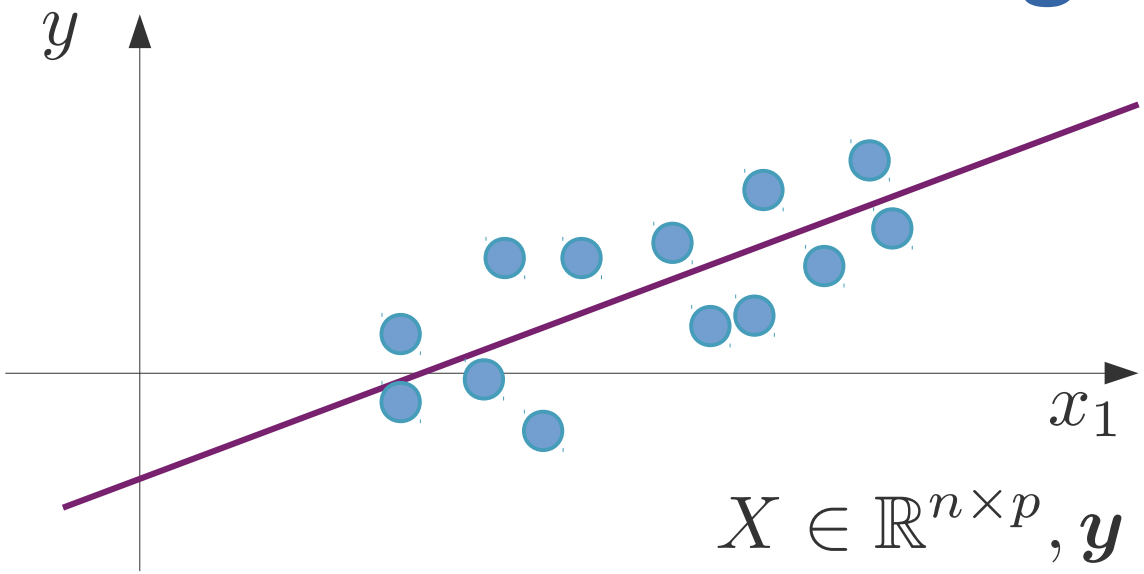
$$X \in \mathbb{R}^{n \times p}, \boldsymbol{y} \in \mathbb{R}^n$$

Least-squares fit (equivalent to MLE under the assumption of Gaussian noise):

$$\hat{\boldsymbol{\beta}} = \arg\min_{\beta} (\boldsymbol{y} - X\boldsymbol{\beta})^\top (\boldsymbol{y} - X\boldsymbol{\beta}) = (X^\top X)^{-1} X^\top \boldsymbol{y}$$

Solution uniquely defined when ?

Linear regression



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Solution uniquely defined when $X^\top X$ **invertible.**

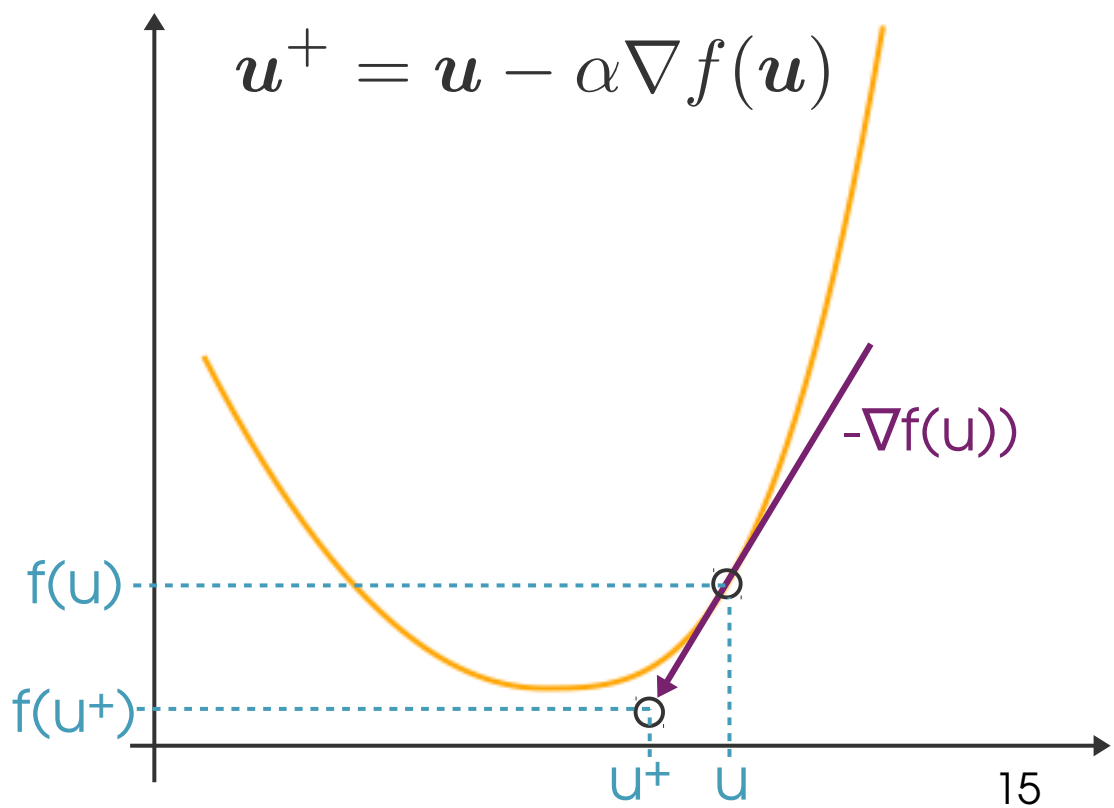
When X^TX not inversible

$$(X^\top X)\hat{\beta} = X^\top y$$

- Pseudo-inverse
- Linear system of p equations:

Numerical methods

- **Gaussian elimination**
- **LU decomposition**
- **Gradient descent**



Properties of the least-squares fit estimate

- **Unbiased** $\mathbb{E}[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$
- **Explicit form** $\hat{\boldsymbol{\beta}} = (X^\top X)^{-1} X^\top \boldsymbol{y}$
- **Computational time** ?

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- **Unbiased** $\mathbb{E}[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$
- **Explicit form** $\hat{\boldsymbol{\beta}} = (X^\top X)^{-1} X^\top \boldsymbol{y}$
- **Computational time:** $\mathcal{O}(p^3 + np)$

computation of $(X^\top X)^{-1} = \mathcal{O}(p^3)$

computation of $X^\top \boldsymbol{y} = \mathcal{O}(np)$

When p is large

- **p > n:** $X^\top X$?

When p is large

- **p > n:** $X^\top X$ not invertible

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  - Multiple possible solutions
  - High variance of the estimator.

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- **Multicollinearity:**

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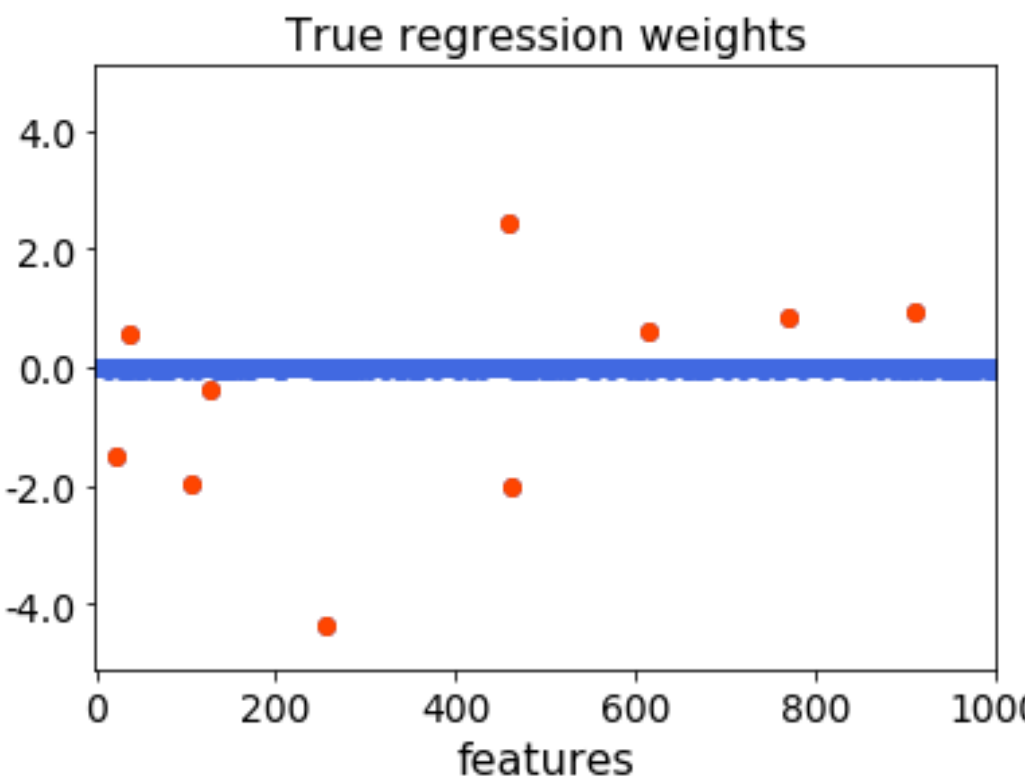
  - Multiple possible solutions
  - High variance of the estimator.
- **Multicollinearity:** $X^\top X$ not invertible
- Large p **reduces interpretability** of the model

Would prefer a small subset of features with strong effects (= large coefficients).

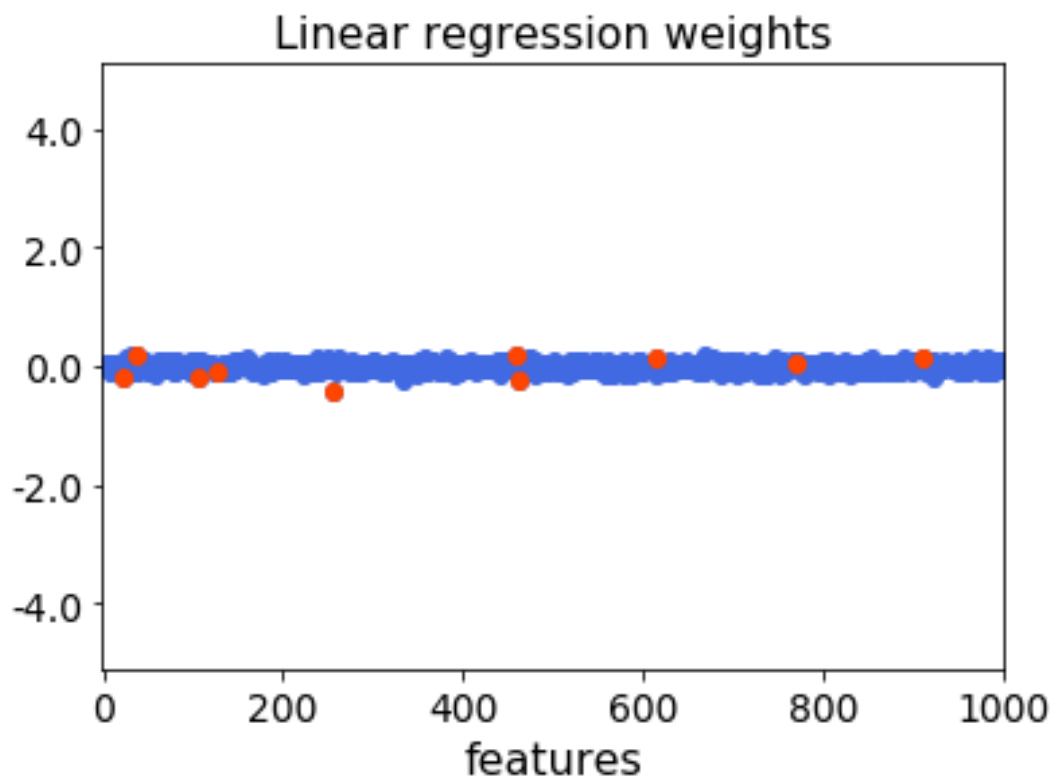
Linear regression when p >> n

Simulated data: p=1000, n=100, 10 causal features

True coefficients



Predicted coefficients



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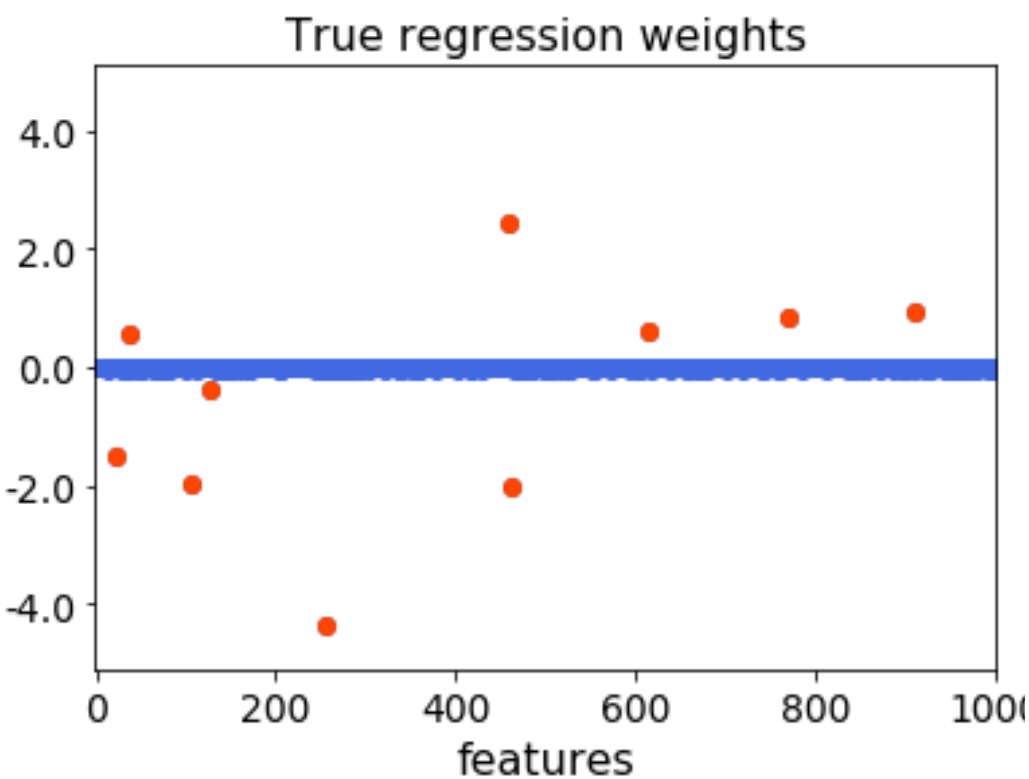

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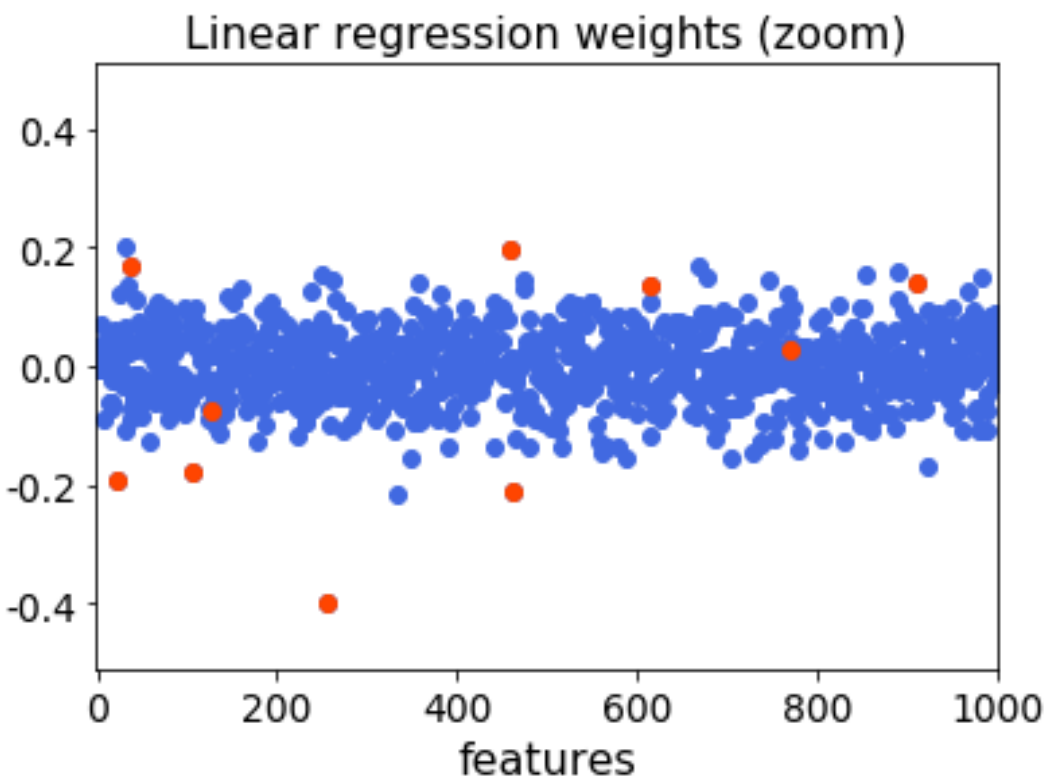

Regularization

Regularization

- Minimize

  Prediction error + **λ** penalty on model complexity

- **Biased estimator** when $\lambda \neq 0$.
- Trade bias for a smaller variance.
- λ can be set by cross-validation.

- Simpler model ≈ fewer parameters

  → **shrinkage:** drive the coefficients of the parameters towards 0.

Ridge regression

Ridge regression

- **Sum-of-squares penalty**

$$\hat{\boldsymbol{\beta}}_{\text{ridge}} = \arg\min_{\boldsymbol{\beta}} ||y - X\boldsymbol{\beta}||_2^2 + \lambda||\boldsymbol{\beta}||_2^2$$

- **Ridge regression estimator:** ?

Ridge regression

- **Sum-of-squares penalty**

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- **Ridge regression estimator:**

$$\hat{\boldsymbol{\beta}}_{\text{ridge}} = (X^\top X + \lambda I)^{-1} X^\top \boldsymbol{y}$$

$$\text{if } (X^\top X + \lambda I) \text{ invertible.}$$

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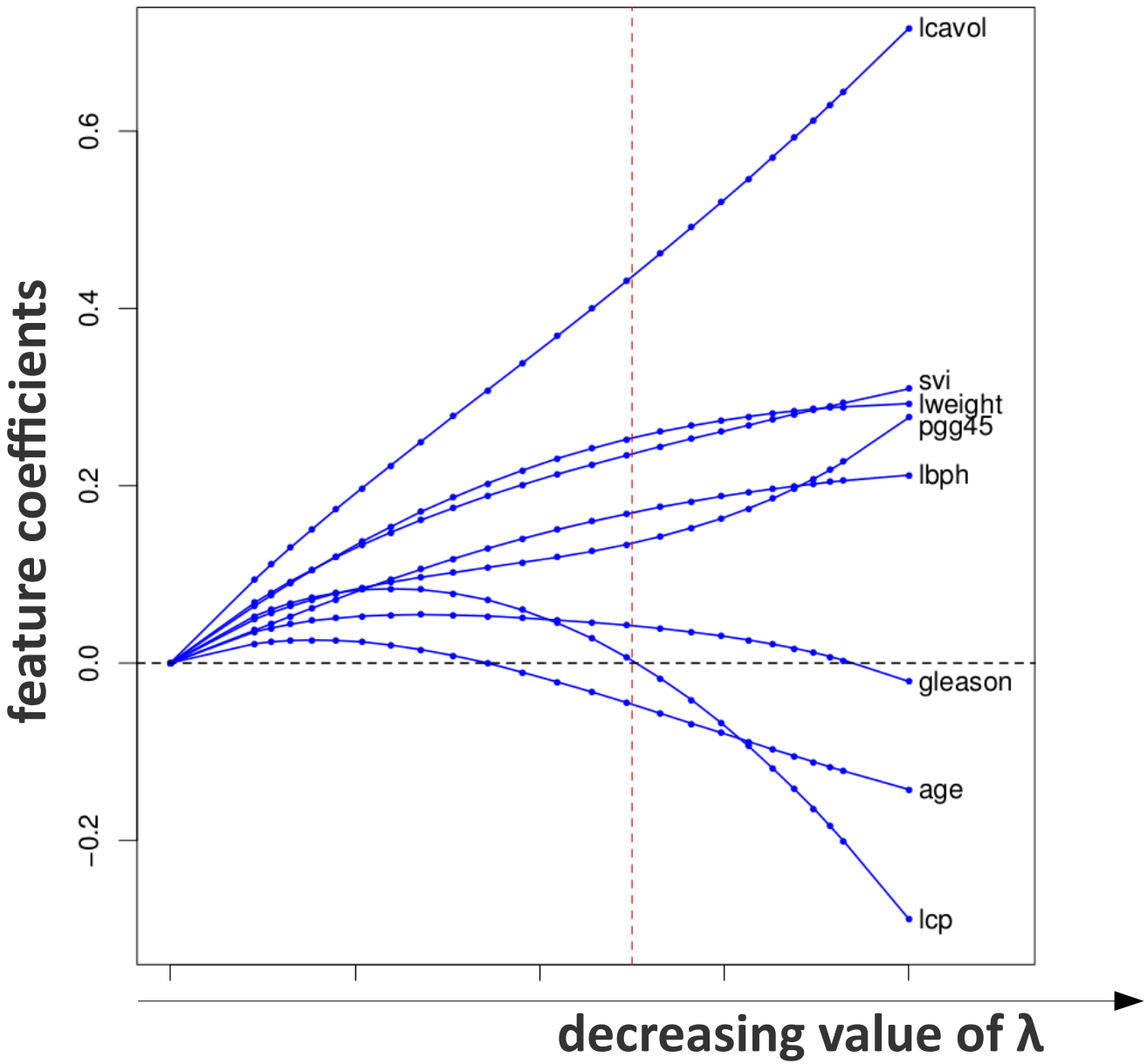
Always!

Ridge regression solution path



Standardization

- **Multiply x_j by a constant:**
  - For **standard linear regression**
  - For **ridge regression**

Standardization

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$$\hat{\beta}_j \rightarrow \frac{1}{c}\hat{\beta}_j$$

  - For **ridge regression:** ?

Standardization

- **Multiply x_j by a constant:**
  - For **standard linear regression:**

$$\hat{\beta}_j \to \frac{1}{c}\hat{\beta}_j$$

  - For **ridge regression:**

    Not so clear, because of the penalization term $\lambda\beta_j^2$
- Need to **standardize** the features

$$\tilde{x}_j^i = \frac{x_j^i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_j^i - \bar{x}_j)^2}}$$

average value of x_j

Ridge regression

- **Grouped selection:**
  - correlated variables get similar weights
  - identical variables get identical weights
- Ridge regression shrinks coefficients towards 0 but does not result in a **sparse model**.
- **Sparsity:**
  - many coefficients get a weight of 0
  - they can be eliminated from the model.

Lasso

Lasso

$$||\boldsymbol{\beta}||_1 = \sum_{j=1}^{p} |\beta_p|$$

- **L1 penalty**

$$\hat{\boldsymbol{\beta}}_{\text{lasso}} = \arg\min_{\boldsymbol{\beta}} ||y - X\boldsymbol{\beta}||_2^2 + \lambda ||\boldsymbol{\beta}||_1$$

- aka **basis pursuit** (signal processing)
- no closed-form solution
- Equivalent to

$$\hat{\boldsymbol{\beta}}_{\text{lasso}} = \arg\min_{\boldsymbol{\beta}} ||y - X\boldsymbol{\beta}||_2^2 \quad \text{s.t.} \quad ||\boldsymbol{\beta}||_1 \leq t$$

for a unique one-to-one match between t and λ.

Optimization problem: ?

Lasso

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for a unique one-to-one match between t and λ.

QP: maximize a quadratic form under linear constraints.

Geometric interpretation

Minimize f(β) under the constraint g(β) ≤ 0

$$g(\boldsymbol{\beta}) = ||\boldsymbol{\beta}||_1 - t$$

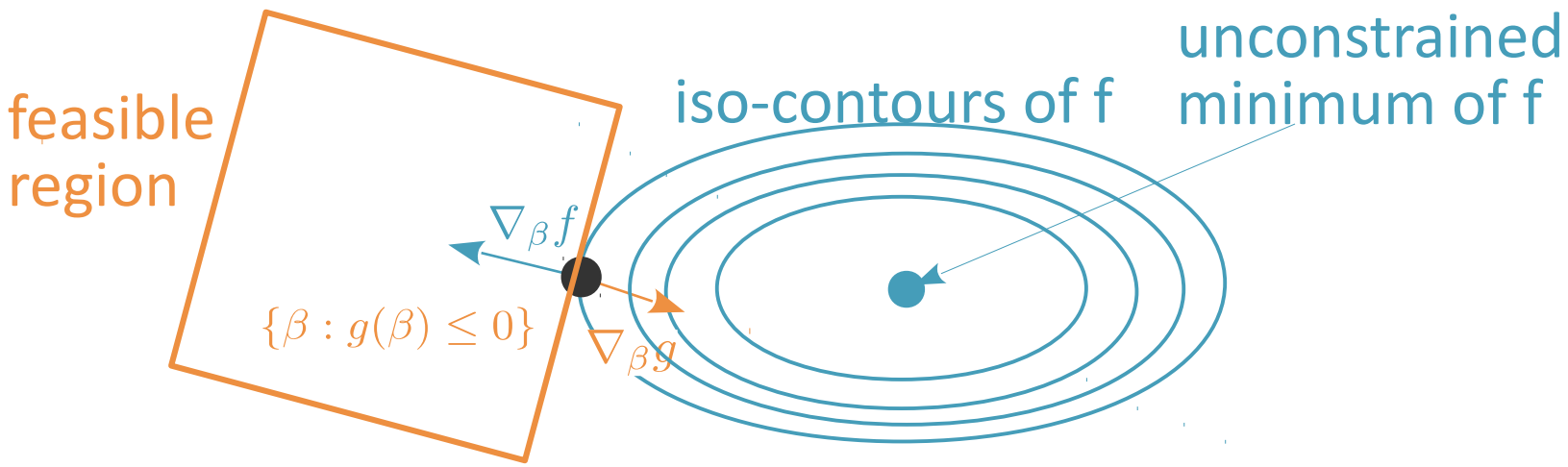
$$f(\boldsymbol{\beta}) = ||\boldsymbol{y} - X\boldsymbol{\beta}||_2^2$$

Geometric interpretation

Minimize f(β) under the constraint g(β) ≤ 0

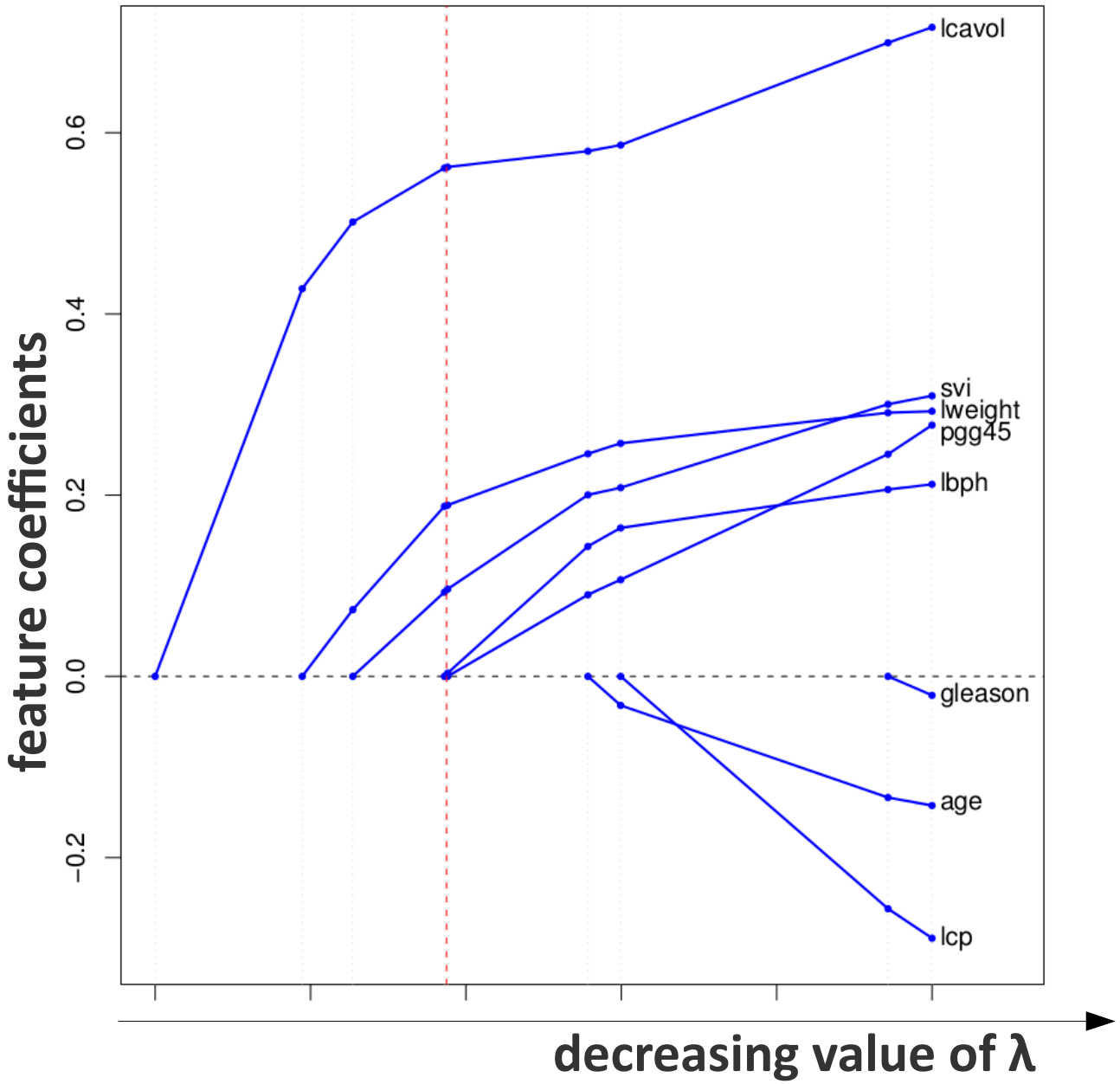
- **Case 1:** the unconstrained minimum lies in the feasible region.
- **Case 2:** it does not.

$$g(\boldsymbol{\beta}) = ||\boldsymbol{\beta}||_1 - t \qquad f(\boldsymbol{\beta}) = ||\boldsymbol{y} - X\boldsymbol{\beta}||_2^2$$



The gradient is orthonormal to the iso-contours and points towards the direction of maximum increase.

Lasso solution path



Forward stepwise regression

- Build model **sequentially**, adding one variable at a time
  - Start with the intercept
  - At each step, add the variable that **most improves the fit**
  - **Stop when** $||\boldsymbol{\beta}||_1 \leq t$
- Greedy solution

Least Angle Regression

At each step, add "only as much of a variable as needed"

1. Standardize the predictors to have mean zero and unit norm. Start with the residual $\mathbf{r} = \mathbf{y} - \bar{\mathbf{y}}$, $\beta_1, \beta_2, \ldots, \beta_p = 0$.

Least Angle Regression

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3. Move β_j from 0 towards its least-squares coefficient $\langle \mathbf{x}_j, \mathbf{r} \rangle$, until some other competitor $\mathbf{x}_k$ has as much correlation with the current residual as does $\mathbf{x}_j$.

$$\boxed{\begin{aligned} \beta_j &\leftarrow \beta_j + \alpha \frac{1}{\sum_{i=1}^n (x_j^i)^2} \sum_{i=1}^n x_j^i r^i \\ &= \beta_j + \alpha (x_j^\top x_j)^{-1} x_j^\top r \\ &= \beta_j + \alpha \langle x_j^\top, x_j \rangle^{-1} \langle x_j, r \rangle \end{aligned}}$$

$$\boxed{r = (y - \bar{y}) - \beta_j x_j}$$

step size

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4. Move β_j and β_k in the direction defined by their joint least squares coefficient of the current residual on $(\mathbf{x}_j, \mathbf{x}_k)$, until some other competitor $\mathbf{x}_l$ has as much correlation with the current residual.

$$\boxed{r = (y - \bar{y}) - \beta_j x_j - \beta_k x_k}$$

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5. Continue in this way until all p predictors have been entered.

Least Angle Regression

At each step, add "only as much of a variable as needed"

1. Standardize the predictors to have mean zero and unit norm. Start with the residual $\mathbf{r} = \mathbf{y} - \bar{\mathbf{y}}$, $\beta_1, \beta_2, \ldots, \beta_p = 0$.
2. Find the predictor $\mathbf{x}_j$ most correlated with $\mathbf{r}$.
3. Move β_j from 0 towards its least-squares coefficient $\langle \mathbf{x}_j, \mathbf{r} \rangle$, until some other competitor $\mathbf{x}_k$ has as much correlation with the current residual as does $\mathbf{x}_j$.
4. Move β_j and β_k in the direction defined by their joint least squares coefficient of the current residual on $(\mathbf{x}_j, \mathbf{x}_k)$, until some other competitor $\mathbf{x}_l$ has as much correlation with the current residual.

4a. If a non-zero coefficient hits zero, drop its variable from the active set of variables and recompute the current joint least squares direction.

5. Continue in this way until all p predictors have been entered.

Maximum number of steps: max(n-1, p)

Approaches to dimensionality reduction

- **Feature selection**

  Choose m < p features, ignore the remaining (p-m)

  - **Filtering** approaches

    Apply a statistical measure to assign a score to each feature (correlation, χ^2-test).

  - **Wrapper** approaches

    Search problem: Find the best set of features for a given predictive model.

  - **Embedded** approaches

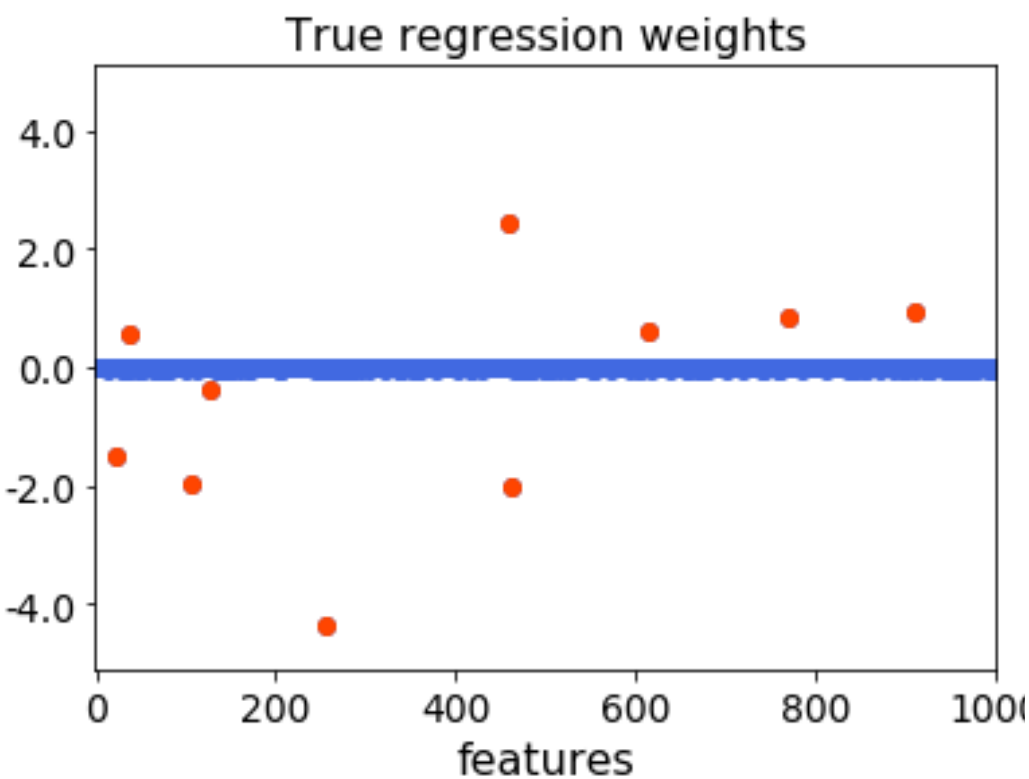
    Simultaneously fit a model and learn which features should be included.

All these feature selection approaches are **supervised.**

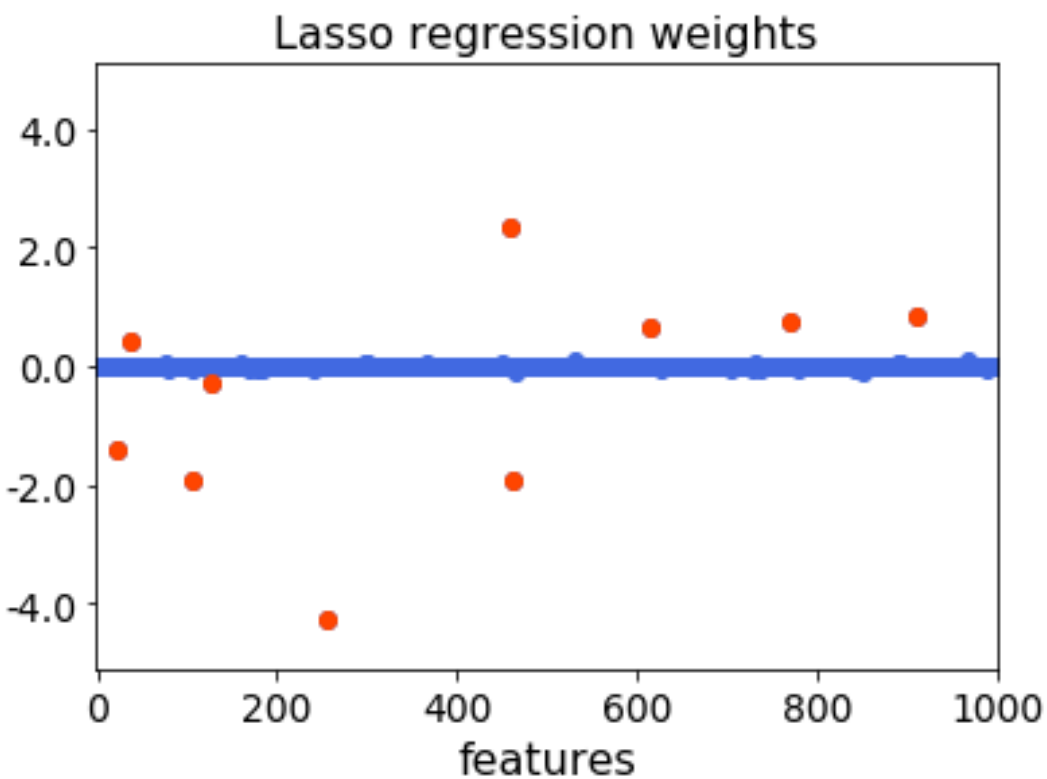
Linear regression when p >> n

Simulated data: p=1000, n=100, 10 causal features

True coefficients



Predicted coefficients



Elastic Net

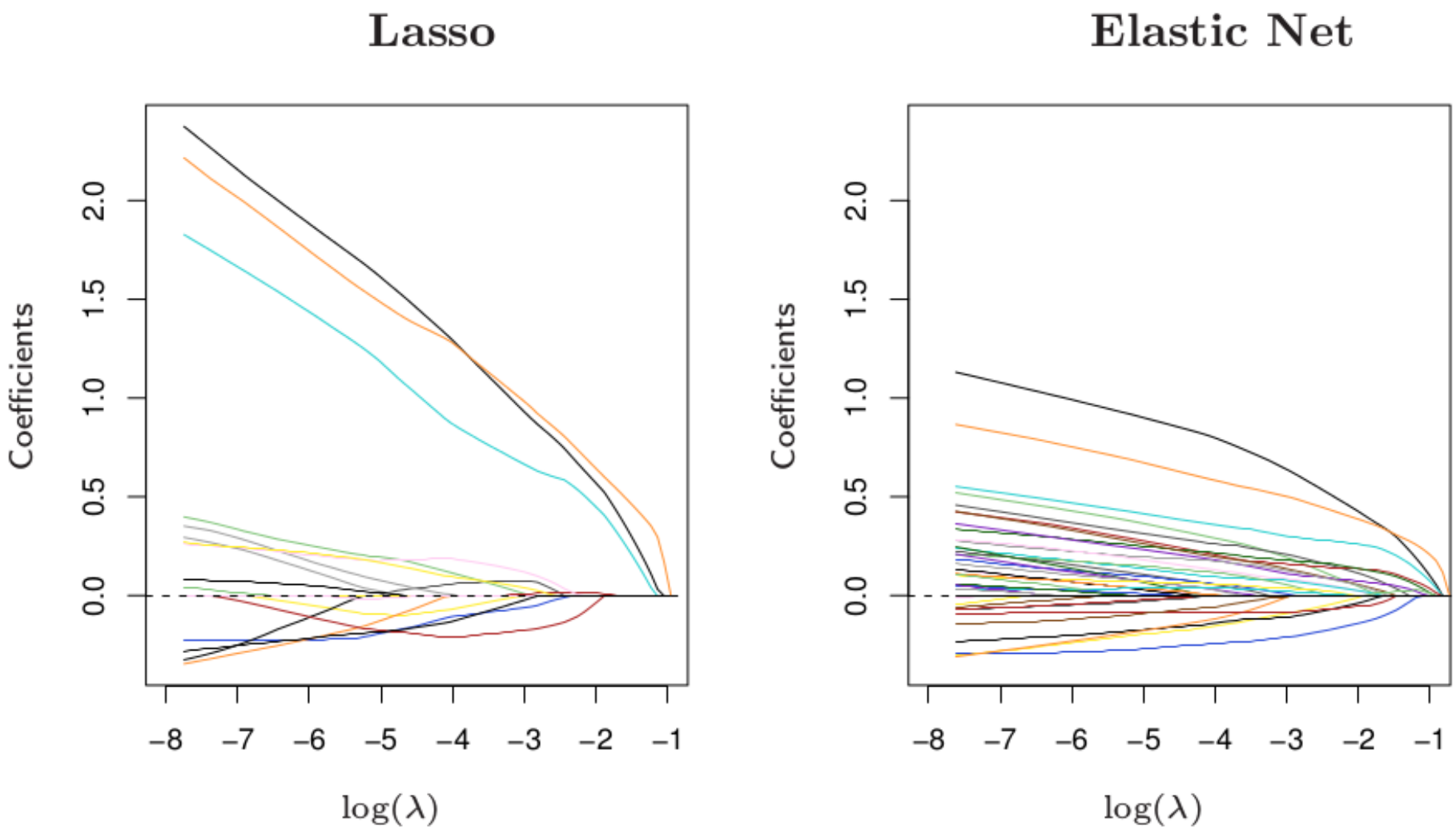
Elastic Net

- **Combine lasso** and **ridge regression**

$$\hat{\boldsymbol{\beta}}_{\text{enet}} = \arg\min_{\boldsymbol{\beta}} ||y - X\boldsymbol{\beta}||_2^2 + \lambda \left(\alpha||\boldsymbol{\beta}||_2^2 + (1-\alpha)||\boldsymbol{\beta}||_1\right)$$

  - **Select variables** like the lasso.
  - **Shrinks together coefficients of correlated variables** like the ridge regression.

E.g. Leukemia data



Elastic Net results in more non-zero coefficients than Lasso, but with smaller amplitudes.

Lq-norm regularization

Lq-norm regularization

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} ||Y - X\boldsymbol{\beta}||_2^2 + \lambda ||\boldsymbol{\beta}||_q^q \qquad ||\boldsymbol{\beta}||_q = \left(\sum_{j=1}^{p} |\beta_j|^q \right)^{1/q}$$

Equivalently:

$$\hat{\boldsymbol{\beta}} = \arg\min_{\beta} ||Y - X\boldsymbol{\beta}||_2^2 \text{ s. t. } ||\boldsymbol{\beta}||_q^q \leq s$$

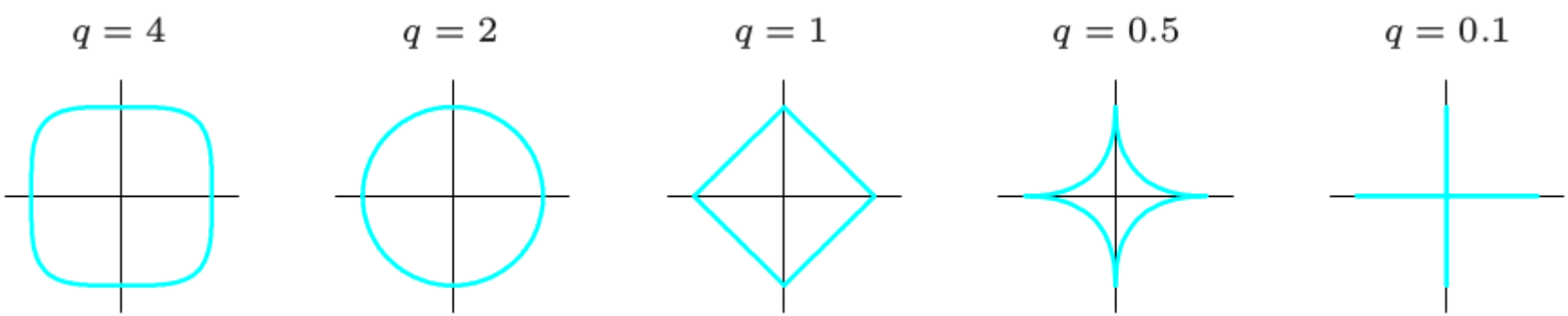
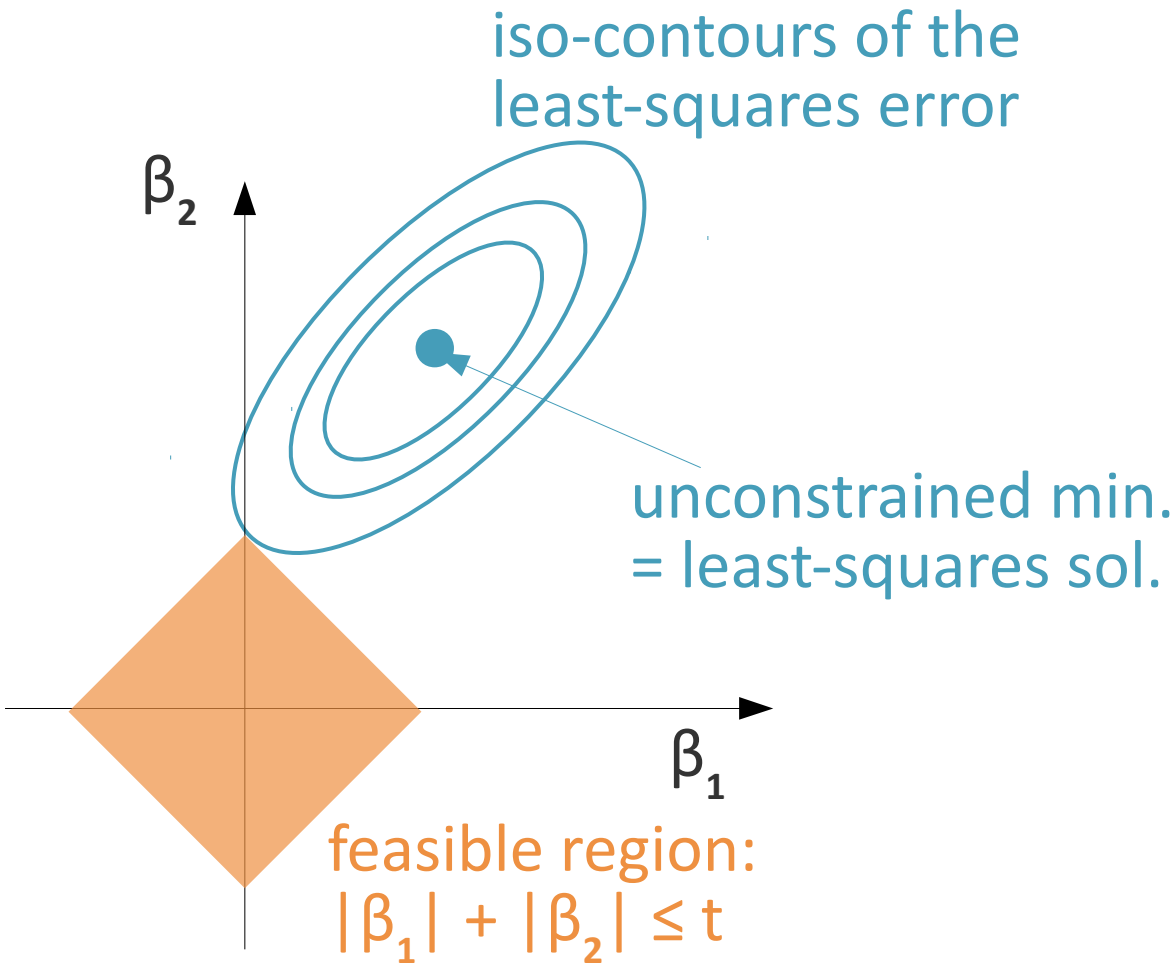


FIGURE 3.12. *Contours of constant value of* $\sum_j |\beta_j|^q$ *for given values of* q.

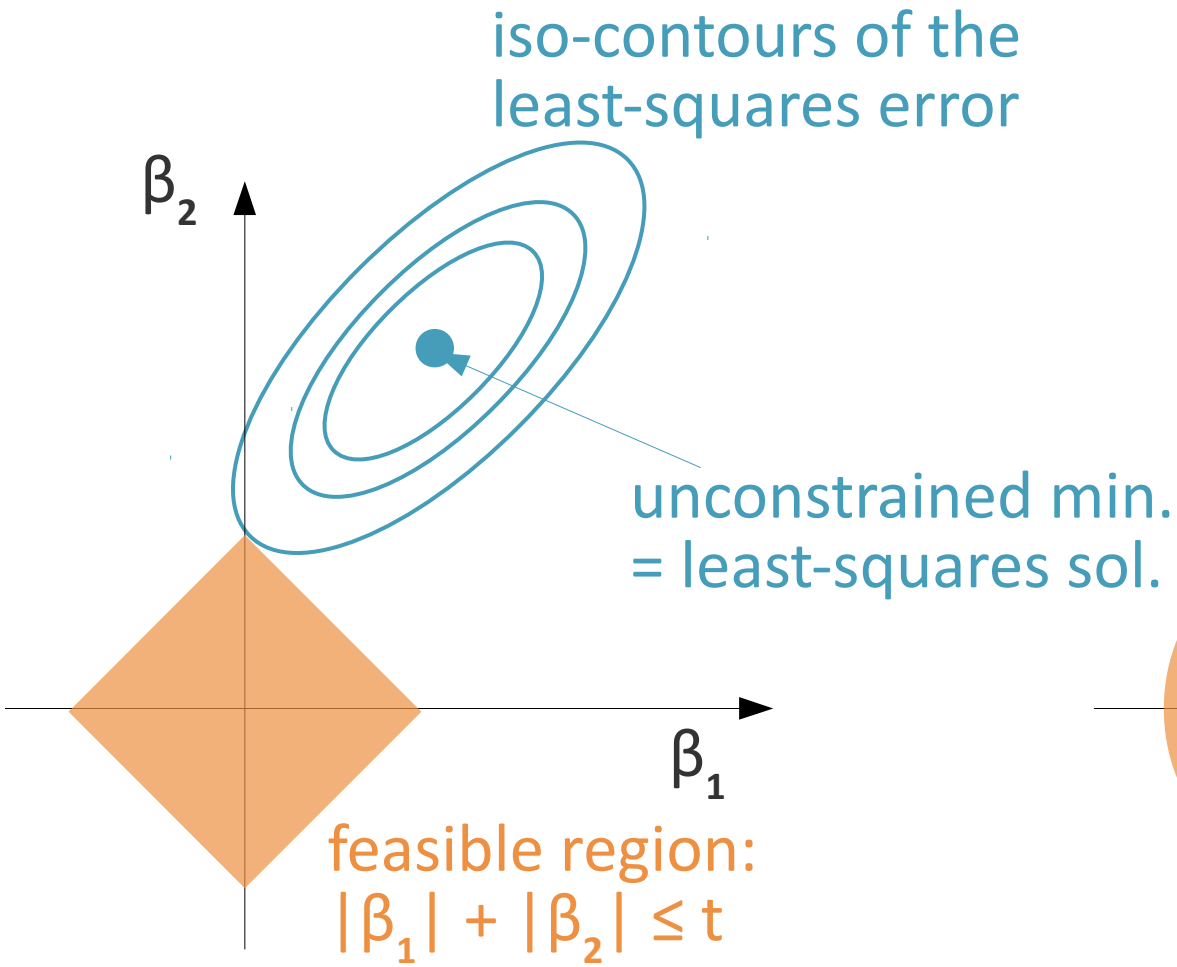
Lasso vs. ridge

L1 norm

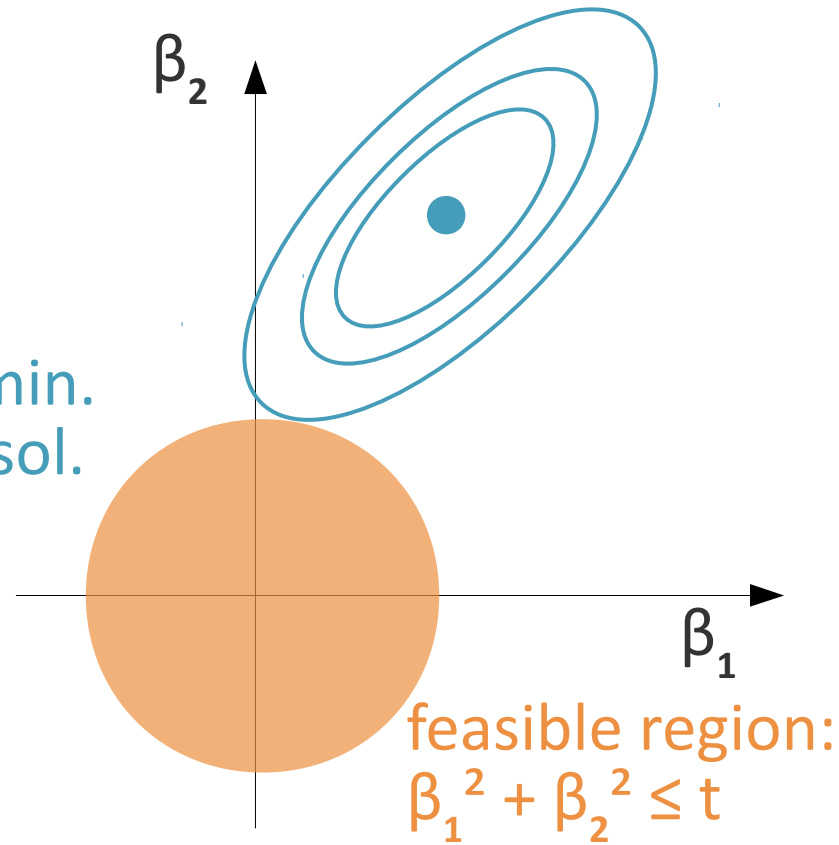


Lasso vs. ridge

L1 norm



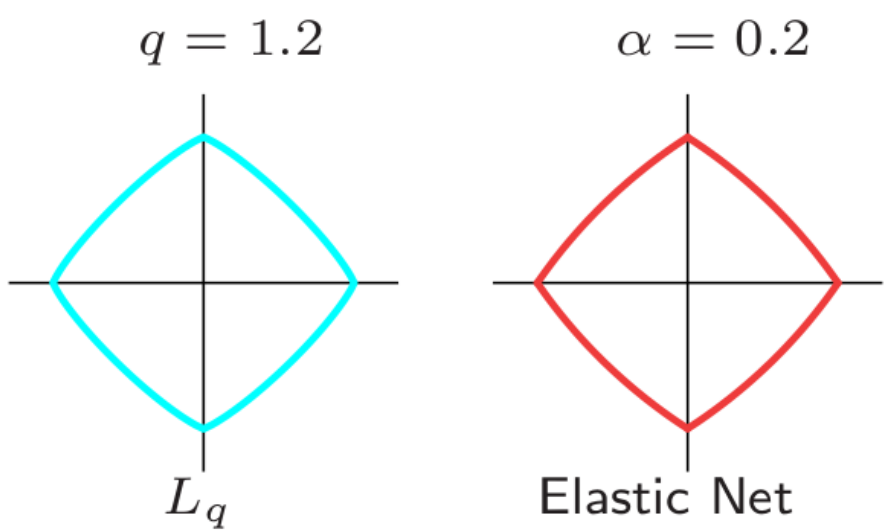
L2 norm



Elastic net

- **Elastic penalty**

$$\hat{\beta} = \arg\min_{\beta} ||y - X\beta||_2^2 + \lambda \left(\alpha||\beta||_2^2 + (1-\alpha)||\beta||_1\right)$$



Structured regularization

Group lasso

Use K predefined groups of variables that are known to "work" together and expected to be either all active or all inactive together.

E.g.: genes belonging to the same biological pathway.

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} ||y - \sum_{k=1}^{K} X_k \boldsymbol{\beta}_k||_2^2 + \lambda \sum_{k=1}^{K} \sqrt{p_k}\, ||\boldsymbol{\beta}_k||_2$$

Size of group k

Features belonging to group k

Other examples of structured penalties

- **Overlapping groups**

  Jacob et al. (2009). Group lasso with overlap and graph lasso. *ICML.*

- **Graphs**

  Li & Li (2010). Variable selection and regression analysis for graph-structured covariates with an application to genomics. *Ann. App. Stats.*

- **Trees**

  Zhao et al. (2006). Grouped and hierarchical model selection through composite absolute penalties. *Ann. Stat.*

- **Multiple related tasks**

  Obozinski et al. (2006). Multitask feature selection. *Technical Report, UC Berkeley.*

Minimize SSE + λ x regularizer

- **Ridge**
  - gives similar weights to similar variables
  - not very sparse
  - analytical solution
- **Lasso**
  - randomly picks one of several correlated variables
  - sparse
  - LAR algorithm
- **Elastic net**
  - selects variables like the lasso
  - shrinks together the coefficients of correlated variables.
- **Many other regularizers** are possible

  Lp norms, groups, graphs, trees...

References

- *A Course in Machine Learning.* `http://ciml.info/dl/v0_99/ciml-v0_99-all.pdf`
  - **Regularization:** Chap 7.2–7.3
- *The Elements of Statistical Learning.* `http://web.stanford.edu/~hastie/ElemStatLearn/`
  - **Regularization:** Chap 10.12
  - **Ridge regression:** Chap 3.4.1
  - **Lasso:** Chap 3.4.2
  - **LAR:** Chap 3.4.4
  - **Elastic Net:** Chap 4.2

Lab 5

- Logistic regression with no scaling

```
# Logistic regression (no regularization, no scaling)
from sklearn import linear_model
clf_logreg = linear_model.LogisticRegression(C=1e6) # large C = no regularization

# Train the model
clf_logreg.fit(Xtr, ytr)

# Predict on the test set
# Predicted probabilities of belonging to the positive class
pos_idx = list(clf_logreg.classes_).index(1)
ypred_logreg = clf_logreg.predict_proba(Xte)[:, pos_idx]

# Predicted binary labels
ypred_logreg_b = np.where(ypred_logreg > 0.5, 1, 0)

from sklearn import metrics
print("No regularization: accuracy = %.3f" % metrics.accuracy_score(yte, ypred_logreg_b))
print("AUC = %.3f" % (metrics.roc_auc_score(yte, ypred_logreg)))
```

```
No regularization: accuracy = 0.636
AUC = 0.642
```

- Logistic regression with scaling

```
# Logistic regression (no regularization, scaling)
clf_logreg_s = linear_model.LogisticRegression(C=1e6)

# Train the model
clf_logreg_s.fit(Xtr_scaled, ytr)

# Predict on the test set
# Predicted probabilities of belonging to the positive class
pos_idx = list(clf_logreg_s.classes_).index(1)
ypred_logreg_s = clf_logreg_s.predict_proba(Xte_scaled)[:, pos_idx]
# Predicted binary labels
ypred_logreg_s_b = np.where(ypred_logreg_s > 0.5, 1, 0)

print("Scaled, no regularization: accuracy = %.3f" % metrics.accuracy_score(yte, ypred_logreg_s_b))
print("AUC = %.3f" % (metrics.roc_auc_score(yte, ypred_logreg_s)))
```

```
Scaled, no regularization: accuracy = 0.782
AUC = 0.859
```

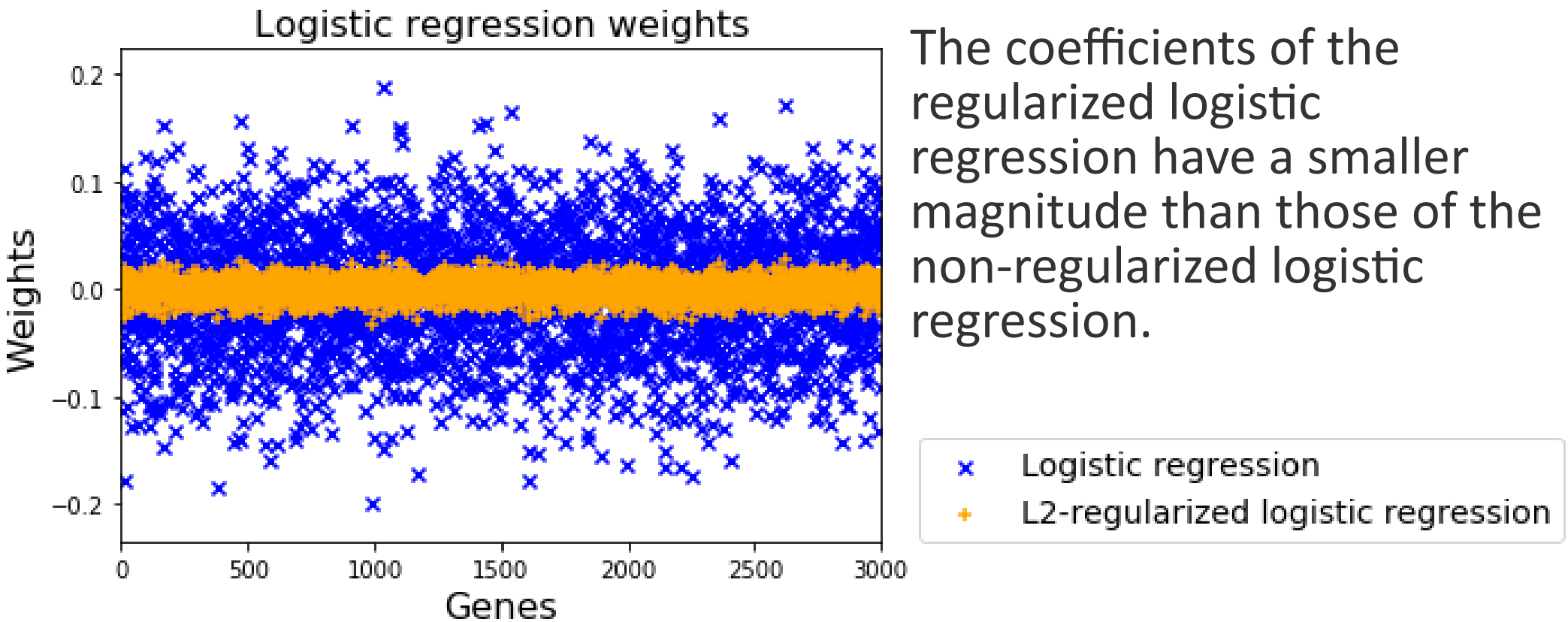
- L2-regularized logistic regression with scaling

```
cvalue = 0.01
clf_logreg_l2_s = linear_model.LogisticRegression(C=cvalue, penalty='l2')

# Train the model
clf_logreg_l2_s.fit(Xtr_scaled, ytr)

# index of positive class
pos_idx = list(clf_logreg_l2_s.classes_).index(1)
# predict probability of being positive
ypred_logreg_l2_s = clf_logreg_l2_s.predict_proba(Xte_scaled)[:, pos_idx]
# predict binary labels
ypred_logreg_l2_s_b = np.where(ypred_logreg_l2_s > 0.5, 1, 0)

print("Scaled, l2 regularization (C=%.2e): accuracy = %.3f" % (cvalue,
                                                                metrics.accuracy_score(yte,
                                                                                       ypred_logreg_l2_s_b)))
print("AUC = %.3f" % (metrics.roc_auc_score(yte, ypred_logreg_l2_s)))
```

```
Scaled, l2 regularization (C=1.00e-02): accuracy = 0.782
AUC = 0.858
```

- The performance did not change.
- Did the regression coefficients change?

```python
# Effect of l2-regularization on the weights
num_features = X_clf.shape[1]
plt.scatter(range(num_features), clf_logreg_s.coef_,
            color='blue', marker='x', label='Logistic regression')
plt.scatter(range(num_features), clf_logreg_l2_s.coef_,
            color='orange', marker='+', label='L2-regularized logistic regression')

plt.xlabel('Genes', fontsize=16)
plt.ylabel('Weights', fontsize=16)
plt.title('Logistic regression weights', fontsize=16)
plt.legend(fontsize=14, loc=(1.05, 0))
plt.xlim([0, num_features])
```



The coefficients of the regularized logistic regression have a smaller magnitude than those of the non-regularized logistic regression.

- Optimization of the parameter C

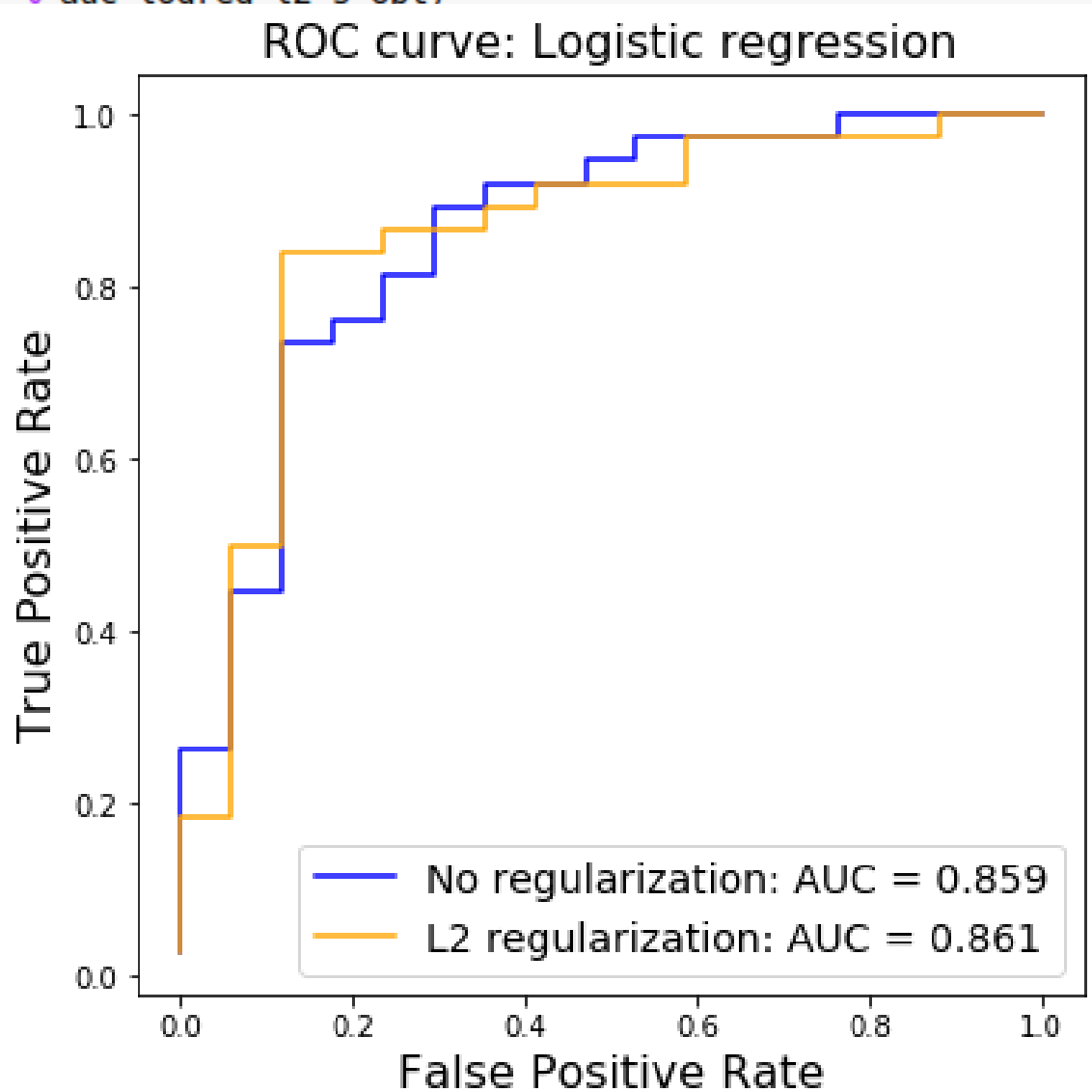
```
# Optimize cvalue
classifier = linear_model.LogisticRegression(penalty='l2')
param_grid = {'C': cvalues_list}
clf_logreg_l2_s_opt = model_selection.GridSearchCV(classifier, param_grid, cv=3)

# Train the model
clf_logreg_l2_s_opt.fit(Xtr_scaled, ytr)

# index of the positive class
pos_idx = list(clf_logreg_l2_s_opt.best_estimator_.classes_).index(1)
# predict probability of being positive
ypred_logreg_l2_s_opt = clf_logreg_l2_s_opt.best_estimator_.predict_proba(Xte_scaled)[:, pos_idx]
# predict binary label
ypred_logreg_l2_s_opt_b = np.where(ypred_logreg_l2_s_opt > 0.5, 1, 0)

cvalue_opt = clf_logreg_l2_s_opt.best_estimator_.C
print("Scaled, l2 regularization (C=%.2e): accuracy = %.3f" % (cvalue_opt,
                                                                metrics.accuracy_score(yte,
                                                                                       ypred_logreg_l2_s_opt_b)))
print("AUC = %.3f" % (metrics.roc_auc_score(yte, ypred_logreg_l2_s_opt)))
```

```
Scaled, l2 regularization (C=3.79e-04): accuracy = 0.818
AUC = 0.861
```

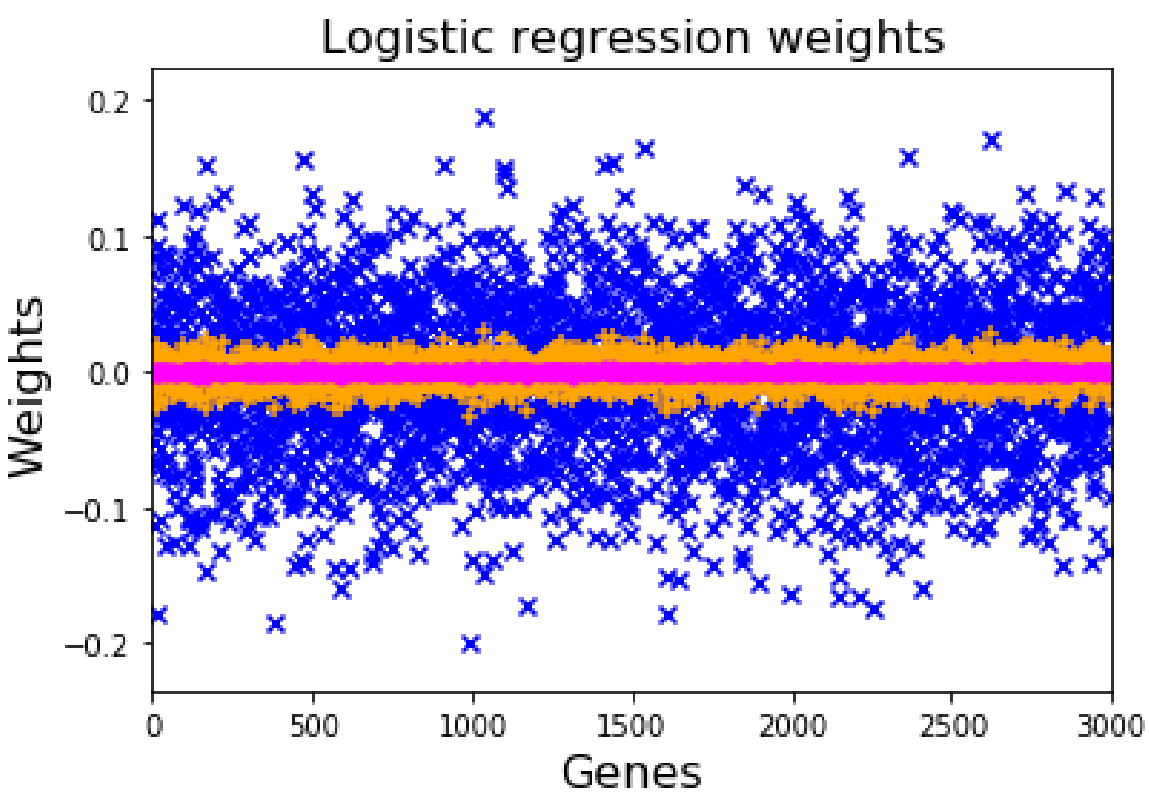
- Now C = 0.000379 (smaller), ie λ has increased. There is *more* regularization. We expect smaller regression coefficients.
- The performance has slightly improved

```
fig = plt.figure(figsize=(6, 6))
fpr_logreg_s, tpr_logreg_s, t = metrics.roc_curve(yte, ypred_logreg_s, pos_label=1)
auc_logreg_s = metrics.auc(fpr_logreg_s, tpr_logreg_s)
plt.plot(fpr_logreg_s, tpr_logreg_s, color='blue',
         label='No regularization: AUC = %0.3f' % auc_logreg_s)

fpr_logreg_l2_s_opt, tpr_logreg_l2_s_opt, t = metrics.roc_curve(yte, ypred_logreg_l2_s_opt, pos_label=1)
auc_logreg_l2_s_opt = metrics.auc(fpr_logreg_l2_s_opt, tpr_logreg_l2_s_opt)
plt.plot(fpr_logreg_l2_s_opt, tpr_logreg_l2_s_opt, color='orange',
         label='L2 regularization: AUC = %0.3f' % auc_logreg_l2_s_opt)

plt.xlabel('False Positive Rate', fontsize=16)
plt.ylabel('True Positive Rate', fontsize=16)
plt.title('ROC curve: Logistic regression', fon
plt.legend(fontsize=14)
```

- The ROC curves are not very different from each other (the AUC are quite close to each other as well).



```python
# Effect of l2-regularization on the weights
num_features = X_clf.shape[1]
plt.scatter(range(num_features), clf_logreg_s.coef_,
            color='blue', marker='x', label='Logistic regression')
plt.scatter(range(num_features), clf_logreg_l2_s.coef_,
            color='orange', marker='+', label='L2-regularized logistic regression')
plt.scatter(range(num_features), clf_logreg_l2_s_opt.best_estimator_.coef_,
            color='magenta', marker='.', label='L2-regularized logistic regression (opt)')

plt.xlabel('Genes', fontsize=16)
plt.ylabel('Weights', fontsize=16)
plt.title('Logistic regression weights', fontsize=16)
plt.legend(fontsize=14, loc=(1.05, 0))
plt.xlim([0, num_features])
```



- As expected, the optimized l2-regularized logistic regression has coefficients of even smaller magnitude than the non-optimized one.

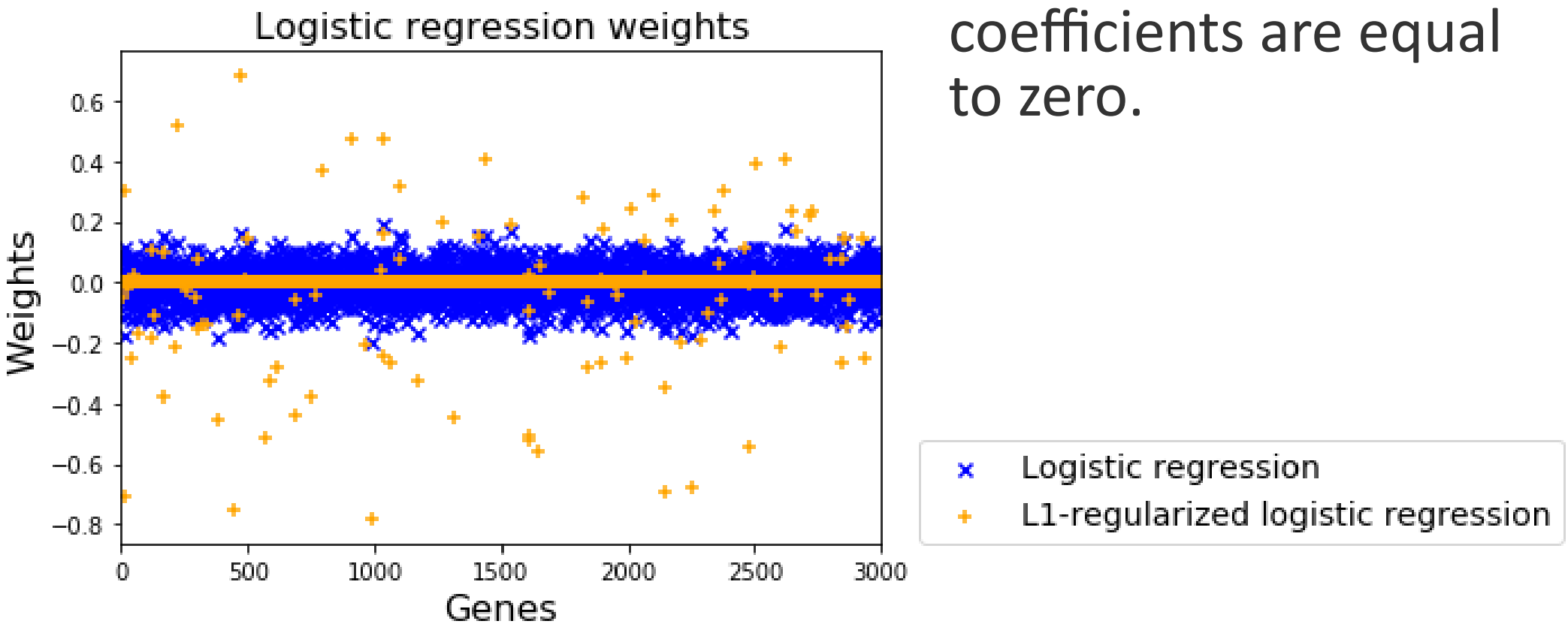
- L1-regularized logistic regression with scaling

```
cvalue = 10.
clf_logreg_l1_s = linear_model.LogisticRegression(C=cvalue, penalty='l1')

# train model
clf_logreg_l1_s.fit(Xtr_scaled, ytr)

# index of the positive class
pos_idx = list(clf_logreg_l1_s.classes_).index(1)
# predict the probability of belonging to the positive class
ypred_logreg_l1_s = clf_logreg_l1_s.predict_proba(Xte_scaled)[:, pos_idx]
# predict binary labels
ypred_logreg_l1_s_b = np.where(ypred_logreg_l1_s > 0.5, 1, 0)

print("Scaled, l1 regularization (C=%.2e): accuracy = %.3f" % (cvalue,
                                                                metrics.accuracy_score(yte,
                                                                                       ypred_logreg_l1_s_b)))
print("AUC = %.3f" % (metrics.roc_auc_score(yte, ypred_logreg_l1_s)))
```

```
Scaled, l1 regularization (C=1.00e+01): accuracy = 0.727
AUC = 0.748
```

- The performance is now noticeably worse!
- What are the regression coefficients like?

```python
num_features = X_clf.shape[1]
plt.scatter(range(num_features), clf_logreg_s.coef_,
            color='blue', marker='x', label='Logistic regression')
plt.scatter(range(num_features), clf_logreg_l1_s.coef_,
            color='orange', marker='+', label='L1-regularized logistic regression')

plt.xlabel('Genes', fontsize=16)
plt.ylabel('Weights', fontsize=16)
plt.title('Logistic regression weights', fontsize=16)
plt.legend(fontsize=14, loc=(1.05, 0))
plt.xlim([0, num_features])
```

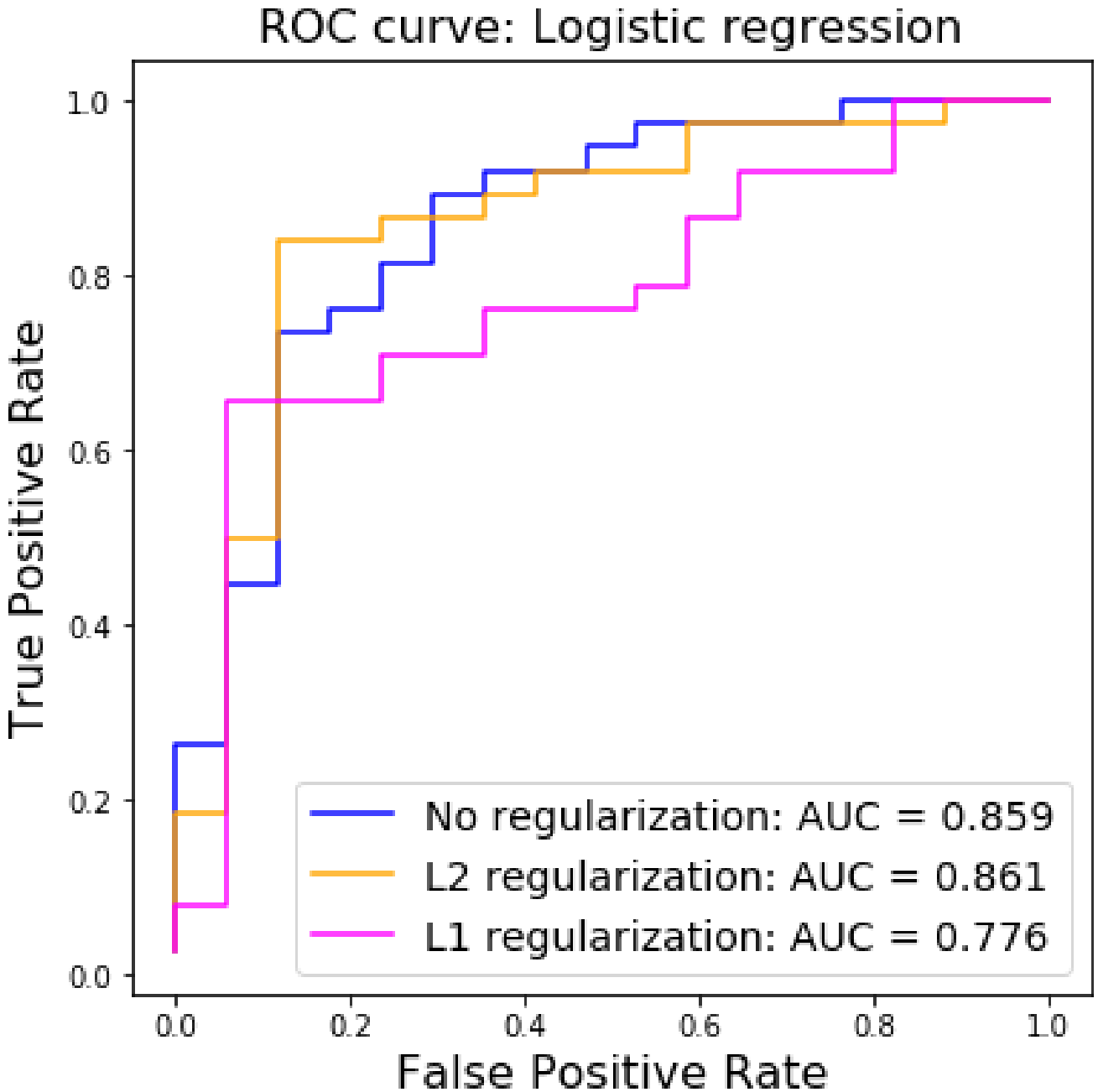


- It looks like many coefficients are equal to zero.

```python
# Number of selected features
print("The non-regularized logistic regression uses %d features" % \
      len(np.where(clf_logreg_s.coef_ !=0)[1]))
print("The L2-regularized logistic regression uses %d features" % \
      len(np.where(clf_logreg_l2_s.coef_ !=0)[1]))
print("The L1-regularized logistic regression uses %d features" % \
      len(np.where(clf_logreg_l1_s.coef_ !=0)[1]))

print("Number of features discarded by the L1-regularization: %d" % \
      (Xtr.shape[1] - len(np.where(clf_logreg_l1_s.coef_ !=0)[1])))
```

```
The non-regularized logistic regression uses 3000 features
The L2-regularized logistic regression uses 3000 features
The L1-regularized logistic regression uses 106 features
Number of features discarded by the L1-regularization: 2894
```

- Indeed, the performance got worse but the l1-regularized model only uses **106** features!
- Notice that if you train the l1-regularized logistic regression you very likely get a different performance and a different number of features: the l1 regularization is **unstable.**



- The ROC curve of the l1-regularized logistic regression is quite below the other curves, confirming a worse performance.

- For me, the optimal value of C is 88.6
- A larger C means a smaller λ, hence less regularization. I am expecting *more* non-zero weights (i.e. selected features).
- Indeed, my l1-regularized model with optimized C uses 289 features. These 289 features include the previous features (remember the lasso regularization path: once a feature is included it stays included).