FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

First Order ODEs

- Types and Techniques of solution of first order ODE’s
  * Integration Methods
    - Direct Integration
    - Separation of Variables
    - Integrating Factor
  * Substitution Methods
    - Direct Substitution
    - Homogeneous 1^{st} order equations
    - Bernoullis’s Equations
- Exact Differential Equations
- Numerical methods for solving ODEs
  * Euler’s Method
  * Improved Euler’s Method
  * Runge-Kutta Method
  * Picard’s iterative method
- Physical applications of first order ODE.

2.0 INTRODUCTION

What is a first order linear differential equation.

This is a linear differential equation where the highest power of the derivative is 1 (first order).y(x) is

$$y' = \frac{dy}{dx} = f(x, y)$$

2.1 DIFFERENTIAL FORM OF LINEAR ODE

If $f(x, y) = \frac{M(x, y)}{-N(x, y)}$

$$\frac{dy}{dx} = \frac{M(x, y)}{-N(x, y)}$$

$M(x,y) = -N(x,y)$

$M(x,y) + N(x,y) = 0$

The general standard form for a first order ODE in the function

Sample questions.

$y(yy' - x) = x$

$y^2 y' - y = x$

$y' = \frac{x+y}{y^2}$

$\frac{dy}{dx} = \frac{x+y}{y^2}$

$y^2 dy = (x+y)dx$

Write the differential equation

2.2 STANDARD FORM OF FIRST ORDER LINEAR ODE

$y' + p(x)y = q(x)$

2.3 METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATIONS

1. Direct Integration
2. Means of Separating Variables
3. Integrating Factor Method
4. Subsitution Methods
5. Picard's Iterative Method

2.3.1 DIRECT INTEGRATION METHOD

You have an independent variable x and the function, y, is in terms of x

If $\frac{dy}{dx} = f(x)$, $f(x)$ is a function of the independent variable only

$y = \int f(x)dx$

Examples:

1. Solve the initial value problem

$$\frac{dy}{dx}=3\cos 2x-8e^{-4x}+5$$

By integrating,

$$y=\int 3\cos 2x-8e^{-4x}+5$$

$$y=\frac{3}{2}\sin 2x+2e^{-4x}+5x+C$$

On solving for the initial problem,

$$C=-5$$

$$y=\frac{3}{2}\sin 2x+2e^{-4x}+5x-5$$

2. Solve the IVP

$$\frac{dy}{dx}=y+3$$

$$\frac{dx}{dy}=\frac{1}{y+3}$$

Integrate with respect to y

$$x=\int \frac{1}{y+3}dy=\ln|y+3|+c$$

$$\ln|y+3|=x-c$$

$$y+3=\pm e^{x-c}$$

$$y+3=\pm e^{-c}e^{x}=Ke^{x}$$

$$y=Ke^{x}-3$$

Using IC(Initial Conditions)

$$-2=Ke^{1}-3$$

$$Ke^{1}=1$$

$$K=e^{-1}$$

$$y=e^{-1}\cdot e^{x}-3$$

$$y=e^{x-1}-3$$

2.3.2 METHOD OF SEPARATING VARIABLES

This method is used to solve separable differential equations.

$$\frac{dy}{dx}=f(x,y)=f(x)g(y)$$

You'll see that the function of x and y, $f(x,y)$ is actually a product of the **function of x** and the **function of y**.
Given the function:

$$\frac{dy}{dx}=f(x)g(y)$$

2.3.2.1 STEPS

1. **Separate variables**: On one side, we want only **y-variables** and **dy**. On the other side, we want only **x-variables** and **dx**'s.

$$\frac{dy}{g(y)}=f(x)dx$$

2. Integrate both sides.

$$\int\frac{dy}{g(y)}=\int f(x)dx$$

3. Add the constant of integration to the x-side of the integrals
4. Solve and make y the subject of the formula

The formula you get is called the **general solution**. If you are given an initial value, $y(1)=2$, then you find the **specific solution**

Solve the following

1. $\frac{dy}{dx}=\frac{x^2}{y^2}$. Given the initial position $y(1)=2$, then you should find the specific solution
Answers: $y=\sqrt[3]{x^3+k}$and $y=\sqrt[3]{x^3+7}$
2. $y'=xy$At point $y(0)=5$ Answer: $y=5e^{\frac{1}{2}x^2}$
3. $\frac{dy}{dx}=y^2+1$At $y(1)=0$
4. $e^{12x+5y}dx+e^{3x-4y}dy=0$
5. Solve $\frac{dy}{dx}=y-4xy$, $y(0)=e$: General Solution: $y=Ae^{x-2x^2}$

Final solution: $y=e^{x-2x^2+1}$

6. Solve $x\frac{dx}{dt}=3e^{t-x}$ Implicit solution: $xe^x-e^x=3e^t+c$

2.3.3 INTEGRATING FACTOR METHOD

This is also used to solve the first order linear differential equations.

1. Write the equation in the **standard form** the first order linear differential equation.

$y'+p(x)y=q(x)$

2. Identify the functions p(x) and q(x)
3. Determine the integrating factor

$I(x)=e^{\int p(x)dx}$

4. Multiply through by the integrating factor.

$I(x)y'+I(x)p(x)y=I(x)q(x)$

5. This can be reduced to the form:

$\frac{d}{dx}[yI(x)]=q(x)I(x)$

6. On integrating,

$yI(x)=\int q(x)I(x)dx+C$

7. Write the general solution

$y=\frac{1}{I(x)}\left[\int I(x)q(x)dx+c\right]$

Solve the following questions

1. $y'+2y=2e^x$ Answer: $y=\frac{2}{3}e^x+Ce^{-2x}$
2. $xy'+4y=2x^3$ Answer: $y=\frac{2}{7}x^3+\frac{C}{x^4}$
3. $(x-2)y'+y=x^2-4$
4. $(4y-3x)dx+5xdy=0$ Answer: $y=\frac{1}{3}x^{\frac{61}{20}}+C$
5. $\frac{dy}{dx}-3y=e^{3x}\sin 4x$
6. $t\frac{dy}{dt}-2y=t^3e^{-2t}$

2.3.4 SUBSTITUTION METHODS

2.3.4.1 DIRECT SUBSTITUTION

Let u = “inside function”

Rewrite the DE using the new variable

For example, Solve:

$$\frac{dy}{dx}=(y+4x-3)^2, \quad y(0)=4$$

Let $u=y+4x-3$

$$\frac{du}{dx}=\frac{dy}{dx}+4$$

$$\frac{dy}{dx}=\frac{du}{dx}-4$$

$$u^2=\frac{du}{dx}-4$$

$$\frac{du}{dx}=u^2+4$$

$$\frac{du}{u^2+4}=dx$$

$$\int\frac{du}{u^2+4}=\int dx$$

$$\tan^{-1}u=x$$

$$u=\tan(x+c)$$

$$u=y+4x-3=\tan(x+c)$$

$$y=3-4x+\tan(x+c)$$

$$y(0)=4$$

$$c=\frac{\pi}{4}$$

$$y=3-4\pi+\tan\left(x+\frac{\pi}{4}+k\pi\right)$$

2.3.4.2 HOMOGENEOUS EQUATIONS

A homogeneous equation is an equation that can be written in the form: $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$

A homogeneous differential equation is a type of differential equation that can be written in the form $\frac{dy}{dx}=f(x,y)$, where f is a function of the ratio $\frac{y}{x}$. This means that both sides of the equation are homogeneous functions of the same degree of x and y.

Steps to solve:

1. Let $v=\frac{y}{x}$

Therefore, $y=vx$

$\frac{dy}{dx}=v+xv'$

2. Rewrite the equation in v and x variables
3. Solve for v first, and then use y=vx to solve for y.

Examples:

1. $xy\frac{dy}{dx}=x^2+y^2$, $y(1)=\sqrt{2}$

Dividing by xy

$$\frac{dy}{dx}=\frac{x^2+y^2}{xy}$$

$$\frac{dy}{dx}=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$\frac{dy}{dx}=\frac{x}{y}+\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{1}{\frac{y}{x}}+\frac{y}{x}$$

Let $v=\frac{y}{x}$

$$y'=v+xv'$$

$$v+xv'=\frac{1}{v}+v$$

$$xv'=\frac{1}{v}$$

$$\frac{dv}{dx}x=\frac{1}{v}$$

$$vdv = \frac{dx}{x}$$

$$\int vdv = \int \frac{1}{x} dx$$

$$\frac{v^2}{2} = \ln|x| + c$$

But $v = \frac{y}{x}$

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln|x| + c$$

$$y^2 = 2x^2(\ln|x| + c)$$

Using $y(1) = \sqrt{2}$

Example 1: $\frac{dy}{dx} = \frac{2x+y}{x-2y}$

This is a homogeneous differential equation, since both sides are homogeneous functions of degree one. We can solve it by substituting $y = vx$ and separating the variables.

$$\frac{dy}{dx} = \frac{2x+y}{x-2y}$$

$$\frac{dy}{dx} = \frac{2x+vx}{x-2vx}$$

$$\frac{dy}{dx} = \frac{x(2+v)}{x(1-2v)}$$

$$\frac{d(vx)}{dx} = \frac{2+v}{1-2v}$$

$$x\frac{dv}{dx} + v\frac{dx}{dx} = \frac{2+v}{1-2v}$$

$$x\frac{dv}{dx} + v = \frac{2+v}{1-2v}$$

$$x\frac{dv}{dx} = \frac{2+v}{1-2v} - v$$

$$x\frac{dv}{dx} = \frac{2+v}{1-2v}$$

Example 2: $\frac{dy}{dx} = \frac{(x^2+y^2)}{(xy)}$

2.3.4.3 The Bernoulli’s Equation

This is an extension of the first order linear equation

$\frac{dy}{dx}+p(x)y=q(x)y^n$

If $n=0, \frac{dy}{dx}+p(x)y=q(x)$ and you use the integrating factor method

If $n=1$, $\frac{dy}{dx}+p(x)y=q(x)y$, there are two ways to go about this:

a. $\frac{dy}{dx}+[p(x)-q(x)]y=0$, then we use the integrating factor method.

b. $\frac{dy}{dx}=y[q(x)-p(x)]$, then use separation method

If $n\neq 0,1$, dividing by y^n

$\frac{1}{y^n}\frac{dy}{dx}+p(x)y^{1-n}=q(x)$

Then use the subsitution:

$v=y^{1-n}$

Taking the derivatives,

$v’=(1-n)y^{-n}y'$From the chain rule

$y^{-n}y'=\frac{v’}{(1-n)}$

Where $y’=\frac{dy}{dx}$

$\frac{1}{y^n}\frac{dy}{dx}+p(x)y^{1-n}=q(x)$

$\rightarrow\frac{1}{1-n}v'+p(x)v=q(x)$

This can be seen to be the first order linear equation in v

Then we solve for v using the integrating factor method

Use $v=y^{1-n}$ to find y.

Example: Solve $2x\frac{dy}{dx}-y=6x^2y^3$

Dividing by xy^3

$2y^{-3}\frac{dy}{dx}-\frac{1}{x}y^{-2}=6x$

$v=y^{-2}$

$v' = -2y^{-3}y'$

$y^{-3}y' = -\frac{1}{2}v'$

Substituting

$2 \cdot -\frac{1}{2}v' - \frac{1}{x}v = 6x$

$v' + \frac{1}{x}v = -6x \rightarrow$ 1st order linear in v

Multiplying through by x

$xv' + v = -6x^2$

$\frac{d}{dx}(vx) = -6x^2$

On integrating

$vx = -2x^3 + C$

$v = -2x^2 + \frac{C}{x}$

Using $v = y^{-2}$

$y^{-2} = -2x^2 + \frac{C}{x}$

$y^{-2} = \frac{C - 2x^2}{x}$

$y^2 = \frac{x}{C - 2x^2}$

$y = \pm\sqrt{\frac{x}{C - 2x^2}}$

2.4 EXACT AND ALMOST EXACT DIFFERENTIAL EQUATIONS

2.4.1 EXACT DIFFERENTIAL EQUATITIONS

The standard form of a differential equation is written as

$M(x,y)\,dx + N(x,y)\,dy = 0$

It can also be written in the form by dividing through by dx

if $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$, then the equation is called EXACT differential equation, and there is a function

$f(x,y) = C$, such that $\frac{\delta f}{\delta x} = M$and $\frac{\delta f}{\delta y} = N$.

Solve the function, $f(x,y)$ for y. Then the equation of y is the solution

2.4.1.1 METHODS OF SOLVING

There are two major methdos to solving the exact differential equations

1. Solution Method:
2. Shortcut Method

2.4.1.1.1 SOLUTION METHOD

$$\frac{\delta f}{\delta x}=M(x,y)$$

Integrate with respect to x.

$$f(x,y)=\int M(x,y)dx+g(y)$$

Since it is coming from a partial derivative, there is a possibility that there will be a function of x that will be considered as a constant, so we add it back to the equation when integrating.

Next, differentiate with respect to y

$$\frac{\delta f}{\delta y}=\frac{\delta}{\delta y}\int M(x,y)dx+g'(y)$$

Next we want to find the equation g(y). Therefore, set $g'(y)=N(x,y)$and then find $g'(y)$, then integrate to find $g(y)$

Substituting $f(x,y)=\int M(x,y)dx+g(y)=C$

Solve for y, if possible. This is because there may not be a solution for y. If there is, that will be the implicit

2.4.1.1.2 SHORTCUT METHOD

$$\frac{\delta f}{\delta x}=M(x,y)$$

Integrate with respect to x

Then you get the function $f(x,y)=\int M(x,y)dx$

$$\frac{\delta f}{\delta y}=N$$

Integrate with respect to y

$f(x,y)=\int N(x,y)\,dy$

f(x,y) – Write common terms and uncommon terms once

f(x,y)= C

Solve for y, if possible.

Examples:

Solve: $(3e^x - y^2)dx+(3-2xy+\sin y)dy=0$

$M(x,y)=3e^x - y^2$

$\frac{\delta M}{\delta y}=-2y$

$N(x,y)=3-2xy+\sin y$

$\frac{\delta N}{\delta x}=-2y$

Since, $\frac{\delta M}{\delta y}=\frac{\delta N}{\delta x}$, the equation is an exact differential equation.

This means there must be a function z = f(x, y) = C such that

$\frac{\delta f}{\delta x}=M(x,y)=3e^x - y^2$

Integrate with respect to x

$f(x,y)=3e^x - xy^2+g(y)$

Differentiate with respect to y.

$\frac{\delta f}{\delta x}=0-2xy+g'(y)=N(x,y)$

$-2xy+g'(y)=3-2xy+\sin y$

It can be seen that

$g'(y)=\sin y+3$

Now we integrate with respect to y

$\int g'(y)\,dy=\int \sin y+3\,dy$

$g(y)=3y-\cos y$

$f(x,y)=3e^x - xy^2+3y-\cos y=C$

Since there is no way you can solve for y, the implicit solution is

$3e^x - xy^2+3y-\cos y=C$

2.4.2 ALMOST EXACT EQUATION

$M(x,y)dx+N(x,y)dy=0$

If $\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$, then the equation is not exact so…

We will use the integrating factor method to solve.

There are two cases to consider:

1. **The MNN case**: If $\frac{M_y - N_x}{N}$ is a function of x only, then the integrating factor is given as $I(x)=e^{\int \frac{M_y - N_x}{N} dx}$
2. **The NMM case**: If $\frac{N_x - M_y}{M}$ is a function of y only, then the integrating factor is given as $I(y)=e^{\int \frac{N_x - M_y}{M} dy}$

Then if you multiply the standard form by the integrating factors, you get…

$(IM)dx+(IN)dy=0$ → This is going to form an exact equation.

Solve: $xy\,dx+(2x^2+3y^2-1)dy=0$

$M=xy$

$\frac{\delta M}{\delta y}=x$

$N=2x^2+3y^2-1$

$\frac{\delta N}{\delta x}=4x$

Since they are not the same, it is not an exact equation.

$\frac{M_y - N_x}{N}=\frac{x-4x}{2x^2+3y^2-1}$→ Since this is not a function of a single variable, we cannot use this expression.

$\frac{N_x - M_y}{M}=\frac{4x-x}{xy}=\frac{3x}{xy}=\frac{3}{y}$→ This is a function of y only

$I(y)=e^{\int \frac{N_x - M_y}{M} dy}=e^{\int \frac{3}{y} dy}=e^{3\ln y}=y^3$

Multiply by the integrating factor y^3

$xy^4dx+(2x^2y^3+3y^5-y^3)dy=C$

Since this is exact, we use the shortcut method.

$\frac{\delta f}{\delta x}=xy^4$

On integrating

$f(x,y)=\frac{x^2}{2}y^4$

$\frac{\delta f}{\delta y}=2x^2y^3+3y^5-y^3$

On integrating with respect to y:

$f(x,y)=\frac{1}{2}x^2y^4+\frac{1}{2}y^6-\frac{1}{4}y^4$

If you look at both equations you'll see that the term $\frac{1}{2}x^2y^4$is common to both.

So we write the common terms first

$f(x,y)=\frac{1}{2}x^2y^4+\ldots$

Next we write the uncommon terms to both

$f(x,y)=\frac{1}{2}x^2y^4+\frac{1}{2}y^6-\frac{1}{4}y^4=0$

Since this is not easy to solve for y, this is the implicit solution for y.

2.5 NUMERICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

1. Euler's Method
2. Improved Euler's Method
3. Runge-Kutta Method
4. Picard's Iteration Method

2.5.1 EULER'S METHOD

You are going to look at the slopes at each point and use the tangent lines to approximate the next point.

$\frac{dy}{dx}=f(x,y)$, $y(x_o)=y_o$

$x_{n+1}=x_n+h$

$y_{n+1}=y_n+hf(x_n,y_n)$

h → gap between x values

We are given the slope at every point (the derivatives), and the initial values

Example:

Approximate y(2) using eulers method if given that

$\frac{dy}{dx}=3y-x$, $y(0)=-1$, $h=0.5$

Since the step is 0.5 and we are looking for x=2

$x_0=0$

$x_1=0.5$

$x_2=1$

$x_3=1.5$

$x_4=2$

So similarly, we are looking for y_1, y_2, y_3 and y_4

$\frac{dy}{dx}=3y-x$

$f(x,y)=3y-x$

$x_{n+1}=x_n+h=x_n+0.5$

$y_{n+1}=y_n+hf(x_n,y_n)=y_n+0.5(3y_n-x_n)=2.5y_n-0.5x_n$

When $n=0$, $x_1=0.5$, $y_1=-2.5$

$n=3$, $x_4=2.0$, $y_4=-42.625$

$y(2)=y_4\approx-42.6$

2.5.2 RUNGE-KUTTA METHOD

This method produces the best result

$y'=f(x,y)$ with $y(x_0)=y_0$

$h=gap$

For $n=0,1,2,3$

$x_{n+1}=x_n+h$

$k_1=f(x_n,y_n)$

$k_2=f\left(x_n+\frac{h}{2},y_n+\frac{h}{2}k_1\right)$

$k_3=f\left(x_n+\frac{h}{2},y_n+hk_2\right)$

$k_4=f(x_n+h,y_n+hk_3)$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

This is called the fourth order Runge-Kutta (RK) Method

ODE45 Algorithm in Matlab use an improved version of this method

2.5.3 PICARDS ITERATION METHOD

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_o) = y_o$$

$$y_{n+1} = y_o + \int_{x_o}^{x} f(x, y_n) dx$$

Questions:

1. Solve by Picard method. Find successive approximation. Solve up to fourth order of initial value problem $y' = 1 + xy$, $y(0) = 1$.

Solution:

$$y' = 1 + xy$$

$$\frac{dy}{dx} = 1 + xy$$

$$y'' + p(x)y + q(x)y = f(x)$$

$$f(x, y) = 1 + xy$$

$$y'' + p(x)y + q(x)y = f(x)$$

$$y(0) = 1$$

$x_o = 0$, $y_o = 1$

Put n=0,

$$y_1 = y_0 + \int_{x_o}^{x} f(x, y_o) dx$$

$$y_1 = 1 + \int_{x_o}^{x} (1 + x y_o) dx$$

$$y_1 = 1 + \int_{x_o}^{x} (1 + x) dx$$

$$y_1 = 1 + x + \frac{x^2}{2}\Big|_0^x$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

For n=1

$$y_2 = y_o + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = 1 + \int_0^x f(x, y_1)\,dx$$

$$y_2 = 1 + \int_0^x 1 + x\left(1 + x + \frac{x^2}{2}\right)dx$$

$$y_2 = 1 + \int_0^x 1 + x + x^2 + \frac{x^3}{2}\,dx$$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\Big|_0^x$$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

Omo this thing long oh. So person wan dey solve this thing for exam. God forbid oh. Well… let’s continue

For n=2

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x_3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

For n=3

$$y_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384}$$

Thanks for watching. Please subscribe and don’t forget to hit the like button. Lmaoo