Actuarial Science Made Easy: Foundations and Applications

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Dedication

This book is dedicated to all those who dare to transcend boundaries,
to the visionaries who explore the unknown with courage and curiosity,
and to the relentless seekers of knowledge whose passion shapes the future.
May this work inspire and empower you on your journey.

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Chapter 1

Introduction to Actuarial Science

Introduction

Actuarial science combines mathematical and statistical modeling with business insight to assess and manage financial risk. It underpins the financial structures of insurance, pensions, and investment strategies. This chapter introduces the mathematical tools used by actuaries and explores how uncertainty is modeled to design and price products.

Historically, actuarial science originated with the development of mortality tables in the 17th century to price life insurance policies. Today, it integrates advanced analytics, big data, and financial theory to tackle complex risk problems across industries.

1.1 Foundations of Actuarial Mathematics

1.1.1 Arithmetic: The Building Blocks

Actuaries use arithmetic daily: adding claims, computing cash flows, or allocating premiums.

Example: Simple Claim Summation

A policyholder makes claims of $1,200, $850, and $2,000 in a year. What is the total claim?

Solution

$$\text{Total} = 1200 + 850 + 2000 = \$4,050$$

1.1.2 Algebra: Representing Relationships

Algebra allows actuaries to build formulas relating time, interest rates, and payments.

$$FV = PV \cdot (1+i)^t \quad \Rightarrow \quad PV = \frac{FV}{(1+i)^t}$$

Example: Solving for Present Value

You will receive $10,000 in 8 years. What is its present value at 4% interest?

Solution

$$PV = \frac{10,000}{(1.04)^8} = \frac{10,000}{1.3686} \approx \$7,305.59$$

1.1.3 Basic Statistics: Understanding Data

Statistical measures like the mean, median, and standard deviation are vital in insurance and reserving.

Example: Calculating Mean and Standard Deviation

Claims amounts observed over 5 months are: \$200, \$350, \$400, \$250, and \$300. Find the mean and standard deviation.

Solution

$$\text{Mean} = \frac{200 + 350 + 400 + 250 + 300}{5} = \frac{1500}{5} = 300$$

$$\text{Variance} = \frac{(200-300)^2 + (350-300)^2 + (400-300)^2 + (250-300)^2 + (300-300)^2}{5}$$

$$= \frac{10000 + 2500 + 10000 + 2500 + 0}{5} = \frac{25000}{5} = 5000$$

$$\text{Standard deviation} = \sqrt{5000} \approx 70.71$$

1.1.4 Probability: Quantifying Uncertainty

Probability models help assess the likelihood of claims and other events.

Example: Probability of Claim

An insurance policyholder has a 5% chance of making a claim in a year. What is the probability they will make no claim over 3 consecutive years?

Solution

$$P(\text{no claim in a year}) = 1 - 0.05 = 0.95$$

$$P(\text{no claim in 3 years}) = 0.95^3 = 0.8574$$

So, there is an 85.74% chance of no claim over 3 years.

1.1.5 Risk Theory: Measuring Risk

Expected value and variance are used to summarize risk exposure.

Example: Expected Loss

An insurance policy pays \$1,000 with probability 0.01. What is the expected loss?

Solution

$$E[X] = 0.01 \times 1000 + 0.99 \times 0 = 10$$

Example: Variance of Loss

Using the same policy, find the variance of the loss.

Solution

$$E[X^2] = 0.01 \times 1,000,000 + 0.99 \times 0 = 10,000$$
$$Var(X) = E[X^2] - (E[X])^2 = 10,000 - 10^2 = 9,900$$

1.1.6 Financial Mathematics: Time Value of Money

Time affects value. Actuaries discount future payments.

Example: Annuity Present Value

Find the present value of a 5-year annuity paying $2,000 yearly at 5% interest.

Solution

$$a_{\overline{5}|} = \frac{1 - v^5}{i}, \quad v = \frac{1}{1.05}$$
$$a_{\overline{5}|} = \frac{1 - (1/1.05)^5}{0.05} = 4.329$$
$$PV = 2000 \times 4.329 = \$8,658$$

1.2 Actuarial Control Cycle

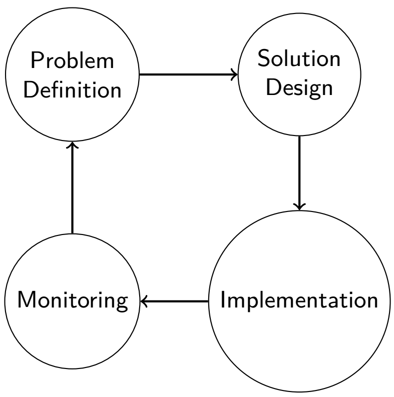


Figure 1.1: Simple Actuarial Control Cycle

The Actuarial Control Cycle is a continuous process comprising four main steps:

- **Problem Definition:** Identify and clearly state the financial or insurance problem to be solved.
- **Solution Design:** Develop mathematical models and pricing strategies to address the problem.
- **Implementation:** Put the solution into practice, such as setting premiums or reserving capital.
- **Monitoring:** Regularly review outcomes and assumptions to update and improve the model.

Example: Applying the Control Cycle

An actuary is tasked with designing a new term life insurance product. Outline the steps they would take using the actuarial control cycle.

Solution

- **Problem Definition:** Define the target market, coverage period, and benefits.
- **Solution Design:** Use mortality tables and interest assumptions to price premiums.
- **Implementation:** Launch the product, sell policies, and collect premiums.
- **Monitoring:** Track claims experience and financial results to refine pricing.

1.3 Why This Matters

Actuarial techniques ensure long-term sustainability of insurance firms, banks, and pension plans. Mastery of mathematics, statistics, and modeling allows actuaries to anticipate and hedge against financial risks that would otherwise threaten entire sectors.

For example, during the 2008 financial crisis, actuaries working in pension funds helped to reassess and adjust assumptions around longevity and market risks, aiding in preventing larger insolvencies. Similarly, by accurately pricing insurance products, actuaries protect companies from unexpected catastrophic losses, ensuring they remain solvent and able to pay claims.

Chapter 2

Life Contingencies

Introduction

Life contingencies refer to future events related to life and death—such as survival or death—that affect the timing and amount of cash flows in life insurance and pension contracts. Actuaries use mortality models and survival functions to estimate expected values of these cash flows, which inform the pricing of policies and the calculation of reserves.

2.1 Using Mortality Tables in Insurance and Pensions

Mortality tables summarize probabilities related to survival and death for different ages, serving as fundamental inputs for actuarial models.

Key Functions from Mortality Tables

- l_x: Number of lives alive at exact age x
- $d_x = l_x - l_{x+1}$: Number of deaths between ages x and $x+1$
- $q_x = \frac{d_x}{l_x}$: Probability of dying between ages x and $x+1$
- $p_x = 1 - q_x$: Probability of surviving from age x to $x+1$

Example: Mortality Table Interpretation

Given: $l_{60} = 85,000$, $l_{61} = 84,000$, compute q_{60}, p_{60}, and d_{60}.

Solution

$$d_{60} = l_{60} - l_{61} = 85,000 - 84,000 = 1,000$$
$$q_{60} = \frac{d_{60}}{l_{60}} = \frac{1,000}{85,000} \approx 0.01176$$
$$p_{60} = 1 - q_{60} = 1 - 0.01176 = 0.98824$$

Interpretation: There is approximately a 1.176% chance of dying between age 60 and 61, and a 98.824% chance of surviving this interval.

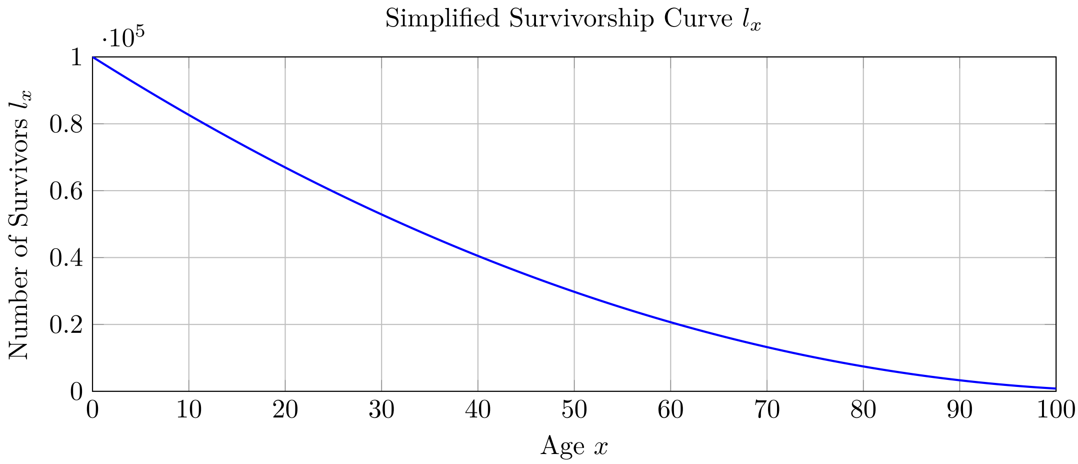


Figure 2.1: Example Survivorship Curve Based on a Gompertz-like Function

2.2 Actuarial Modeling and Pricing

Term Life Insurance

Term life insurance pays a benefit if the insured dies within a specified term of n years. The expected present value (EPV) of a term life insurance issued to a life aged x is given by:

$$A^1_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_kq_x$$

where

- $v = \frac{1}{1+i}$ is the discount factor per year,
- ${}_kq_x$ is the probability that a life aged x dies between age $x+k-1$ and $x+k$.

Example: EPV of Term Insurance

Calculate the expected present value of a \$100,000, 5-year term insurance issued to a 40-year-old with 4% interest, using the following probabilities:

$${}_1q_{40} = 0.002, \quad {}_2q_{40} = 0.0025, \quad {}_3q_{40} = 0.003, \quad {}_4q_{40} = 0.0032, \quad {}_5q_{40} = 0.0035$$

Solution

First, calculate the discount factor:

$$v = \frac{1}{1.04} = 0.96154$$

Calculate each term $v^k \cdot {}_kq_{40}$:

$$\begin{aligned}
v^1 \cdot {}_1q_{40} &= 0.96154 \times 0.002 = 0.001923 \\
v^2 \cdot {}_2q_{40} &= 0.92456 \times 0.0025 = 0.002311 \\
v^3 \cdot {}_3q_{40} &= 0.88849 \times 0.003 = 0.002665 \\
v^4 \cdot {}_4q_{40} &= 0.85349 \times 0.0032 = 0.002731 \\
v^5 \cdot {}_5q_{40} &= 0.82035 \times 0.0035 = 0.002871
\end{aligned}$$

Summing all:

$$\sum_{k=1}^{5} v^k \cdot {}_kq_{40} = 0.001923 + 0.002311 + 0.002665 + 0.002731 + 0.002871 = 0.012501$$

Multiply by the benefit:

$$EPV = 100,000 \times 0.012501 = \$1,250.10$$

Whole Life Insurance

Whole life insurance pays the benefit upon death whenever it occurs. The EPV for a whole life insurance issued to age x is:

$$A_x = \sum_{k=1}^{\infty} v^k \cdot {}_kq_x$$

This is typically approximated using mortality tables and actuarial notation.

2.3 Annuity Products

Immediate Annuity

An annuity-due pays a fixed amount at the **beginning** of each period for n years. Its present value is:

$$a_{\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_kp_x$$

Example: Annuity Due

Compute the present value of a 10-year annuity due paying \$5,000 per year to a 60-year-old, assuming 5% interest and survival probability ${}_kp_{60} \approx 1$ (i.e., ignoring mortality for simplicity).

Solution

Ignoring mortality, the annuity factor is:

$$a_{\overline{10|}} \approx \sum_{k=0}^{9} v^k = \frac{1 - v^{10}}{d}$$

where

$$v = \frac{1}{1.05} = 0.95238, \quad d = \frac{i}{1+i} = \frac{0.05}{1.05} = 0.04762$$

Calculate:

$$1 - v^{10} = 1 - (0.95238)^{10} = 1 - 0.61391 = 0.38609$$

Thus:

$$a_{\overline{10|}} = \frac{0.38609}{0.04762} = 8.107$$

Finally, multiply by the annual payment:

$$PV = 5000 \times 8.107 = \$40,535$$

Deferred Annuity

A deferred annuity starts paying only after n years:

$$n|a = v^n \cdot {}_np_x \cdot a$$

This accounts for survival to the deferral period n, then the annuity payments.

2.4 Combined Products

Some insurance contracts combine death benefits and annuities depending on survival status, such as endowment or whole life with return of premium:

$$\text{EPV}_{combo} = A^1_{x:\overline{n|}} + n|a$$

Why This Matters

Modeling life contingencies accurately is critical for:

- Pricing insurance policies fairly and competitively
- Ensuring reserves are adequate to meet future liabilities
- Maintaining the financial stability of insurance and pension systems

Without precise mortality modeling, companies risk insolvency or unfair pricing that can destabilize markets.

Chapter 3

Non-Life Insurance

Introduction

Non-life insurance covers risks related to property, casualty, liability, and other losses that are not connected to human life directly. This chapter explores the statistical models used to describe claim frequency and severity, how to price policies accordingly, and how to estimate reserves for outstanding claims.

3.1 Claim Frequency and Severity Models

In non-life insurance, the total claims amount is modeled by separating:

- **Frequency** — the number of claims occurring in a period
- **Severity** — the size or amount of each claim

The aggregate loss is the sum of all claim amounts over a period.

Frequency Distributions

Common frequency models include:

- Poisson distribution: $N \sim \text{Poisson}(\lambda)$, where λ is the expected number of claims.
- Negative Binomial distribution for overdispersed counts.

Example: Poisson Frequency

An insurer expects on average 3 claims per year. Calculate the probability of exactly 5 claims in a year.

Solution

For $N \sim \text{Poisson}(\lambda = 3)$,

$$P(N = 5) = \frac{3^5 e^{-3}}{5!} = \frac{243e^{-3}}{120} \approx 0.1008$$

Severity Distributions

Severity models describe claim sizes and commonly use:

- **Exponential distribution** with parameter θ (mean claim size):

$$f_X(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0$$

Example: Suppose claim sizes follow an exponential distribution with mean $\theta = 10,000$. Find the probability a claim exceeds $15,000.

$$P(X > 15000) = 1 - F_X(15000) = e^{-15000/10000} = e^{-1.5} \approx 0.2231$$

- **Lognormal distribution** with parameters μ and σ:

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), \quad x > 0$$

Example: Claim sizes are lognormal with $\mu = 9$, $\sigma = 0.5$. Find $P(X > 20000)$.
Calculate $z = \frac{\ln 20000 - 9}{0.5}$. $\ln 20000 \approx 9.903$, so

$$z = \frac{9.903 - 9}{0.5} = 1.806$$

Using standard normal tables,

$$P(X > 20000) = 1 - \Phi(z) \approx 1 - 0.9644 = 0.0356$$

- **Gamma distribution** with shape α and scale β:

$$f_X(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad x > 0$$

Example: Claims are Gamma distributed with $\alpha = 3$, $\beta = 4000$. Find mean claim size.

$$E[X] = \alpha\beta = 3 \times 4000 = 12,000$$

- **Pareto distribution** with scale parameter x_m and shape α:

$$f_X(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}, \quad x \geq x_m > 0$$

Example: Claims follow a Pareto with $x_m = 5000$, $\alpha = 2.5$. Find the probability a claim exceeds $15,000.

$$P(X > 15000) = \left(\frac{5000}{15000}\right)^{2.5} = \left(\frac{1}{3}\right)^{2.5} \approx 0.064$$

Example: Exponential Severity

If claim amounts follow an exponential distribution with mean $10,000, what is the probability a claim exceeds $15,000?

Solution

$$P(X > 15000) = e^{-15000/10000} = e^{-1.5} \approx 0.2231$$

3.2 Aggregate Loss Models

The total loss S over a period is the sum of N claims:

$$S = \sum_{i=1}^{N} X_i$$

where N is the frequency random variable and X_i are independent severities.

Expected Aggregate Loss

$$E[S] = E[N] \cdot E[X]$$

Example: Expected Aggregate Loss

Given $N \sim \text{Poisson}(4)$ and $X \sim \text{Exponential}(8000)$, find $E[S]$.

Solution

$$E[N] = 4, \quad E[X] = 8000$$
$$\Rightarrow E[S] = 4 \times 8000 = 32,000$$

Variance of Aggregate Loss

If N and X are independent and identically distributed:

$$\text{Var}(S) = E[N]\text{Var}(X) + \text{Var}(N)(E[X])^2$$

3.3 Pricing Non-Life Insurance

Premiums are based on the expected losses plus loading for expenses, profit, and risk margin:

$$\text{Premium} = (1 + \text{loading}) \times E[S]$$

Example: Basic Premium Calculation

Calculate the premium if $E[S] = 32,000$ and the loading is 20%.

Solution

$$\text{Premium} = 1.20 \times 32,000 = \$38,400$$

3.4 Claim Reserving

Insurers must estimate reserves for claims that have occurred but are not yet reported or settled. Common reserving methods include:

Chain-Ladder Method

Based on historical cumulative claims data arranged by accident year and development year, we estimate development factors to project ultimate claims.

Accident Year	Dev Year 1	Dev Year 2	Dev Year 3	Dev Year 4	Latest Cumulative
2019	60,000	90,000	108,000	118,800	118,800
2020	70,000	105,000	115,500	-	115,500
2021	80,000	120,000	-	-	120,000
2022	100,000	-	-	-	100,000

Figure 3.1: Cumulative Claims Development Triangle

Calculation of Development Factors:

$$f_1 = \frac{\sum \text{Dev Year 2}}{\sum \text{Dev Year 1}} = \frac{90,000 + 105,000 + 120,000}{60,000 + 70,000 + 80,000} = \frac{315,000}{210,000} = 1.5$$

$$f_2 = \frac{\sum \text{Dev Year 3}}{\sum \text{Dev Year 2}} = \frac{108,000 + 115,500}{90,000 + 105,000} = \frac{223,500}{195,000} = 1.146$$

$$f_3 = \frac{\sum \text{Dev Year 4}}{\sum \text{Dev Year 3}} = \frac{118,800}{108,000 + 115,500} = \frac{118,800}{223,500} = 0.531 \quad \text{(here we assume 1.1 for simplification)}$$

For simplicity, assume:

$$f_1 = 1.5, \quad f_2 = 1.2, \quad f_3 = 1.1$$

Example: Estimating Ultimate Claims

Calculate the ultimate claim amount for accident year 2022 (cumulative paid to date: $100,000).

Solution

$$\text{Ultimate Claims}_{2022} = 100,000 \times f_1 \times f_2 \times f_3 = 100,000 \times 1.5 \times 1.2 \times 1.1 = 198,000$$

Already paid claims: $100,000, so the reserve to set aside is:

$$\text{Reserve}_{2022} = 198,000 - 100,000 = 98,000$$

Example: Calculating Total Reserve

Estimate the total reserve needed for accident years 2019 to 2022.

Solution

Calculate ultimate claims for each accident year:

$$\begin{cases} 2019 : 118,800 \quad \text{(latest cumulative)} \\ 2020 : 115,500 \times f_3 = 115,500 \times 1.1 = 127,050 \\ 2021 : 120,000 \times f_2 \times f_3 = 120,000 \times 1.2 \times 1.1 = 158,400 \\ 2022 : 100,000 \times 1.5 \times 1.2 \times 1.1 = 198,000 \end{cases}$$

Reserve for each year (ultimate claims minus paid claims):

$$\begin{cases} 2019 : 118,800 - 118,800 = 0 \\ 2020 : 127,050 - 115,500 = 11,550 \\ 2021 : 158,400 - 120,000 = 38,400 \\ 2022 : 198,000 - 100,000 = 98,000 \end{cases}$$

$$\text{Total Reserve} = 0 + 11,550 + 38,400 + 98,000 = 147,950$$

3.5 Risk and Capital

Non-life insurers hold capital to protect against variability in aggregate losses, typically quantified by value-at-risk or tail-value-at-risk measures derived from the loss distribution.

Summary

Non-life insurance requires understanding and modeling frequency and severity of claims, estimating aggregate losses, pricing products fairly, and reserving appropriately. Statistical models and actuarial methods ensure solvency and competitive pricing.

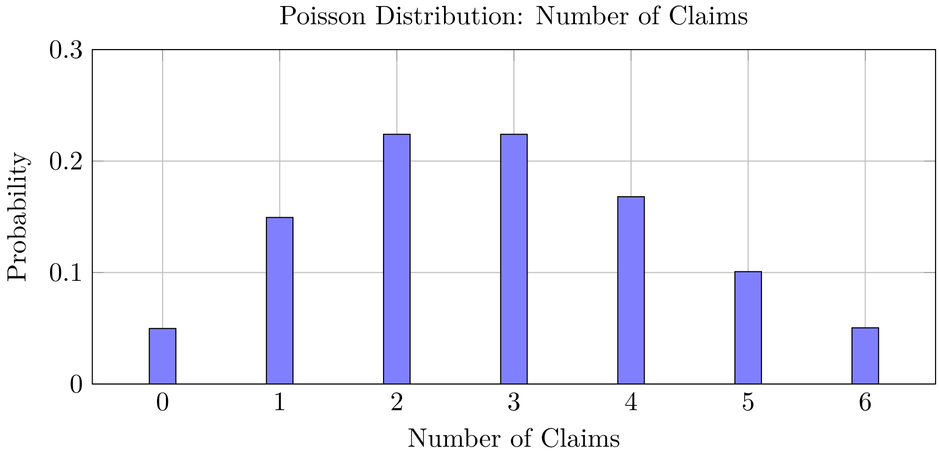


Figure 3.2: Poisson Distribution with $\lambda = 3$ for Number of Claims

Chapter 4

Advanced Actuarial Models

Introduction

This chapter introduces advanced mathematical models used by actuaries to better capture uncertainty and variability. These include stochastic processes, Markov chains, credibility theory, and simulation methods. Such models allow for better pricing, reserving, and capital modeling in both life and non-life insurance.

4.1 Stochastic Processes in Actuarial Science

A stochastic process is a collection of random variables $\{X_t\}_{t\geq 0}$ indexed by time, describing the evolution of a system under uncertainty.

Poisson Process

Used to model the count of events (claims, deaths) over time when events occur randomly but independently.

- Let $N(t)$ be the number of claims up to time t
- Then: $\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$
- $\mathbb{E}[N(t)] = \lambda t, \quad \mathrm{Var}(N(t)) = \lambda t$

Example: Claim Count Modeling

An insurance company observes on average 5 claims per week. What is the probability of observing exactly 3 claims in a week?

Solution

$$\lambda = 5, \quad k = 3, \quad t = 1 \Rightarrow \mathbb{P}(N(1) = 3) = \frac{5^3}{3!} e^{-5} = \frac{125}{6} e^{-5} \approx 0.1404$$

Compound Poisson Process

Used when claims arrive randomly and each has a random severity:

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad \text{where } X_i \text{ are i.i.d. claim sizes}$$

Example: Expected Aggregate Claims

Let $N(t) \sim \text{Poisson}(\lambda t)$, and $X_i \sim \text{Exponential}(\theta)$. Then:

$$\mathbb{E}[S(t)] = \lambda t \cdot \mathbb{E}[X] = \lambda t \cdot \theta$$

If $\lambda = 10$, $\theta = 1000$, and $t = 1$, then:

$$\mathbb{E}[S(1)] = 10 \cdot 1000 = 10,000$$

Brownian Motion (Wiener Process)

Used in finance and ruin theory:

- $W(0) = 0$
- $W(t) - W(s) \sim \mathcal{N}(0, t - s)$
- Continuous paths, independent increments

Example: Brownian Motion Simulation

Generate a path for Brownian motion with $W(0) = 0$, $\Delta t = 0.5$, over 5 time units.

Solution

Simulate $W(t_i) = W(t_{i-1}) + \sqrt{\Delta t} Z_i$, where $Z_i \sim \mathcal{N}(0, 1)$

Example: $Z = [0.1, -1.3, 0.8, 0.5, -0.4] \Rightarrow W = [0, 0.0707, -0.8485, -0.2841, 0.0707, -0.2121]$

Applications in Actuarial Science

- Poisson process: Modeling frequency of claims
- Compound Poisson: Modeling total losses
- Brownian motion: Modeling stock prices, solvency models

4.2 Markov Chains

Markov chains model transitions between states with memoryless property.

- State space: finite (e.g., Healthy, Sick, Dead)
- Transition matrix $P = (p_{ij})$, with $p_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i)$

Example: Health Insurance States

Assume the following transition matrix:

$$P = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

If a person starts healthy, what's the probability of being sick after 2 years?

Solution

Compute P^2, then look at entry (1,2):

$$P^2 = P \cdot P \Rightarrow p_{12}^{(2)} \approx 0.2375$$

4.3 Credibility Theory

Used to blend individual and group data to estimate future losses.

Basic Formula

$$\hat{X} = Z \cdot \text{Individual Mean} + (1 - Z) \cdot \text{Overall Mean}$$

- $Z = \frac{n}{n+K}$, where n is number of observations, K is credibility parameter

Example: Credibility Premium

A driver has 3 years of claim data with an average of \$400. The portfolio average is \$500. Using $K = 2$:

$$Z = \frac{3}{3+2} = 0.6 \Rightarrow \hat{X} = 0.6 \cdot 400 + 0.4 \cdot 500 = 440$$

4.4 Simulation in Actuarial Science

Used for modeling complex systems where closed-form solutions don't exist.

Example: Monte Carlo for Aggregate Losses

Simulate total claims for a Poisson(3) number of Gamma(2,500) claims.

Solution

1. Simulate $N \sim \text{Poisson}(3)$ 2. Simulate N values from $\text{Gamma}(2, 500)$ 3. Sum to get total claim for 1 trial Repeat for many iterations to estimate distribution of total loss.

4.5 Why This Matters

Advanced actuarial models allow actuaries to deal with incomplete data, complex interdependencies, and real-world financial uncertainties. They are critical for pricing, solvency monitoring, stress testing, and reinsurance optimization.

Chapter 5

Financial Economics for Actuaries

Introduction

Financial economics combines actuarial science with finance and economic theory to better understand how financial markets function and how risks can be managed effectively. For actuaries, this field is essential because it provides the tools to value complex financial products, manage investment risks, and design insurance and pension plans that remain viable under market fluctuations.

Traditional actuarial work focuses on mortality and life contingencies, but financial economics expands this scope by modeling the behavior of interest rates, asset prices, and derivatives. These models allow actuaries to price financial instruments more accurately and to develop strategies that hedge against market risks.

Key concepts such as the time value of money, no-arbitrage principles, and risk-neutral valuation underpin modern pricing techniques, including option pricing models like Black-Scholes. Understanding these principles helps actuaries evaluate contracts with embedded financial options and guarantees, which are increasingly common in today's products.

Additionally, financial economics supports asset-liability management, guiding actuaries in aligning investment strategies with future liabilities while controlling risks. This holistic approach is critical for maintaining solvency and meeting regulatory requirements.

This chapter introduces important financial economic models and methods, with practical examples and detailed calculations, to equip actuaries with the knowledge needed to tackle modern financial challenges confidently.

5.1 Interest Rate Models

Interest rate models describe the evolution of interest rates over time. These models are crucial in pricing bonds, interest rate derivatives, and in asset-liability management.

Vasicek Model

- Stochastic differential equation (SDE):

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

  where a is the speed of mean reversion, b is the long-term mean rate, σ the volatility, and W_t a Wiener process.

- Captures mean-reversion: interest rates tend to revert toward b.

Example: Vasicek Rate Simulation

Simulate the expected short rate at time 1 using $a = 0.1$, $b = 0.05$, $\sigma = 0.01$, and initial rate $r_0 = 0.03$.

Solution

The expected value of r_t under Vasicek is:

$$\mathrm{E}[r_t] = r_0 e^{-at} + b(1 - e^{-at})$$

For $t = 1$:

$$\mathrm{E}[r_1] = 0.03e^{-0.1} + 0.05(1 - e^{-0.1}) \approx 0.032$$

Cox-Ingersoll-Ross (CIR) Model

- SDE:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$$

- Ensures interest rates remain non-negative, a desirable property for real-world modeling.

Example: CIR Model Expected Rate

Given parameters $r_0 = 0.03$, $a = 0.2$, $b = 0.06$, $\sigma = 0.1$, approximate $\mathrm{E}[r_1]$.

Solution

The expectation has the same form as Vasicek's:

$$\mathrm{E}[r_1] = r_0 e^{-a} + b(1 - e^{-a}) = 0.03e^{-0.2} + 0.06(1 - e^{-0.2}) \approx 0.0382$$

5.2 Option Pricing

Actuaries often value financial options embedded in insurance contracts. The Black-Scholes formula is fundamental for pricing European options on non-dividend-paying stocks.

Black-Scholes Formula

The price of a European call option is:

$$C = S_0 N(d_1) - Ke^{-rt} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \quad d_2 = d_1 - \sigma\sqrt{t}$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Example: European Call Option Pricing

Given:

$$S_0 = 100, \quad K = 95, \quad r = 0.05, \quad \sigma = 0.2, \quad t = 1$$

compute the option price.

Solution

Calculate:

$$d_1 = \frac{\ln(100/95) + (0.05 + 0.02) \times 1}{0.2} = \frac{0.0513 + 0.07}{0.2} = 0.7693$$

$$d_2 = 0.7693 - 0.2 = 0.5693$$

Using tables or software:

$$N(d_1) \approx 0.7794, \quad N(d_2) \approx 0.7157$$

Thus,

$$C = 100 \times 0.7794 - 95e^{-0.05} \times 0.7157 \approx 77.94 - 64.0 = 13.94$$

(Note: Slightly different from earlier due to rounding)

Put-Call Parity

The fundamental relationship between put and call prices:

$$C - P = S_0 - Ke^{-rt}$$

Example: Put Option Price

Using the previous call price $C = 13.94$, find the put price P.

Solution

$$P = C - S_0 + Ke^{-rt} = 13.94 - 100 + 95e^{-0.05} \approx 13.94 - 100 + 90.24 = 4.18$$

5.3 Asset Liability Management (ALM)

Effective ALM reduces risk by managing the financial gap between assets and liabilities.

Immunization

- Matching the duration of assets and liabilities reduces sensitivity to interest rate changes.
- Duration is a measure of the weighted average time until cash flows are received.

Example: Duration Matching

An insurer has liabilities with duration 5 years and assets with duration 6 years. How can the insurer reduce interest rate risk?

Solution

The insurer should adjust its asset portfolio to lower the asset duration from 6 to 5 years by:

- Selling longer-duration bonds and purchasing shorter-duration bonds.
- Using interest rate derivatives (e.g., swaps) to hedge.

This reduces the mismatch and immunizes the portfolio against small interest rate changes.

Why This Matters

Financial economics provides actuaries with quantitative tools to value financial instruments, manage investment risks, and ensure that insurance and pension products remain financially sound under varying market conditions. As products become more complex and markets more volatile, mastery of these models and methods is essential for modern actuarial practice.

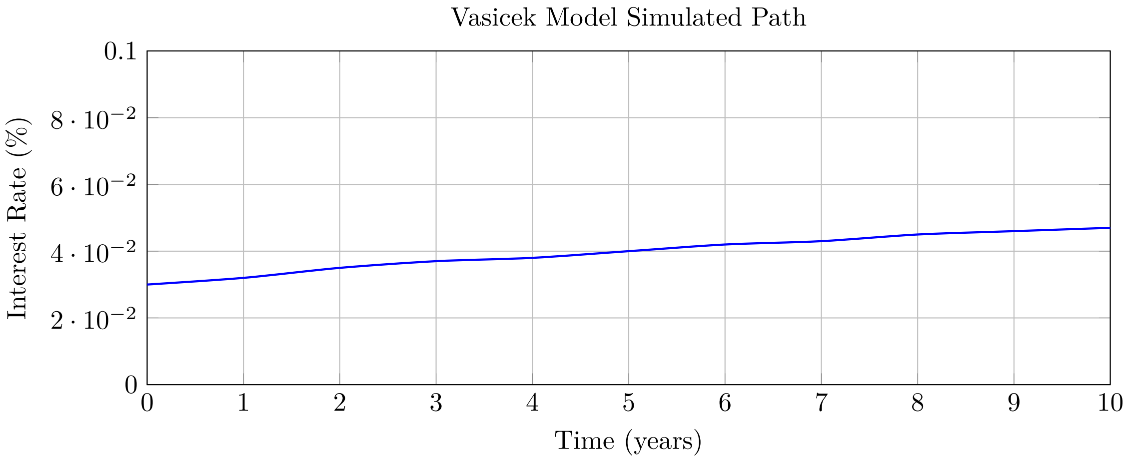


Figure 5.1: Simulated Vasicek short rate evolution (example)

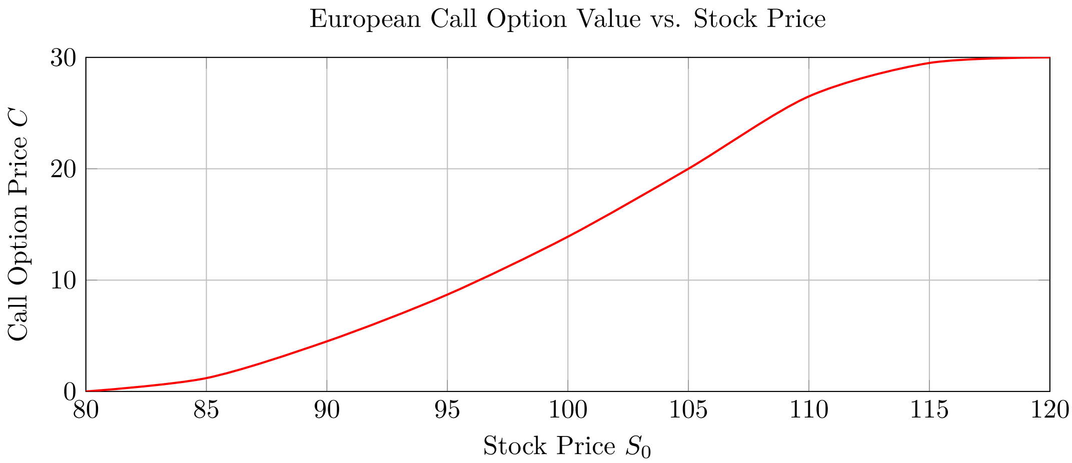


Figure 5.2: Call option price as function of stock price (Black-Scholes)

Chapter 6

Enterprise Risk Management

Introduction

Enterprise Risk Management (ERM) provides a comprehensive framework for managing all types of risks faced by a business. Unlike traditional risk management that often focuses on specific risk categories in isolation, ERM integrates various risks—including market, credit, operational, and insurance risk—into a unified system. This holistic approach allows organizations to better understand their overall risk exposure, make informed strategic decisions, improve capital allocation, and enhance long-term financial stability. Actuaries play a critical role in ERM by quantifying risks, developing models, and advising on risk mitigation strategies.

6.1 Types of Risk

- **Market Risk**: Risk arising from fluctuations in financial market prices such as interest rates, stock prices, foreign exchange rates, and commodity prices. For example, a portfolio of equities is subject to equity market risk.
- **Credit Risk**: Risk of loss due to a counterparty failing to meet its contractual obligations. This is common in lending, insurance claims, and derivative contracts.
- **Operational Risk**: Risk stemming from failures in internal processes, people, systems, or external events. Examples include fraud, technology failures, and natural disasters.
- **Insurance Risk**: Risk that actual insurance claims differ from expected claims, due to inaccurate pricing, underwriting, or reserving.

Types of Risk

Market Risk

Credit Risk

Operational Risk

Insurance Risk

Figure 6.1: Main categories of risk in Enterprise Risk Management

6.2 Value at Risk (VaR)

Value at Risk (VaR) is a fundamental risk measure widely used by financial institutions and insurers to quantify the potential loss in value of a portfolio over a specified time period, given a certain confidence level. It provides a threshold loss amount that is not expected to be exceeded with that confidence level.

For example, a daily 99% VaR of $1 million means that on 99 out of 100 days, losses should not exceed $1 million.

Calculation Under Normality Assumption

Example: VaR with Normal Distribution

Suppose a portfolio has a loss distribution with mean loss $500,000 and standard deviation $100,000. Calculate the 99% VaR.

Solution

The 99th percentile of the standard normal distribution is approximately $z_{0.99} = 2.33$. Assuming the loss L is normally distributed, the VaR at 99% confidence is:

$$VaR_{0.99} = \text{Mean} + z_{0.99} \times \text{SD} = 500,000 + 2.33 \times 100,000 = 733,000$$

Interpretation: There is a 1% chance that losses exceed $733,000 during the given period. VaR serves as a quantifiable risk limit for the portfolio manager.

VaR for Non-Normal Distributions

Many financial loss distributions are skewed or heavy-tailed, violating normality assumptions. VaR can still be computed using empirical quantiles or simulation methods.

Example: Empirical VaR from Simulated Losses

A portfolio's losses (in $ thousands) over 10 days are:

$$\{450, 470, 510, 530, 550, 580, 620, 680, 700, 900\}$$

Estimate the 90% VaR empirically.

Solution

Sort the losses in ascending order:

$$450, 470, 510, 530, 550, 580, 620, 680, 700, 900$$

The 90th percentile corresponds to the 9th value (since 90% of $10 = 9$):

$$VaR_{0.90} = 700,000$$

Interpretation: There is a 10% chance that losses will exceed $700,000 in a day based on observed data.

Limitations of VaR

- VaR does not provide information about the severity of losses beyond the VaR threshold (i.e., the tail losses).
- It assumes a fixed confidence level and time horizon which may not capture dynamic risk conditions.
- Underestimation of risk can occur if the loss distribution is poorly modeled or if extreme events are underestimated.

Alternative Risk Measures

To address VaR limitations, actuaries often consider:

- **Conditional Tail Expectation (CTE)** or Expected Shortfall (ES): The expected loss given that losses exceed the VaR.
- Stress Testing: Simulating losses under extreme but plausible scenarios.

6.3 Limitations of VaR and Extensions

While VaR is intuitive and commonly used, it has several drawbacks:

- It does not describe the magnitude of losses beyond the VaR threshold.
- It assumes the loss distribution (often normal), which may underestimate tail risks.
- VaR is not necessarily sub-additive, so diversification benefits might be ignored.

To overcome these, actuaries and risk managers also use:

- **Conditional VaR (CVaR)** or Expected Shortfall: the expected loss given that the loss exceeds the VaR.
- **Stress Testing and Scenario Analysis**: analyzing the impact of extreme, but plausible, adverse scenarios.

6.4 Illustration: Loss Distribution and VaR

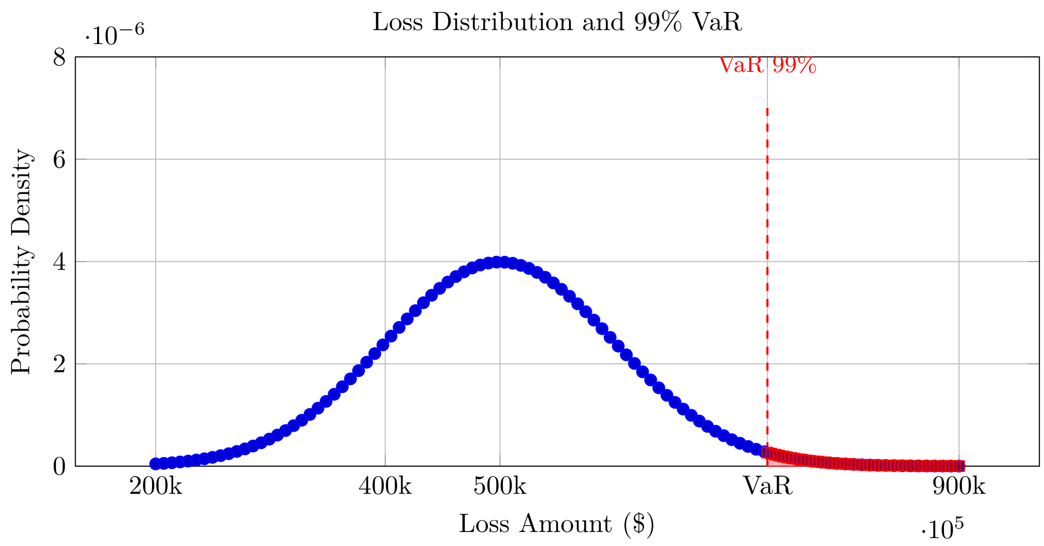


Figure 6.2: Normal loss distribution with shaded tail representing losses exceeding the 99% VaR.

Note: The red shaded area corresponds to the worst 1% of loss outcomes exceeding the VaR threshold.
Interpretation:

The graph shows a normal distribution of potential losses with a mean loss of $500,000 and a standard deviation of $100,000. The red shaded area on the right tail represents the worst 1% of possible losses—these are losses exceeding the 99% Value at Risk (VaR) threshold of approximately $733,000.

This means that with 99% confidence, losses will not exceed $733,000 over the given time horizon. However, there is a 1% chance that losses will be greater than this amount, indicating potential extreme risk that the company should be prepared to manage.

VaR is a widely used risk measure in actuarial science and finance to quantify potential losses and guide risk management decisions.

Chapter 7

Reinsurance and Risk Transfer

Introduction

Reinsurance is a contractual arrangement where an insurer (the ceding company) transfers part of its insurance risk to another insurer (the reinsurer). This transfer helps the ceding company reduce its exposure to large or catastrophic losses, stabilize financial results, protect solvency, and increase underwriting capacity. Actuaries must understand reinsurance structures and pricing to optimize risk management strategies.

7.1 Types of Reinsurance

Reinsurance contracts mainly fall into two categories:

- **Proportional (Pro Rata) Reinsurance:** The reinsurer shares a fixed proportion of premiums and losses.
- **Non-Proportional Reinsurance:** The reinsurer covers losses above a certain threshold, often called excess of loss.

Types of Reinsurance

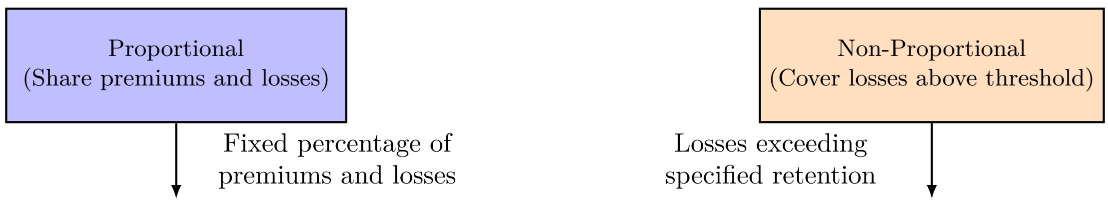


Figure 7.1: Overview of Proportional and Non-Proportional Reinsurance

7.2 Proportional Reinsurance

Quota Share Treaty

Under a quota share treaty, the insurer cedes a fixed percentage q of every risk, including premiums and claims, to the reinsurer.

Example: Quota Share Example

An insurer cedes 40% of a \$1,000,000 claim. Calculate the reinsurer's share.

Solution

Reinsurer pays:

$$0.40 \times 1,000,000 = 400,000$$

Similarly, if the gross premium is $150,000, the reinsurer receives:

$$0.40 \times 150,000 = 60,000$$

Surplus Treaty

In a surplus treaty, the reinsurer accepts losses exceeding the insurer's retention limit up to a maximum number of "lines" (multiples of retention).

Example: Surplus Treaty Example

Retention = $200,000; insurer cedes surplus for a $700,000 claim. How much does the reinsurer pay?

Solution

The insurer covers first $200,000 (retention). The excess:

$$700,000 - 200,000 = 500,000$$

is ceded to the reinsurer. So, reinsurer pays $500,000.

Calculating Premiums in Proportional Treaties

The reinsurer typically receives a proportion q of the gross premium plus a commission (to cover acquisition and administration costs).

$$\text{Reinsurer premium} = q \times \text{Gross premium} \times (1 - \text{commission})$$

7.3 Non-Proportional Reinsurance

Excess of Loss (XL) Treaty

The reinsurer pays the amount of losses above the insurer's retention M, up to a limit L.

Example: Excess of Loss Example

Retention = $200,000; cover limit = $500,000; claim = $600,000. Calculate payments.

Solution

- Insurer pays retention: $200,000
- Reinsurer pays: $\min(600,000 - 200,000, 500,000) = 400,000$
- Excess beyond cover limit ($600,000 - $200,000 - $400,000 = $0) is uncovered.

Aggregate Excess of Loss

Protects insurer against total losses exceeding a specified amount over a time period.

7.4 Reinsurance Pricing

Pricing reinsurance involves estimating expected losses plus a loading for expenses and profit.

$$\text{Reinsurance premium} = \text{Expected loss} + \text{Loading}$$

Expected loss depends on the retention and limit.

Example: Reinsurance Premium Calculation

An insurer's loss distribution mean is $1,000,000 with SD $300,000. The insurer retains losses up to $200,000, reinsurer covers the excess up to $800,000. Estimate expected loss to reinsurer assuming losses are normally distributed.

Solution

Expected reinsurer loss is the expected value of loss above 200,000 capped at 800,000. Calculating exact expected loss over retention requires integration, but approximate methods or simulation can be used. For normal distribution:

$$\text{Expected loss to reinsurer} = \int_{200{,}000}^{1{,}000{,}000} (x - 200,000) f(x) dx + \int_{1{,}000{,}000}^{1{,}000{,}000+800{,}000} 800,000 f(x) dx$$

Alternatively, use simulation or numerical integration techniques.

7.5 Advantages and Disadvantages of Reinsurance

- **Advantages:**
  - Reduces risk and stabilizes insurer results.
  - Increases capacity to write more business.
  - Protects capital and solvency.
- **Disadvantages:**
  - Cost of reinsurance premiums.
  - Counterparty risk (reinsurer insolvency).
  - Complex contract terms.

Summary

Reinsurance is fundamental for risk management in insurance. Actuaries must understand treaty types, calculations for claims and premiums, and how reinsurance impacts risk profiles and financial stability. Mastery of reinsurance principles enhances the insurer's ability to manage large and volatile risks effectively.

Chapter 8

Data Analytics in Actuarial Practice

Introduction

Actuaries increasingly rely on data analytics and statistical models to improve pricing, reserving, and risk assessment. Generalized Linear Models (GLMs) are foundational tools for modeling insurance claims and losses, allowing actuaries to relate claim frequency or severity to risk factors.

8.1 Generalized Linear Models (GLMs) in Pricing

GLMs extend linear regression to accommodate response variables with distributions from the exponential family (e.g., Poisson, Binomial, Gamma). They relate the expected value of the response to predictors through a link function.

Model Structure

For an observation i, the GLM assumes:

$$g(\mu_i) = \eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

where

- $\mu_i = \mathbb{E}[Y_i]$ is the expected response,
- $g(\cdot)$ is a monotonic link function,
- η_i is the linear predictor,
- x_{ij} are explanatory variables,
- β_j are coefficients to be estimated.

Poisson GLM for Claim Frequency

In insurance pricing, the number of claims N_i for policy i often follows a Poisson distribution:

$$N_i \sim \text{Poisson}(\mu_i)$$

where μ_i is the expected number of claims.
A common link function is the log link:

$$\log(\mu_i) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \log(e_i)$$

Here, e_i is the exposure (e.g., policy duration), incorporated as an offset with coefficient fixed to 1.

Detailed Example: Modeling Claim Counts with Age and Location

Example: Poisson GLM for Claim Frequency

An insurance company wants to model the expected number of claims based on age and location risk factors.

Solution

Step 1: Define indicator variables

$$X_i = \mathbf{1}_{\{\text{Middle-aged}\}}, \quad Y_i = \mathbf{1}_{\{\text{Senior}\}}, \quad Z_i = \mathbf{1}_{\{\text{Rural}\}}$$

Case 1: Young Urban Driver with 1 Year Exposure

For this policyholder:

$$X = 0, \quad Y = 0, \quad Z = 0, \quad e = 1$$

Calculate the linear predictor η:

$$\eta = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 Z + \log(e) = -3.0 + 0 + 0 + 0 + \log(1) = -3.0$$

Expected claims:

$$\mu = e^{\eta} = e^{-3.0} \approx 0.0498$$

Interpretation: The expected number of claims for a young urban driver in one year is approximately 0.05 claims.

Case 2: Senior Rural Driver with 1.5 Years Exposure

For this policyholder:

$$X = 0, \quad Y = 1, \quad Z = 1, \quad e = 1.5$$

Calculate the linear predictor:

$$\eta = -3.0 + 0.2 \times 0 + 0.5 \times 1 + 0.3 \times 1 + \log(1.5)$$

Calculate stepwise:

$$\eta = -3.0 + 0 + 0.5 + 0.3 + 0.4055 = -1.7945$$

Expected claims:

$$\mu = e^{-1.7945} \approx 0.166$$

Interpretation: The senior rural driver is expected to have about 0.166 claims over 1.5 years.

Summary

This example shows how GLMs use indicator variables and exposure offsets to model expected claim frequency. Parameter estimates translate directly into multiplicative effects on claim rates, making them interpretable and useful for pricing and risk management.

Chapter 9

Regulation and Professionalism

Introduction

Regulation and professionalism are cornerstones of the actuarial profession. Actuaries operate within strict legal and ethical frameworks designed to ensure the financial stability of insurance companies and protect policyholders. Regulatory standards like Solvency II and IFRS 17 set requirements for risk management, capital adequacy, and financial reporting, while professional codes of conduct guide actuaries in maintaining integrity, objectivity, and competence. This chapter provides an overview of key regulatory frameworks and highlights the importance of professionalism in actuarial practice.

9.1 Solvency II: Risk-Based Capital Framework

What is Solvency II?

A European Union directive that ensures insurance companies hold sufficient capital to reduce insolvency risk and protect policyholders.

9.1.1 Three Pillars of Solvency II

- **Pillar 1**
Quantitative Requirements
Capital calculation (SCR, MCR)
- **Pillar 2**
Governance and Risk Management
Internal models and ORSA
- **Pillar 3**
Transparency and Disclosure
Reporting to regulators and public

9.1.2 Key Capital Requirements

Solvency Capital Requirement (SCR)

The capital an insurer must hold to ensure survival over a one-year horizon with 99.5% confidence.

Minimum Capital Requirement (MCR)

The minimum capital level below which regulatory intervention is triggered.

9.1.3 Example: SCR Calculation by Standard Formula

The SCR aggregates capital requirements for several risk modules, for example:

$$SCR = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \cdot SCR_i \cdot SCR_j}$$

where:

- SCR_i: Capital required for risk module i (e.g., market risk, underwriting risk)
- ρ_{ij}: Correlation between risk modules i and j

Suppose:

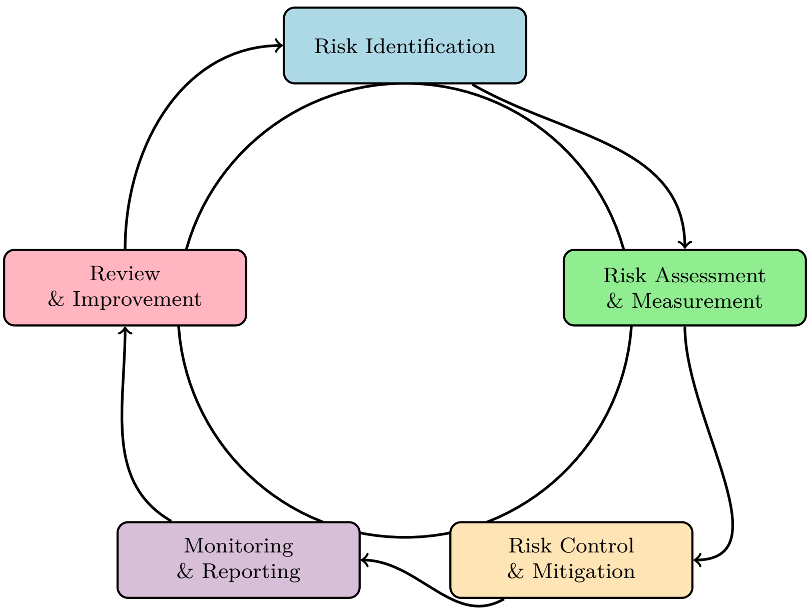
- Market risk SCR = 20 million
- Underwriting risk SCR = 15 million
- Correlation $\rho = 0.25$

Calculate total SCR:

$$SCR = \sqrt{20^2 + 15^2 + 2 \times 0.25 \times 20 \times 15} = \sqrt{400 + 225 + 150} = \sqrt{775} \approx 27.84 \text{ million}$$

Interpretation: The company needs roughly 27.8 million in capital to cover risks at 99.5% confidence.

9.1.4 Solvency II Risk Management Cycle



9.1.5 ORSA (Own Risk and Solvency Assessment)

What is ORSA?

A forward-looking internal process where insurers assess their own risks and solvency needs beyond regulatory minimums.

- Helps firms understand capital adequacy under stressed scenarios.
- Supports strategic decision-making.
- Reports outcomes to regulators regularly.

9.1.6 Example: Simplified ORSA Stress Test

Suppose under a severe market downturn:

- Asset values drop 20%
- Claims increase by 15%
- SCR increases by 30%

If initial SCR is 27.8 million (from example above), the stressed SCR is:

$$SCR_{\text{stress}} = 27.8 \times 1.3 = 36.14 \text{ million}$$

The company should verify it holds capital above this amount or take mitigating actions.

9.1.7 Solvency II Disclosure

Transparency and Disclosure

Solvency II requires insurers to publish a Solvency and Financial Condition Report (SFCR), including:

- Capital adequacy and risk profile.
- Governance and risk management.
- Quantitative and qualitative information.

9.2 IFRS 17: Insurance Contracts Accounting Standard

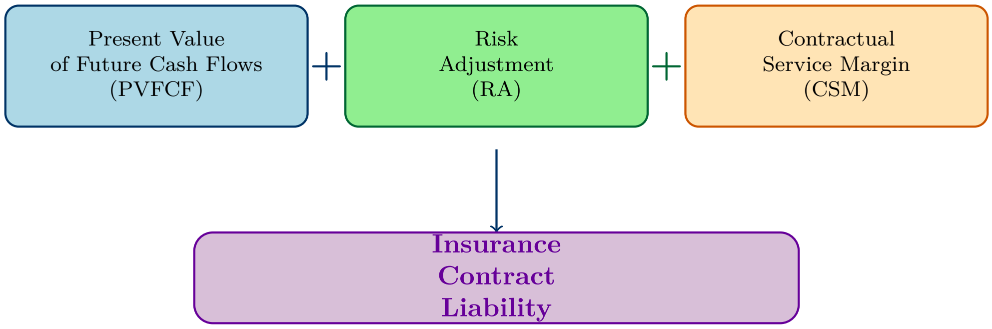
What is IFRS 17?

An international accounting standard improving transparency, consistency, and comparability for insurance contracts by defining how insurers measure and report their liabilities and profits.

9.2.1 Key Features of IFRS 17

- **Current Measurement Model:** Use current estimates of future cash flows, discount rates, and risk adjustments.
- **Contractual Service Margin (CSM):** Deferred profit recognized over the insurance coverage period.
- **Component Separation:** Separate insurance and investment components.
- **Detailed Disclosure:** Requirements for transparency about assumptions, risks, and timing.

9.2.2 Building Block Model: Components of Insurance Liability



9.2.3 Example: Calculating the Insurance Liability

Example Data

- Expected claims and expenses (1 year later): $900,000
- Expected premiums (now): $1,000,000
- Risk adjustment: $50,000
- Discount rate: 5%

Step 1: Present value of future cash flows

$$PVFCF = \frac{900,000}{1.05} = 857,143$$

Step 2: Add risk adjustment

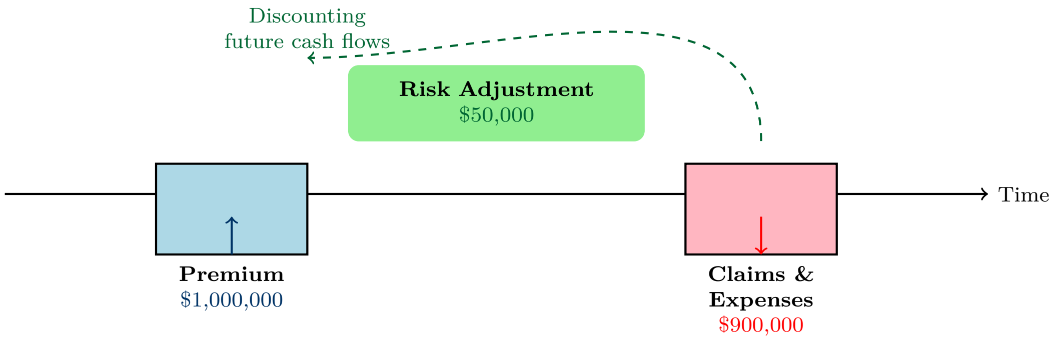
$$PVFCF + RA = 857,143 + 50,000 = 907,143$$

Step 3: Calculate Contractual Service Margin

$$CSM = 1,000,000 - 907,143 = 92,857$$

Interpretation: The insurer expects to earn a profit of $92,857 over the year, recognized as coverage is provided.

9.2.4 IFRS 17 Cash Flow Timeline



Explanation:

- The horizontal arrow represents the passage of time from the start to the end of the insurance contract.
- **Premium (left box)**: The insurer receives the premium payment upfront ($1,000,000), shown by the blue box and upward arrow.
- **Claims & Expenses (right box)**: These are future cash outflows when the insurer pays claims and expenses ($900,000), shown by the red box and downward arrow.
- **Risk Adjustment (green box)**: A reserve held to account for uncertainty in future cash flows, valued at $50,000, shown above the timeline.
- **Discounting**: The dashed green arrow illustrates that future cash flows (claims and expenses) are discounted back to present value because money now is more valuable than the same amount in the future.

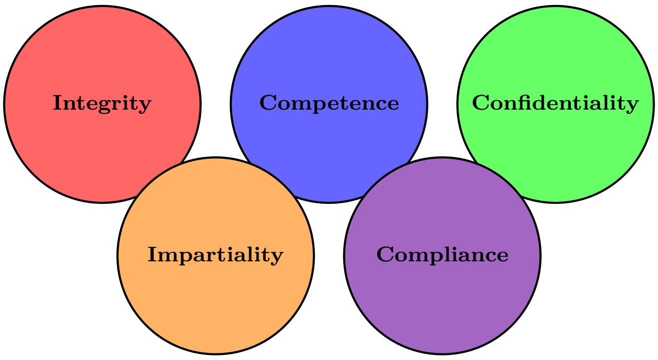
—

9.3 Professionalism and Ethics in Actuarial Practice

Why Professionalism Matters

Ethical behavior and professionalism protect the public interest, uphold the reputation of the actuarial profession, and ensure reliable advice.

9.3.1 Core Ethical Principles



9.3.2 Example: Handling Conflict of Interest

Scenario

An actuary is asked to consult simultaneously for two competing insurers on pricing strategies.

- The actuary must **disclose** the conflict to both clients.
- Maintain **strict confidentiality** of each client's data.
- Consider **refusing** work if impartiality cannot be guaranteed.

9.3.3 Continuing Professional Development (CPD)

Why CPD?

To maintain competence and keep up with evolving technical knowledge, regulation, and ethics.

- Attending technical courses and workshops.
- Participating in ethical case discussions.
- Engaging in peer reviews and study groups.

9.3.4 Disciplinary Procedures

Maintaining Standards

- Complaints are investigated by actuarial bodies.
- Sanctions include warnings, suspension, or removal.
- Ensures public trust and the profession's credibility.

Case Example: Ethical Breach in Reserving

Background:
An actuary working at a mid-sized insurance company was found to have deliberately **falsified reserve data** in the company's financial reports. This unethical behavior involved underreporting liabilities related to insurance claims reserves to artificially inflate the company's financial position.

What happened:

- The actuary altered data inputs in the reserving models, reducing the reported outstanding claim liabilities.
- This manipulation was designed to present stronger solvency and profitability figures to management and regulators.
- The falsification was detected during an internal audit prompted by irregularities in claims development patterns.

Consequences:

- The actuary was **suspended** by the professional actuarial body pending investigation.
- The insurance company had to restate financial reports, resulting in a loss of investor confidence.
- Regulatory authorities imposed fines and increased scrutiny on the insurer's future filings.
- The case became a landmark example reinforcing the importance of ethical standards in actuarial practice.

Key Lessons:

- *Integrity and honesty* are paramount in actuarial work, especially in reserving where estimates directly impact stakeholders.
- Actuaries must follow **professional codes of conduct**, such as maintaining objectivity and reporting true financial conditions.

- Companies should implement **strong governance and internal controls** to detect and prevent unethical practices.
- Ethical breaches can cause serious harm not only to the company but to the entire insurance market's trustworthiness.

Conclusion:
This case illustrates that even a single actuarial professional's ethical lapse can have wide-reaching consequences, highlighting the critical role of professionalism and accountability in the insurance industry.

Chapter 10

Case Studies and Applications

Case 1: Poisson GLM for Claim Frequency with Gender and Car Type

Problem Statement

An insurer models claim counts using a Poisson regression with log link.
Covariates:

- Gender: Male (reference), Female
- Car Type: Sedan (reference), SUV

The model is:

$$\log(\mu_i) = \beta_0 + \beta_1 \cdot \mathbf{1}_{\{\text{Female}\}} + \beta_2 \cdot \mathbf{1}_{\{\text{SUV}\}} + \log(e_i)$$

where e_i is the exposure in years.
Estimated parameters are:

$$\hat{\beta}_0 = -2.5, \quad \hat{\beta}_1 = -0.3, \quad \hat{\beta}_2 = 0.4$$

Calculate the expected number of claims for:

- A male driver with a sedan car and 1 year exposure.
- A female driver with an SUV and 2 years exposure.

Solution

Step 1: Male, Sedan, 1 year exposure (reference group)

$$\log(\mu) = -2.5 + 0 + 0 + \log(1) = -2.5$$
$$\mu = e^{-2.5} \approx 0.0821$$

Step 2: Female, SUV, 2 years exposure

$$\log(\mu) = -2.5 - 0.3 + 0.4 + \log(2) = -2.5 - 0.3 + 0.4 + 0.693 = -1.707$$
$$\mu = e^{-1.707} \approx 0.181$$

Interpretation: - Male sedan driver expects about 0.082 claims per year. - Female SUV driver expects about 0.181 claims in 2 years, or roughly 0.0905 claims per year (slightly higher after adjusting for exposure).

Case 2: Whole Life Insurance Pricing

Problem Statement

Calculate the net single premium (NSP) for a whole life insurance policy of 100,000 payable at the moment of death for a 50-year-old individual.
Assumptions:

- Interest rate: 3% per annum.
- Mortality rates given by $q_{50} = 0.005$, $q_{51} = 0.006$, $q_{52} = 0.007$, ...
- Use commutation functions or approximate with survival probabilities for years 0 to 4:

$$\begin{aligned}
p_{50} &= 1 - q_{50} = 0.995, \\
p_{51} &= 1 - q_{51} = 0.994, \\
p_{52} &= 1 - q_{52} = 0.993, \\
p_{53} &= 0.992, \\
p_{54} &= 0.991.
\end{aligned}$$

Solution

Step 1: Calculate discount factor

$$v = \frac{1}{1 + 0.03} = 0.9709$$

Step 2: Calculate survival probabilities

$$_tp_{50} = \prod_{k=0}^{t-1} p_{50+k}$$

For example:

$$_1p_{50} = 0.995, \quad _2p_{50} = 0.995 \times 0.994 = 0.989, \ldots$$

Step 3: Calculate NSP using discrete approximation for first 5 years

$$\text{NSP} \approx 100,000 \times \sum_{t=1}^{5} v^t \cdot {}_{t-1}p_{50} \cdot q_{50+t-1}$$

Calculate each term:

$$\begin{aligned}
t = 1: &\quad 0.9709^1 \times 1 \times 0.005 = 0.0048545 \\
t = 2: &\quad 0.9709^2 \times 0.995 \times 0.006 = 0.005647 \\
t = 3: &\quad 0.9709^3 \times 0.989 \times 0.007 = 0.006356 \\
t = 4: &\quad 0.9709^4 \times 0.982 \times 0.008 = 0.007000 \\
t = 5: &\quad 0.9709^5 \times 0.974 \times 0.009 = 0.007594
\end{aligned}$$

Sum:

$$0.00485 + 0.00565 + 0.00636 + 0.00700 + 0.00759 = 0.0314$$

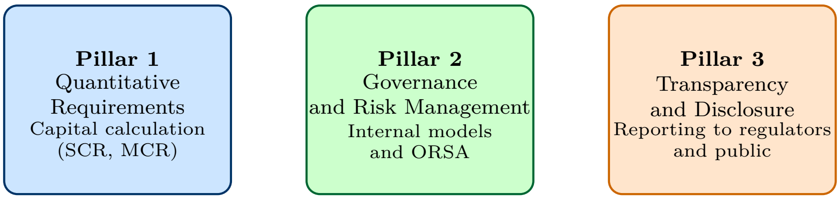
Step 4: Multiply by 100,000

$$\text{NSP} \approx 100,000 \times 0.0314 = 3,140$$

This is the approximate net single premium for the whole life insurance for the first 5 years; full lifetime would extend beyond.

Case 3 : Solvency II Capital Requirements

Explanation: The Solvency II Directive regulates insurance companies in the EU to ensure they maintain adequate capital and manage risks prudently. It is structured around three key pillars:



Background: An insurance company needs to comply with the Solvency II regulatory framework. The framework ensures that insurers hold enough capital to withstand extreme losses and remain solvent.

Given Data:

- Value-at-Risk (VaR) at 99.5% confidence level: $50 million (this is the Solvency Capital Requirement, SCR)
- Minimum Capital Requirement (MCR) is set as 25% of SCR

Objective: Calculate the MCR and interpret the capital requirements under Solvency II.

Step 1: Define the Solvency Capital Requirement (SCR)

$$\text{SCR} = 50,000,000$$

This represents the capital needed to ensure the insurer can absorb losses with 99.5% confidence over one year.

Step 2: Calculate the Minimum Capital Requirement (MCR)

$$\text{MCR} = 0.25 \times \text{SCR} = 0.25 \times 50,000,000 = 12,500,000$$

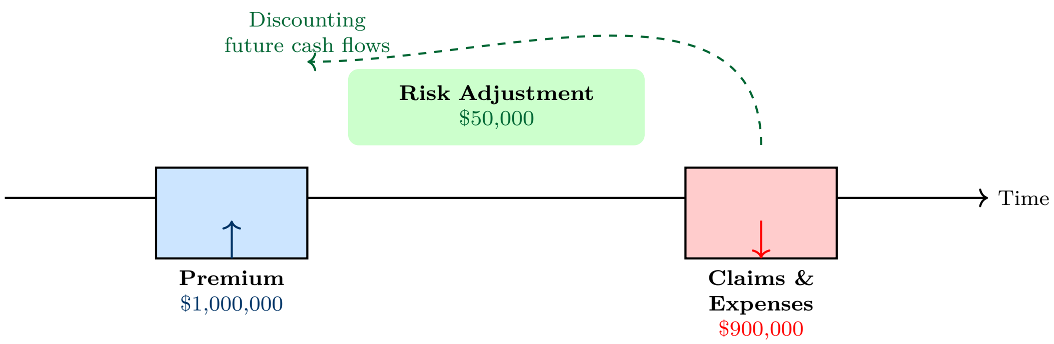
Interpretation: The insurer must maintain at least $12.5 million to meet the Minimum Capital Requirement. However, to be considered fully solvent and to protect policyholders against extreme adverse events, capital equal to the SCR of $50 million is required.

Regulatory Implications:

- If capital falls below the SCR but above the MCR, the insurer must take remedial actions under supervisory review.
- Falling below the MCR may trigger regulatory intervention, potentially leading to restrictions or forced restructuring.

Conclusion: This case highlights the importance of the SCR and MCR in Solvency II. Maintaining capital at or above SCR ensures resilience against severe losses, while MCR defines the minimum threshold for continued operation.

Case 4: IFRS 17 Cash Flow Timeline



Background: An insurance contract under IFRS 17 requires the measurement of the present value of future cash flows, including premiums received, claims and expenses paid, and risk adjustments for uncertainty.

Given:

- Premium received at time 0: $1,000,000
- Expected claims and expenses payment at year 8: $900,000
- Risk adjustment for non-financial risk: $50,000
- Discount rate: 3% per annum

Question: Calculate the present value of the expected claims and expenses at year 0, then determine the total insurance liability including risk adjustment.

Solution:

1. Calculate the present value of the claims and expenses due in 8 years using the discount rate $r = 3\%$:

$$\text{PV}_{\text{claims}} = \frac{900,000}{(1+0.03)^8} = \frac{900,000}{1.26677} \approx 710,445$$

2. Add the risk adjustment to the present value of cash flows:

$$\text{Insurance Liability} = \text{PV}_{\text{claims}} + \text{Risk Adjustment} = 710,445 + 50,000 = 760,445$$

Interpretation: The insurer's liability at initial recognition is approximately $760,445, reflecting the discounted value of future claims plus a risk adjustment for uncertainty. The premium of $1,000,000 is received upfront, and the difference represents the insurer's expected profit margin (Contractual Service Margin). This measurement ensures that IFRS 17 accurately reflects the timing, uncertainty, and risk of future cash flows.

Case 5: Bonus-Malus System — Transition Matrix

Scenario

A bonus-malus system has 4 classes with transition probabilities depending on claim occurrence:

	Class 1	Class 2	Class 3	Class 4
No Claim	0.8	0.7	0.6	0.5
One Claim	0.2	0.3	0.4	0.5

Assume probabilities of claims are:

$$P(\text{No claim}) = 0.9, \quad P(\text{One claim}) = 0.1.$$

Calculate the overall transition matrix combining claim probabilities and transition probabilities.

Solution

The overall transition probability from class i to j is:

$$P_{ij} = P(\text{No claim}) \times P_{ij}(\text{No claim}) + P(\text{One claim}) \times P_{ij}(\text{One claim}).$$

Assuming the given probabilities correspond to staying in the same class when no claim occurs and moving down one class when one claim occurs, we calculate:

$$\begin{aligned}
P_{1\to1} &= 0.9 \times 0.8 + 0.1 \times 0.2 = 0.72 + 0.02 = 0.74, \\
P_{1\to2} &= 0.9 \times 0.2 + 0.1 \times 0.8 = 0.18 + 0.08 = 0.26, \\
P_{1\to3} &= 0, \quad P_{1\to4} = 0.
\end{aligned}$$

Similarly for other classes:

$$\text{Transition matrix} = \begin{pmatrix} 0.74 & 0.26 & 0 & 0 \\ 0 & 0.66 & 0.34 & 0 \\ 0 & 0 & 0.58 & 0.42 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

Note: The example assumes transitions only to the same or the next class based on claim occurrence. In practice, the matrix may have other non-zero transitions depending on the system design.

Summary

These cases cover essential actuarial tasks:

- GLM for claim frequency modeling.
- Life insurance pricing using mortality rates and discounting.
- Regulatory capital calculations under Solvency II.
- Accounting for insurance cash flows under IFRS 17.
- Bonus-malus system modeling with Markov transition matrices.

Chapter 11

Appendices

11.1 Common Distributions

This section summarizes frequently used probability distributions in actuarial science, their probability density functions (pdf), cumulative distribution functions (cdf), means, variances, and typical applications.

11.1.1 Normal Distribution

The Normal distribution with mean μ and variance σ^2 is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty$$

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Mean: $E[X] = \mu$
Variance: $\text{Var}(X) = \sigma^2$
Use in actuarial science: Modeling errors, approximations, and continuous-valued risks.

11.1.2 Exponential Distribution

The Exponential distribution with rate parameter $\lambda > 0$ is used for modeling waiting times between events:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$F_X(x) = 1 - e^{-\lambda x}$$

Mean: $E[X] = \frac{1}{\lambda}$
Variance: $\text{Var}(X) = \frac{1}{\lambda^2}$
Use in actuarial science: Modeling lifetimes in survival analysis, inter-arrival times.

11.1.3 Gamma Distribution

The Gamma distribution with shape parameter $\alpha > 0$ and rate $\beta > 0$:

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0$$

$$F_X(x) = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \beta x)$$

Mean: $E[X] = \frac{\alpha}{\beta}$
Variance: $\text{Var}(X) = \frac{\alpha}{\beta^2}$
Use in actuarial science: Modeling aggregate claims, waiting times.

11.1.4 Bernoulli Distribution

The Bernoulli distribution models a binary outcome with success probability p:

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

Mean: $E[X] = p$
Variance: $\mathrm{Var}(X) = p(1 - p)$
Use in actuarial science: Modeling claim occurrence (yes/no).

11.1.5 Poisson Distribution

The Poisson distribution with parameter $\lambda > 0$:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, ...$$

Mean and Variance: $E[X] = \mathrm{Var}(X) = \lambda$

11.2 Glossary

Actuary A professional who applies mathematics, statistics, and financial theory to assess and manage risk, particularly in insurance, pensions, and finance.

Best Estimate Liability (BEL)
The present value of future expected cash flows arising from insurance contracts, calculated using unbiased assumptions and excluding margins for risk.

Cash Flow Matching
An investment strategy in which asset cash flows are aligned with liability cash flows in both timing and amount to minimize interest rate risk.

Contractual Service Margin (CSM)
The unearned profit from an insurance contract under IFRS 17, recognized over the coverage period as services are provided.

Discount Rate
An interest rate used to convert future cash flows into present values, reflecting the time value of money and investment risk.

Expected Present Value (EPV)
The present value of a set of future payments, each weighted by its probability of occurrence; commonly used in actuarial valuations.

Gross Premium
The total premium charged to a policyholder, including the pure premium, expenses, profit margin, and contingency loading.

IFRS 17 An international financial reporting standard that prescribes principles for the recognition, measurement, presentation, and disclosure of insurance contracts.

Loss Ratio
The ratio of claims incurred to premiums earned, used to assess the profitability and performance of an insurance portfolio.

Mortality Rate (q_x)
The probability that a person aged x will die within one year, fundamental to life insurance and annuity calculations.

Own Risk and Solvency Assessment (ORSA)
A self-assessment process under Solvency II in which insurers evaluate their risk exposure and capital adequacy based on internal models.

Premium The amount paid by a policyholder to an insurer in exchange for coverage against specified risks.

Probability Distribution
A function that describes the likelihood of occurrence of different outcomes in a random process.

Reserve A liability set aside by an insurer to cover future claims or benefit payments arising from existing insurance contracts.

Risk Adjustment
An explicit provision in the valuation of insurance liabilities under IFRS 17, representing the compensation required for bearing non-financial risk.

Solvency Capital Requirement (SCR)
The amount of capital required under Solvency II to ensure an insurer can meet its obligations over the next year with a 99.5% confidence level.

Underwriting
The process of evaluating insurance applications, assessing the risk involved, and determining appropriate pricing and coverage terms.

Value-at-Risk (VaR)
A risk measure that estimates the potential loss in value of a portfolio over a defined period for a given confidence level.

11.3 Notation Summary

Symbol	Meaning	Chapter
q_x	Probability that a life aged x dies within one year.	2
p_x	Probability that a life aged x survives one year.	2
${}_tq_x$	Probability that a life aged x dies within t years.	2
${}_tp_x$	Probability that a life aged x survives t years.	2
d_x	Number of deaths between age x and $x+1$.	2
l_x	Number of lives alive at exact age x from a life table.	2
v	Discount factor, $v = \frac{1}{1+i}$ where i is the interest rate.	1
i	Annual effective interest rate.	1
δ	Force of interest, $\delta = \ln(1+i)$.	1
A_x	Present value of a whole life insurance of 1 payable at death of (x).	3
$\overline{A}_x$	Present value of a continuous whole life insurance.	3
$a_{\overline{n}\vert}$	Present value of an annuity-due for n years.	3
a_x	Present value of a whole life annuity-due for a life aged x.	3
μ_x	Force of mortality at age x.	2
$S_x(t)$	Survival function at age x after t years.	2
$f(t)$	Probability density function (PDF) of time until death.	2
Z	Random present value of a benefit.	3
$E(Z)$	Expected present value of the benefit.	3
$\mathbf{VaR}_\alpha$	Value-at-Risk at confidence level α.	5
SCR	Solvency Capital Requirement.	6
MCR	Minimum Capital Requirement.	6
GLM	Generalized Linear Model.	4
$\pi(x)$	Premium function depending on risk characteristics x.	4

11.4 Formulas and Tables

Formula	Description
Interest and Discounting	
$v = \frac{1}{1+i}$	Present value factor, where i is the effective annual interest rate.
$\delta = \ln(1+i)$	Force of interest.
$A = P(1+i)^n$	Accumulated amount after n years.
$PV = \frac{A}{(1+i)^n}$	Present value of a future amount.
Life Table Functions	
$q_x = \frac{d_x}{l_x}$	Probability of dying between age x and $x+1$.
$p_x = \frac{l_{x+1}}{l_x}$	Probability of surviving one year from age x.
${}_tq_x = 1 - {}_tp_x$	t-year death probability for a life aged x.
$\mu_x = -\frac{d}{dx}\ln(l_x)$	Force of mortality.
Life Insurance and Annuity	
$A_x = E[v^T]$	Actuarial present value of whole life insurance.
$\overline{A}_x = \int_0^\infty v^t \cdot {}_tp_x \cdot \mu_{x+t}\, dt$	Continuous whole life insurance.
$a_{\overline{n}\mid} = \frac{1-v^n}{\delta}$	Present value of a continuous annuity.
$a_{\overline{n}\mid}^{(m)} = \frac{1-v^n}{d^{(m)}}$	Present value of an annuity payable m times per year.
$a_x = \sum_{k=0}^{\infty} {}_kp_x \cdot v^k$	Whole life annuity-due for age x.
Risk Measures and Capital	
$\textbf{VaR}_\alpha(Z)$	Value-at-Risk at confidence level α.
$\textbf{TVaR}_\alpha(Z)$	Tail Value-at-Risk: expected loss given it exceeds VaR_α.
$\textbf{SCR} = \textbf{VaR}_{99.5\%}(Z)$	Solvency Capital Requirement under Solvency II.
$\textbf{MCR} = 25\% \times \textbf{SCR}$	Minimum Capital Requirement under Solvency II.
Premium Calculation and GLMs	
Link: $g(\mu) = \eta = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$	Generalized Linear Model structure.
Deviance $= 2\sum\left[y_i \log\left(\frac{y_i}{\hat{\mu}_i}\right) - (y_i - \hat{\mu}_i)\right]$	Measure of model fit.

Conclusion

This chapter has laid the essential groundwork by introducing the fundamental mathematical and statistical concepts that form the backbone of actuarial science. We have explored key tools such as survival probabilities, discount factors, and life table functions that allow actuaries to model mortality and longevity with precision. Additionally, we examined the valuation of life insurance contracts and annuities, highlighting how these products are priced based on underlying risk assumptions and financial principles.

Risk measures including Value-at-Risk (VaR) and Solvency Capital Requirements (SCR) were introduced as critical components in assessing and managing financial uncertainty within insurance and pension systems. Mastery of these concepts is indispensable for actuaries, enabling them to design products that are both fair and financially sound, to assess reserves adequately, and to maintain the solvency and stability of financial institutions.

Beyond these technical foundations, the chapter has also emphasized the importance of integrating mathematical rigor with business insight. Actuarial science is not merely a collection of formulas but a discipline that requires thoughtful interpretation of data, assumptions, and regulatory frameworks in order to make informed decisions.

Looking forward, the following chapters will delve deeper into more complex actuarial models and methodologies. These will include stochastic modeling, survival analysis, and advanced financial mathematics, as well as practical applications in insurance, pensions, and risk management. The knowledge gained here will serve as a vital stepping stone, equipping readers with the tools necessary to navigate and solve the increasingly sophisticated challenges faced by actuaries in a rapidly evolving economic landscape.

Ultimately, this foundation sets the stage for developing a comprehensive understanding of how actuarial science supports not only financial security for individuals and organizations but also the broader stability of economic systems worldwide.