rV

μ = population mean!!

factorial trt combinations! when

$= V_1 V_2 \dots V_n$

V_j is the number of levels of factor

Consider an experiment involving ~~n observations~~, v treatment factors and b block factors. The number of levels ~~for the i~~ of th treatment factor and jth block factor are denoted by t_i and m_j, $(i = 1, 2, \dots, v; j = 1, 2, \dots, b)$ respectively. Let $\boldsymbol{y}$ be an $n \times 1$ vector of responses, then the linear mixed-effects model for the experiment can be written in matrix notation as

$$\boldsymbol{y} = \mathbf{1}\mu + \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\beta} + \epsilon, \tag{1}$$

where $\mathbf{1}$ is an $n \times 1$ vector whose elements are all unity, μ denotes the grand mean ~~of the data~~, and $\epsilon \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_n)$ is ~~a~~ an $n \times 1$ vector of unobserved random experimental errors. ~~where~~ The

Matrix $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. ~~The subscript n in $\mathbf{I}_n$ is omitted for clarity resulting in $\mathbf{I}$.~~ ← When?

The treatment parameter vector α of length ~~$t = t_1 \times t_2 \times \cdots \times t_v$~~ is defined as

$$\boldsymbol{\alpha} = (\alpha_{11\dots11}, \alpha_{11\dots12}, \dots, \alpha_{11\dots1t_v}, \dots, \alpha_{11\dots t_{v-1}t_v}, \dots, \alpha_{t_1t_2\dots t_{v-1}t_v}), \tag{2}$$

where $\alpha_{f_1\dots f_v}$ denotes the effect of treatment combination $f_1 \dots f_v$, $(f_i = 1, \dots, t_i; i = 1, \dots, v)$. The $n \times b$ treatment design matrix, $\mathbf{X}$, in (1), defines the allocation of treatment combinations to experimental units. ~~The matrix $\mathbf{X}$ comprises n rows, corresponding to the number of observations, and t columns, corresponding to the number of factorial treatment combinations.~~ The vector of block parameters of length $m = m_1 + m_2 + \cdots + m_b$ is defined as

This doesn't explain how! How does it indicate the allocation?

$$\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_b), \tag{3}$$

where

$$\boldsymbol{\beta}_j = (\beta_{j1}, \beta_{j2}, \dots, \beta_{jm_j})$$

$(x_1, x_2 \dots x_n)$: n-tuple $x_i \in \{0, \dots, V_i\}$

and $\beta_{jk} \sim \mathcal{N}(0, \sigma_j^2)$, $(j = 1, 2, \dots, b; k = 1, 2, \dots, m_j)$. The block design matrix, $\mathbf{Z}$, in (1) can then be expressed as

$$\mathbf{Z} = [\mathbf{Z}_1 | \mathbf{Z}_2 | \dots | \mathbf{Z}_b], \tag{4}$$

β_j is not a block factor!

Same

where $\mathbf{Z}_j$ is the design matrix corresponding to jth block factor, ~~$\boldsymbol{\beta}_j$~~. Thus, the matrix $\mathbf{Z}$ consists of n rows, corresponding to the number of observations, and m columns, corresponding to the sum of the length of each vector parameter, $\boldsymbol{\beta}_j$, in the vector $\boldsymbol{\beta}$. The treatment and block design matrices ~~presented~~ defined here are shown to be essential components of the decomposition method described in this section.

Why to q?

2.2. Null decomposition using projection matrices

The vector of responses, $\boldsymbol{y}$, in (1), spans ~~an~~ n-dimensional Euclidean space, commonly denoted by $\mathbb{R}^n$. A vector space, $\mathbb{V}$, is a *subspace* of $\mathbb{R}^n$, i.e. $\mathbb{V} \subset \mathbb{R}^n$, if every vector in $\mathbb{V}$ is also in $\mathbb{R}^n$ (Hadi 1996). The information decomposition of $\boldsymbol{y}$ involves its projection from the $\mathbb{R}^n$ space into its constituent vector subspace components. These vector subspaces correspond to what are commonly referred to as the *strata* of the ANOVA. This can be mathematically expressed as

$$\mathbb{R}^n = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_q, \tag{5}$$

what??

where $\oplus$ denotes the addition operator of vector spaces, and $\mathbb{V}_l$ $(l = 1, 2, \dots, q)$. ~~Thus, the $q + 1$ terms in (5) is directly related to the number of strata, and this depends on the experimental design.~~ For example, an experiment arranged according to a row-column design generates ~~four terms of~~ strata, namely "Grand Mean", "Between Rows", "Between Columns" and "Within Rows and Columns". For any designed experiment, the first and last elements,

Doesn't follow. Flow?

You need to connect q to something before you write equation (5)

$P_0 = Z_0(Z_0'Z)^{-1}Z_0'$
$P_1 =$

$Z_i(Z_i'Z_i)^{-1}Z_i'$

elements in (5), i.e. $\mathbb{V}_0$ and $\mathbb{V}_q$, always denote the vector subspace corresponding to ~~for~~ the grand mean and experimental error, respectively.

Since the decomposition of what? is the separation of the known variation in the data, the variance structure of the data, $\boldsymbol{y}$, can be expressed in spectral form as,

This is like saying because — it follows that (6). Not the case! Re-word.

$$\mathrm{Var}(\boldsymbol{y}) = \sum_{l=0}^{q} \xi_l \mathbf{Q}_l, \tag{6}$$

or simply projection matrix)

where $\mathbf{Q}_l$ is an $n \times n$ *orthogonal projector* matrix, which projects ~~transforms vector~~ $\boldsymbol{y}$ from ~~vector space~~ $\mathbb{R}^n$ onto the corresponding vector subspace $\mathbb{V}_l$ (i.e. stratum l) ~~and ξ_l is the lth~~ stratum variance (i.e. $\xi_l = \mathrm{Var}(\mathbf{Q}_l\boldsymbol{y})$). ~~Since the orthogonal projector is used to linear transform a vector onto a specific vector subspace, the orthogonal projector is a~~ the *projection matrix*, ~~which~~ has the properties of being symmetric (i.e. $\mathbf{Q}_l' = \mathbf{Q}_l$) orthogonal (i.e. $\mathbf{Q}_l\mathbf{Q}_{l'} = 0$ if $l \neq l'$) and idempotent, i.e. $\mathbf{Q}_l^2 = \mathbf{Q}_l$ (Hadi 1996). no italics Q_l

It follows from (6) that ~~The~~ total ~~unadjusted~~ sum of squares (SS) of $\boldsymbol{y}$, ~~i.e. $\boldsymbol{y}'\boldsymbol{y}$~~, can be decomposed into $q+1$ components ~~of the SS~~, i.e. $\boldsymbol{y} \in \mathbb{R}^n$ into orthogonal SS, ignoring treatments)

$Q_l y$?

$$\boldsymbol{y}'\boldsymbol{y} = \sum_{l=0}^{q} \boldsymbol{y}'\mathbf{Q}_l\boldsymbol{y}, \tag{7}$$

where $\boldsymbol{y}'\mathbf{Q}_l\boldsymbol{y}$ denotes the total SS in the lth stratum. This ~~Equations (5), (6) and (7) thus give a basic illustration of the~~ decomposition of ~~the data vector space~~, total variability and SS without the treatment, which we refer to as the *null decomposition*. The remainder of this section describes each step of the null decomposition, ~~particularly~~ showing how the orthogonal projector, ~~denoted by~~ $\mathbf{Q}_l$, is computed in stratum l.

For any designed experiment, the initial step of the null decomposition is to separate out the first term of the block structure formula, i.e. the grand mean, from the data vector, $\boldsymbol{y}$. Since μ, in (1), is a vector of length 1, the grand mean ~~vector~~ spans a 1-dimensional grand mean ~~vector subspace, denoted by $\mathbb{V}_0$ in (5)~~. Meanwhile, the ~~orthogonal projector~~ projection matrix, $\mathbf{Q}_l$ ~~is the projection matrix to~~ projects a vector onto vector subspace $\mathbb{V}_l$. ~~This vector subspace can be generated by a set of vectors or a matrix.~~ The grand mean vector subspace, $\mathbb{V}_0$, is thus generated by a $n \times n$ averaging matrix, denoted by $\mathbf{K}_n$, with all elements equal to n^{-1}. The subscript n in $\mathbf{K}_n$ is again omitted for simplicity resulting in $\mathbf{K}$. Hence, the projection of the vector $\boldsymbol{y}$ on to the grand mean vector subspace is given by

is a scalar, not a vector.

define What is K_n? Repetitive.

$$\mathbf{Q}_0\boldsymbol{y} = \mathbf{P}_{\mathbf{K}}\boldsymbol{y}$$

which one? No!

$P_0 = K$ and its other $P_0^{\perp} = I - K$.

P_0

where $\mathbf{P}_{\mathbf{K}}$ denotes the projection matrix of matrix $\mathbf{K}$ and is defined as

$$\mathbf{P}_{\mathbf{K}} = \mathbf{K}(\mathbf{K}'\mathbf{K})^{-1}\mathbf{K}'.$$

Do all K matrices have the same subscript for the rest of this section? If not, you cannot drop them

Since $\mathbf{P}_{\mathbf{K}}$ can be shown to be the same as $\mathbf{K}$, ~~for simplicity~~, $\mathbf{P}_{\mathbf{K}}\boldsymbol{y}$ = Ky ~~is rewritten as $\mathbf{K}\boldsymbol{y}$~~.

The ~~next step is to obtain the~~ orthogonal complement of $\mathbf{K}\boldsymbol{y}$ by subtracting the $\mathbf{K}\boldsymbol{y}$ from $\boldsymbol{y}$, i.e.

$$\boldsymbol{y} - \mathbf{K}\boldsymbol{y} = (\mathbf{I} - \mathbf{K})\boldsymbol{y},$$

where $(\mathbf{I}-\mathbf{K})\boldsymbol{y}$ denotes the *mean corrected observational vector*, which spans vector $\mathbb{V}_0^{\perp}$ with dimension of $(n-1)$. The vector subspace $\mathbb{V}_0^{\perp}$ is also the orthogonal complement of $\mathbb{V}_0$. Furthermore, the adjusted total SS is computed by pre-multiplying the $(\mathbf{I}-\mathbf{K})\boldsymbol{y}$ by its transpose, i.e.

$$[(\mathbf{I}-\mathbf{K})\boldsymbol{y}]'[(\mathbf{I}-\mathbf{K})\boldsymbol{y}] = \boldsymbol{y}'(\mathbf{I}-\mathbf{K})\boldsymbol{y}. \tag{8}$$

$V_0^{\perp}$, and

spans the n-1 dimensional orthogonal complement of V_0.

We refer to (7) ~~the decomposition~~ which ignores treatment effects, as a null decomposition.

$(Py)'(Py) = y'(P'P)y = y'Py$

$Py \rightarrow y = z$ $\quad (z'z)$ Project into

block structure formulae?

adj for the grand mean is

The total ~~adjusted~~ SS ~~can also be~~ computed by subtracting the grand mean of data, $\boldsymbol{y}'\mathbf{K}\boldsymbol{y}$, from the total unadjusted total SS, $\boldsymbol{y}'\boldsymbol{y}$, i.e.

$(I-K)$

$$\cancel{\boldsymbol{y}'\boldsymbol{y} - \boldsymbol{y}'\mathbf{K}\boldsymbol{y} =} \ \boldsymbol{y}'(\mathbf{I}-\mathbf{K})\boldsymbol{y}.$$

The vector $(\mathbf{I}-\mathbf{K})\boldsymbol{y}$ is then projected onto the vector subspace of stratum 1, i.e. $\mathbb{V}_1$, which results in $\mathbf{Q}_1\boldsymbol{y}$, where $\mathbf{Q}_1$ is the orthogonal projector of the stratum that corresponds to the second term of the expanded block structure formula. The vector subspace, $\mathbb{V}_1$, is generated by the ~~block design matrix~~ corresponding to the second term of the expanded block structure formula, ~~denoted by~~ $\mathbf{Z}_1$. The orthogonal projector $\mathbf{Q}_1$ is thus given by the $\mathbf{I}-\mathbf{K}$ pre-multiplied by $\mathbf{P}_{\mathbf{Z}_1}$, i.e. $\mathbf{P}_{\mathbf{Z}_1}(\mathbf{I}-\mathbf{K})$, which can be re-written as

$I_K \otimes K$ $\rightarrow$

$$\mathbf{P}_{\mathbf{Z}_1}(\mathbf{I}-\mathbf{K})\cancel{\boldsymbol{y}} = (\mathbf{P}_{\mathbf{Z}_1}-\mathbf{K})\cancel{\boldsymbol{y}} \tag{9}$$

$Q_1 = P_{Z_1}(I-K)$

where vector $(\mathbf{P}_{\mathbf{Z}_1}-\mathbf{K})\boldsymbol{y}$ represents estimates of effects associated with the second term of the expanded block structure formula. The orthogonal complement of $(\mathbf{P}_{\mathbf{Z}_1}-\mathbf{K})\boldsymbol{y}$ is then derived by subtraction from the vector $(\mathbf{I}-\mathbf{K})\boldsymbol{y}$ as

$$(\mathbf{I}-\mathbf{K})\boldsymbol{y} - (\mathbf{P}_{\mathbf{Z}_1}-\mathbf{K})\boldsymbol{y} = (\mathbf{I}-\mathbf{P}_{\mathbf{Z}_1})\boldsymbol{y}, \tag{10}$$

which corresponds to the elimination of the effects from the second term of the expanded block structure formula.

The SS are derived by pre-multiplying the vectors in (10) by their transpose, as described in (8), i.e.

$$\boldsymbol{y}'(\mathbf{I}-\mathbf{K})\boldsymbol{y} - \boldsymbol{y}'(\mathbf{P}_{\mathbf{Z}_1}-\mathbf{K})\boldsymbol{y} = \boldsymbol{y}'(\mathbf{I}-\mathbf{P}_{\mathbf{Z}_1})\boldsymbol{y}.$$

If the expanded block structure formula contains additional terms, the vector $(\mathbf{I}-\mathbf{P}_{\mathbf{Z}_1})\boldsymbol{y}$ is further projected onto the next vector subspace, $\mathbb{V}_2$, corresponding to the third term of the expanded block structure formula. Thus, in general, the projection of the data vector, $\boldsymbol{y}$ from $\mathbb{V}_l$ onto $\mathbb{V}_{l+1}$ can be written as $\mathbf{P}_{\mathbf{Z}_{l+1}}\mathbf{Q}_l\boldsymbol{y}$. The orthogonal complement of $\mathbf{P}_{\mathbf{Z}_{l+1}}\mathbf{Q}_l\boldsymbol{y}$ can be derived by subtraction from $\mathbf{Q}_l\boldsymbol{y}$, i.e.

$$\rightarrow \ \mathbf{Q}_l\boldsymbol{y} - \mathbf{P}_{\mathbf{Z}_{l+1}}\mathbf{Q}_l\boldsymbol{y} = (\mathbf{I}-\mathbf{P}_{\mathbf{Z}_{l+1}})\mathbf{Q}_l\boldsymbol{y} = \mathbf{Q}_{l+1}\boldsymbol{y}, \ (l = 0, 1, \ldots, q; \mathbf{Q}_0 = \mathbf{K}). \tag{11}$$

The SS thus are derived by pre-multiplying the vector in (11) by their transpose, as follows:

$$\boldsymbol{y}'\mathbf{Q}_l\boldsymbol{y} - \boldsymbol{y}'\mathbf{Q}_l\mathbf{P}_{\mathbf{Z}_{l+1}}\mathbf{Q}_l\boldsymbol{y} = \boldsymbol{y}'\mathbf{Q}_l(\mathbf{I}-\mathbf{P}_{\mathbf{Z}_{l+1}})\mathbf{Q}_l\boldsymbol{y} = \boldsymbol{y}'\mathbf{Q}_{l+1}\boldsymbol{y}, \ (l = 0, 1, \ldots, q; \mathbf{Q}_0 = \mathbf{K}).$$

Providing the SS for each stratum is defined using the orthogonal projectors, the EMS can be computed for the theoretical ANOVA table. The expected sum of squares (ESS) of the lth stratum ~~without the~~ treatment effects ~~can be shown as~~

ignoring   is given by

$$\mathrm{E}(\boldsymbol{y}'\mathbf{Q}_l\boldsymbol{y}) = \mathrm{tr}(\mathbf{Q}_l)\,\mathrm{cov}(\boldsymbol{y}), \tag{12}$$

where $\mathrm{cov}(\boldsymbol{y})$ is the variance-covariance matrix of y and $\mathrm{tr}(\mathbf{Q}_l)$ is the *trace* of the matrix $\mathbf{Q}_l$ (Searle 1982).

Consider an experiment arranged in RCBD, then the null decomposition of the total SS can be expressed as

$$\boldsymbol{y}'\boldsymbol{y} = \boldsymbol{y}'\mathbf{K}\boldsymbol{y} + \boldsymbol{y}'(\mathbf{P}_{\mathbf{Z}_1}-\mathbf{K})\boldsymbol{y} + \boldsymbol{y}'(\mathbf{I}-\mathbf{P}_{\mathbf{Z}_1})\boldsymbol{y}, \tag{13}$$

12

where $\mathbf{K}$, $(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})$ and $(\mathbf{I} - \mathbf{P}_{\mathbf{Z}_1})$ denote the orthogonal projectors of the grand mean, Between and Within Blocks strata, i.e. $\mathbb{V}_0$, $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively.

Since the SS of the Between Blocks stratum is $\boldsymbol{y}'(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})\boldsymbol{y}$, the ESS of the Between Blocks stratum can be shown as

$$\begin{aligned} \mathrm{E}[\boldsymbol{y}'(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})\boldsymbol{y}] &= \mathrm{tr}(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})\,\cancel{\mathrm{cov}}^{\text{var}}(\boldsymbol{y}) = \\ &\cancel{= \mathrm{tr}(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})\,\mathrm{cov}(\boldsymbol{\epsilon}) + \mathrm{tr}[\mathbf{Z}'(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})\mathbf{Z}]\,\mathrm{cov}(\boldsymbol{\beta})} \\ &= \cancel{(m_1 - 1)\sigma^2 + (m_1 - 1)m_2\sigma_1^2} \\ &= (m_1 - 1)(\sigma^2 + m_2\sigma_1^2), \end{aligned} \tag{14}$$

where m_1 and m_2 denote the Number of block number and block size, respectively. ~~Subsequently,~~ Thus the EMS is calculated by dividing the ESS by the corresponding DF. It follows that ~~Hence, the EMS~~ of the Between Blocks stratum is $\sigma^2 + m_2\sigma_1^2$. Similarly, the EMS of the Within Blocks stratum can be shown ~~as~~ to be σ^2. Table 1 shows the theoretical ANOVA table ~~without~~ ignoring ~~the~~ treatment ~~components~~ effects for RCBD.

Table 1: ANOVA with the coefficients of variance components of the EMS from an experiment arranged with RCBD.

Source of Variation	Vector (Sub)space	DF	SS	EMS
Between Blocks	$\mathbb{V}_1$	$m_1 - 1$	$\boldsymbol{y}'(\mathbf{P}_{\mathbf{Z}_1} - \mathbf{K})\boldsymbol{y}$	$\sigma^2 + m_2\sigma_1^2$
Within Blocks	$\mathbb{V}_1^{\perp} = \mathbb{V}_2$	$m_1(m_2 - 1)$	$\boldsymbol{y}'(\mathbf{I} - \mathbf{P}_{\mathbf{Z}_1})\boldsymbol{y}$	σ^2
Adjusted Total	$\mathbb{V}_0^{\perp}$	$n - 1$	$\boldsymbol{y}'(\mathbf{I} - \mathbf{K})\boldsymbol{y}$	
Grand Mean	$\mathbb{V}_0$	1	$\boldsymbol{y}'\mathbf{K}\boldsymbol{y}$	
Total	$\mathbb{R}^n$	n	$\boldsymbol{y}'\boldsymbol{y}$	

Use $m_1 - 1$.

2.3. Computing the treatment SS

~~The previous~~ Section 2.2 presented an overview of the process used to compute the SS and EMS involved in ~~from~~ the null decomposition. The next step is to compute the treatment SS and EMS within each stratum. To do this ~~compute the treatment SS~~, the treatment design matrix, $\mathbf{X}$, in (1) must be defined ~~in (1) with another set of matrices. This set of matrices are known as~~ the treatment contrast matrices, and are denoted by $\mathbf{C}_x$ (John ~~and Williams~~ 1987). These matrices ~~can be~~ are generated based on the treatment structure and *yield identity*, which describes how the treatment effects are partitioned into orthogonal components. Rather

Consider an example where the treatment structure comprises a single treatment factor, with ~~the~~ effect of treatment f_1 ~~is~~ denoted by α_{f_1}, $f_1 = 1, \ldots, t_1$, then the yield identity is given by

$$\alpha_{f_1} = \overline{\alpha}_{\cdot} + (\alpha_{f_1} - \overline{\alpha}_{\cdot}) \tag{15}$$

The dot in the subscript denotes the summation over the subscript it replaces and the over-line, ~~also known as *bar*,~~ indicates the average of the terms associated with the nominal subscript. Thus, $\overline{\alpha}_{\cdot}$ denotes the ~~overall~~ mean ~~of the α_{f_1}~~ over all treatments (trt levels) and $\alpha_{f_1} - \overline{\alpha}_{\cdot}$, denotes the effect of treatment f_1 corrected for the grand mean mean. In matrix notation, (15) can be written as

$$\boldsymbol{\alpha} = \mathbf{C}_0\boldsymbol{\alpha} + \mathbf{C}_1\boldsymbol{\alpha}$$

where

$$\begin{aligned}\mathbf{C}_0 &= \mathbf{K}_{t_1}\\ \mathbf{C}_1 &= \mathbf{I}_{t_1} - \mathbf{K}_{t_1}\end{aligned}$$

where $\mathbf{I}_{t_1}$ is the $t_1 \times t_1$ identity matrix and $\mathbf{K}_{t_1}$ is the $t_1 \times t_1$ averaging matrix with all elements equal t_1^{-1}. The contrast matrix $\mathbf{C}_0\boldsymbol{\alpha}$, corresponding to $\overline{\alpha}_{.}$, can be seen as an operation that averages over the α_i. The contrast matrix $\mathbf{C}_1\boldsymbol{\alpha}$, corresponds to $\alpha_{f_1} - \overline{\alpha}_{.}$, represents the effect of treatment i after correction for the mean.

Consider a factorial experiment with two factors, f_1 and f_2, with levels t_1 and t_2, respectively, where the yield identity of $\alpha_{f_1 f_2}$, $f_1 = 1, \ldots, t_1$; $f_2 = 1, \ldots, t_2$, can be written as,

$$\alpha_{f_1 f_2} = \overline{\alpha}_{..} + (\overline{\alpha_{f_1.}} - \overline{\alpha}_{..}) + (\overline{\alpha}_{.f_2} - \overline{\alpha}_{..}) + (\alpha_{f_1 f_2} + \overline{\alpha_{f_1.}} + \overline{\alpha}_{.f_2} - \overline{\alpha}_{..}), \tag{16}$$

where $\overline{\alpha}_{..}$ denotes the overall mean of $\alpha_{f_1 f_2}$, $\overline{\alpha_{f_1.}} - \overline{\alpha}_{..}$ denotes the main effects of factor f_1, $\overline{\alpha}_{.f_2} - \overline{\alpha}_{..}$ represents the main effects of factor f_2 and $\alpha_{f_1 f_2} + \overline{\alpha_{f_1.}} + \overline{\alpha}_{.f_2} - \overline{\alpha}_{..}$ is the interaction of two treatment factors. In the matrix notation, (16) can be written as

$$\boldsymbol{\alpha} = \mathbf{C}_{00}\boldsymbol{\alpha} + \mathbf{C}_{10}\boldsymbol{\alpha} + \mathbf{C}_{01}\boldsymbol{\alpha} + \mathbf{C}_{11}\boldsymbol{\alpha}$$

where

$$\begin{aligned}\mathbf{C}_{00} &= & \mathbf{K}_{t_1} \otimes \mathbf{K}_{t_2} &= \mathbf{K}_{t_1 t_2}\\ \mathbf{C}_{10} &= & \mathbf{I}_{t_1} \otimes \mathbf{K}_{t_2} - \mathbf{K}_{t_1 t_2} &= (\mathbf{I}_{t_1} - \mathbf{K}_{t_1}) \otimes \mathbf{K}_{t_2}\\ \mathbf{C}_{01} &= & \mathbf{K}_{t_1} \otimes \mathbf{I}_{t_2} - \mathbf{K}_{t_1 t_2} &= \mathbf{K}_{t_1} \otimes (\mathbf{I}_{t_2} - \mathbf{K}_{t_2})\\ \mathbf{C}_{11} &= & \mathbf{I}_{t_1 t_2} - \mathbf{I}_{t_1} \otimes \mathbf{K}_{t_2} - \mathbf{K}_{t_1} \otimes \mathbf{I}_{t_2} + \mathbf{K}_{t_1 t_2} &= (\mathbf{I}_{t_1} - \mathbf{K}_{t_1}) \otimes (\mathbf{I}_{t_2} - \mathbf{K}_{t_2}),\end{aligned}$$

where denotes $\otimes$ an operator for the Kronecker product.

To summarise, for a v-factor experiment with each treatment factor f_i at t_i levels, $i = 1, \ldots, v$, the treatment structure is given by

$$\boldsymbol{\alpha} = \sum_x \mathbf{C}_x \boldsymbol{\alpha} \tag{17}$$

where x is a string of binary numbers, $(x_1 x_2 \ldots x_v)$, and

$$\mathbf{C}_x = \mathbf{C}_{x_1} \otimes \mathbf{C}_{x_2} \otimes \cdots \otimes \mathbf{C}_{x_v} = \bigotimes_{i=1}^{v} \mathbf{C}_{x_i},$$

where

$$\mathbf{C}_{x_i} = \begin{cases}\mathbf{K}_{t_i}, & x_i = 0\\ \mathbf{I}_{t_i} - \mathbf{K}_{t_i}, & x_i = 1,\end{cases} \tag{18}$$

0 or 1

where x_i is ~~a binary number~~. For example, in ~~a 2~~-factor $t_1 \times t_2$ factorial experiment, the treatment contrast matrix is represented by

the

$$\mathbf{C}_x = \mathbf{C}_{x_1 x_2} = \mathbf{C}_{x_1} \otimes \mathbf{C}_{x_2}.$$

Thus, the main effects of factors f_1 and f_2 are represented by the two sets of contrasts $\mathbf{C}_{10}\boldsymbol{\alpha}$ and $\mathbf{C}_{01}\boldsymbol{\alpha}$, respectively, and the $f_1 f_2$ interaction is represented by the set of contrasts $C(11)\alpha$.

$C_{11}\alpha$