PMATH 347: Groups and Rings

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Week 1: Groups

1: Binary operations and definition of a group

Binary operations

Definition — binary operation

A **binary operation** on a set X is a function $b\colon X \times X \to X$.

Notation:

- We can use any letter (b, m) or symbol ($+$, $\cdot$).
- We can use function notation (typically for symbols)

$$b\colon X \times X \to X : (x, y) \mapsto b(x, y)$$

  or inline notation (typically for letters)

$$+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (x, y) \mapsto x + y.$$

- Some symbols: $a + b$, $a \times b$, $a \cdot b$, $a \circ b$, $a \oplus b$, $a \otimes b$, $a \odot b$, $a \diamond b$, $a * b$, $a \bullet b$, $a \boxplus b$, $a \boxtimes b$.
- If not ambiguous, can drop the symbol:

$$X \times X \to X : (a, b) \mapsto ab.$$

Example

- Addition $+$ is a binary operation on $\mathbb{N}$, but subtraction $-$ is not since $a - b$ is not necessarily in $\mathbb{N}$.
- Subtraction is a binary operation on $\mathbb{Z}$, *i.e.*, it defines a function $-\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.
- If $(V, +, \cdot)$ is a vector space over a field $\mathbb{K}$, then $+$ is a binary operation on V, but $\cdot$ is not since $\cdot$ is a function $\mathbb{K} \times V \to V$.

Definition — k-ary operation

A **k-ary operation** on a set X is a function

$$\underbrace{X \times X \times \cdots \times X}_{k \text{ times}} \to X.$$

A 1-ary operation is called a **unary operation**.

Example

- Negation $\mathbb{Z} \to \mathbb{Z} : x \mapsto -x$ is a unary operation.
- Taking the multiplicative inverse $x \mapsto 1/x$ is not a unary operation on $\mathbb{Q}$, since $1/0$ is not defined, but it is a unary operation on

$$\mathbb{Q}^\times := \{a \in \mathbb{Q} : a \neq 0\}.$$

Associative operations

Definition — associative

A binary operation $\boxtimes\colon X \times X \to X$ is **associative** if

$$a \boxtimes (b \boxtimes c) = (a \boxtimes b) \boxtimes c$$

for all $a, b, c \in X$.

Many operations mentioned so far are associative:

- Addition and multiplication for $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, polynomials, and functions;
- Vector addition, matrix addition and multiplication;
- Modular addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$;
- Function composition (homework).

Subtraction and division are not associative:

$$10 - (5 - 1) = 6 \neq 4 = (10 - 5) - 1.$$

Subtraction is adding negative numbers; similarly for division. So we aren't as interested in subtraction and division, thus we can focus on associative operations.

A **bracketing** of a sequence $a_1, \ldots, a_n \in X$ is a way of inserting brackets into $a_1 \boxtimes \cdots \boxtimes a_n$ so that the expression can be evaluated (with binary steps).

Example

Bracketings of $a_1, \ldots, a_4$ are:

- $a_1 \boxtimes (a_2 \boxtimes (a_3 \boxtimes a_4))$
- $a_1 \boxtimes ((a_2 \boxtimes a_3) \boxtimes a_4)$
- $(a_1 \boxtimes a_2) \boxtimes (a_3 \boxtimes a_4)$
- $(a_1 \boxtimes (a_2 \boxtimes a_3)) \boxtimes a_4$
- $((a_1 \boxtimes a_2) \boxtimes a_3) \boxtimes a_4$

Proposition

A binary operation $\boxtimes\colon X \times X \to X$ is associative if and only if for all finite sequences $a_1, \ldots, a_n \in X$ with $n \geq 1$, every bracketing of $a_1, \ldots, a_n$ evaluates to the same element of X.

Meaning if $\boxtimes$ is associative, then the notation $a_1 \boxtimes \cdots \boxtimes a_n$ is unambiguous.

Proof.

($\Longleftarrow$) The two bracketings $a \boxtimes (b \boxtimes c)$ and $(a \boxtimes b) \boxtimes c$ of a, b, c evaluate to the same element of X for all sequences of length 3. So $\boxtimes$ is associative by definition.

($\Longrightarrow$) By induction. Base cases are $n = 1, 2, 3$. For $n = 1, 2$, there is only one bracketing. For $n = 3$, follows from the definition of associativity.

Suppose the proposition is true for all sequences of length $1 \leq k < n$.

Let w be a bracketing of $a_1, \ldots, a_n$. Then $w = w_1 \boxtimes w_2$ where w_1 is a bracketing of $a_1, \ldots, a_k$ and w_2 is a bracketing of $a_{k+1}, \ldots, a_n$ for some $k < n$. By induction,

$$\begin{aligned} w_1 &= (\cdots((a_1 \boxtimes a_2) \boxtimes a_3) \cdots \boxtimes a_k) \\ w_2 &= (a_{k+1} \boxtimes \cdots (a_{n-2} \boxtimes (a_{n-1} \boxtimes a_n)) \cdots) \end{aligned}$$

So by repeatedly applying associativity,

$$\begin{aligned} w &= (\cdots((a_1 \boxtimes a_2) \boxtimes a_3) \cdots \boxtimes a_k) \boxtimes (a_{k+1} \boxtimes \cdots (a_{n-1} \boxtimes a_n) \cdots) \\ &= (\cdots(a_1 \boxtimes a_2) \cdots \boxtimes a_{k-1}) \boxtimes (a_k \boxtimes (a_{k+1} \boxtimes \cdots \boxtimes a_n) \cdots) \\ &= \ldots \\ &= (a_1 \boxtimes (a_2 \boxtimes \cdots (a_{n-1} \boxtimes a_n)) \cdots) \end{aligned}$$

□

Commutative (abelian) operations

Definition — commutative (abelian)

A binary operation $\boxtimes\colon X \times X \to X$ is **commutative** or **abelian** if $a \boxtimes b = b \boxtimes a$ for all $a, b \in X$.

Many familiar operations are commutative:

- Addition and multiplication on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- Vector and matrix addition
- Modular addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$

The following operations are **not** commutative:

- Subtraction and division: $3 - 1 \neq 1 - 3$
- Function composition
- Matrix multiplication

Note:

1. Subtraction and division are not commutative or associative
2. Function composition and matrix multiplication are not commutative, but are associative

We won't study operations like (1), but we are interested in those like (2).

The first half of this course is group theory: single associative operation, not necessarily commutative.

The second half of this course is ring theory: two associative operations, focus on the both commutative case.

Identities

Definition — identity

Let $\boxtimes$ be a binary operation on a set X. An element $e \in X$ is an **identity** for $\boxtimes$ if

$$e \boxtimes x = x \boxtimes e = x$$

for all $x \in X$.

Example

- The zero element 0 of $\mathbb{Z}$ is an identity for $+$, since $0 + x = x + 0 = x$ for all $x \in \mathbb{Z}$.
- $1 \in \mathbb{Q}$ is an identity for $\cdot$, since $1 \cdot x = x \cdot 1 = x$ for all $x \in \mathbb{Q}$.
- $0 \in \mathbb{Q}$ is not an identity for $\cdot$, since $0 \cdot x = 0 \neq x$ for all $x \in \mathbb{Q}$.

Lemma

If $e, e' \in X$ are both identities for $\boxtimes$, then $e = e'$.

Proof.

$e = e \boxtimes e' = e'$. □

Inverses

Definition — inverse

Let $\boxtimes$ be a binary operation on X with an identity element e. An element y is a **left inverse** for x (with respect to $\boxtimes$) if $y \boxtimes x = e$, a **right inverse** if $x \boxtimes y = e$, and an **inverse** if $x \boxtimes y = y \boxtimes x = e$.

Example

- $-n$ is an inverse for $n \in \mathbb{Z}$ with respect to $+$, since $n + (-n) = (-n) + n = 0$.
- $n \in \mathbb{Z}$ does not have an inverse with respect to $\cdot$ unless $n = \pm 1$.
- If $x \in \mathbb{Q}$ is non-zero, then $1/x$ is an inverse of x with respect to $\cdot$. The element 0 does not have an inverse, since there is no element y with $0 \cdot y = 1$.

Lemma

Let $\boxtimes$ be an associative binary operation with an identity e. If y_L and y_R are left and right inverses of x respectively, then $y_L = y_R$.

Proof.

$y_L = y_L \boxtimes e = y_L \boxtimes (x \boxtimes y_R) = (y_L \boxtimes x) \boxtimes y_R = e \boxtimes y_R = y_R$. □

Corollaries:

- If x has both a left and a right inverse, then x has an inverse.
- Inverses are unique: if y and y' are both inverses of x, then $y = y'$.

An element a is **invertible** if it has an inverse, in which case the inverse is denoted by a^{-1}.

Exercise

Show it is possible to have a left (resp. right) inverse, but not be invertible. Also show left and right inverses are not necessarily unique (unless an element has both).

Properties of inverses

Lemma

1. If $\boxtimes$ has an identity e, then e is invertible, and $e^{-1} = e$.
2. If a is invertible, then so is a^{-1}, and $(a^{-1})^{-1} = a$.
3. If $\boxtimes$ is associative, and a and b are invertible, then so is $a \boxtimes b$, and $(a \boxtimes b)^{-1} = b^{-1} \boxtimes a^{-1}$.

Proof.

1. $e \boxtimes e = e$.
2. $a \boxtimes a^{-1} = a^{-1} \boxtimes a = e$, so a is an inverse to a^{-1}.
3. $(a \boxtimes b) \boxtimes (b^{-1} \boxtimes a^{-1}) = a \boxtimes (b \boxtimes b^{-1}) \boxtimes a^{-1} = a \boxtimes e \boxtimes a^{-1} = a \boxtimes a^{-1} = e$, and similarly $(b^{-1} \boxtimes a^{-1}) \boxtimes (a \boxtimes b) = e$.

□

Inverses and solving equations

Proposition

Let $\boxtimes$ be an associative binary operation on X with an identity e, and let x and y be variables taking values in X.
An element $a \in X$ is invertible if and only if the equations $a \boxtimes x = b$ and $y \boxtimes a = b$ have unique solutions for all $b \in X$.

Proof.

($\Longleftarrow$) A solution to $a \boxtimes x = e$ is a right inverse of a, and a solution to $y \boxtimes a = b$ is a left inverse. Since both solutions exist, a has an inverse.

($\Longrightarrow$) Suppose a is invertible. Then

$$a \boxtimes (a^{-1} \boxtimes b) = (a \boxtimes a^{-1}) \boxtimes b = e \boxtimes b = b$$

so $a^{-1} \boxtimes b$ is a solution to $a \boxtimes x = b$.

If x_0 is a solution to $a \boxtimes x = b$, then

$$a^{-1} \boxtimes b = a^{-1} \boxtimes (a \boxtimes x_0) = (a^{-1} \boxtimes a) \boxtimes x_0 = e \boxtimes x_0 = x_0$$

so $a^{-1} \boxtimes b$ is the unique solution to $a \boxtimes x = b$.

Similarly, $b \boxtimes a^{-1}$ is the unique solution to $y \boxtimes a = b$.

□

Left and right cancellation property

Proposition

Let $\boxtimes$ be an associative binary operation and let $a \in X$. Then:

1. If a has a left inverse and $a \boxtimes u = a \boxtimes v$, then $u = v$.
2. If a has a right inverse and $u \boxtimes a = v \boxtimes a$, then $u = v$.

Proof.

1. $u = a_L \boxtimes a \boxtimes u = a_L \boxtimes a \boxtimes v = v$.
2. Similar.

□

(1) and (2) also hold for $n \in \mathbb{Z}$ with respect to $\cdot$ if $n \neq 0$, even though n is not invertible for $n \neq \pm 1$.

Groups

Definition — group

A **group** is a pair $(G, \boxtimes)$ where

1. G is a set, and
2. $\boxtimes$ is an associative binary operation on G such that
   (a) $\boxtimes$ has an identity e, and
   (b) every element $g \in G$ is invertible with respect to $\boxtimes$.

A group is **abelian** (or **commutative**) if $\boxtimes$ is abelian.

A group is **finite** if G is a finite set. The **order** of G is the number of elements in G if G is finite, or $+\infty$ if G is infinite.

The order of G is denoted by $|G|$.

Terminology:

- Usually we refer to $(G, \boxtimes)$ simply as G, and just assume the operation is given. (Note: we still need to clearly specify the operation for each group we work with.)
- It's cumbersome to write $\boxtimes$, so usually we use one of the following options:
  - Use $\cdot$ as the standard symbol: $g \cdot h$ is the product of $g, h \in G$.
  - Drop the symbol entirely: gh is the product of $g, h \in G$.
- The identity of G is denoted by e (or e_G for clarity). Also used are 1 and 1_G.
- g^{-1} is defined for all $g \in G$. The function $G \to G : g \mapsto g^{-1}$ can be regarded as a unary operation on G.
- Consider $\iota \colon G \to G : g \mapsto g^{-1}$. Since $(g^{-1})^{-1} = g$, $\iota \circ \iota = \mathrm{Id}_G$, the identity map $G \to G$. In particular, ι is a bijection (injective and surjective).
- If $g \in G$, then
$$g^n := \underbrace{g \cdots g}_{n \text{ times}}$$
and
$$g^{-n} := (g^{-1})^n = (g^n)^{-1}$$
where $g^0 := e$. Exercise: if $m, n \in \mathbb{Z}$, then $(g^n)^m = g^{mn}$.
- If $g, h \in G$, then
$$(gh)^n = gh \cdots gh,$$
which is not necessarily the same as $g^n h^n$ if G is not abelian.

Example

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are all (abelian) groups under operation $+$. The identity is 0 and the inverse of n is $-n$. These groups have infinite order.
- $\mathbb{Z}/n\mathbb{Z}$ is also a group under $+$ (and also abelian). The identity is $0 = [0]$ and the inverse of $[m]$ is $-[m] = [-m]$. This group is finite with order $|\mathbb{Z}/n\mathbb{Z}| = n$.
- If $(V, +, \cdot)$ is a vector space, then $(V, +)$ is a group. The identity is 0 and the inverse of v is $-v$.
- $\mathbb{Z}$ is not a group with respect to $\cdot$, since most elements do not have an inverse.
- $\mathbb{Q}$ is also not a group with respect to $\cdot$, since 0 does not have an inverse.
- $\mathbb{Q}^\times$ is a group with respect to $\cdot$.
- Every group has to contain at least one element, the identity. So the simplest possible group is 1 with operation $1 \cdot 1 = 1$. This is the **trivial group**.

A non-abelian example

All the previous examples are abelian.

Let $\mathrm{GL}_n(\mathbb{K})$ denote the invertible $n \times n$ matrices over a field $\mathbb{K}$.

Proposition

$\mathrm{GL}_n(\mathbb{K})$ is a group under matrix multiplication (called the **general linear group**). For $n \geq 2$, $\mathrm{GL}_n(\mathbb{K})$ is non-abelian.

Proof.

If A and B are invertible matrices, then AB is also invertible, so matrix multiplication is an associative binary operation on $\mathrm{GL}_n(\mathbb{K})$. The identity matrix is an identity and every element has an inverse by definition, so $\mathrm{GL}_n(\mathbb{K})$ is a group.

Exercise: find matrices A, B such that $AB \neq BA$. □

Additive notation

Standard notation for a group operation is gh. This is called **multiplicative notation**.

For groups like $(\mathbb{Z}, +)$, it is confusing to write mn instead of $m + n$ since mn already has another meaning.

For abelian groups G, we can also use **additive notation**. In additive notation, we write the group operation as $g + h$. The identity is denoted by 0 or 0_G. Inverses are denoted by $-g$.

Writing g^n in additive notation gives

$$\underbrace{g + \cdots + g}_{n \text{ times}}$$

so instead of g^n we use ng. Similarly g^{-n} is $-ng$.

Multiplicative notation	Additive notation
$g \cdot h$ or gh	$g + h$
e_G or 1_G	0_G
g^{-1}	$-g$
g^n	ng

For non-abelian groups we always use multiplicative notation. For abelian groups, we can choose either. Note the conventions may conflict, so we should be clear about which we choose.

For a group like $(\mathbb{Z}, +)$, we could use mn, but it is clearer to use $m + n$.

For a group like $(\mathbb{Q}^\times, \cdot)$, we could use $x + y$, but it is clearer to use $x \cdot y$ or xy.

Multiplication table

Definition — multiplication table

The **multiplication table** of a group G is a table with rows and columns indexed by the elements of G. The cell for row g and column h contains the product gh.

The multiplication table contains the complete information of the group (even for infinite groups).

Example

For $\mathbb{Z}/2\mathbb{Z}$:

	0	1
0	0	1
1	1	0

Order of elements

Definition — order of a group element

If G is a group, then the order of $g \in G$ is

$$|g| := \min\{k \geq 1 : g^k = e_G\} \cup \{+\infty\}.$$

Easy properties:

- $|g| = 1$ if and only if $g = e_G$.
- If $g^n = 1$, then $g^{n-1}g = gg^{n-1} = g^n = 1$, so $g^{n-1} = g^{-1}$. In particular, if $|g| = n < \infty$, then $g^{-1} = g^{n-1}$.

Example

We use additive notation for $\mathbb{Z}/n\mathbb{Z}$, so g^n is written as ng and $e = 0$. For this group, $k1 = 0$ if and only if $n \mid k$, so $|1| = n$.

Lemma

$g^n = e$ if and only if $g^{-n} = e$, so in particular, $|g| = |g^{-1}|$.

Proof.

We have $g^{-n} = (g^n)^{-1}$. Since $g \mapsto g^{-1}$ is a bijection, $g^n = e$ if and only if $(g^n)^{-1} = e^{-1} = e$.

But $g^{-n} = (g^{-1})^n$ also, so $\{k \geq 1 : g^k = e\} = \{k \geq 1 : (g^{-1})^k = e\}$ which implies $|g| = |g^{-1}|$. □