ODE WITH VARIABLE COEFFICIENTS

0.1 CAUCHY-EULER EQUATIONS

These are also called **Homogeneous linear differential equations**.

The general form is:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

Note the following of homogeneous linear differential equations

1. A differential equation is called Homogeneous Linear Differential Equation with variable coefficients if the powers of x are equal to the orders of the dervative associated with them.
2. The dependent variable y and its derivatives with respect to the independent variable x appear in their first degree and are not multiplied together.
3. These DE are also known as Cauchy-Euler equations.

0.1.1 EXAMPLES

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = \sin x$$

This is homogeneous because the order of the derivatives and the power of x preceding it are the same.

$$x \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = x^2 + 1$$

$$x^4 \frac{d^2 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$$

These two above are not homogeneous because the powers of the x and the order are not the same.

0.2 METHODS OF SOLVING

1. Reduction of Orders
2. D-factor method

0.2.1 REDUCTION OF ORDERS

Given t^2 is a solution to $t^2 y''+3t y'-8y=0$, find the general solution of the differential equation.

The general solution is a sum of the unique independent variables of the two possible solutions y_1 and y_2 of the differential equation:

$y=c_1 y_1+c_2 y_2$

$y_1=t^2$

The other solution y_2 will be a product of a function of t $v(t)$ and the first solution y_1

$y_2=v(t)t^2=t^2\cdot v$

$y'_2=t^2 v'+2vt$

$y''_2=t^2 v''+2v't+2v+2tv'$

$y''_2=t^2 v''+4tv'+2v$

On substituting to these into the differential equation,

$t^2 y''+3t y'-8y=0$

$t^2(t^2 v''+4tv'+2v)+3t(t^2 v'+2vt)-8(t^2 v)$

$t^4 v''+4t^3 v'+2t^2 v+3t^3 v'+6t^2 v-8t^2 v=0$

$t^4 v''+7t^3 v'=0$

Dividing through by t^4

$v''+\frac{7}{t}v'=0$

Let $w=v'$, $w'=v''$

$w'+\frac{7}{t}w=0$

$$\frac{dw}{dt}=\frac{-7}{t}w$$

$$\frac{1}{w}dw=\frac{-7}{t}dt$$

$$\int\frac{1}{w}dw=\int\frac{-7}{t}dt$$

$$\ln|w|=-7\ln|t|+k_1$$

$$e^{\ln|w|}=k_2e^{\ln|t^{-7}|}$$

$$w=k_2t^{-7}$$

$$w=v'$$

$$\int v'dt=\int k_2t^{-7}dt$$

$$v=k_2\frac{t^{-6}}{-6}+k_4$$

$$v=k_3t^{-6}+k_4$$

Recall,

$$y_2=v(t)t^2=t^2\cdot v$$

$$y_2=(k_3t^{-6}+k_4)t^2$$

$$y_2=k_3t^{-4}+k_4t^2$$

Now the first solution is t^2. The second solution

The unique functions in the first one and the second one are t^2, common to both and t^{-4} which is in the second solution.

$$y=c_1y_1+c_2y_2$$

$$y=c_1t^2+c_2t^{-4}$$

0.2.2 D-OPERATOR METHOD

Cauchy-Euler equations can be easily converted to equations with constant coefficients by changing the independent variable by the transformation

$$x=e^z$$

$$z=\log x$$

$$\frac{dz}{dx}=\frac{1}{x}$$

$$\frac{dy}{dx}=\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)$$

$$\frac{dy}{dx}=\frac{1}{x}\left(\frac{dy}{dz}\right)$$

$$x\frac{dy}{dx}=\frac{dy}{dz}$$

$$x\frac{d}{dx}\equiv\frac{d}{dz}$$

$$x\frac{d}{dx}\equiv D$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=\frac{1}{x}\left[\frac{d}{dx}\left(\frac{dy}{dz}\right)\right]+\left(\frac{dy}{dx}\right)\left[\frac{d}{dx}\left(\frac{1}{x}\right)\right]$$

$$\frac{d^2y}{dx^2}=\frac{1}{x}\left[\frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)\right]+\left(\frac{dy}{dz}\right)\left[-\frac{1}{x^2}\right]=\frac{1}{x}\left[\left(\frac{d^2y}{dz^2}\right)\left(\frac{1}{x}\right)\right]+\left(\frac{dy}{dz}\right)\left[-\frac{1}{x^2}\right]$$

$$\frac{d^2y}{dx^2}=\frac{1}{x^2}\left[\frac{d^2y}{dz^2}-\frac{dy}{dz}\right]$$

$$x^2\frac{d^2y}{dx^2}=\left[\frac{d^2y}{dz^2}-\frac{dy}{dz}\right]=\left[\frac{d}{dz}\left(\frac{dy}{dz}\right)-\frac{d}{dz}y\right]=\frac{d}{dz}\left[\frac{d}{dz}-1\right]y=D[D-1]y$$

So from the above it can be seen and

$$x^2\frac{d^2}{dx^2}\equiv D[D-1]$$

Similarly,

$$x^3\frac{d^3}{dx^3}\equiv D(D-1)(D-2)$$

Steps to solving:

1. Check if the equation is homogeneous or not
2. Make it homogeneous if not
3. Substitute

$$x=e^z$$

$$z=\log x$$

$$x\frac{dy}{dx}\equiv Dy$$

$x^2\frac{d^2y}{dx^2}\equiv D[D-1]y$

$D\equiv\frac{d}{dz}$

4. Obtained differential equation will be a linear differential equation with constant coefficients in terms of D.

5. Find the complementary function (CF) and the particular integral(PI)

6. Find the general solution, $y=CF+PI$

7. Finally, substitute $x=e^z$and $z=\log x$

QUESTIONS

1. Solve $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+4y=0$

On substituting,

$x=e^z$

$z=\log x$

$x\frac{dy}{dx}\equiv Dy$

$x^2\frac{d^2y}{dx^2}\equiv D[D-1]y$

$D\equiv\frac{d}{dz}$,

We have

$x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+4y=0\rightarrow\ [D(D-1)+5D+4]y=0$

$(D^2+4D+4)y=0$

$(D+2)^2y=0$

The auxiliary equation is given as

$(r+2)^2=0$

On solving this,

$r=-2$ twice

The general solution for this solution is given as

$y_c=(c_1+c_2z)e^{-2z}$

Since it is also a homogeneous equation with constant coefficients.

$y_p=0$

$y=y_c+y_p$

$y_c=(c_1+c_2 z)(e^z)^{-2}$

Substituting back,

$y=(c_1+c_2\log x)x^{-2}$

2. Solve $x^3\frac{d^3 y}{d x^3}+2x^2\frac{d^2 y}{d x^2}+3x\frac{dy}{dx}-3y=0$

On changing to the D-operator,

$(D-1)(D^2+3)y=0$

The Auxiliary Equation is

$(r-1)(r^2+3)=0$

$r=1,0\pm i\sqrt{3}$

$CF=c_1e^z+[c_2\cos(z\sqrt{3})+c_3\sin(z\sqrt{3})]$

$y=c_1e^z+[c_2\cos(z\sqrt{3})+c_3\sin(z\sqrt{3})]$

$y=c_1x+c_2\cos(\sqrt{3}\log x)+c_3\sin(\sqrt{3}\log x)$

3. $x^2\frac{d^2 y}{d x^2}-4x\frac{dy}{dx}+6y=x$

$(D-2)(D-3)y=e^z$

The auxiliary function:

$(r-2)(r-3)=0$

$r=2,3$

$CF=c_1e^{2z}+c_2e^{3z}$

$PI=\frac{1}{(D-2)(D-3)}e^z=\frac{1}{(1-2)(1-3)}e^z$

$PI=\frac{1}{2}e^z$

y = CF + PI

$y=c_1e^{2z}+c_2e^{3z}+\frac{1}{2}e^z$

$y=c_1x^2+c_2x^3+\frac{1}{2}x$

4. $x^3\frac{d^2y}{dx^2}+7x^2\frac{dy}{dx}+13xy=x\log x$

Answer: $y=x^{-3}[c_1\cos(2\log x)+c_2\sin(2\log x)]+\frac{1}{13}\left[\log x-\frac{6}{13}\right]$

EQUATION REDUCIBLE TO HOMOGENEOUS FORM

$$(a+bx)^n\frac{d^2y}{dx^n}+P_1(a+bx)^{n-1}\frac{d^{n-1}y}{ax^{n-1}}+\ldots+P_{n-1}(a+bx)\frac{dy}{dx}+P_ny=Q$$

This can be reduced to homogeneous differential equation with constant coefficients by substituting:

$a+bx=e^z$

$z=\log(a+bx)$

$\frac{d}{dz}=D$

$(a+bx)\frac{dy}{dx}=bDy$

$(a+bx)^2\frac{d^2y}{dx^2}=b^2D(D-1)y$

$(a+bx)^3\frac{d^2y}{dx^3}=b^3D(D-1)(D-2)y$