APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

7.0 NEWTON'S LAW OF COOLING

$T(t) = T_s + (T_o - T_s)e^{-kt}$

$T(t)$ – Temperature of the object at time t

T_s – Temperature of the surroundings

T_o – Original temperature of the object

t – time

k – A constant to be found in the question

7.1 PROOF OF NEWTON'S LAW

$\frac{dT}{dt} = -k(T_f - T_s)$

From the above equation, it can be sait that as the object approaches the surrounding temperature, the rate of temmperature change decreases.

$\frac{dT}{dt} = -k(T_f - T_s)$

$dT = -k(T_f - T_s)dt$

$\frac{dT}{T_f - T_s} = -k\,dt$

$\ln(T - T_s) = -kt + c$

$e^{\ln(T_f - T_s)} = e^{-kt+c}$

$T_f - T_s = e^{-kt} \cdot e^c$

$T_f - T_s = Ce^{-kt}$

$T_f = T_s + Ce^{-kt}$

$T(t) = T_s + Ce^{-kt}$

At t=0,

$T(t) = T(0) = T_o(initial\,temperature\,of\,the\,object)$

$T_o = T_s + Ce^{-k(0)}$

$T_o = T_s + Ce^0$

$T_o = T_s + C(1)$

$T_o = T_s + C$

$C = T_o - T_s$

Putting that back into the original equation

$T(t) = T_s + Ce^{-kt}$

$T(t) = T_s + (T_o - T_s)e^{-kt}$

7.1.1 QUESTIONS

It takes 12mins for an object at 100C to cool to 80C in a room at 50C. How much longer will it take for its temperature to decrease to 70C.

Answer: 9.408mins

7.2 Exponential growth and Decay Calculus, Relative Growth Rate, Differential equations

$$\frac{dP}{dt}=kP$$

The above implies that the population grows at a rate that is proportional to the population size.

k– Relative growth rate

$$\frac{dP}{dt}=kP$$
$$dP=kP\,dt$$
$$\frac{dP}{P}=kdt$$
$$\int\frac{dP}{P}=\int k\,dt$$
$$\ln(P)=kt+c$$
$$e^{\ln(P)}=e^{kt+c}$$
$$P=e^{kt}\cdot e^{c}$$
$$P=C\,e^{kt}$$
$$P(t)=C\,e^{kt}$$

At $t=0$

$$P(0)=C\,e^{k(0)}$$
$$P(0)=C\times 1$$
$$C=P(0)$$
$$C=P_o$$
$$P(t)=P_o e^{kt}$$

7.3 MIXING PROBLEMS

$$\frac{dA}{dt}=(rate\,coming\,in)-(rate\,going\,out)$$
$$\frac{dA}{dt}=flowing\,in\,rate\times concentration-flowing\,out\,rate\times outward\,concentration$$
$$conc=\frac{amount}{volume}$$

A vat with 500 gallons of bear contains 4% alcohol (by volume). Beer with 7% alcohol is pumped into the vat at a rate of 5gal/min and the mixture is pumpedout at the same rate.

a. What is the amount of alcohol after an hour

b. What is the percentage of alcohol after an hour.

$$\frac{dA}{dt}=(rate\,coming\,in)-(rate\,going\,out)$$

$$\frac{dA}{dt}=0.35-0.01\,A$$
$$dA=(0.35-0.01\,A)dt$$
$$\frac{dA}{0.35-0.01\,A}=dt$$
$$\int\frac{dA}{0.35-001\,A}=\int dt$$
$$\frac{1}{-0.01}\ln|0.35-0.01\,A|=t+c_1$$
$$\ln|0.35-0.01\,A|=-0.01(t+c_1)$$
$$\ln|0.35-0.01\,A|=-0.01t+c_2$$
$$e^{\ln|0.35-0.01\,A|}=e^{-0.01t+c_2}$$
$$0.35-0.01\,A=\pm e^{-0.01t+c_2}$$
$$0.35-0.01\,A=\pm e^{-0.01t}\cdot e^{c}$$
$$0.35-0.01\,A=\pm C\,e^{-0.01\,t}$$
$$-0.01\,A=-0.35+C\,e^{-0.01\,t}$$

Dividing through by -0.01

$$A=35+C\,e^{-0.01\,t}$$

On solving with the initial conditions $A(0)=20$, $C=-15$

$$A(t)=35-15\,e^{-0.01\,t}$$
$$A(60)\approx 26.77\,gal$$

$$\frac{A(60)}{500}\approx 0.0535=5.35\,\%$$

7.4 SPRING

The general formula for the second order differential equation for a spring system is

$$m\,y''+c\,y'+ky=F$$
$m\rightarrow mass$
$c\rightarrow damping\,related$
$k\rightarrow spring\,constant$
$F\rightarrow external\,force$
$y\rightarrow Displacement$

$$F=ma=-kx$$
$$ma=-kx$$
$$a=\frac{d^2x}{dt^2}$$
$$m\frac{d^2x}{dt^2}+kx=0$$

To solve the homogeneous equation,

$$m\,r^2+k=0$$

$r^2 = \frac{-k}{m}$

$r = \sqrt{\frac{-k}{m}}$

$r = 0 \pm \sqrt{\frac{k}{m}} i$

But $\omega = \sqrt{\frac{k}{m}}$

$x = c_1 \cos \omega t + c_2 \sin \omega t$

Using initial conditions,

$c_1 = a_o$

$x = a_o \cos(\omega t + \phi)$

1. An object stretches a spring 6 inches in equilibrium
a. Setup and solve a DE for its motion
b. Find the displacement given it is initially displaced by 18 inches with a velocity of 3ft/s
Assume there's no damping

From the question, there is no damping and there is no statement about an external force. So our general equation, $m\,y'' + c\,y' + ky = F$, can be reduced to

$m\,y'' + ky = 0$

$y'' + \frac{k}{m} y = 0$

But $mg = k\,\Delta l$

$\Delta l \rightarrow Stretch\,at\,equilibrium$

$\frac{k}{m} = \frac{g}{\Delta l}$

$g = 32$

$\Delta L = 0.5 ft$

$\frac{k}{m} = 64$

$y'' + 64 y = 0$

$r^2 + 64 = 0$

$r = \pm 8 i$

$y = c_1 \cos(8t) + c_2 \sin(8t)$

$y \rightarrow Displacement$

From the question

$y(0) = 3/2 ft$

$y'(0) = 3 ft/s$

On solving, $c_1 = \frac{3}{2}$, $c_2 = \frac{3}{8}$

$y = c_1 \cos(8t) + c_2 \sin(8t)$

2. A 10kg mass is attached to a spring with $k = 180 N/m$. The mass is given and an initial velocity of 2m/s upwards with an exteral force of $F(t) = 3\cos t$. The resistance due to damping is $-110\,y'\,N$

7.5 FIRST ORDER R-C CIRCUITS

7.6 FIRST ORDER R-L CIRCUITS

7.7 SECOND ORDER R-L-C CIRCUITS

RLC Circuits are used as tuning circuits in radio communications

Are used as voltage multipliers

7.7.1 SERIES RLC CIRCUITS

A voltage source, a key, a resistor, an inductor and a capacitor are connected in series.

Before time t=0 i.e. t<0, the switch is open and therefore no enenergy is stored in the elements.

At time t = 0, the switch is closed.

Applying KVL

$V_T = V_R + V_L + V_C$---i

The current flowing through the circuit $i$is the same as the current flowing through the capacitor i_c

$i = i_c$

$i_c = C\frac{dV_c}{dt}$

$V_R = iR = RC\frac{dV_c}{dt}$

$V_L = L\frac{di}{dt} = LC\frac{d^2V_c}{dt^2}$

$V_T = V_R + V_L + V_C$

$V_T = RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c$

Dividing through by LC

$\frac{V_T}{LC} = \frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC}$

Now, we have gotten a second order linear differential equation.

$Total\,solution = CF + PI$

The complementary function is the transient response of the circuit.

The Particular Integral is the steady state response of the circuit.

$V_C(\infty) = V_T$

For the transient response $\frac{V_T}{LC} = 0$

$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = 0$

The characteristic equation is given as:

$r^2 + \frac{R}{L}r + \frac{1}{LC} = 0$

$$r_1, r_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$r_1 = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$r_2 = \frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

In an RLC circuit, L and C induce some kind of oscillation in the circuit. The resistor has a tendency to dampen/suppress the oscillation.

The oscillation generated, ω is given as
$$\omega = \frac{1}{\sqrt{LC}}$$
The frequencty of this oscillation is known as the natural frequency.
$$\alpha = \frac{R}{2L} = Damping\, Coefficient$$
$$\frac{\alpha}{\omega} = \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = Damping\, Factor$$
Damping Factor is the normalized damping coefficient and this defines the circuit responds to different excitations.

Cases to consider:
Case 1: $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$, Overdamped Response
It takes time to reach its maximum value slowly because the response is sluggish
$$r_1 = -\alpha + \beta$$
$$r_2 = -\alpha - \beta$$
In that case,
$$V_c(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$$

Case 2, $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$, Critically damped Response:
Here it is fast to reach its maximum value
Roots will be negative, real and equal
$$r_1 = r_2 = -\alpha$$
$$V_c(t) = e^{-\alpha t}(c_1 + c_2 t)$$

Case 3: $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$, Under damped Response
For this it reaches a maximum response, and then there is a slight oscillation and then gets to rest.
Roots will be coomplex conjugate
$$r_1 = -\alpha + j\beta$$
$$r_2 = -\alpha - j\beta$$
The solution is given as
$$V_c(t) = e^{-\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right]$$

Case 4: $R = 0$
Roots will be imaginary

$$r_1 = j\sqrt{\frac{1}{LC}} = j\omega$$
$$r_1 = -j\sqrt{\frac{1}{LC}} = -j\omega$$
The solution is given as
$$V_c(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

7.7.2 PARALLEL RLC CIRCUIT
A current source is connected in parallel with a resistor, inductor and capacitor.

7.8 RADIOACTIVE DECAY

$$Amount\,remaining = Initial \times e^{-\lambda t_{1/2}}$$

$$N = N_o e^{-\lambda t_{1/2}}$$
$$\frac{dN}{dt} = -\lambda N$$
$$\frac{dN}{N} = -k\,dt$$
$$\int \frac{dN}{N} = \int -k\,dt$$
$$\ln N = -kt + C$$
$$N = C e^{-kt}$$
$$t_{1/2} = \frac{\ln 2}{\lambda}$$

A certain radioactive material is known to decay at a rate proportional to the amount present. If initally there is 50mg of the material present and after two hours it is observed that the material has lost 10 % of its original mass. Find
I. the mass of the material after four hours and
ii. The time at which the material has decayed to ½ its original mass
A. 40.5mg,t