STRESSES IN THIN CYLINDERS AND SPHERES

3.1 THIN CYLINDERS

A cylinder is curved geometrical shape, with the surface formed by particle-point at a fixed distance from a given line segment, known as the axis of the cylinder.

A thin cylinder is one that has the internal diameter at least 20 times the thickness.

$d > 20t$

They could also be said to be cylinders with wall thickness to radius ratio, as follows,

$t/r < 0.1$ (r is the internal radius)

Examples of cylinders include, pressure vessel used in storage tanks and containers, water pipes, boilers, submarine hulls, roof domes etc.

3.1.1 TYPES OF STRESS IN A THIN WALLED CYLINDER

When a thin-walled cylinder is subjected to internal pressure, there will be three mutually perpendicular stresses acting within the cylinders wall thickness. These stresses are:

1. σ_L- Longitudinal stress: Related with length
2. σ_h – Hoop / Circumferential stress: Related with Diameter
3. σ_r – radial stress

3.1.2 ASSUMPTIONS IN THE ANALYSIS OF STRESSES IN CYLINDERS

The following assumptions are made in the analysis of thin cylinders

1. There is no shear stress acting in the wall

2. The longitudinal and hoop stresses are constant, and do not vary within the wall thickness

3. The radial stress (which acts normal to the curved plane of the isolated-elemental) is negligibly small compared to other two stresses acting within the wall thickness, especially when $\frac{t}{r} < \frac{1}{20}$

4. For the fact that the radial stress is negligible, the state of stress of an element of thin cylinder is considered bi-axial (i.e. only longitudinal and hoop stress)

5. P, the internal pressure acting within the cylinder, is constant and equal in all directions.

3.1.3 HOOP OR CIRCUMFERENTIAL STRESS IN CYLINDERS

Hoop stress, σ_h , describes the stresses acting along the circumference of the cylinder.

Force due to internal pressure = internal force within the thin-walled cylinder

$$\sigma_h \times 2l \times t = P \times d \times l$$

$$\sigma_h = \frac{Pd}{2t}$$

3.1.4 LONGITUDINAL STRESS IN CYLINDERS

This is the stress acting along the longitudinal thickness of thin cylinder wall, and can be determined by cutting through the vertical plane of the cylinder.

Force due to internal pressure = internal force within the thin-walled cylinder

$$P \times \pi \frac{d^2}{4} = \sigma_l \pi d t$$

$$\sigma_l = \frac{Pd}{4t}$$

$$\sigma_L = \frac{1}{2} \sigma_h$$

P – Pressure

d – Diameter

t – Thickness

3.1.5 STRESS-STRAIN DEFORMATION IN THIN CYLINDER

When a cylinder is subjected to internal pressure, the corresponding strain is due to volume or capacity change and it is known as volumetric strain

$$\varepsilon_V = \frac{\Delta V}{V_o}$$

$$\varepsilon_V = \frac{V_f - V_o}{V_o}$$

$$V_o = \pi dtl$$

$$V_f = \pi(d + \Delta d)t(l + \Delta l)$$

$$\Delta V = V_f - V_o = \pi(d + \Delta d)t(l + \Delta l) - \pi dtl$$

$$\varepsilon_v = \frac{\Delta l}{l} + \frac{\Delta d}{d}$$

Since a cylinder is considered bi-axial, from poisson's ratio,

$$\varepsilon_x = \frac{\sigma_x - v\sigma_y}{E}$$

Now expressing strain in terms of longitudinal strain and hoop strain.

$$\varepsilon_h = \varepsilon_x$$

$$\varepsilon_l = \varepsilon_y$$

$$\varepsilon_L = \frac{\sigma_L - v\sigma_h}{E}$$

$$\varepsilon_h = \frac{\sigma_h - v\sigma_L}{E}$$

But recall that…

$$\sigma_h = \frac{Pd}{2t}$$

$$\sigma_L = \frac{Pd}{4t}$$

$$\varepsilon_L = \frac{\sigma_L - v\sigma_h}{E}$$

$$\varepsilon_L = \frac{\frac{Pd}{4t} - v\frac{Pd}{2t}}{E}$$

$$\varepsilon_L = \frac{\frac{Pd}{4t} - 2v\frac{Pd}{4t}}{E}$$

$$\varepsilon_L = \frac{Pd}{4tE}(1-2v)$$

Also,

$$\varepsilon_L = \frac{\Delta L}{L}$$

$$\varepsilon_h = \frac{Pd}{4tE}(2-v)$$

Also,

$$\varepsilon_h = \frac{\Delta D}{D}$$

$$\frac{\Delta D}{D} = \frac{Pd}{4tE}(2-v)$$

$$\Delta D = \frac{Pd}{4tE}(2-v)d$$

$$\Delta D = \frac{Pd^2}{4tE}(2-v)$$

Also,

$$\frac{\Delta L}{L} = \frac{Pd}{4tE}(1-2v)$$

$$\Delta L = \frac{Pd}{4tE}(1-2v)L$$

$$\frac{\Delta V}{V} = 2\varepsilon_h + \varepsilon_L$$

$$\frac{\Delta V}{V} = 2\left[\frac{Pd}{4tE}(1-2v)\right] + \left[\frac{Pd}{4tE}(2-v)\right]$$

$$\Delta V = \frac{Pd}{4tE}(5-4V)V$$

$$V_i = \frac{\pi d^2}{4} \times L_i$$

$$V_f = \frac{\pi d_f^2}{4} \times L_f$$

3.2 SPHERES

A sphere is an object with no longitudinal description but radially shaped in nature. Therefore the internal stresses acting within the spherical wall is circumferential or hoop stress in all direction.

Considering the equilibrium of the cut section, then
Force due to internal pressure = internal force within the spherical wall

$$P \times \pi \times \frac{d^2}{4} = \sigma \pi d t$$

$$\sigma = \frac{Pd}{4t}$$

3.2.1 STRESS-STRAIN IN SPHERES

Calculating the volumetric strain in the sphere:

$$\varepsilon_V = \frac{\Delta V}{V_o}$$

$$\varepsilon_V = \frac{V_f - V_o}{V_o}$$

$$V_O = \left(\frac{4}{3}\right) \pi r^2 t$$

$$V_f = \left(\frac{4}{3}\right) \pi t (r + \Delta r)^2$$

$$V_f - V_o = \left(\frac{4}{3}\right) \pi t (r^2 + 2 r \Delta r + \Delta r^2 - r^2)$$

We ignore Δr^2 because it is very small.

$$\Delta V = \left(\frac{4}{3}\right) \pi t 2 r \Delta r$$

$$\varepsilon_V = \frac{\Delta V}{V_o}$$

$$\varepsilon_v = \frac{\left(\frac{4}{3}\right) \pi t\, 2 r \Delta r}{\left(\frac{4}{3}\right) \pi r^2 t} = 2 \frac{\Delta r}{r} = 2 \varepsilon_\sigma$$

Since ε_σ is the bi-axial strain for thin-walled spherical objects.

Applying poisson's ratio,

$$\varepsilon_\sigma = \frac{\sigma - \gamma\varepsilon}{E} = (1-\gamma)\frac{\sigma}{E}$$

For a cylinder with spherical ends closure the additional volumetric effect due the ends is

$$\varepsilon_v = 2\varepsilon_\sigma = 2(1-\gamma)\frac{\sigma}{E} = (1-\gamma)\frac{2Pd}{4tE}$$

$$\sigma = \frac{Pd}{4t}$$

3.3 SAMPLE QUESTIONS

1. Calculate ΔD, ΔL and ΔV of a thin cylinder of diameter $100\,cm$, thickness is $1\,cm$, length(L) is 5m and pressure (P) is 3N/mm^2. Young's modulus of $2\times10^5\,N/mm^2$ and poisson ratio is 0.3.

$$Engineering\,strain = \frac{+\Delta l}{l_0}$$