ODE WITH CONSTANT COEFFICIENTS

4.1 INTRODUCTION

Given the general form of a second order linear differential equation:

$a(x)y''+b(x)y'+c(x)y=d(x)$

$a(x)$, $b(x)$, $c(x)$ are usually continuous functions of x. In that case, it is called a differential equation with variable coefficients

If $d(x)=0$, we have a homogeneous linear equation

If $d(x)\neq 0$, we have a non-homogeneous linear equation

If $a(x)$, $b(x)$, amd $c(x)$ are constants, it will give an ODE with constant coefficients equation

4.2 SOLVING HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

$ay''+by'+cy=0$

A typical solution of this is:

$y=e^{rx}$

$y'=re^{rx}$

$y''=r^2e^{rx}$

On replacing the equation, we have:

$ar^2e^{rx}+bre^{rx}+ce^{rx}=0$

$e^{rx}[ar^2+br+c]=0$

$ar^2+br+c=0$

Next we solve for r:

$r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Now you need to consider three cases when solving for r:

Case 1: $b^2-4ac>0$

Here, You will have two values for r i.e. r_1 and r_2

The general solution for this case is:

$y=c_1 e^{r_1 x}+c_2 e^{r_2 x}$

Case 2: $b^2-4ac=0$,

Here you get one value for r

The general solution is given as

y = $y=c_1 e^{rx}+c_2 x e^{rx}$

Case 3: $b^2-4ac<0$

Here you would have to use the quadratic formula to solve.

You will also get two values of r

$r_1=\alpha+\beta i$

$r_2=\alpha-\beta i$

The general solution will be

$y=e^{\alpha x}\left[c_1\cos(\beta x)+c_2\sin(\beta x)\right]$

4.3 SOLVING NONHOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

4.3.1 METHOD OF UNDETERMINED COEFFICIENTS

For non homogeneous 2nd ORDER Differential equations

$a y'' + b y' + cy = d(x)$

This is a nonhomogeneous general equation.

To solve we have the general solution

$y(x)=y_p(x)+y_c(x)$

$y_p(x)$ is the particular solution of the non-homogeneous equation $a y'' + b y' + cy = d(x)$

Next, we write the homogeneous equation of the differential equation

$a y'' + b y' + cy = 0$

The solution of this homogeneous equation is the value of $y_c(x)$.

Questions

1. Solve the equations $y''+5y'+6y=x^2$.

First step is to solve the homogeneous equation of the differential equation.

$y''+5y'+6y=0$

On solving, the general solution is

$y_c(x)=c_1e^{-2x}+c_2e^{-3x}$

Next, we look at the degree of the function on the right.

$y''+5y'+6y=x^2$

Since the power of the function on the right is 2 $d(x)=x^2$, the general solution for this non-homogeneous equation will have a general form

$y_p(x)=Ax^2+Bx+C$

Then if we find the following derivatives:

$y'_p(x)=2Ax+B$

$y''_p(x)=2A$

Next we substitute the equations into the general form

$y''+5y'+6y=x^2$

$2A+5(2Ax+B)+6(Ax^2+Bx+C)=x^2$

$2A+10Ax+5B+6Ax^2+6Bx+6C=x^2$

$(6A)x^2+(10A+6B)x+2A+5B+6C=1x^2+0x+0$

6A = 1

$A=\frac{1}{6}$

$10A+6B=0$

$10\left(\frac{1}{6}\right)+6B=0$

Multiplying through by 6

$10+36B=0$

$B = \frac{-10}{36}$

$B = \frac{-5}{18}$

$2A + 5B + 6C = 0$

$2\left(\frac{1}{6}\right) + 5\left(\frac{-5}{18}\right) + 6C = 0$

Multipying through by 18

$6 - 25 + 108C = 0$

$C = \frac{19}{108}$

$y_p(x) = Ax^2 + Bx + C$

$y_p(x) = \frac{1}{6}x^2 - \frac{5}{18}x + \frac{19}{108}$

But,

$y(x) = y_p(x) + y_c(x)$

$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{6}x^2 - \frac{5}{18}x + \frac{19}{108}$

2. Solve the differential equation $y'' + 9y = e^{2x}$

Answers

$r_1 = 0 + 3i$

$r_2 = 0 - 3i$

$\alpha = 0$

$\beta = 3$

$y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$

$y_p = Ae^{2x}$

$y_p(x) = \frac{1}{13}e^{2x}$

$y = y_c + y_p$

$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{13}e^{2x}$

3. Solve the differential equation $y'' + 3y' + 2y = \cos(x)$

Answers:

$r_1 = -1$

$r_2 = -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y_p = A\cos x + B\sin x$

Note for help that $\cos x = 1\cos x + 0\sin x$

$B = \frac{3}{10}$

$A = \frac{1}{10}$

$y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x$

$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{10}[\cos x + 3\sin x]$

4. Solve the differential equation

$y'' + y = \cos x$

$y_c = c_1 \cos x + c_2 \sin x$

Looking at this, y_p would normally be $y_p = A\cos x + B\sin x$. However if you use this, y'' and y will cancel out and we will have $0 = \cos x$ and we can't do anything with that.

So we will use

$y_p = Ax\cos x + Bx\sin x$

$A = 0$

$B = \frac{1}{2}$

$y_p = \frac{1}{2} x\sin x$

$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} x\sin x$

5. $y'' - 9y = xe^x + \sin 2x$

$y_c = c_1 e^{3x} + c_2 e^{-3x}$

For y_p, we are going to have two values: y_{p1} and y_{p2}. This is because for the homogeneous equation solution, the function on the right is a summation of two functions xe^x and $\sin 2x$

$y = y_c + y_{p1} + y_{p2}$

So for the first solution,

$y''-9y=xe^{x}$

$y_{p1}=(Ax+B)e^{x}$

$y'_{p1}=Ae^{x}+(Ax+B)e^{x}$

$y''_{p1}=2Ae^{x}+(Ax+B)e^{x}$

$A_1=\frac{-1}{8}$

$B_1=\frac{-1}{32}$

$y_{p1}=-\left(\frac{1}{8}x+\frac{1}{32}\right)e^{x}$

$y_{p2}=A_2\cos 2x+B_2\sin 2x$

$A_2=0$

$B_2=\frac{-1}{13}$

$y_{p2}=\frac{-1}{13}\sin 2x$

4.3.2 METHOD OF VARIATION OF PARAMETERS

$y''+y=sec\,x$

First you solve the homogeneous equation

$y''+y=0$

On solving we see that

$r^2=-1$

$r=\pm i$

$r_1=0+i$

$r_2=0-i$

$y_c=c_1\cos x+c_2\sin x$

When use the method of variation of parameters, our solution to the non-homogeneous equation will be

$y_p=u_1y_1+u_2y_2$

When using variation of parameters, c_1 and c_2 from the homogeneous equation will be converted to functions u_1 and u_2.

$y_1 = \cos x \quad y_2 = \sin x$.

Next we write the condition that we want to achieve

$u_1' y_1 + u_2' y_2 = 0$

$u_1' \cos x + u_2' \sin x = 0$

Next we find y'_p and $y''_p$and we plug it into the expression.

$y_p = u_1 y_1 + u_2 y_2$

Using the above gotten equation and the other equation $u_1' \cos x + u_2' \sin x = 0$, we can solve for u'_1 and u'_2. Then we integrate to get u_1 and u_2.

Then we plug that into $y_p = u_1 y_1 + u_2 y_2$and we get the solution of the non-homogeneous equation

$y_p = u_1 \cos x + u_2 \sin x$

$y'_p = u'_1 \cos x - u_1 \sin x + u'_2 \sin x + u_2 \cos x$

Recall from our condition that $u_1' \cos x + u_2' \sin x = 0$

$y'_p = u_2 \cos x - u_1 \sin x$

$y''_p = u'_2 \cos x - u_2 \sin x - u'_1 \sin x - u_1 \cos x$

$y'' + y = sec\, x$

On solving

$y'' + y = u'_2 \cos x - u'_1 \sin x = sec\, x$

Solving by system of equations:

$u_1' \cos x + u_2' \sin x = 0 - - - - \rightarrow \times \sin x$

$u'_2 \cos x - u'_1 \sin x = sec\, x - - - - \rightarrow \times \cos x$

$u_1' \cos x \sin x + u_2' \sin^2 x = 0$

$-u'_1 \sin x \cos x + u'_2 \cos^2 x = 1$

On adding,

$u'_2 \sin^2 x + u'_2 \cos^2 x = 1$

$u'_2 (\sin^2 x + \cos^2 x) = 1$

$u'_2 = 1$

$\int u'_2 \, dx = \int 1 \, dx$

$u_2 = x$

$u_1' \cos x + u_2' \sin x = 0$

$u_1' \cos x + \sin x = 0$

$u_1' = \frac{-\sin x}{\cos x}$

$u_1' = -\tan x$

$u_1 = \ln(\cos x)$

Recall that:

$y_p = u_1 \cos x + u_2 \sin x$

$y_p = \ln(\cos x) \cos x + x \sin x$

$y_c = c_1 \cos x + c_2 \sin x$

$y = y_c + y_p$

$y = (c_2 + x) \sin x + [c_1 + \ln(\cos x)] \cos x$