SECOND ORDER CIRCUITS

- A second order circuit is characterised by a second-order differential equation.
- Therefore, the circuits which contain two storage elements are called second order networks
- Second order circuits are typically called RLC circuits
- Second order circuit may have two storage elements of different type or the same type (Provided elements of the same type cannot be represented by an equivalent single element)

FINDING INITIAL AND FINAL VALUES

It is easy to get initial and final values of v and i but not for $\frac{dV}{dt}$ and $\frac{di}{dt}$

Two important points in determining the initial conditions

1. Polarity of voltage v(t) across the capacitor and the direction of current through the inductor
2. v(0+)=v(0-)

SOURCE FREE SERIES RLC CIRCUIT

Given a charged capacitor, A charged inductor and a resistor in series,

The energy is represented by initial capacitor voltage, V and initial inductor current

$$V(0)=\frac{1}{C}\int i\,dt=V_o$$

$$i(0)=I_o$$

Applying KVL

$$V_R+V_L+V_C=0$$

$$iR+L\frac{di}{dt}+\frac{1}{C}\int i\,dt=0$$

On differentiating and then rearranging,

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0$$

Using a characteristic equation:

$$r^2+\frac{R}{L}r+\frac{1}{LC}=0$$

$$r_1,r_2=\frac{-\frac{R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{LC}}}{2}$$

$$r_1,r_2=\frac{-R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

Usually to solve the differential equation, we need some initial conditions: I_o , V_o , $\frac{di}{dt}$, $\frac{dV}{dt}$

Applying the initial conditions to this:

$$iR+L\frac{di}{dt}+\frac{1}{C}\int i\,dt=0$$,

We get:

$$I_O R+L d\frac{I_o}{dt}+V_o=0$$

$$\frac{dI_o}{dt}=\frac{-1}{L}(R I_o+V_o)$$

$$\frac{d^2V_c}{dt^2}+\frac{R}{L}\frac{dV_c}{dt}+\frac{V_c}{LC}=0$$

The characteristic equation is given as:

$$r^2+\frac{R}{L}r+\frac{1}{LC}=0$$

$$r_1, r_2=\frac{-R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

$$r_1=\frac{-R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

$$r_2=\frac{-R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

In an RLC circuit, L and C induce some kind of oscillation in the circuit. The resistor has a tendency to dampen/suppress the oscillation.

The oscillation generated, ω is given as

$$\omega=\frac{1}{\sqrt{LC}}$$

The frequencty of this oscillation is known as the natural frequency.

$$\alpha=\frac{R}{2L}=Damping\,Coefficient$$

$$\frac{\alpha}{\omega}=\zeta=\frac{R}{2}\sqrt{\frac{C}{L}}=Damping\,Factor$$

$$r_1=-\alpha+\sqrt{\alpha^2-\omega^2}$$
$$r_2=-\alpha-\sqrt{\alpha^2-\omega^2}$$

Damping Factor is the normalized damping coefficient and this defines the circuit responds to different excitations.

Cases to consider:

Case 1: Overdamped Response

$$\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

$$\alpha > \omega_o$$

$$C > \frac{4L}{R^2}$$

It takes time to reach its maximum value slowly because the response is sluggish. $C > \frac{4L}{R^2}$

This occurs when

For this case, the roots are real and negative.

$$r_1 = -\alpha + \beta$$

$$r_2 = -\alpha - \beta$$

In that case,

$$V_c(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$$

You’ll see that

Case 2, Critically damped Response:

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

$$\alpha = \omega_o$$

$$C = \frac{4L}{R^2}$$

Here it is fast to reach its maximum value

Roots will be negative, real and equal

$$r_1 = r_2 = -\alpha$$

$$V_c(t) = e^{-\alpha t}(c_1 + c_2 t)$$

Case 3: Under damped Response

$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

For this it reaches a maximum response, and then there is a slight oscillation and then gets to rest.

Roots will be complex conjugate

$$r_1 = -\alpha + j\beta$$

$$r_2 = -\alpha - j\beta$$

The solution is given as

$$V_c(t) = e^{-\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right]$$

Case 4: $R = 0$

Roots will be imaginary

$$r_1 = j\sqrt{\frac{1}{LC}} = j\omega$$

$$r_1 = -j\sqrt{\frac{1}{LC}} = -j\omega$$

The solution is given as
$V_c(t)=c_1\cos(\omega t)+c_2\sin(\omega t)$

7.7.1 FORCED SERIES RLC CIRCUITS
A voltage source, a key, a resistor, an inductor and a capacitor are connected in series.
Before time $t=0$ i.e. $t<0$, the switch is open and therefore no enenergy is stored in the elements.

At time t = 0, the switch is closed.
Applying KVL
$V_T=V_R+V_L+V_C$---i

The current flowing through the circuit $i$is the same as the current flowing through the capacitor i_c
$i=i_c$
$i_c=C\frac{dV_c}{dt}$
$V_R=iR=RC\frac{dV_c}{dt}$
$V_L=L\frac{di}{dt}=LC\frac{d^2V_c}{dt^2}$

$V_T=V_R+V_L+V_C$
$V_T=RC\frac{dV_c}{dt}+LC\frac{d^2V_c}{dt^2}+V_c$
Dividing through by LC
$\frac{V_T}{LC}=\frac{d^2V_c}{dt^2}+\frac{R}{L}\frac{dV_c}{dt}+\frac{V_c}{LC}$

Now, we have gotten a second order linear differential equation.
$Total\,solution=CF+PI$

The complementary function is the transient response of the circuit. This transient state response is more or less the source free response gotten earlier
The Particular Integral is the steady state response or th\\e of the circuit.
$PI=V_C(\infty)=V_T$
For the transient response $\frac{V_T}{LC}=0$

The above have been seen to be the transient state responses. The forced response will be the steady state response

For the following cases:
1. Overdamped Response

$V_c(t)=c_1 e^{-r_1 t}+c_2 e^{-r_2 t}+V_T$

2. Critically damped response:

$V_c(t)=e^{-\alpha t}(c_1+c_2 t)+V_T$

3. Under-damped response:

$V_c(t)=e^{-\alpha t}[c_1\cos(\beta t)+c_2\sin(\beta t)]+V_T$

7.7.2 FORCED PARALLEL RLC CIRCUIT

A current source is connected in parallel with a resistor, inductor and capacitor.

At first, the key is open at t<0.
At t=0, the key is then closed

Applying KCL

$$i=i_R+i_L+i_C$$

$$I=\frac{V}{R}+\frac{1}{L}\int Vdt+C\frac{dV_c}{dt}$$

On differentiating,

$$C\frac{d^2V}{dt^2}+\frac{1}{L}V+\frac{1}{R}\frac{dV}{dt}=0$$

$$\frac{d^2V}{dt^2}+\frac{1}{RC}\frac{dV}{dt}+\frac{1}{LC}V$$

The characteristic equation is given as:

$$r^2+\frac{1}{RC}r+\frac{1}{LC}=0$$

The roots are,

$$r_1,r_2=\frac{-1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

We can say that

$$\alpha=\frac{1}{2RC}$$

$$\beta=\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

$$r_1=-\alpha+\beta$$

$$r_2=-\alpha-\beta$$

Similarly to the Series, there are cases to consider:

Case 1: $\left(\frac{1}{2RC}\right)^2>\frac{1}{LC}$. The case of an **Overdamped Response**.

The roots will be negative and real

The roots

$$r_1=-\alpha+\beta$$

$$r_2=-\alpha-\beta$$

The solution will be in the form

$$V_c(t)=c_1 e^{-r_1 t}+c_2 e^{-r_2 t}$$

Case 2: $\left(\frac{1}{2RC}\right)^2=\frac{1}{LC}$: This is the Critically Damped Response

The roots will be real and equal

$r_1=r_2=-\alpha$

Solution:

$V_c(t)=(c_1+c_2t)e^{-\alpha t}$

Case 3: $\left(\frac{1}{2RC}\right)^2<\frac{1}{LC}$: Under-damped response

The roots will be complex conjugate

$r_1=-\alpha+j\beta$

$r_2=-\alpha-j\beta$

$V(t)=e^{-\alpha t}[c_1\cos(\beta t)+c_2\sin(\beta t)]$

Case 4:

Roots will be imaginary.

$r_1=j\omega$

$r_2=-j\omega$

$\omega=\frac{1}{\sqrt{LC}}$

$V_c(t)=c_1\cos(\omega t)+c_2\sin(\omega t)$

USES OF RLC CIRCUITS

RLC Circuits are used as tuning circuits in radio communications

Are used as voltage multipliers