Math 202A Lecture Notes

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1 Metric Spaces

1.1 Basic Notions

Definition 1.1. Let X be a set. A *metric* on X is a function $\rho : X \times X \to \mathbb{R}^+$ such that

- $\rho(x, y) = \rho(y, x)$.
- (Triangle inequality) $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all $y \in X$.
- $\rho(x, y) = 0$ if and only if $x = y$.

Together, (X, ρ_X) is referred to as a metric space. If a function ρ satisfies the first two properties but only reverse implication in the third property holds, then ρ is a *semimetric* or a *pseudometric*. $\diamond$

Observe that if $S \subseteq X$ and ρ is a metric on X, the restriction of ρ to S is a metric on S.

Definition 1.2. A *norm* on a real/complex vector space V is a function $\|\cdot\| : V \to \mathbb{R}^+$ such that

- $\|\alpha v\| = |\alpha| \, \|v\|$ for $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$
- $\|x + y\| \le \|x\| + \|y\|$
- $\|x\| = 0$ if and only if $x = 0$.

If only the first two properties hold, then $\|\cdot\|$ is a *seminorm*. $\diamond$

Observe that having a norm on a vector space allows us to define a metric on the space as $\rho(x, y) = \|x - y\|$.

We now define the functions between metric spaces that are 'interesting'. In the following discussion, let (X, ρ_X), (Y, ρ_Y) be metric spaces.

Definition 1.3. Let $f : X \to Y$ be a function. Then, we say that f is an *isometry* if $\rho_Y(f(x_1), f(x_2)) = \rho_X(x_1, x_2)$ for any $x_1, x_2 \in X$. If f is onto, then f is viewed as an isomorphism between the two metric spaces. $\diamond$

Definition 1.4. Let $f : X \to Y$ be a function. If there exists some constant K such that

$$\rho_Y(f(x_1), f(x_2)) \leq K\rho_X(x_1, x_2)$$

for any $x_1, x_2 \in X$, then f is said to be *K-Lipschitz*, and the smallest K for which the inequality holds above for all pairs of points is referred to as the *Lipschitz constant*. $\diamond$

Definition 1.5. Let $f : X \to Y$ be a function. Then, f is *uniformly continuous* if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\rho_X(x_1, x_2) < \delta \implies \rho_Y(f(x_1), f(x_2)) < \varepsilon$$

for any $x_1, x_2 \in X$. $\diamond$

Definition 1.6. Let $f : X \to Y$ be a function. Then, f is *continuous* if for every x_1 and for all $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\rho_X(x_1, x) < \delta \implies \rho_Y(f(x_1), f(x)) < \varepsilon. \qquad \diamond$$

Definition 1.7. Let X be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then, we say that (x_n) *converges to* x, or $(x_n) \to x$, if for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$n > N \implies \rho(x, x_n) < \varepsilon. \qquad \diamond$$

Proposition 1.1. *Let X, Y be metric spaces, and let $f : X \to Y$ be a continuous function. Furthermore, let (x_n) be a sequence in X that converges to x. Then, $(f(x_n))$ is convergent, and converges to $f(x)$.*

Definition 1.8. Let X be a metric space, and $S \subseteq X$. Then, S is *dense* in X if for every point $x \in X$ and every $\varepsilon > x$, $S \cap B_\varepsilon(x)$ is non-empty. $\diamond$

Proposition 1.2. *Let X, Y be metric spaces and let S be a dense subset of X. Furthermore, let f, g be continuous functions from X to Y. Then, if f and g agree on S, then $f = g$.*

1.2 Cauchy Sequences and Completion

Definition 1.9. Let X be a metric space. Then, a sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$m, n > N \implies \rho(x_m, x_n) < \varepsilon. \qquad \diamond$$

Definition 1.10. If every Cauchy sequence in a metric space X converges, then X is *complete.* $\diamond$

As an example of a metric space that is not complete, consider $\mathbb{Q}$ with the metric inherited from $\mathbb{R}$.

Proposition 1.3. *Let X, Y be a metric space, and $f : X \to Y$ a uniformly continuous function. Then, if (x_n) is a Cauchy sequence, then $(f(x_n))$ is also a Cauchy sequence.*

Definition 1.11. Let X be a metric space. A *completion* of X is a complete metric space Y together with an isometry $i : X \to Y$ such that $i(X)$ is dense in Y. $\diamond$

Proposition 1.4. *If $((Y_1, \rho_{Y_1}), i_1)$ and $((Y_2, \rho_{Y_2}), i_2)$ are completions of X, then there exists an isometric onto function $g : Y_1 \to Y_2$ such that $f_2 = g \circ f_1$.*

Proposition 1.5. *If $((\tilde{X}_1, \rho_{\tilde{X}_1}), j_1)$ and $((\tilde{X}_2, \rho_{\tilde{X}_2}), j_2)$ are both completions of some metric space X, there exists some isometry i between the two completions such that $j_2 = i \circ j_1$.*

Definition 1.12. Let (a_n), (b_n) be Cauchy sequences. Then, we say that the two sequences are *equivalent* if $\rho(a_n, b_n) \to 0$ as $n \to \infty$. $\diamond$

Proposition 1.6. *Let X and Y be metric spaces, with Y complete. Furthermore, let S be a dense subset of X with the inherited metric. Let $f : S \to Y$ be a uniformly continuous function. Then, there exists a unique and uniformly continuous extension of f to X.*

Theorem 1.1. *Every metric space has a completion.*

The theorem above gives us a completion of every metric space (X, ρ), namely, the set of Cauchy sequences in X identified under the equivalence relation

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} \rho(x_n, y_n) = 0.$$

The metric η on the completion is then defined as

$$\eta([(x_n)], [(y_n)]) = \lim_{n\to\infty} \rho(x_n, y_n) = 0,$$

and the injection j_X is defined such that $x \in X$ is mapped to the constant sequence $[(x)]$ in the completion.

Consider the set of continuous functions $C([0, 1])$, with the supremum norm

$$\|f\|_\infty = \sup \{|f(t)| \mid t \in [0, 1]\}.$$

This metric space can be shown to be complete. Other possible norms on this space include

$$\|f\|_1 = \int_0^1 f(x)\, \mathrm{d}x \quad \|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 \, \mathrm{d}x}.$$

Under the L^1 norm, the space above is not complete[1].

Assume V is a vector space equipped with a norm. Then, we can give an easier description of the completion of V. In particular, let Cauchy(V) denote the set of Cauchy sequences in V, and let $\mathcal{N}(V)$ denote the set of sequences in V that converge to 0. Then, the completion can be described as Cauchy$(V)/\mathcal{N}(V)$.

[1] Completion gives us the set of Lebesgue-integrable functions on $[0, 1]$!

2 General Topology

Definition 2.1. A set is *open* in a metric space if it can expressed as the (possibly arbitrary) union of open balls. ◇

We collect a few useful facts relating to preimages below:

- $f^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f^{-1}(A_\alpha)$.
- $f^{-1}(\bigcap_\alpha A_\alpha) = \bigcap_\alpha f^{-1}(A_\alpha)$
- $f^{-1}(A_1 \setminus A_2) = f^{-1}(A_1) \setminus f^{-1}(A_2)$

Definition 2.2. Let X be a set. A *topology* on X is a collection of sets $\mathcal{T}$ such that $\mathcal{T}$ is closed under finite intersection, arbitrary unions, and contains both X and $\varnothing$. ◇

The sets in the topology are referred to as *open sets*. Complements of open sets are *closed sets*.

Definition 2.3. Let $A \subseteq X$. Then, the *closure* of A is the intersection of all closed sets that contain A. ◇

If we let the topology on X be the power set of X, then this topology is referred to as the *discrete topology*. Otherwise, if we let the topology be just X and the empty set, the the topology is referred to as the *indiscrete* or *trivial topology*.

Definition 2.4. Let X be a set with two topologies $\mathcal{T}_1, \mathcal{T}_2$. Then, we say that $\mathcal{T}_1$ is *coarser* or *weaker* if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Otherwise, we say that $\mathcal{T}_1$ is *finer* or *stronger* if $\mathcal{T}_1 \supseteq \mathcal{T}_2$. ◇

Given a collection of topologies for a set X, the intersection of all these topologies gives a topology that is weaker than all of the topologies in the intersection. It then follows that given a collection S of subsets of X, there exists a smallest topology containing S. Namely, it is the intersection of all topologies containing S.

Definition 2.5. Let $(X, \mathcal{T})$ be a topological space. Then, $\mathcal{B}$ is a basis for $\mathcal{T}$ if every set in $\mathcal{T}$ can be expressed as a union of sets in $\mathcal{B}$. ◇

Proposition 2.1. *Let $\mathcal{B}$ be a collection of subsets of X such that for any $U, V \in \mathcal{B}$, $U \cap V$ can be expressed as the union of elements of $\mathcal{B}$. Then, the collection of the unions of elements of $\mathcal{B}$ is a topology, and $\mathcal{B}$ is a base for this generated topology.*

Definition 2.6. Let X be a set, and let S be a collection of subsets of X such that the union of all sets in S is all of X. Then, the collection of finite intersections of S forms a base for a topology, and S is a *subbase* for that topology. ◇

Definition 2.7. Let X, Y be topological spaces. Then, a function f is *continuous* is the preimage of every open set in Y is open in X, as determined by the topologies on the two spaces. ◇

Proposition 2.2. *The composition of continuous functions is continuous.*

Definition 2.8. Let X, Y be topological spaces. A bijective function $f : X \to Y$ is a *homeomorphism* if both f and f^{-1} are continuous. ◇

Proposition 2.3. *Let X, Y be a topological spaces where the topology of Y is generated by some basis $\mathcal{B}$. Then, f is continuous if and only if the preimage of every set in $\mathcal{B}$ is open.*

Observe that the previous proposition also holds if we replace basis by subbasis.

Proposition 2.4. *Let X be a set, and let $\{Y_\alpha\}_{\alpha \in I}$ a collection of topological spaces. For each Y_α, define a function $f_\alpha : X \to Y_\alpha$. Then, there exists a weakest topology on X such that f_α is continuous for every α, namely the topology generated by the subbase*

$$\mathcal{B} = \bigcup_{\alpha \in I} \left\{ f_\alpha^{-1}(U) \;\middle|\; U \in \mathcal{T}_\alpha \right\}.$$