CO 456: Introduction to Game Theory

University of Waterloo
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Contents

Cooperative Games

Lecture 1: Course Administration

Overview

Planned topics:

1. Combinatorial games
2. Strategic games
3. Mechanism design
4. Cooperative games

Instructor

Martin Pei (`mpei`).

Assignments

Roughly 30-40 assignment problems, up to 4 per week. Due Fridays at 11:59 pm Eastern on Crowdmark. All equally weighted, marked out of 10.

Exams

Three term tests and a final exam.

- Term test 1: Wednesday, October 7, 12:01 am to 11:59 pm EDT
- Term test 2: Wednesday, November 11, 12:01 am to 11:59 pm EST
- Term test 3: Wednesday, December 2, 12:01 am to 11:59 pm EST
- Final exam: scheduled during final exam period (December 9-23)

Each term test is allotted 2.5 hours within the 24-hour window (student's choice). Final exam is allotted 3 hours within its 24-hour window (student's choice). Open book.

Grading

- 40% assignment problems (lowest 7 dropped)
- Best 3 of 4 assessments:
  - 20% term test 1
  - 20% term test 2
  - 20% term test 3
  - 20% final exam

Combinatorial Games

Lecture 2: Impartial Games

Nim

We are given some piles of chips. Two players play alternately. On a player's turn, they pick a pile and remove at least 1 chip from it. The first player who cannot make a move loses.

Example

- 1, 1, 2.
  - Player I removes 2 chips from the last pile.
  - Player II removes a 1-chip pile.
  - Player I removes the last chip.
  - Player II has no move and loses.
  - Player I has a winning strategy.

  This is a **winning game** or **winning position**.
- 5, 5.
  - Regardless of Player I's move, Player II can mirror it on the other pile.
  - Player II always has a move, so Player I loses.
  - Player I always loses (*i.e.* Player II has a winning strategy).

  This is a **losing game** or **losing position**.
- 5, 7.
  - Player I first equalizes piles (here, removing 2 from the pile of 7).
  - Player II loses (by the previous case).

  This is a winning game.

Lemma 2.1

In instances of Nim with two piles of n, m chips, it is a winning game if and only if $n \neq m$.

Impartial games

Nim is an impartial game.

Definition — impartial game

Conditions for an **impartial game**:

1. There are two players, Player I (who starts) and Player II.
2. There are several positions and a starting position.
3. A player performs one of a set of allowable moves, which depends only on the current position and not on the player (hence "impartial"). Each possible move generates an **option**.
4. The players move alternately.
5. There is complete information.
6. There are no chance moves.
7. The first player with no available move loses.
8. The rules guarantee that games end.

Example

Games which are not impartial:

- Tic-tac-toe (violates 7—may tie)
- Chess (violates 3—may only move your pieces)
- Poker (violates 5—cards hidden)
- Monopoly (violates 6—relies on dice rolls)

Example

Let $G = (1, 1, 2)$ be a Nim game. There are 4 possible moves, hence 4 possible options.

$$\begin{tikzcd}
 & \boxed{H_1 = (1,1,1)} \\
 & \boxed{H_2 = (1,1,0)} \\
\boxed{G = (1,1,2)} \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & \boxed{H_3 = (1,0,2)} \\
 & \boxed{H_4 = (0,1,2)}
\end{tikzcd}$$

Each H_i is itself another Nim game.

Note: we can define an impartial game by its position and options recursively.

Definition — game simplicity

A game H that is reachable from game G by a sequence of allowable moves is **simpler** than G.

Example

Other impartial games:

- Subtraction game
  - One pile of chips.
  - Valid move: remove 1, 2, or 3 chips.
- Rook game
  - $m \times n$ chess board with a rook at (i, j).
  - Valid move: move the rook any number of spaces up or left.
- Green hackenbush game
  - Graph connected to the floor at some vertices.
  - Valid move: remove an edge of the graph, then any components no longer connected to the floor.

Spoiler: all impartial games are essentially Nim games.

Winning strategy

Lemma 2.2

In any game G, either Player I or Player II has a winning strategy.

Proof.

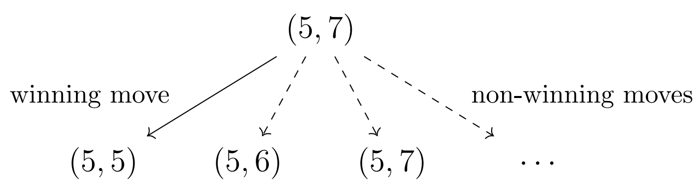
By induction on simplicity of G.

If G has no allowable moves, then Player I loses, so Player II has a winning strategy. Assume G has allowable moves and the lemma holds for all games simpler than G. Among all options of G, if Player I has a winning strategy in one of them, Player I will move to that option and win. Otherwise, Player II has a winning strategy for all options, so Player II wins regardless of Player I's move. □

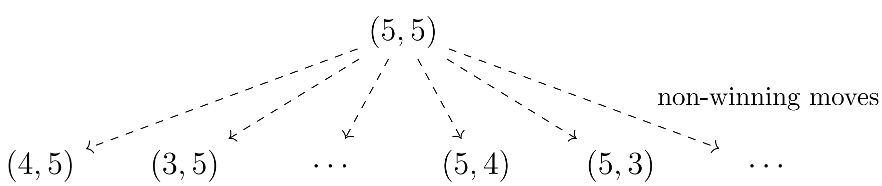
That is, every impartial game G is either a winning game or a losing game.

Example

Winning game of Nim (at least one winning move):



Losing game of Nim (no winning moves):



Note: we assume players play perfectly. If there is a winning move, then they will take it.

Lecture 3: Equivalent Games (I)

Game sums

Definition — game sum

Let G and H be two games with respective options $G_1, \ldots, G_m$ and $H_1, \ldots, H_n$. We define $G + H$ as the game with options

$$G_1 + H, \ldots, G_m + H, G + H_1, \ldots, G + H_n.$$

Example

We denote $*n$ to be a game of Nim with one pile of n chips. Then $*1 + *1 + *2$ is a Nim game with three piles of 1, 1, and 2 chips.

Example

Let $\#n$ be the subtraction game with n chips. Then $*5 + \#7$ is the game where a move is either to remove at least 1 chip from the pile of 5 (Nim) or to remove 1, 2, or 3 chips from the pile of 7 (subtraction game).

Lemma 3.1

Let $\mathcal{G}$ be the set of all impartial games. Then for all $G, H, J \in \mathcal{G}$,

1. $G + H \in \mathcal{G}$ (closure)
2. $(G + H) + J = G + (H + J)$ (associativity)
3. There exists an identity $0 \in \mathcal{G}$ (the game with no options) where $G + 0 = 0 + G = G$.
4. $G + H = H + G$ (symmetry)

Note: $\mathcal{G}$ is an abelian group, except for an inverse element.

Game equivalences

Definition — game equivalence

Two games G and H are **equivalent** if for any game J, $G+J$ and $H+J$ have the same outcome (*i.e.*, both are winning games or both are losing games). Notation: $G \equiv H$.

Example

$*3 \equiv *3$. Since $*3 + J$ is the same game as $*3 + J$, they have the same outcome.

$*3 \not\equiv *4$. $*3 + *3$ is a winning game but $*4 + *3$ is a losing game (Lemma 2.1).

Lemma 3.2

$*n \equiv *m$ if and only if $n = m$.

Lemma 3.3

The relation $\equiv$ is an equivalence relation. That is, for all $G, H, K \in \mathcal{G}$,

1. $G \equiv G$ (reflexivity).
2. $G \equiv H$ if and only if $H \equiv G$ (symmetry).
3. If $G \equiv H$ and $H \equiv J$, then $G \equiv J$ (transitivity).

Exercise 3.4

Prove that if $G \equiv H$, then $G + J \equiv H + J$ for any game J.

Proof.

Consider any games J and K. Let $M = J + K$. Then $G + J + K = G + M$ and $H + J + K = H + M$. Since $G \equiv H$, $G + M$ and $H + M$ have the same outcome. But then $(G + J) + K$ and $(H + J) + K$ have the same outcome by Lemma 3.1. K was arbitrary, so $G + J \equiv H + J$. $\square$

Losing games and empty Nim

Nim with one pile $*n$ is a losing game if and only if $n = 0$.

Theorem 3.5

G is a losing game if and only if $G \equiv *0$.

Corollary 3.6

If G is a losing game, then J and $J + G$ have the same outcome for any game J.

Proof.

Since G is a losing game, $G \equiv *0$ by Theorem 3.5. Then $J + G \equiv J + *0 \equiv J$ by Exercise 3.4 and Lemma 3.1, so J and $J + G$ have the same outcome. $\square$

Example

1. Recall $*5 + *5$ and $*7 + *7$ are losing games. Corollary 3.6 says $*5 + *5 + *7 + *7$ is also a losing game. (Player I moves in either $*5 + *5$ or $*7 + *7$. Player II plays a winning move in the same part by equalizing piles.)
2. $\underbrace{*1 + *1 + *2}_{\text{winning}} + \underbrace{*5 + *5}_{\text{losing}}$ is a winning game by Corollary 3.6. (Player I removes $*2$, leaving a similar game to the previous.)

Proof (Theorem 3.5).

($\Longleftarrow$) If $G \equiv *0$, then $G + *0$ has the same outcome as $*0 + *0$. But $*0 + *0$ is a losing game, so G is a losing game.

($\Longrightarrow$) Suppose G is a losing game. We show $G + J$ and $*0 + J \equiv J$ have the same outcome.

1. Suppose J is a losing game. We show "if G and J are both losing games, then $G + J$ is a losing game" by induction on simplicity of $G + J$.

   When $G + J$ has no options, G and J have no options, so G, J, and $G + J$ are all losing games. Assume $G + J$ has some otpions and the statement holds for simpler games. WLOG, Player I moves on G, resulting in $G' + J$. G is a losing game, so G' is a winning game. Player II makes a winning move from G' to G'', resulting in $G'' + J$. Then G'' is a losing game, so by induction $G'' + J$ is a losing game. Player I loses, so $G + J$ is a losing game.

(2) Suppose J is a winning game, so J as a winning move to J'. Player I moves from $G + J$ to $G + J'$. G and J' are both losing games, so by (1) $G + J'$ is a losing game. Player II loses, so Player I wins and $G + J$ is a winning game.

$\square$

Lecture 4: Equivalent Games (II)

Lemma 4.1 — copycat principle

For any game G, $G + G \equiv *0$.

Proof.

By induction on the simplicity of G.

When G has no options, $G + G$ has no options, so $G + G \equiv *0$ by Lemma 4.1.

Suppose G has options, and WLOG suppose Player I moves from $G + G$ to $G' + G$. Then Player II can move to $G' + G'$. By induction, $G' + G' \equiv *0$, so it is a losing game for Player I. Therefore, $G + G$ is a losing game, and $G + G \equiv *0$. □

Aside: this means G is its own "inverse".

Lemma 4.2

$G \equiv H$ if and only if $G + H \equiv *0$.

Proof.

($\implies$) From $G \equiv H$, we add H to both sides to get $G + H \equiv H + H \equiv *0$ by the copycat principle.

($\impliedby$) From $G + H \equiv *0$, we add H to both sides to get $G + H + H \equiv *0 + H \equiv H$. But $G + H + H \equiv G + *0 \equiv G$ by the copycat principle, so $G \equiv H$.

□

Example

$*1 + *2 + *3$ is a losing game, so $*1 + *2 + *3 \equiv *0$. By Lemma 4.2, $*1 + *2 \equiv *3$. Or, $*1 + *3 \equiv *2$.

Another way to prove game equivalence is by showing that they have equivalent options.

Lemma 4.3

If the options of G are equivalent to the options of H, then $G \equiv H$. (More precisely: there is a bijection between options of G and H where paired options are equivalent.)

Example

We can show $*1 + *2 \equiv *3$ by Lemma 4.3:

$$\begin{array}{ccc} *1 + *2 & & *3 \\ \hline *2 & \equiv & *2 \\ *1 & \equiv & *1 \\ *1 + *1 & \equiv & *0 \end{array}$$

Note: the converse of Lemma 4.3 is false.

Proof (Lemma 4.3).

It suffices to show that $G + H \equiv *0$ (by Lemma 4.2), *i.e.*, $G + H$ is a losing game. This is true when G and H both have no options. Suppose that G and H have options and suppose WLOG Player I moves to $G' + H$. By assumption, there exists an option of H, say H', where $H' \equiv G'$. So Player II can move to $G' + H'$. Since $G' \equiv H'$, $G' + H' \equiv *0$ by Lemma 4.2. So $G' + H'$ is a losing game for Player I. Hence $G + H$ is a losing game. $\square$

Lecture 5: Nim and Nimbers (I)

Goal: show that every Nim game is equivalent to a Nim game with a single pile.

Nimbers

Definition — nimber

If G is a game such that $G \equiv *n$ for some n, then n is the **nimber** of G.

Example

Any losing game has nimber 0 (Theorem 3.5).

Exercise 5.1

Show that the notion of a nimber is well-defined. (That is, every impartial game has exactly one nimber.)

Proof (see P3(b)). □

Theorem 5.2

If $n = 2^{a_1} + 2^{a_2} + \cdots$ where $a_1 > a_2 > \cdots$, then $*n \equiv *2^{a_1} + *2^{a_2} + \cdots$.

(The link to powers of 2 is hard to explain; we'll revisit this later.)

Example

$11 = 2^3 + 2^1 + 2^0$ and $13 = 2^3 + 2^2 + 2^0$. Using Theorem 5.2, $*11 \equiv *2^3 + *2^1 + *2^0$ and $*13 \equiv *2^3 + *2^2 + *2^0$. Then

$$\begin{aligned}
*11 + *13 &\equiv (*2^3 + *2^1 + *2^0) + (*2^3 + *2^2 + *2^0) \\
&\equiv (*2^3 + *2^3) + *2^2 + *2^1 + (*2^0 + *2^0) && \text{(associativity, commutativity)} \\
&\equiv *0 + *2^2 + *2^1 + *0 && \text{(copycat principle)} \\
&\equiv *2^2 + *2^1 \\
&\equiv *(2^2 + 2^1) && \text{(Theorem 5.2)} \\
&\equiv *6.
\end{aligned}$$

So the nimber of $*11 + *13$ is 6.

In general, how can we find the nimber of $*b_1 + *b_2 + \cdots + *b_n$? We look at the binary expansions of each b_i. The copycat principle will cancel any pair of identical powers of 2. So, we should look for powers of 2 that appear in an odd number of expansions of b_i's.

We can use binary numbers: 11 in binary is 1011, 13 in binary is 1101. Take the XOR of the binary representations:

$$\begin{array}{r} \texttt{1011} \\ \oplus\ \texttt{1101} \\ \hline \texttt{0110} \end{array} = 6$$

So $11 \oplus 13 = 6$.

Example

Consider $*25 + *21 + *11$. The nimber of this game is given by the binary XOR of the numbers:

$$\begin{array}{r} \texttt{11001} \\ \texttt{10101} \\ \oplus\ \texttt{01011} \\ \hline \texttt{00111} \end{array}$$

So $*25 + *21 + *11 \equiv *7$ (the nimber is 7).

Corollary 5.3

$*b_1 + *b_2 + \cdots + *b_n \equiv *(b_1 \oplus b_2 \oplus \cdots \oplus b_n)$.

This shows that every Nim game has a nimber.

Winning strategy for Nim

Example

$*11 + *13 \equiv *6$. This is a winning game. How can we find a winning move?

We want to move to a game that is equivalent to $*0$. We can add $*6$ to both sides: $*11 + *13 + *6 \equiv *6 + *6 \equiv *0$ (copycat principle). But this isn't a valid move.

Consider $*11 + (*13 + *6)$. We see $13 \oplus 6 = 11$. So this is equivalent to $*11 + *11$, a losing game.

The winning move: remove 2 chips from the pile of 13.

Example

$*25 + *21 + *11 \equiv *7$. Add $*7$ to both sides. Consider $*25 + (*21 + *7) + *11$. We see $21 \oplus 7 = 18$, so this is equivalent to $*25 + *18 + *11$.

The winning move: remove 3 chips from the pile of 21.

Why did we pair $*7$ with $*21$ instead of $*25$ or $*11$? Those would be invalid moves: $25 \oplus 7 = 30 > 25$ and $11 \oplus 7 = 12 > 11$.

Can we always pair the nimber with a pile such that the resulting equivalent game is simpler? Yes.

Lemma 5.4

If $*b_1 + \cdots + *b_n \equiv *s$ where $s > 0$, then there exists some b_i where $b_i \oplus s < b_i$.

Proof idea: look for the largest power of 2 in s. Consider $*25 + *21 + *11 \equiv *7$.

$$
\begin{array}{cccccc}
 & 1 & 1 & \boxed{0} & 0 & 1 \\
 & 1 & 0 & \boxed{1} & 0 & 1 \\
\oplus & 1 & 1 & \boxed{0} & 0 & 1 \\
\hline
 & 0 & 0 & \boxed{1} & 1 & 1 \\
 & & & \uparrow & \uparrow & \uparrow \\
 & & & 4 & 2 & 1
\end{array}
\qquad
\begin{array}{rl}
25 & 25 \oplus 7\text{: } 4 \text{ is added} \\
21 & 21 \oplus 7\text{: } 4 \text{ is subtracted} \\
11 & 11 \oplus 7\text{: } 4 \text{ is added} \\
7 & \\
 & \\
\leftarrow & 4 > 2 + 1 \text{ so } \oplus \text{ reduces 21 and increases 25 or 11}
\end{array}
$$

Proof (Lemma 5.4).

Suppose $s = 2^{a_1} + 2^{a_2} + \cdots$ where $a_1 > a_2 > \cdots$. Then 2^{a_1} appears in the binary expansions of $b_1, \ldots, b_n$ an odd number of times (in particular, at least once). Let b_i be one of them. Suppose $*b_i + *s \equiv *t$ for some t. Since 2^{a_1} is in the binary expansions of b_i and s, 2^{a_1} is not in the binary expansion of t. For $2^{a_2}, 2^{a_3}, \ldots$, at worst none of them are in the binary expansion of b_i, so all of them are in the binary expansion of t. So $t \leq b_i - 2^{a_1} + 2^{a_2} + 2^{a_3} + \cdots < b_i$ since $2^{a_i} > 2^{a_2} + 2^{a_3} + \cdots$. $\square$

Finding winning moves in a winning Nim game: Say a game has nimber s. Look at the largest power of 2 in the binary expansion of s. Pair it up with any pile $*b_i$ containing this power of 2. By Lemma 5.4, $s \oplus b_i < b_i$. So a winning move is taking away $b_i - (s \oplus b_i)$ chips from the pile $*b_i$.

Lecture 6: Nim and Nimbers (II)

Lemma 6.2

Let $0 \le p, q < 2^a$ and suppose Theorem 5.2 holds for all values less than 2^a. Then $p \oplus q < 2^a$.

Proof (exercise: see P3(c))

□

Illustration of proof of Theorem 5.2: Consider $*7$. $7 = 4 + 2 + 1$. We want to prove $*7 \equiv *4 + *2 + *1$; by induction, $*2 + *1 \equiv *3$. We want to show $*7 \equiv *4 + *3$. By Lemma 4.3, we can show the two sides have equivalent options.

Options of $*7$: $*0, *1, \ldots, *6$.

Options of $*4 + *3$: (1) Move on $*4$. (2) Move on $*3$.

(1) Options are $*0 + *3$, $*1 + *3$, $*2 + *3$, $*3 + *3$. Each part is < 4, so by Lemma 6.2 each option is < 4. (Calculating them, we have $*3, *2, *1, *0$, so they are also distinct.)

(2) Options are $*4 + *2$, $*4 + *1$, $4 + *0$. Here, each first part is 4 and each second part is < 4, so each power of 2 appears at most once among the two parts. We can apply induction here. (Calculating them, we have $*6, *5, *4$, which are exactly the remaining options of $*7$.)

Proof (Theorem 5.2).

By induction on n.

When $n = 1$, $n = 2^0$ and $*1 \equiv *2^0$.

Suppose $n = 2^{a_1} + 2^{a_2} + \cdots$ where $a_1 > a_2 > \cdots$. Let $q = n - 2^{a_1} = 2^{a_2} + 2^{a_3} + \cdots$. If $q = 0$, then $n = 2^{a_1}$, so $*n \equiv *2^{a_1}$ (Lemma 3.1). Assume $q \ge 1$. Since $q < n$, by induction, $*q \equiv *2^{a_2} + *2^{a_3} + \cdots$. It remains to show that $*n \equiv *2^{a_1} + *q$.

The options of $*n$ are $*0, *1, \ldots, *(n-1)$. The options of $*2^{a_1} + *q$ can be partitioned into two types.

(1) Consider options of the form $*i + *q$ where $0 \le i < 2^{a_1}$. Since $i, q < n$, by induction, the theorem holds for i and q. So $*i$ and $*q$ are equivalent to sums of Nim piles of their binary expansions. Using arguments from the last lecture (cancellation by copycat principle, as in Corollary 5.3), $*i + *q \equiv *r_i$ where $r_i = i \oplus q$. Since $i, q < 2^{a_1}$, we have $r_i < 2^{a_1}$ (Lemma 6.2). So $0 \le r_0, r_1, \ldots, r_{2^{a_1}-1} < 2^{a_1}$.

We now show these r_i's are distinct. Suppose $r_i = r_j$ for some $i \ne j$. Then $*r_i \equiv *r_j$ (Lemma 3.2), so $*i + *q \equiv *j + *q$. Adding $*q$ on both sides, we get

$*i \equiv *j$ (copycat principle), so $i = j$. Contradiction.

Finally, there are 2^{a_1} of the r_i's and there are 2^{a_1} possible values (0 to $2^{a_1} - 1$). By pigeonhole, for each $0 \leq j \leq 2^{a_1} - 1$, there is exactly one r_i with $r_i = j$. So the options of this type are equivalent to $\{*0, *1, \ldots, *2^{a_1} - 1\}$.

2. Consider options of the form $*2^{a_1} + *i$ where $0 \leq i < q$. Suppose $i = 2^{b_1} + 2^{b_2} + \cdots$ where $b_1 > b_2 > \cdots$. Then no b_i is equal to a_1. So $2^{a_1} + 2^{b_1} + 2^{b_2} + \cdots$ is a sum of distinct powers of 2. Then

$$\begin{aligned} *2^{a_1} + *i &\equiv *2^{a_1} + *2^{b_1} + *2^{b_2} + \cdots && \text{(applying induction on } i\text{)} \\ &\equiv *(2^{a_1} + 2^{b_1} + 2^{b_2} + \cdots) && \text{(applying induction on } 2^{a_1} + i\text{)} \\ &\equiv *(2^{a_1} + i). \end{aligned}$$

Since $0 \leq i < q$, the options of this type are equivalent to

$$\{*2^{a_1}, *(2^{a_1} + 1), \ldots, *\underbrace{(2^{a_1} + q - 1)}_{n-1}\}.$$

Combining the two types of options, we see that the options of $*2^{a_1} + *q$ are equivalent to the options of $*n$. By Lemma 4.2, $*2^{a_1} + *q \equiv *n$. $\square$

Lecture 7: Sprague–Grundy Theorem

So far: all Nim games are equivalent to a Nim game of a single pile.

Goal: extend this to all impartial games.

Poker nim

Being equivalent does not mean that two games play the same way.

Example

$*11 + *13 \equiv *6$. LHS: we want to move to $*11 + *11 \equiv *0$ by removing 2 chips from $*13$. RHS: remove all 6 chips.

There are other moves in the game, however. Say we move to $*11 + *8 \equiv *15$. LHS: we remove 5 chips from $*13$. RHS: add 9 chips.

Or, starting with $*11 + *11 \equiv *0$, any move on $*11 + *11$ will increase $*0$. Technically, adding chips is not allowed in Nim.

Definition — poker nim

A variation on Nim: **poker nim** consists of a regular Nim game plus a bag of B chips. We now allow regular Nim moves and adding $B' \leq B$ chips to one pile.

Example

In poker nim, $*3 + *4$ can move to $*53 + *4$ (assuming $B \geq 50$).

How does this change the game of Nim? It doesn't.

Say we face a losing game, so any regular Nim moves would lead to a loss. In poker nim, we now add some chips to one pile. The opposing play will simply remove the chips we placed, so nothing has changed. Adding chips is not beneficial to any player.

When we say that a game is equivalent to a Nim game with one pile, it is actually a *poker nim* game with one pile.

Mex

Suppose a game G has options equivalent to $*0, *1, *2, *5, *10, *25$. We claim that $G \equiv *3$.

The options of $*3$, which are $*0, *1, *2$, are all available to G. If we add chips to $*3$, then the opposing player can just remove them to get back to $*3$.

How did we get $*3$? It is the smallest non-negative integer that is not an option of G.

Definition — mex

Given a set of non-negative integers S, $\text{mex}(S)$ is the smallest non-negative integer not in S. (Mex stands for "minimum excluded integer".)

Example

$\text{mex}(\{0, 1, 2, 5, 15, 25\}) = 3$.

The mex function is the critical link between any impartial games and Nim games.

Theorem 7.1

Let G be an impartial game and let S be the set of integers n such that there exists an option of G equivalent to $*n$. Then $G \equiv *(\text{mex}(S))$.

Example

$*1 + *1 + *2$ has options

- $*1 + *2 \equiv *3$,
- $*1 + *1 \equiv *0$, and
- $*1 + *1 + *1 \equiv *1$.

By Theorem 7.1, $*1 + *1 + *2 \equiv *(\text{mex}(\{0, 1, 3\})) \equiv *2$. (This is obvious by the copycat principle.)

Exercise 7.2

Prove that a game cannot be equivalent to one of its options.

Proof (Theorem 7.1).

Let $m = \text{mex}(S)$. It suffices to show that $G + *m \equiv *0$ (Lemma 4.2).

1. Suppose we move to $G + *m'$ where $m' < m$. Since $m = \text{mex}(S)$, there exists an option G' of G such that $G' \equiv *m'$. Player II moves to $G' + *m'$, which is a losing game since $G' \equiv *m'$ (Lemma 4.2).
2. Suppose we move to $G' + *m$ where G' is an option of G. Then $G' \equiv *k$ for some $k \in S$. So $G' + *m \equiv *k + *m \not\equiv *0$ since $k \neq \text{mex}(S)$. So $G' + *m$ is a winning game for Player II. Then $G + *m$ is a losing game for Player I, so $G + *m \equiv *0$.

□

Theorem 7.3 — Sprague–Grundy Theorem

Any impartial game G is equivalent to a poker nim game $*n$ for some n.

Slightly sketchy proof.

If G has no options, then $G \equiv *0$. Suppose G has options $G_1, \ldots, G_k$. By induction, $G_i \equiv *n_i$ for some n_i. By Theorem 7.1, $G \equiv *(\text{mex}(\{n_1, \ldots, n_k\}))$. □

So any impartial game has a nimber. How does this help?

Lecture 8: Finding Nimo

Finding nimbers is recursive:

- Games with no options have nimber 0.
- Move backwards and use mex to determine other nimbers.

Example

Rook game.

	1	2	3	4	5
1	$*0$	$*1$	$*2$	$*3$	$*4$
2	$*1$	$*0$	$*3$	$*2$	$*5$
3	$*2$	$*3$	$*0$	$*1$	$*6$
4	$*3$	$*2$	$*1$	$*0$	$*7$

Winning move: move to $(4, 4)$, an option with nimber 0.

This is like a 2-pile Nim game.

Example

Substraction game (remove 1, 2, or 3 chips). Let s_n be the nimber of a subtraction game with n chips. Then $s_n = \text{mex}(\{s_{n-1}, s_{n-2}, s_{n-3}\})$ (if they exist).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
s_n	0	1	2	3	0	1	2	3	0	1	2	3	0	$\cdots$

We see a game is a losing game if and only if $n \equiv 0 \pmod 4$. When $n \not\equiv 0 \pmod 4$, a winning move is to remove just enough chips to get to the next multiple of 4. For example, if $n = 7$, remove 3 chips. (Equivalently, remove s_n chips.)

Example

Subtraction game with removing 2, 5, or 6 chips. Then $s_n = \text{mex}(\{s_{n-2}, s_{n-5}, s_{n-6}\})$ (if they exist).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
s_n	0	0	1	1	0	2	1	3	0	2	1	0	0	1	1	$\cdots$

repeats (not proved here)

A game is a losing game if and only if $n \equiv 0, 1, 4, 8 \pmod{11}$.

For example, a winning move from 9 chips is to move to 4.

Example: combining games

Let G be the 4×5 rook game at $(4, 2)$. Let H be the second subtraction game with $n = 7$.

	1	2	3	4	5
1	$*0$	$*1$	$*2$	$*3$	$*4$
2	$*1$	$*0$	$*3$	$*2$	$*5$
3	$*2$	$*3$	$*0$	$*1$	$*6$
4	$*3$	$*2$	$*1$	$*0$	$*7$

n	0	1	2	3	4	5	6	7
s_n	0	0	1	1	0	2	1	3

Then $G \equiv *2$ and $H \equiv *3$, so $G + H \equiv *2 + *3 \equiv *1$, which is a winning game.

The winning moves:

- From H, $3 \oplus 1 = 2$. Move to $*2$. Remove 2 chips in the subtraction game.
- From G, $2 \oplus 1 = 3$. Move to $*3$ (this may be surprising since the nimber increases, but it is entirely legal here). Move to $(4, 1)$ or $(3, 2)$.

Notes:

- In general, there may not be a pattern for the nimbers of impartial games.
- Because of the recursive nature of nimbers, the search space becomes too large for many games.
- For impartial games in which we can find the nimbers, we can find winning moves by considering the nimbers.

Strategic Games

Lecture 9: Strategic Games

Prisoner's dilemma

Game show version: 2 players won \$10,000. They each need to make a final decision: "share" or "steal".

- If both pick "share", then they each win \$5,000.
- If one picks "steal" and the other picks "share", then the one who picked "steal" gets \$10,000 and the other gets nothing.
- If both pick "steal", then they each get a small consolation prize worth \$10.

This is an example of a strategic game.

How should the players behave? The benefit a player receives is dependent on their own decision and the decisions of other players.

Definition — strategic game

A **strategic game** is defined by specifying a set $N = \{1, \ldots, n\}$ of players where for each player $i \in N$, there is a set of possible strategies S_i to play and a utility function $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$.

Example

With the prisoner's dilemma above, $s_1 = s_2 = \{\text{share}, \text{steal}\}$. Samples of the utility functions: $u_1(\text{share}, \text{share}) = 5000$, $u_2(\text{steal}, \text{share}) = 0$. We can summarize the utility functions in a **payoff table**.

		P II	
		share	steal
P I	share	5000, 5000	0, 10000
	steal	10000, 0	10, 10

Each cell records the utilities of P I, P II in that order given the strategies played in that row (P I) and column (P II).

1. All players are rational and selfish (want to maximize their own utility).
2. All players have knowledge of all game parameters (including rationality and selfishness).
3. All players move simultaneously.
4. Player i plays a strategy $s_i \in S_i$, forming a strategy profile $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$. Player i earns $u_i(s)$.

Resolving the prisoner's dilemma

Given a strategic game, what are we looking for? One answer is we want to know how the players are expected to behave.

In the prisoner's dilemma, what would a rational and selfish player choose to play?

1. If you know that the other player chooses "share", then choosing "share" gives 5000 while choosing "steal" gives 10000. "Steal" is better.

2. If you know that the other player chooses "steal", then choosing "share" gives 0 while choosing "steal" gives 10. "Steal" is better.

In both cases, it is better to steal than to share. So we expect both player to choose "steal".

This is an example of a **strictly dominating** strategy: regardless of how other players behave, this strategy gives the best utility over all other possible strategies. If a strictly dominating strategy exists, then we expect the players to play it.

In this case, playing a strictly dominating strategy ("steal") yields very little benefit. The players could get more if there is some cooperation (both share). So even though we expect the strictly dominating strategy to be played, it might not have the best "social welfare" (total utility of the players).

Nash equilibrium (NE)

There are many games with no strictly dominating strategies.

Example: Bach or Stravinsky?

Two players want to go to a concert. Player I likes Bach while Player II likes Stravinsky, but they both prefer to be with each other. The payoff table looks like this:

		P II	
		Bach	Stravinsky
P I	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

No strictly dominating strategy exists for either player. What do we expect to happen? If both players choose "Bach", then there is no reason for one player to switch their strategy (which gives utility 0). The result is similar if both choose "Stravinsky".

These are steady states, which we call **Nash equilibria**: strategy profiles where no player is incentivized to change strategy.

Mixed strategies

There are many games with "no" Nash equilibria.

Example: rock, paper, scissors

R beats S, S beats P, P beats R. Let utilities be 1 for a win, 0 for a tie, and -1 for a loss.

		P II		
		R	P	S
P I	R	$0, 0$	$-1, 0$	$1, -1$
	P	$1, -1$	$0, 0$	$-1, 1$
	S	$-1, 1$	$1, -1$	$0, 0$

Lecture 10: Nash Equilibrium and the Best Response Function

Notation

Let $S = S_1 \times \cdots S_n$ be the set of all strategy profiles. We will often compare the utilities of a player's strategies when we fix the strategies of the remaining players. Let S_{-i} be the set of all strategy profiles of all players except player i (we drop S_i from the cartesian product $S_1 \times \cdots \times S_n$). If $s \in S$, then the profile obtained from s by dropping s_i is denoted $s_{-i} \in S_{-i}$. If player i switches their strategy from s_i to s_i', then the new strategy profile is denoted $(s_i', s_{-i}) \in S$.

Nash equilibrium

Recall: a Nash equilibrium is a strategy profile where no player is incentivized to switch strategies.

Definition — Nash equilibrium

A strategy profile $s^* \in S$ is a **Nash equilibrium** if $u_i(s^*) \geq u_i(s_i', s_{-i}^*)$ for all $s_i' \in S_i$ and for all $i \in N$.

Example: prisoner's dilemma revisited

		P II share	P II steal
P I	share	5000, 5000	0, 10000
	steal	10000, 0	10, 10

Let $s^* = (\text{steal}, \text{steal})$.

For Player I: $u_1(s^*) = 10$, $u_1(\underbrace{\text{share}}_{s_1'}, \underbrace{\text{steal}}_{s_{-1}^*}) = 0 < u_1(s^*)$.

Similar for Player II. Thus s^* is an NE.

Example: guessing $\frac{2}{3}$ average game

There are three players and a positive integer k. The players each simultaneously pick an integer from $\{1, \ldots, k\}$, producing the strategy profile $s = (s_1, s_2, s_3)$. There is \$1 which is split among all players whose choices are closest to $\frac{2}{3}$ of the average of the three numbers. The other players get \$0.

If $s = (5, 2, 4)$, then the average is $\frac{11}{3}$, and $\frac{2}{3}$ of the average is $\frac{22}{9} = 2 + \frac{4}{9}$. Player II is the closest, so $u_2(s) = 1$ and $u_1(s) = u_3(s) = 0$.

Is s an NE? No: if Player II switches to 2, then $u_1(2, s_{-1}) = u_1(2, 2, 4) = \frac{1}{2}$ ($\frac{2}{3}$ of the average is now $\frac{16}{9}$, which is closer to 2 than to 4).

Is there an NE in this game? Idea: lowering your guess generally pulls $\frac{2}{3}$ of the average closer to your guess. Try $(1, 1, 1)$. If a player switches to $t \geq 2$, then $\frac{2}{3}$ of the average is $\frac{4+2t}{9} = \frac{4}{9} + \frac{2}{9}t$, which is closer to 1 than to t.

Exercise 10.1

Prove that $(1, 1, 1)$ is the only NE of this game.

Best response function (BRF)

For an NE, a player does not want to switch. If you fix the strategies of the remaining players, then you play a strategy that maximizes utility for yourself, *i.e.*, it is a "best response" to the fixed strategies.

Definition — best response function (BRF)

Player i's **best response function** for $s_{-i} \in S_{-i}$ is given by

$$B_i(s_{-i}) = \{s_i' \in S_i : u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \ \forall s_i \in S_i\}.$$

That is, the utility of a best response is greater than or equal to the utilities of all possible responses to s_{-i}.

Example

In the prisoner's dilemma: $B_1(\text{share}) = \{\text{steal}\}$, $B_1(\text{steal}) = \{\text{steal}\}$.

Example

In the $\frac{2}{3}$ average game, $B_1(5,5) = \{1,2,3,4\}$:

$$u_1(x,5,5) = \begin{cases} 1 & \text{if } \ x < 5 \\ \frac{1}{3} & \text{if } \ x = 5 \\ 0 & \text{if } \ x > 5 \end{cases}$$

If s^* is an NE, then each player i must have played a best response to s^*_{-i}. Changing s^*_i cannot increase the utility for i. The converse is also true.

Lemma 10.2

$s^* \in S$ is a Nash equilibrium if and only if $s^*_i \in B_i(s^*_{-i})$ for all $i \in N$.

Proof (exercise: see P8(a)).

($\Longrightarrow$) Here, $s^* \in S$ is a Nash equilibrium. By definition of a Nash equilibrium, $u_i(s^*) \geq u_i(s_i, s^*_{-i})$ for all $s_i \in S_i$ and for all $i \in N$. Then note $s^* = (s^*_i, s^*_{-i})$, so equivalently $u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$ for all $s_i \in S_i$ and for all $i \in N$. By definition of the BRF, $s^*_i \in B_i(s^*_{-i})$ for all $i \in N$.

($\Longleftarrow$) Here, $s^*_i \in B_i(s^*_{-i})$ for all $i \in N$. By definition of the BRF, every s^*_i satisfies $u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \ \forall s_i \in S_i$. But $(s^*_i, s^*_{-i}) = s^*$, so $u_i(s^*) \geq u_i(s_i, s^*_{-i})$ for all $s_i \in S_i$ and for all $i \in N$. By definition of a Nash equilibrium, s^* is a Nash equilibrium.

□

This lemma helps us find NEs by looking for strategies in the BRF.

Example

In the prisoner's dilemma payoff table, we mark best responses for each player:

$\star$: $B_1(\text{share}) = \{\text{steal}\}$, $B_1(\text{steal}) = \{\text{steal}\}$

$\dagger$: $B_2(\text{share}) = \{\text{steal}\}$, $B_2(\text{steal}) = \{\text{steal}\}$

		P II	
		share	steal
P I	share	$5000, 5000$	$0, 10000^\dagger$
	steal	$10000^\star, 0$	$10^\star, 10^\dagger$

$(\text{steal}, \text{steal})$ is a strategy profile where all strategies are best responses to each other, so it is an NE.

Example

Consider this arbitrary game. Find the BRFs and NEs.

		P II		
		X	Y	Z
P I	A	$1, 2$	$2, 1$	$1, 0$
	B	$2, 1$	$0, 1$	$0, 0$
	C	$0, 1$	$0, 0$	$1, 2$

BRFs:

$\star$: $B_1(X) = \{B\}, \quad B_1(Y) = \{A\}, \quad B_1(Z) = \{A, C\}$

$\dagger$: $B_2(A) = \{X\}, \quad B_2(B) = \{X, Y\}, \quad B_2(C) = \{Z\}$

		P II		
		X	Y	Z
P I	A	$1, 2^\dagger$	$2^\star, 1$	$1^\star, 0$
	B	$2^\star, 1^\dagger$	$0, 1^\dagger$	$0, 0$
	C	$0, 1$	$0, 0$	$1^\star, 2^\dagger$

The NEs are (B, X) and (C, Z) as the strategies are best responses to each other. The rest are not NEs as one strategy is not a best response to the other.

Lecture 11: Cournot's Oligopoly Model

We have a set $N = \{1, \ldots, n\}$ of n firms producing a single type of goods sold on the common market. Each firm i needs to decide the number of units of goods q_i to produce. Production cost is $C_i(q_i)$ where C_i is a given increasing function. Given a strategy profile $q = (q_1, \ldots, q_n)$, a unit of the goods sell for the price of $P(q)$, where P is a given non-increasing function on $\sum_i q_i$ (that is, more goods means a lower price). The utility of firm i in the strategy profile q is $u_i(q) = q_i P(q) - C_i(q_i)$.

Szidarovsky and Yakowitz proved that a Nash equilibrium always exists under some continuity and differentiability assumptions on P and C.

Special case: linear costs and prices

Suppose we assume $C_i(q_i) = cq_i$ $\forall i \in N$ (the cost is linear with the same unit cost c for all firms) and $P(q) = \max\{0, \alpha - \sum_j q_j\}$ (price starts at α, decreases by 1 for each unit produced, and has minimum 0) where $0 < c < \alpha$.

The utility is

$$u_i(q) = q_i P(q) - C_i(q_i) = \begin{cases} q_i(\alpha - c - \sum_j q_j) & \text{if } \alpha - \sum_j q_j \geq 0 \\ -cq_i & \text{if } \alpha - \sum_j q_j < 0 \end{cases}.$$

When is it possible to make a profit? When $\alpha - c - \sum_j q_j > 0$.

Separate q_i from the sum: $\alpha - c - q_i - \sum_{j \neq i} q_j > 0$, so $q_i < \alpha - c - \sum_{j \neq i} q_j$. This does not make sense for q_i if RHS ≤ 0, so assume RHS > 0.

The utility is then $q_i(\alpha - c - q_i - \sum_{j \neq i} q_j)$. Treating q_i as the variable, this utility is maximized when $q_i = \frac{1}{2}(\alpha - c - \sum_{j \neq i} q_j)$ (use calculus). The best response function for firm i given the production of the other firms q_{-i} is then

$$B_i(q_{-i}) = \begin{cases} \{\frac{1}{2}(\alpha - c - \sum_{j \neq i} q_j)\} & \text{if } \alpha - c - \sum_{j \neq i} q_j > 0 \\ \{0\} & \text{otherwise} \end{cases}.$$

Two-firm case

Suppose we simplify to 2 firms. Suppose $q^* = (q_1^*, q_2^*)$ is a Nash equilibrium. By Lemma 10.2, a player's choice must be the best response to the other player's choice. So $q_1^* \in B_1(q_2^*)$ and $q_2^* \in B_2(q_1^*)$. Verify that we may assume $q_1^*, q_2^* > 0$. Then $q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$ and $q_2^* = \frac{1}{2}(\alpha - c - q_1^*)$. Solving this gives $q_1^* = q_2^* = \frac{1}{2}(\alpha - c)$. This is the amount we expect each firm to produce at equilibrium. The price at equilibrium is then

$$P(q^*) = \alpha - q_1^* - q_2^* = \alpha - \frac{2}{3}(\alpha - c) = \frac{1}{3}\alpha + \frac{2}{3}c$$

and the profit at equilibrium is

$$u_i(q^*) = q_i^*(\alpha - c - q_1^* - q_2^*) = \frac{1}{9}(\alpha - c)^2.$$

Note 1: Suppose the two firms can collude, and together they produce Q units in total. The total profit is then $Q(\alpha - c - Q)$, which is maximized at $Q = \frac{1}{2}(\alpha - c)$. The profit is then $\frac{1}{2}(\alpha - c)(\alpha - c - \frac{1}{2}(\alpha - c)) = \frac{1}{4}(\alpha - c)^2$. Each firm gets $\frac{1}{8}(\alpha - c)^2 > \frac{1}{9}(\alpha - c)^2$.

Note 2: In the general case with n firms, if q^* is an NE, then $q_i^* = \frac{1}{2}(\alpha - c - \sum_{j \neq i} q_j^*)$. Solving this system gives $q_i^* = \frac{\alpha - c}{n+1}$. The price is $P(q^*) = \alpha - \sum_j q_j^* = \alpha - \frac{n}{n+1}(\alpha - c) = \frac{1}{n+1}\alpha + \frac{n}{n+1}c$. As $n \to \infty$, $P(q^*) \to c$. As more firms are involved, the expected market price gets closer to the production cost.

Lecture 12: Strict Dominance

Definition — strict dominance

For two strategies $s_i^{(1)}, s_i^{(2)} \in S_i$ for player i, we say that $s_i^{(1)}$ **strictly dominates** $s_i^{(2)}$ if for all $s_{-i} \in S_{-i}$, $u_i(s_i^{(1)}, s_{-i}) > u_i(s_i^{(2)}, s_{-i})$.

If there exists a strategy that strictly dominates s_i, then s_i is **strictly dominated**. If s_i strictly dominates all strategies $s_i' \in S_i \setminus \{s_i\}$, then s_i is a **strictly dominating strategy**.

In prisoner's dilemma, "steal" is a strictly dominating strategy for both players.

Lemma 12.1

If $s_i \in S_i$ is a strictly dominating strategy for player i and $s^* \in S$ is a Nash equilibrium, then $s_i^* = s_i$.

In other words: in any NE, the strictly dominating strategy is played whenever it exists. A game is easy to play if such a strategy exists.

We now look at strictly dominated strategies.

Example

Consider the following strategic game:

	X	Y	Z
A	$4,2$	$1,3$	$2,1$
B	$2,3$	$0,1$	$3,1$

Z is strictly dominated by X since $u_2(A,X) > u_2(A,Z)$ and $u_2(B,X) > u_2(B,Z)$. Z is a strictly dominated strategy, so there is no reason to play it.

Lemma 12.2

If $s^* \in S$ is a Nash equilibrium, then s_i^* is not strictly dominated for any $i \in N$.

Iterated elimination of strictly dominated strategies (IESDS)

Example

	X	Y	Z
A	$4,2$	$1,3$	$2,1$
B	$2,3$	$0,1$	$3,1$

Z is strictly dominated, so it will not appear in any NE.

$\downarrow$ eliminate Z

	X	Y
A	$4,2$	$1,3$
B	$2,3$	$0,1$

B is strictly dominated. (Notice this is new in this smaller game.)

$\downarrow$ eliminate B

	X	Y
A	$4,2$	$1,3$

X is strictly dominated.

$\downarrow$ eliminate X

	Y
A	$1,3$

Is this an NE? (Yes.)

IESDS: Repeatedly eliminate strictly dominated strategies until we have only one strategy profile. We claim that if this works, then the surviving profile is the unique NE of the game.

Example: facility location game

Two firms are each given a permit to open one store in one of six towns along a highway. Firm I can open in A, C, or E; Firm II can open in B, D, or F.

A — B — C — D — E — F

Assume towns are equally spaced and equally populated. Customers in a town will go to the closest store. Where should the firms open the stores?

Apply IESDS:

		Firm II		
		B	D	F
Firm I	A	$1,5$	$2,4$	$3,3$
	C	$4,2$	$3,3$	$4,2$
	E	$3,3$	$2,4$	$5,1$

Firm I: A is strictly dominated by C.
Firm II: F is strictly dominated by D.

$$\downarrow \text{ eliminate A and F}$$

Firm II

Firm I	B	D
C	$4,2$	$3,3$
E	$3,3$	$2,4$

Firm I: E is strictly dominated by C.
Firm II: B is strictly dominated by D.

$$\downarrow \text{ eliminate B and E}$$

Firm II

Firm I	D
C	$3,3$

(C, D) is an NE.

Note: We can extend this to 1000 towns with alternating options. The two ends are strictly dominated by the centre towns. eliminate them to get 998 towns. Repeat. End with the two towns in the centre as the NE.

Results in IESDS

Theorem 12.3

Suppose G is a strategic game. If IESDS ends with only one strategy profile s^*, then s^* is the unique Nash equilibrium of G.

This is a consequence of the following result.

Theorem 12.4

Let G be a strategic game where s_i is a strictly dominated strategy for player i. Let G' be obtained from G by removing s_i from S_i. Then s^* is a Nash equilibrium of G if and only if s^* is a Nash equilibrium of G'.

Proof sketch.

($\Longrightarrow$) Suppose s^* is an NE of G. Since s_i is strictly dominated, it cannot appear in s^* (Lemma 12.2). So s^* is a valid strategy in G'. If s^* is not an NE of G', then a player can deviate to get a higher utility. However, all strategies in G' are available in G, so such a player can do it in G as well. This contradicts s^* being an NE of G.

($\Longleftarrow$) Suppose s^* is an NE of G'. Suppose s^* is not an NE in G. Then a player can deviate to get a higher utility. This can be replicated in G' (which results in a contradiction) unless it is player i switching to strategy s_i (the only strategy in G not in G'). Then player i could switch to the strategy that strictly dominates s_i (available in G') to get a higher utility in G'. This contradicts s^* being an NE in G'.

$\square$

Lecture 13: Weak Dominance

Definition — weak dominance

For two strategies $s_i^{(1)}, s_i^{(2)} \in S_i$ for player i, we say that $s_i^{(1)}$ **weakly dominates** $s_i^{(2)}$ if for all $s_{-i} \in S_{-i}$, $u_i(s_i^{(1)}, s_{-i}) \geq u_i(s_i^{(2)}, s_{-i})$, and this inequality is strict for at least one $s_{-i} \in S_{-i}$.

If there exists a strategy that weakly dominates s_i, then s_i is **weakly dominated**. If s_i weakly dominates all strategies $s_i' \in S_i \setminus \{s_i\}$, then s_i is a **weakly dominating strategy**.

Example

Consider this strategic game:

	X	Y	Z
A	$3,3$	$1,1$	$4,1$
B	$2,1$	$0,1$	$3,1$

Z is weakly dominated by X since $u_2(A,X) > u_2(A,Z)$ and $u_2(B,X) \geq u_2(B,Z)$.

Z is not weakly dominated by Y since there is no case with strict inequality.

Iterated elimination of weakly dominated strategies (IEWDS)

Remove weakly dominated strategies until there is only one strategy profile.

Example

	X	Y	Z
A	$3,3$	$1,1$	$4,1$
B	$2,1$	$0,1$	$3,1$

Y and Z are weakly dominated by X.

$\downarrow$ eliminate Y and Z

	X
A	$3,3$
B	$2,1$

B is weakly dominated by A.

$\downarrow$ eliminate B

	X
A	$3,3$

(A, X) is an NE.

Theorem 13.1

Suppose G is a strategic game. If IEWDS ends with only one strategy profile s^*, then s^* is a Nash equilibrium of G.

Note: Compare with Theorem 12.3—here, we can no longer claim that the NE is unique. A different sequence of eliminations can result in a different NE.

Example

In the payoff table above, eliminating Y then B then X gives a different NE, (A, Z).

Key difference: unlike strictly dominated strategies, weakly dominated strategies can appear in an NE.

Some NE cannot be found through IEWDS: *e.g.*, Bach or Stravinsky has no weakly dominated strategies.

	B	S
B	$2,1$	$0,0$
S	$0,0,$	$2,1$

Weakly dominating strategies

Just like strictly dominating strategies, weakly dominating strategies are good to play.

Lemma 13.2

If for all players i, s_i^* is a weakly dominating strategy, then s^* is a Nash equilibrium.

Lecture 14: Auctions

Setup of an auction: A seller puts one item up for an auction. Potential buyers put in bids to buy the item. The seller decides who wins (usually highest bidder) and the price they pay.

Typical auction: open bid auction. Buyers bid repeatedly until no one else bids. Highest bid wins and pays their bid price.

Another type: closed bid auction. Each buyer submits one secret bid to the seller. (Easier to analyze.)

First price auction: highest bid wins, winner pays their bid. (If three bidders bid 150, 100, and 200, the third bidder wins and pays 200.) Does this simulate an open auction? No—in the open auction setting, the winner will bid slightly over 150 and win, so they would actually pay closer to 150.

Second price auction: highest bid wins, winner pays second highest bid. (If three bidders bid 150, 100, and 200, the third bidder wins and pays 150.) This better simulates an open auction.

We will analyze second price closed bid auctions.

Setup: we have buyers $N = \{1, \ldots, n\}$. Buyer i thinks the item has value v_i ("valuation"). Suppose buyer i submits bid b_i, giving strategy profile $b = (b_1, \ldots, b_n)$. The winner is the buyer who submits the highest bid; they pay a price equal to the second highest bid. If there is a tie, then the winner is the buyer with the lowest index i among all tied buyers.

Given a strategy profile b, the utility for buyer i is

$$u_i(b) = \begin{cases} v_i - \max\limits_{j \neq i} b_j & i \text{ wins in } b \\ 0 & \text{otherwise} \end{cases}.$$

Suppose your valuation of the item is 100. Would you bid anything other than 100?

1. Your bid wins.

   24, 69, 75, **100**

   Pay 75, get utility 25.

   If you bid more than 75, you still win, you still pay 75. If you bid less than 75, you lose and get utility 0.

2. Your bid loses.

   24, 69, 75, **100**, 121

   Your utility is 0.

   If you bid less than 121, you still lose and you still get utility 0. If you bid more than 121, you win, but you pay 121 and get utility -21.

In all cases, your utility does not increase if you bid anything else than your valuation.

Theorem 14.1

In the second price auction, v_i is a weakly dominating strategy for player $i \in N$.

Proof.

We first show that $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all $b_i \in S_i$ and $b_{-i} \in S_{-i}$. There are two cases.

1. v_i is a winning bid in (v_i, b_{-i}).

   Let b_j be the second highest bid (which could equal v_i). The utility for player i is $u_i(v_i, b_{-i}) = v_i - b_j \geq 0$. Suppose player i changes their bid to b_i.

   If $b_i > b_j$ or ($b_i = b_j$ and $i < j$), then b_i is still the winning bid in (b_i, b_{-i}). Payment is still b_j, so the utility remains the same.

   Otherwise, b_i is a losing bid, so the utility is 0, which is at most $u_i(v_i, b_{-i})$.

   So $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for any b_i.

2. v_i is a losing bid in (v_i, b_{-i}).

   Let b_j be the winning bid (so $b_j \geq v_i$). The utility for player i is $u_i(v_i, b_{-i}) = 0$. Suppose player i changes their bid to b_i.

   If $b_i < b_j$ or ($b_i = b_j$ and $i > j$), then b_i is still a losing bid in (b_i, b_{-i}). The utility is still 0.

   Otherwise, b_i is a winning bid with payment b_j. The utility is $u_i(b_i, b_{-i}) = v_i - b_j \leq 0$ (since $b_j \geq v_i$).

   So $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$.

In both cases, bidding v_i gives the highest utility among all possible bids of player i.

We still need to show that for all $b_i \neq v_i$, there exists $s_{-i} \in S_{-i}$ such that $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$. There are two cases.

1. Suppose $b_i < v_i$.

   Let k be in $b_i < k < v_i$. Set $b_j = k$ for all $j \neq i$. When v_i is played against b_{-i}, player i wins ($v_i > k$) and pays k. The utility is then $u_i(v_i, b_{-i}) = v_i - k > 0$. When b_i is played against b_{-i}, player i loses ($b_i < k$) and the utility is $u_i(b_i, b_{-i}) = 0$. So $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$.

2. Suppose $b_i > v_i$.

   Let k be in $v_i < k < b_i$. Set $b_j = k$ for all $j \neq i$. When v_i is played against b_{-i}, player i loses ($v_i < k$) and the utility is $u_i(v_i, b_{-i}) = 0$. When b_i is played against

b_{-i}, player i wins ($b_i > k$) and pays k. The utility is $u_i(b_i, b_{-i}) = v_i - k < 0$. So $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$.

Therefore, playing v_i is a weakly dominating strategy. □

Note: The way we play this game does not depend on knowing how other players value the item, so it is easy to play—simply bid your valuation.

Usually, strategic games require you to know all the information to play perfectly. However, this game does not.

Exercise 14.2

Suppose buyer 1 has the highest valuation v_1 and buyer 2 has the second highest valuation v_2. Show that $(v_2, v_1, 0, 0, \ldots, 0)$ is a NE.

Lecture 15: Mixed Strategies

Definitions

Example: matching pennies

Two players each have a penny. They simultaneously show heads or tails. If they match, then player I gains the penny from player II. If they don't match, then player II gets the penny from player I.

		P II: H	P II: T
P I	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

There is no Nash equilibrium here (in the way described so far).

Let's allow the players to play this probabilistically. For example, player I might play H 1/3 of the time and T 2/3 of the time. Player II might play 3/4 on H and 1/4 on T.

Is there an equilibrium here? If player I plays 1/3 H and 2/3 T, then player II wants to play H more often than T. Then player I wants to play H more often than T. Then player II wants to play T more often than H, etc. It seems the strategy profile is stable only if both players play 1/2 H and 1/2 T.

Definition — mixed strategy, mixed strategy profile

A **mixed strategy** for player i is a vector $x^i \in \mathbb{R}_+^{S_i}$ such that $\sum_{s \in S_i} x_s^i = 1$ (intuition: a probability distribution over possible strategies).

The set of all mixed strategies for player i is denoted Δ^i.

A **mixed strategy profile** is a vector $x = (x^1, \ldots, x^n)$ where $x^i \in \Delta^i$ is a mixed strategy for player i.

The set of all mixed strategy profiles is denoted $\Delta = \Delta^1 \times \cdots \times \Delta^n$.

The mixed strategy profile with player i removed is $x^{-i} \in \Delta^{-i}$.

Notes:

1. If we play a strategy with probability 1, then it is a **pure strategy** (this is the way we've played games before this lecture).
2. As convention for this course, we use s's to represent pure strategies and x's to represent mixed strategies.

Example

In matching pennies, if we order the pure strategies in the order (H, T), then we had $x^1 = (x^1_{\mathrm{H}}, x^1_{\mathrm{T}}) = (\frac{1}{3}, \frac{3}{3})$ and $x^2 = (x^2_{\mathrm{H}}, x^2_{\mathrm{T}}) = (\frac{3}{4}, \frac{1}{4})$ as mixed strategies.

The (mixed) strategy profile is $x = (x^1, x^2) = ((\frac{1}{3}, \frac{2}{3}), (\frac{3}{4}, \frac{1}{4}))$.

Why mixed strategies?

1. Introduce unpredictability in games that are played repeatedly. Examples: in penalty kicks, you do not always want to kick to the same side; in politics, you do not always want to make major announcements on Tuesdays.
2. Think of a player as representing a population, with probability of a strategy being proportional to the portion of the population who prefer it. Example: say 55% like donkeys and 45% like elephants, then perhaps there will be more donkeys in zoos.

Utility

We will use expected value as utility.

Example

Consider matching pennies again, with strategies $x^1 = (\frac{1}{3}, \frac{2}{3})$ and $x^2 = (\frac{3}{4}, \frac{1}{4})$.

		P II	
		H	T
P I	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

Two cases for player I:

1. If player I plays H as a pure strategy, then we get 1 with 3/4 chance and -1 with 1/4 chance; expected utility is $3/4 \cdot 1 + 1/4 \cdot (-1) = 1/2$.
2. If player I plays T as a pure strategy, then we get -1 with 3/4 chance and 1 with 1/4 chance; expected utility is $3/4 \cdot (-1) + 1/4 \cdot 1 = -1/2$.

Overall, player I plays H 1/3 of the time and T 2/3 of the time. So the expected utility is $1/3 \cdot (1/2) + 2/3 \cdot (-1/2) = -1/6$.

Definition — expected utility

We are given a strategy profile $x = (x^1, \ldots, x^n) \in \Delta$. The **expected utility of a pure strategy** $s_i \in S_i$ for player i is

$$u_i(s_i, x^{-i}) = \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \prod_{j \neq i} x^j_{s_j}$$

where $u_i(s_i, s_{-i})$ is the utility from the pure strategy game.

The **expected utility** of player i in x is

$$u_i(x) = \sum_{s_i \in S_i} x^i_{s_i} \cdot u_i(s_i, x^{-i}).$$

Example

For matching pennies above, $u_1(\mathrm{H}, x^2) = \frac{1}{2}$, $u_1(\mathrm{T}, x^2) = \frac{-1}{2}$, and $u_1(x) = \frac{-1}{6}$.

Example

Suppose 3 players each make a choice between A and B. A \$1 prize is split among players who pick the majority choice. Suppose $x^1 = (p, 1-p)$, $x^2 = (\frac{1}{2}, \frac{1}{2})$, and $x^3 = (\frac{2}{5}, \frac{3}{5})$. What is the expected utility for player I?

When player I plays A, there are 4 cases:

- $u_1(A, A, A) = \frac{1}{3}$ with probability $x^2_A \cdot x^3_A = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$.
- $u_1(A, A, B) = \frac{1}{2}$ with probability $x^2_A \cdot x^3_B = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$.
- $u_1(A, B, A) = \frac{1}{2}$ with probability $x^2_B \cdot x^3_A = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$.
- $u_1(A, B, B) = 0$ with some probability (not relevant for calculations).

The utility for playing A is $u_1(A, x^{-1}) = \frac{1}{5} \cdot \frac{1}{3} + \frac{3}{10} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2} + 0 = \frac{19}{60}$.

The utility for playing B is $\frac{7}{20}$ (check this).

The expected utility for player I is $u_1(x) = p \cdot \frac{19}{60} + (1-p) \cdot \frac{7}{20} = \frac{7}{20} - \frac{1}{15}p$.

It would make sense to pick $p = 0$, so player I always picks B. (Player III is more likely to pick B, letting us form a majority more often.)

Lecture 16: Mixed Equilibria

Definitions

Definition — mixed Nash equilibrium

A mixed strategy profile $\overline{x} \in \Delta$ is a **mixed Nash equilbrium** if for each player $i \in N$, $u_i(\overline{x}) \geq u_i(x^i, \overline{x}^{-i})$ for all $x^i \in \Delta^i$.

(We often omit the word "mixed", so it is also a Nash equilibrium.)

Definition — best response function

Given a profile $\overline{x}^{-i} \in \Delta^{-i}$, the **best response function** for player i, $B_i(\overline{x}^{-i})$, is the set of all mixed strategies of player i that have maximum utility against $\overline{x}^{-i}$, *i.e.*, $B_i(\overline{x}^{-i}) = \{\overline{x}^i \in \Delta^i : u_i(\overline{x}^i, \overline{x}^{-i}) \geq u_i(x^i, \overline{x}^{-i})\ \forall x^i \in \Delta^i\}$.

Proposition 16.1

$\overline{x} = (\overline{x}^1, \ldots, \overline{x}^n) \in \Delta$ is a Nash equilibrium if and only if $\overline{x}^i \in B_i(\overline{x}^{-i})$ for all $i \in N$.

Example: matching pennies

		P II	
		H	T
P I	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

Suppose $x^1 = (p, 1-p)$ and $x^2 = (q, 1-q)$.

For player I, the expected utility for playing H is $q \cdot 1 + (1-q) \cdot (-1) = 2q - 1$. The expected utility for playing T is $q \cdot (-1) + (1-q) \cdot q = 1 - 2q$. Utility for player I is $p \cdot (2q-1) + (1-p) \cdot (1-2q) = p(-2+4q) + (1-2q)$.

Given q, which p maximizes this utility? Here, $1 - 2q$ is constant, so we maximize $p(-2+4q)$. Three cases:

1. If $q < \frac{1}{2}$, then $-2 + 4q < 0$. Maximize with $p = 0$.
2. If $q = \frac{1}{2}$, then $-2 + 4q = 0$. Then any p maximizes it, so $p \in [0, 1]$.
3. If $q > \frac{1}{2}$, then $-2 + 4q > 0$. Maximize with $p = 1$.

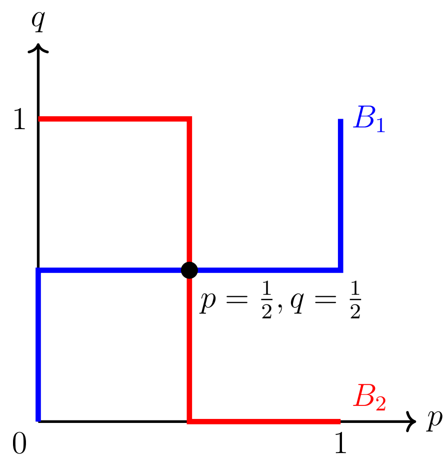
The BRF for player I is:

$$B_1(x^2) = \begin{cases} \{(0,1)\} & \text{if } \ q < \frac{1}{2} \\ \{(p, 1-p) : p \in [0,1]\} & \text{if } \ q = \frac{1}{2} \\ \{(1,0)\} & \text{if } \ q > \frac{1}{2} \end{cases}$$

For player II, the utility is $q(2-4p)+(2p-1)$ (check this). Divide cases with $p = \frac{1}{2}$ like above. The BRF for player II is:

$$B_2(x^1) = \begin{cases} \{(1,0)\} & \text{if } \ p < \frac{1}{2} \\ \{(q, 1-q) : q \in [0,1]\} & \text{if } \ p = \frac{1}{2} \\ \{(0,1)\} & \text{if } \ p > \frac{1}{2} \end{cases}$$

We look for p, q such that x^1, x^2 are best responses to each other. In this example, we can draw B_1, B_2 on a "graph".



The intersection of B_1 and B_2 is where they are best responses simultaneously, hence a Nash equilibrium. In this case, we have $x^1 = (\frac{1}{2}, \frac{1}{2})$ and $x^2 = (\frac{1}{2}, \frac{1}{2})$ and (x^1, x^2) is an NE.

Example: Bach or Stravinsky

	B	S
B	$2,1$	$0,0$
S	$0,0,$	$2,1$

Suppose $x^1 = (p, 1-p)$ and $x^2 = (q, 1-q)$.

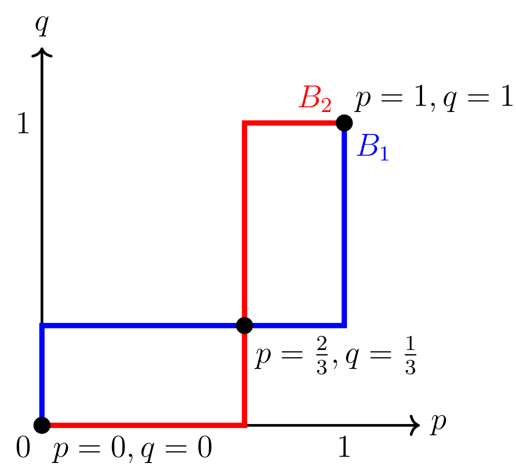
Then $u_1(B, x^2) = 2q$ and $u_1(S, x^2) = 1 - q$. So $u_1(x) = p \cdot 2q + (1-p) \cdot (1-q) = p(-1+3q) + (1-q)$. Break into cases depending on $q = \frac{1}{3}$. The BRF is:

$$B_1(x^2) = \begin{cases} \{(0,1)\} & \text{if } \ q < \frac{1}{3} \\ \{(p, 1-p) : p \in [0,1]\} & \text{if } \ q = \frac{1}{3} \\ \{(1,0)\} & \text{if } \ q > \frac{1}{3} \end{cases}$$

For player II, $u_2(x) = q(-2+3p) + 2 - 2p$. Break into cases depending on $p = \frac{2}{3}$. The BRF is:

$$B_2(x^1) = \begin{cases} \{(0,1)\} & \text{if } \ p < \frac{2}{3} \\ \{(q, 1-q) : q \in [0,1]\} & \text{if } \ p = \frac{2}{3} \\ \{(1,0)\} & \text{if } \ p > \frac{2}{3} \end{cases}$$

The "graph" here is:



There are 3 NE here:

- 2 pure strategies: $((0,1),(0,1))$ and $((1,0),(1,0))$
- 1 mixed strategy: $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$

With two players, it is easy to find Nash equilibria; with more players, in general it is not easy.

Lecture 17: Support Characterization

Recall: BRF maximizes a player's utility $u_i(x) = \sum_{s \in S_i} x_s^i u_i(s, x^{-i})$.

Suppose $\overline{x}^{-i}$ is fixed. Which $x^i \in \Delta^i$ maximizes $u_i(x^i, \overline{x}^{-i})$? Write an LP:

$$\begin{aligned} \max \quad & \sum_{s \in S_i} x_s^i u_i(s, \overline{x}^{-i}) \\ \text{s.t.} \quad & \sum_{s \in S_i} x_s^i = 1 \\ & x^i \geq \mathbb{0} \end{aligned} \tag{P}$$

Variables: x_s^i for each $s \in S_i$. What is the dual? We have one constraint in (P), so one dual variable y.

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y \geq u_i(s, \overline{x}^{-i}) \quad \text{for all } s \in S_i \end{aligned} \tag{D}$$

Both (P) and (D) are feasible: for (P), set x^i to any probability distribution; for (D), set y to be the maximum $u_i(s, \overline{x}^{-i})$. Therefore, (P) and (D) have optimal solutions and their optimal values are equal.

(D) is easy to solve: $y = \max_{s \in S_i} u_i(s, \overline{x}^{-i})$. This is the maximum utility when pure strategies are played against $\overline{x}^{-i}$.

(P) also has optimal value y. So the maximum utility of all mixed strategies is equal to the max utility of pure strategies.

Complementary slackness conditions: $x_s^i = 0$ or $y = u_i(s, \overline{x}^{-i})$ for all $s \in S_i$.

Equivalently, $x_s^i > 0$ implies $y = u_i(s, \overline{x}^{-i})$. Translation: only pure strategies with maximum utility could have positive probabilties in a best response.

Theorem 17.1 — support characterization

Given $\overline{x}^{-i} \in \Delta^{-i}$, a mixed strategy $x^i \in B_i(\overline{x}^{-i})$ if and only if $x_s^i > 0$ implies $s \in S_i$ is a pure strategy of maximum utility against $\overline{x}^{-i}$.

Definition — support

For a mixed strategy $x^i \in \Delta^i$, the **support** is the set of strategies with positive probability in x^i.

Rephrasing Theorem 17.1: x^i is in the BRF if and only if the support of x^i are strategies with maximum utility.

Example: Bach or Stravinsky

	B	S
B	2, 1	0, 0
S	0, 0,	2, 1

Suppose player II plays $x^2 = (q, 1-q)$. The utilities of player I using pure strategies are $u_1(\mathrm{B}, x^2) = 2q$ and $u_1(\mathrm{S}, x^2) = 1 - q$. Depending on q, the strategies with maximum utility are different.

1. If $2q < 1 - q$, then $q < \frac{1}{3}$, and B is not in the support and gets probability 0. BRF is $\{(0, 1)\}$.
2. If $2q = 1 - q$, then $q = \frac{1}{3}$, and both B and S could be in the support. Any combination works, so BRF is $\{(p, 1-p) : p \in [0, 1]\}$.
3. If $2q > 1 - q$, then $q > \frac{1}{3}$, and S is not in the support and gets probability 0. BRF is $\{(1, 0)\}$.

This matches the BRF we calculated in the last lecture.

Example

Consider a 2-player game with this payoff table:

	D	E	F
A	2, 2	3, 3	1, 1
B	3, 1	0, 4	2, 1
C	3, 4,	5, 1	0, 7

Suppose player II plays $x^2 = (0, \frac{1}{3}, \frac{1}{2})$. What is $B_1(x^2)$?

- $u_1(A, x^2) = 0 \cdot 2 + \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
- $u_1(B, x^2) = 0 \cdot 3 + \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
- $u_1(C, x^2) = 0 \cdot 3 + \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$

Playing A or C gives the maximum utility. By support characterization, $x^1_B = 0$. Any distribution over x^1_A and x^1_C works.

The BRF is $B_1(x^2) = \{(p, 0, 1-p) : p \in [0, 1]\}$.

The maximum utility for player I is $p \cdot \frac{5}{3} + (1-p) \cdot \frac{5}{3} = \frac{5}{3}$, which is equal to the maximum utility for a pure strategy.

Any strategy in $B_1(x^2)$ maximizes utility for player I. Which of these maximizes utility for player II? This will give a Nash equilibrium.

Suppose $x^1 = (p, 0, 1-p)$. Then:

- $u_2(D, x^1) = 4 - 2p$
- $u_2(E, x^1) = 1 + 2p$
- $u_2(F, x^1) = 7 - 6p$

If $x^2 = (0, \frac{1}{3}, \frac{2}{3})$ is in the best response, then E and F must have (the same) maximum utility. That is, $1 + 2p = 7 - 6p$ so $p = \frac{3}{4}$. The utility for E and F is then $\frac{5}{2}$; the utility for D is also $\frac{5}{2}$, so E and F indeed have maximum utility. (So does D, but this is fine.)

So $x^1 = (\frac{3}{4}, 0, \frac{1}{4})$ and $x^2 = (0, \frac{1}{3}, \frac{2}{3})$ are best responses to each other, and (x^1, x^2) is a Nash equilibrium.

Note: one "algorithm" for finding an NE is by looking at possible combinations of the supports for each player. In the example above, if we say "suppose player I's support is {A, C} and player II's support is {E, F}" then we can use support characterization to find an NE or prove none exist for these supports. Problem: this grows exponentially (2^k possible supports for each player if there are k pure strategies), so it is not practical.

Exercise 17.2

Show that in the game of rock paper scissors, both players playing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the only Nash equilibrium.

Lecture 18: Voting Game

Downs paradox: Voting has costs. The probability that one vote is a decisive vote is very small. Costs outweigh benefits.

Expectation: people don't vote. Reality: people do vote.

Model for voter participation: Suppose there are candidates A, B and the number of supporters are a, b, respectively. WLOG, assume $a \geq b$. Each person can choose to "vote" or "abstain". If they vote, then they incur a cost of c where $0 < c < 1$. Regardless of voting or abstaining, each person gets a payoff of 2 if their supporting candidate wins, 1 for a tie, or 0 for a loss.

Pure Nash equilibria

Suppose $a = b = 1$.

		P II (B)	
		A	V
P I (A)	A	$1, 1$	$0, 2-c$
	V	$2-c, 0,$	$1-c, 1-c$

This is similar to prisoner's dilemma: if both players vote, they get lower utility than if both players abstain. (Both voting is an NE.)

Suppose $a = b \geq 2$. There are 4 cases:

1. Everyone votes.

   Tie: everyone has utility $1 - c$, switching gives 0. NE.

2. Not everyone votes, but there is a tie.

   One who abstains can vote, moving utility from 1 to $2 - c > 1$. Not NE.

3. One candidate wins by 1 vote.

   One who abstains for the loser can vote, moving utility from 0 to $1 - c > 0$. Not NE.

4. One candidate wins by at least 2 votes.

   One for votes for the winning candidate can abstain, moving utility from $2 - c$ to $2 > 2 - c$. Not NE.

In a close election, we expect more people to vote.

Exercise 18.1

Show that when $a > b$, there is no pure Nash equilibrium.

Mixed Nash equilibria

One possible scenario for a mixed NE.

Suppose $a > b$. Suppose among all A supporters, b of them will vote and $a - b$ of them will abstain. Suppose every B supporter will vote with the same probability p. So the best that B can do is a tie. It is easy to check that $p = 0$ or $p = 1$ is not an NE. Assume $p \in (0, 1)$.

Consider a B supporter. If they abstain, then B cannot win. So utility of "abstain" as a pure strategy is 0. If they vote, then B ties only if all other B supporters vote (utility $1 - c$), otherwise B loses (utility $-c$). The expected utility of "vote" as a pure strategy is $p^{b-1}(1 - c) + (1 - p^{b-1})(-c) = p^{b-1} - c$.

When is it possible that this is in an NE? $p \in (0, 1)$, so both strategies have positive probabilities. To be in the best response, support characterization implies that the two utilities are equal. So $0 = p^{b-1} - c$, or $p = c^{\frac{1}{b-1}}$.

Given this p, are A supporters incentivized to change their mixed strategies? Currently, all of them are playing pure strategies. In order to switch, the utility of switching to the other pure strategy must be greater.

1. Consider an A supporter who abstained. Expected utility is $p^b \cdot 1 + (1 - p^b) \cdot 2 = 2 - p^b$.

   Expected utility of voting is $2 - c$ (A is guaranteed to win). $2 - c = 2 - p^{b-1} \leq 2 - p^b$ since $0 < p < 1$. Switching to a pure strategy does not increase utility. So switching to any mixed strategy does not increase utility. There is no reason to switch.

2. Consider an A supporter who voted. Expected utility is $p^b \cdot (1 - c) + (1 - p^b) \cdot (2 - c) = 2 - p^b - c$.

   If they abstain, then A loses if all B supporters vote, A ties if $b - 1$ B supporters vote, or A wins otherwise. The expected utility of abstaining is $p^b \cdot 0 + b \cdot p^{b-1} \cdot (1 - p) \cdot 1 + (1 - p^b - b \cdot p^{b-1} \cdot (1 - p)) \cdot 2 = 2 - 2p^b - bp^{b-1}(1 - p)$. Check: $2 - p^b - c \geq 2 - 2p^b - bp^{b-1}(1 - p)$. There is no reason to switch.

When $p = c^{\frac{1}{b-1}}$, this is a mixed NE.

What happens to voter participation as cost increases? If c increases, then p increases, so more voters will vote.

Lecture 19: Two-player Zero-sum Games

Definition — zero-sum game

A strategic game is a **zero-sum game** if for all strategy profiles $s \in S$, $\sum_{i \in N} u_i(s) = 0$.

Examples: matching pennies and rock paper scissors.

For a two-player zero-sum game, let $S_1 = \{1, \ldots, m\}$ and $S_2 = \{1, \ldots, n\}$. Define such a game with a payoff matrix $A \in \mathbb{R}^{m \times n}$ where $u_1(i,j) = A_{ij}$ and $u_2(i,j) = -A_{ij}$.

Example

Payoff for player I:

$$\begin{array}{cc|c|c|c|}
 & \multicolumn{4}{c}{\text{P II}} \\
 & & 1 & 2 & 3 \\
\cline{3-5}
\text{P I} & 1 & 3 & 5 & -2 \\
\cline{3-5}
 & 2 & -5 & 7 & 1 \\
\cline{3-5}
\end{array} = A$$

Payoff for player II is $-A$.

Note: for a mixed strategy profile $x = (x^1, x^2)$, we have $u_1(x^1, x^2) = -u_2(x^1, x^2)$.

Min-maxing argument for finding a Nash equilibrium

Given a strategy that we play, the opposing player will maximize their utility, which minimizes our utility. Knowing how they would play, what can we do to maximize our own utility?

Player I's perspective: Suppose player I plays x^1. They expect player II to play from their best response.

Example

The expected utilities for player II's 3 strategies above are $u_2(1, x^1) = -3x_1^1 + 5x_2^1$, $u_2(2, x^1) = -5x_1^1 - 7x_2^1$, and $u_3(3, x^1) = 2x_1^1 - x_2^1$.

We look for

$$\begin{aligned}
&\max\{-3x_1^1 + 5x_2^1, -5x_1^1 - 7x_2^1, 2x_1^1 - x_2^1\} \\
= &\min\{3x_1^1 - 5x_2^1, 5x_1^1 + 7x_2^1, -2x_1^1 + x_2^1\}.
\end{aligned}$$

We wish to maximize:

$$\begin{aligned} \max \quad & u_1 \\ \text{s.t.} \quad & u_1 \le 3x_1^1 - 5x_2^1 \\ & u_1 \le 5x_1^1 + 7x_2^1 \\ & u_1 \le -2x_1^1 + x_2^1 \\ & x_1^1 + x_2^1 = 1 \\ & x_1 \ge \mathbb{0} \end{aligned}$$

Player II's expected utility for playing pure strategy j is $-(x^1)^T A_{\bullet j}$ (where $A_{\bullet j}$ is the j-th column of A). The utility of player II's best response is equal to the maximum of these values:

$$\max_{j \in \{1,\ldots,n\}} -(x^1)^T A_{\bullet j} = - \min_{j \in \{1,\ldots,n\}} (x^1)^T A_{\bullet j}.$$

So utility for player I is $\min_{j \in \{1,\ldots,n\}} (x^1)^T A_{\bullet j}$. Player I wants to maximize this.

$$\begin{aligned} \max \quad & \min_{j \in \{1,\ldots,n\}} (x^1)^T A_{\bullet j} \\ \text{s.t.} \quad & \sum_{i=1}^m x_i^1 = 1 \\ & x_1 \ge \mathbb{0} \end{aligned} \quad \longrightarrow \quad \begin{aligned} \max \quad & u_1 \\ \text{s.t.} \quad & u_1 \le (x^1)^T A_{\bullet j} \ \forall j \in \{1,\ldots,n\} \\ & \sum_{i=1}^m x_i^1 = 1 \\ & x^1 \ge \mathbb{0} \end{aligned}$$

Player II's perspective: Suppose player II plays x^2. Then player I will play from their best response.

Example

Player I's best response has utility

$$\max\{3x_1^2 + 5x_2^2 - 2x_3^2, -5x_1^2 + 7x_2^2 + x_3^2\}.$$

We wish to minimize:

$$\begin{aligned} \min \quad & u_2 \\ \text{s.t.} \quad & 3x_1^2 + 5x_2^2 - 2x_3^2 \le u_2 \\ & -5x_1^2 + 7x_2^2 + x_3^2 \le u_2 \\ & x_1^2 + x_2^2 + x_3^2 = 1 \\ & x_2 \ge \mathbb{0} \end{aligned}$$

The utility of player I's best response is $\max_{i \in \{1,\ldots,m\}} (x^2)^T A_{i\bullet}$ (where $A_{i\bullet}$ is the i-th row of A). Player II's utility is $-\max_{i \in \{1,\ldots,m\}} (x^2)^T A_{i\bullet}$. Maximizing this is equivalent to minimizing

$\max_{i \in \{1,\ldots,m\}} (x^2)^T A_{i\bullet}$:

$$\begin{array}{ll} \max & \min_{i \in \{1,\ldots,m\}} (x^2)^T A_{i\bullet} \\ \text{s.t.} & \sum_{j=1}^{n} x_j^2 = 1 \\ & x_2 \geq \mathbb{0} \end{array} \longrightarrow \begin{array}{ll} \max & u_2 \\ \text{s.t.} & u_2 \leq (x^2)^T A_{i\bullet} \ \forall i \in \{1,\ldots,m\} \\ & \sum_{j=1}^{n} x_j^2 = 1 \\ & x^2 \geq \mathbb{0} \end{array}$$

Main result: the LPs for player I and player II are duals of each other. (Exercise: verify this in general.)

Both LPs are feasible (take x^1, x^2 to be any probability distributions and u_1, u_2 as max/min values). So both have optimal solutions with the same objective value. (Note: objective value of player I's LP is the utility of player I, so the objective value of player II's LP is the negative of the utility of player II.) The optimal solutions are best responses to each other, so they form a Nash equilibrium. Solve for this using simplex (a modified version of simplex is provably polynomial time).

Theorem 19.1

Any (finite) two-player zero-sum game has a mixed Nash equilibrium, and this can be efficiently computed.

Example

For our two LPs above, an optimal solution is

- Player I: $x^1 = (\frac{6}{11}, \frac{5}{11})$ and $u_1 = \frac{-7}{11}$
- Player II: $x^2 = (\frac{3}{11}, 0, \frac{8}{11})$ and $u_2 = \frac{-7}{11}$

where player I's utility is u_1 and player II's utility is $-u_2$.

Note: computing NEs in general is difficult. Even in the 3-player zero-sum game or 2-player general-sum game, no polynomial time algorithm is known.

Lecture 20: Nash's Theorem (I)

Theorem 20.1 — Nash's theorem

Every strategic game with finitely many players and pure strategies has a Nash equilibrium.

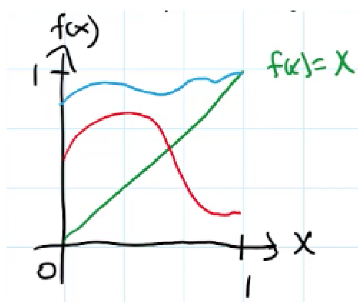
Brouwer's fixed point theorem

Theorem — Brouwer's fixed point theorem

Let X be a convex and compact set in a finite-dimensional Euclidean space, and let $f\colon X \to X$ be a continuous function. Then there exists $x_0 \in X$ such that $f(x_0) = x_0$ ("fixed point").

Example

Let $X = [0, 1]$. Consider any continuous function $f\colon [0, 1] \to [0, 1]$.



The graph of f will always intersect $f(x) = x$, producing a fixed point. This is a consequence of the intermediate value theorem (applied to $f(x) - x$).

Terminology from the theorem:

- We will think of a Euclidean space as $\mathbb{R}^n$ with the standard dot product, which defines how we measure distance and angle.
- A set is convex if for any two points in the set, the line segment joining them is also in the set. Precise definition: S is convex if for all $u, v \in S$, $\lambda u + (1 - \lambda)v \in S$ for all $\lambda \in [0, 1]$. Note that the convex combination of any set of points is convex: $S = \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_1, \ldots, \lambda_n \geq 0, \lambda_1 + \cdots + \lambda_n = 1\}$ is convex.
- A set is compact if it is closed and bounded. Closed: all "boundary" points are in the set, *e.g.* $[0, 1]$ is closed, $(0, 1)$ is not closed. Bounded: there is a constant that bounds the distance between any two points.

Note: this is a deep theorem from analysis. We will not prove it here, though there are many fascinating proofs of it (suggestion: look into the combinatorial proof using Sperner's Lemma). None of the proofs are constructive: we know that a fixed point exists, but the proofs do not tell us how to find one.

Illustrations:

1. Print a world map and place it on your desk. This is a continuous mapping from the surface of the Earth to the part of the surface occupied by the map on your desk. The theorem implies there is a fixed point: some point on the map is directly on top of the point it represents on your desk.
2. Take a cup of tea and stir it. Let it settle. Then some part of the liquid is in the same spot before the stir.

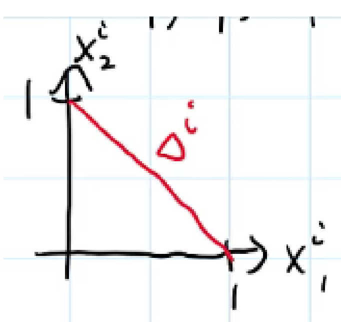
Relation to strategic games

We want to use Brouwer's fixed point theorem when X is the set of all mixed strategy profiles of a finite strategic game. Need to verify that Δ is convex and compact.

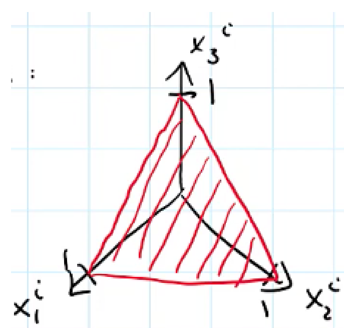
Start with just one player i and their set of mixed strategies Δ^i. If the set of pure strategies is $\{1, \dots, k\}$, then

$$\Delta^i = \{(x_1^i, \dots, x_k^i) : x_j^i \geq 0, x_1^i + \cdots + x_k^i = 1\}.$$

$k = 2$: $\Delta^i = \{(p, 1-p) : p \in [0, 1]\}$



$k = 3$: $\Delta^i = \{(p, q, r) : p + q + r = 1, p, q, r \geq 0\}$



We can see (without proof) that Δ^i is compact: it is closed, and any 2 points have distance at most $\sqrt{2}$. Δ^i is convex: it is the convex combination of the standard basis vectors $e_1, \dots, e_k$. (An element of Δ^i has the form $x_1^i e_1 + \cdots + x_k^i e_k$ where $x_1^i + \cdots + x_k^i = 1$, $x_j^i \geq 0$.) These $e_1, \dots, e_k$ are the pure strategies of player i.

The set of all strategy profiles is $\Delta = \Delta^1 \times \cdots \times \Delta^n$. We can "pretend" that this is a set in $\mathbb{R}^{|S_1| + \cdots + |S_n|}$. It is still compact (a result from analysis is that the cartesian product of compact sets is compact). It is also convex (exercise). So we can use Δ as the set in Brouwer's fixed point theorem.

To do: find a continuous function $f \colon \Delta \to \Delta$ that relates fixed points to Nash equilibria.

Lecture 21: Nash's Theorem (II)

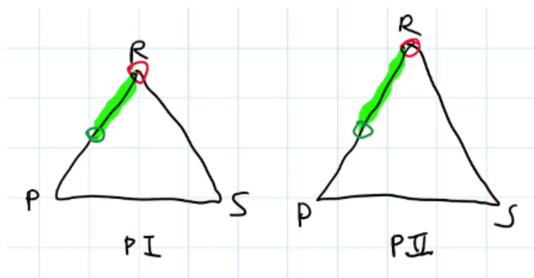
Recall: the set of all strategy profiles Δ is convex and compact. Need $f\colon \Delta \to \Delta$ to relate fixed points and Nash equilibria.

Main idea

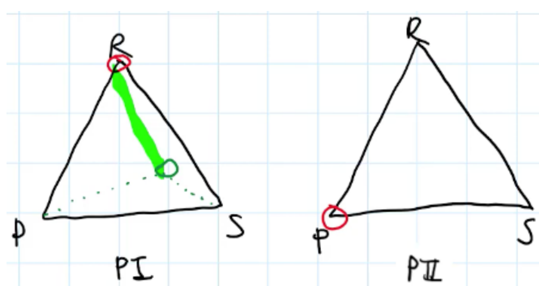
Given a strategy profile $x = (x^1, \ldots, x^n)$, a player i will look at possibly switching to a pure strategy to gain utility against x^{-i}. If pure strategy s improves utility, then player i wants to shift the probability distribution so that s receives higher probability. The function will take x and map it to another strategy profile where each player improves their utility.

Example: rock paper scissors

Suppose both play rock as a pure strategy. They can increase utility by moving toward paper.



Suppose player I plays rock and player II plays paper. Player II cannot improve utility. Player I can improve utility by 1 if they move to paper, or improve utility by 2 if they move to scissors. Player I will move more toward scissors than paper.



What is the meaning of a fixed point? No player can improve their utility. So it must be a Nash equilibrium.

Defining the function

First, define Φ which records the improvement of a player in switching to a pure strategy. Given strategy profile $x \in \Delta$, a player i, and a pure strategy $s \in S_i$, define

$$\Phi^i_s(x) = \max\{0, u_i(s, x^{-i}) - u_i(x)\}.$$

If playing s increases utility for player i, then $\Phi_s^i(x)$ represents this increase; otherwise $\Phi_s^i(x) = 0$.

For player i and strategies s where $\Phi_s^i(x) > 0$, we want to increase the probability assigned to s. We want to replace x_s^i by $x_s^i + \Phi_s^i(x)$, but the sum of the probabilities would be greater than 1. We can normalize this by dividing by $\sum_{s' \in S_i} (x_{s'}^i + \Phi_{s'}^i(x)) = 1 + \sum_{s' \in S_i} \Phi_{s'}^i(x)$.

We define $f\colon \Delta \to \Delta$ by $f(x) = \overline{x}$ where for each player i and strategy $s \in S_i$,

$$\overline{x}_s^i = \frac{x_s^i + \Phi_s^i(x)}{1 + \sum_{s' \in S_i} \Phi_{s'}^i(x)}.$$

Verify that $f(x) \in \Delta$.

Example

In rock paper scissors where player I plays rock and player II plays paper, the strategy profile is $x = ((1,0,0),(0,1,0))$. For player II, $\Phi_s^2(x) = 0$ for each $s \in \{\mathrm{R}, \mathrm{P}, \mathrm{S}\}$. For player I, $\Phi_{\mathrm{R}}^1(x) = 0$, $\Phi_{\mathrm{P}}^1(x) = 1$, and $\Phi_{\mathrm{S}}^1(x) = 2$. So the new strategy for player I is

$$\overline{x}_{\mathrm{R}}^1 = \frac{1+0}{1+3} = \frac{1}{4}, \quad \overline{x}_{\mathrm{P}}^1 = \frac{0+1}{1+3} = \frac{1}{4}, \quad \overline{x}_{\mathrm{S}}^1 = \frac{0+2}{1+3} = \frac{1}{2}.$$

Then $f(x) = ((\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (0,1,0))$.

Completing the proof of Nash's theorem

Given $x \in \Delta$, consider Φ and $f\colon \Delta \to \Delta$ defined above. We see that f is continuous since Φ is continuous. By Brouwer's fixed point theorem, there exists $\hat{x} \in \Delta$ such that $f(\hat{x}) = \hat{x}$. We prove that $\hat{x}$ is a NE by showing $\hat{x}^i \in B_i(\hat{x}^{-i})$.

For player i, let $s \in S_i$ be a pure strategy such that $\hat{x}_s^i > 0$ and $u_i(s, \hat{x}^{-i}) \le u_i(\hat{x})$. (Exercise: show such s exists.) Then $\Phi_s^i(\hat{x}) = 0$. Since $\hat{x}$ is a fixed point, $\hat{x}_s^i = (f(\hat{x}))_s^i = \hat{x}_s^i / (1 + \sum_{s' \in S_i} \Phi_{s'}^i(\hat{x}))$. Since $\hat{x}_s^i > 0$, the denominator must be 1. So $\sum_{s' \in S_i} \Phi_{s'}^i(\hat{x}) = 0$. But Φ is non-negative, so $\Phi_{s'}^i(\hat{x}) = 0$ for all $s' \in S_i$. This means that $u_i(s', \hat{x}^{-i}) \le u_i(\hat{x})$ for all $s' \in S_i$. So playing $\hat{x}^i$ gives the highest utility against $\hat{x}^{-i}$, so $\hat{x}^i \in B_i(\hat{x}^{-i})$. Since this holds for all players, $\hat{x}$ is a Nash equilibrium. $\square$

Note: this proves that a NE always exists, but the proof does not show us how to find such a NE, as it depends on Brouwer's fixed point theorem.

Cooperative Games

Lecture 22: Cooperative Games

There are games where groups of players can work together to obtain higher utility.

Example

Alice (A), Bob (B), and Carol (C) want to buy ice cream. There are three sizes of ice cream: 1L, 1.5L, and 2L with costs \$7, \$9, \$11 respectively. A has \$3, B has \$4, C has \$5. On their own, they cannot buy any. But if they pool money together, they can get some ice cream. (For example, B + C can buy 1.5L.)

Definition — cooperative game

A **cooperative game** is given by a set of players N and a characteristic function $v\colon 2^N \to \mathbb{R}$ that assigns a value $v(S)$ to each coalition $S \subseteq N$ of players. We use (N, v) to represent this game. The set N is the **grand coalition**.

Example

In the ice cream game, $N = \{\text{A}, \text{B}, \text{C}\}$ and v is defined by:

S	$\varnothing, \{\text{A}\}, \{\text{B}\}, \{\text{C}\}$	$\{\text{A}, \text{B}\}$	$\{\text{A}, \text{C}\}$	$\{\text{B}, \text{C}\}$	$\{\text{A}, \text{B}, \text{C}\}$
$v(S)$	0	1	1	1.5	2

General assumptions: $v(\varnothing) = 0$, $v(S) \geq 0$ for all $S \subseteq N$.

Example

A country has 101-member parliament. There are 4 parties A, B, C, D with 40, 22, 30, 9 members, respectively. They need to decide how to spend a \$1B winfall but they need to form a majority to spend it.

$$N = \{A, B, C, D\}, \qquad v(S) = \begin{cases} 10^9 & \text{if parties in } S \text{ have at least 51 members} \\ 0 & \text{otherwise} \end{cases}$$

Example

In a matching game, we are given a graph $G = (V, E)$ and edge weights $w\colon E \to \mathbb{R}$. The players are the vertices, $V = N$. The weight of an edge represents the benefits if the two vertex players work together. For any subset $S \subseteq N$, the value is the maximum weight of a matching using vertices in S.

Outcomes of cooperative games

Outcomes of strategic games: strategy profiles (pure or mixed). Which strategy is played by each player?

Outcomes of cooperative games:

1. Divide the players into groups, called coalitions ("coalition structure"). Each coalition will generate their assigned value.
2. Distribute the value that each coalition generates among its members ("payoff vector").

Definition — coalition structure, payoff vector, efficient

Given a cooperative game (N, v), a **coalition structure** is a partition π of N, *i.e.* $\pi = (c^1, \ldots, c^k)$ where each $c^i \subseteq N$, $c^i \cap c^j = \varnothing$ whenever $i \neq j$, and $c^1 \cup \cdots \cup c^k = N$.

A **payoff vector** is a vector $x \in \mathbb{R}^N$ such that $x \geq \mathbb{0}$ and $\sum_{i \in c^j} x_i \leq v(c^j)$ for all $j = 1, \ldots, k$.

Notation: for any $T \subseteq N$, $x(T) = \sum_{i \in T} x_i$. So the inequality here is $x(c_j) \leq v(c^j)$.

An outcome consists of (π, x). Such an outcome is **efficient** if $x(c^j) = v(c^j)$ for all j.

Example

An outcome of the ice cream game is (π, x) where $\pi = (\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{C}\})$ and $x_{\mathrm{A}} = 0.5$, $x_{\mathrm{B}} = 0.5$, and $x_{\mathrm{C}} = 0$. This outcome is efficient: $v(\{\mathrm{A}, \mathrm{B}\}) = 1 = x_{\mathrm{A}} + x_{\mathrm{B}}$ and $v(\{\mathrm{C}\}) = x_{\mathrm{C}}$.

Some classes of games

1. **Monotone games**: $S \subseteq T \implies v(S) \leq v(T)$ ("more people produce more value").
2. **Superadditive games**: for disjoint S, T, $v(S) + v(T) \leq v(S \cup T)$ ("forming coalitions is always better").

   Superadditivity implies monotonicity, but the converse is not true. We usually only consider the grand coalition: $\pi = (N)$.
3. **Convex games**: for any S, T, $v(S) + v(T) \leq v(S \cup T) + v(s \cap T)$ (supermodularity inequality).

   Convexity implies superadditivity, but the converse is not true.

Proposition 22.1

A game (N, v) is convex if and only if for every S, T where $S \subseteq T \subseteq N$ and for every player $i \in N \setminus T$, $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$.

(Proof as exercise.)

This proposition gives the intuition "a player is more useful in larger coalitions".

Lecture 23: Shapley Values (I)

Two desirable properties of an outcome in cooperative games:

1. Fairness. The payoff vector should reflec the contribution of the playres to their coalitions.
2. Stability. We want to incentivize the players to stay in their assigned coalition in the coalition structure.

Shapley values

Shapley values deal with the fairness of the payoff vector. Assume players form the grand coalition. (If not, look at individual coalitions separately.)

Question: how can we quantify a player's contribution?

Idea 1: compare the value of the coalition before and after a player joins it.

> **Example**
>
> In the ice cream game, the contribution of A is $v(\{\mathrm{A,B,C}\}) - v(\{\mathrm{B,C}\}) = 0.5$. The contribution of B is 1 and the contribution of C is 1.
>
> Can a payoff vector take these values? No.

Problem: the sum of the payoffs may exceed the value of the coalition ($x(N) > v(N)$).

Idea 2: fix a sequence of the players and see their contribution sequentially.

> **Example**
>
> Consider the ice cream game with sequence A, B, C. $v(\{\mathrm{A}\}) = 0$, so A contributes 0. $v(\{\mathrm{A,B}\}) = 1$, so B contributes 1. $v(\{\mathrm{A,B,C}\}) = 2$, so C contributes 1.
>
> This is efficient: $x(N) = v(N)$.

Problem: different ordering produce different results.

Shapley's idea: look at all possible orderings of players and average a player's contributions.

Notation:

- A permutation of N has the form $\sigma = (\sigma_1, \ldots, \sigma_n)$ where each σ_1 is a distinct element of N.
- The element σ_i is at the i-th position of σ.
- The set of all permutations of N is denoted S_N.

Definition — marginal contribution, Shapley value

Given a permutation $\sigma \in S_N$, the **marginal contribution** of player σ_i is $\Delta_\sigma(\sigma_i) = v(\{\sigma_1, \ldots, \sigma_i\}) - v(\{\sigma_1, \ldots, \sigma_{i-1}\})$.

The **Shapley value** of player i is

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i).$$

Example

In the ice cream game, $N = \{\text{A, B, C}\}$ and six permuations:

$$\begin{aligned} \sigma^1 &= (\mathrm{A}, \mathrm{B}, \mathrm{C}) & \sigma^4 &= (\mathrm{B}, \mathrm{C}, \mathrm{A}) \\ \sigma^2 &= (\mathrm{A}, \mathrm{C}, \mathrm{B}) & \sigma^5 &= (\mathrm{C}, \mathrm{A}, \mathrm{B}) \\ \sigma^3 &= (\mathrm{B}, \mathrm{A}, \mathrm{C}) & \sigma^6 &= (\mathrm{C}, \mathrm{B}, \mathrm{A}) \end{aligned}$$

We calculate the marginal contribution of A in each of the permutations:

$$\begin{aligned} \Delta_{\sigma^1}(\mathrm{A}) &= v(\{\mathrm{A}\}) - v(\varnothing) = 0 & \Delta_{\sigma^4}(\mathrm{A}) &= v(\{\mathrm{B}, \mathrm{C}, \mathrm{A}\}) - v(\{\mathrm{B}, \mathrm{C}\}) = 0.5 \\ \Delta_{\sigma^2}(\mathrm{A}) &= v(\{\mathrm{A}\}) - v(\varnothing) = 0 & \Delta_{\sigma^5}(\mathrm{A}) &= v(\{\mathrm{C}, \mathrm{A}\}) - v(\{\mathrm{C}\}) = 1 \\ \Delta_{\sigma^3}(\mathrm{A}) &= v(\{\mathrm{B}, \mathrm{A}\}) - v(\{\mathrm{B}\}) = 1 & \Delta_{\sigma^6}(\mathrm{A}) &= v(\{\mathrm{C}, \mathrm{B}, \mathrm{A}\}) - v(\{\mathrm{C}, \mathrm{B}\}) = 0.5 \end{aligned}$$

So the Shapley value for A is $\varphi_\mathrm{A} = \frac{1}{6}(0 + 0 + 1 + 0.5 + 1 + 0.5) = \frac{1}{2}$.

The other Shapley values are $\varphi_\mathrm{B} = \varphi_\mathrm{C} = \frac{3}{4}$.

Lecture 24: Shapley Values (II)

Four good properties of Shapley values

(1) Efficiency: it distributes $v(N)$ to all players.

Proposition 24.1

$$\sum_{i\in N}\varphi_i = v(N).$$

Proof.

For any $\sigma \in S_N$, the sum of all marginal contributions is

$$\begin{aligned}\sum_{i\in N}\Delta_\sigma(i) &= \sum_{i=1}^{n}\Delta_\sigma(\sigma_i) \qquad \text{(since a permutation is a bijection)}\\ &= [v(\{\sigma_1\}) - v(\varnothing)] + [v(\{\sigma_1,\sigma_2\}) - v(\{\sigma_1\})] + \cdots + \\ &\qquad [v(\{\sigma_1,\ldots,\sigma_n\}) - v(\{\sigma_1,\ldots,\sigma_{n-1}\})]\\ &= v(\{\sigma_1,\ldots,\sigma_n\}) - v(\varnothing)\\ &= v(N).\end{aligned}$$

So the sum of Shapley values is

$$\sum_{i\in N}\varphi_i = \sum_{i\in N}\frac{1}{n!}\sum_{\sigma\in S_N}\Delta_\sigma(i) = \frac{1}{n!}\sum_{\sigma\in S_N}\sum_{i\in N}\Delta_\sigma(i) = \frac{1}{n!}\sum_{\sigma\in S_N}v(N) = \frac{1}{n!}(n!)v(N) = v(N).$$

$\square$

(2) Symmetric: two players i, j are **symmetric** if $v(c\cup\{i\}) = v(c\cup\{j\})$ for all $c \subseteq N\backslash\{i,j\}$.

Example

In the ice cream game, B and C are symmetric: $v(\varnothing\cup\{\text{B}\}) = v(\varnothing\cup\{\text{C}\}) = 0$ and $v(\{\text{A}\}\cup\{\text{B}\}) = v(\{\text{A}\}\cup\{\text{C}\}) = 1$.

Proposition 24.2

If i, j are symmetric players, then $\varphi_i = \varphi_j$.

Proof idea: consider S_N and swap i, j in each permutation to yield S_N again. Calculate the marginal contribution of i before the swap and of j after the swap. The values should be equal.

Proof (Proposition 24.2).

Define $f\colon S_N \to S_N$ where $f(\sigma)$ is obtained from σ by swapping i and j. This is a bijection ($f^{-1} = f$). We claim $\Delta_\sigma(i) = \Delta_{f(\sigma)}(j)$. Two cases:

- Suppose i precedes j in σ. Let C be all elements preceding i. In $f(\sigma)$, C is also the elements that precede j. So $\Delta_\sigma(i) = v(C \cup \{i\}) - v(C)$ and $\Delta_{f(\sigma)}(j) = v(C \cup \{j\}) - v(C)$. Since $C \subseteq N \setminus \{i, j\}$ and i, j are symmetric, $v(C \cup \{i\}) = v(C \cup \{j\})$. So $\Delta_\sigma(i) = \Delta_{f(\sigma)}(j)$.
- Suppose j precedes i in σ. Let C be all elements preceding i except j. In $f(\sigma)$, C is also the elements preceding j except i. So $\Delta_\sigma(i) = v(C \cup \{j\} \cup \{i\}) - v(C \cup \{j\})$ and $\Delta_{f(\sigma)}(j) = v(C \cup \{i\} \cup \{j\}) - v(C \cup \{i\})$. Since $C \subseteq N \setminus \{i, j\}$ and i, j are symmetric, $v(C \cup \{j\}) = v(C \cup \{i\})$. So $\Delta_\sigma(i) = \Delta_{f(\sigma)}(j)$.

Therefore,

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i) = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_{f(\sigma)}(j) = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(j) = \varphi_j.$$

$\square$

Example: unanimny game

Suppose $|N| = n$, and $v(S) = 1$ if $S = N$ and $v(S) = 0$ otherwise. Any pair of players is symmetric, so $\varphi_i = \varphi_j$ for any i, j. Since φ is efficient, the sum is $v(N) = 1$. So $\varphi_i = \frac{1}{n}$ for each $i \in N$.

3. Dummy player: i is a **dummy player** if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. The player does not contribute to any coalition.

Example

In the 101-seat parliament with parties A, B, C, D with 40, 22, 30, 9 seats, party D is a dummy player. No combination of A, B, C exists where it is not a majority, but adding 9 seats gives a majority.

Proposition 24.3

If i is a dummy player, then $\varphi_i = 0$.

Proof.

For any $\sigma \in S_N$, say i is at the j-th position ($\sigma_j = i$). The marginal contribution of i is

$\Delta_\sigma(i) = v(\{\sigma_1, \ldots, \sigma_{j-1}, i\}) - v(\{\sigma_1, \ldots, \sigma_{j-1}\}) = 0$ by definition of a dummy player. So

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i) = 0.$$

$\square$

Note: the converse is not true in general. If a game is monotone, then the converse is true.

(4) Additivity: suppose there are multiple games with the same set of players. We add the values together to get a new game. Then the Shapley values are also added together.

Proposition 24.4

Let (N, v^1) and (N, v^2) be two cooperative games. Define $v^3(S) = v^1(S) + v^2(S)$ for all $S \subseteq N$. Let φ_i^j be the Shapley values of player i in (N, v^j) for each $j = 1, 2, 3$. Then $\varphi_i^3 = \varphi_i^1 + \varphi_i^2$ for all i.

(Proof as exercise.)

The Shapley values satisfy 4 good properties: efficiency, symmetric, dummy player, and additivity.

The deep result about Shapley values: the Shapley value function is the only one that satisfies all 4 properties. (If f is a function that maps (N, v) to a real vector $\mathbb{R}^N$ and all 4 properties hold, then f gives the Shapley values.)