TRIGONOMETRIC INTEGRATION AND TIPS

TIPS FOR SIN AND COS

1. $\int \sin^m x \cos^n x \, dx$
If n = odd and m = even,
We can say that $n=2k+1$
$\int \sin^m x \cos^{2k+1} x \, dx$
$\int \sin^m x \cos^{2k} x \cos x \, dx$
$\int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx$
$\int \sin^m x \left(1-\sin^2 x\right)^k \cos x \, dx$
From this point on, we can then decide to do the u-substitution with
$u=\sin x$

2. A similar method can be applied if m is odd and n is even
For that, we are going to end up with something like:
$\int \left(1-\cos^2 x\right)^k \cos^n x \sin x \, dx$

3. When m and n are even (m=n=even)
We will use the half-angle formula
$\frac{1}{2}\sin 2x = \sin x \cos x$

4. If both are odd
$\int \sin^m x \cos^n x \, dx$
$\int \sin^{m-1} x \cos^{n-1} x \sin x \cos x \, dx$
$\int \sin^{m-1} x \left(\cos^2 x\right)^{\frac{n-1}{2}} \sin x \cos x \, dx$
$\int \sin^{m-1} x \left(1-\sin^2 x\right)^{\frac{n-1}{2}} \sin x \cos x \, dx$
$u=\sin x$
$du=\cos x \, dx$

5. $\int \frac{a\cos x+b\sin x}{c\cos x+d\sin x} dx = \left(\frac{ac+bd}{c^2+d^2}\right)x + \left(\frac{ab-bc}{c^2+d^2}\right)\ln\left|c\cos x+d\sin x\right| + k$

QUESTIONS

1. $\int \sin^5 x\, dx$ Answer: $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$
2. $\int \sin^6 x\, dx$
3. $\int \sin^2 x \cos^2 x\, dx$
4. $\int \sin^2 x\, dx$
5. $\int \sin^3 x\, dx$
6. $\int \sin^4 x\, dx$
7. $\int \cos^2 x \sin^3 x\, dx$
8. $\int \cos^3 x\, dx$
9. $\int \cos^3 x \sin^2 x\, dx$
10. $\int \sin x (\cos x)^{\frac{3}{2}} dx$
11. $\int sec^2 x\, csc^2 x\, dx$
12. $\int \frac{\sin x + 4\cos x}{-3\sin x - \cos x} dx$

TIPS FOR SEC AND TAN

1. If both are even $u = \tan x$ and bring out $sec^2 x$
2. If both are odd $u = secx$ and bring out $sec\, x \tan x$

1. $\int \tan^3 x\, sec\, x\, dx$

TRIGONOMETRIC SUBSTITUTION

Recall $SOH \quad CAH \quad TOA$

$\sin\theta = \frac{Opposite}{Hypotenuse}$

if $z = a\sin\theta$

$\sin\theta = \frac{z}{a}$

$Adjacent = \sqrt{a^2 - z^2}$

Given an example, $\sqrt{a^2 - z^2}$, $z = a\sin\theta$

$dz = a\cos\theta\, d\theta$

$\theta = \arcsin\left(\frac{x}{a}\right)$

$1 - \sin^2 x = \cos^2 x$

1. When you have a question containing $\sqrt{a^2 - z^2}$, z = $a\sin\theta$

2. When you have a question containing 1 over $\sqrt{a^2+z^2}$, $z=a\sinh\theta$
3. When you have a question containing $\sqrt{z^2-a^2}$, $z=a\cosh\theta$
4. When you have a question containing root a^2+z^2, $z=a\tan\theta$
5. When you have a question containing z^2-a^2, $z=a\,sec\,\theta$

$$\int\sqrt{a^2-z^2}\,dz=\frac{a^2}{2}\left[\sin^{-1}\left(\frac{z}{a}\right)+\frac{z\sqrt{a^2-z^2}}{a^2}\right]+c$$

$$\int\sqrt{z^2+a^2}\,dz=\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{z}{a}\right)+\frac{z\sqrt{z^2+a^2}}{a^2}\right]+c$$

$$\int\sqrt{z^2-a^2}\,dz=\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{z}{a}\right)+\frac{z\sqrt{z^2-a^2}}{a^2}\right]+c$$

<u>REDUCTION FORMULAE</u>

Note the recursion formulae of the following

1. The reduction formula for sine $\int\sin^n x\,dx=\frac{-1}{n}\cos(x)\sin^{n-1}(x)+\frac{n-1}{n}\int\sin^{n-2}(x)dx$. To prove the reduction formula of sin,
$\int\sin^n dx=\int\sin^{n-1}x\sin x\,dx$
The you solve by parts
2. The reduction formula for cosine $\int\cos^n x\,dx=\frac{1}{n}\sin(x)\cos^{n-1}(x)+\frac{n-1}{n}\int\cos^{n-2}(x)\,dx$
3. The reduction formula for secant $\int sec^n x=\frac{sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}\int sec^{n-2}x\,dx$
It should be noted that the reduction formula for secant doesn't work if n=1

4. The reduction formula for tangent $\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx$

<u>Questions</u>

1. $\int\tan x\,dx$ Answer: $\ln|sec\,x|+c$
2. Find the integral of cot (x)
3. Find the integral: $\int\tan^5(x)\,dx$

Solutions

1. $\int\frac{\sin x}{\cos x}dx$

Let $u=\cos x$

$\ln(\cos x)^{-1}+c$

$\ln\left|\frac{1}{\cos x}\right|+c$

$\ln|sec\,x|+c$

2. $\int \cot(x)\,dx$

$\cot(x)=\frac{\cos(x)}{\sin(x)}$

$\int \cot(x)\,dx=\int \frac{\cos(x)}{\sin(x)}dx$

Let $u=\sin(x)$

$\frac{du}{dx}=\cos(x)$

$dx=\frac{du}{\cos(x)}$

3. Typically, you will want to bring out a $\tan^2 x$. This is because of the identity

$\tan^2(x)=sec^2(x)-1$

$\int \tan^5(x)\,dx=\int \tan^3(x)\tan^2(x)\,dx$

$\int \tan^3(x)\tan^2(x)\,dx=\int \tan^3(x)\left[sec^2(x)-1\right]dx$

$\int \tan^3(x)sec^2(x)\,dx-\int \tan^3(x)\,dx$

$\int \tan^3(x)sec^2(x)\,dx-\int \tan(x)\tan^2(x)\,dx$

$\int \tan^3(x)sec^2(x)\,dx-\int \tan(x)\left(sec^2(x)-1\right)dx$

$\int \tan^3(x)sec^2(x)\,dx-\int \tan(x)\,sec^2(x)\,dx+\int \tan(x)\,dx$

Next we use the substitution method.

For the first 2, let $u=\tan(x)$

$\frac{du}{dx}=sec^2(x)$

$dx=\frac{du}{sec^2(x)}$

$\int \tan^3(x)sec^2(x)\,dx-\int \tan(x)\,sec^2(x)\,dx+\int \tan(x)\,dx=\int u^3 sec^2(x)\frac{du}{sec^2(x)}-\int u\,sec^2(x)\frac{du}{sec^2(x)}+\ln|sec(x)|$

Recall that $\int \tan(x)\,dx=\ln|sec\,x|+c$

$\int u^3du-\int u\,du+\ln|sec(x)|$

After solving sha, you should have an answer:

$\frac{1}{4}\tan^4(x)-\frac{1}{2}\tan^2(x)+\ln|sec(x)|+c$

7. $\int x\sqrt{1+x^2}\,dx$

WEIERSTRASS SUBSTITUTION

This is also called the **t-sub** method

$$t = \tan\left(\frac{x}{2}\right)$$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = 2\frac{t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan x = \frac{2t}{1-t^2}$$

QUESTIONS

1. $\int \frac{1}{2+\cos x}\,dx$