5 Optimal Control Problems

5.1 Basic Problem in Optimal Control:

$$\min J(x,u) = \int_{t_0}^{t_f} L(t, x(t), u(t))dt + \varphi(x(t_f), t_f)$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0.$$

To work up to solving this, we'll follow the pattern of what we did with algebraic optimization problems. Instead of having objective functions that map from $\Re^n$ into $\Re$, the objective functions can be functionals. Functionals map functions to real numbers. The calculus required to find minima for functionals is called the calculus of variations (COV).

Fundamental Problem in the Calculus of Variations: Given a functional J and a set of admissible functions A, determine which function (or functions) provide a minimum value for J.

Example problems:

1. Find the function f with minimum arclength in the set A of all continuously differentiable functions on $a \le x \le b$ with $f(a) = f_1$, $f(b) = f_2$. Define

$$J(f) = \int_a^b \sqrt{1 + f'(x)^2}dx$$

Then, we want to find the f that minimizes J.

2. A bead with mass m and initial velocity $v(0) = 0$ slides with no friction under the force of gravity from point (x_1, y_1) to (x_2, y_2) along a wire defined by a curve $f(x)$. What is the shape that has the shortest travel time?

$$T = \int_0^T dt = \int_0^{s_1} \frac{dt}{ds}ds = \int_0^{s_1} \frac{1}{v}ds = \int_{x_1}^{x_2} \frac{\sqrt{1 + f'(x)^2}}{v}dx.$$

To determine v in terms of f or x, use the fact that energy is conserved. Energy in the system:

$$\frac{1}{2}mv^2 + mgf = 0 + mgy_1$$

at the starting point; this implies that

$$v = \sqrt{2g(y_1 - f(x))}.$$

So for any curve, the time to traverse the curve is given by

$$T(f) = \int_{x_1}^{x_2} \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2g(y_1 - f(x))}}dx$$

3. Linear Quadratic Regulator with sensed measurements

$$\min \int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$y(t) = Cx(t)$$

$$x(t) \in \Re^n, \quad u(t) \in \Re^m, \quad A \in \Re^{n\times n}, \quad B \in \Re^{n\times m}, \quad C \in \Re^{n\times p}$$

Q positive semidefinite, R positive definite.

To minimize these problems, we need to develop necessary and sufficient conditions. There are many intricacies to this, and we will just scratch the surface.

Key point to remember: Analogously to the calculus used for basic optimization problems, we would like to have a condition like "$J'(f) = 0$" as a way to find candidate minimizers f^*. We will return to this idea repeatedly to develop the ideas that can be used to solve optimal control problems.

Suppose f^* minimizes J locally. Then we can generate other admissible functions "nearby f^*" in the set A by $f = f^* + \epsilon h$ where ϵ is a small real number and h is a function. If we pick h carefully and keep ϵ small, f will remain in A.

If J is minimized at f^* then $\mathcal{J}(\epsilon) = J(f^*+\epsilon h)$ is minimized when $\epsilon = 0$. So, related to $J(f)$, we have created a related function $\mathcal{J}(\epsilon)$. We can apply standard calculus to minimizing $\mathcal{J}$ and thus uncover necessary conditions for $J(f)$. In particular, we know that $\mathcal{J}(\epsilon)$ is minimized when $\epsilon = 0$. Therefore the first order necessary condition for a minimum for $\mathcal{J}$ is $\mathcal{J}'(0) = 0$. The function f^* is called the *extremizer* (or candidate minimizer) for J and J is called *stationary* at f^*.

5.2 Simplest problem in the calculus of variations:

Find a local minimum for

$$J(f) = \int_a^b L(x, f, f')dx$$

where $f \in C^2[a,b]$, $f(a) = y_1$, $f(b) = y_2$, L is a given function that is C^2 on $[a,b] \times \Re^2$. Note that f is smoother than needed if a more general approach is used. It is easier to develop the necessary conditions with this assumptions, but not applicable to as wide a variety of problems.

Form $\mathcal{J}(\epsilon) = J(f^* + \epsilon h)$. We know that $\mathcal{J}$ is optimized when $\epsilon = 0$ and so the first order necessary condition is $\mathcal{J}'(0) = 0$.

$$\mathcal{J}(\epsilon) = J(f^* + \epsilon h) = \int_a^b L(x, (f^* + \epsilon h), (f^* + \epsilon h)')dx.$$

First order necessary condition: $\frac{d}{d\epsilon}\mathcal{J}(0) = 0$ corresponds to the candidate minimizer f^*, and J is stationary at f^*. Assume that h is chosen to be twice continuously differentiable with $h(a) = 0 = h(b)$. Then for ϵ small enough, $f^* + \epsilon h$ remains in A and is called an *admissible variation.*

$$\frac{d}{d\epsilon}\mathcal{J}(\epsilon)|_{\epsilon=0} = \frac{d}{d\epsilon}J(f^*+\epsilon h)|_{\epsilon=0} = \int_a^b \frac{\partial}{\partial\epsilon}L(x,(f^*+\epsilon h),(f^*+\epsilon h)')dx|_{\epsilon=0}$$

$$= \int_a^b \left(\frac{\partial L}{\partial x}(x,(\cdot),(\cdot)')\frac{dx}{d\epsilon} + \frac{\partial L}{\partial f}(x,(\cdot),(\cdot)')\frac{df}{d\epsilon} + \frac{\partial L}{\partial f'}(x,(\cdot),(\cdot)')\frac{df'}{d\epsilon}\right)dx|_{\epsilon=0}$$

$$= \int_a^b \left(\frac{\partial L}{\partial f}(x,f^*,(f^*)')h + \frac{\partial L}{\partial f'}(x,f^*,(f^*)')h'\right)dx = 0$$

This last line is the first order initial condition and will hold for all $h \in C^2[a,b]$ with $h(a) = 0 = h(b)$. To put this in a more usable form, integrate by parts on the second term to change h' to h.

Set $u = \frac{\partial L}{\partial f'}$ and $dv = h'dx$. Then $du = \frac{d}{dx}\frac{\partial L}{\partial f'}dx$ and $v = h$. This implies that

$$\begin{aligned} 0 &= \int_a^b \left(\frac{\partial L}{\partial f}(x,f^*,(f^*)')h + \frac{\partial L}{\partial f'}(x,f^*,(f^*)')h'\right)dx \\ &= \int_a^b \left(\frac{\partial L}{\partial f}(x,f^*,(f^*)') - \frac{d}{dx}\frac{\partial L}{\partial f'}(x,f^*,(f^*)')\right)hdx \qquad (1) \\ &\quad + \frac{\partial L}{\partial f'}(x,f^*,(f^*)')h|_a^b \end{aligned}$$

To use this, we need to apply the
Fundamental Lemma of the Calculus of Variations: If f is continuous on $[a,b]$ and if $\int_a^b f(x)h(x)dx = 0$ for every $h \in C^2[a,b]$ with $h(a) = 0 = h(b)$, then $f(x) = 0$ for $x \in [a,b]$.

From this, we can conclude that

$$\frac{\partial L}{\partial f}(x,f^*,(f^*)') - \frac{d}{dx}\frac{\partial L}{\partial f'}(x,f^*,(f^*)') = 0.$$

To summarize this, we can write the following theorem which represents the First Order Necessary Condition for the Simplest Optimal Control problem:

Theorem: If f^* provides a local minimum to

$$J(f) = \int_a^b L(x,f,f')dx$$

where $f \in C^2[a,b]$ and $f(a) = y_1$, $f(b) = y_2$ then f must satisfy

$$\frac{\partial L}{\partial f}(x,f^*,(f^*)') - \frac{d}{dx}\frac{\partial L}{\partial f'}(x,f^*,(f^*)') = 0 \quad \text{for} \quad x \in [a,b]. \qquad (2)$$

Equation (**??**) is called the *Euler-Lagrange Equation*. Solutions f^* to the Euler-Lagrange equation are called *extremals* and J is *stationary* at such functions.
Examples:

5.3 Variation on the simplest problem: higher derivatives of f in the objective function

If

$$J(f) = \int_a^b L(x, f(x), f'(x), ..., f^{(n)}(x))dx$$

with boundary conditions

$$f(a) = y_1, f'(a) = y_2, ..., f^{(n-1)}(a) = y_n$$

$$f(b) = z_1, f'(b) = z_2, ..., f^{(n-1)}(b) = z_n,$$

then the 1st order necessary condition becomes

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial f''} - \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial L}{\partial f^{(n)}} = 0 \quad \text{for} \quad x \in [a, b].$$

where each of the terms in the necessary condition are evaluated at $(x, f^*, (f^*)', \ldots, (f^{*(n)})$. This equation is solved to find the *candidate minimizers* or *extremals*, f^*, for the objective function.

Example

5.4 Variation 2 on the simplest problem: multiple functions f, g in the objective function

If

$$J(f) = \int_a^b L(x, f(x), f'(x), g(x), g'(x))dx$$

with boundary conditions

$$f(a) = y_1, f(b) = y_2, g(a) = z_1, g(b) = z_2,$$

then the 1st order necessary condition takes the form of a system of Euler-Lagrange equations for f and g:

$$\frac{\partial L}{\partial f}(x, f^*, (f^*)', g^*, (g^*)') - \frac{d}{dx}\frac{\partial L}{\partial f'}(x, f^*, (f^*)', g^*, (g^*)') = 0$$

$$\frac{\partial L}{\partial g}(x, f^*, (f^*)', g^*, (g^*)') - \frac{d}{dx}\frac{\partial L}{\partial g'}(x, f^*, (f^*)', g^*, (g^*)') = 0,$$

for $x \in [a, b]$. These equations (typically coupled) are solved to find the *candidate minimizers* or *extremals*, f^*, g^*, for the objective function.

Example

5.5 Variation 3 on the simplest problem: unspecified boundary condition

Instead of having $f(a)$ and $f(b)$ specified, assume $f(a) = y_1$ and $f(b)$ is unspecified. This means that the admissible variations, h, have $h(a) = 0$ and $h(b)$ is unspecified. Then in order for (**??**) to reduce to the form to which we can apply the FLCOV, we need to assume that

$$\frac{\partial L}{\partial f'}(b, f^*(b), (f^*(b))') = 0$$

and the Euler-Lagrange equation will still hold.

Example

5.6 Hamiltonian Theory

One method to obtain equations of motion for a physical system is to use the idea that the system should evolve along a path of least resistance. The concept of *action* was proposed (where action has the units of energy $\times$ time), and the motion is to minimize the action (a.k.a. Principle of Least Action).

Define $y_1, y_2, \ldots, y_n$ to be the coordinates of a given system dynamics. Then $\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n$ are the corresponding velocities. Denote the kinetic and potential energies of the system as T and V respectively.

Hamilton's principle states that the time evolution of a mechanical system takes place so that the integral of the difference of kinetic and potential energy is *stationary.* Or, the motion of the system from time t_0 to t_1 provides an extremal for the functional

$$J(y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n) = \int_{t_0}^{t_1} (T - V) dt$$

The functions $y_1(t), \ldots, y_n(t)$ can also be thought of as parametric equations defining the paths that the system takes over time.

Examples