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1 Source

Book: How to Prove It: A Structured Approach 3rd Edition
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Exercise: 3.3.4

2 Task

Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$

3 Scratch Work

Assume $A \subseteq \mathscr{P}(A)$

Givens	Goal
$A \subseteq \mathscr{P}(A)$	$\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$

Apply the definition of a subset

Givens	Goal
$A \subseteq \mathscr{P}(A)$	$\forall x(x \in \mathscr{P}(A) \implies x \in \mathscr{P}(\mathscr{P}(A)))$

Let x be an arbitrary element

Givens	Goal
$A \subseteq \mathscr{P}(A)$	$x \in \mathscr{P}(A) \implies x \in \mathscr{P}(\mathscr{P}(A))$

Assume x $\in \mathscr{P}(A)$

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$	$x \in \mathscr{P}(\mathscr{P}(A))$

By the definition of a power set

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$	$x \subseteq \mathscr{P}(A)$

Apply the definition of a subset

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$	$\forall y(y \in x \implies y \in \mathscr{P}(A))$

Let y be an arbitrary element of x

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$	$y \in x \implies y \in \mathscr{P}(A)$

Assume y $\in x$

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$ y $\in x$	$y \in \mathscr{P}(A)$

By the definition of a power set

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$ y $\in x$	$y \subseteq A$

By the definition of a subset

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$ y $\in x$	$\forall z(z \in y \implies z \in A)$

Let z be an arbitrary element of y

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$ y $\in x$	$z \in y \implies z \in A$

Assume z $\in y$

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\in \mathscr{P}(A)$ y $\in x$ z $\in y$	$z \in A$

By the definition of a power set

Givens	Goal
$A \subseteq \mathscr{P}(A)$ x $\subseteq A$ y $\in x$ z $\in y$	$z \in A$

By the definition of a subset

Givens	Goal
$A \subseteq \mathscr{P}(A)$ $\forall k(k \in x \implies k \in A)$ y $\in x$ z $\in y$	$z \in A$

Universal instantiation (k = y)

Givens	Goal
$A \subseteq \mathscr{P}(A)$ $\forall k(k \in x \implies k \in A)$ $y \in x \implies y \in A$ y $\in x$ z $\in y$	$z \in A$

Modus Ponens (y $\in x$)

Givens	Goal
$A \subseteq \mathscr{P}(A)$ $\forall k(k \in x \implies k \in A)$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

By the definition of a subset ($A \subseteq \mathscr{P}(A)$)

Givens	Goal
$\forall m(m \in A \implies m \in \mathscr{P}(A))$ $\forall k(k \in x \implies k \in A)$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

Universal instantiation (m = y)

Givens	Goal
$\forall m(m \in A \implies m \in \mathscr{P}(A))$ y $\in A \implies y \in \mathscr{P}(A)$ $\forall k(k \in x \implies k \in A)$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

Modus Ponens ($y \in A$)

Givens	Goal
$\forall k(k \in x \implies k \in A)$ $\forall m(m \in A \implies m \in \mathscr{P}(A))$ y $\in \mathscr{P}(A)$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

By the definition of a power set

Givens	Goal
$\forall k(k \in x \implies k \in A)$ $\forall m(m \in A \implies m \in \mathscr{P}(A))$ y $\subseteq A$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

By the definition of a subset

Givens	Goal
$\forall k(k \in x \implies k \in A)$ $\forall m(m \in A \implies m \in \mathscr{P}(A))$ $\forall n(n \in y \implies n \in A)$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

Universal Instantiation (n = z)

Givens	Goal
$\forall k(k \in x \implies k \in A)$ $\forall m(m \in A \implies m \in \mathscr{P}(A))$ $\forall n(n \in y \implies n \in A)$ $z \in y \implies z \in A$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

Modus Ponens (z $\in y$)

Givens	Goal
$\forall k(k \in x \implies k \in A)$ $\forall m(m \in A \implies m \in \mathscr{P}(A))$ $\forall n(n \in y \implies n \in A)$ $z \in A$ $y \in A$ y $\in x$ z $\in y$	$z \in A$

Now the goal follows from the givens

4 Final Solution

4.1 Task

Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$

4.2 Solution

Assume $A \subseteq \mathscr{P}(A)$. Let x be an arbitrary element. Assume $x \in \mathscr{P}(A)$. Let y be an arbitrary element of x. Assume $y \in x$. Let z be an arbitrary element of y. Assume $z \in y$. Since $x \in \mathscr{P}(A)$, $x \subseteq A$. Since $y \in x$, $y \in A$. Since $y \in \mathscr{P}(A)$, $y \subseteq A$. Since $z \in y$, $z \in A$. Since z is an arbitrary element of y and $z \in A$, $y \subseteq A$ and $y \in \mathscr{P}(A)$. Since y is an arbitrary element of x and $y \in \mathscr{P}(A)$, $x \subseteq \mathscr{P}(A)$ and $x \in \mathscr{P}(\mathscr{P}(A))$. Since x is an arbitrary element of $\mathscr{P}(A)$ and $x \in \mathscr{P}(\mathscr{P}(A))$, it follows that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.