SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Let y be the dependent variable and x be the independent variable.
In theory, a second order differential equation should have two solutions.

3.2 The general form of the second order linear differential equation is given as:

$a(x)\,y'' + b(x)\,y' + c(x)\,y = d(x)$

3.3 STANDARD FORM

Dividing by a(x)

$y'' + \frac{b(x)}{a(x)} y' + \frac{c(x)}{a(x)} y = \frac{d(x)}{a(x)}$

This can then be reduced to the standard form

$y'' + p(x)\,y + q(x)\,y = f(x)$

3.4 HOMOGENEOUS AND NON-HOMOGENEOUS DE

Given the standard form of a second-order linear differential equation: $y'' + p(x)\,y + q(x)\,y = f(x)$,

The above is said to be homogeneous if $f(x)=0$. If $f(x)\neq 0$ and it is a function of x, then it is non-homogeneous

3.5 SUPERPOSITION THEOREM

This theorem is going to be used a lot especially when solving questions relating to non-homogeneous differential equations.

If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation

$y'' + p(x)y' + q(x) = 0$, then

$y = c_1 y_1 + c_2 y_2$ is also a solution for any c_1 and c_2

3.5.1 PROOF

Define $L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$(This is a linear operator)

y_1 and y_2 are solutions.

The operator is linear because it meets the requirement of linearity: That is

1. $L(y_1 + y_2) = Ly_1 + Ly_2$
2. $L(cy_1) = cLy_1$

Then considering

$Ly = L(c_1 y_1 + c_2 y_2) = L(c_1 y_1) + L(c_2 y_2) = c_1 L y_1 + c_2 L y_2 = 0$

Therefore, y is also a solution.

You see this side above of linearity and shit, when I wrote it initially it was God and I that understood what the fuck that was; but rn that I am reviewing it, only God understands oh.

3.6 LINEAR INDEPENDENCE

3.6.1 DEFINITION

If $c_1 y_1(x) + c_2 y_2(x) = 0$ for any value of x i.e. if the equation on the left actually equals zero, then this implies that $c_1 = c_2 = 0$, we say that y_1 and y_2 are **linear independent**. If the functions are not linearly independent, they are linearly dependent

Show that

$y_1(x) = x$ and $y_2(x) = \cos x$

Answer: Suppose $c_1 x + c_2 \cos x = 0$ for any x.

Substitute, x=0,

$c_1 \cdot 0 + c_2 \cdot \cos 0 = 0$

$c_2 = 0$

This implies that

$c_1 x = 0$

Now, let $x=1$, $c_1 = 0$

Thus, $c_1 = c_2 = 0$.

Therefore by the definition y_1 and y_2 are linearly independent.

3.6.2 TEST FOR LINEAR INDEPENDENCE

3.6.2.1 WROSKIAN TEST FOR LINEAR INDEPENDENCE

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$$

For some x_o, then y_1 and y_2 are linearly independent.

Example

Show that $y_1 = \sin 2x$ and $y_2 = \cos 3x$ are linearly independent

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix}$$

$$\rightarrow -3\sin^2 3x - 3\cos^2 3x = -3\left(\sin^2 3x + \cos^2 3x\right) = -3$$

Since the answer $-3 \neq 0$for any x, then y_1 and y_2 are linearly independent

Note: The following sets and their subsets are linearly independent

1. $1, x, x^2, x^3, x^4, \ldots$
2. $1, e^x, e^{2x}, e^{3x}, \ldots$
3. $1, e^{\alpha}, e^{2\alpha}, e^{3\alpha}, \ldots$
4. $1, x, e^{\alpha x}, x e^{\alpha x}, x^2 e^{\alpha x}, \ldots$
5. $e^{\alpha_1 x}, e^{\alpha_1 x}, e^{\alpha_1 x}, \ldots \alpha_1 \neq \alpha_2 \neq \alpha_n$
6. $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$

7. $1, \sin x, \cos x, \sin^2 x, \cos^2 x, \sin x \cos x, \ldots$

8. $x, \sin x, x \sin x, x^2 \sin^2 x, \ldots$

Theorem to note:

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation

$y'' + p(x) y' + q(x) y = 0$

The general solution is given by

$y = c_2 y_2 + c_1 y_1$

c_1 and c_2 are constants

Shit!!!! WTF did I write here. I’m so clueless

Theorem: Existence and Uniqueness theorem:

If p,q and f are continuous on the on some interval containing a, then $y'' + p(x) y' + q(x) y = f(x)$, has a unique solution satisfying

$y(a) = b$ and $y'(a) = b_1$