CO 456: Introduction to Game Theory

University of Waterloo
Martin Pei
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Marco Yang

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Lecture 2: Impartial Games

Nim

We are given some piles of chips. Two players play alternately. On a player's turn, they pick a pile and remove at least 1 chip from it. The first player who cannot make a move loses.

Example

- 1, 1, 2.
  - Player I removes 2 chips from the last pile.
  - Player II removes a 1-chip pile.
  - Player I removes the last chip.
  - Player II has no move and loses.
  - Player I has a winning strategy.

  This is a **winning game** or **winning position**.
- 5, 5.
  - Regardless of Player I's move, Player II can mirror it on the other pile.
  - Player II always has a move, so Player I loses.
  - Player I always loses (*i.e.* Player II has a winning strategy).

  This is a **losing game** or **losing position**.
- 5, 7.
  - Player I first equalizes piles (here, removing 2 from the pile of 7).
  - Player II loses (by the previous case).

  This is a winning game.

Lemma 2.1

In instances of Nim with two piles of n, m chips, it is a winning game if and only if $n \neq m$.

Impartial games

Nim is an impartial game.

Definition — impartial game

Conditions for an **impartial game**:

1. There are two players, Player I (who starts) and Player II.
2. There are several positions and a starting position.
3. A player performs one of a set of allowable moves, which depends only on the current position and not on the player (hence "impartial"). Each possible move generates an **option**.
4. The players move alternately.
5. There is complete information.
6. There are no chance moves.
7. The first player with no available move loses.
8. The rules guarantee that games end.

Example

Games which are not impartial:

- Tic-tac-toe (violates 7—may tie)
- Chess (violates 3—may only move your pieces)
- Poker (violates 5—cards hidden)
- Monopoly (violates 6—relies on dice rolls)

Example

Let $G = (1,1,2)$ be a Nim game. There are 4 possible moves, hence 4 possible options.

$$\begin{tikzcd}
 & \boxed{H_1 = (1,1,1)} \\
 & \boxed{H_2 = (1,1,0)} \\
\boxed{G = (1,1,2)} \arrow[ruu] \arrow[ru] \arrow[r] \arrow[rd] & \boxed{H_3 = (1,0,2)} \\
 & \boxed{H_4 = (0,1,2)}
\end{tikzcd}$$

Each H_i is itself another Nim game.

Note: we can define an impartial game by its position and options recursively.

Definition — game simplicity

A game H that is reachable from game G by a sequence of allowable moves is **simpler** than G.

Example

Other impartial games:

- Subtraction game
  - One pile of chips.
  - Valid move: remove 1, 2, or 3 chips.
- Rook game
  - $m \times n$ chess board with a rook at (i, j).
  - Valid move: move the rook any number of spaces up or left.
- Green hackenbush game
  - Graph connected to the floor at some vertices.
  - Valid move: remove an edge of the graph, then any components no longer connected to the floor.

Spoiler: all impartial games are essentially Nim games.

Winning strategy

Lemma 2.2

In any game G, either Player I or Player II has a winning strategy.

Proof.

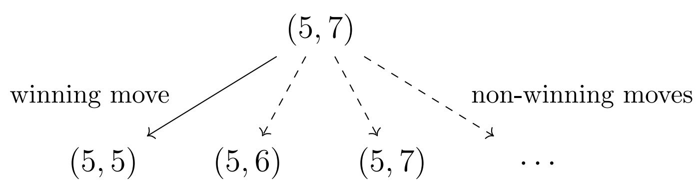
By induction on simplicity of G.

If G has no allowable moves, then Player I loses, so Player II has a winning strategy. Assume G has allowable moves and the lemma holds for all games simpler than G. Among all options of G, if Player I has a winning strategy in one of them, Player I will move to that option and win. Otherwise, Player II has a winning strategy for all options, so Player II wins regardless of Player I's move. $\square$

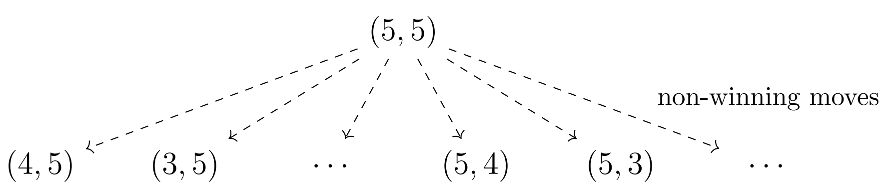
That is, every impartial game G is either a winning game or a losing game.

Example

Winning game of Nim (at least one winning move):



Losing game of Nim (no winning moves):



Note: we assume players play perfectly. If there is a winning move, then they will take it.

Lecture 3: Equivalent Games (I)

Game sums

Definition — game sum

Let G and H be two games with respective options $G_1, \ldots, G_m$ and $H_1, \ldots, H_n$. We define $G + H$ as the game with options

$$G_1 + H, \ldots, G_m + H, G + H_1, \ldots, G + H_n.$$

Example

We denote $*n$ to be a game of Nim with one pile of n chips. Then $*1 + *1 + *2$ is a Nim game with three piles of 1, 1, and 2 chips.

Example

Let $\#n$ be the subtraction game with n chips. Then $*5 + \#7$ is the game where a move is either to remove at least 1 chip from the pile of 5 (Nim) or to remove 1, 2, or 3 chips from the pile of 7 (subtraction game).

Lemma 3.1

Let $\mathcal{G}$ be the set of all impartial games. Then for all $G, H, J \in \mathcal{G}$,

1. $G + H \in \mathcal{G}$ (closure)
2. $(G + H) + J = G + (H + J)$ (associativity)
3. There exists an identity $0 \in \mathcal{G}$ (the game with no options) where $G + 0 = 0 + G = G$.
4. $G + H = H + G$ (symmetry)

Note: $\mathcal{G}$ is an abelian group, except for an inverse element.

Game equivalences

Definition — game equivalence

Two games G and H are **equivalent** if for any game J, $G + J$ and $H + J$ have the same outcome (*i.e.*, both are winning games or both are losing games). Notation: $G \equiv H$.

Example

$*3 \equiv *3$. Since $*3 + J$ is the same game as $*3 + J$, they have the same outcome.

$*3 \not\equiv *4$. $*3 + *3$ is a winning game but $*4 + *3$ is a losing game (Lemma 2.1).

Lemma 3.2

$*n \equiv *m$ if and only if $n = m$.

Lemma 3.3

The relation $\equiv$ is an equivalence relation. That is, for all $G, H, K \in \mathcal{G}$,

1. $G \equiv G$ (reflexivity).
2. $G \equiv H$ if and only if $H \equiv G$ (symmetry).
3. If $G \equiv H$ and $H \equiv J$, then $G \equiv J$ (transitivity).

Exercise 3.4

Prove that if $G \equiv H$, then $G + J \equiv H + J$ for any game J.

Proof.

Consider any games J and K. Let $M = J + K$. Then $G + J + K = G + M$ and $H + J + K = H + M$. Since $G \equiv H$, $G + M$ and $H + M$ have the same outcome. But then $(G + J) + K$ and $(H + J) + K$ have the same outcome by Lemma 3.1. K was arbitrary, so $G + J \equiv H + J$. $\square$

Losing games and empty Nim

Nim with one pile $*n$ is a losing game if and only if $n = 0$.

Theorem 3.5

G is a losing game if and only if $G \equiv *0$.

Corollary 3.6

If G is a losing game, then J and $J + G$ have the same outcome for any game J.

Proof.

Since G is a losing game, $G \equiv *0$ by Theorem 3.5. Then $J + G \equiv J + *0 \equiv J$ by Exercise 3.4 and Lemma 3.1, so J and $J + G$ have the same outcome. □

Example

1. Recall $*5 + *5$ and $*7 + *7$ are losing games. Corollary 3.6 says $*5 + *5 + *7 + *7$ is also a losing game. (Player I moves in either $*5 + *5$ or $*7 + *7$. Player II plays a winning move in the same part by equalizing piles.)
2. $\underbrace{*1 + *1 + *2}_{\text{winning}} + \underbrace{*5 + *5}_{\text{losing}}$ is a winning game by Corollary 3.6. (Player I removes $*2$, leaving a similar game to the previous.)

Proof (Theorem 3.5).

($\Longleftarrow$) If $G \equiv *0$, then $G + *0$ has the same outcome as $*0 + *0$. But $*0 + *0$ is a losing game, so G is a losing game.

($\Longrightarrow$) Suppose G is a losing game. We show $G + J$ and $*0 + J \equiv J$ have the same outcome.

1 Suppose J is a losing game. We show "if G and J are both losing games, then $G + J$ is a losing game" by induction on simplicity of $G + J$.

When $G+J$ has no options, G and J have no options, so G, J, and $G+J$ are all losing games. Assume $G+J$ has some otpions and the statement holds for simpler games. WLOG, Player I moves on G, resulting in $G' + J$. G is a losing game, so G' is a winning game. Player II makes a winning move from G' to G'', resulting in $G'' + J$. Then G'' is a losing game, so by induction $G'' + J$ is a losing game. Player I loses, so $G + J$ is a losing game.

(2) Suppose J is a winning game, so J as a winning move to J'. Player I moves from $G+J$ to $G+J'$. G and J' are both losing games, so by (1) $G+J'$ is a losing game. Player II loses, so Player I wins and $G+J$ is a winning game.

$\square$