THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 (2016/17 Term 1)
Financial Mathematics
Assignment 3 solution

Note: If you have any questions about the solution or assignment score, please let me know by sending an Email to kckchan@math.cuhk.edu.hk

1. ([2;p55]) Draw the expiry payoff diagrams for each of the following portfolios:

(a) Short one share, long two calls with exercise price E (this combination is called a straddle);

(b) Long one call and one put, both with exercise price E (this is also a straddle: why?);

(c) Long one call and two puts, all with exercise price E (a strip);

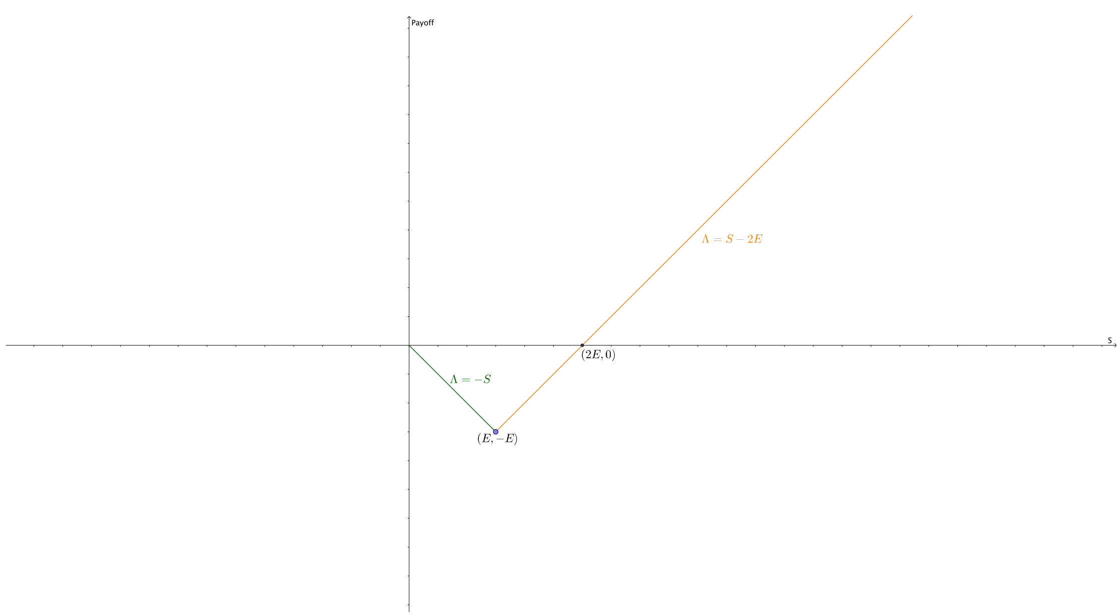
(d) Long one put and two calls, all with exercise price E (a strap);

(e) Long one call with exercise price E_1 and one put with exercise E_2. Compare the three cases $E_1 > E_2$ (known as a strangle), $E_1 = E_2$, $E_1 < E_2$.

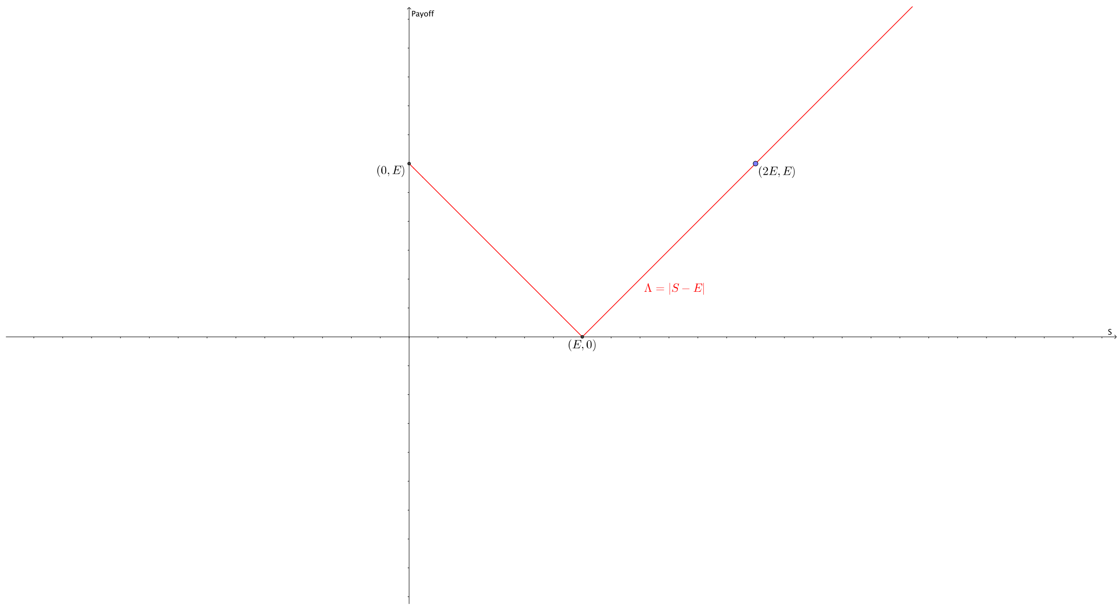
(f) As (e) but also short one call and one put with exercise price E (when $E_1 < E < E_2$, this is called a butterfly spread).

Solution:

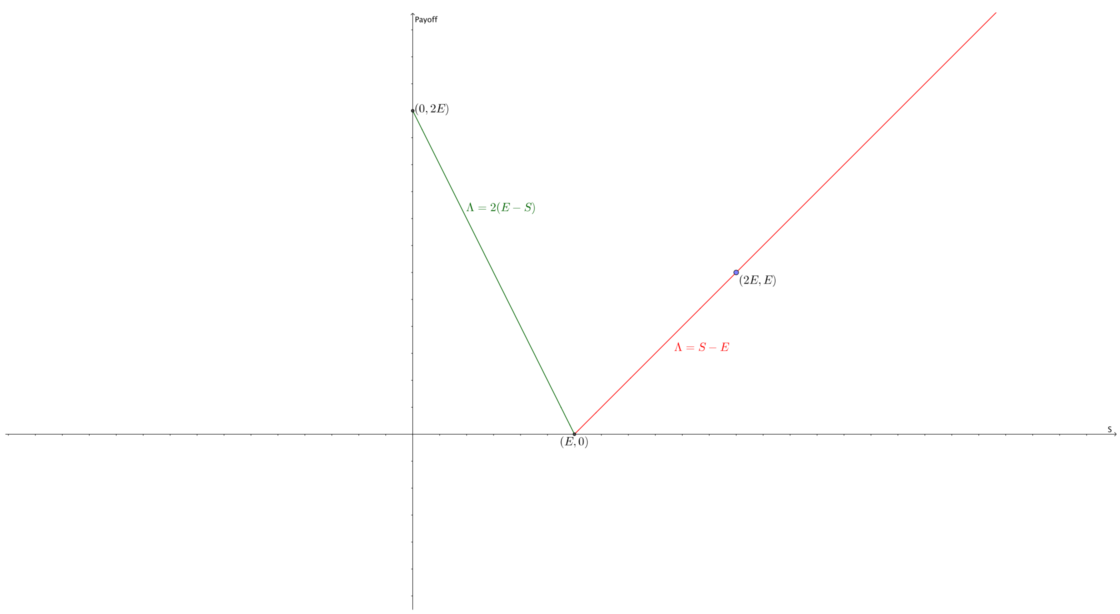
(a) $\Lambda = \begin{cases} -S & S \le E \\ S - 2E & S > E \end{cases}$



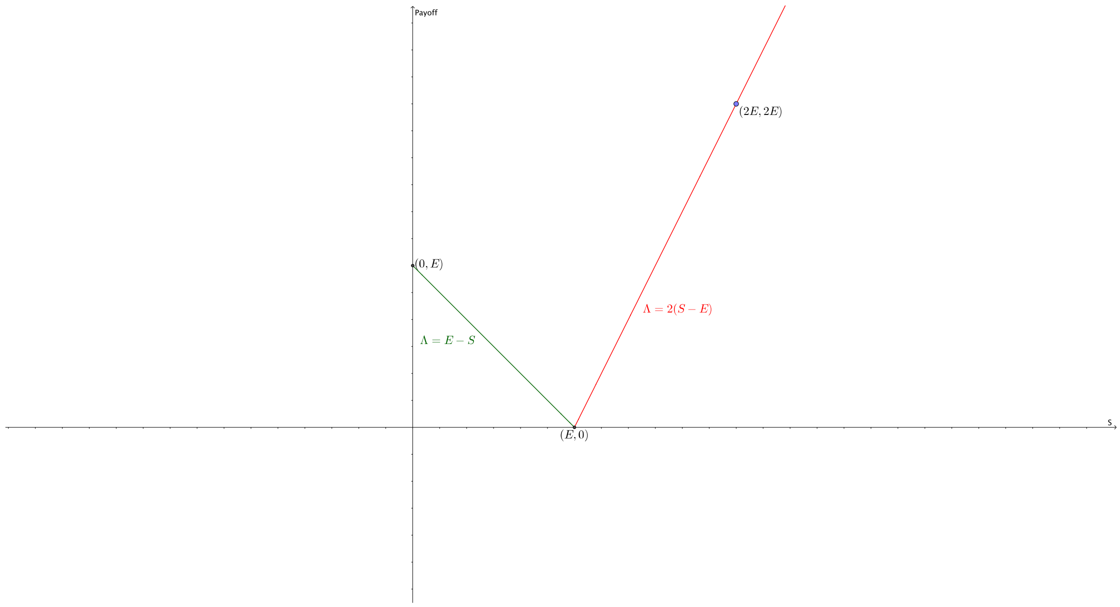
(b) $\Lambda = |E - S|$



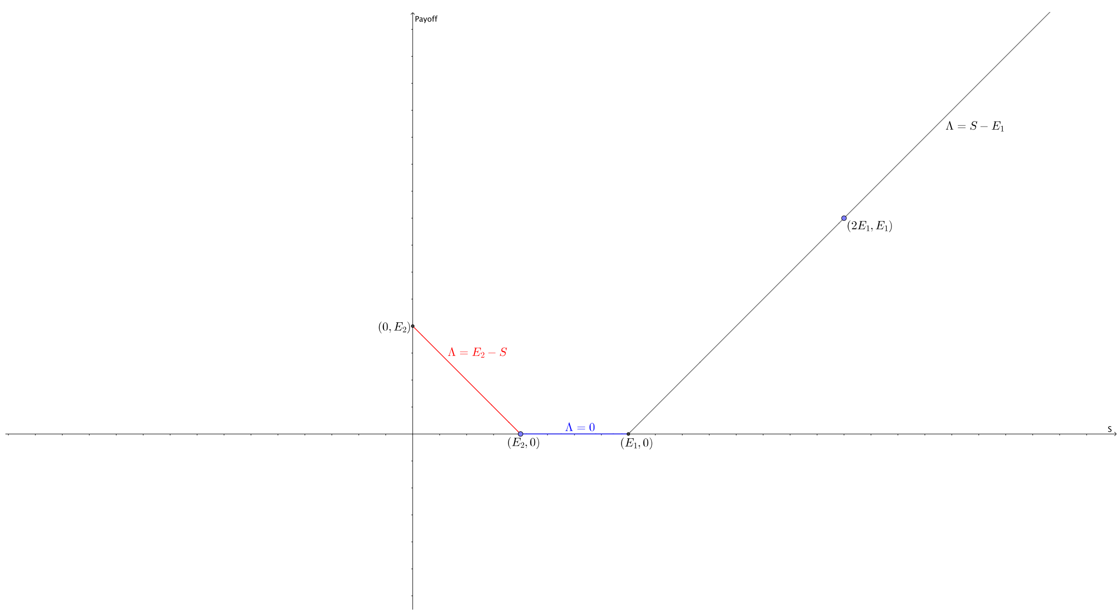
(c) $\Lambda = \begin{cases} 2(E-S) & S \le E \\ S-E & S > E \end{cases}$



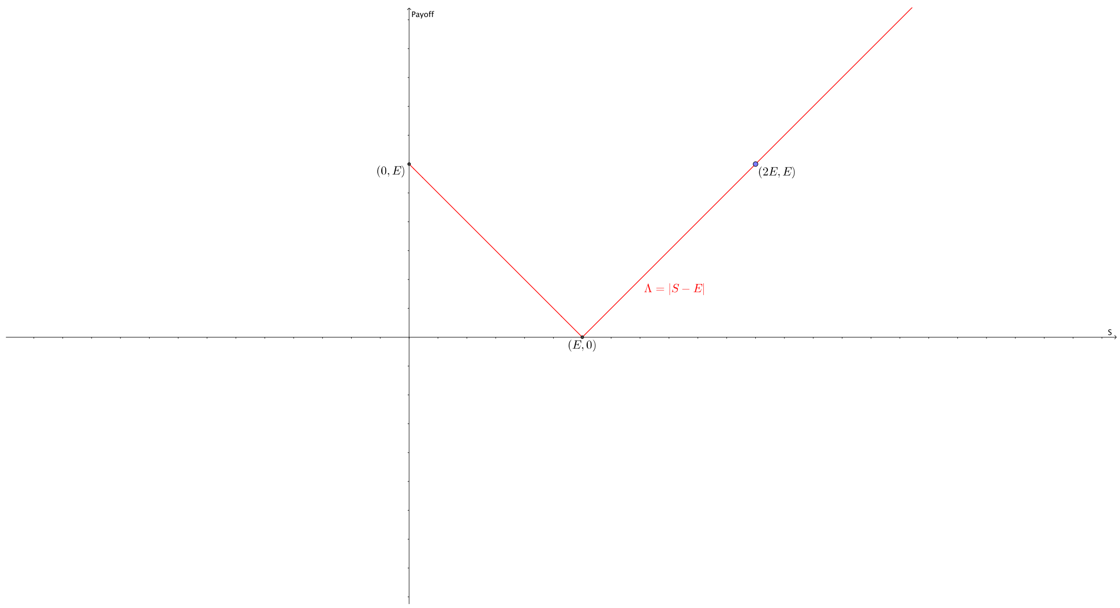
(d) $\Lambda = \begin{cases} E-S & S \le E \\ 2(S-E) & S > E \end{cases}$



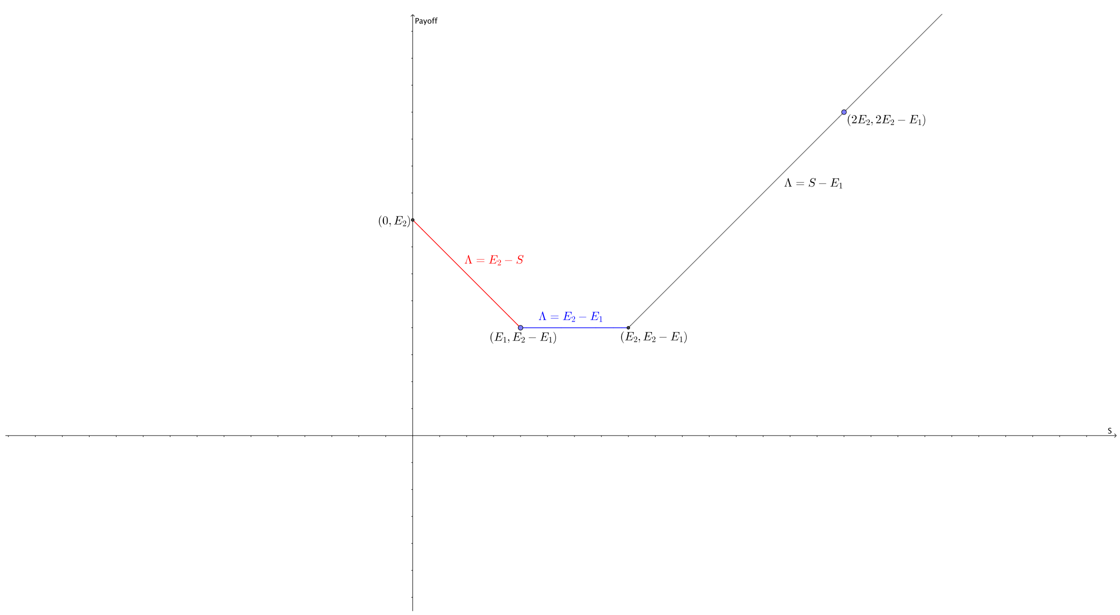
(e) (i) If $E_1 > E_2$, $\Lambda = \begin{cases} E_2 - S & S \leq E_2 \\ 0 & E_2 < S < E_1 \\ S - E_1 & S > E_1 \end{cases}$



(ii) If $E_1 = E_2 = E$, $\Lambda = |E - S|$



(iii) If $E_1 > E_2$, $\Lambda = \begin{cases} E_2 - S & S \leq E_1 \\ E_2 - E_1 & E_2 < S < E_1 \\ S - E_1 & S > E_2 \end{cases}$



(f) We have

$$\Lambda = \max(S - E_1, 0) + \max(E_2 - S, 0) - \max(S - E, 0) - \max(E - S, 0)$$

Here we have 9 cases:

(a) $E_1 > E_2 > E$

(b) $E_1 > E_2 = E$

(c) $E_1 > E > E_2$

(d) $E_2 > E_1 > E$

(e) $E_2 > E_1 = E$

(f) $E_2 > E > E_1$
(g) $E > E_2 > E_1$
(h) $E > E_2 = E_1$
(i) $E > E_1 > E_2$

By explicitly writing down the equation, we can obtain the 9 graphs.

Remark: When you are asked to draw a diagram, please don't just show me the line without indicating any information. You have to do more than stating the equation of the graph. Please state additional information, such as x-intercept, y-intercept and slope, to let people know what you are drawing. This time marks are given to you, but please make sure you draw a graph with sufficient information provided in exams.

2. Derive the price formula of an European put based on the Black-Scholes model.
Solution: Observe that the rescaled payoff function of an European option is

$$\Lambda(z) = \max\{1 - e^z, 0\}$$

So we have

$$v(x,\tau) = e^{-r\tau} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(x-z+(r-\frac{\sigma^2}{2})\tau^2)^2}{2\sigma^2\tau}} (1 - e^z)\, dz$$

By direct integration, we get

$$p(S,t) = Ee^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1)$$

3. Show that the payoff function of a portfolio $c-p$ is $S-E$. From this and the Black-Scholes formula, show the formula of the put-call parity.

Solution: $\Pi = \max\{S - E,\, 0\} - \max\{E - S,\, 0\} = S - E$
Next consider the rescaled payoff function of the above portfolio:

$$\Lambda(z) = e^z - 1$$

By using equation 3.15 in lecture notes, we have

$$v(x,\tau) = e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(x-z+(r-\frac{\sigma^2}{2})\tau^2)^2}{2\sigma^2\tau}} (e^z - 1)\, dz$$

By direct integration and change of variable, we get

$$c(s,t) - p(s,t) = S(t) - Ee^{-r(T-t)}$$

which is the **put-call parity**.

4. ([6;p56]) Find the most general solution of the Black-Scholes equation that has the special form

(a) $V = V(S)$
(b) $V = A(t)B(S)$

Solution:

(a) Note that $\dfrac{\partial V}{\partial t} = 0$. So we obtain the following ODE:

$$\frac{1}{2}\sigma^2 S \frac{d^2 S}{dS^2} + rS\frac{dV}{dS} - rV = 0$$

The above equation has solutions of the form $V(S) = S^p$ for some p, we will then solve for the p. By putting $V = S^p$, we have

$$\frac{1}{2}\sigma^2 p(p01) + rp - r = 0$$

which gives $p = 1$ or $p = \dfrac{-2r}{\sigma^2}$. Hence the solution is a linear combination of these two. i.e.

$$V(S) = AS + BS^{(-2r)/\sigma^2}$$

where A and B are constants, which could be determined if initial conditions are given.

(b) By simple differentiation we could obtain the following equation:

$$-\frac{A'(t)}{A(t)} = \frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} + rS\frac{B'(S)}{B(S)} - r$$

Since left hand side depends only on t and right hand side depends only on S, we then have

$$-\frac{A'(t)}{A(t)} = \lambda = \frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} + rS\frac{B'(S)}{B(S)} - r$$

where λ is a constant. For the t-related ODE, we have

$$\ln A(t) = -\lambda t + C' \quad \text{and} \quad A(t) = Ce^{-\lambda t}$$

where C is a constant. For the S-related ODE, using similar idea as (a), we have

$$p_{\pm} = \frac{-(r - \frac{\sigma^2}{2}) \pm \sqrt{(r + \frac{\sigma^2}{2})^2 + 2\sigma^2\lambda}}{\sigma^2}$$

Hence $B(S) = Ae^{(p_+)S} + Be^{(p_-)S}$ and $V(S) = A(t)B(S) = e^{-\lambda t}(Ae^{(p_+)S} + Be^{(p_-)S})$

5. ([9;p57]) Suppose that a share price S is currently \$100, and that tomorrow it will be either \$101, with probability p, or \$99 , with probability $1-p$. A call option, with value c, has exercise price \$100.

(a) Set up a Black-Scholes hedged portfolio and hence find the value of c. (Ignore interest rates.)

(b) Now repeat the calculation for a cash-or-nothing call option with payoff \$100 if the final asset price is above \$100, zero otherwise. What difference do you notice?

Solution:

(a) Construst the following portfolio: $\Pi = c - \Delta S$. Observe that

$$d\Pi = \begin{cases} 1 - 101\Delta & \text{with probaility } p \\ -99\Delta & \text{with probaility } 1-p \end{cases}$$

and hence we should have

$$1 - 101\Delta = 0 = -99\Delta$$

Solving gives $\Delta = 0.5$ and $\Pi = -49.5$ tomorrow. Ignoring interest rate we have

$$-49.5 = c - 0.5(100)$$

and hence $c = \$0.5$.

(b) Using the same construction as (a), we have

$$d\Pi = \begin{cases} 100 - 101\Delta & \text{with probaility } p \\ -99\Delta & \text{with probaility } 1 - p \end{cases}$$

Solving gives $\Delta = 50$ and $\Pi = -4950$. Ignoring interet rate we have

$$-4950 = c - 50(100)$$

and hence $c = \$50$.

6. ([5;p86])Suppose that in the Black-Scholes equation, $r(t)$ and (t) are both non-constant but known functions of t. Show that the the following procedure reduces the Black-Scholes equation to the diffusion equation.
Solution: (a) Observe that

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\frac{\partial}{\partial t'} \\ \frac{\partial}{\partial S} &= \frac{1}{Ee^x}\frac{\partial}{\partial x} \\ \frac{\partial^2}{\partial S^2} &= \frac{e^{-2x}}{E^2}\frac{\partial^2}{\partial x^2} - \frac{e^{-2x}}{E^2}\frac{\partial}{\partial x} \end{aligned}$$

Result follows by subsituting the above formulas into the Black-Scholes equation

(b) Note that $d\hat{\tau} = \frac{1}{2}\sigma^2(t')dt'$, hence,

$$\frac{1}{2}\sigma^2(t')\frac{\partial v}{\partial \hat{\tau}} = \frac{1}{2}\sigma^2(t')\frac{\partial^2 v}{\partial x^2} + (r(t') - \frac{1}{2}\sigma^2(t'))\frac{\partial v}{\partial x} - r(t')v$$

and result directly follows by dividing the above equation by $\frac{1}{2}\sigma^2(t')$.

(c) By direct differentiation, we have

$$\begin{aligned} \frac{\partial v}{\partial \hat{\tau}} &= F'(x + A(\hat{\tau})) \cdot a(\hat{\tau}) \cdot e^{-B(\hat{\tau})} - F(x + A(\hat{\tau})) \cdot b(\hat{\tau}) \cdot e^{-B(\hat{\tau})} \\ \frac{\partial v}{\partial x} &= F'(x + A)e^{-B(\hat{\tau})} \end{aligned}$$

Then we can verify that

$$\frac{\partial v}{\partial \hat{\tau}} = a(\hat{\tau})\frac{\partial v}{\partial x} - b(\hat{\tau})v$$

which confirms that $v = Fe^{-B}$ is the general solution.

(d) Define $v(x, \hat{\tau}) = e^{-B(\hat{\tau})}V(x + A(\hat{\tau}), \hat{\tau})$, we obtain

$$\begin{aligned} v_x &= e^{-B(\hat{\tau})}V_{\hat{x}} \\ v_{xx} &= e^{-B(\hat{\tau})}V_{\hat{x}\hat{x}} \\ v_{\hat{\tau}} &= -b(\hat{\tau})e^{-B(\hat{\tau})}V + e^{-B(\hat{\tau})}V_{\hat{x}}a(\hat{\tau}) + e^{-B(\hat{\tau})}V_{\hat{\tau}} \end{aligned}$$

Put back into the equation we obtain $V_{\hat{\tau}} = V_{\hat{x}\hat{x}}$, and hence we have the following close form solution:

$$V(\hat{x}, \hat{\tau}) = \frac{1}{\sqrt{4\pi\hat{\tau}}}\int_{-\infty}^{\infty} V_0(s)e^{-(\hat{x}-s)^2/4\hat{\tau}}ds$$

and hence

$$v(\hat{x}, \hat{\tau}) = \frac{e^{-B(\hat{\tau})}}{\sqrt{4\pi\hat{\tau}}}\int_{-\infty}^{\infty} V_0(s)e^{-(x+A(\hat{\tau})-s)^2/4\hat{\tau}}ds$$

(e) They have same initial condition. i.e.

$$V_0(s) = v_0(s)$$

7. ([1;p104]) What is the put-call parity relation for options on an asset that pays a constant continuous dividend yield?

Solution: Note that both options satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0$$

By linearity, $V = P - C$ also satisfies the above equation, with terminal condition $V(S, T) = E - S$. In order to make use of BS equation, we let

$$V_1(S, t) = e^{D_0(T-t)}V(S, t)$$

Hence V_1 satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - (r - D_0)V = 0$$

which gives

$$V_1(S, t) = Ee^{-(r-D_0)(T-t)} - S \quad \text{and} \quad P - C = Ee^{-r(T-t)} - Se^{-D_0(T-t)}$$

9. ([10;p105])Derive the put-call parity result for the forward/futures price in the form

$$C - P = (F - E)e^{-r(T-t)}$$

What is the corresponding version when the asset pays a constant continuous dividend yield?

Solution: Consider two portfolio given by

$$\Lambda_1(F, t) = C(F, t) + Ee^{-r(T-t)} \quad \text{and} \quad \Lambda_2(F, t) = P(F, t) + Fe^{-r(T-t)}$$

At $t = T$, we have

$$\Lambda_1(F, T) = \max\{F - E, 0\} + E = \max\{F, E\} = \Lambda_2(F, T)$$

By no arbitrage assumption, we have

$$C - P = (F - E)e^{-r(T-t)}$$

For the dividend case, using simliar idea we can obtain

$$C - P = (F - E)e^{-(r-D_0)(T-t)}$$

10. ([11;p105])What is the forward price for an asset that pays a single dividend $d_y S(t_d)$ at time t_d?

Solution: For $T \geq t \geq t_d^+$, we have $V(S, t) = S(t) - Fe^{-r(T-t)}$. By continuity, we have

$$V(S, t_d^-) = V(S(1 - d_y), t_d^+) = (1 - d_y)\left[S - \frac{Fe^{-r(T-t_d^-)}}{1 - d_y}\right]$$

Assuming zero cost in entering the forward contract, we obtain

$$V(S(1 - d_y), t_d^+) = (1 - d_y)\left[S - \frac{Fe^{-r(T-t_d-)}}{1 - d_y}\right] = 0 \quad \text{and} \quad F = (1 - d_y)Se^{-r(T-t)}$$