COMPLEX NUMBERS

Complex numbers can be expressed in different forms:
1. Cartesian form or rectangular form
2. Polar form
3. Exponential form

COMPLEX NUMBERS IN RECTANGULAR FORM

$z = a + bi$

To graph a complex number in rectangular form, the x-axis is the real axis (that is the values for a) while the y-axis will be the imaginary axis (that is the values for b)

The absolute value of a complex number $|z|$ is given as

$|z| = \sqrt{a^2 + b^2}$

The value of this absolute value is always positive

Trying to draw this as a right angled triangle, the hypotenuse will be the absolute value of z.

The angle between |z| and a is θ

COMPLEX NUMBERS IN POLAR FORM

$z = r\left[\cos(\theta) + i\sin(\theta)\right]$

When represented in polar form graphically, we have to take note of the r-values. These r values are usually the circles in the graph. The closest circle has an r value of 1, the second closest has an r-value of 2 and so on.

$r = \sqrt{a^2 + b^2}$

$r = |z|$

Trying to draw this as a right angled triangle, the hypotenuse will be r.

$a = r\cos(\theta)$

$b = r\sin(\theta)$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

θ is said as the argument of z and it is written as $Arg_{(z)}$ or the **amplitude** of the complex number

Remember that when considering the angle, that we have to take note of which quadrant it is found.

For quadrant 1

$\theta = \theta_{ref}$

For quadrant 2

$\theta = 180 - \theta_{ref}$

For quadrant 3

$\theta = 180 + \theta_{ref}$

For quadrant 4
$\theta = 360 - \theta_{ref}$
Therefore the answer for this will be given as
$z = 4\left[\cos 300 + i\sin 300\right]$
You can still decide to change it to radians
$\theta \times \frac{\pi}{180}$
We often use a shorthand version $rL\theta$to denote polar form

MULTIPLYING COMPLEX NUMBERS IN POLAR FORM

$z_1 = r_1\left[\cos(\theta)_1 + i\sin(\theta)_1\right]$
$z_2 = r_2\left[\cos(\theta)_2 + i\sin(\theta)_2\right]$
$z_1 \cdot z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$

QUOTIENT OF TWO COMPLEX NUMBERS

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$
So from the above, it can be noted that the angle used for calculation depends on the reference angle and the position of the reference angle.

INVERSE OF A COMPLEX NUMBER

$z^{-1} = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right]$

COMPLEX NUMBERS IN EXPONENTIAL FORM

In the polar form of complex numbers,
$z = r\left[\cos\theta + i\sin\theta\right]$
According to the Euler's Formula
$e^{i\theta} = \cos\theta + i\sin\theta$
$z = re^{i\theta}$

Complex Numbers in Exponential Form
In the polar form of complex numbers,
$z = r\left[\cos\theta + i\sin\theta\right]$

According to the Euler's Formula
$e^{i\theta} = \cos\theta + i\sin\theta$

$z = r e^{i\theta}$
r is the distance between the polar coordinates (r, θ) and the pole (or origin) (0, 0)
θ is said as the argument of z and it is written as $Arg_{(z)}$
In polar form, we get an infinite number of possible exponential form of a given complex number.
Each θ differs by a multiple of 2π

Like…
$\theta_2 = \theta_1 + 2\pi$
$\theta_3 = \theta_2 + 2\pi = \theta_1 + 4\pi$
Therefore, the exponential form can be written as
$z = r e^{i(\theta + 2\pi n)}$
$n \geqslant 0$
If $\theta = \frac{2\pi}{3}$, then

$$\frac{r e^{i\frac{2\pi}{3}}}{z_1} = \frac{r e^{i\frac{2\pi}{3}+2\pi}}{z_1} = \frac{r e^{i\frac{2\pi}{3}+4\pi}}{z_1} = \frac{r e^{i\frac{2\pi}{3}+6\pi}}{z_1} = \frac{r e^{i\frac{2\pi}{3}+8\pi}}{z_1} = \cdots = \frac{r e^{i\frac{2\pi}{3}+2\pi n}}{z_1}$$

From this questions
$\theta_1 = \frac{2\pi}{3} \theta_n = \frac{2\pi}{3} + 2\pi n$
For every value of theta, you will get the same thing
z_1 , z_2 , z_3 , z_4 … z_n will all represent the same complex number

CONVERTING FROM EXPONENTIAL TO POLAR

MULTIPLICATIVE INVERSE

In exponential form:
$z = r e^{i\theta}$
then,
$z^{-1} = (r e^{i\theta})^{-1} = r^{-1} e^{i(-\theta)}$
$z^{-1} = \frac{1}{r} e^{i(-\theta)}$
This is the multiplicative index in exponential form
Now, by Euler's formula
$e^{i(-\theta)} = \cos(-\theta) + i \sin(-\theta)$
$z^{-1} = \frac{1}{r}[\cos(-\theta) + i \sin(-\theta)]$
Multiplication in exponential form
Let
$z_1 = r_1 e^{i\theta_1}$
$z_2 = r_2 e^{i\theta_1}$
$z_1 \times z_2 = r_1 e^{i\theta_1} \times r_1 e^{i\theta_1}$

$$z_1 \times z_2 = r_1 r_2 \times e^{i\theta_1} e^{i\theta_2}$$
$$z_1 \times z_2 = r_1 r_2 e^{(i\theta_1 + i\theta_2)}$$
$$z_1 \times z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

DE MOIVRE'S THEOREM

Finding the nth power of a complex number

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$r^n[\cos(n\theta) + i\sin(n\theta)]$$

Finding Complex roots

$$Z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

ROOTS OF UNITY

The roots of unity is a number which is complex in nature and gives 1 when raised to the power of a positive integer n. It is also called as "de Moivre system"

The cube roots of unity are

$$1 \quad , \quad \frac{-1}{2} + i\frac{\sqrt{3}}{2} \quad , \quad \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

To prove,

The cubed roots of unity are the cubed roots of 1

$$\sqrt[3]{1} = a$$
$$a^3 = 1$$
$$a^3 - 1 = 0$$
$$a^3 - 1^3 = 0$$

Recall,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$a^3 - 1^3 = 0$$
$$(a - 1)(a^2 + a + 1) = 0$$

PROPERTIES OF CUBE ROOTS OF UNITY

1. One imaginary cube root of unity is the square of the other

If the cubed roots of unity are

$$1, w_1, w_2$$

Then,

$$w_1 = w_2^2$$
$$w_2 = w_1^2$$

Therefore, the cubed roots of unity can be written as:

$$1, w, w^2$$

Also,

$(w^2)^2 = w$

2. If the two imaginary cubed roots are multiplied, then the product we get is equal to 1.
If the cubed roots of unity are

$1, w_1, w_2$

Then,

$w_1 \times w_2 = 1$

$w \times w^2 = 1$

$w^3 = 1$

3. The sum of the cubed roots of unity is equal to zero.
If the cubed roots of unity are

$1, w_1, w_2$

$1 + w_1 + w_2 = 0$

$1 + w + w^2 = 0$

Solve the following questions:

1. Plot each complex number
A. z = 4 + 3i
B. z = -2 – 3i
C. z = 2
D. z = -3
E. z = 4i
F. Z = -3i

2. Calculate the absolute value of each complex number shown below
A. z = 3 + 4i Answer: 5
B. z = 4 – 6i Answer: $2\sqrt{13}$
C. z = 3 Answer: 3
D. z = -4 Answer: 4
E. z = -8i Answer: 8

3. Write the complex number in polar form
A. $z = 3 + 2i$
Solution:

$z = 3 + 2i$

$r = \sqrt{a^2 + b^2}$

$r = 3\sqrt{2}$

$\theta = 45$

$z = 3\sqrt{2}[\cos 45 + i \sin 45]$

To convert it to radian forms,

$$z = 3\sqrt{2}\left[\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right]$$

B. $z=2-2\sqrt{3}i$
Solution:
$r=4$
$\theta_{ref}=\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$
$\theta=60$
Notice that even though the value of b is negative, we used it as a positive value.
When the graph is drawn, it can be seen that the value of r is the fourth quadrant.
However, in our calculation, we want the angle with the positive x-axis
That will be $360-60=300$.

For the last example
$r=4$, $\theta=300$
It can be expressed like
$z=4L300$
C. $z=-3+5i$. Answer: $\sqrt{34}L121$
D. $z=3$. This can be easily solved:
If $z=3$, then $r=3$
Then if we place this on the graph, we will see that the angle is 0. Therefore if given just one value, the angle will either be 0, 90, 180 or 270
E z = -4
For this, r = 4 and $\theta=180$
F. $z=-2i$ Answer: r = 2, $\theta=270$
G. $z=5i$ Answer: r = 5, $\theta=90$

4. Write the complex number in rectangular form
This can easily be done by plugging in the values
A. $z=4[\cos 90+i\sin 90]$Answer: z = 4i
B. $z=10[\cos 30+i\sin 30]$Answer: $z=5\sqrt{3}+5i$
C. $z=20\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right]$Answer: $z=-10-10\sqrt{3}i$
5. Write the complex number in rectangular form. Round your answer to the nearest hundredth
A. $z=15[\cos 4.2+i\sin 4.2]$This calculation should be done in radian mode in your calculator
Answer: $z=-7.35-13.07i$To the nearest hundredth.

6. Find the product of the two complex numbers. Write the answer in polar form
A.
$z_1=5[\cos 30+i\sin 30]$
$z_1=7[\cos 45+i\sin 45]$
Answer: $z_1\cdot z_2=35[\cos 75+i\sin 75]$
B.
$z_1=8[\cos 60+i\sin 60]$
$z_2=5[\cos 100+i\sin 100]$
Answer: $z_1\cdot z_2=40[\cos 160+i\sin 160]$
C. $z_1=6\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$
$z_2=3\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

Answer: $z_1 \cdot z_2 = 18\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right]$

D.

$z = 3 - 4i$

$z_2 = 5 + 12i$

Answer: r = 65, $\theta = 14.25$

7. Find the quotient $\frac{z_1}{z_2}$ of the complex numbers shown below. Write the final answer in polar form

A.

$z_1 = 12[\cos 80 + i\sin 80]$

$z_2 = 3[\cos 30 + i\sin 30]$

Answer: 4[{cos{50}}+i{sin{50}}]

B.

$z1 = 15\left[\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right]$

$z2 = 3\left[\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right]$

Answer: $5\left[\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right]$

C.

$z_1 = 5[\cos 60 + i\sin 60]$

$z_2 = 35[\cos 190 + i\sin 190]$

$\frac{1}{7}[\cos - 130 + i\sin - 130]$

$\frac{1}{7}[\cos 230 + i\sin 230]$

8. Find the quotient z1/z2 of the complex numbers shown below. Write the final answer in polar form using an angle between 0 and 360 degrees

$z_1 = 3\sqrt{3} - 3i$

$z_2 = -1 + \sqrt{3}i$

z = 3L210

QUESTIONS

1. Given that$z_1 = 2 + 3i z_2 = 3 + 2i z_3 = a + biand\frac{z_1 z_3}{z_2} = \frac{17}{13} - \frac{7}{13}i$. Find the values of a and b

2. If z is a complex number such that $|z+1| = z + 2(1+i)$, then find z

3. If $z_1 = 2 + 8i$and $z_2 = 1 - i$, evaluate $\left|\frac{z_1}{z_2}\right|$.

4. If $z = \frac{50}{3+4i}$, find z^2

5. Find the values of x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$

6. If $z=\frac{50}{3+4i}$, find $2z^2+8z$

AGENDA

1. The real and complex numbers
2. Representations and Algebra of Complex numbers
3. Complex Functions
4. Roots of Unity
5. De Moivre’s Theorem and Applications