ENERGY STORAGE ELEMENTS

6.1 CAPACITORS

A capacitor is a device that stores electrical charge. It is two metal plates separated by an insulator. It basically stores charge by taking electrons from one side and pumping it to the other side.

$Q = CV$

Q – Charge in Coulombs (C)

C – Capacitance in Farads (F)

V – Voltage in Volts(V)

Recall also that:

$Q = It$

$C = \frac{Q}{V}$

Using the units,

$1F = \frac{1C}{V}$

If a capacitance has a capacitance of 10F, if we charge it up to 1V, it can store 10C of charge.
If we charge it up to 2V, it can store up to 20C.

Capacitance can be seen as charge efficiency because the higher the capacitance, the more charges it can store per volt.

Since, $Q = CV$, the higher the voltage, the higher the charge stored. However, even though mathematically, it looks like Capacitance is inversely proportional to Voltage, it really is not. This is because, the capacitance does not depend on the voltage rather it depends on the material used, and other factors.

In metals, protons are fixed in place and the charge carriers are electrongs. Therefore, the total charge can also be described as

$Q = ne$

n – number of electrons.

e – Charge of the electron

$e = -1.6 \times 10^{-19} C$

$Electric\ Potential = \frac{Electric\ Potential\ Energy}{Charge}$

$Voltage = \frac{W}{q}$
$W = q \Delta V$

Voltage and electric potential are not necessarily the same but they are similar
Voltage is the difference in the electric potentials of two points.
$Voltage = Change \in Electric\ Potential$
$\Delta V = V_B - V_A$

FACTORS THAT AFFECT CAPACITANCE
1. Area of the plates
2. Distance between plates
3. Nature of the Dielectric: With a dielectric you can store more charges by increasing a capacitance, however the voltage will also reduce with the same proportion.

$C = \frac{K \varepsilon_o A}{d}$
K is the dielectric constant, and for air, $K = 1.0006\ldots$ and in a vaccuum, $K = 1$
E.g. For Quartz, $K \approx 4.3$
For water, $K \approx 80$

For a vaccuum, $C_o = \frac{\varepsilon_o A}{d}$
When there is a dielectric, $C = \frac{\varepsilon A}{d}$

$\varepsilon_o - Permitivity\ of\ free\ space$
$\varepsilon_o = 8.85 \times 10^{-12} C^2/N m^2$
ε = Permitivity of the material
$\varepsilon = K \varepsilon_o$

From the above it can be seen that
$C = K C_o$
$V = \frac{V_o}{K}$

When adding a dielectric:
1. Charge the capacitor first
2. Disconnect the capacitor
3. Add the dielectric.

When a capacitor is being charged, there will be an electric field between the plates.

$$E=\frac{V}{d}$$

The electric field is also given as

$$E=\frac{\sigma}{\varepsilon_o}$$

σ – Surface charge density

$$\sigma=\frac{Q}{A}$$

$$C=\frac{Q}{V}=\frac{\sigma A}{Ed}=\frac{E\varepsilon_o A}{Ed}$$

$$C_o=\frac{\varepsilon_o A}{d}$$

ELECTRIC POTENTIAL ENERGY STORED IN A CAPACITOR

$$U=\frac{1}{2}QV$$

$$U=\frac{1}{2}CV^2$$

$$U=\frac{1}{2}\frac{Q^2}{C}$$

6.2 RC CIRCUITS

If you have a battery, a resistor, a capacitor and a switch. When the switch is open, the voltage across the capacitor is 0.

6.2.1 SOURCE FREE RESPONSE

If just a charged capacitor is connected in series with a resistor, energy will be transferred from the capacitor into the circuit. The response gotten from this is called the **natural response** or **source free response**.

From KVL

$$V_R+V_C=0$$

$$iR+V_c=0$$

$$R\left[C\frac{dV_c}{dt}\right]+V_c=0$$

$$\frac{dV_c}{dt}+\frac{1}{RC}V_C=0$$

This is a first order linear differential equation.
Since it is homogeneous, its solution is going to be the Complementary Function. This CF is simply the **source free response.**
Given the general equation,

$$y'+py=q$$

$$y'+py=0$$

$$CF=Ae^{-px}$$

Similarly for this $\frac{dV_c}{dt}+\frac{1}{RC}V_C=0$,

$V_C(t)=Ae^{\frac{-t}{RC}}$

Using the initial conditions,

$V_C(0)=Ae^{-0}$

$V=A$

$V_C(t)=V_o e^{\frac{-t}{RC}} \rightarrow$ Here the capacitor is discharging.

Time constant, $\tau=RC$

6.2.2 FORCED RESPONSE IN RC CIRCUITS

If you have a battery, a resistor, a capacitor and a switch. When the switch is open, the voltage across the capacitor is 0.

When the switch is closed at time t=0, the capacitor begins to charge. Initially the capacitor will have a voltage of 0V or an initial voltage V_o and the resistor will have a voltage of 12V. At the end of the charge, the capacitor will have a voltage of 12V and the resistor will have a voltage of 0V. This is because when the capacitor is fully charged, current stops flowing in the circuit and the resistor can’t have a voltage.

Applying KVL

$V=V_R+V_C$

V = iR + V_C

$V=R\left[C\frac{dV_C}{dt}\right]+V_C$

$\frac{dV_C}{dt}+\frac{1}{RC}V_C=\frac{V}{RC}$

You’ll see that the general solution of this kind of differential equation is:

$Total\,solution=CF+PI$

$CF=Ae^{\frac{-1}{RC}t}$

$PI=V_C(t=\infty)=V_{max}$

$V_C(t)=V_{max}+Ae^{\frac{-t}{RC}}$

At the initial conditions,

$V_o=V_{max}+Ae^{-0}$

$A=V_o-V_{max}$

$V_C(t)=V_{max}+(V_o-V_{max})e^{\frac{-t}{RC}}$

The above is called the **step response of the circuit**.

If $V_o=0$

$V(t)=EMF\left[1-e^{\frac{-t}{RC}}\right]$

The formula for the voltage across the capacitor when it is charging is given as

$$V(t)=EMF\left[1-e^{\frac{-t}{RC}}\right]$$

$$t=-RC\ln\left[1-\frac{V(t)}{EMF}\right]$$

The graph of this is a progressively increasing graph until it gets to its maximum voltage (voltage of the battery) at infinity.

Time Constant, $\tau=RC$

If you are asked in the question, how much will the capacitor have charged after n time constants, then:

$$t=n\tau=nRC$$

Number of time constants

$$n=\frac{t}{\tau}$$

7.3 INDUCTORS

7.4 RL CIRCUITS

In a series connection of a voltage source, a key, a resistor and an inductor:

7.4.1 SOURCE FREE RESPONSE

When the key has been closed for a long time and then this charged inductor is now in series with an inductor only.

Applying KVL

$$V_R+V_L=0$$

$$iR+L\frac{di}{dt}=0$$

$$\frac{di}{dt}+\frac{R}{L}i=0$$

Solving the differential equation

$$i(t)=Ae^{\frac{-R}{L}t}$$

Applying initial conditions,

$$i(0)=i_o$$

$$i(t)=i_oe^{\frac{-R}{L}t}$$

Sometimes,

$$i(t)=i_{max}e^{\frac{-R}{L}t}$$

Time constant, $\tau = \frac{L}{R}$

The voltage across the inductor at that point

$$V_L = L\frac{di}{dt}$$

$$V_L = L\frac{d}{dt}\left[i_{max}\, e^{\frac{-R}{L}t}\right]$$

$$V_L = L \times I \times e^{\frac{-R}{L}t} \times \left(\frac{-R}{L}\right)$$

$$V_L = -i_{max} R\, e^{\frac{-R}{L}t}$$

When the key is open, no current is being transferred in the circuit.

7.4.2 FORCED RESPONSE OF RL CIRCUIT

As soon as the key is close, current starts to increase from its initial I_o in the circuit until it reaches its maximum I_{max}. As the current increases, the circuit is in transient state. When the key has been closed for a long time, $t=(\infty)$, I = I_{max}, $L=0$ because inductance depends on change in current so when the current is constant, there is no change in current.

Applying KVL

$$V = V_R + V_L$$

$$V = iR\, L\frac{di}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

Looking at it, the solution should be in the form

Solution = CF + PI

$$PI = i\, at\, t = \infty$$

At $t=\infty$, the inductor will act as a short circuit and the current will be maximum

$$I_{max} = \frac{V}{R}$$

$$PI = \frac{V}{R}$$

CF = Source free response

$$CF = A\, e^{\frac{-R}{L}t}$$

$$i(t) = PI + CF$$

$$i(t) = \frac{V}{R} + A\, e^{\frac{-R}{L}t}$$

$$i(t) = I_{max} + A\, e^{\frac{-R}{L}t}$$

Applying the initial conditions,

$$i(0) = i_o$$

$$i_o = i_{max} + A\, e^{-0}$$

$A = i_o - i_{max}$

$i(t) = i_{max} + (i_o - i_{max}) e^{\frac{-R}{L}t}$

If $i_o = 0$,

$i(t) = i_{max}(1 - e^{\frac{-R}{L}t})$

IN SUMMARY, TO SOLVE FOR RL AND RC CIRCUITS,

1. Draw the circuit at time t=0+
2. Find initial conditions: $V_c(0)$ or $i_L(0)$
3. Find R_{eq} and (C_{eq} or L_{eq}) of the circuit
4. Find the time constant
5. Find the final values of V_c or i_L
6. Total solution