MECHANICS OF MATERIALS

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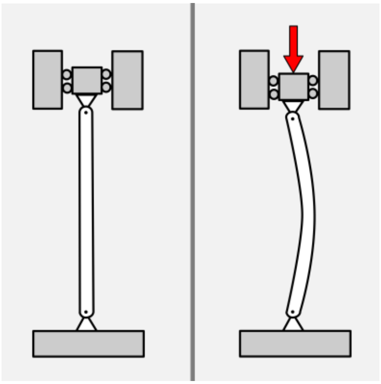
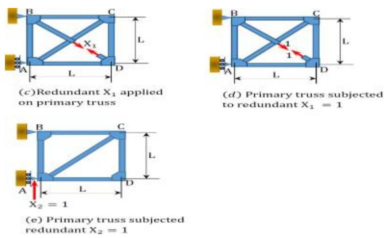
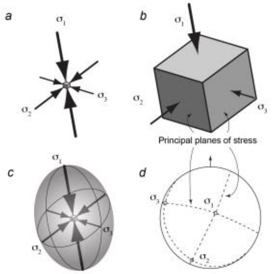
Introduction



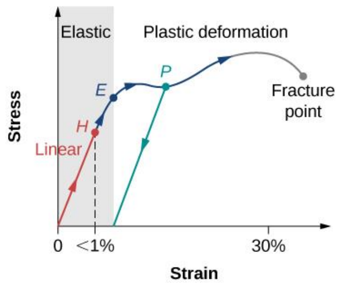
Buckling Instability of Struts/ Columns



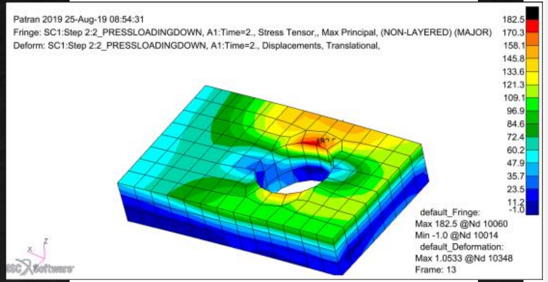
Energy methods



Stress concentration, Stress-Strain transformation,

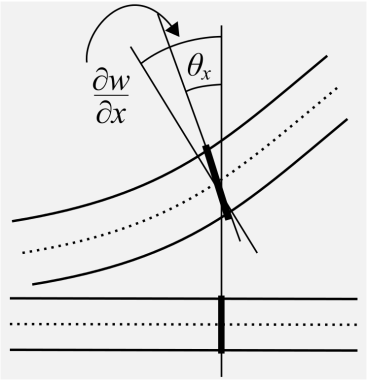


Elementary plasticity



Fracture Mechanics

Viscoelasticity



Thin Plates and Shells Application

Summary

Mechanics of Material II

Stress-Strain

Course Outline

Textbooks

- MECHANICS OF MATERIALS: An Integrated Learning System by Timothy A. Philpot

Introduction

The three fundamental areas of engineering mechanics are

- statics,
- dynamics, and
- mechanics of materials.

Statics and dynamics are devoted primarily to the study of external forces and motions associated with particles and rigid bodies (i.e., idealized objects in which any change of size or shape due to forces can be neglected).

Mechanics of materials is the study of the internal effects caused by external loads acting on real bodies that deform

Introduction

Regardless of the application, however, a safe and successful design must address the following three mechanical concerns:

1. *Strength*: Is the object strong enough to withstand the loads that will be applied to it? Will it break or fracture? Will it continue to perform properly under repeated loadings?

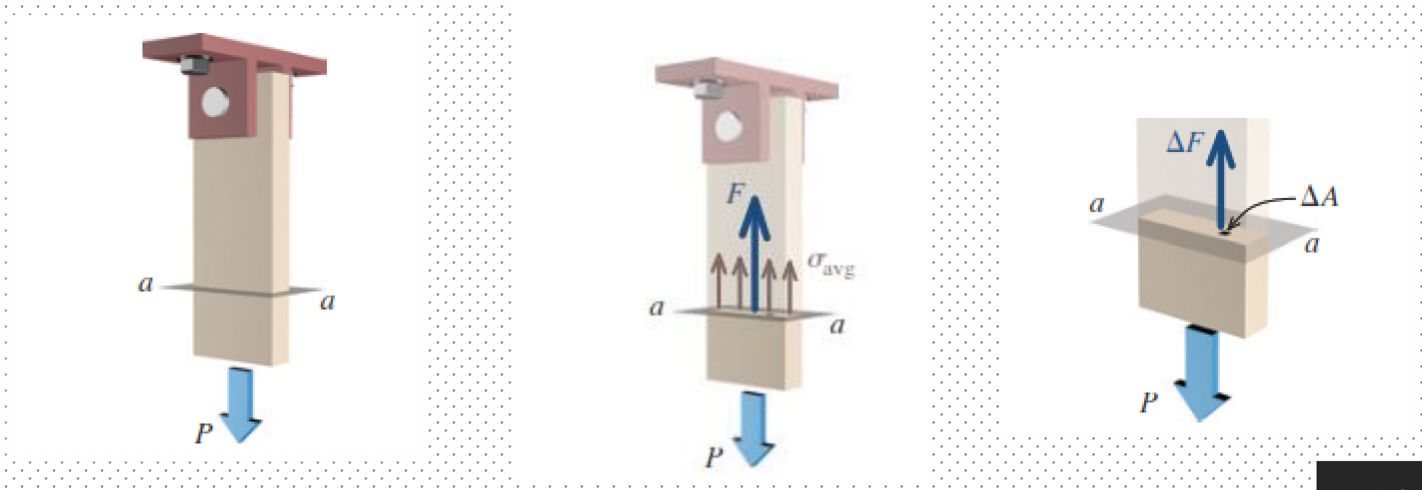
2. *Stiffness*: Will the object deflect or deform so much that it cannot perform its intended function?

3. *Stability*: Will the object suddenly bend or buckle out of shape at some elevated load so that it can no longer continue to perform its function?

Normal Stress Under Axial Loading

Stress is the intensity of internal force. Force is a vector quantity and as such has both magnitude and direction. Intensity implies an area over which the force is distributed. Therefore, stress can be defined as

$$Stress = \frac{Force}{Area}$$



Normal Stress Under Axial Loading

$$\sigma_{avg} = \frac{F}{A}$$

The sign convention for normal stresses is defined as follows:

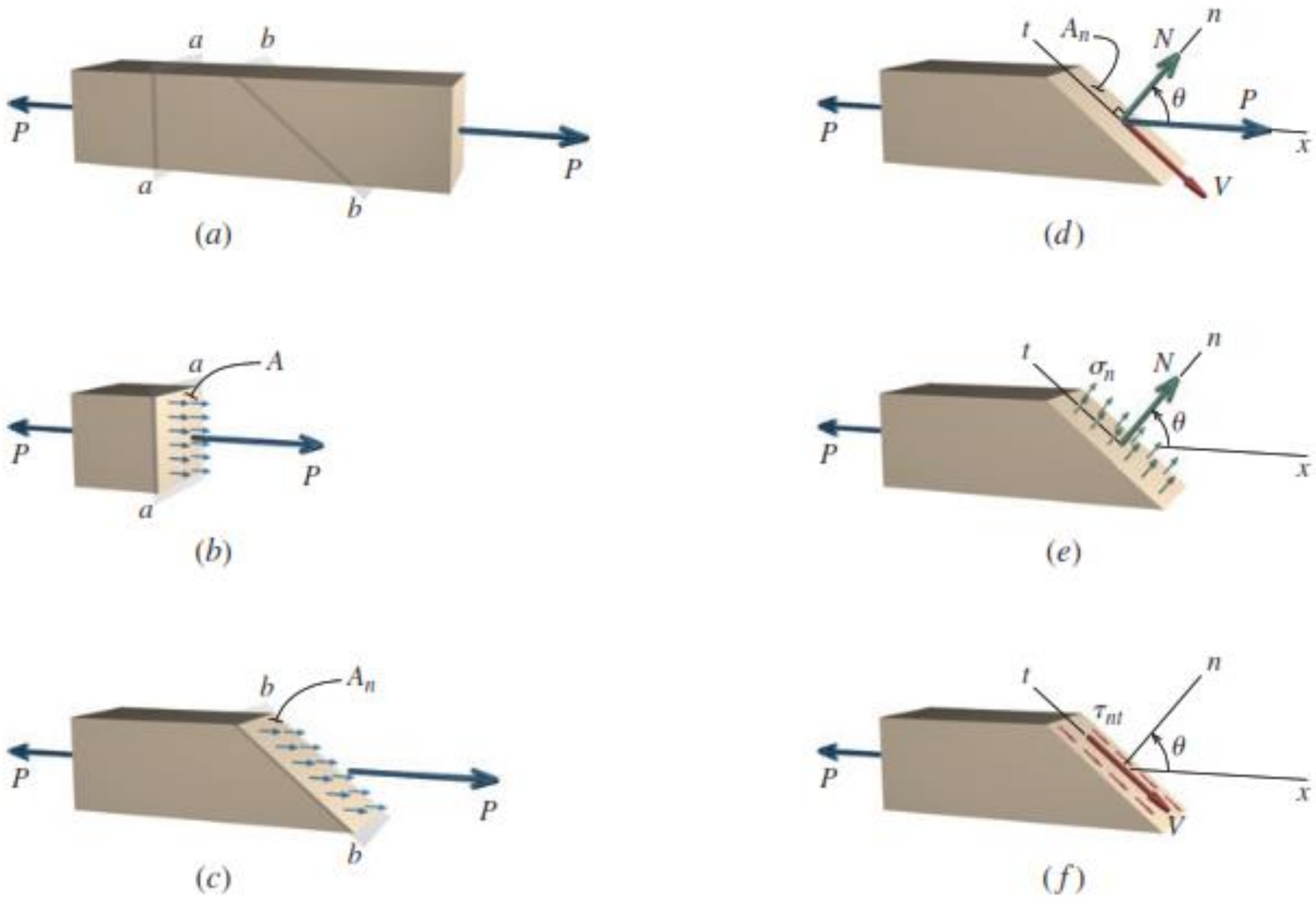
- A positive sign indicates a tension normal stress, and
- a negative sign denotes a compression normal stress.

Stresses on Inclined Sections

$$\sigma_{avg} = \frac{F}{A}$$

Consider a prismatic bar subjected to an axial force P applied to the centroid of the bar.
Loading of this type is termed uniaxial since the force applied to the bar acts in one direction.

Stresses on Inclined Sections



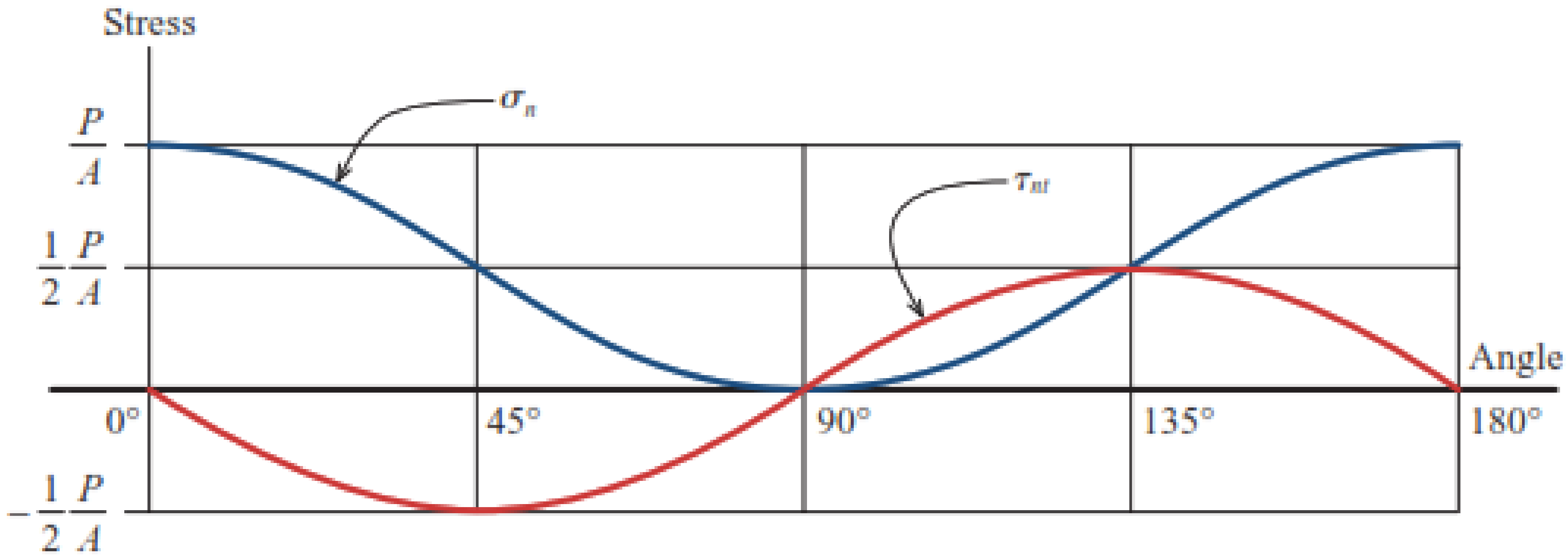
$$\sigma_n = \frac{N}{A_n} = \frac{P\cos\theta}{A/\cos\theta}$$

$$= \frac{P}{A}\cos^2\theta = \frac{P}{2A}\left(1+\cos 2\theta\right)$$

$$\tau_{nt} = \frac{V}{A_n} = \frac{-P\sin\theta}{A/\cos\theta}$$

$$= -\frac{P}{A}\sin\theta\cos\theta = -\frac{P}{2A}\sin 2\theta$$

Variation of normal and shear stress as a function of inclined plane orientation.



Stresses on Inclined Sections

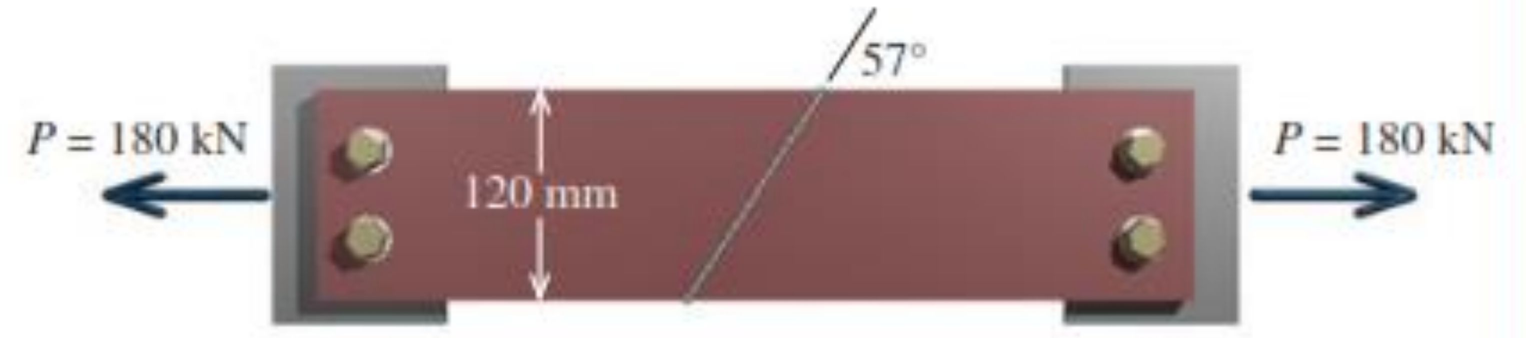
Therefore, the maximum normal and shear stresses in an axial member that is subjected to an uniaxial tension or compression force applied through the centroid of the member (termed a centric loading) are

$$\sigma_{\max} = \frac{P}{A}, \qquad \tau_{\max} = \frac{P}{2A}$$

Note that the normal stress is either maximum or minimum on planes for which the shear stress is zero. It can be shown that the shear stress is always zero on the planes of maximum or minimum normal stress.

Example

A 120-mm-wide steel bar with a butt-welded joint, as shown, will be used to carry an axial tension load of P 180 kN. If the normal and shear stresses on the plane of the butt weld must be limited to 80 MPa and 45 MPa, respectively, determine the minimum thickness required for the
bar.



Stresses on Inclined Sections

$$\sigma_n = \frac{N}{A_n}, A_n \geq \frac{180 \times \cos 33^0 \times 1000}{80} = A_n \geq 1887.008mm^2$$

$$\tau_{nt} = \frac{V}{A_n}, A_n \geq \frac{180 \times \sin 33^0 \times 1000}{45} = A_n \geq 2178.56mm^2$$

$$\cos 33^0 = \frac{120}{L_n}, L_n = 143.084mm$$

The minimum thickness is computed as

$$t \geq \frac{2178.56}{143.084} = 15.23mm$$

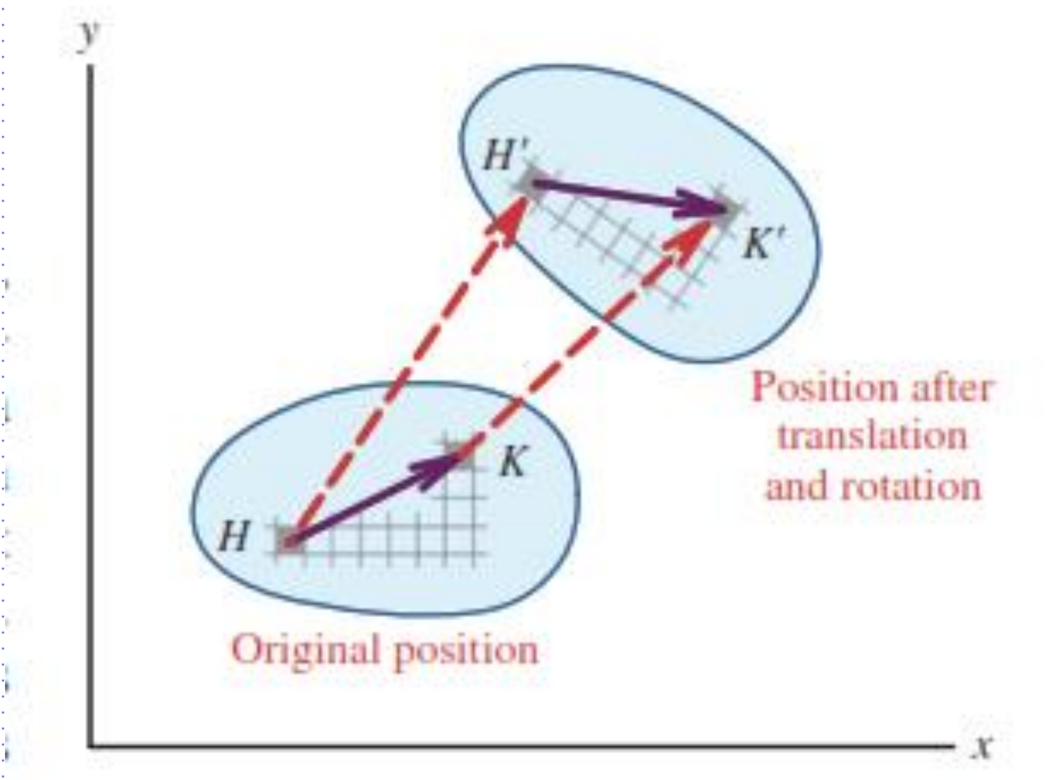
Strain: Introduction

In the design of structural elements or machine components, the deformations experienced by the body because of applied loads often represent a design consideration equally as important as stress. For this reason, the nature of the deformations experienced by a real deformable body as a result of internal stress will be studied, and methods to measure or compute deformations will be established.

Strain

The movement of a point with respect to some convenient reference system of axes is a vector quantity known as a displacement.

In some instances, displacements are associated with a translation and/or rotation of the body as a whole

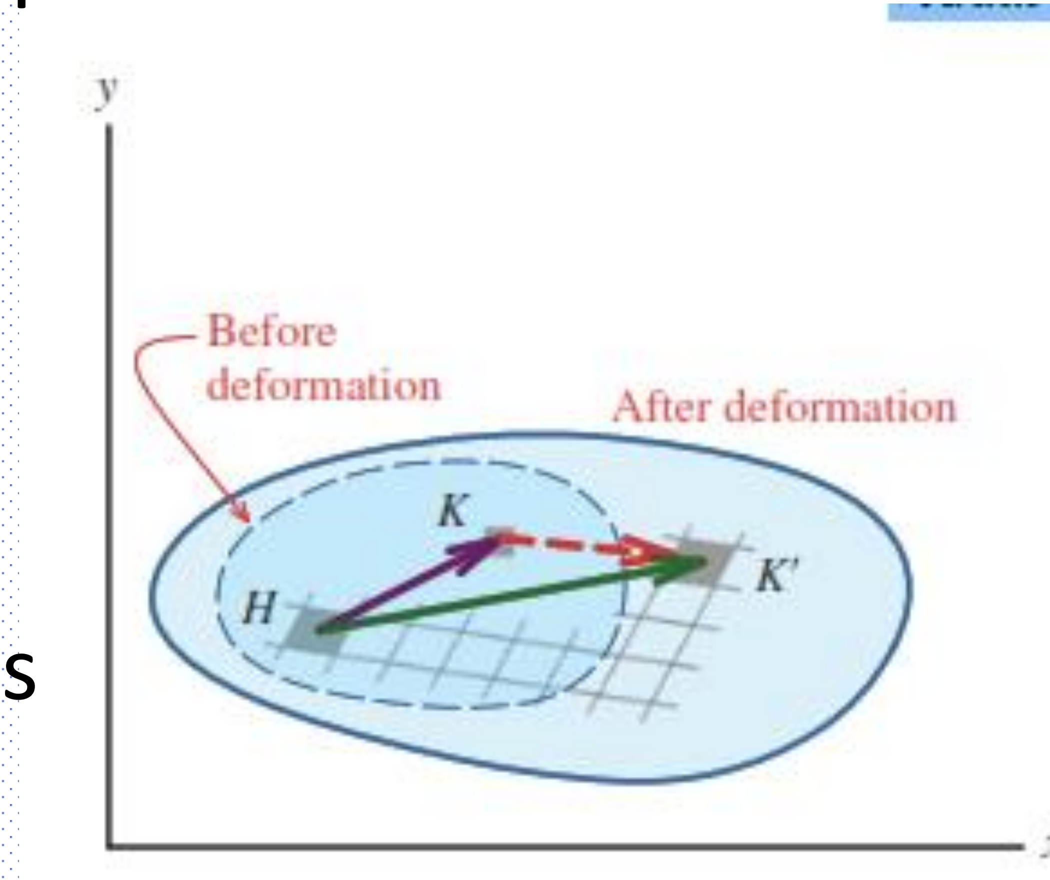


Strain: Displacement

The size and shape of the body are not changed by this type of displacement, which is termed a *rigid-body displacement*

Strain: Displacement

When displacements are caused by an applied load or a change in temperature, individual points of the body move relative to each other. The change in any dimension associated with these load- or temperature-induced displacements is known as *deformation*.



Strain: Displacement

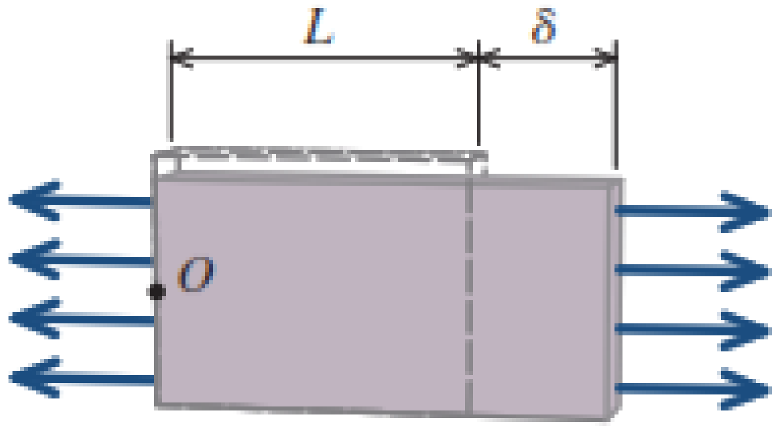
Under general conditions of loading, deformations will not be uniform throughout the body. Some line segments will experience extensions, while others will experience contractions. Different segments (of the same length) along the same line may experience different amounts of extension or contraction.

Similarly, angle changes between line segments may vary with position and orientation in the body.

Strain

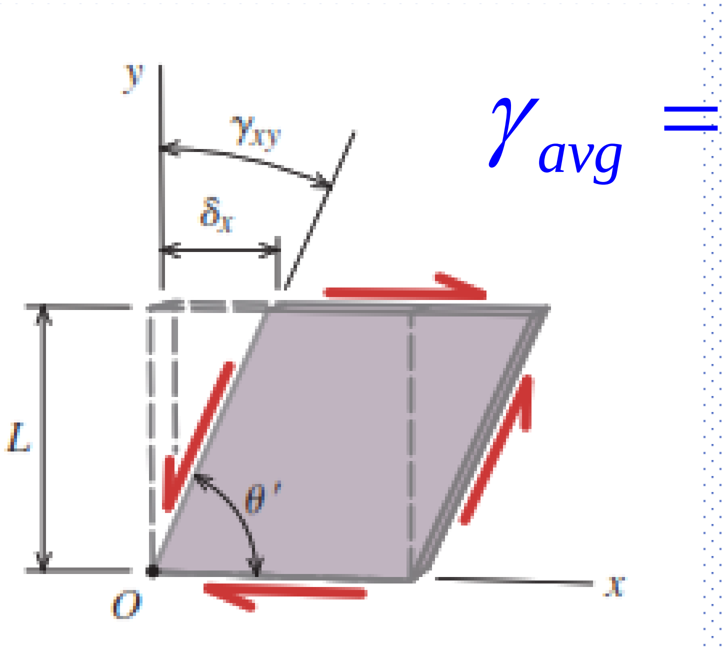
Strain is a quantity used to provide a measure of the intensity of a deformation (deformation per unit length)

The average normal strain ε_{avg} over the length of the bar is obtained by dividing the axial deformation of the bar by its initial length *L*;



$$\varepsilon_{avg} = \frac{\delta}{L}$$

Shear Strain

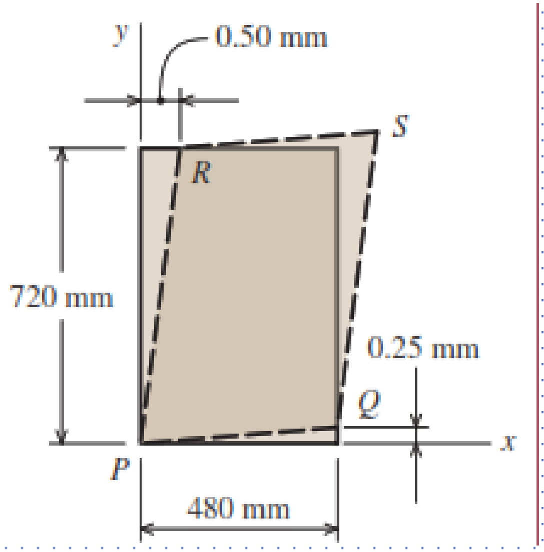


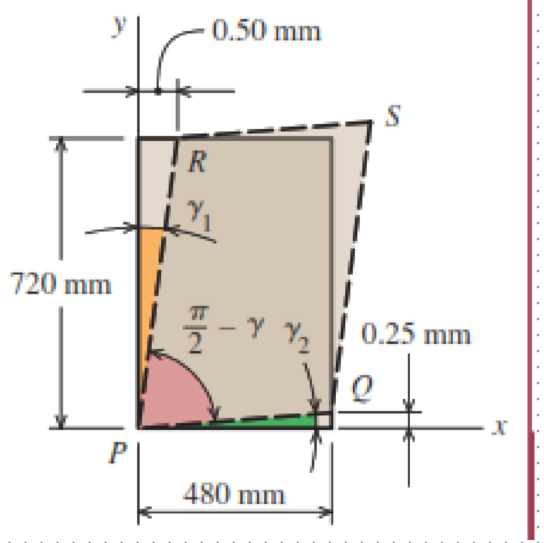
$$\gamma_{avg} = \frac{\delta_x}{L}$$

A deformation involving a change in shape (distortion) can be used to illustrate a shear strain. An average shear strain γ_{avg} associated with two reference lines that are orthogonal in the undeformed state

Example

A thin rectangular plate is uniformly deformed as shown. Determine the shear strain γ_{xy} at P.





$$\gamma_1 = \frac{0.5}{720} = 6.94 \times 10^{-4}$$

$$\gamma_2 = \frac{0.25}{480} = 5.21 \times 10^{-4}$$

$$\gamma = \gamma_1 + \gamma_2 = 1.215 \times 10^{-3}\, rad$$

The small angle approximation will be used here; therefore, sin $\gamma \approx \gamma$, tan $\gamma \approx \gamma$

Thermal Strain

When unrestrained, most engineering materials expand when heated and contract when cooled. The thermal strain caused by a one-degree (1°) change in temperature is designated by the alpha and is known as *the coefficient of thermal expansion*.

$$\varepsilon_T = \alpha \Delta T$$

The total normal strain in a body acted on by both temperature changes and applied load is given by

$$\varepsilon_{Total} = \varepsilon_{\alpha} + \varepsilon_T$$

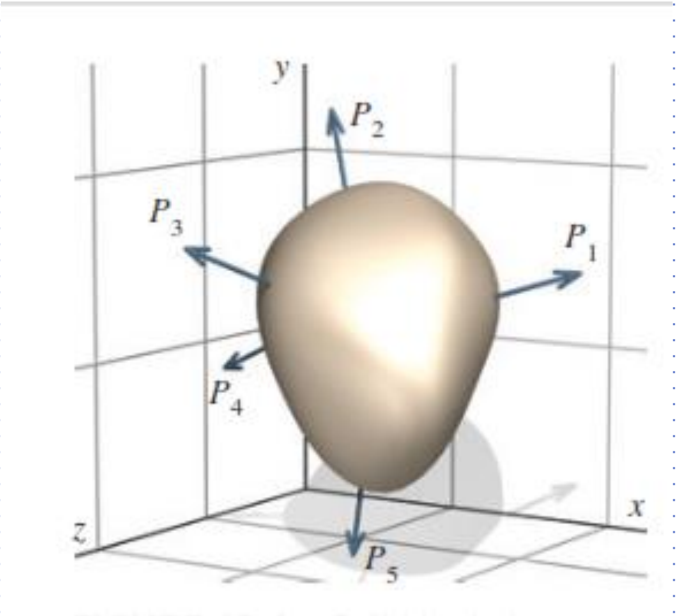
Stress Transformations

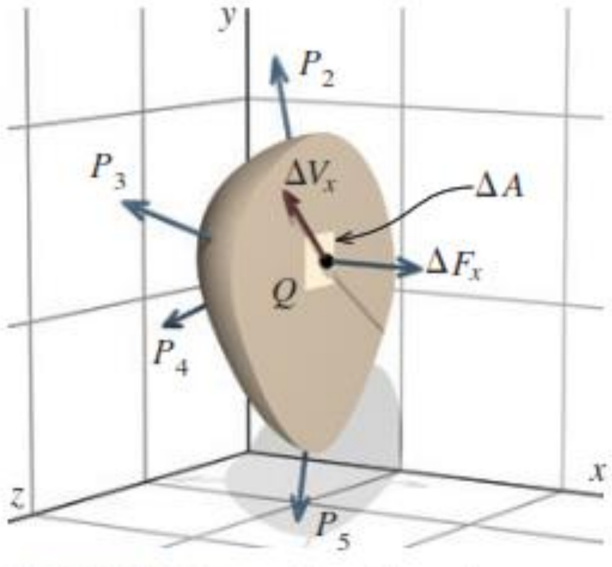
Previously, formulas were developed for normal and shear stresses that act on specific planes in axially loaded bars. This approach, while instructive, is not efficient for the determination of maximum normal and shear stresses, which are often required in a stress analysis.

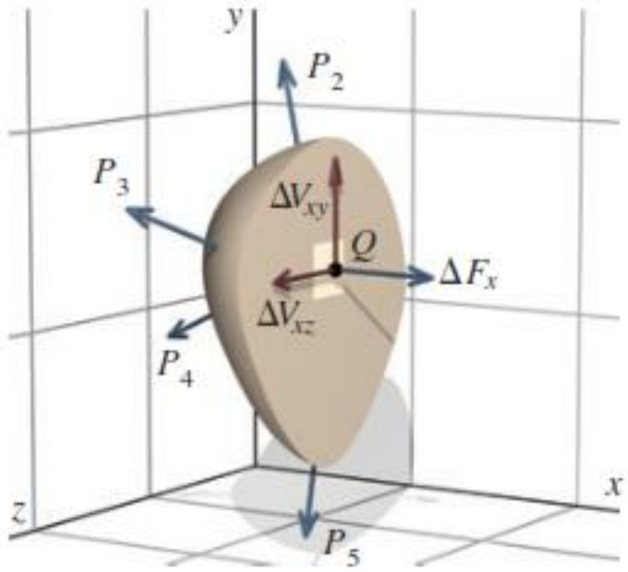
(a) normal and shear stresses acting on any specific plane passing through a point of interest, and

(b) maximum normal and shear stresses acting at any possible orientation at a point of interest.

Stress Transformations

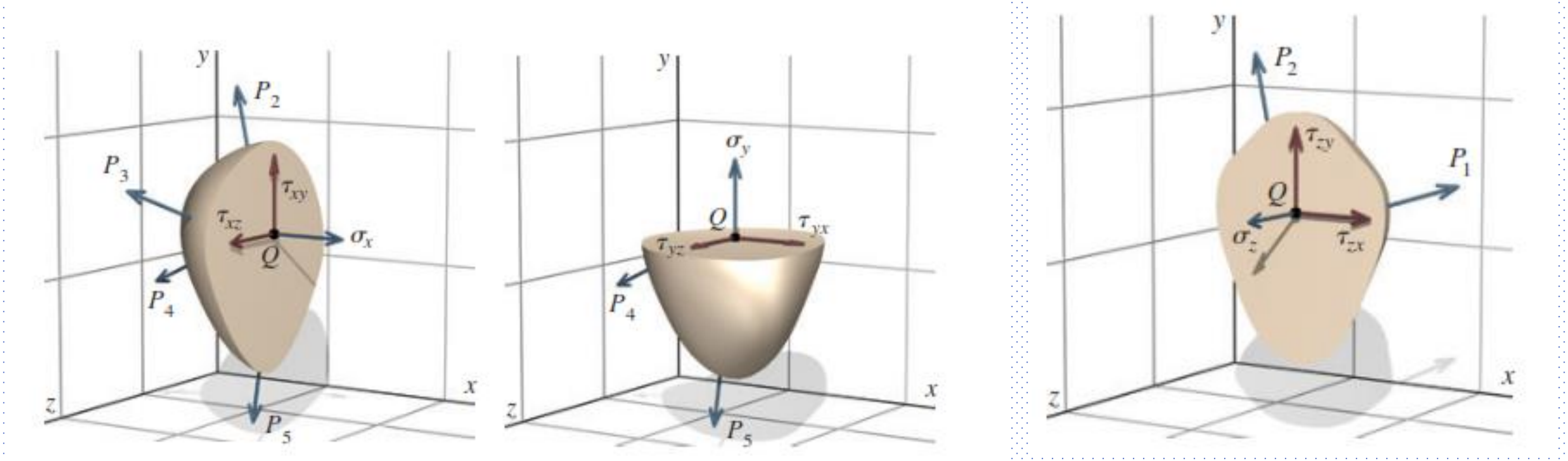






$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}, \quad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_{xy}}{\Delta A}, \quad \sigma_x = \lim_{\Delta A \to 0} \frac{\Delta V_{xz}}{\Delta A}$$

Stress Transformations



Stress Transformations

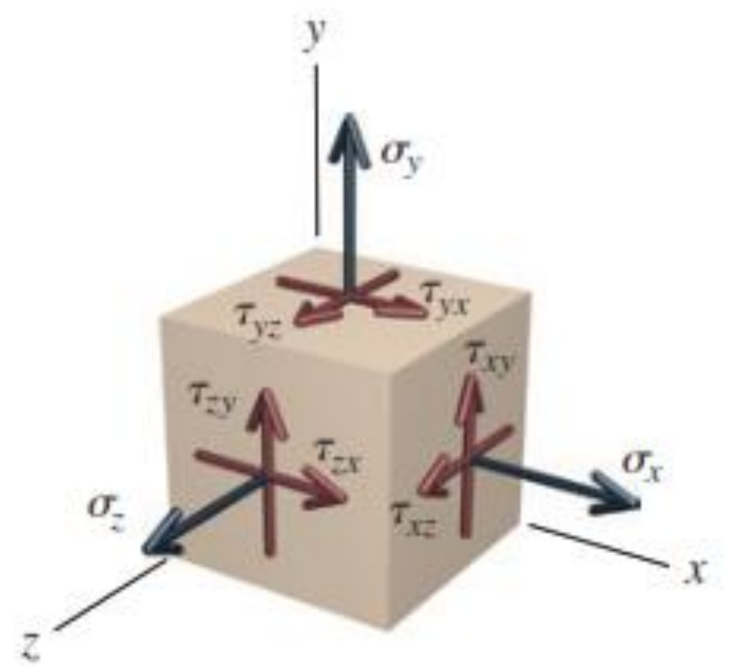
If a different set of coordinate axes (say, x'–y'–z') had been chosen in the previous discussion, then the stresses found at point Q would be different from those determined on the x, y, and z planes.

Stresses in the x'–y'–z' coordinate system, however, are related to those in the x–y–z coordinate system, and through a mathematical process called stress transformation, stresses can be converted from one coordinate system to another.

Stress Transformations

If the normal and shear stresses on the *x, y,* and *z* planes at point *Q* are known ,then the normal and shear stresses on any plane passing through point Q can be determined. For this reason, the stresses on these planes are called the state of stress at a point.

The state of stress can be uniquely defined by three stress components acting on each of three mutually perpendicular planes.



Stress Transformations

Normal stresses are indicated by the symbol σ and a single subscript that indicates the plane on which the stress acts. The normal stress acting on a face of the stress element is positive if it points in the outward normal direction. In other words, normal stresses are positive if they cause tension in the material. Compression normal stresses are negative.

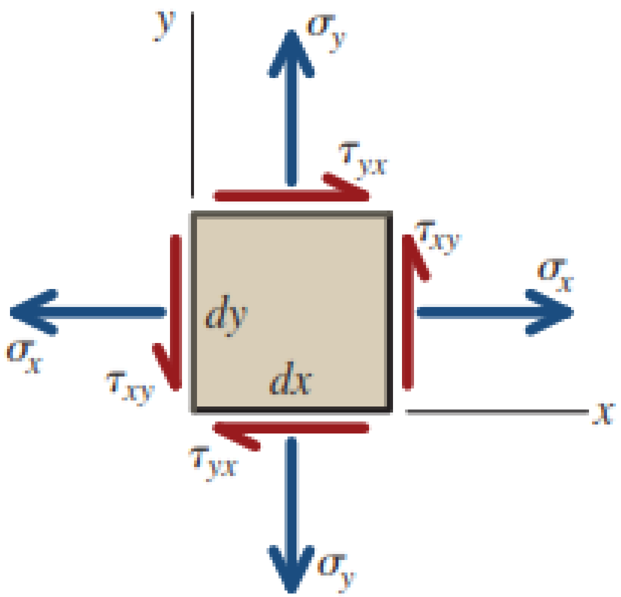
Shear stresses are denoted by the symbol τ followed by two subscripts. The first subscript designates the plane on which the shear stress acts. The second subscript indicates the direction in which the stress acts.

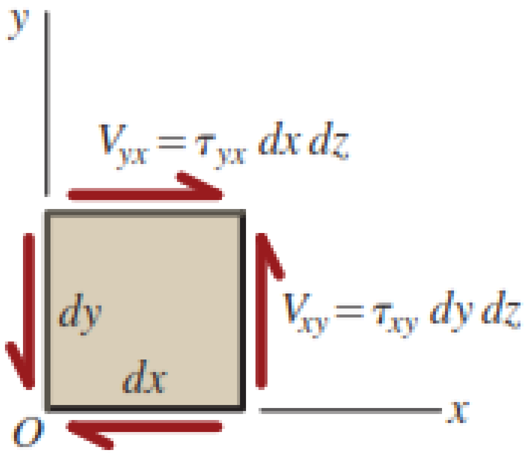
Equilibrium of the Stress Element

Two-dimensional projection of a stress element having width *dx* and height *dy*. The thickness of the stress element perpendicular to the *x*–*y* plane is *dz*.

The stress element represents an infinitesimally small portion of a physical object. If an object is in equilibrium, then any portion of the object that one chooses to examine must also be in equilibrium, no matter how small that portion may be. Consequently, the stress element must be in equilibrium.

Equilibrium of the Stress Element





$$\sum M_0 = V_{xy} dx - V_{yx} dy = \left(\tau_{xy} dy \cdot dz\right) dx - \left(\tau_{yx} dx \cdot dz\right) dy = 0$$

$$\tau_{xy} = \tau_{yx}$$

Equilibrium of the Stress Element

The result of this simple equilibrium analysis produces a significant conclusion:

- *If a shear stress exists on any plane, there must also be a shear stress of the same magnitude acting on an orthogonal plane (i.e., a perpendicular plane).*

From this conclusion, we can also assert that

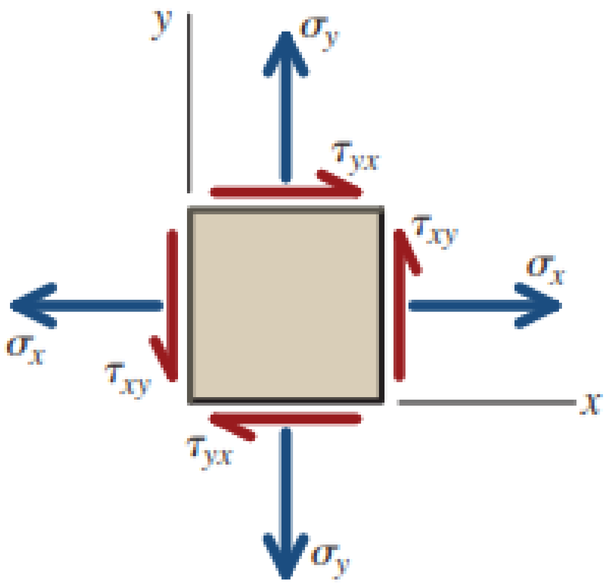
$$\tau_{yx} = \tau_{xy} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{xz} = \tau_{zx}$$

Plane Stress

Significant insight into the nature of stress in a body can be gained from the study of a state known as two-dimensional stress or plane stress. For purposes of analysis, assume that the faces perpendicular to the z axis are free of stress. Thus,

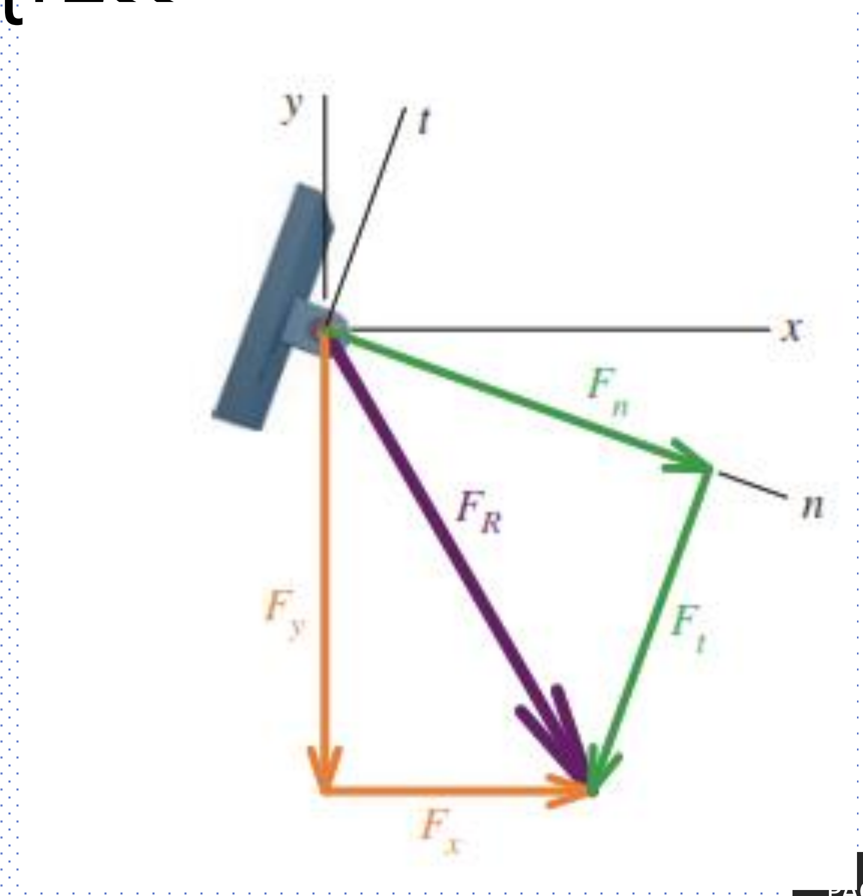
$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

$$\tau_{xz} = \tau_{yz} = 0$$



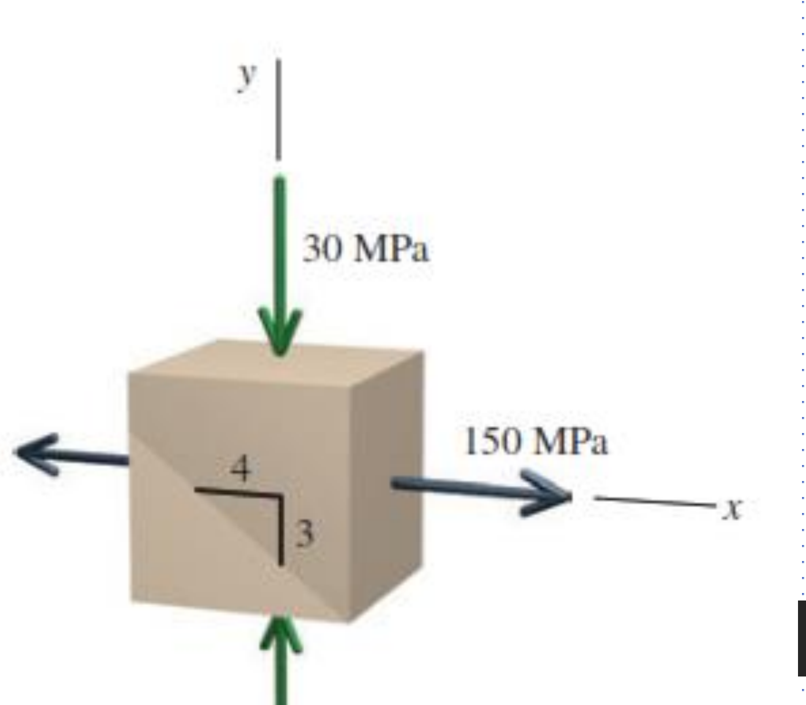
Equilibrium Method for Plane Stress Transformations

The process of changing stresses from one set of coordinate axes to another is termed stress transformation.
In some ways, the concept of stress transformation is analogous to vector addition.



Example

At a given point in a machine component, the following stresses were determined:150 MPa (T) on a vertical plane, 30 MPa (C) on a horizontal plane, and zero shear stress. Determine the stresses at this point on a plane having a slope of 3 vertical to 4 horizontal.



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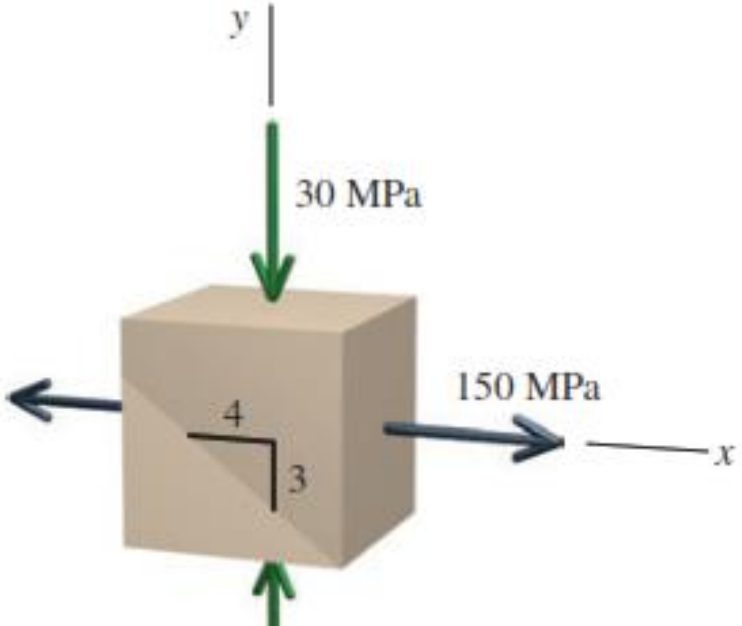
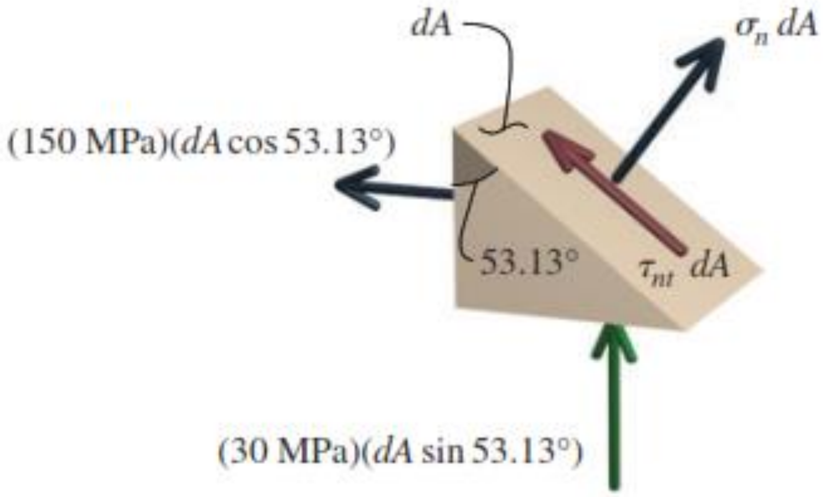




Example

The area of the inclined surface will be designated dA. Accordingly, the area of the vertical face can be expressed as *dA* cos 53.13^0, and the area of the horizontal face can be expressed as *dA* sin 53.13^0.

$$\sum F_n = \sigma_n dA + \left(30 \times dA \sin 53.13\right)\sin 53.13 - \left(150 \times dA \cos 53.13\right)\cos 53.13$$

$$\sigma_n = 34.80 MPa \left(\mathrm{T}\right)$$

$$\sum F_t = \tau_{nt} dA + \left(30 \times dA \sin 53.13\right)\cos 53.13 - \left(150 \times dA \cos 53.13\right)\sin 53.13 = 0$$

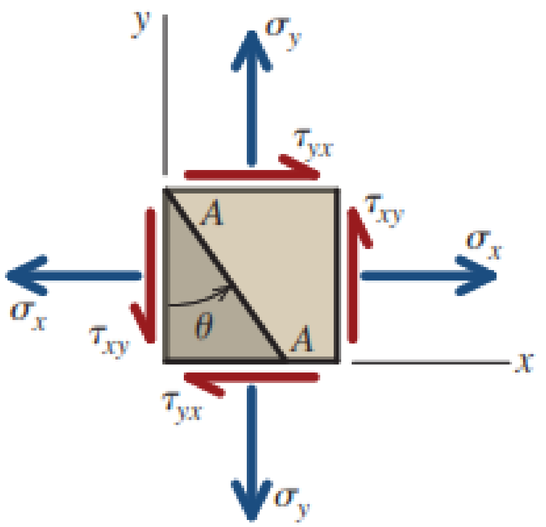
$$\tau_{nt} = -86.4 MPa$$

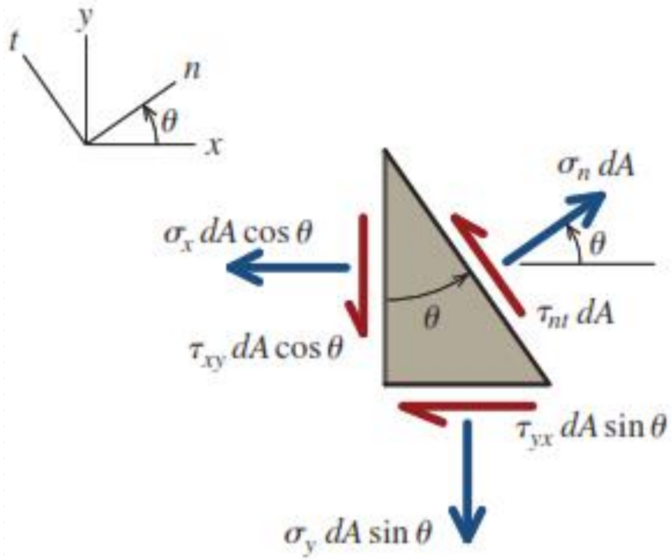
The negative sign indicates that the shear stress really acts in the negative *t* direction on the positive *n* face.
Note that the normal stress should be designated as tension or compression.

General Equations of Plane Stress Transformation

For a successful design, an engineer must be able to determine critical stresses at any point of interest in a material object. By the mechanics of materials theory developed for axial members, torsion members, and beams, normal and shear stresses at a point in a material object can be computed in reference to a particular coordinate system, such as an *x*–*y* coordinate system.

General Equations of Plane Stress Transformation





$$\tau_{yx} = \tau_{xy}$$

$$\sum F_n = \sigma_n dA - \tau_{yx}(dA\sin\theta)\cos\theta - \tau_{xy}(dA\cos\theta)\sin\theta$$
$$-\sigma_x(dA\cos\theta)\cos\theta - \sigma_y(dA\sin\theta)\sin\theta = 0$$

$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sum F_t = \tau_{nt} dA - \tau_{xy}(dA\cos\theta)\cos\theta + \tau_{yx}(dA\sin\theta)\sin\theta$$
$$+\sigma_x(dA\cos\theta)\sin\theta - \sigma_y(dA\sin\theta)\cos\theta = 0$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

General Equations of Plane Stress Transformation

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\tau_{nt} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

These equations provide a means for determining normal and shear stresses on any plane whose outward normal is

(a) perpendicular to the *z* axis (i.e., the out-of-plane axis), and

(b) oriented at an angle θ with respect to the reference *x* axis.

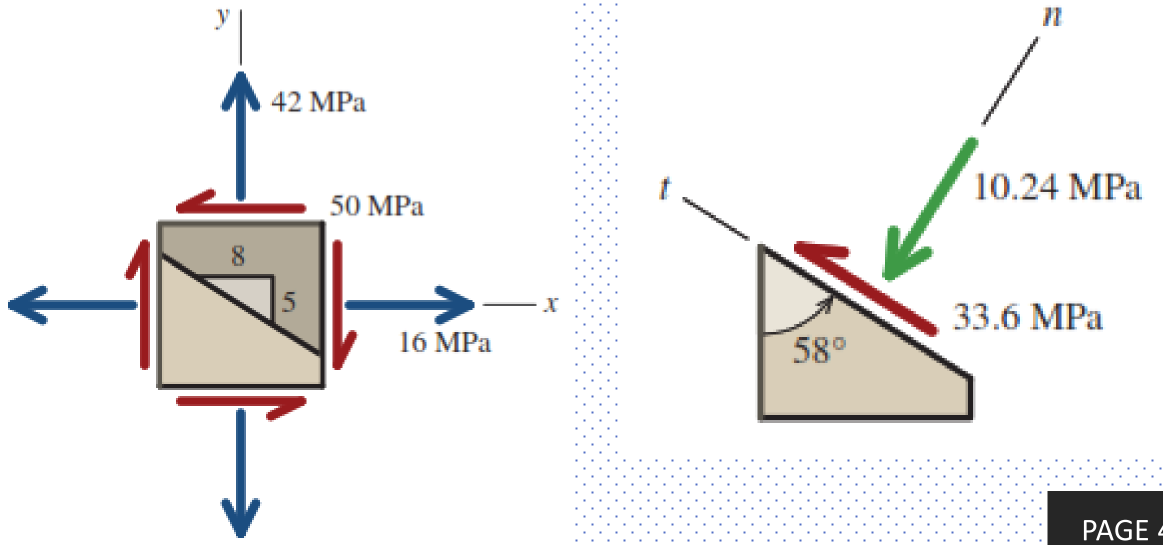
Example

At a point on a structural member subjected to plane stress, normal and shear stresses exist on horizontal and vertical planes through the point as shown. Use the stress transformation equations to determine the normal and shear stress on the indicated plane surface.

$$\sigma_x = 16MPa \qquad \tau_{xy} = -50MPa$$

$$\sigma_y = 42MPa \qquad \theta = 58^0$$

$$\sigma_n = -10.24MPa \qquad \tau_{nt} = +33.6MPa$$



Principal Planes

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_n}{d\theta} = -\frac{\sigma_x - \sigma_y}{2} 2\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tau_{nt} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$-\frac{\sigma_x - \sigma_y}{2} 2\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Shear stress vanishes on planes where maximum and minimum normal stresses occur.

Magnitude of Principal Stresses & Maximum In-Plane Shear Stress Magnitude

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

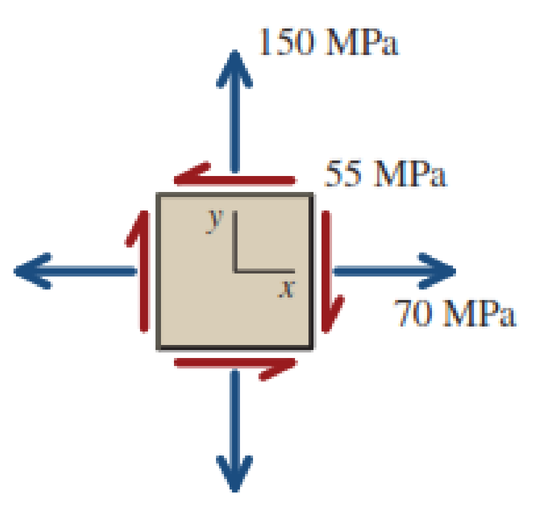
$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

$$\tau_{\max} = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

Example

Consider a point in a structural member that is subjected to plane stress. Normal and shear stresses acting on horizontal and vertical planes at the point are shown.

(a) Determine the principal stresses and the maximum in-plane shear stress acting at the point.
(b) Show these stresses in an appropriate sketch.
(c) Determine the absolute maximum shear stress at the point.



Example

(a) From the given stresses, the values to be used in the stress transformation equations are $\sigma_x = +70$ MPa, $\sigma_y = +150$ MPa, and $\tau_{xy} = -55$ MPa. The *in-plane principal stresses* can be calculated from Equation (12.12):

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} \pm \sqrt{\left(\frac{70 \text{ MPa} - 150 \text{ MPa}}{2}\right)^2 + (-55 \text{ MPa})^2}$$

$$= 178.0 \text{ MPa}, 42.0 \text{ MPa}$$

The *maximum in-plane shear stress* can be computed from Equation (12.15):

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{70 \text{ MPa} - 150 \text{ MPa}}{2}\right)^2 + (-55 \text{ MPa})^2}$$

$$= \pm 68.0 \text{ MPa}$$

On the planes of maximum in-plane shear stress, the normal stress is simply the *average normal stress*, as given by Equation (12.17):

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} = 110 \text{ MPa} = 110 \text{ MPa (T)}$$

Example

On the planes of maximum in-plane shear stress, the normal stress is simply the *average normal stress*, as given by

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} = 110 \text{ MPa} = 110 \text{ MPa (T)}$$

(b) The principal stresses and the maximum in-plane shear stress must be shown in an appropriate sketch. The angle θ_p indicates the orientation of one principal plane relative to the reference x face. From

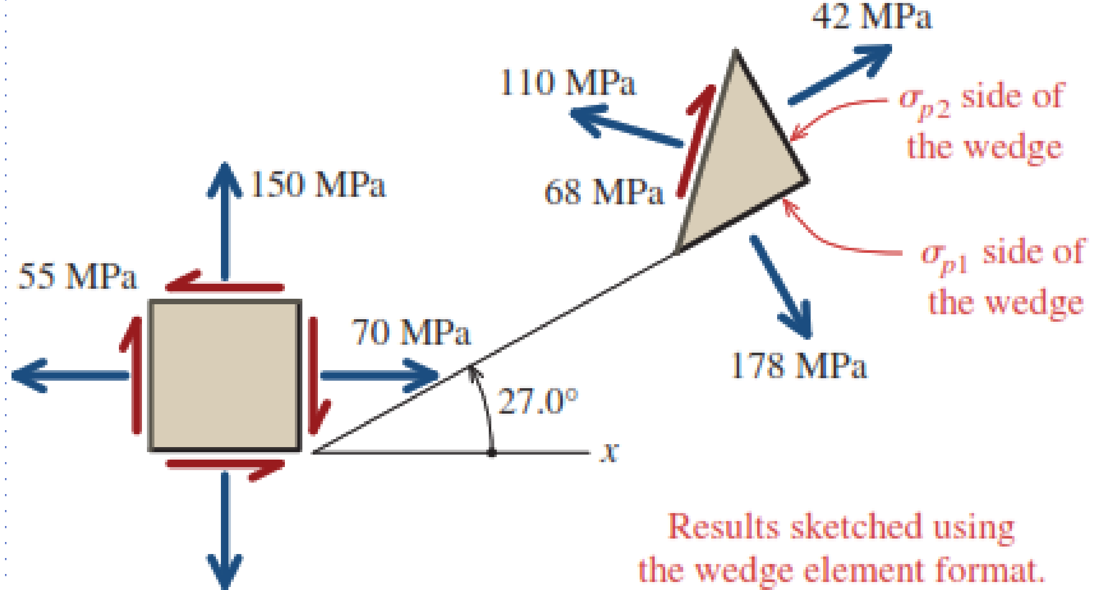
$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{-55 \text{ MPa}}{(70 \text{ MPa} - 150 \text{ MPa})/2} = \frac{-55 \text{ MPa}}{-40 \text{ MPa}}$$

$$\therefore \theta_p = 27.0°$$

Example

(c) Since σ_{p1} and σ_{p2} are both positive values, the absolute maximum shear stress will be greater than the maximum in-plane shear stress. In this example, the three principal stresses are $\sigma_{p1} = 178$ MPa, $\sigma_{p2} = 42$ MPa, and $\sigma_{p3} = 0$. The maximum principal stress is $\sigma_{max} = 178$ MPa, and the minimum principal stress is $\sigma_{min} = 0$. The absolute maximum shear stress can be computed from Equation (12.18):

$$\tau_{abs\ max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{178\ \text{MPa} - 0}{2} = 89.0\ \text{MPa}$$ **Ans.**



Results sketched using the wedge element format.

Mohr's Circle for Plane Stress

The process of changing stresses from one set of coordinate axes (i.e., *x–y–z*) to another set of axes (i.e., *n–t–z*) is termed stress transformation, and the general equations for plane stress transformation were developed earlier

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Mohr's Circle for Plane Stress

$$\sigma_n - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Both equations can be squared, then added together, and simplified to give

$$\boxed{\left(\sigma_n - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{nt}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \Rightarrow \boxed{(\sigma_n - C)^2 + \tau_{nt}^2 = R^2}$$

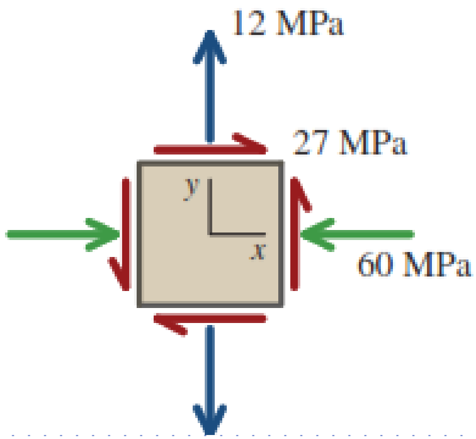
This is the equation of a circle in terms of the variables σ_n and τ_{nt}. The center of the circle is located on the σ axis (i.e., $\tau = 0$) at

$$\boxed{C = \frac{\sigma_x + \sigma_y}{2}}$$

The radius of the circle is given by the right-hand side of I

$$\boxed{R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

Mohr's Circle for Plane Stress

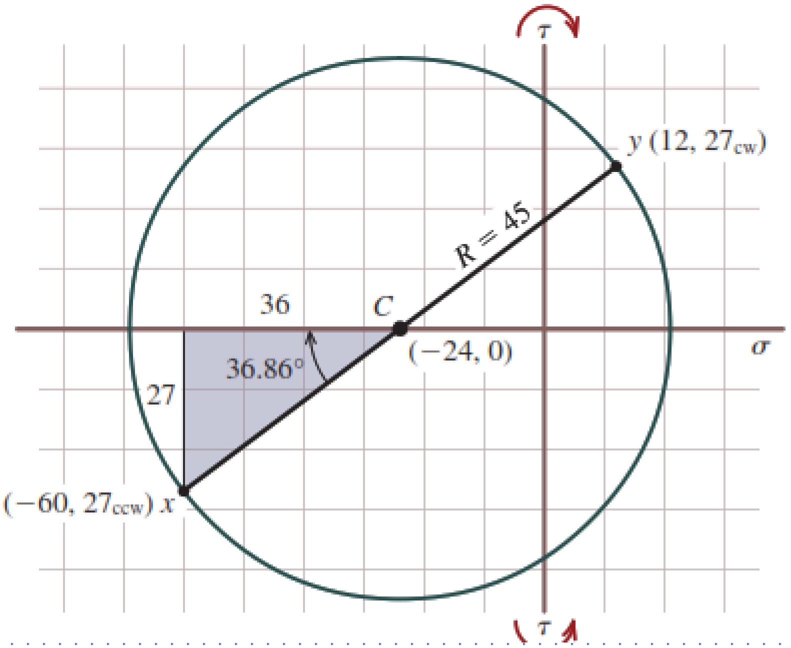


Principal and Maximum In-Plane Shear Stresses

Consider a point in a structural member that is subjected to plane stress. Normal and shea
stresses acting on horizontal and vertical planes at the point are shown.

(a) Determine the principal stresses and the maximum in-plane shear stress acting at the point.

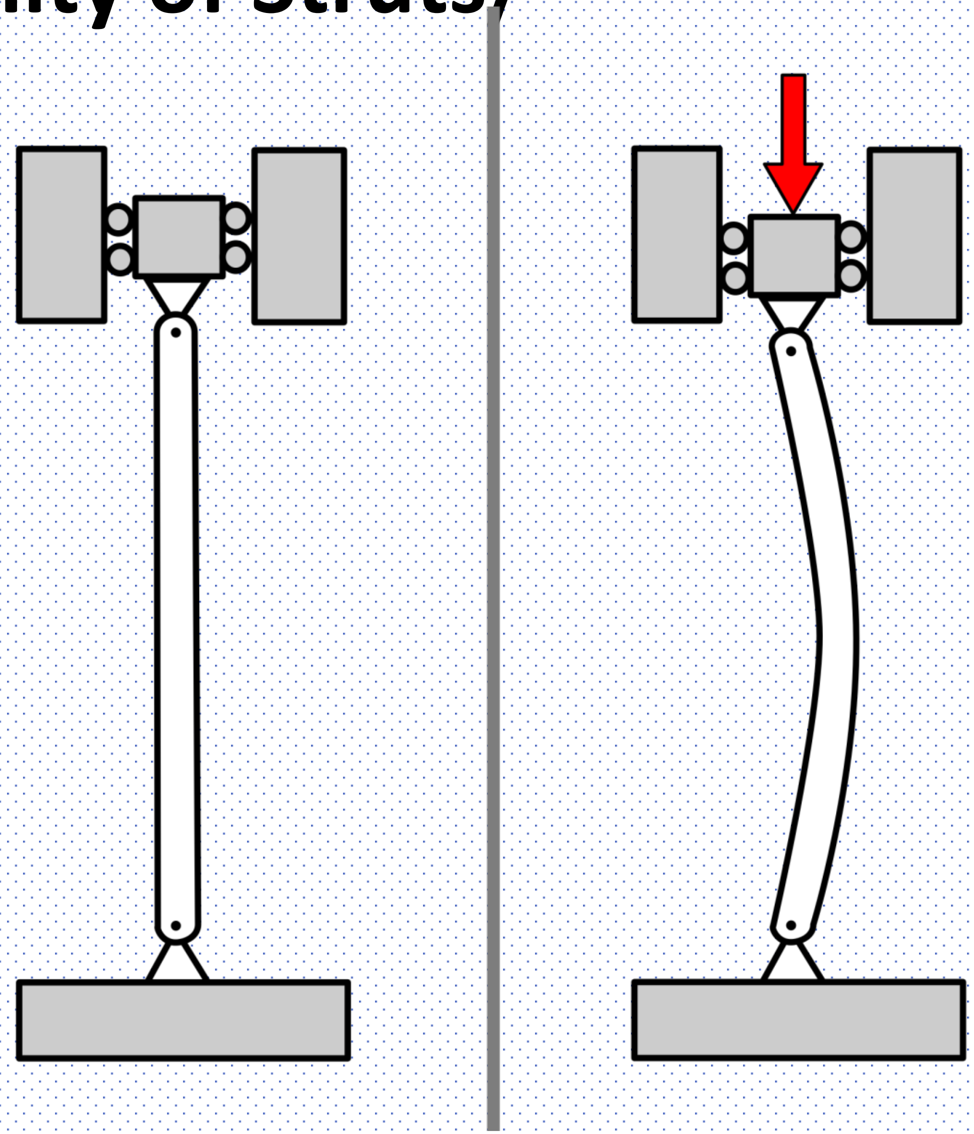
(b) Show these stresses in an appropriate sketch.



Further Readings

Shear Stresses on Principal Planes
The Third Principal Stress
Maximum In-Plane Shear Stress Magnitude
Absolute Maximum Shear Stress

Buckling Instability of Struts/ Columns



Introduction

Failures of many engineering structures fall into one of two simple categories:

- material failure : equilibrium conditions or equations of motion that are written for the initial, undeformed configuration of the structure
- structural instability: equations of equilibrium or motion to be formulated on the basis of the deformed configuration of the structure

Introduction

Dynamic stability analysis is essential for structures subjected to nonconservative loads, such as wind or pulsating forces. Structures loaded in this manner may falsely appear to be stable according to static analysis while in reality they fail through vibrations of ever increasing amplitude or some other accelerated motion.

Introduction

A physical phenomenon of a reasonably straight, slender member (or body) bending laterally (usually abruptly) from its longitudinal position due to compression is referred to as buckling.

There are two kinds of buckling:

- bifurcation-type buckling; and
- deflection-amplification-type buckling.

In fact, most, if not all, buckling phenomena in the real-life situation are the deflection amplification type

Introduction

Structural members resisting tension, shear, torsion, or even short stocky columns fail when the stress in the member reaches a certain limiting strength of the material. Therefore, once the limiting strength of material is known, it is a relatively simple matter to determine the load carrying capacity of the member. Buckling, both the bifurcation and the deflection-amplification type, does not take place as a result of the resisting stress reaching a limiting strength of the material.

Introduction

The stress at which buckling occurs depends on a variety of factors ranging from the dimensions of the member to the boundary conditions to the properties of the material of the member. Determining the buckling stress is a fairly complex undertaking.

Introduction

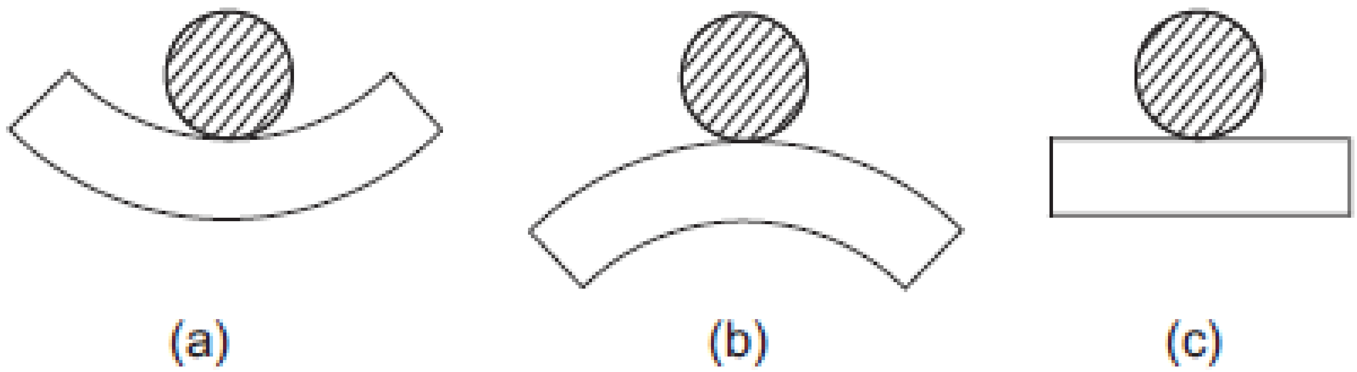
If buckling does not take place because certain strength of the material is exceeded, then, does a compression member buckle?

Introduction

A slender column shortens when compressed by a weight applied to its top, and, in so doing, lowers the weight's position. The tendency of all weights to lower their position is a basic law of nature. It is another basic law of nature that, whenever there is a choice between different paths, a physical phenomenon will follow the easiest path. Confronted with the choice of bending out or shortening, the column finds it easier to shorten for relatively small loads and to bend out for relatively large loads. In other words, when the load reaches its buckling value the column finds it easier to lower the load by bending than by shortening. *Structure in Architecture*

Introduction: Neutral Equilibrium

The concept of the stability of various forms of equilibrium of a compressed bar is frequently explained by considering the equilibrium of a ball (rigid body) in various positions



stable equilibrium unstable equilibrium neutral equilibrium

Introduction: Neutral Equilibrium

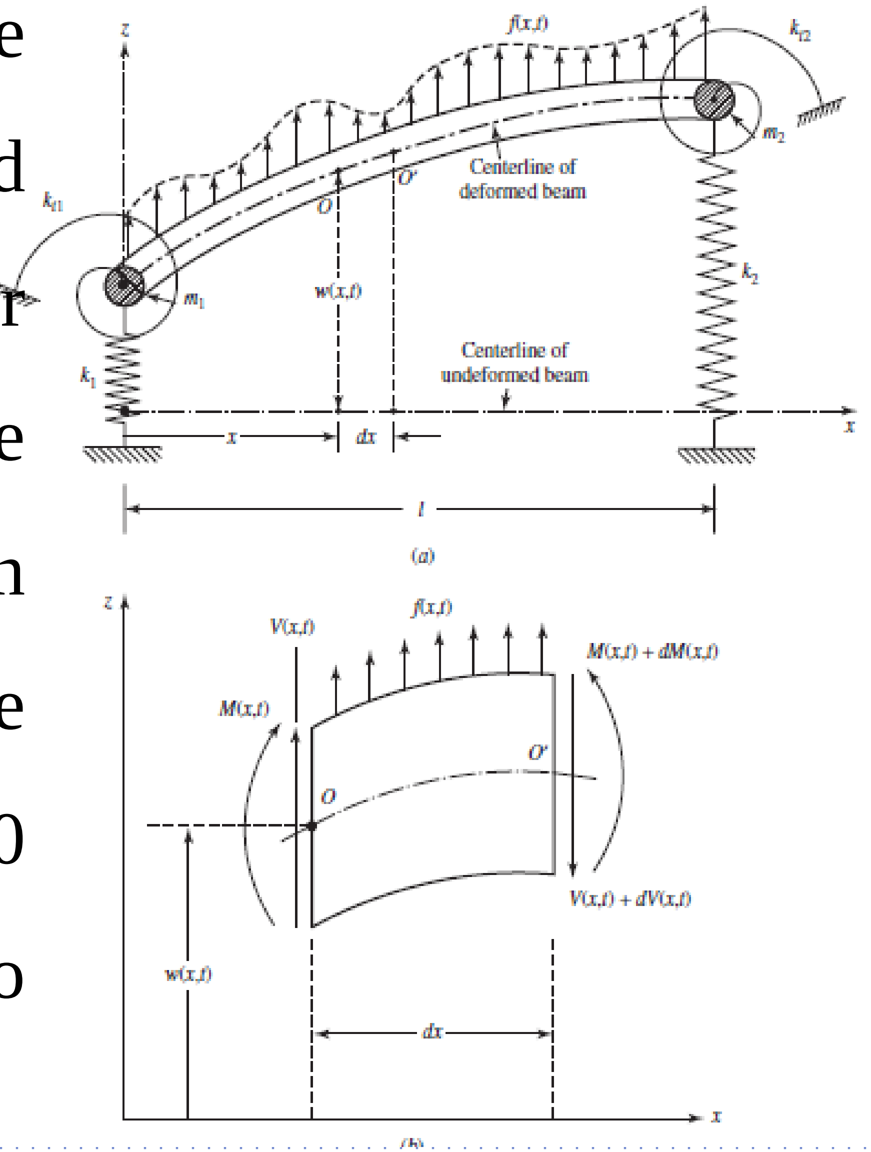
The straight configuration of the column is stable at small loads, but it is unstable at large loads. It is assumed that a state of neutral equilibrium exists at the transition from stable to unstable equilibrium in the column.

Then the load at which the straight configuration of the column ceases to be stable is the load at which neutral equilibrium is possible. This load is usually referred to as the *critical load*.

Euler Theory

Euler Theory: Theory of Bending

In the Euler–Bernoulli or thin beam theory, the rotation of cross sections of the beam is neglected compared to the translation. The angular distortion due to shear is considered negligible compared to the bending deformation. The thin beam theory is applicable to beams for which the length is much larger than the depth (at least 10 times) and the deflections are small compared to the depth.



Euler Theory: Assumptions

This theory is based upon the following assumptions:

The column is perfectly elastic

The compressive load is ideally axial

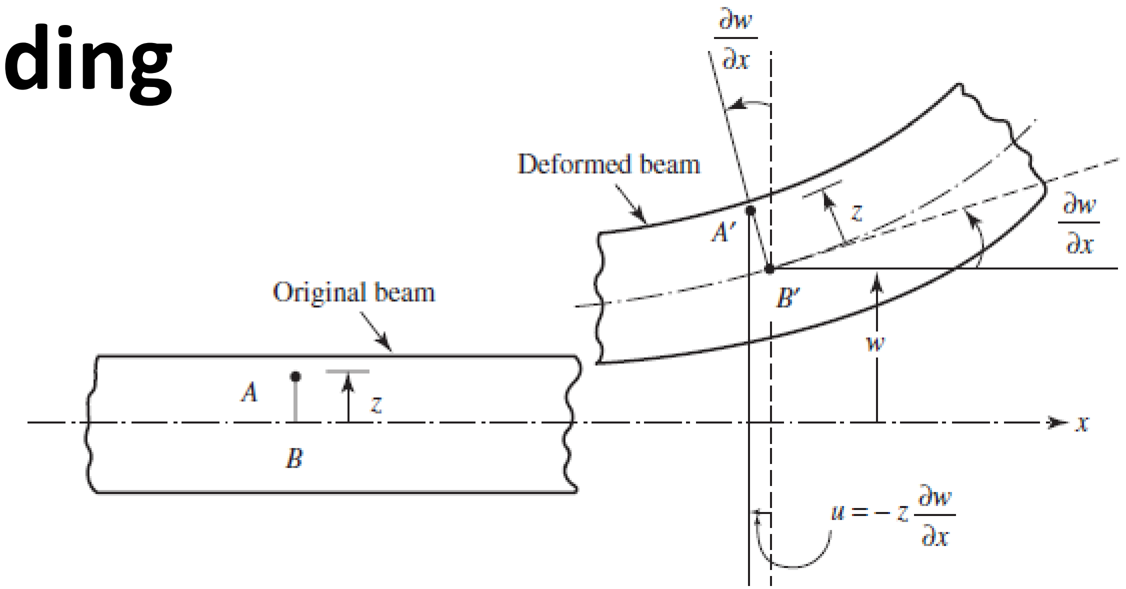
The length of the column is very long compared to its transverse dimensions.

Euler Theory: Theory of Bending

$$u = -z\frac{\partial w(x,t)}{\partial x},$$

$$v = 0,$$

$$w = w(x,t)$$



where *u*, *v*, and w denote the components of displacement parallel to *x*, *y*, and *z* directions, respectively. The components of strain and stress corresponding to this displacement field are given by

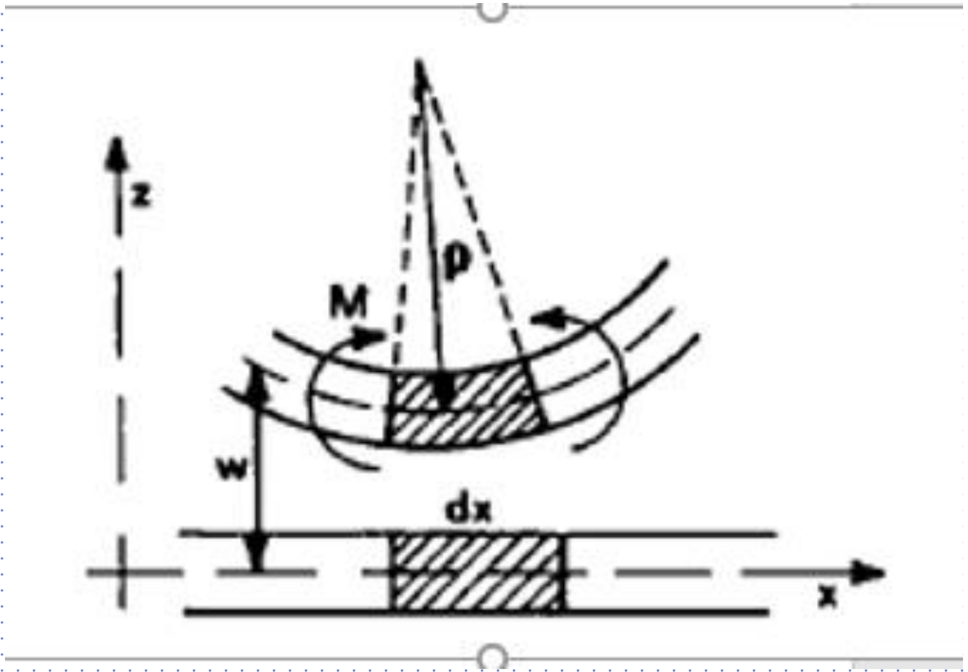
$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2},$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

$$\sigma_{xx} = -Ez\frac{\partial^2 w}{\partial x^2}$$

Euler Theory: Theory of Bending

$$\varepsilon = -\frac{z}{\rho}$$



$$\sigma = E\varepsilon = -E\frac{z}{\rho}$$

$$M = -\int_A \sigma z dA$$

$$M = \int_A E\frac{z^2}{\rho} dA = \frac{E}{\rho}\int_A z^2 dA = \frac{EI}{\rho}$$

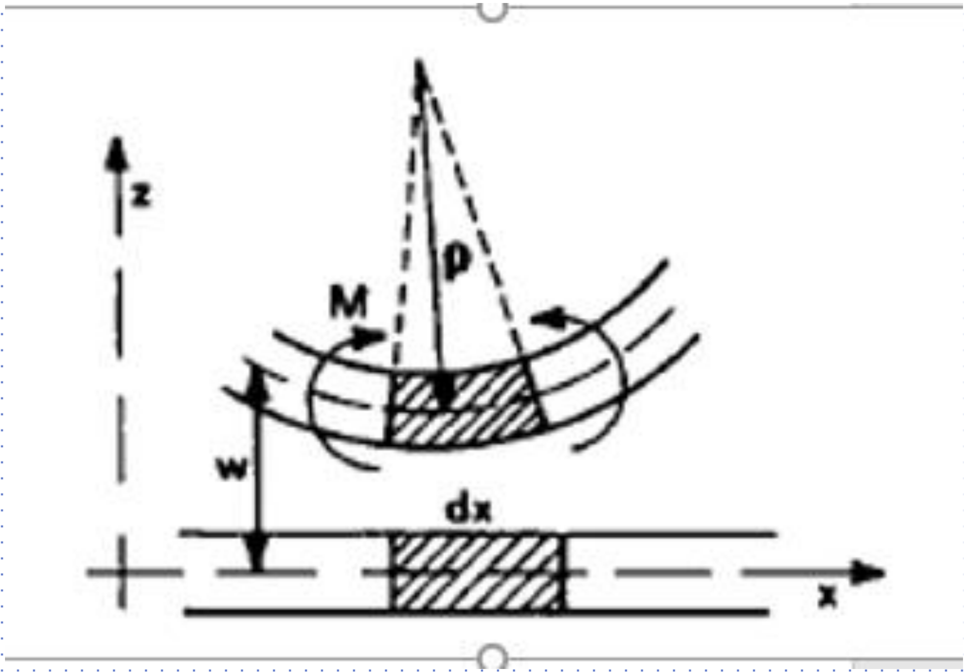
$$M = \frac{EI}{\rho}$$

Euler Theory: Theory of Bending

$$\varepsilon = -\frac{z}{\rho}$$

$$\sigma = E\varepsilon = -E\frac{z}{\rho}$$

$$M = -\int_A \sigma z dA$$



$$M = \int_A E\frac{z^2}{\rho}dA = \frac{E}{\rho}\int_A z^2 dA = \frac{EI}{\rho}$$

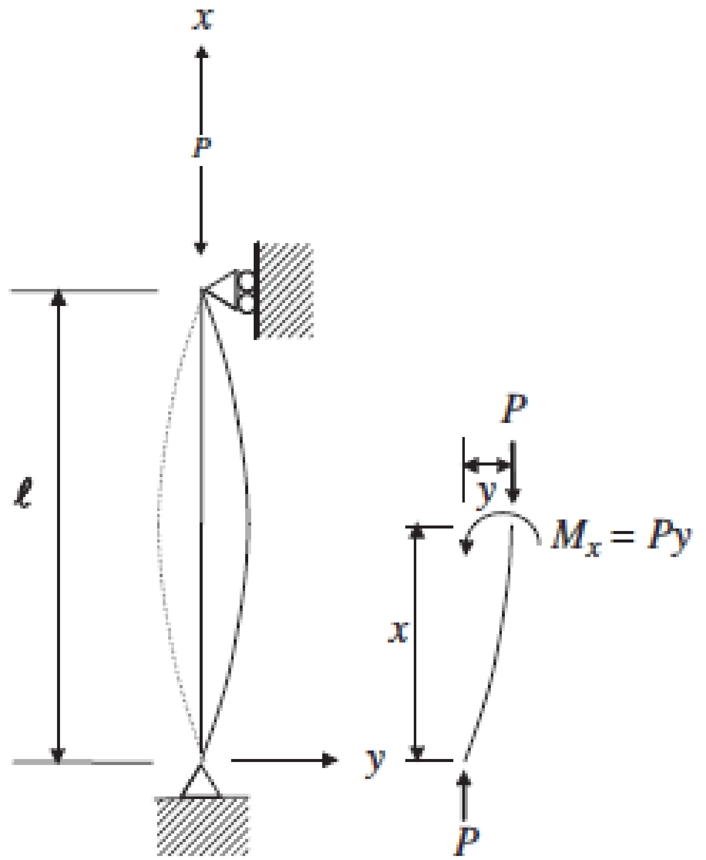
$$M = \frac{EI}{\rho}$$

Euler Theory: Theory of Bending

$$\frac{M}{EI} = \frac{1}{\rho} = w''$$

$$M = EIw''$$

$$EIy'' + Py = 0$$



Equation above is a second-order linear differential equation with constant coefficients. Its boundary conditions are

$$y = 0 \quad \text{at} \quad x = 0, \qquad x = l$$

Euler Theory: Theory of Bending

$$EIy'' + Py = 0$$

$$EIy'' + \frac{P}{EI}y = 0$$

$$y'' + k^2 y = 0$$

$$k^2 = \frac{P}{EI}$$

The general solution for *P>O* (compression) is

$$y = A\cos kx + B\sin kx$$

$$y = 0, x = 0 \rightarrow A = 0$$

$$y = 0, x = l \rightarrow B\sin kl = 0$$

Now we observe that the last equation allows a nonzero deflection if and only if

$$kl = \pi, 2\pi, 3\pi, ...$$

Euler Theory: Theory of Bending

$$kl = \pi, 2\pi, 3\pi, \ldots\ldots = n\pi$$

$$k = \frac{n\pi}{l}$$

$$n^2\pi^2 = \frac{Pl^2}{EI} \rightarrow P = \frac{n^2\pi^2 EI}{l^2}, n = 1, 2, 3\ldots\ldots.$$

$$\boxed{P = \frac{n^2\pi^2 EI}{l^2}, \qquad n = 1, 2, 3\ldots\ldots.}$$

$$y = B\sin\frac{n\pi x}{l}$$

If a pinned prismatic column of length L is going to buckle, it will buckle at *n* = 1 unless external bracings are provided in between the two ends.

Euler Theory: Theory of Bending

Critical stress

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(l/r)^2}$$

slenderness ratio $\frac{l}{r}$

radius of gyration of the cross section $r = \sqrt{\frac{I}{A}}$

$$\text{eigen pair}\begin{cases} \text{eigenvalue}: P_{cr} = \dfrac{n^2\pi^2 EI}{l^2} = \dfrac{n^2\pi^2 EI}{l_e^2} \\ \text{eigenvector}: y = B\sin\dfrac{n\pi x}{l} \end{cases}$$

Euler Theory

The Euler's buckling or critical load can be calculated for all these cases proceeding similarly as explained in the case of a column with both ends pin jointed. These results are:

For both ends fixed,

$$P_c = \frac{4\pi^2 EI}{L^2}$$

For one end fixed, other end free,

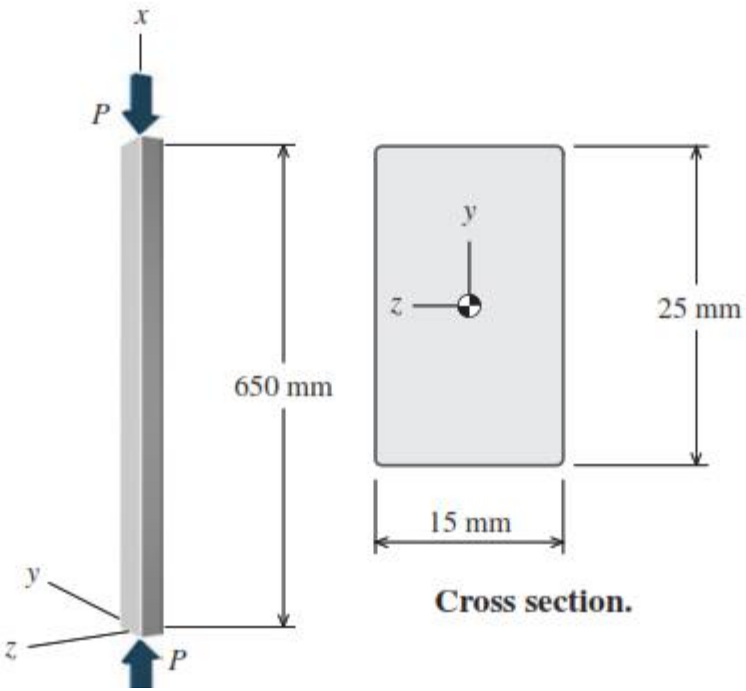
$$P_c = \frac{\pi^2 EI}{4L^2}$$

For one end fixed, other end pin jointed,

$$P_c = \frac{2\pi^2 EI}{L^2}$$

Euler Theory

A 15-mm by 25-mm rectangular aluminum bar is used as a 650-mm-long compression member. The ends of the compression member are pinned. Determine the slenderness ratio and the Euler buckling load for the compression member. Assume that $E = 70$ GPa.



Cross section.

$$I_y = \frac{(25 \text{ mm})(15 \text{ mm})^3}{12} = 7{,}031.25 \text{ mm}^4$$

The slenderness ratio is equal to the length of the column divided by its radius of gyration. The radius of gyration for this cross section with respect to the y axis is

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{7{,}031.25 \text{ mm}^4}{375 \text{ mm}^2}} = 4.330 \text{ mm}$$

and therefore, the slenderness ratio for buckling about the y axis is

$$\frac{L}{r_y} = \frac{650 \text{ mm}}{4.330 \text{ mm}} = 150.1$$

Euler Theory

A member of a pin jointed structure is 1.5m long with a cross section 10mm by 25mm. Determine the compressive force at which it will buckle. Take $E = 70\ N/mm^2$

$$I = \frac{bd^3}{12} = 2.08 \text{ x } 10^3 mm^4 \;\; ; \;\; A = 25 \text{ x } 10 = 0.25 \text{ x } 10^3 mm^2$$

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{2.08 \text{ X } 10^3}{0.25 \text{ X } 10^3}} = 2.88mm; \qquad \frac{L}{r} = \frac{1.5 \text{ X } 10^3}{2.88} = 520.8$$

$$L_e = L = 1.5m$$

$$P_c = \frac{\pi^2 EI}{{L_e}^2} = \frac{\pi^2 \text{x } 70 \text{ x } 10^3 \text{x } 2.08 \text{ x } 10^3}{1500^2}$$

$$= 638N$$

Euler Theory

A member of a pin jointed structure is 1.5m long with a cross section 10mm by 25mm. Determine the compressive force at which it will buckle. Take $E = 70\,N/mm^2$

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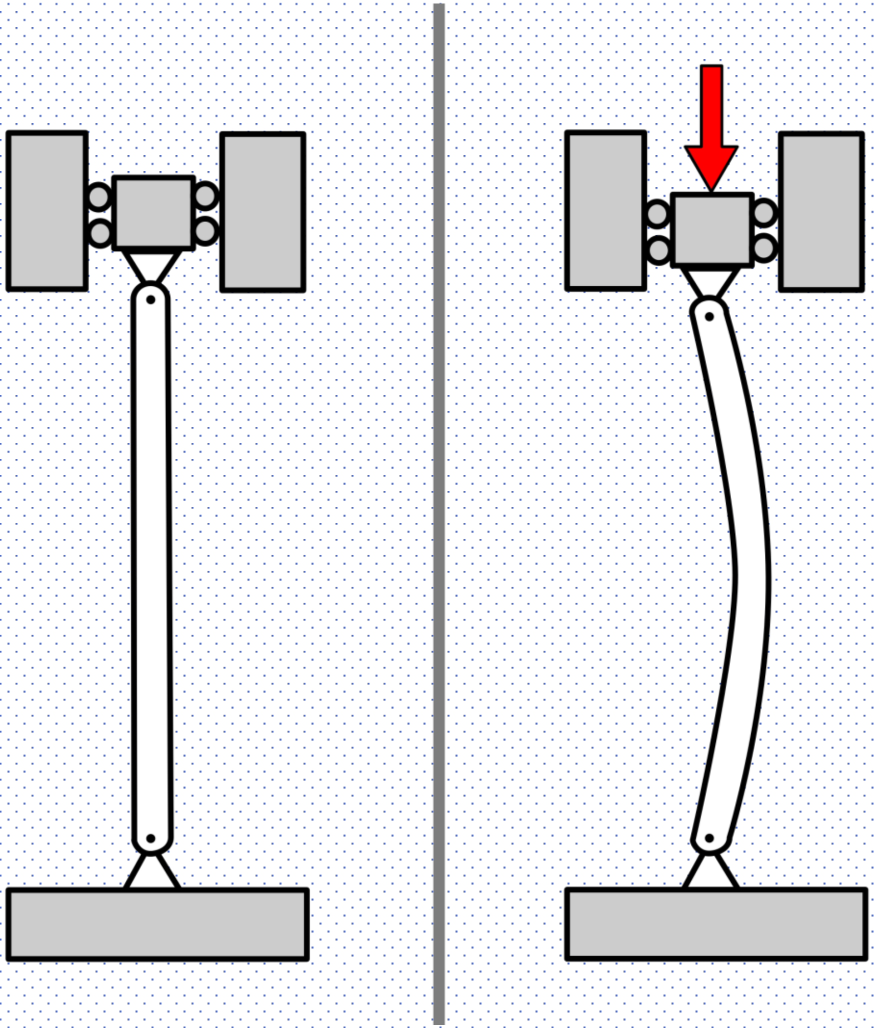
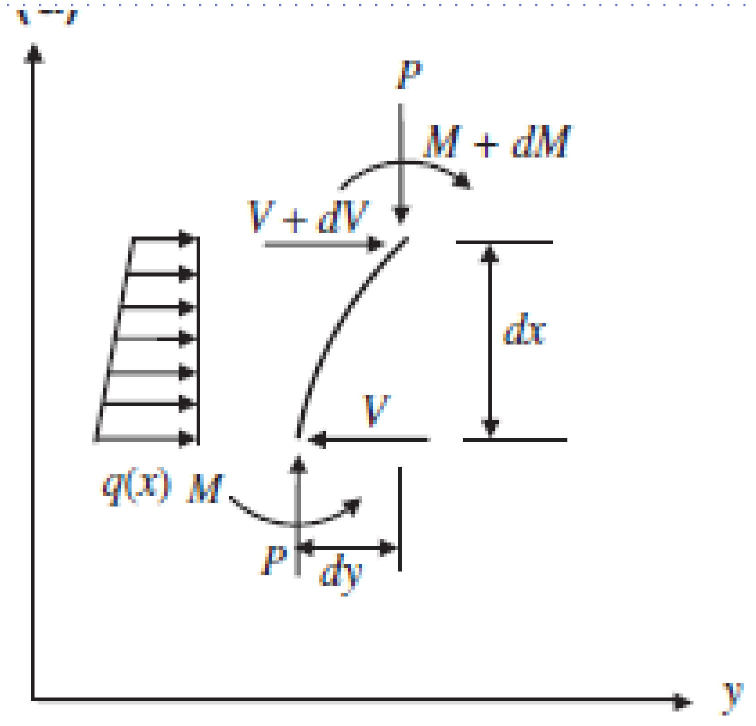
$$= 638N$$



Buckling Instability of Struts/ Columns

Bifurcation-type buckling is essentially flexural behaviour. Therefore, the free-body diagram must be based on the deformed configuration as the examination of equilibrium is made in the neighbouring equilibrium position. Summing the forces in the horizontal direction in



$$\sum F_y = 0 = (V + dV) - V + qdx$$

$$dV = V' = -qdx$$

Buckling Instability of Struts/ Columns

Summing the moment at the top of the free body gives

$$\sum M_{top} = (M + dM) - M + Vdx + Pdy - qdx\frac{dx}{2}$$

Neglecting the second-order term leads to

$$dM + Vdx + Pdy = 0 \Rightarrow \frac{dM}{dx} + P\frac{dy}{dx} = -V$$

$$M'' + (Py')' = -V'$$

Buckling Instability of Struts/ Columns

$$M'' + (Py')' = -V' \qquad V' = -qdx \qquad M = EIy''$$

Fundamental beam-column governing differential equation.

$$EIy^{iv} + Py'' = qdx$$

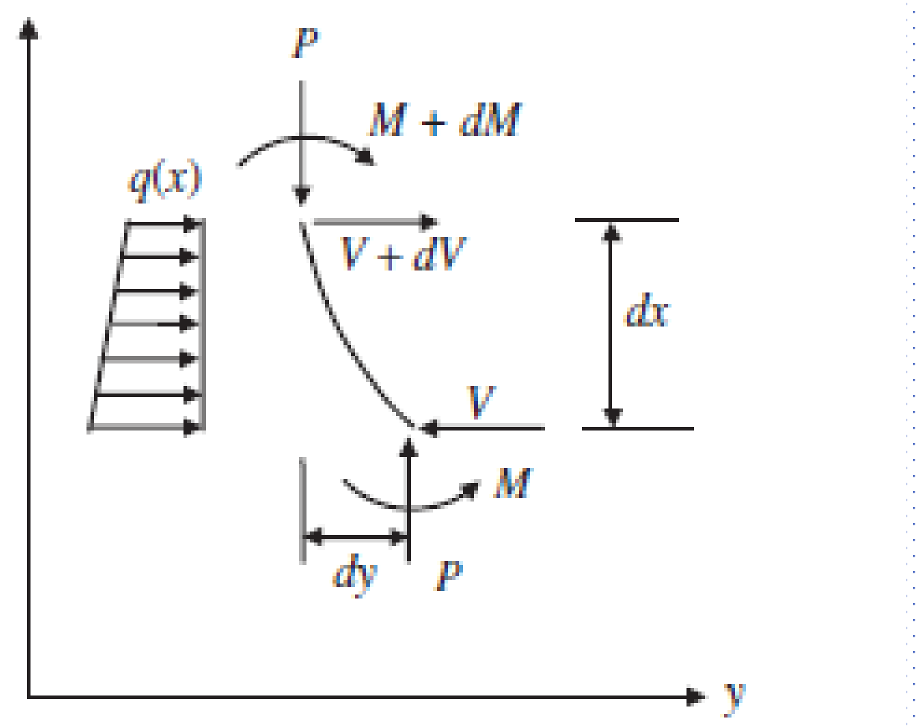
Buckling Instability of Struts/ Columns

$$EIy^{iv} + Py'' = qdx \qquad y^{iv} + k^2 y'' = 0, \quad k^2 = \frac{P}{EI}$$

The homogeneous solution of governs the bifurcation buckling of a column (characteristic behaviour). The concept of geometric imperfection (initial crookedness), material heterogeneity, and an eccentricity is equivalent to having nonvanishing q(x) terms

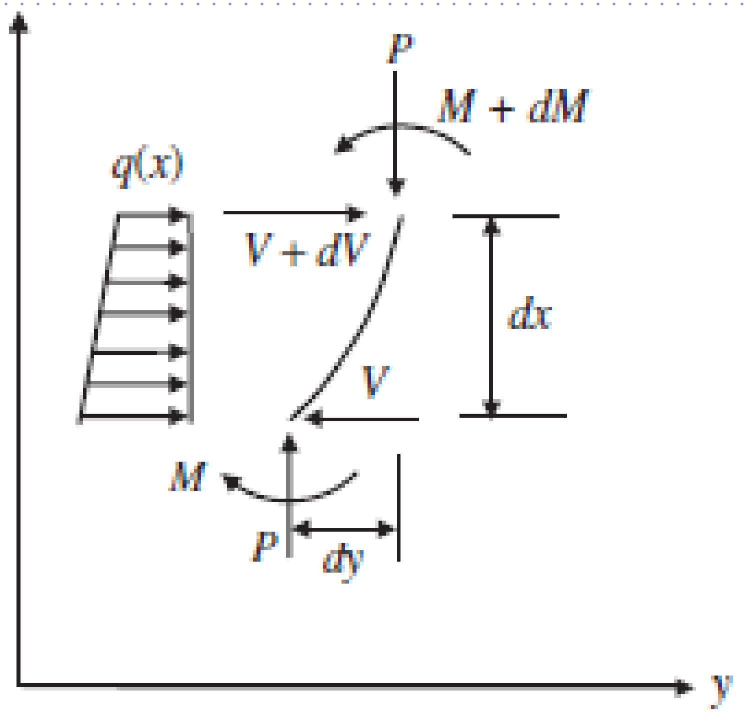
Assignment : Civil

Derive the beam-column governing differential equation of the following free body diagram.



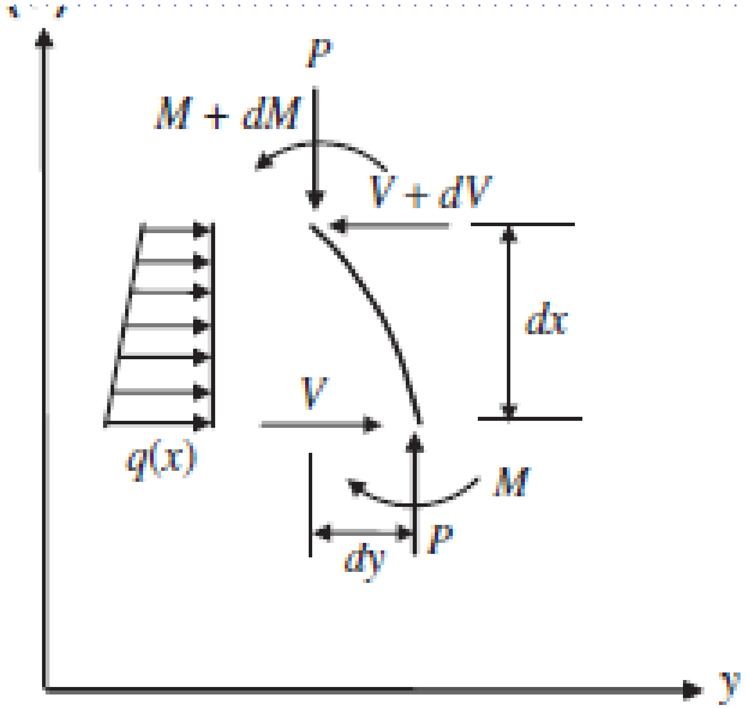
Assignment: System

Derive the beam-column governing differential equation of the following free body diagram.



Assignment: Building/QS

Derive the beam-column governing differential equation of the following free body diagram.



Buckling Instability of Struts/ Columns

$$y^{iv} + k^2 y'' = 0, \quad k^2 = \frac{P}{EI}$$

Assuming the solution to be of a form $y = \alpha e^{mx}$

$$y'' = \alpha m^2 e^{mx}; y^{iv} = \alpha m^4 e^{mx}$$

$$\alpha e^{mx}\left(m^4 + m^2 k^2\right) = 0$$

$$\alpha \neq 0; \quad e^{mx} \neq 0; \quad \rightarrow \quad \left(m^4 + m^2 k^2\right) = 0$$

Buckling Instability of Struts/ Columns

$$m^2\left(m^2+k^2\right)=0$$

$$m=\pm 0, \quad m=\pm ik$$

$$y_h=c_1e^{kix}+c_2e^{-kix}+c_3xe^0+c_4e^0$$

$$y_h=c_1e^{kix}+c_2e^{-kix}+c_3x+c_4$$

$$=A\sin kx+B\cos kx+Cx+D$$

Buckling Instability of Struts/ Columns

$$m^2\left(m^2+k^2\right)=0$$

$$m=\pm 0,\quad m=\pm ik$$

$$y_h=c_1e^{kix}+c_2e^{-kix}+c_3xe^0+c_4e^0$$

$$y_h=c_1e^{kix}+c_2e^{-kix}+c_3x+c_4$$

$$=A\sin kx+B\cos kx+Cx+D$$

Buckling Instability of Struts/ Columns

For a nontrivial solution for A, B, C, and D (or the stability condition equation), the determinant of coefficients must vanish. Hence,

$$D = \begin{vmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kl & \cos kl & l & 1 \\ k\cos kl & -k\sin kl & 1 & 0 \end{vmatrix} = 0$$

$$2\left(\cos kl - 1\right) + kl \sin kl = 0$$

Buckling Instability of Struts/ Columns

$$2(\cos kl - 1) + kl\sin kl = \sin\frac{kl}{2}\left(\frac{kl}{2}\cos\frac{kl}{2} - \sin\frac{kl}{2}\right) = 0$$

$$\sin\frac{kl}{2} = 0; \qquad \tan\frac{kl}{2} = \frac{kl}{2}$$

$$k = \frac{2n\pi}{l}; k^2 = \frac{P}{EI}$$

$$\frac{kl}{2} = \pi, 2\pi = n\pi$$

$$\left(\frac{2n\pi}{l}\right)^2 = \frac{P}{EI} \rightarrow P = \frac{4n^2\pi^2}{l^2}EI$$

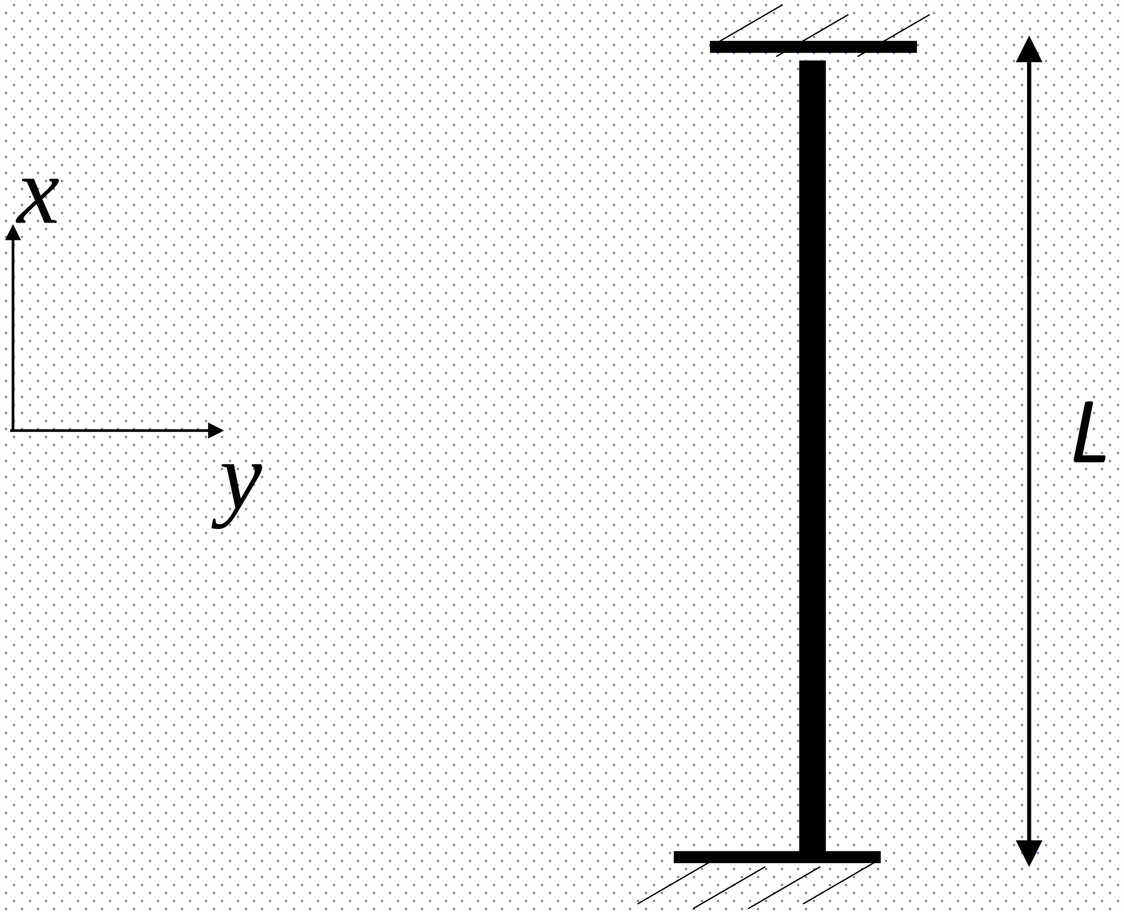
Buckling Instability of Struts/ Columns

$$y_h = A\sin kx + B\cos kx + Cx + D$$

$$y_h = A\sin\frac{2n\pi}{l}x + B\cos\frac{2n\pi}{l}x + Cx + D \quad \left.\right\} \quad \begin{aligned} &y = 0 @ x = 0;\\ &y = 0 @ x = l;\\ &y' = 0 @ x = 0;\\ &y' = 0 @ x = l \end{aligned}$$

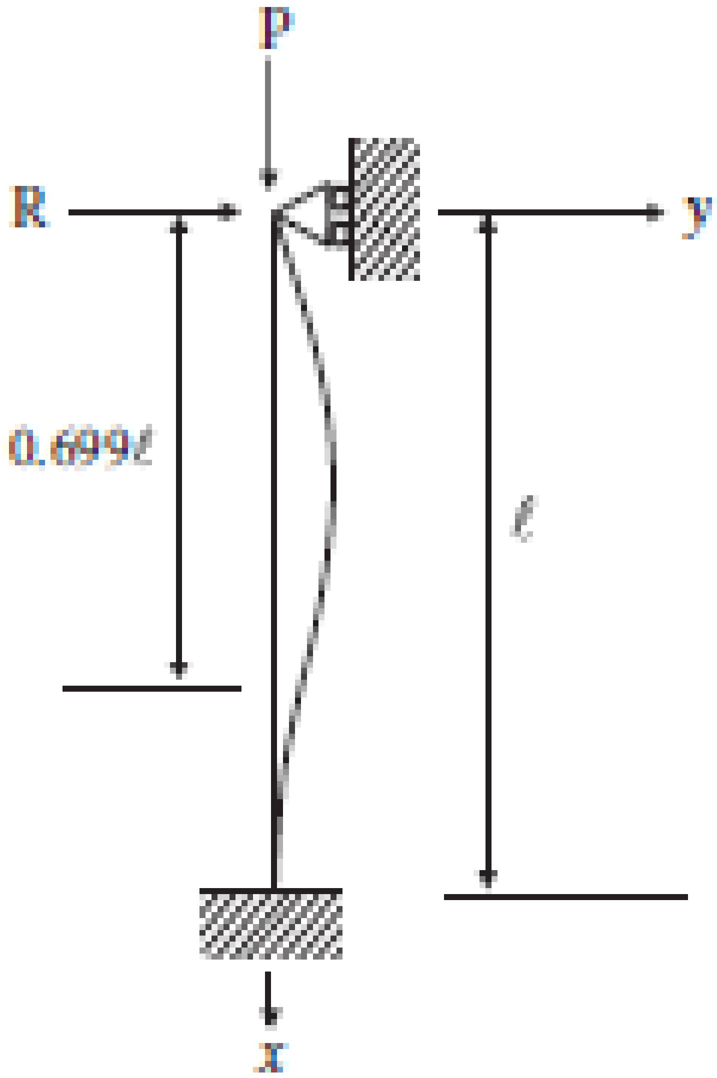
$$\text{eigen pair}\begin{cases} \text{eigenvalue}: P_{cr} = \dfrac{4n^2\pi^2}{l^2}EI = \dfrac{EI\pi^2}{(l/2)^2} = \dfrac{EI\pi^2}{l_e^2} \\ \text{eigenvector}: y = B\left(\cos\dfrac{2n\pi}{l}x - 1\right) \end{cases}$$

Short test

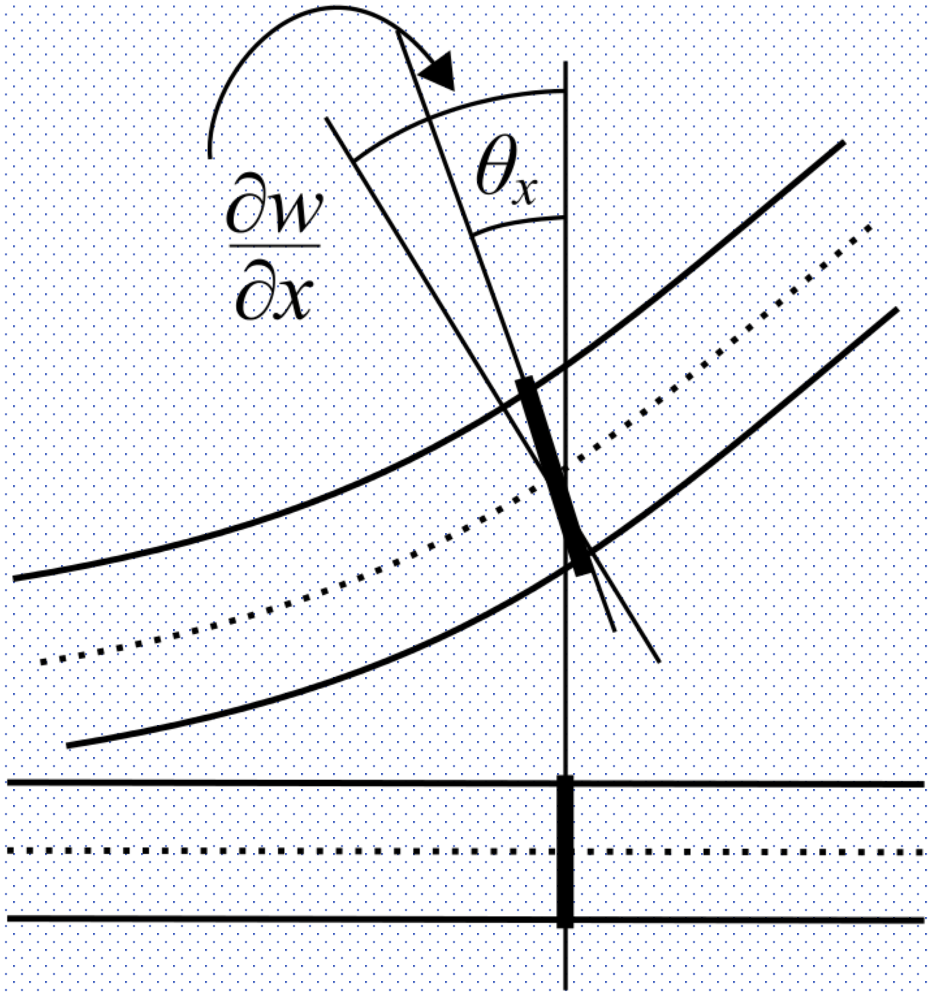


Determine the coefficients of the clamped clamped of the beam column shown

Assignment



Determine the eigenvector and eigen value for the propped column shown


$\frac{\partial w}{\partial x}$
θ_x

Thin Plates and Shells: Application

A plate is a structural element that is relatively thin in one direction compared with the other two, and is flat.
Plate has bending stiffness, whereas the membrane does not. Typically, the flexural (bending) stiffness arises because a plate is considerably thicker than a membrane relative to its other dimensions.

Thin Plates and Shells: Application

Plates are important structural elements. They may exist in many applications. In civil engineering, flat panels exist in various steel or concrete structures (e.g., floor slabs). They may be of various shapes (rectangular, circular, rhombic, triangular, trapezoidal, and others).

Thin Plates and Shells: Application

Plates also occur in aerospace (e.g., aircraft, missile) and naval (e.g.,ship, submarine) structures.

Thin Plates and Shells: Application

In mechanical engineering, plates can be seen as rotor disks in brake systems, parts of various clutch and other components. They can also exist as flat panels in machine housings.

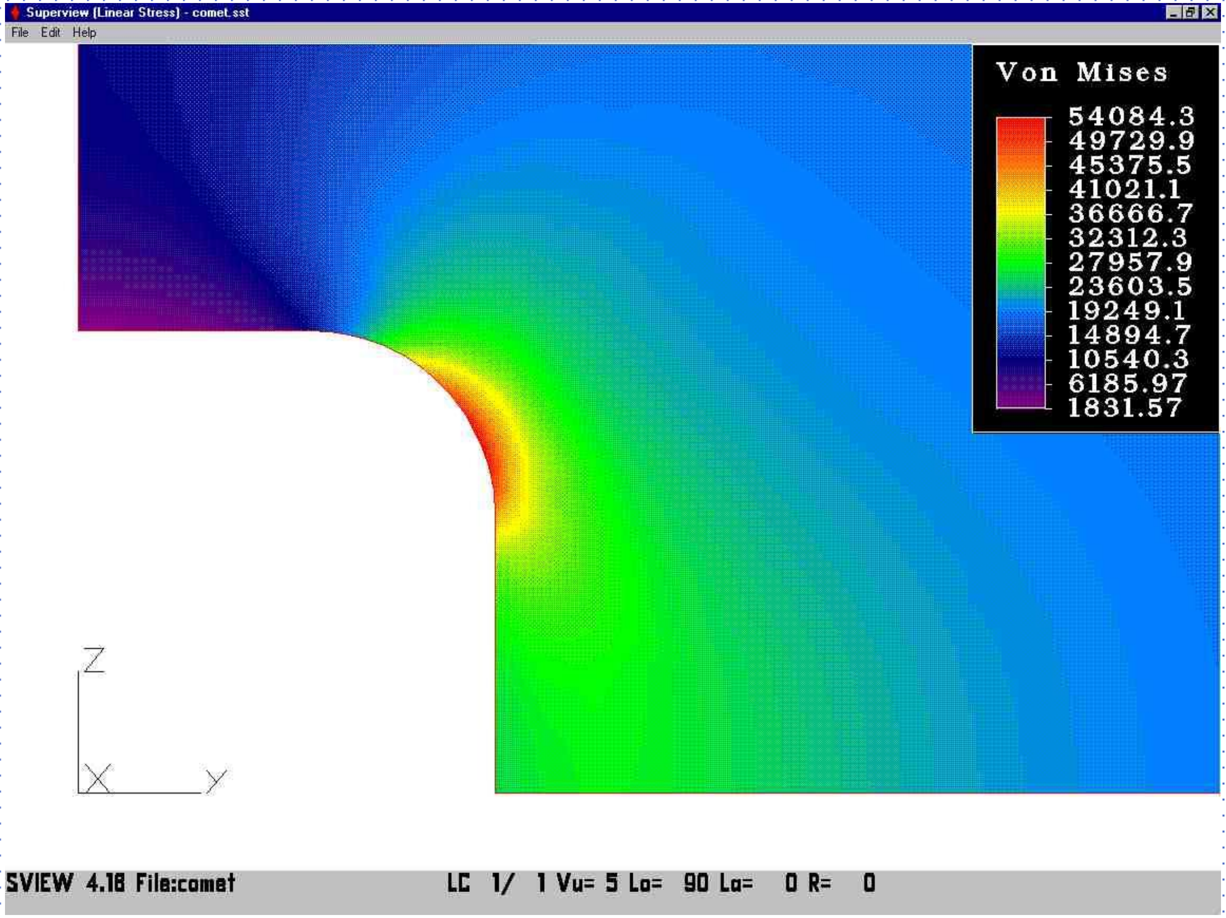
Thin Plates and Shells: Application

Geometrically viewed, a shell is like a plate, except that it has curvature. Whereas a plate is flat, a shell is not. Nevertheless, like a plate it has one dimension, which we call its thickness (h), which is small compared to its other dimensions. The thickness need not be constant, but in many practical applications it is. And like a plate, deformation of a shell is characterized entirely by what happens at its mid surface and the normal to the mid-surface.

Thin Plates and Shells: Application

Shell theory represents the deformations of a three dimensional body by equations which are mathematically two dimensional.

That is, only two independent space variables are needed to unequivocally define what is occurring at every point within the shell, instead of three.



Superview (Linear Stress) - comet.sst
File Edit Help
Von Mises
54084.3
49729.9
45375.5
41021.1
36666.7
32312.3
27957.9
23603.5
19249.1
14894.7
10540.3
6185.97
1831.57
Z
X
Y
SVIEW 4.18 File:comet
LC 1/ 1 Vu= 5 Lo= 90 La= 0 R= 0

Introduction

Materials used for structural applications of practical interest may exhibit viscoelastic behavior that has a profound influence on the performance of that material.

Materials used in engineering applications may exhibit viscoelastic behavior as an unintentional side effect.

In applications, one may deliberately make use of the viscoelasticity of certain materials in the design process, to achieve a particular goal.

Introduction

The mathematics underlying viscoelasticity theory is of interest within the applied mathematics community.

Viscoelasticity is of interest in some branches of materials science, metallurgy, and solid state physics because it is causally linked to a variety of microphysical processes and can be used as an experimental probe of those processes.

Viscoelastic Phenomena

Most solid materials are described, for small strains, by Hooke's law of linear elasticity: stress is proportional to strain . In one dimension, Hooke's law is as follows:

$$\sigma = E\varepsilon$$

$$\varepsilon = J\sigma$$

$$J = \frac{1}{E}$$

Consequently, the elastic compliance J *is the inverse of the modulus E:*

In contrast to elastic materials, a viscous fluid under shear stress obeys

$$\sigma = \eta \frac{d\varepsilon}{dt}$$

In reality, all materials deviate from Hooke's law in various ways, for example, by exhibiting, both viscous-like and elastic characteristics.

Viscoelastic materials are those for which the relationship between stress and strain depends on time

Viscoelastic Phenomena

The response to step strain is stress relaxation, and the response to step stress is creep. Some phenomena in viscoelastic materials are:

- if the stress is held constant, the strain increases with time (creep);
- if the strain is held constant, the stress decreases with time (relaxation);
- the effective stiffness depends on the rate of application of the load;
- if cyclic loading is applied, hysteresis (a phase lag) occurs, leading to a dissipation of mechanical energy;
- acoustic waves experience attenuation;
- rebound of an object following an impact is less than 100 percent; and
- during rolling, frictional resistance occurs.

Viscoelastic Phenomena

All materials exhibit some viscoelastic response.

In common metals, such as steel or aluminum, as well as in quartz, *at room temperature* and at *small strain*, the behavior does not deviate much from the behavior of *linearly elastic materials*.

Synthetic polymers, wood, and human tissue, as well as metals, at *high temperature display large viscoelastic effects*.

In some applications, even a small viscoelastic response can be significant.

To be complete, an analysis or design involving such materials must incorporate their viscoelastic behavior.

Transient Properties: Creep and Relaxation

Creep is a progressive deformation of a material under constant stress. In one dimension, suppose the history of stress σ as it depends on time to be a step function with the magnitude σ_0, beginning at time zero:

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Transient Properties: Creep

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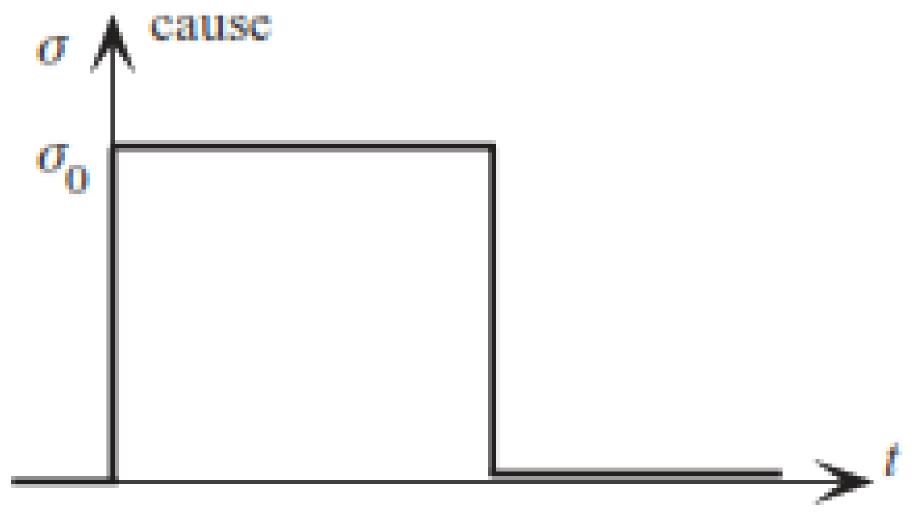
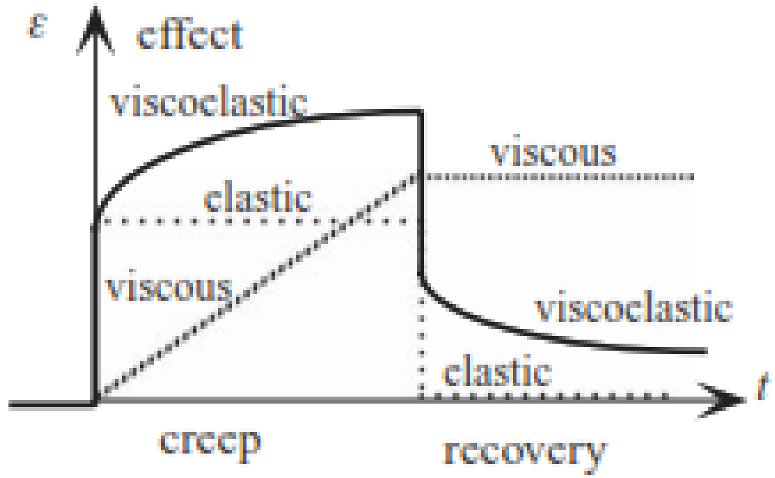


$$\sigma(t) = \sigma_0 H(t)$$

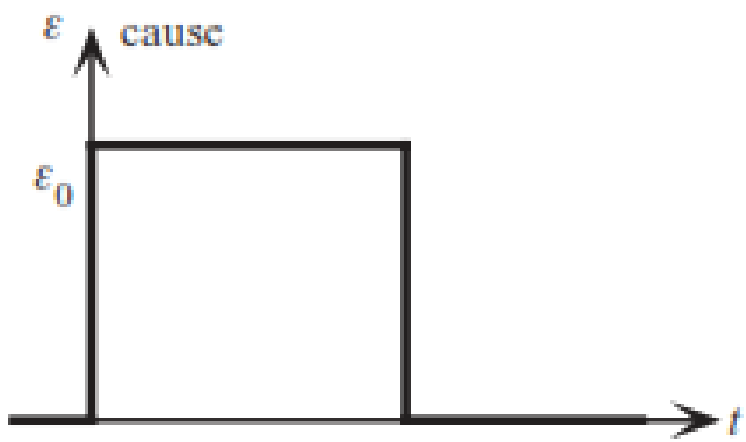


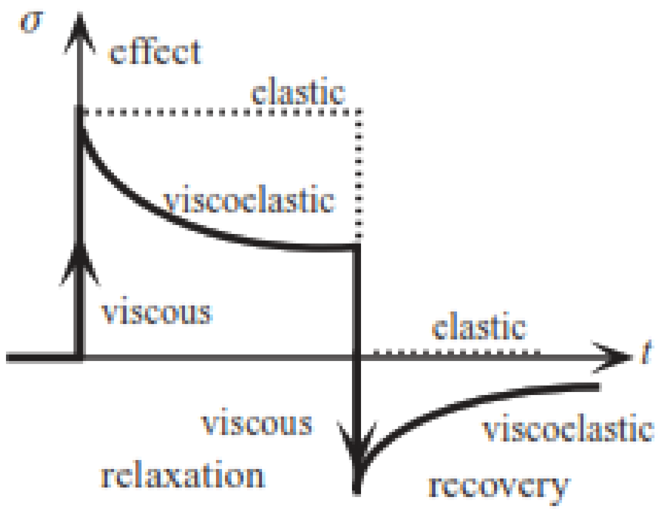
$$J(t) = \frac{\varepsilon(t)}{\sigma_0}$$

J(t) = creep compliance

Transient Properties: Relaxation

Stress relaxation is the gradual decrease of stress when the material is held at constant strain. If we suppose the strain history to be a step function of magnitude ε_0 beginning at time zero: $\varepsilon(t) = \varepsilon_0\mathcal{H}(t)$, the stress $\sigma(t)$ in a viscoelastic material will decrease. The ratio,





$$E(t) = \frac{\sigma(t)}{\varepsilon_0}$$

J(t) = relaxation modulus

NOTHING IS

IMPOSSIBLE

THE WORD

ITSELF SAYS

"I'M POSSIBLE"

~ Audrey Hepburn ~