Classroom Bingo

An answer to the probability puzzle in Significance Magazine (December 2021)

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GitHub: `https://github.com/kchaz/ClassroomBingo`

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The Problem

"A primary schoolteacher is playing a game with her class. She has two identical dice, with the numbers on the six faces of each die being 1, 2, 2, 3, 3 and 3. The teacher tells the class that she will throw the pair of dice, add up the two numbers showing, and call out that number in a game of bingo. She then asks each member of the class to make their own bingo card consisting of five numbers of their own choosing. The teacher explains that the children can repeat a number on their card if they wish (and then delete just one occurrence of the number whenever it is called). Most of the class chose the five possible different totals as their bingo numbers, but one clever pupil made a better selection. Can you figure out what her five numbers might have been? And is this the best possible selection?" [3]

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Notation

In this problem, it is important to distinguish bingo cards from roll sequences and to be careful about order. The dice can in theory be rolled infinitely many times but the bingo cards only ever contain 5 spaces. Throughout, we represent bingo cards and sequences of rolls via the possible dice sum outcomes (2, 3, 4, 5, and 6 - see Table 1 below), encoded in two possible ways:

1. **Outcome set representation**: we use set notation to represent the *unordered* set of outcomes present in a bingo card or roll sequence. For example, $\{3,3,5,5,6\}$ represents a bingo card with two 3's, two 5's, and one 6. For clarity, we adopt the convention of writing outcomes least to greatest, but the numbers can be placed on the card in any order (see Figure 1). Similarly, $\{3,3,3,3,3,4,4,5,5,5,6,6\}$ represents 12 rolls in which five 3's, two 4's, three 5's, and two 6's occurred – but not necessarily in that order. The set really represents a large number of possible ordered **roll sequences** that could have produced the set of outcomes.

2. **Outcome count representation:** rather than write out increasingly long sequences, it is convenient to represent any card or set of roll outcomes with a length-5 vector of counts for each outcome. The bingo card $\{3,3,5,5,6\}$ is written $(0,2,0,2,1)$ and the aforementioned set of roll outcomes is written $(0,5,2,3,2)$.

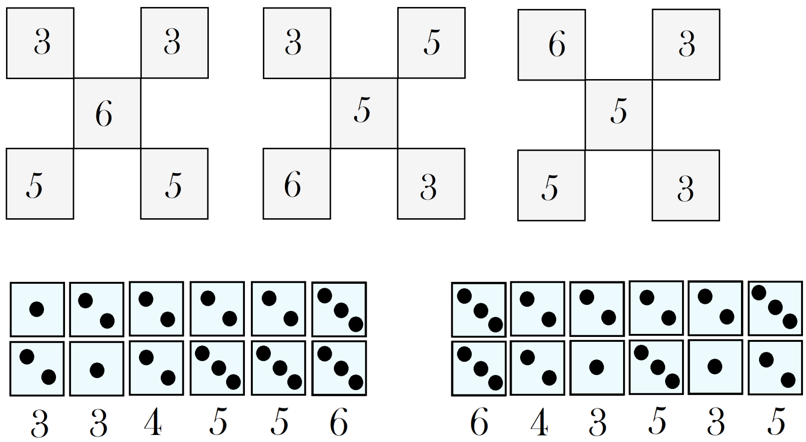

Figure 1: ***Top:*** *Three example cards that are entirely equivalent in our notation. They are all* $\{3,3,5,5,6\}$ *or* $(0,2,0,2,1)$*. Order never matters for bingo cards.* ***Bottom:*** *example of two 6-roll sequences that are equivalent in our notation. We do need to account for these different sequences below as they are different ways of winning for the given card.*

The Best Card in Five Rolls

What does it mean for a bingo card to be best? Intuitively, we want the card with the greatest probability of winning, but since the number of times the teacher rolls is unlimited, each card's probability of winning approaches 1 as the number of rolls grows larger. However, given the presence of other players, we might wish for a high probability of winning in few rolls. To start, then, we ask which bingo card has the greatest probability of winning in 5 rolls – the quickest way to win and the only way to both win and be certain another student will not win first.

Table 1 gives the probability of each possible outcome. A sequence of all 5's is the highest probability ordered sequence. Should we make our card all 5's? Not necessarily. This fails to take into account that bingo cards are unordered. There are many more ordered roll sequences that would yield a win with a card such as $(1, 2, 3, 4, 5)$ ($5! = 120$ sequences) than with the card of all 5's (one sequence).

Dice Sum	Dice Outcomes	Probability
2	11	1/36
3	12 21	4/36
4	13 31 22	10/36
5	23 32	12/36
6	33	9/36

Table 1: *Probabilities for the sum of two dice tossed independently, where each die has one 1, two 2's, and three 3's. For example, the probability of a 4 is calculated as* $2(\frac{1}{6})(\frac{3}{6}) + (\frac{2}{6})(\frac{2}{6})$.

To determine the greatest probability card, we must consider the number of possible winning roll sequences for each card. This number depends on the number of doubles, triples etc. that occur within a card, and using it, we can group cards into 7 equivalence classes shown in Table 2. We represent each class by a letter sequence such as *bcdee* to indicate a card that has one outcome twice and the rest once. Cards within these classes do not necessarily have the same probabilities but they do have the same number of orderings of their outcomes. We call these **multinomial equivalence classes**.

Class	Description	Num. Orders	Best Card	Best Card Probability
abcde	no repeats	120	$\{2, 3, 4, 5, 6\}$	$0.0086 = (120 * 12 * 10 * 9 * 4 * 1)/36^5$
bcdee	one double	60	$\{3, 4, 5, 5, 6\}$	$0.0514 = (60 * 12^2 * 10 * 9 * 4)/36^5$
cddee	two doubles	30	$\{4, 4, 5, 5, 6\}$	$\mathbf{0.0643} = (30 * 12^2 * 10^2 * 9)/36^5$
cdeee	one triple	20	$\{4, 5, 5, 5, 6\}$	$0.0514 = (20 * 12^3 * 10 * 9)/36^5$
ddeee	one triple and one double	10	$\{5, 5, 5, 4, 4\}$	$0.0286 = (10 * 12^3 * 10^2)/36^5$
deeee	one quadruple	5	$\{4, 5, 5, 5, 5\}$	$0.0171 = (5 * 12^4 * 10)/36^5$
eeeee	all the same	1	$\{5, 5, 5, 5, 5\}$	$0.0041 = (12^5)/36^5$

Table 2: *Equivalence classes defined by cards that have the same number of possible orderings of their elements. For example, the card* $\{4, 4, 5, 5, 6\}$ *has* $\binom{5}{0\ 0\ 2\ 2\ 1} = \frac{5!}{0!0!1!2!2!} = 30$ *possible winning 5-roll sequences. The card* $\{2, 2, 3, 3, 4\}$ *does, too, and since these two cards have the same multinomial coefficient, we can go back to comparing the probabilities of each card's outcomes.*

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Within each class, the card with the greatest probability of winning can be found by choosing as many 5's as possible, followed by as many 4's, as many 6's and so on. The results and their probabilities are shown in the last two columns of Table 2. The last column shows that, in five rolls, the card with two 4's, two 5's, and one 6 has the highest overall probability of winning (Figures 2 and 3). This card reflects a tradeoff between choosing high probability outcomes and choosing different outcomes so that there are more ways to win.

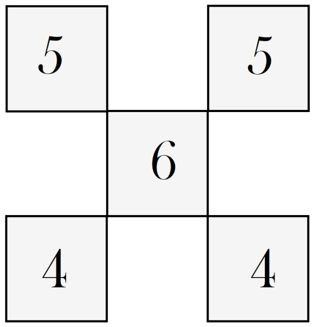

Figure 2: *A card with the highest probability of winning in 5 rolls.*

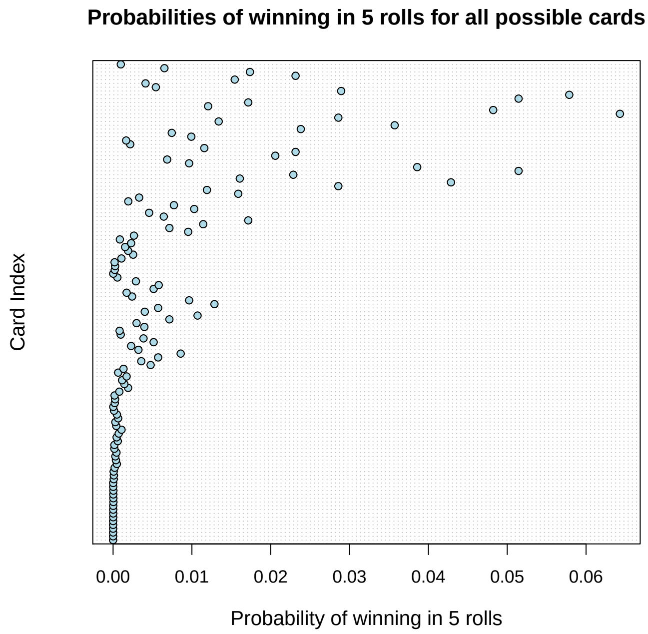

Figure 3: *Win probabilities for all 126 possible bingo cards. The greatest probability cards have a distinct advantage over the lower probability ones.*

Considering More Than Five Rolls

What happens when the teacher rolls six, seven, eight or more times? In this situation, the combinatoric picture changes. It was not clear to us at the outset whether this would change our card choice. For example, the all-5's card has only one possible winning roll sequence in 5 rolls. However, in 10 rolls, there are many more sequences that include five 5's and would thus win for that card...perhaps enough to overcome its initial combinatoric disadvantage? It still seems like a good idea to pick a card that has a high chance of winning early, but we now look to better understand how the probability of winning in n or fewer rolls changes as n increases.

To do this, we plot the trajectory of each card's probability p_n of winning in n or fewer rolls as n increases. While $p_n \to 1$ for all cards as $n \to \infty$, we look for cards that go to 1 faster and ideally, for a card that consistently has the greatest probability. Our code for calculating these probabilities is provided on GitHub. In short, using the `combinat`[1] package in `R`, we generated all possible cards and all possible sets of roll outcomes for each value of n, both represented by length-5 outcome count vectors [2]. Then, for each card and value of n, we identified which sets of roll outcomes win for that card, calculated their multinomial probabilities, and summed these probabilities.[1] Figure 4 (left) shows the cumulative probability trajectories for all 126 cards. As it turns out, the $\{4, 4, 5, 5, 6\}$ bingo card (in blue) consistently has the greatest probability for the entire range of n in which p_n is appreciably less than 1.[2] The right plot shows the probability trajectories for winning in exactly n rolls. Even with our best card, it is more likely to win in 6 or 7 rolls than in exactly 5, but it is unlikely to win in, say, exactly 25 rolls because it is so likely to win before we reach that many rolls.

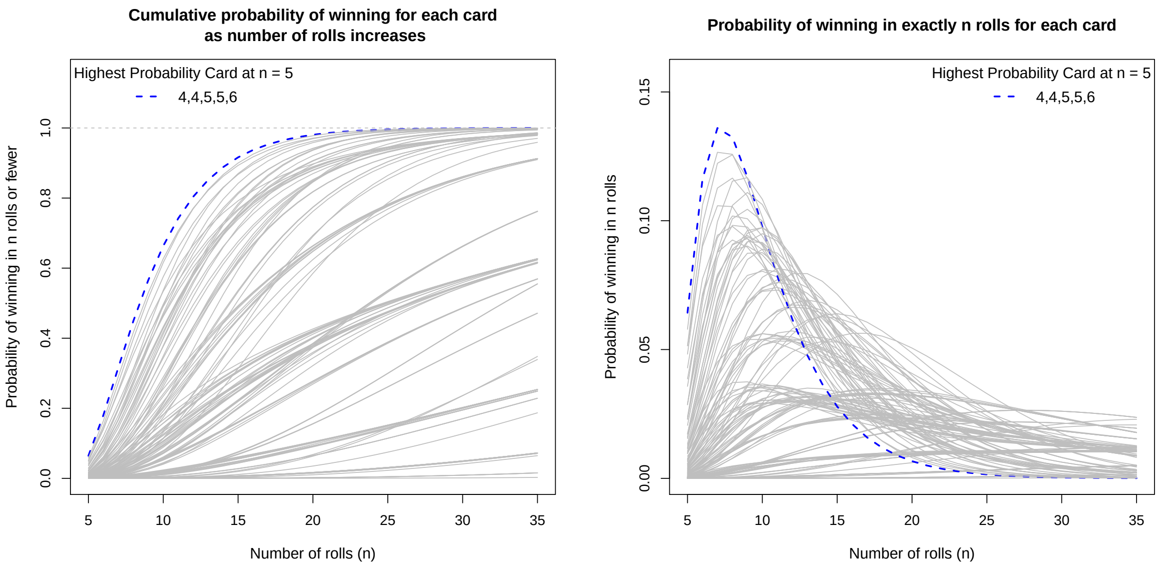

Figure 4: ***Left:*** *Trajectory of p_n for each of the 126 possible cards as n increases.* ***Right:*** *Trajectory of the probability of winning in exactly n rolls as n increases.*

Changing the Outcome Probabilities

Figure 4 has another interesting feature: some of the lines intersect, indicating that, as we hypothesized, it is possible for the ranking of bingo cards by their win probabilities to change with n. Though in the original case, one card is superior for all n where the cards meaningfully differ in cumulative probability of winning, we find that for other sets of outcome probabilities, order changes can occur earlier. For example, suppose that the probabilities are as in Table 3.

[1] Observe that given roll outcome count vector $(n_1, n_2, n_3, n_4, n_5)$ for $n = \sum_{i=1}^{5} n_i$ rolls and given a card with outcome count vector $(b_1, b_2, b_3, b_4, b_5)$, the roll includes a win for the card if $\sum_{i=1}^{5} I(n_i \geq b_i) = 5$, meaning the roll has at least b_i of each outcome i. We use this in our computations.

[2] In fact, the card $\{4, 5, 5, 5, 6\}$ eventually has a greater probability but only at the 27th roll, where both have essentially probability 1 of a win in $n \leq 27$ rolls.

Outcome	Probability
2	$1/36$
3	$4/36$
4	$6/36$
5	$20/36$
6	$5/36$

Table 3: *Alternative set-up with outcome probabilities that more strongly favor 5.*

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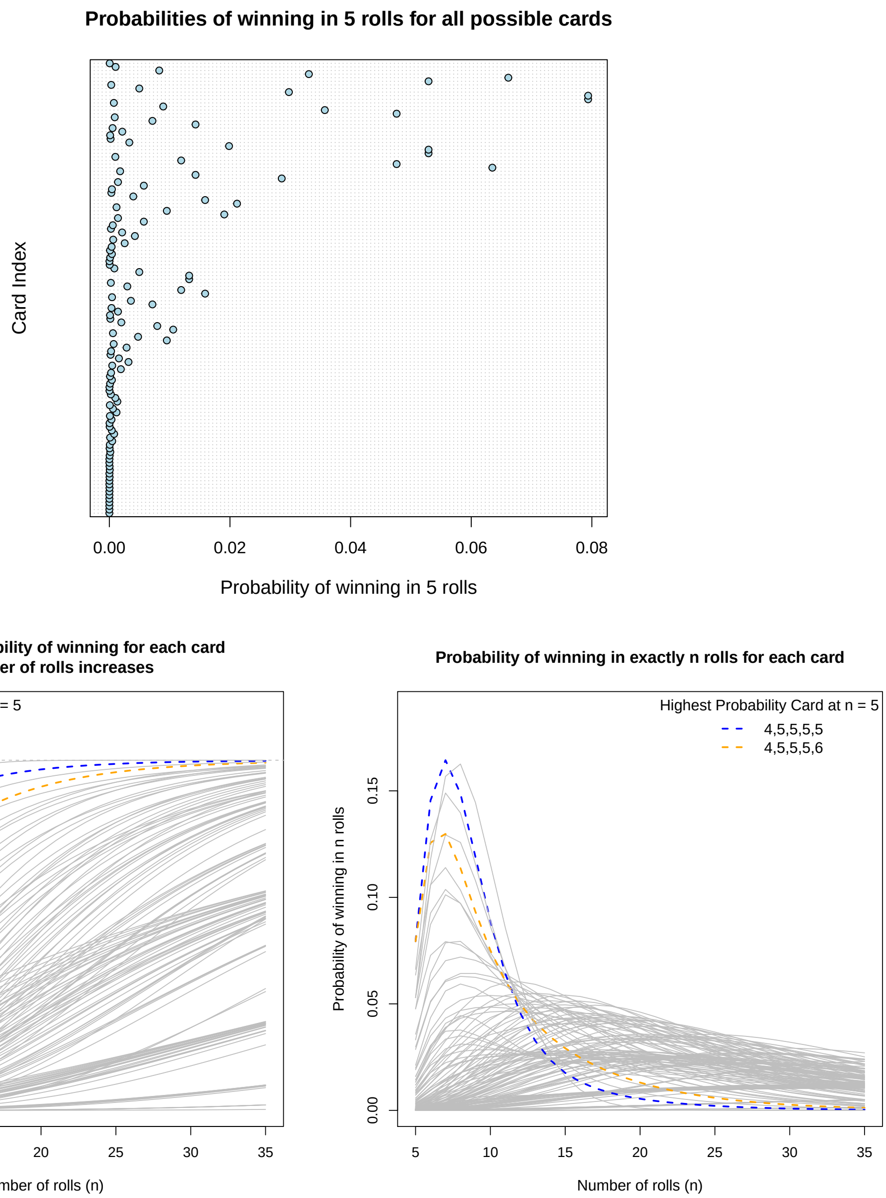

Figure 5: *The same set of plots as in Figures 3 and 4 but for the probabilities in Table 3.*

Two interesting dynamics emerge. First, at $n = 5$ rolls, there is a tie for best bingo card (blue and orange lines) between $\{4, 5, 5, 5, 5\}$ and $\{4, 5, 5, 5, 6\}$. But as n increases, the $\{4, 5, 5, 5, 5\}$ card is clearly superior; its combinatoric disadvantage from having only two different outcomes lessens. At $n = 5$, these cards were indistinguishable; for larger n, one is clearly better. Second, starting at $n = 10$, even $\{4, 5, 5, 5, 5\}$ no longer has the greatest probability of winning. In fact, the $\{5, 5, 5, 5, 5\}$ card (top grey line on the left plot) has the greatest probability of winning in $n \leq 10$ rolls.

In this case, it becomes harder to decide which card is best. Given that it is still more likely to not win in five rolls than to win, it makes sense to consider probabilities for $n > 5$ as well. On the other hand, since p_n increases towards 1 for all cards, it makes little sense to select a card that has a greater probability of winning only at a large number of rolls when it is so likely that someone will have won by then, ending the game.

For the original case, we would have gotten away with looking just for the card with the greatest probability in 5 rolls. In the general case, that is not guaranteed to produce the best card.

The Role of Other Players

Above, we know nothing of the other bingo players, but consider now how this knowledge might change our reasoning. Given 126 players with unique bingo cards, we would always choose the card with the highest probability in 5 rolls – the game would end after that. If playing alone, say in a game that allowed players to bet on winning in n or fewer rolls, then one would want to choose the card with highest cumulative probability at n. As shown above, this is not in every set-up the card with highest probability in 5 rolls. If playing with other students who have been randomly dealt x other distinct bingo cards, it is possible to estimate by simulation an average probability that a given card wins in n rolls without any of the other cards winning first. If playing with other statisticians...we'll leave that for future work.

References

[1] Chasalow, S. *combinat: combinatorics utilities*, 2012. R package version 0.0-8. https://cran.r-project.org/web/packages/combinat/index.html.

[2] Chasalow, S. D., and Brand, R. J. Algorithm AS 299: Generation of simplex lattice points. *Applied Statistics 44*, 4 (1995), 534.

[3] Fletcher, M. Probability puzzle: Classroom bingo. *Significance Magazine* (2021). https://www.significancemagazine.com/culture/712-probability-puzzle-classroom-bingo.