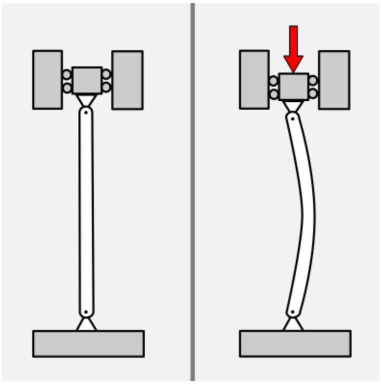
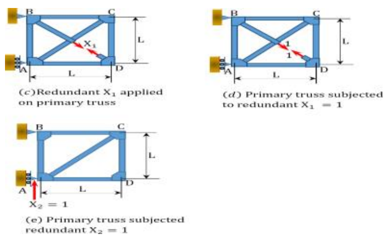
MECHANICS OF MATERIALS

OYELADE, Akintoye Olumide
Department of Civil and Environmental Engineering

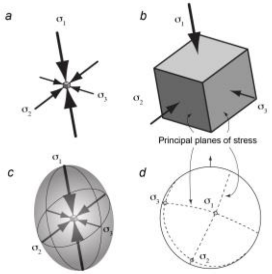
Introduction



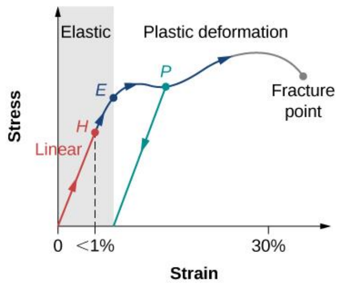
Buckling Instability of Struts/ Columns



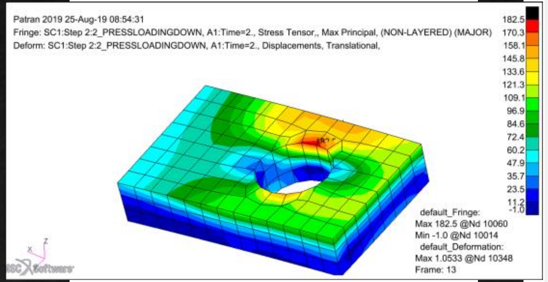
Energy methods



Stress concentration, Stress-Strain transformation,

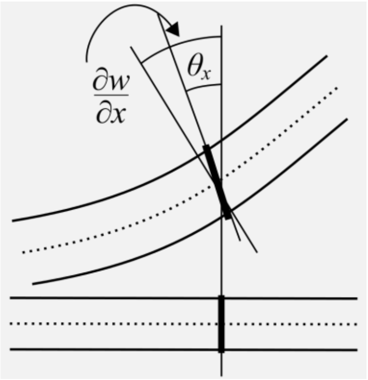


Elementary plasticity



Fracture Mechanics

Viscoelasticity



Thin Plates and Shells Application

Summary

Mechanics of Material II

Stress-Strain

Course Outline

- Stress
- Strain
- Stress concentration
- Stress-Strain transformation
- Conclusion

Textbooks

- MECHANICS OF MATERIALS: An Integrated Learning System by Timothy A. Philpot

Introduction

The three fundamental areas of engineering mechanics are

- statics,
- dynamics, and
- mechanics of materials.

Statics and dynamics are devoted primarily to the study of external forces and motions associated with particles and rigid bodies (i.e., idealized objects in which any change of size or shape due to forces can be neglected).

Mechanics of materials is the study of the internal effects caused by external loads acting on real bodies that deform

Introduction

Regardless of the application, however, a safe and successful design must address the following three mechanical concerns:

1. *Strength*: Is the object strong enough to withstand the loads that will be applied to it? Will it break or fracture? Will it continue to perform properly under repeated loadings?

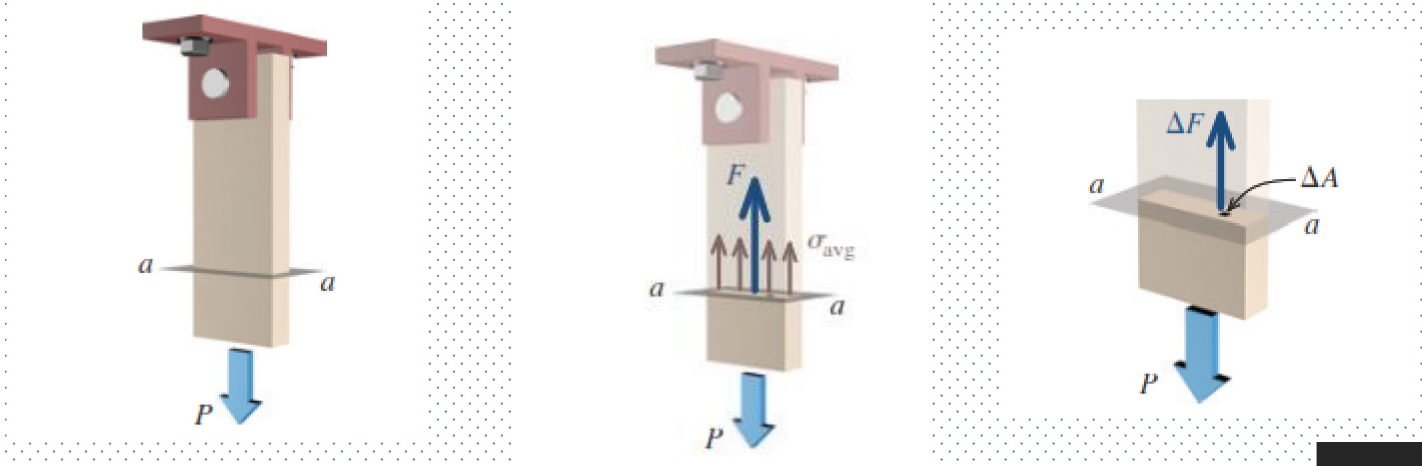
2. *Stiffness*: Will the object deflect or deform so much that it cannot perform its intended function?

3. *Stability*: Will the object suddenly bend or buckle out of shape at some elevated load so that it can no longer continue to perform its function?

Normal Stress Under Axial Loading

Stress is the intensity of internal force. Force is a vector quantity and as such has both magnitude and direction. Intensity implies an area over which the force is distributed. Therefore, stress can be defined as

$$Stress = \frac{Force}{Area}$$



Normal Stress Under Axial Loading

$$\sigma_{avg} = \frac{F}{A}$$

The sign convention for normal stresses is defined as follows:

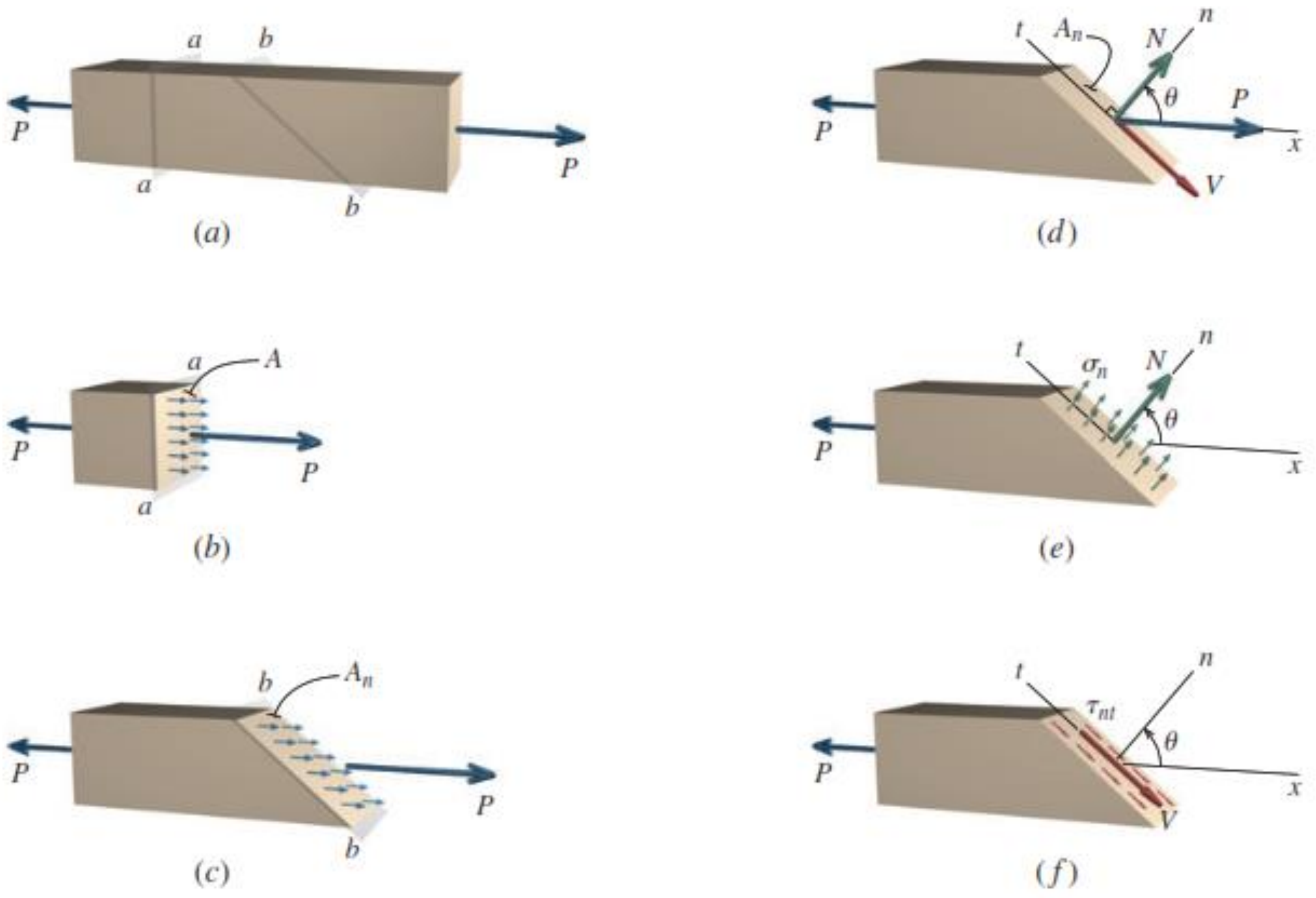
- A positive sign indicates a tension normal stress, and
- a negative sign denotes a compression normal stress.

Stresses on Inclined Sections

$$\sigma_{avg} = \frac{F}{A}$$

Consider a prismatic bar subjected to an axial force P applied to the centroid of the bar.
Loading of this type is termed uniaxial since the force applied to the bar acts in one direction.

Stresses on Inclined Sections



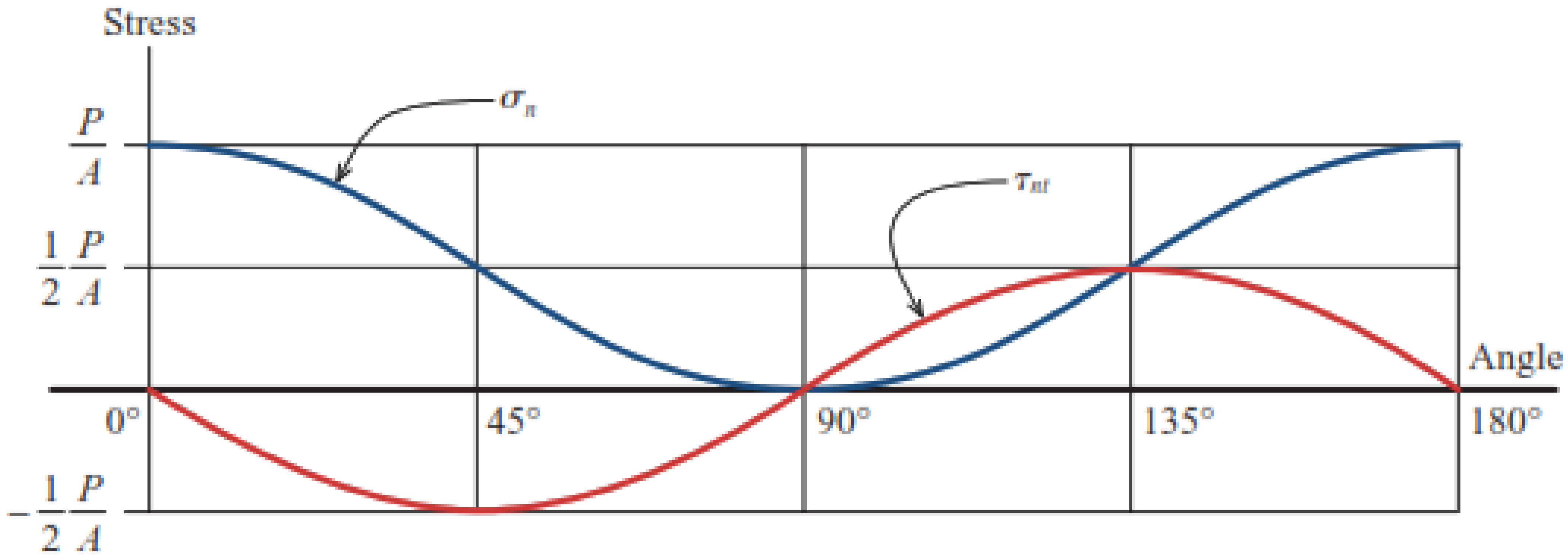
$$\sigma_n = \frac{N}{A_n} = \frac{P\cos\theta}{A/\cos\theta}$$

$$= \frac{P}{A}\cos^2\theta = \frac{P}{2A}(1+\cos 2\theta)$$

$$\tau_{nt} = \frac{V}{A_n} = \frac{-P\sin\theta}{A/\cos\theta}$$

$$= -\frac{P}{A}\sin\theta\cos\theta = -\frac{P}{2A}\sin 2\theta$$

Variation of normal and shear stress as a function of inclined plane orientation.



Stresses on Inclined Sections

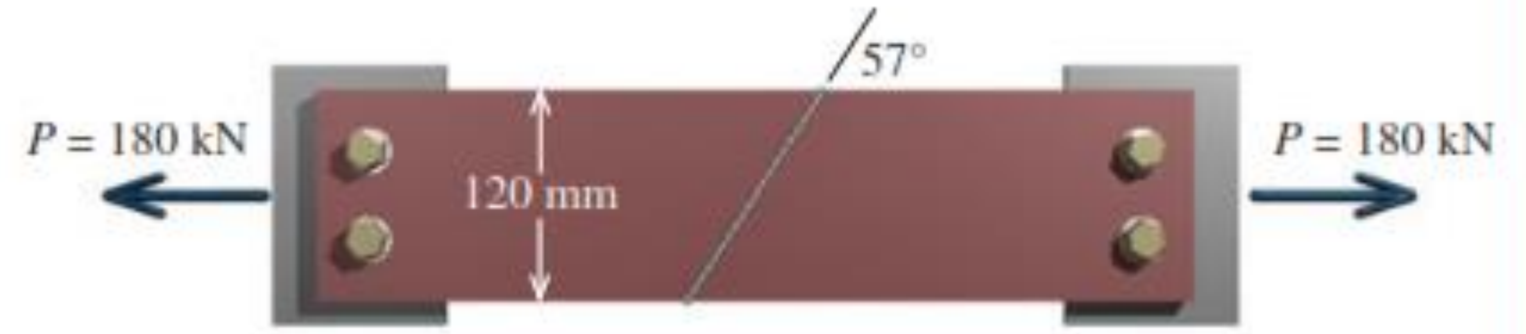
Therefore, the maximum normal and shear stresses in an axial member that is subjected to an uniaxial tension or compression force applied through the centroid of the member (termed a centric loading) are

$$\sigma_{\max} = \frac{P}{A}, \qquad \tau_{\max} = \frac{P}{2A}$$

Note that the normal stress is either maximum or minimum on planes for which the shear stress is zero. It can be shown that the shear stress is always zero on the planes of maximum or minimum normal stress.

Example

A 120-mm-wide steel bar with a butt-welded joint, as shown, will be used to carry an axial tension load of P 180 kN. If the normal and shear stresses on the plane of the butt weld must be limited to 80 MPa and 45 MPa, respectively, determine the minimum thickness required for the bar.



Stresses on Inclined Sections

$$\sigma_n = \frac{N}{A_n}, A_n \geq \frac{180 \times \cos 33^0 \times 1000}{80} = A_n \geq 1887.008mm^2$$

$$\tau_{nt} = \frac{V}{A_n}, A_n \geq \frac{180 \times \sin 33^0 \times 1000}{45} = A_n \geq 2178.56mm^2$$

$$\cos 33^0 = \frac{120}{L_n}, L_n = 143.084mm$$

The minimum thickness is computed as

$$t \geq \frac{2178.56}{143.084} = 15.23mm$$

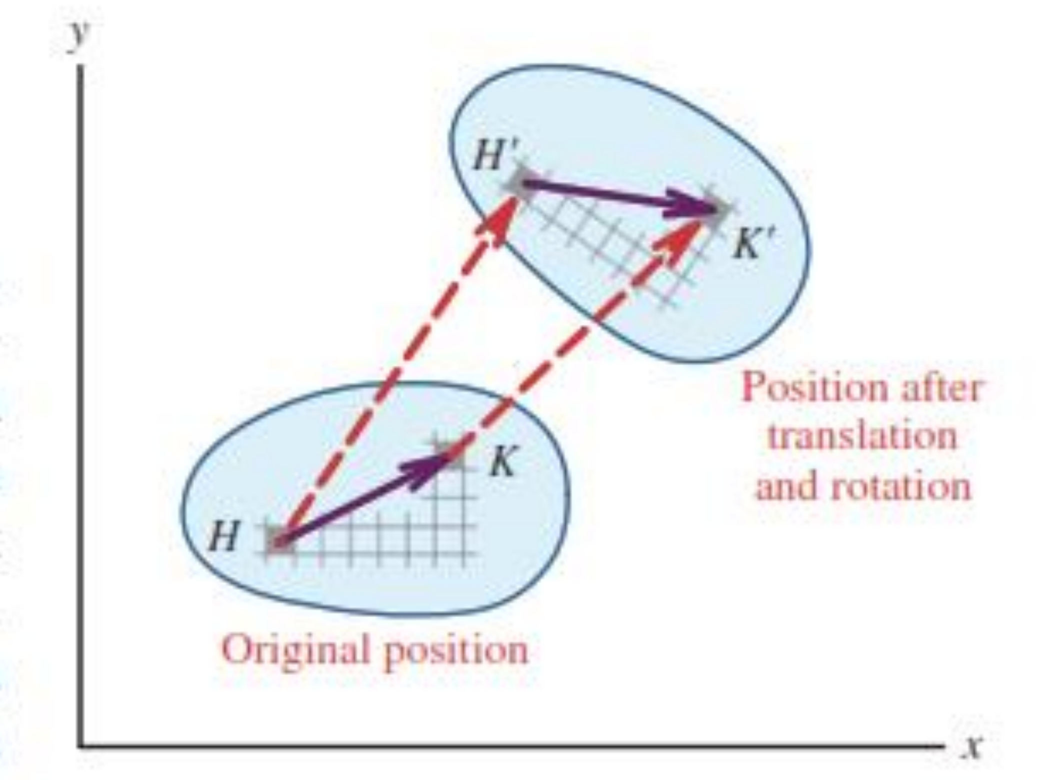
Strain: Introduction

In the design of structural elements or machine components, the deformations experienced by the body because of applied loads often represent a design consideration equally as important as stress. For this reason, the nature of the deformations experienced by a real deformable body as a result of internal stress will be studied, and methods to measure or compute deformations will be established.

Strain

The movement of a point with respect to some convenient reference system of axes is a vector quantity known as a displacement.

In some instances, displacements are associated with a translation and/or rotation of the body as a whole

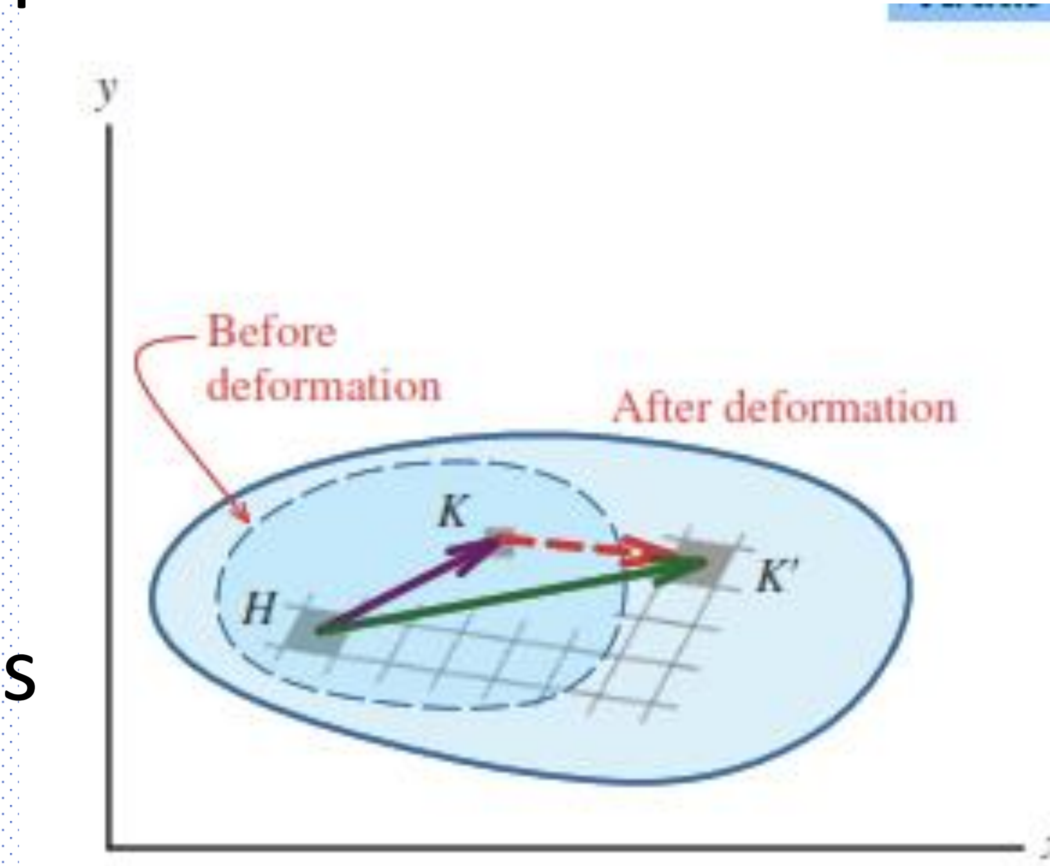


Strain: Displacement

The size and shape of the body are not changed by this type of displacement, which is termed a *rigid-body displacement*

Strain: Displacement

When displacements are caused by an applied load or a change in temperature, individual points of the body move relative to each other. The change in any dimension associated with these load- or temperature-induced displacements is known as *deformation*.



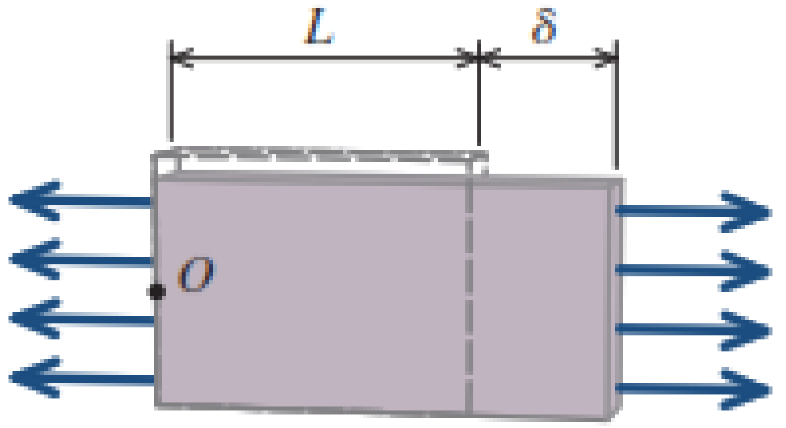
Strain: Displacement

Under general conditions of loading, deformations will not be uniform throughout the body. Some line segments will experience extensions, while others will experience contractions. Different segments (of the same length) along the same line may experience different amounts of extension or contraction.
Similarly, angle changes between line segments may vary with position and orientation in the body.

Strain

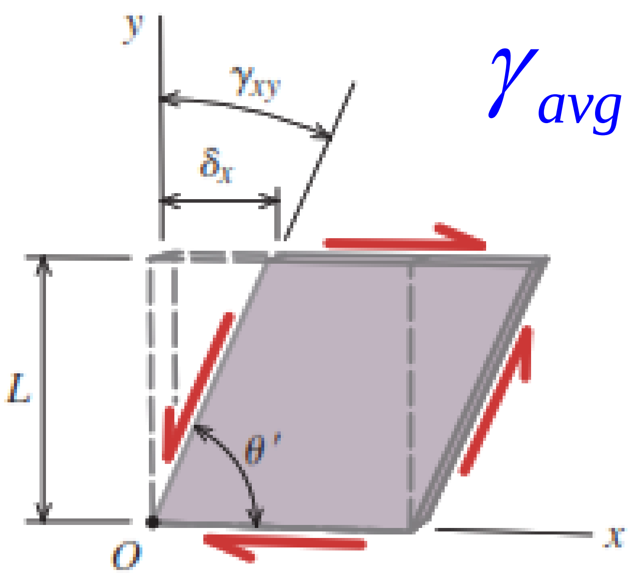
Strain is a quantity used to provide a measure of the intensity of a deformation (deformation per unit length)

The average normal strain ε_{avg} over the length of the bar is obtained by dividing the axial deformation of the bar by its initial length *L*;



$$\varepsilon_{avg} = \frac{\delta}{L}$$

Shear Strain

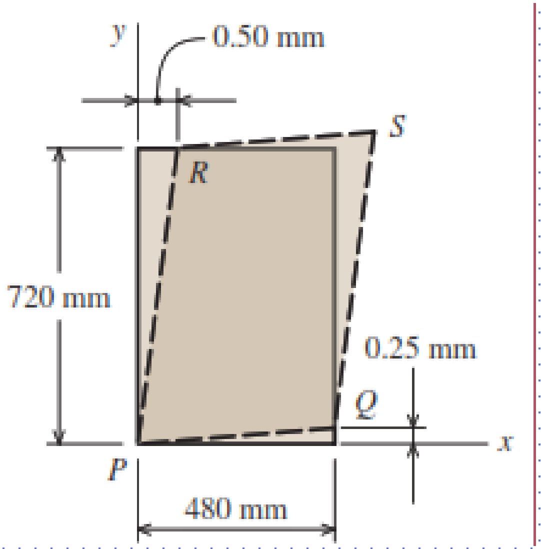


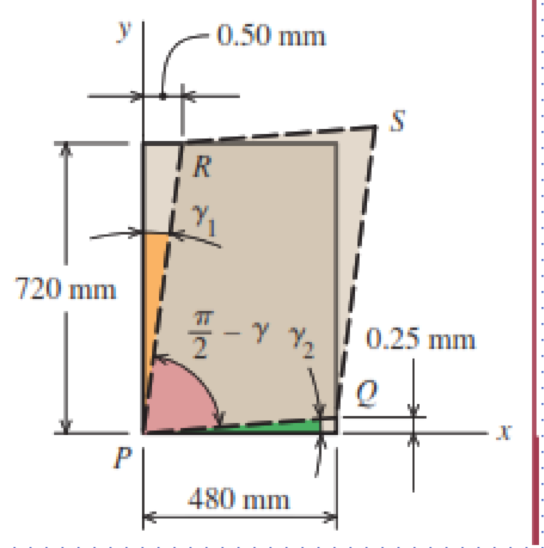
$$\gamma_{avg} = \frac{\delta_x}{L}$$

A deformation involving a change in shape (distortion) can be used to illustrate a shear strain. An average shear strain γ_{avg} associated with two reference lines that are orthogonal in the undeformed state

Example

A thin rectangular plate is uniformly deformed as shown. Determine the shear strain γ_{xy} at P.





$$\gamma_1 = \frac{0.5}{720} = 6.94 \times 10^{-4}$$

$$\gamma_2 = \frac{0.25}{480} = 5.21 \times 10^{-4}$$

$$\gamma = \gamma_1 + \gamma_2 = 1.215 \times 10^{-3}\, rad$$

The small angle approximation will be used here; therefore, $\sin\gamma \approx \gamma$, $\tan\gamma \approx \gamma$

Thermal Strain

When unrestrained, most engineering materials expand when heated and contract when cooled. The thermal strain caused by a one-degree (1°) change in temperature is designated by the alpha and is known as *the coefficient of thermal expansion*.

$$\varepsilon_T = \alpha \Delta T$$

The total normal strain in a body acted on by both temperature changes and applied load is given by

$$\varepsilon_{Total} = \varepsilon_{\alpha} + \varepsilon_T$$

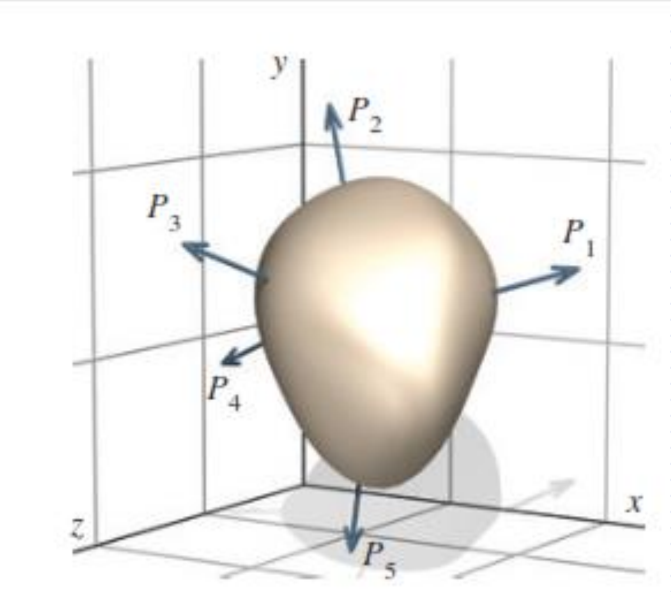
Stress Transformations

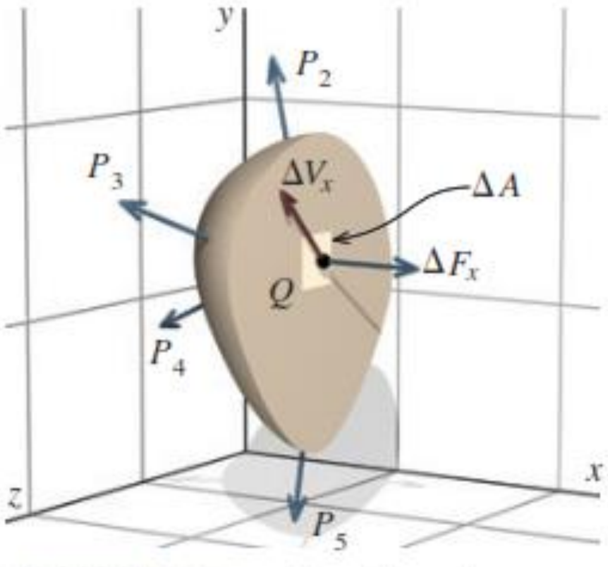
Previously, formulas were developed for normal and shear stresses that act on specific planes in axially loaded bars. This approach, while instructive, is not efficient for the determination of maximum normal and shear stresses, which are often required in a stress analysis.

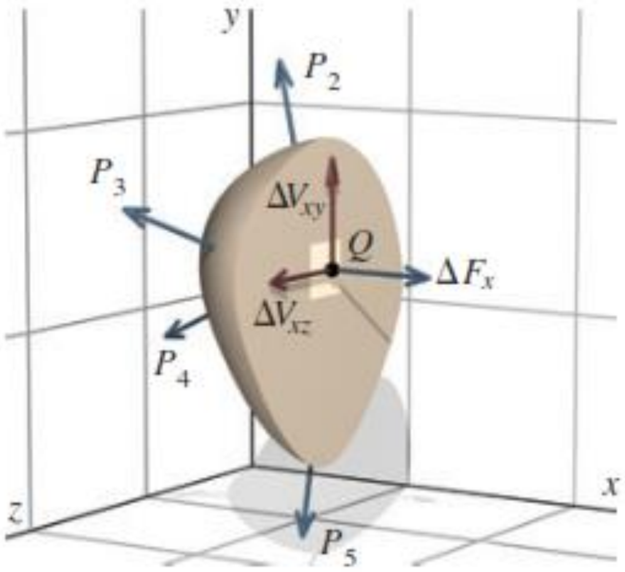
(a) normal and shear stresses acting on any specific plane passing through a point of interest, and

(b) maximum normal and shear stresses acting at any possible orientation at a point of interest.

Stress Transformations

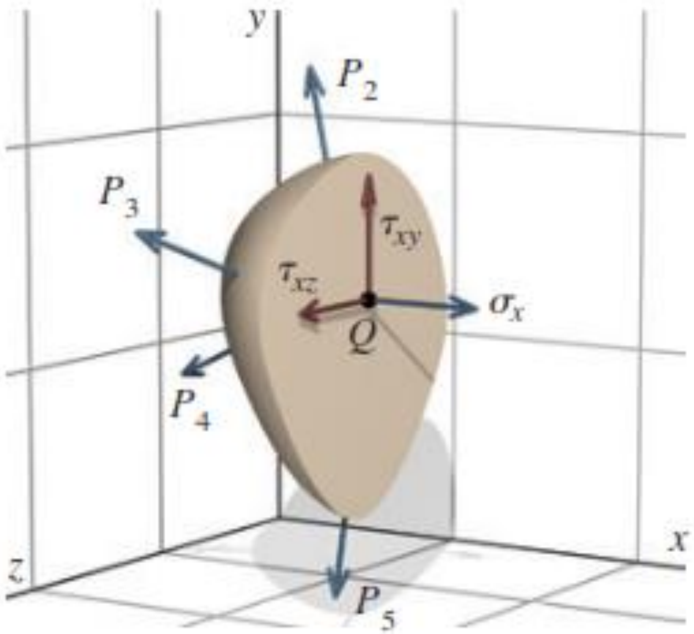


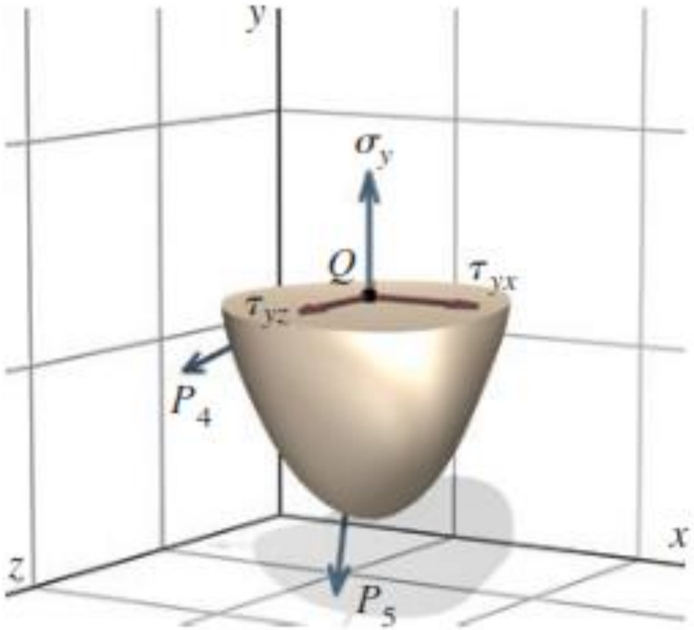


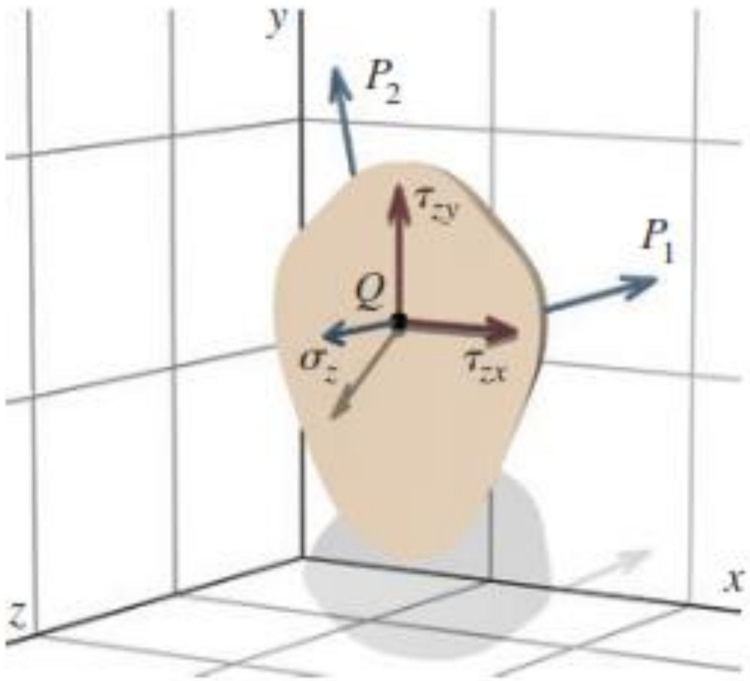


$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}, \quad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_{xy}}{\Delta A}, \quad \sigma_x = \lim_{\Delta A \to 0} \frac{\Delta V_{xz}}{\Delta A}$$

Stress Transformations







Stress Transformations

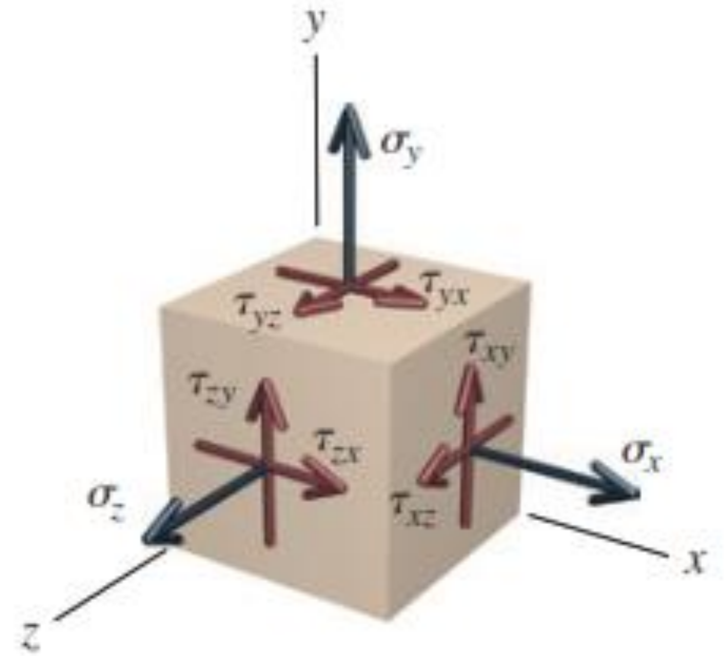
If a different set of coordinate axes (say, $x'–y'–z'$) had been chosen in the previous discussion, then the stresses found at point Q would be different from those determined on the x, y, and z planes.

Stresses in the $x'–y'–z'$ coordinate system, however, are related to those in the $x–y–z$ coordinate system, and through a mathematical process called stress transformation, stresses can be converted from one coordinate system to another.

Stress Transformations

If the normal and shear stresses on the *x, y,* and *z* planes at point *Q* are known ,then the normal and shear stresses on any plane passing through point Q can be determined. For this reason, the stresses on these planes are called the state of stress at a point.

The state of stress can be uniquely defined by three stress components acting on each of three mutually perpendicular planes.



Stress Transformations

Normal stresses are indicated by the symbol σ and a single subscript that indicates the plane on which the stress acts. The normal stress acting on a face of the stress element is positive if it points in the outward normal direction. In other words, normal stresses are positive if they cause tension in the material. Compression normal stresses are negative.

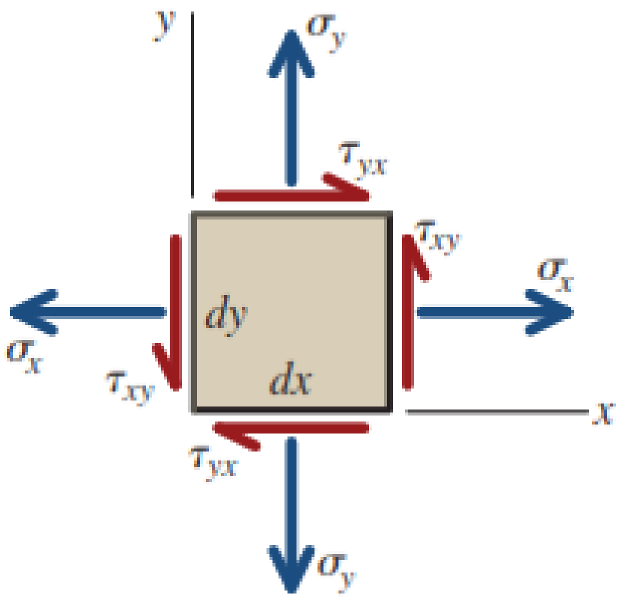
Shear stresses are denoted by the symbol τ followed by two subscripts. The first subscript designates the plane on which the shear stress acts. The second subscript indicates the direction in which the stress acts.

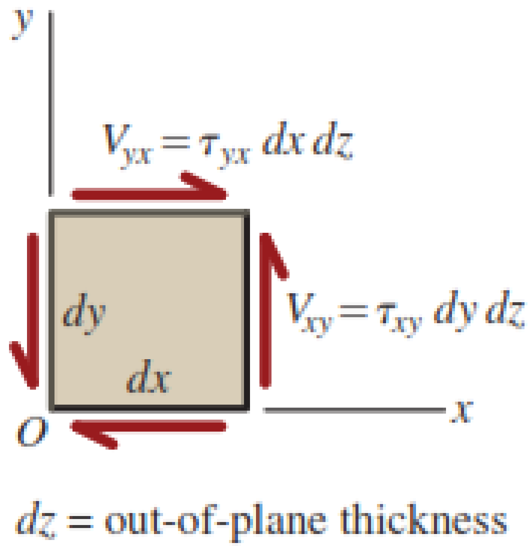
Equilibrium of the Stress Element

Two-dimensional projection of a stress element having width *dx* and height *dy*. The thickness of the stress element perpendicular to the *x*–*y* plane is *dz*.

The stress element represents an infinitesimally small portion of a physical object. If an object is in equilibrium, then any portion of the object that one chooses to examine must also be in equilibrium, no matter how small that portion may be. Consequently, the stress element must be in equilibrium.

Equilibrium of the Stress Element





$$\sum M_0 = V_{xy}dx - V_{yx}dy = \left(\tau_{xy}dy \cdot dz\right)dx - \left(\tau_{yx}dx \cdot dz\right)dy = 0$$

$$\tau_{xy} = \tau_{yx}$$

Equilibrium of the Stress Element

The result of this simple equilibrium analysis produces a significant conclusion:

- *If a shear stress exists on any plane, there must also be a shear stress of the same magnitude acting on an orthogonal plane (i.e., a perpendicular plane).*

From this conclusion, we can also assert that

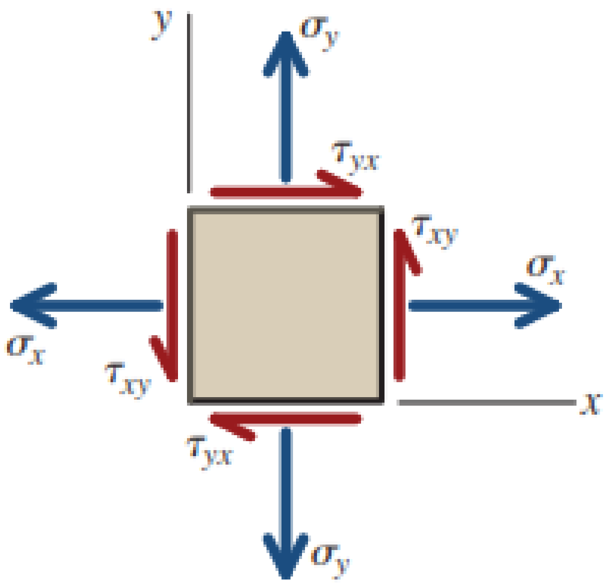
$$\tau_{yx} = \tau_{xy} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{xz} = \tau_{zx}$$

Plane Stress

Significant insight into the nature of stress in a body can be gained from the study of a state known as two-dimensional stress or plane stress. For purposes of analysis, assume that the faces perpendicular to the z axis are free of stress. Thus,

$$\sigma_x = \tau_{zx} = \tau_{zy} = 0$$

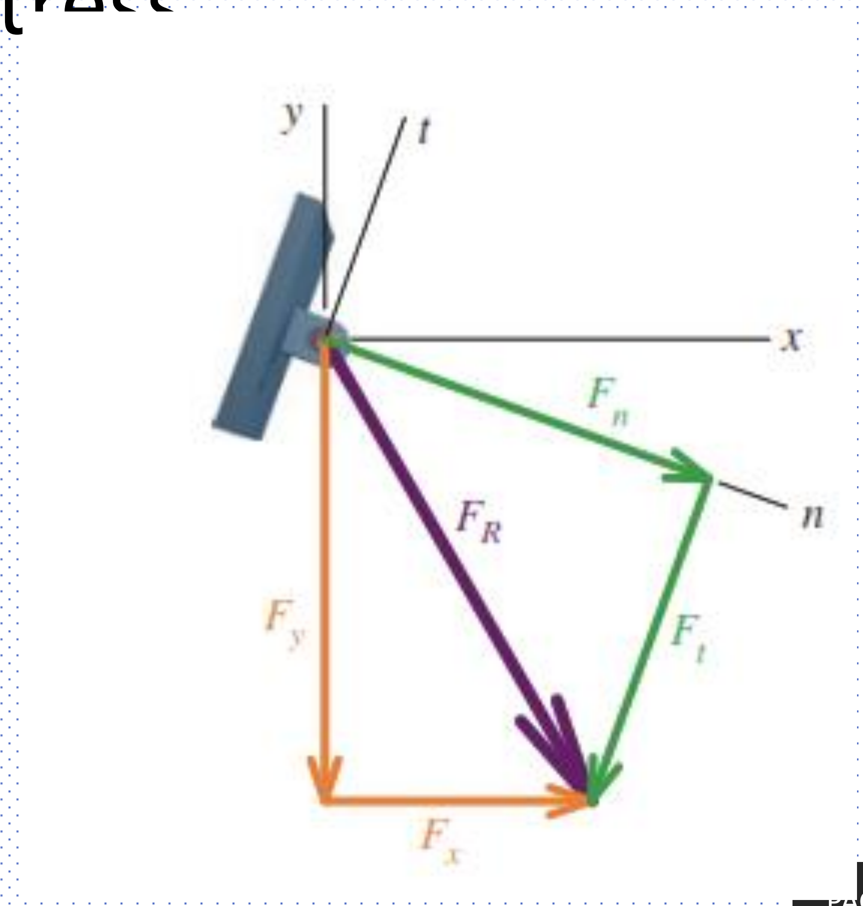
$$\tau_{xz} = \tau_{yz} = 0$$



Equilibrium Method for Plane Stress Transformations

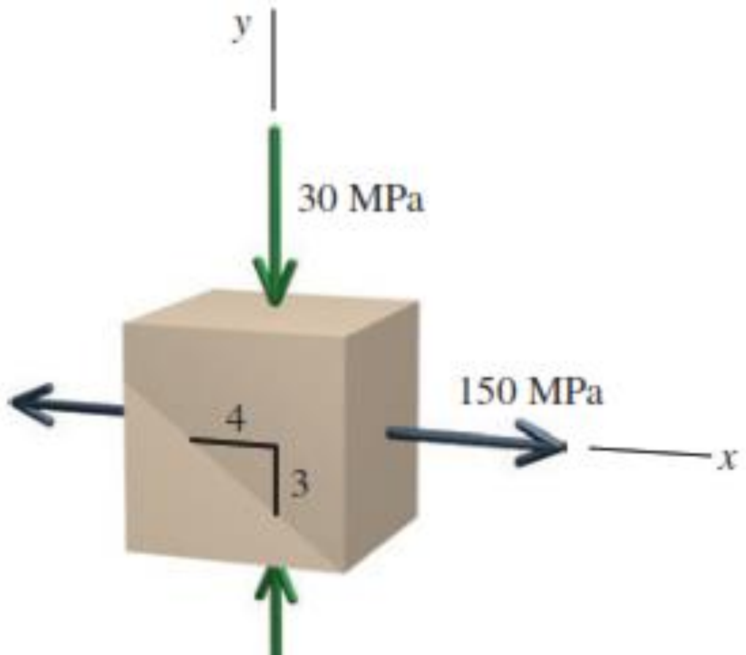
The process of changing stresses from one set of coordinate axes to another is termed stress transformation.

In some ways, the concept of stress transformation is analogous to vector addition.



Example

At a given point in a machine component, the following stresses were determined:150 MPa (T) on a vertical plane, 30 MPa (C) on a horizontal plane, and zero shear stress. Determine the stresses at this point on a plane having a slope of 3 vertical to 4 horizontal.



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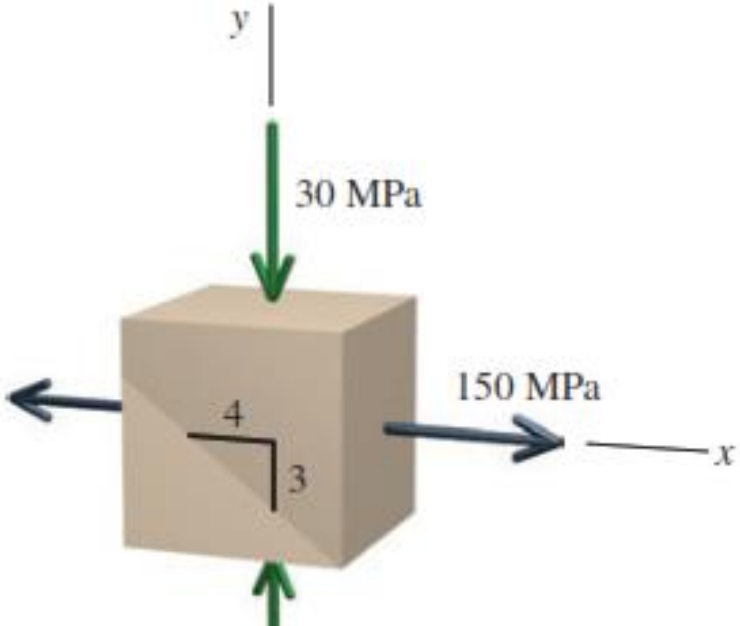
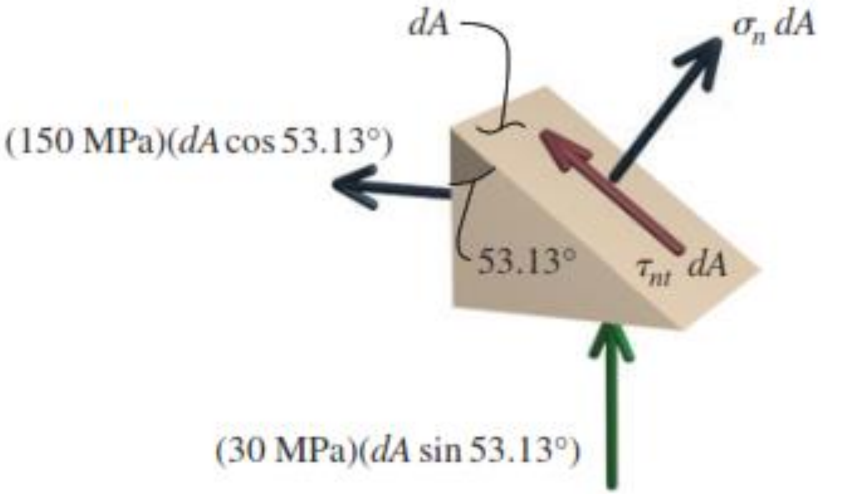




Example

The area of the inclined surface will be designated dA. Accordingly, the area of the vertical face can be expressed as *dA* cos 53.13^0, and the area of the horizontal face can be expressed as *dA* sin 53.13^0.

$$\sum F_n = \sigma_n dA + (30 \times dA \sin 53.13)\sin 53.13 - (150 \times dA \cos 53.13)\cos 53.13$$

$$\sigma_n = 34.80 MPa\,(\mathrm{T})$$

$$\sum F_t = \tau_{nt} dA + (30 \times dA \sin 53.13)\cos 53.13 - (150 \times dA \cos 53.13)\sin 53.13 = 0$$

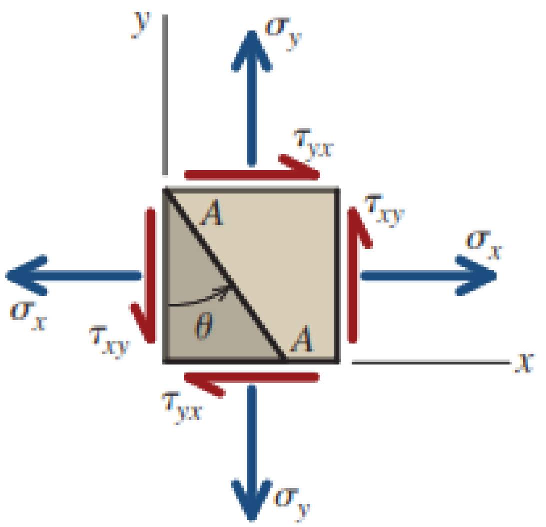
$$\tau_{nt} = -86.4 MPa$$

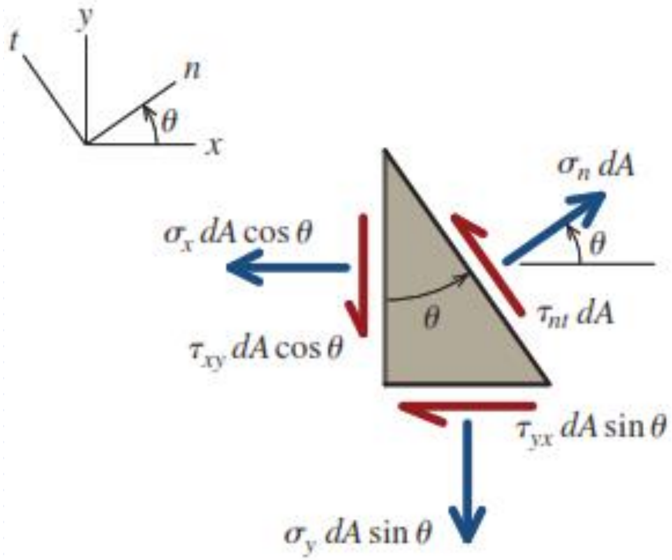
The negative sign indicates that the shear stress really acts in the negative *t* direction on the positive *n* face.
Note that the normal stress should be designated as tension or compression.

General Equations of Plane Stress Transformation

For a successful design, an engineer must be able to determine critical stresses at any point of interest in a material object. By the mechanics of materials theory developed for axial members, torsion members, and beams, normal and shear stresses at a point in a material object can be computed in reference to a particular coordinate system, such as an *x*–*y* coordinate system.

General Equations of Plane Stress Transformation





$$\tau_{yx}=\tau_{xy}$$

$$\sum F_n=\sigma_n dA-\tau_{yx}\left(dA\sin\theta\right)\cos\theta-\tau_{xy}\left(dA\cos\theta\right)\sin\theta$$
$$-\sigma_x\left(dA\cos\theta\right)\cos\theta-\sigma_y\left(dA\sin\theta\right)\sin\theta=0$$

$$\sigma_n=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+2\tau_{xy}\sin\theta\cos\theta$$

$$\sum F_t=\tau_{nt}dA-\tau_{xy}\left(dA\cos\theta\right)\cos\theta+\tau_{yx}\left(dA\sin\theta\right)\sin\theta$$
$$+\sigma_x\left(dA\cos\theta\right)\sin\theta-\sigma_y\left(dA\sin\theta\right)\cos\theta=0$$

$$\tau_{nt}=-\left(\sigma_x-\sigma_y\right)\sin\theta\cos\theta+\tau_{xy}\left(\cos^2\theta-\sin^2\theta\right)$$

General Equations of Plane Stress Transformation

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\tau_{nt} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

These equations provide a means for determining normal and shear stresses on any plane whose outward normal is

(a) perpendicular to the *z* axis (i.e., the out-of-plane axis), and

(b) oriented at an angle θ with respect to the reference *x* axis.

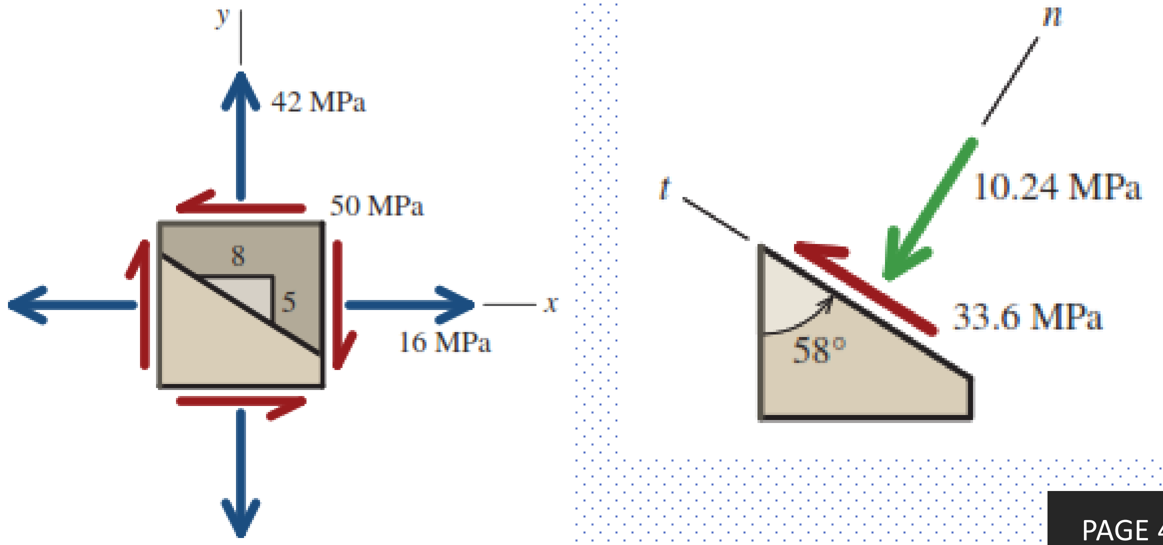
Example

At a point on a structural member subjected to plane stress, normal and shear stresses exist on horizontal and vertical planes through the point as shown. Use the stress transformation equations to determine the normal and shear stress on the indicated plane surface.

$$\sigma_x = -16MPa \qquad \tau_{xy} = -50MPa$$

$$\sigma_y = -42MPa \qquad \theta = 58^0$$

$$\sigma_n = -10.24MPa \qquad \tau_{nt} = +33.6MPa$$



Principal Planes

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_n}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}2\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tau_{nt} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$-\frac{\sigma_x - \sigma_y}{2}2\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Shear stress vanishes on planes where maximum and minimum normal stresses occur.

Magnitude of Principal Stresses & Maximum In-Plane Shear Stress Magnitude

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

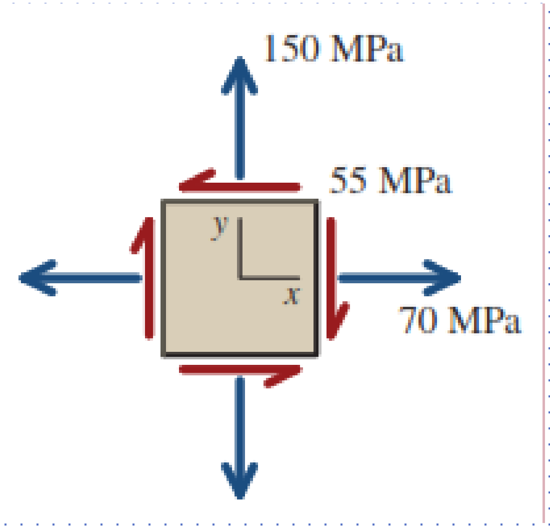
$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

$$\tau_{\max} = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

Example

Consider a point in a structural member that is subjected to plane stress. Normal and shear stresses acting on horizontal and vertical planes at the point are shown.

(a) Determine the principal stresses and the maximum in-plane shear stress acting at the point.
(b) Show these stresses in an appropriate sketch.
(c) Determine the absolute maximum shear stress at the point.



Example

(a) From the given stresses, the values to be used in the stress transformation equations are $\sigma_x = +70$ MPa, $\sigma_y = +150$ MPa, and $\tau_{xy} = -55$ MPa. The *in-plane principal stresses* can be calculated from

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} \pm \sqrt{\left(\frac{70 \text{ MPa} - 150 \text{ MPa}}{2}\right)^2 + (-55 \text{ MPa})^2}$$

$$= 178.0 \text{ MPa}, 42.0 \text{ MPa}$$

The *maximum in-plane shear stress* can be computed from

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2} = \pm\sqrt{\left(\frac{70 \text{ MPa} - 150 \text{ MPa}}{2}\right)^2 + (-55 \text{ MPa})^2}$$

$=$

On the planes of maximum in-plane shear stress, the normal stress is simply the *average normal stress*, as given by Equation (12.17):

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} = 110 \text{ MPa} = 110 \text{ MPa (T)}$$

Example

On the planes of maximum in-plane shear stress, the normal stress is simply the *average normal stress*, as given by

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} = 110 \text{ MPa} = 110 \text{ MPa (T)}$$

(b) The principal stresses and the maximum in-plane shear stress must be shown in an appropriate sketch. The angle θ_p indicates the orientation of one principal plane relative to the reference x face. From

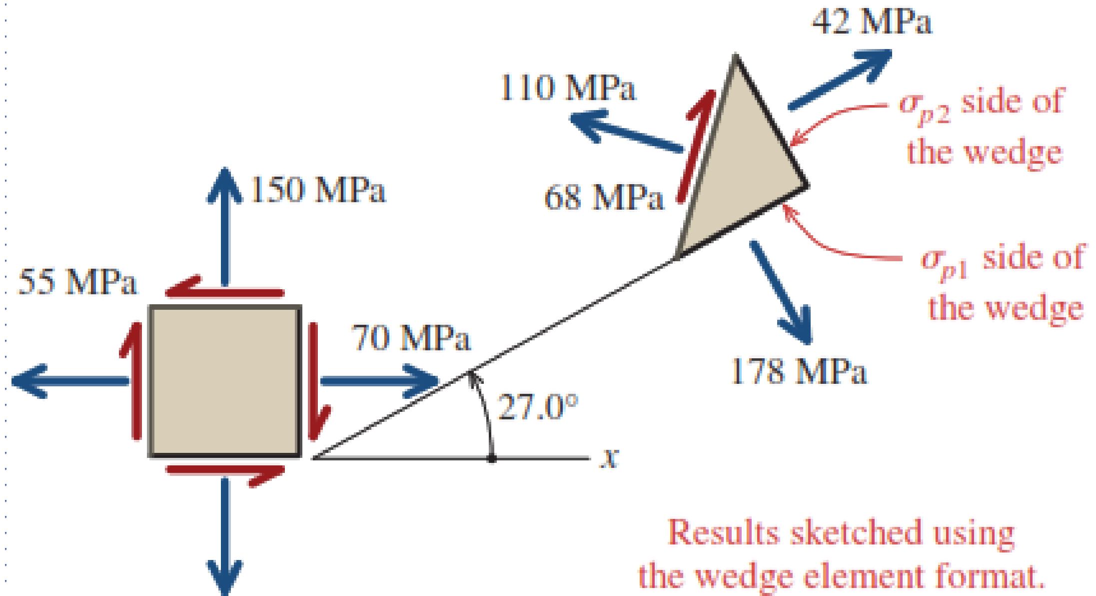
$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{-55 \text{ MPa}}{(70 \text{ MPa} - 150 \text{ MPa})/2} = \frac{-55 \text{ MPa}}{-40 \text{ MPa}}$$

$$\therefore \theta_p = 27.0°$$

Example

(c) Since σ_{p1} and σ_{p2} are both positive values, the absolute maximum shear stress will be greater than the maximum in-plane shear stress. In this example, the three principal stresses are $\sigma_{p1} = 178$ MPa, $\sigma_{p2} = 42$ MPa, and $\sigma_{p3} = 0$. The maximum principal stress is $\sigma_{max} = 178$ MPa, and the minimum principal stress is $\sigma_{min} = 0$. The absolute maximum shear stress can be computed from Equation (12.18):

$$\tau_{abs\ max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{178\ \text{MPa} - 0}{2} = 89.0\ \text{MPa} \qquad \textbf{Ans.}$$



Further Readings

Shear Stresses on Principal Planes
The Third Principal Stress
Maximum In-Plane Shear Stress Magnitude
Absolute Maximum Shear Stress