Game Theory for Graph Coloring

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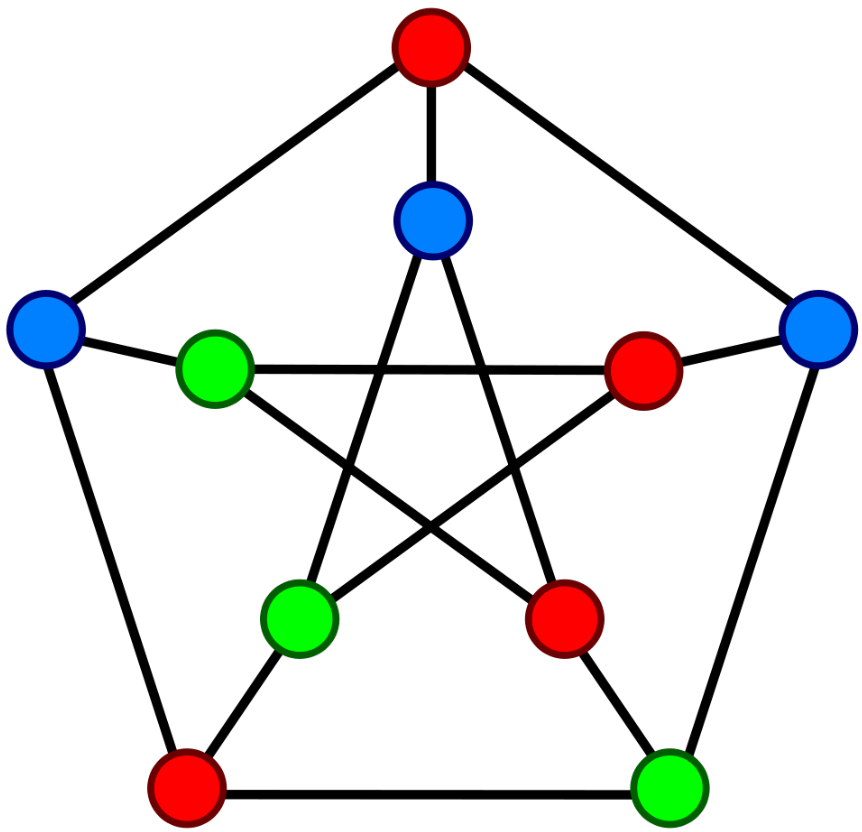
Graph Coloring?

Statement: What is the minimum number of “colors” needed so that no two neighboring nodes have the same color?



Why?

- Topologically equivalent to edge coloring, face coloring
- “Colors” represent labels of any kind.
- Applications to Task Scheduling, Bandwidth Allocation, Games

Graph Theory

A **graph** $G(V, E)$ has

- a set of vertices or nodes V
- a set of edges (undirected) E that connect certain nodes
- Two nodes are *neighbors* if there is an edge between them

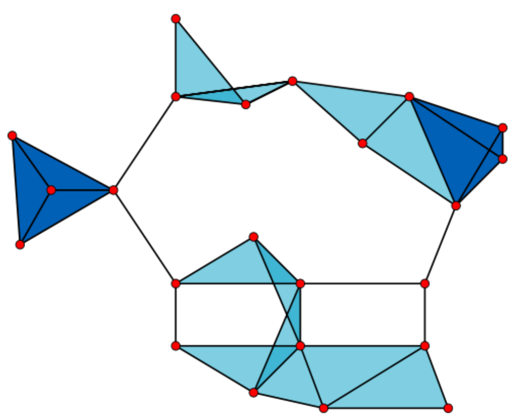
Other concepts:

Clique A subset of vertices such that all vertices are connected. A *complete* subgraph

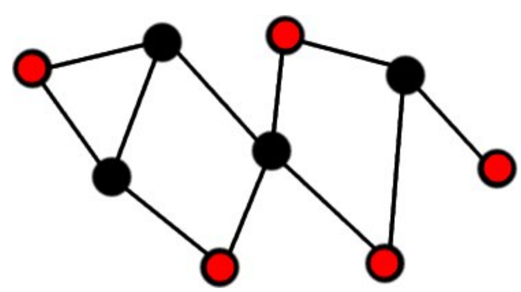
Independent set A subset of vertices such that none are neighbors.

Degree The number of neighbors of a vertex.

Chromatic number The minimum number of colors necessary to properly color a graph.



Cliques shaded. Maximal cliques shaded **darker**.



Maximal independent set indicated in red (shaded nodes).

Original Casting as a Game

Panagopoulou and Spirakis. "A game theoretic approach for efficient graph coloring."

Why? There is no polynomial-time algorithm for coloring a graph with minimum colors.

Players n nodes of graph G.

Actions Action c_v is any one color x out of a finite set of colors X. Joint action is c.

Preferences Maximize utility:

$$\lambda_v(c) = \begin{cases} 0 & \exists u \in N(v) : c_v = c_u \\ n_{c_v} & \text{else} \end{cases}$$

What are the *advantages* or this construct?

What are the *disadvantages*?

Original Casting as a Game (con't)

Exact potential function

$$\Phi(c) = \frac{1}{2} \sum_{x \in X} n_x^2(c)$$

What does this buy us?

Price of Anarchy tight upper bound

$$\mathsf{R}(G) \leq \frac{\min\left\{\Delta_2(G) + 1, \frac{n+\omega(G)}{2}, \frac{1+\sqrt{1+8m}}{2}, n - \alpha(G) + 1\right\}}{\max\left\{\omega(G), \frac{n}{\alpha(G)}\right\}}$$

Explained:

- Δ_2 — max degree of a node whose neighbor is a node with max degree.
- ω — max clique number.
- m — number of edges: $|E|$.
- α — max independent set size.

Distributed Welfare Games

The approach just detailed took an age-old distributed control problem, cast it as a game, and published meaningful results. So what's the deal?

Marden and Wierman "Distributed Welfare Games" introduces a class of games with suggestions for utility design, guarantees of NE existence, and PoA bounds.

Separable welfare function for Graph Coloring:

$$W(c) = \sum_{x \in X} W^x(c), \text{where}$$

$$W^x(c) = \begin{cases} 0 & n_x = 0 \\ -1 & n_x > 0 \end{cases}$$

DWG—Utility Design

Graph Coloring utility feels ad-hoc.

Alternatives:

- Wonderful Life

$$\begin{aligned}\lambda_v^{\mathrm{WLU}}(c) &= W^{c_v}(c) - W^{c_v}(\emptyset, c_{-v}) \\ &= \begin{cases} 0 & n_{c_v} > 1 \\ -1 & n_{c_v} = 1 \end{cases}\end{aligned}$$

- Shapley Value

$$\begin{aligned}\lambda_v^{\mathrm{SV}}(c) &= \sum_{S \subseteq N_{c_v} \setminus v} \frac{(n_{c_v} - |S| - 1)!|S|!}{n_{c_v}!} \left[W^{c_v}(S \cup v) - W^{c_v}(S)\right] \\ &= \frac{(n_{c_v} - 0 - 1)!(0)!}{n_{c_v}!}(-1) \\ &= -\frac{1}{n_{c_v}}\end{aligned}$$

So now we have to go back and re-derive all of the Panagopoulou and Spirakis results?