GEG 219 (Ordinary Differential Equations) 2units

COURSE OUTLINE

1. Introduction to Differential Equations

- Linear dependence

- Classification of Ordinary Differential Equations

\* Order

\* Degree

\* Linearity.

2. First Order ODEs

- Types and Techniques of solution of first order ODE’s

- Picard’s iterative method

- physical applications of first order ODE.

3. Theory and solutions of higher order linear equations; physical applications.

4. Ordinary differential equations with constant coefficients

- Methods of undetermined coefficients

-Variation of parameters

- D-Operator.

5. Linear Differential Equations with Variable coefficients.

6. Cauchy-Euler’s equations

7. Systems of linear operations

8. Properties of linear operations

9. Series solution

8. First order non-linear equations

- Autonomous

- Equidimensional

- Scale-invariant

**INTRODUCTION TO DIFFERENTIAL EQUATIONS**

A differential equation is one which contains a function and its derivative in the same equation.

It could be said to be an equation with at least one derivative

A differential equation is a relationship between an independent variable x, a dependent variable y and at least one derivative y.

NB:

**TYPES OF DIFFERENTIAL EQUATIONS**

1. Ordinary Differential Equations: This is one in which the unknown function (y) depends only on one independent variable (x).

2. Partial Differential Equations. This is one in which the unknown functions depends on at least two independent variables.

**FORMATION OF A DIFFERENTIAL EQUATION**

A differential equation is formed when arbitrary constants are eliminated from a given function.

**ORDER AND DEGREE OF A DIFFERENTIAL EQUATION**

The order of a DE is the highest derivative appearing in the equation. It is also called differential coefficient.

The degree of a DE is the power of the highest order is the degree

a. . This will have an order of 2 and a degreee of 1.

The order of a differential equation indicates how many initial conditions are needed to find a unique solution.

**INITIAL VALUE PROBLEM AND BOUNDARY VALUE PROBLEM**

In initial value problem, we are given the value of the function and its derivative at the same point (initial point)

and . For initial value problem, we are given a differential equation and an initial condition. For a first order equation, you will need one initial condition. For a second order equation, you will need two separate initial conditions.

Boundary Value Problems: Here we are given the value of function y(x) at two different points i.e. ,

**VERIFYING SOLUTIONS**

Verify that is a solution of . Find and such that , .

Answer: Therefoe, is a solution of the differential equation

**LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS**

A linear differential equation is one where the degree is 1.

**CONDITIONS FOR LINEAR DIFFERENTIAL EQUATION**

1. In front of y and its derivatives must only be pure functions of x and never of y or other variables.

2. The powers of y and its derivatives must be 1. (Degree of 1)

3. g(x) must also be a function of x and neither y or its derivative should be there.

**FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS**

This is a linear differential equation where the highest power of the derivative is 1 (first order).y(x) is

**DIFFERENTIAL FORM OF LINEAR ODE**

if

The general standard form for a first order ODE in the function

Sample questions.

Write the differential equation

**METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATIONS**

1. Direct Integration

2. Means of Separating Variables

3. Integrating Factor Method

4. Subsitution Methods

5. Picard’s Iterative Method

DIRECT INTEGRATION METHOD

This comes in two forms:

Case I: You have an independent variable x and the function is in terms of x

If , is a function of the independent variable only

Case 2:

If , and is a function of the dependent variable

What you’ll do here is to flip it

Examples:

1. Solve the initial value problem

By integrating,

On solving for the initial problem,

2. Solve the IVP

Integrate with respect to y

Using IC(Initial Conditions)

**METHOD OF SEPARATING VARIABLES**

This method is used to solve separable differential equations.

You’ll see that the function of x and y is actually a product of the **function of x** and the **function of y**.

Given the function:

1. Separate variables. On one side, we want only y-variables and dy. On the other side, we want only x-variables and dx’s.

2. Integrate both sides.

3. Add the constant of integration to the x-side of the integrals

4. Solve and make y the subject of the formula

The formula you get is called the **general solution**. If you are given an initial value, y(1)=2, then you find the specific solution

Solve the following

1. . Given the initial position , then you should find the specific solution

Answers: and

2. At point Answer:

3. At

4.

5. Solve , : General Solution:

Final solution:

6. Solve Implicit solution:

**INTEGRATING FACTOR METHOD**

This is used to solve the first order linear differential equations.

1. Write the equation in the **standard form** the first order linear differential equation.

2. Identify the functions p(x) and q(x)

3. Determine the integrating factor

4. Multiply through by the integrating factor.

5. This can be reduced to the form:

6. On integrating,

7. Write the general solution

Solve the following questions

1. Answer:

2. Answer:

3.

4. Answer:

5.

6.

SUBSTITUTION METHODS

Case 1: Direct Substitution

Let u = “inside function”

Rewrite the DE using the new variable

Solve:

,

Let

Case 2:HOMOGENEOUS EQUATIONS

A homogeneous equation is an equation that can be written in the form:

A homogeneous differential equation is a type of differential equation that can be written in the form , where f is a function of the ratio . This means that both sides of the equation are homogeneous functions of the same degree of x and y.

Steps to solve:

1. Let

Therefore,

2. Rewrite the equation in v and x variables

3. Solve for v first, and then use y=vx to solve for y.

Examples:

1. ,

Dividing by xy

Let

But

Using

Example 1:

This is a homogeneous differential equation, since both sides are homogeneous functions of degree one. We can solve it by substituting y=vx and separating the variables.

Example 2:

Case 3: The Bernoulli’s Equation

This is an extension of the first order linear equation

If n=0, and you use the integrating factor method

If n=1, , there are two ways to go about this:

a. , then we use the integrating factor method.

b. , then use separation method

If , dividing by

Then use the subsitution:

Taking the derivatives,

From the chain rule

Where

This can be seen to be the first order linear equation in v

Then we solve for v using the integrating factor method

Use to find y.

Example: Solve

Dividing by

Substituting

→ 1st order linear in v

Multiplying through by x

On integrating

Using

EXACT DIFFERENTIAL EQUATIONS

The standard form of a differential equation is written as

It can also be written in the form by dividing through by dx

if , then the equation is called EXACT differential equation, and there is a function

, such that and .

Solve the function, for y. Then the equation of y is the solution

METHODS OF SOLVING

1. Solution Method:

Integrate with respect to x.

Since it is coming from a partial derivative, there is a possibility that there will be a function of x that will be considered as a constant, so we add it back to the equation when integrating.

Next, differentiate with respect to y

Next we want to find the equation g(y). Therefore, set and then find , then integrate to find

Substituting

Solve for y, if possible. This is because there may not be a solution for y. If there is, that will be the implicit

2. Shortcut Method

Integrate with respect to x

Then you get the function

Integrate with respect to y

f(x,y) – Write common terms and uncommon terms once

f(x,y)= C

Solve for y, if possible.

Examples:

Solve:

Since,, the equation is an exact differential equation.

This means there must be a function z = f(x, y) = C such that

Integrate with respect to x

Differentiate with respect to y.

It can be seen that

Now we integrate with respect to y

Since there is no way you can solve for y, the implicit solution is

ALMOST EXACT EQUATION

If , then the equation is not exact so…

We will use the integrating factor method to solve.

There are two cases to consider:

1. The MNN case:If is a function of x only, then the integrating factor is given as

2. The NMM case. If is a function of y only, then the integrating factor is given as

Then if you multiply the standard form by the integrating factors, you get…

→ This is going to form an exact equation.

Solve:

Since they are not the same, it is not an exact equation.

→ Since this is not a function of a single variable, we cannot use this expression.

→ This is a function of y only

Multiply by the integrating factor

Since this is exact, we use the shortcut method.

On integrating

On integrating with respect to y:

If you look at both equations you’ll see that the term is common to both.

So we write the common terms first

Next we write the uncommon terms to both

Since this is not easy to solve for y, this is the implicit solution for y.

NUMERICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

1. Euler’s Method

2. Improved Euler’s Method

3. Runge-Kutta Method

EULERS

You are going to look at the slopes at each point and use the tangent lines to approximate the next point.

,

h → gap between x values

We are given the slope at every point (the derivatives), and the initial values

Example:

Approximate y(2) using eulers method if given that

, ,

Since the step is 0.5 and we are looking for x=2

So similarly, we are looking for , , and

When , ,

, ,

RUNGE-KUTTA METHOD

This method produces the best result

with

h=gap

For n=0,1,2,3

This is called the fourth order Runge-Kutta (RK) Method

ODE45 Algorithm in Matlab use an improved version of this methos

AUTONOMOUS DIFFERENTIAL EQUATIONS

A differential equation is called autonomous if it does not depend on the independent variable. For example,

, you’ll see that in the expression, there is no expression of t

NOTE: It does not change as time (independent variable) changes

Critical points: These are points where the derivative is equal to zero.

If at , is called a critical point

After that, we may check whether the critical points are stable or not

Example, find the critical points and classify them as stable, half stable or unstable.

{dx over dt}=-{x^2}{(x+1)}{(x-2)}

For critical point, -{x^2}{(x+1)}{(x-2)} = 0

x = 0, -1, 2

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Let y be the dependent variable and x be the independent variable.

The general form of the second order linear differential equation is given as:

Dividing by a(x)

This can then be reduced to the standard form

2nd Order ODE: In theory, there should be two solutions

Case 1: Homogeneous 2nd Order DE

If , DE is homogeneous

Superposition Principle:

If  and are two solutions of the homogeneous equation

, then

is also a solution for any and

Proof: Define (This is a linear operator)

and are solutions.

The operator is linear because it meets the requirement of linearity: That is

1.

2.

Then considering

Therefore, y is also a solution

LINEAR INDEPENDENCE

Definition: If for any value of x, then this implies that , we say that and are linear independent. If the functions are not linearly independent, they are linearly dependent

Show that

and

Answer: Suppose for any x.

Substitute, x=0,

This implies that

Now, let ,

Thus, .

Therefore by the definition y\_1 and y\_2 are linearly independent.

WROSKIAN TEST FOR LINEAR INDEPENDENCE.

For some x\_o, then y\_1 and y\_2 are linearly independent.

Example: Show that and are linearly independent

Since the answer for any x, then and are linearly independent

Note: The following sets and their subsets are linearly independent

1.

2.

3.

4.

5.

6.

7.

8.

Theorem to note:

If {y\_1}(x) and {y\_2}(x) are two linearly independent solutions of the homogeneous equation

The general solution is given bby

and are constants

Case 2: Non-Homogeneous Equations

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How do we solve?

1.We split the problem into 2 parts

→ Homogeneous equation

Find the general

For the non-homogeneous equatiton, we find a particular solution

The general solution of the non-homogeneous

To find y\_p we use two methods

- Undetermined coefficient method

- Variation of parameters method.

To find and , we use the initial conditions

Theorem: Existence and Uniqueness theorem:

If p,q and f are continuous on the on some interval containing a, then , has a unique solution satisfying

and

PICARDS ITERATION METHOD

PROMPTS

1. Hi, I want you to answer like a [Millionaires name]. Use all the knowledge of [their name] and any and every available information about how [name] thinks. My challenge is: []. now provide me a detailed 500-word answer with 3 action points

2.