The Role of Information in Distributed Control*

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1 Our Model

We consider the class of resource allocation problems where there exists a set of agents $N = \{1, 2, ..., n\}$ and a closed and compact mission space $C \subset R^2$ for which the players are to be allocated. The allocation of the agents is represented by the tuple $x = (x_1, ..., x_n)$ where x_i denotes the position of agent of agent i. The set of possible positions for each agent is denoted by the closed and compact set $X_i \subseteq C$. Let $X = \times_{i \in N} X_i$ represent the set of possible allocations. The goal of this problem is to allocate the agents over the mission space to optimize (maximize) a given global objective of the form $G : X \to R^+$. Throughout this work we assume that this objective is *submodular*. Submodularity corresponds to a notion of decreasing marginal returns which is very common in a variety of engineering applications.

The goal of this paper is to analyze the role of information as it pertains to the efficiency of the stable solutions for such systems. To that end, we denote the information set of player i at location x_i by the set $F_i(x_i) \subseteq X_i$. Roughly speaking, for any allocation $x \in X$, each agent is capable of evaluating the global objective for all allocations of the form $\{(x_i', x_{-i}) : x_i' \in F_i(x_i)\}$, i.e., all allocations generated if the player unilaterally deviated to any position $x_i' \in F_i(x_i)$. We adopt the convention that for all players $i \in N$ and locations $x_i \in X_i$ that $x_i \in F_i(x_i)$. Accordingly, we introduce the following notion of equilibrium which adheres to this informational structure:

Definition 1.1 (Local equilibria) *An allocation x^* is an equilibrium with respect to the information sets $\{F_i\}_{i \in N}$ if for each player $i \in N$ we have*

$$G(x_i^*, x_{-i}^*) = \max_{x_i \in F_i(x_i^*)} G(x_i, x_{-i}^*)$$

In the special case when $F_i(x_i) = X_i$ for all players $i \in N$ this definition boils down to that of a pure Nash equilibrium. Alternatively, for any $\epsilon > 0$ define $F_i(x_i) = \{x_i' \in X_i : ||x_i' - x_i|| \leq \epsilon\}$. When $\epsilon \to 0^+$ this definition represents the rest points of a distributed gradient ascent dynamic on the objective G.

For simplicity, let the set of possible positions for each agent i be the entire mission space, i.e., $X_i = C$. Define the *information index* of the information sets $\{F_i\}_{i \in N}$ as follows:

$$d = \min_{x \in X} \min_{y \in C} |\{i \in N : y \in F_i(x_i)\}|.$$

The information index highlights a worst case measure of redundancy in information across the mission space. It turns out that this information index is intimately related to the efficiency of the resulting equilibria as shown in the following theorem.

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Theorem 1.1 *Consider the setup above with an information index $d \geq 1$. The efficiency of the resulting equilibria is bounded above by*

$$\frac{G(x^{\rm opt})}{G(x^{\rm ne})} \leq 1 + \frac{n}{d}$$

Proof: With a slight abuse of notation extend the definition of the global objective G to include all numbers of players ranging from $\{0, 1, ..., 2n\}$. Let $x^{\rm opt}$ and $x^{\rm ne}$ represent an optimal and equilibrium allocation respectively. Define $M_i(x) := G(x) - G(x_{-i})$ as the marginal contribution of agent i in the allocation x. The fact that $x^{\rm ne}$ represents an equilibrium gives us the following: for all $x_i' \in F_i(x_i^{\rm ne})$ we know that $G_i(x^{\rm ne}) \geq G(x_i', x_{-i}^{\rm ne})$ which is equivalent to

$$M_i(x^{\rm ne}) \geq M_i(x_i', x_{-i}^{\rm ne}).$$

Furthermore, since G is submodular we know that for any allocation x

$$G(x) \geq \sum_{i \in N} M_i(x).$$

Without loss of generalities, order players such that $M_1(x^{\rm ne}) \geq M_2(x^{\rm ne}) \geq ... \geq M_n(x^{\rm ne})$. Define $V_i(x) := \max_{x_i' \in F_i(x_i)} G(x_i', x) - G(x)$ as the maximum gain in marginal contribution that can be attributed to adding a new agent in the information set $F_i(x_i)$, i.e., going from n to $n+1$ players in the allocation. Define $V^* := \max_i V_i(x^{\rm ne})$. If $x^{\rm ne}$ represents an equilibrium and G is submodular then we have that for each agent $i \in N$

$$\begin{aligned} V_i(x^{\rm ne}) &= V(x_i^*, x^{\rm ne}) - V(x^{\rm ne}) \\ &\leq V(x_i^*, x_{-i}^{\rm ne}) - V(x_{-i}^{\rm ne}) \\ &\leq V(x_i^{\rm ne}, x_{-i}^{\rm ne}) - V(x_{-i}^{\rm ne}) \\ &= M_i(x^{\rm ne}). \end{aligned}$$

Therefore we have that

$$G(a^{\rm ne}) \geq \sum_{i \in N} V_i(x^{\rm ne})$$

An information index d implies that for any agent $i \in \{1, ..., d\}$

$$M_i(x^{\rm ne}) \geq V^*.$$

To see this, suppose some player $j \in \{1, 2, ..., d\}$ had a marginal contribution $M_j(x^{\rm ne}) < V^*$. Since the information index is d and $j \leq d$, this means that there exists a player $k \in \{j, ..., n\}$ with a position x_k^* such that $V_k(x_k^*, x_{-k}^{\rm ne}) = V^*$. Therefore, $x^{\rm ne}$ is not an equilibrium as

$$M_k(x_k^*, x_{-k}^{\rm ne}) \geq V^* \geq M_k(x_k^{\rm ne}, x_{-k}^{\rm ne}).$$

Therefore, we know that for any equilibrium with information index d we have that

$$G(x^*) \geq \sum_{i \in N} M_i(x^{\rm ne}) \geq dV^*.$$

We can use the priciples derived above to bound the efficiency of an equilibrium as follows:

$$
\begin{aligned}
\frac{G(x^{\mathrm{opt}})}{G(x^{\mathrm{ne}})} &\leq \frac{G(x^{\mathrm{opt}}, x^{\mathrm{ne}})}{G(x^{\mathrm{ne}})}, && (1)\\
&\leq \frac{G(x^{\mathrm{ne}}) + \sum_{i\in N}\Big(G(x_i^{\mathrm{opt}}, x^{\mathrm{ne}}) - G(x^{\mathrm{ne}})\Big)}{G(x^{\mathrm{ne}})}, && (2)\\
&= 1 + \frac{\sum_{i\in N}\Big(G(x_i^{\mathrm{opt}}, x^{\mathrm{ne}}) - G(x^{\mathrm{ne}})\Big)}{G(x^{\mathrm{ne}})}, && (3)\\
&\leq 1 + \frac{\sum_{i\in N}\Big(G(x_i^{\mathrm{opt}}, x^{\mathrm{ne}}) - G(x^{\mathrm{ne}})\Big)}{\sum_i M_i(x^{\mathrm{ne}})}, && (4)\\
&\leq 1 + \frac{nV^*}{\sum_i M_i(x^{\mathrm{ne}})}, && (5)\\
&\leq 1 + \frac{nV^*}{dV^*}, && (6)\\
&= 1 + \frac{n}{d} && (7)
\end{aligned}
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