STRESSES ON INCLINED PLANES AND PRINCIPAL STRESS

2.1 INTRODUCTION

In a stressed medium or solid body, that can be subjected to stresses, in the two orthogonal directions (x and y), there will be stresses in any other planes-inclined, which are the resultant or the aggregate effects of these two orthogonal stresses. Which can be considered as follows:

1. Uni-axial normal force loading systems
2. Bi-axial normal force loading systems
3. Shear stresses acting alone
4. Complex stress system or the combination of shear and normal forces.

2.1.1 UNI-AXIAL NORMAL FORCE LOADING SYSTEM

Consider a uni-axial tensile stress acting on plane AC(hypotenuse/plane) of a bar of uniform cross-section.
AB is the opposite, and BC is the adjacent (height of the triangle).
The stress normal to the plane is given as:

$$\sigma_n = \sigma \cos^2\theta = (1 + \cos 2\theta)\frac{\sigma}{2}$$

The stress tangential to the plane is given as:

$$\tau = \sigma \sin\theta \cos\theta = \alpha \frac{\sin 2\theta}{2}$$

The resultant stress σ_r on the inlcined plane is given as:

$$\sigma_r = \sqrt{\sigma_n{}^2 + \tau^2} = \left(\frac{\sigma}{2}\right)\sqrt{(1+\cos 2\theta)^2 + \sin^2\theta}$$

$$\sigma_r = \sigma \cos\theta$$

$$\tan\phi = \frac{\tau}{\sigma_n} = \tan\theta$$

2.1.2 BI-AXIAL LOADING SYSTEM

$$\sigma_n = \sigma_x \cos\theta \cos\theta + \sigma_y \sin\theta \sin\theta$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2}$$

$$\tau = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2}$$

When $\theta = 45$

$$\tau_{max} = \frac{1}{2}(\sigma_x - \sigma_y)$$

The resultant stress

$$\sigma_r = \sqrt{\sigma_n{}^2 + \tau^2}$$

$$\sigma_r = \sigma_x{}^2 \cos^2\theta + \sigma_y{}^2 \sin^2\theta$$

$$\tan\phi = \frac{\tau}{\sigma_n} = \frac{\sigma_x - \sigma_y}{\sigma_x \cot\theta + \sigma_y \tan\theta}$$

For ϕ to be maximum, $\frac{d(\tan\phi)}{d\theta} = 0$

$$\frac{d}{d\theta}(\sigma_x \cot\theta + \sigma_y \tan\theta) = 0$$

$$-\sigma_x cosec^2\theta + \sigma_y sec^2\theta = 0$$

On solving,

$$\tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}}$$

$$\tan\phi_{max} = \frac{\sigma_x - \sigma_y}{2 \cdot \sqrt{\sigma_x \sigma_y}}$$

2.1.3 SHEAR STRESS ACTING ALONE

Consider a two-dimensional body subjected to shear stresses. Because shear stresses are complementary in nature, so it will be of the same magnitude in x and y directions.

$$\sigma_n = T_{xy} \cos\theta \sin\theta + T_{xy} \sin\theta \cos\theta = T_{xy} \sin 2\theta$$

At $\theta = 45$, $\sigma_n = T_{xy}$

$$\tau = T_{xy} \cos^2\theta + T_{xy} \sin^2\theta = T_{xy}(\cos^2\theta + \sin^2\theta)$$

$$\tau = T_{xy} \cos 2\theta$$

2.1.4 COMPLEX STRESS SYSTEMS

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - 2T_{xy}\sin\theta\cos\theta$$

$$\sigma_n = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + \frac{1}{2}\left(\sigma_x - \sigma_y\right)\cos 2\theta - T_{xy}\sin 2\theta$$

For σ_n to have a maximum or minimum value

$$\frac{d\sigma_n}{d\theta} = 0$$

i.e. $\left(\sigma_x - \sigma_y\right)\sin 2\theta - 2T_{xy}\cos 2\theta = 0$

$$\tan 2\theta = \frac{-2T_{xy}}{\left(\sigma_x - \sigma_y\right)}$$

The tangential stress across the face of the plane is given as:

$$\tau = \sigma_x \sin\theta\cos\theta - \sigma_y \sin\theta\cos\theta + T_{xy}\cos^2\theta - T_{xy}\sin^2\theta$$

$$\tau = \frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + T_{xy}\cos 2\theta$$

2.2 PRINCIPAL STRESSES AND PRINCIPAL PLANES

At any point within a stressed body, no matter how complex the state may be, there always exists three mutually perpendicular planes on which the resultant stress is pure normal stress and there is no shear stress i.e. {%tau} = 0

$$\tau = \frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + T_{xy}\cos 2\theta = 0$$

$$\tan 2\theta = \frac{-2T_{xy}}{\sigma_y - \sigma_x}$$

For the diagram of the principal plane,

The hypotenuse is $\sqrt{\left(\sigma_y - \sigma_x\right)^2 + 4\tau_{xy}^{\ 2}}$

The opposite is the base of the triangle.The value is $2\tau_{xy}$

The adjacent is the height of the triangle. The value is $\left(\sigma_y - \sigma_x\right)$

$$\sin 2\theta = \frac{2T_{xy}}{\pm\sqrt{\left(\sigma_y - \sigma_x\right)^2 + 4T_{xy}^{\ 2}}}$$

$$\cos 2\theta = \frac{\sigma_y - \sigma_x}{\pm\sqrt{\left(\sigma_y - \sigma_x\right)^2 + 4T_{xy}^{\ 2}}}$$

Recall that for a complex system,

$$\sigma_n = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + \frac{1}{2}\left(\sigma_x - \sigma_y\right)\cos 2\theta - T_{xy}\sin 2\theta$$

$$\sigma_{1,2} = \frac{1}{2}\left(\sigma_x + \sigma_y\right) \pm \frac{1}{2}\left[\left(\sigma_x - \sigma_y\right)^2 + 4\,T_{xy}{}^2\right]^2$$

$$\sigma_1 = \sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

On adding the two principal stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

On subtracting,

$$\sigma_1 - \sigma_2 = 2\left[\left(\sigma_x - \sigma_y\right)^2 + 4\,\tau_{xy}{}^2\right]^{\frac{1}{2}}$$

2.3 PLANE OF MAXIMUM SHEAR STRESS AND MAXIMUM SHEAR STRESS

$$\frac{d\,\tau}{d\,\theta} = 0$$

$$\frac{d\left(\frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + T_{xy}\cos 2\theta\right)}{d\,\theta} = 0$$

$$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\,\tau_{xy}}$$

2.3.2 MAXIMUM SHEAR STRESS

$$\tau = \frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + T_{xy}\cos 2\theta = 0$$

But for maximum stress,

$$\cos 2\theta = \frac{2\,T_{xy}}{\pm\sqrt{\left(\sigma_y - \sigma_x\right)^2 + 4\,T_{xy}{}^2}}$$

$$\sin 2\theta = \frac{\sigma_y - \sigma_x}{\pm\sqrt{\left(\sigma_y - \sigma_x\right)^2 + 4\,T_{xy}{}^2}}$$

$$\tau_{max} \pm \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\,\tau_{xy}{}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\tau_{max}=\frac{1}{2}(\sigma_1-\sigma_2)$$

2.3.3 AVERAGE STRESS (When shear stress is maximum)

$$\sigma_{avr}=\frac{\sigma_x+\sigma_y}{2}$$

2.4 COMPOUND STRAINS

Strains developed in a specific direction as a result of orthogonal or perpendicular stresses acting at a point within a stressed material. From the poisson's ratio γ

x-direction, $\varepsilon_x=\frac{\sigma_x}{E}-\frac{\gamma\sigma_y}{E}=\frac{1}{E}(\sigma_x-\gamma\sigma_y)$

y-direction, $\varepsilon_y=\frac{\sigma_y}{E}-\frac{\gamma\sigma_x}{E}=\frac{1}{E}(\sigma_y-\gamma\sigma_x)$