Fair Resource Allocation in a Volatile Marketplace

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Abstract

We consider a setting where a platform must dynamically allocate a collection of goods that arrive to the platform in an online fashion to budgeted buyers, as exemplified by online advertising systems where platforms decide which impressions to serve to various advertisers. Such dynamic resource allocation problems are challenging for two reasons: (a) the platform must strike a balance between optimizing its own revenues and guaranteeing fairness to its (repeat) buyers and (b) the problem is inherently dynamic due to the uncertain, time-varying supply of goods available with the platform. We propose a stochastic approximation scheme akin to a dynamic market equilibrium. Our scheme relies on frequent re-solves of an Eisenberg-Gale convex program, and does not require the platform to have any knowledge about how the goods arrival processes evolve over time. The scheme fully extracts buyer budgets (thus maximizing platform revenues), while at the same time provides a 0.64 approximation of the proportionally fair allocation of goods achievable in the offline case, as long as the supply of goods comes from a wide family of (possibly non-stationary) Gaussian processes.

1 Introduction

The problem of allocating a finite supply of resources, or alternatively goods, to a population of budget-constrained customers, is ubiquitous in a swathe of applications in domains ranging from e-commerce, online advertising and on-demand cloud computing to humanitarian logistics. The specific model we are concerned with in this paper can be described succinctly as follows: a stream of goods of types indexed by i arrive in an online fashion to a platform over a finite time interval. The platform seeks to allocate each arriving good to a customer belonging to a set indexed by j. Customer j has budget $B_{j,0}$ and utility u_{ij} for a unit of a good of type i. We assume that goods are perishable in the sense that allocation decisions must be made immediately or the current good is lost. The platform has two controls: (a) the allocation x_{ij} specifying how much of the inventory of goods of type i is given the agent j and (b) the unit price p_i at which this allocation occurs.

NIPS 2018 Workshop on Challenges and Opportunities for AI in Financial Services: the Impact of Fairness, Explainability, Accuracy, and Privacy, Montréal, Canada. This is a non-archival publication - the authors may submit revisions and extensions of this paper to other publication venues.

In this work, we focus on finding a market-clearing allocation of the goods that is "fair" to the platform's customers. While the bulk of the literature on resource allocation is driven by the objective of maximizing platform revenue (for instance the network revenue management problem from Gallego and Van Ryzin [1997]), we believe fairness is a crucial system objective in two kinds of contexts: (a) those where the platform is explicitly mandated to optimize fairness, such as one might encounter in healthcare or humanitarian settings and (b) those where the platform is a profit maximizer, but due to the repeated nature of the interactions with its customers, the platform must ensure fairness to its customers in order to attract their repeat business over the long term. Such repeated interaction settings are common in online advertising, where advertisers re-negotiate the terms of their advertising campaigns with the ad network even on a daily basis, as is the case with Google's AdWords system.

Offline versus online resource allocation. As we shall see in the sequel, fairness in resource allocation has already been studied in deterministic settings. However, in many practical applications, the inventory of goods available to the platform across the entire time horizon is a highly uncertain quantity. For example, the rates at which impressions with certain features arrive to an ad network are sensitive to 'shocks' due to news or social trends – such dynamics are highly dimensional, non-stationary and inherently difficult to model and forecast. Moreover, decisions are without recourse as impressions must be served their ads in the timeframe of several milliseconds. We call this setting the *online* problem, as opposed to the *offline* problem where we assume clairvoyant access to the inventory of goods. The present work is an attempt to bridge the gap between our understanding of fairness schemes for the offline and online versions of resource allocation problems.

Below, we give an example of an application that fits our model of fair resource allocation: *Allocation of sponsored search or display ads.* Here, the decision-maker is an ad platform whch has access to a stream of user requests, or impressions, and seeks to monetize this inventory by delivering it to various advertiser campaigns. The campaigns typically have time horizons ranging from one day to several weeks. Advertisers set a maximum budget for the campaign to spend and a set of bids for certain impression types; these parameters remain valid for the duration of the campaign. At the end of the time horizon, the campaign may get renegotiated with the platform and renewed. Thus, there is a repeated sequence of interactions between the platform and its advertisers, with each interaction being an allocation problem in itself. In light of this, the platform must balance between two distinct goals: (a) *extracting advertiser budgets* as platform revenues, while (b) *guaranteeing fair allocations* of impressions to the advertisers to encourage them to return and commit budgets to new campaigns over future interactions, as well as attracting new advertisers to the platform encouraging healthier, more efficient competition. We remark that several advertising platforms offer a swathe of analytics tools to help their customers optimize their campaigns with the goal of fostering marketplace growth.

Fairness. In this paper, we focus on *weighted proportional fairness*, a fairness metric which sits at the midway point between the two extremes described above, being less averse to inequality than max-min fairness, while at the same time more averse than utilitarian efficiency. This fairness notion has been studied extensively, see, e.g., Nash Jr [1950], Bertsimas et al. [2011, 2012].

We define weighted proportional fairness in the following way: for a weights vector $\mathbf{w} \in \mathbf{R}_+^J$, an allocation of utilities $\mathbf{U} \in \mathbf{R}_+^J$ is proportionally fair if

$$\sum_{j=1}^{J} w_j \frac{U_j' - U_j}{U_j} \leq 0, \quad \text{for all other feasible utility vectors } \mathbf{U}' \in \mathbf{R}_+^J.$$

Market equilibrium. Pricing and allocating goods in a marketplace with multiple budgeted buyers has been studied extensively as a market equilibrium problem starting with the work of Walras. We focus here on the linear utility case (i.e., the utility for customer j is $U_j = \sum_{i=1}^{I} u_{ij} x_{ij}$), which is known as Fisher market equilibrium. A Fisher market equilibrium is a pair $(\mathbf{x}, \mathbf{p})$, where $\mathbf{x}$ specifies a vector of allocations of goods to customers and $\mathbf{p}$ the unit price for each good type, with the following properties: (a) The market clears both in terms of the inventory of goods and buyer budgets and (b) The optimal allocation $\mathbf{x}$ is such that it maximizes the individual utility of customers given prices $\mathbf{p}$.

The customer utilities garnered via a Fisher allocation implement a proportionally fair allocation with weights given by the customer budgets. Hence, Fisher equilibrium provides a mechanism to arrive at fair allocations while simultaneously clearing the market. The second property suggests that a Fisher allocation is fair in an additional sense: the market clearing prices $\mathbf{p}$ are such that, if each customer was allowed to pick their own selfish allocation at prices $\mathbf{p}$, they would choose exactly the centralized allocation $\mathbf{x}$. From an implementation standpoint, in the case that the supply of goods is known, the pair $(\mathbf{x}, \mathbf{p})$ is efficiently computable Devanur et al. [2008], Jain [2007], Ye [2008], for example by the celebrated Eisenberg-Gale convex program Eisenberg and Gale [1959].

Having posited an *offline* market equilibrium mechanism that fills the fairness and budget clearance conditions we were looking for at the onset, we ask ourselves how to extend it in settings where the inventory of goods evolves dynamically and unpredictably over time. In the following, we present an implementable scheme that can be used in this online setting and that inherits (possibly approximately) the market clearance and fairness properties of static Fisher equilibrium.

1.1 Main Contributions

Online scheme for proportional fairness. We present an online algorithm that, in the presence of uncertainty in the rates at which goods of various types arrive, produces a series of Fisher market allocation and price pairs. The scheme works by solving a static Eisenberg-Gale (EG) convex program at each of a discrete set of times within the horizon; each re-solve assumes that the current rate $\mathbf{\Lambda}_t$ at which inventories of goods arrive to the platform remains constant throughout time, and plugs that rate into an EG program to yield a myopic allocation price pair $(\mathbf{x}_t, \mathbf{p}_t)$. We show that our algorithm satisfies the following properties:

1. **Almost sure budget consumption:** the algorithm is market clearing even in the presence of uncertainty in the inventory of goods. We show that as the number of resolves becomes large, our algorithm is guaranteed to always clear the market as long as the sample paths of the goods arrival process are continuous.

2. **Constant factor proportional fairness guarantee versus offline optimum:** assuming that $\{\mathbf{\Lambda}_t\}$ belongs to a natural family of Gaussian processes with concave volatility structure, we provide a *constant* 0.64 bound on the fairness loss our scheme incurs, relative to the optimal proportionally fair allocation that is achievable with clairvoyant knowledge of $\{\mathbf{\Lambda}_t\}$'s sample path.

   This contrasts results in adversarial models, where a logarithmic approximation is the best possible [Azar et al., 2010]. We emphasize that (a) our bound applies to processes within a large parametric family and of *arbitrarily* large volatility and (b) the family consists of *non-stationary* processes that are inherently difficult to learn: in fact, one can view our stochastic model as in-between an i.i.d./random permutation model and an adversarial one.

Furthermore, we then relax the condition of concave volatility structure and consider a generic family Gaussian processes, to which we obtain a parametrized lower bound depending on the degree of convexity of volatility over time.

3. **Asymptotically optimal proportional fairness guarantee:** additionally, in regimes when the volatility of the $\{\mathbf{\Lambda}_t\}$ process goes to 0, the gap becomes 0; in other words, our scheme is optimal when the platform has reliable forecasts.

These are the first constant factor guarantees for online resource allocation where the objective is to maximize proportional fairness, as well as for online versions of Fisher market equilibrium.

2 Model

Consider a platform which, over a finite selling season $[0, T]$, receives an inventory stream of goods of I different types. At time t, a good of type i arrives with rate $\Lambda_{i,t} \geq 0$ which can only be used instantaneously and cannot be stored for future use, but only sold immediately to a set of J customers. Each customer j is endowed with a budget $B_{j,0} > 0$ at time zero. Customer j gains utility $u_{ij} \geq 0$ from consuming one unit of good type i, where $u_{ij} > 0$ for some $i \in \{1, \cdots, I\}$. At each time point $t \in [0, T)$, the platform determines the fraction, $x_{ij,t}$, of good type i's instantaneous supply allocated to customer j, and the unit price, $p_{i,t}$, for a good of type i.

We assume here that customers are truthful in their declarations of utilities and budgets, but remark that there is a body of literature on game-theoretic equilibria in Fisher markets as well as other market equilibrium models, such as Adsul et al. [2010], Brânzei et al. [2014], Mehta et al. [2014] and Cole and Tao [2015].

Resource dynamics. Here, we present the goods arrival process which is central to our model. We assume that $\{\mathbf{\Lambda}_t\}$ is a *reflected Gaussian process (GP)*, defined as:

Definition 1 *We call a process $\{\mathbf{\Lambda}_t\}$ a* reflected Gaussian process *if*

1. $\Lambda_{i,t} = |\bar{\Lambda}_{i,t}|$, *where $\bar{\Lambda}_{i,t}$ is a Gaussian process with continuous sample paths.*
2. $\mathsf{E}\left[\bar{\Lambda}_{i,t}\right] = \lambda_i > 0$.
3. *The variance of $\bar{\Lambda}_{i,t}$, denoted as $\sigma^2_{i,t}$, is non-decreasing in t.*

In the above, we interpret λ_i as a deterministic forecast of good type i's arrival rate, and $\sigma^2_{i,t}$ as a measure of the forecast uncertainty of good type i's arrival rate at time t. The third assumption that $\sigma^2_{i,t}$ is non-decreasing in t reflects the intuition that the forecast for an event that will happen far from now is more uncertain than the forecast for an event that will happen in a near future. We denote by GP the family of all such processes.

Many frequently used processes are GP. As one example, we consider a *generalized moving average process* $\{\mathbf{\Lambda}_t\}$, defined as

$$\Lambda_{i,t} = |\bar{\Lambda}_{i,t}| = \left|\lambda_i + \int_{s=0}^{t} \phi_i(t-s) dZ_{i,s}\right| \tag{1}$$

where $\lambda_i > 0$, $\phi_i(\cdot) \in \mathcal{C}_1$ and $dZ_{i,s}$ is an increment of Brownian motion. It is evident that this process satisfies the first two conditions above. For this process, following from Ito's isometry, we have $\sigma^2_{i,t} = \int_{s=0}^{t} \phi_i\left(t-s\right)^2 ds$. Therefore, this process satisfies the third condition above, i.e., $\sigma^2_{i,t}$ is non-decreasing in t. Therefore, generalized moving average processes are a special case of GP.

We define the following subset of GP, which is a natural class of Gaussian processes to which we will devote special attention in the sequel:

Definition 2 *We call a process* $\{\mathbf{\Lambda}_t\}$ *a* regular reflected Gaussian process, *or* rGP, *if* $\{\mathbf{\Lambda}_t\} \in \mathrm{GP}$ *and furthermore,* $\sigma_{i,t}^2$ *is concave in t for all* $i \in \{1, \cdots, I\}$*.*

To develop some intuition about this process class, it is useful again to refer to the example of generalized moving average processes. Recall that for such processes, $\sigma_{i,t}^2 = \int_{s=0}^{t} \phi_i\,(t-s)^2\,ds$. Therefore, any such process which additionally satisfies the condition that $|\phi_i(\cdot)|$ is non-increasing is rGP. This monotonicity condition has the interpretation that shocks generated today have a diminishing influence on the future values of the process; essentially, the process eventually 'forgets' shocks that have happened very far in the past. The following special moving average processes satisfy this condition: (1) the Wiener process with $\phi_i(t) = \sigma_i$ for arbitrary $\sigma_i > 0$; (2) the Ornstein-Uhlenbeck (OU) process with mean λ_i, the initial good arrival rate $\Lambda_{i,0} = \lambda_i$, and $\phi_i(t) = \sigma_i \exp(-\beta_i t)$ for arbitrary $\sigma_i > 0$, and $\beta_i > 0$.

Definition 3 *For* $k > 0$*, we call a process* $\{\mathbf{\Lambda}_t\}$ *a* k-th order reflected Gaussian process, *or* $\mathrm{GP(k)}$*, if* $\{\mathbf{\Lambda}_t\} \in \mathrm{GP}$ *and furthermore* $\frac{\sigma_{i,t}^2}{\sigma_{i,t'}^2} \le \max\left\{\left(\frac{t}{t'}\right)^k, 1\right\}$ *for all* $i \in \{1, \cdots, I\}$*,* $t, t' \in [0, T]$*.*

We denote by rGP and, respectively, $\mathrm{GP}(k)$ the family of all regular GP and respectively, k-th order GP processes.

Allocation and pricing policies. Define $\mathbf{\Lambda}^t \triangleq \{\mathbf{\Lambda}_s : s \in [0, t]\}$ and $(\boldsymbol{x}^t, \boldsymbol{p}^t) \triangleq \{x_{ij,s}, p_{i,s} : \forall i, j, s \in [0, t)\}$. Define a filtration $\{\mathcal{F}_t : t \in [0, T)\}$ where $\mathcal{F}_t \triangleq \sigma(\mathbf{\Lambda}^t, \boldsymbol{x}^t, \boldsymbol{p}^t, \boldsymbol{B}_0)$. We consider a family of dynamic allocation and pricing policies Π. Each policy $\pi \triangleq \{x_{ij,t}^\pi, p_{i,t}^\pi : \forall i, j, t \in [0, T)\} \in \Pi$ consists of the platform's allocation and pricing decisions $x_{ij,t}^\pi$ and $p_{i,t}^\pi$. We say π is feasible if it is adapted to the filtration $\{\mathcal{F}_t : t \in [0, T]\}$ and satisfies:

$$
\begin{aligned}
x_{ij,t}^\pi &\ge 0, \qquad \forall i, j, t, \qquad && \sum_{j=1}^{J} x_{ij,t}^\pi \le 1 \qquad \forall i, t, \\
x_{ij,t}^\pi &= 0 \text{ if } B_{j,t} = 0 \qquad \forall i, j, t, \qquad && p_{i,t}^\pi \ge 0 \qquad \forall i, t.
\end{aligned}
$$

Under a policy $\pi \in \Pi$ and conditional on the sample path of $\{\mathbf{\Lambda}_t\}$, customer j's remaining budget at time t is: $B_{j,t} = B_{j,0} - \int_{s=0}^{t}\left(\sum_{i=1}^{I} p_{i,s}^\pi \Lambda_{i,s} x_{ij,s}^\pi\right) ds$. The total utility customer j garners over the entire selling season is: $U_j^\pi = \int_{t=0}^{T}\left(\sum_{i=1}^{I} u_{ij} \Lambda_{i,t} x_{ij,t}^\pi\right) dt$.

2.1 Fair Allocations

As alluded in the introduction, we seek to maximize the budget-weighted proportional fairness of the utilities allocated to customers. Let us define, for any vectors $\boldsymbol{U} \in \mathbf{R}_+^J$ and $\boldsymbol{w} \in \mathbf{R}_+^J$ with $||\boldsymbol{w}||_1 \triangleq \sum_{j=1}^{J} w_j = 1$, the quantity

$$
F(\boldsymbol{U}, \boldsymbol{w}) \triangleq \prod_{j=1}^{J} U_j^{w_j}. \tag{2}
$$

Further, for any policy $\pi \in \Pi$, define the *expected weighted proportional fairness* measure as

$$
\mathrm{FAIR}^\pi(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0) \triangleq \mathsf{E}\left[F\left(\boldsymbol{U}^\pi, \frac{\boldsymbol{B}_0}{||\boldsymbol{B}_0||_1}\right)\right],
$$

where the expectation is with respect to $\{\mathbf{\Lambda}_t\}_{t\geq 0}$. We seek a policy $\pi \in \Pi$ that admits a competitive ratio versus the best achievable offline fairness.

It is worth discussing the relationship between the above definition of the proportional fairness, $F(\boldsymbol{U}, \boldsymbol{w})$, and another widely adopted definition of weighted proportional fairness given by $\ln(F(\boldsymbol{U}, \boldsymbol{w})) = \sum_{j=1}^{J} w_j \ln U_j$. The reason we use the exponentiated fairness metric is to ensure that the performance ratio of our policies is well-defined. Specifically, in the sequel we will use the ratio of the fairness achieved under a heuristic policy π to the optimal fairness level to evaluate the fairness performance of our proposed policy π. To make sure this ratio is meaningful, the fairness measure must be non-negative, which is not guaranteed under the standard definition.

The platform's fairness optimization problem. The platform's problem is to find a policy $\pi^* \triangleq \{x^*_{ij,t}, p^*_{i,t}, \forall i, j, t\}$ that achieves

$$\mathrm{FAIR}^{\pi^*}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0) = \sup_{\pi\in\Pi} \mathrm{FAIR}^{\pi}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0) \triangleq \mathrm{FAIR}^*(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0).$$

There are several challenges to finding such an optimal fairness policy.

Robustness to mis-specification of $\{\mathbf{\Lambda}_t\}$. Computing the optimal fairness policy requires the platform to build up a forecast model of the multi-dimensional process governing the arrival of goods. However, due to its highly unpredictable nature, the platform may not have the ability to do so. Hence, the allocation and pricing policy obtained from solving an optimization problem which assumes a wrong forecast might lead to a poor outcome.

Heavy computational burden. Even assuming that a forecast of the goods arrival process $\{\mathbf{\Lambda}_t\}$ was available, it is still necessary to solve a DP to find the optimal policy. The curse of dimensionality makes it practically infeasible to compute the optimal fairness policy even for small scale problems.

Due to the aforementioned challenges, in the rest of this paper, as opposed to seeking optimality, we consider heuristic policies that can both avoid the above difficulties and achieve good approximations of the best fairness outcome.

3 Benchmark: Offline Problem with Known Goods Inventory

As noted before, a big challenge is unpredictability in the volume of goods that will arrive to the platform. In this section, we consider an offline benchmark problem in which the platform clairvoyantly and precisely knows, at time $t = 0$, the volume of goods to arrive throughout the entire time horizon. We study this offline problem for two reasons. First, the optimal fairness policy in the offline problem will motivate our construction of heuristic policies for the online problem. Second, the optimal fairness value in the offline problem serves as an upper bound on the optimal fairness value in the online problem and can, therefore, be used as a benchmark to evaluate the performance of heuristic policies that we will propose later for the online case.

We begin by introducing the following auxiliary static optimization problem $f(\boldsymbol{B}, \boldsymbol{S})$ with $\boldsymbol{B} \in \mathbb{R}^J_{>0}$ and $\boldsymbol{S} \in \mathbb{R}^I_{>0}$ being the (perfectly predicted or known) supply vector:

$$
\begin{aligned}
\max_{\{x_{ij}\}} \quad & \ln F\left(\boldsymbol{U}, \frac{\boldsymbol{B}}{||\boldsymbol{B}||_1}\right) & \triangleq \qquad & \max_{\{x_{ij}\}} \quad \sum_{j=1}^{j} \frac{B_j}{||\boldsymbol{B}||_1} \cdot \ln U_j \\
\text{s.t.} \quad & U_j = \sum_{i=1}^{I} u_{ij} S_i x_{ij} \; \forall j, & & \text{s.t.} \quad U_j = \sum_{i=1}^{I} u_{ij} S_i x_{ij} \; \forall j, \\
& \sum_{j=1}^{J} x_{ij} \leq 1 \; \forall i, & & \qquad\;\; \sum_{j=1}^{J} x_{ij} \leq 1 \; \forall i, \\
& x_{ij} \geq 0 \; \forall i, j, & & \qquad\;\; x_{ij} \geq 0 \; \forall i, j.
\end{aligned} \tag{3}
$$

We abuse our notation to denote the optimal value of this program as $f(\boldsymbol{B}, \boldsymbol{S})$. We denote the optimal solution as $\{x_{ij}(\boldsymbol{B}, \boldsymbol{S}), \forall i, j\}$ and the optimal dual variables associated with constraints $\sum_{j=1}^{J} x_{ij} \leq 1$ as $\{\tilde{p}_i(\boldsymbol{B}, \boldsymbol{S}), \forall i\}$. We define $p_i(\boldsymbol{B}, \boldsymbol{S}) \triangleq \frac{\tilde{p}_i(\boldsymbol{B}, \boldsymbol{S})}{S_i} \cdot ||\boldsymbol{B}||_1$, which can be interpreted as the 'physical' price per unit of good of type i, and is simply a renormalization of the $\tilde{p}_i(\boldsymbol{B}, \boldsymbol{S})$ dual variables. We also note that (3) is better known as the Eisenberg-Gale convex program [Eisenberg and Gale, 1959] that solves for a Fisher market equilibrium.

Under the clairvoyance assumption, solving the above auxiliary static optimization problem yields a (stationary) allocation $\boldsymbol{x}(\boldsymbol{B}_0, \int_{t=0}^{T} \boldsymbol{\Lambda}_t dt)$ and $\boldsymbol{p}(\boldsymbol{B}_0, \int_{t=0}^{T} \boldsymbol{\Lambda}_t dt)$ (hereafter written in a compact form as $(\boldsymbol{x}, \boldsymbol{p})(\boldsymbol{B}_0, \int_{t=0}^{T} \boldsymbol{\Lambda}_t dt))$ which maximizes the proportional fairness level while also clearing the market. This is a standard result for Fisher equilibrium in the *offline* problem.

4 Re-Optimization Policy with Continuous Reviews for Online Problem

In this section, we propose and analyze a heuristic policy for the online version of our resource allocation problem. We propose a policy which simply repeatedly solves, at each point of time, for the optimal policy given the assumption that the prevailing supply rate $\boldsymbol{\Lambda}_t$ remains constant until the end of the season (i.e., the outstanding supply is $\boldsymbol{\Lambda}_t \cdot (T - t)$). Formally, $\pi^{re} \in \Pi$ is a *re-optimization policy with continuous updates* if it satisfies the following conditions:

$$
\begin{aligned}
x_{ij,t}^{\pi^{re}} &\triangleq x_{ij}(\boldsymbol{B}_t, \boldsymbol{\Lambda}_t \cdot (T - t)), \qquad && \forall i, j, t, \\
p_{i,t}^{\pi^{re}} &\triangleq p_i(\boldsymbol{B}_t, \boldsymbol{\Lambda}_t \cdot (T - t)), \qquad && \forall i, t.
\end{aligned}
$$

The re-optimization policy π^{re} is appealing for several reasons:

1. The information the platform needs to update its control at time t only consists of *prevailing* goods arrival rates $\boldsymbol{\Lambda}_t$ and customers' remaining budgets $\boldsymbol{B}_t$. Therefore, the platform does not need to keep track of any history of past good arrivals or decisions. In addition, the scheme requires no supply forecast. Therefore, the platform avoids the risk of building a misspecified forecast model. In contrast, one must know the future sample path of $\{\boldsymbol{\Lambda}_t\}$ to solve the program in Section 3.

2. The basic computational block is solving a convex optimization problem with linear constraints, for which efficient algorithms exist (see Boyd and Vandenberghe [2004]). Implementing the re-optimization policy additionally requires the platform to frequently re-solve convex programs, which is computationally tractable.

4.1 Performance Analysis of The Re-Optimization Policy with Continuous Reviews

This subsection is devoted to analyzing the performance of the re-optimization policy π^{re}.

Market clearance. First, we characterize the dynamics of each customer's budget consumption and each good type's utilization.

Theorem 4 *The re-optimization policy π^{re} clears the markets, namely,*

1. *Budgets are consumed smoothly: $B_{j,t} = B_{j,0}\frac{T-t}{T},\ \forall j, t.$*
2. *All goods inventory is consumed: $\int_{t=0}^{T} \Lambda_{i,t} \left(\sum_{j=1}^{J} x_{ij,t}^{\pi^{re}}\right) dt = \int_{t=0}^{T} \Lambda_{i,t} dt,\ \forall i.$*

Additionally, our policy must guarantee good fairness performance. We establish a constant uniform lower bound on the fairness performance of our re-optimization policy for reflected Gaussian processes.

Theorem 5 *Under the re-optimization policy π^{re},*

1. *For all regular reflected Gaussian processes $\{\mathbf{\Lambda}_t\} \in \mathrm{rGP}$,*
$$\frac{\mathrm{FAIR}^{\pi^{re}}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0)}{\mathrm{FAIR}^{*}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0)} \quad \geq \quad 0.64.$$
2. *For all k-th order reflected Gaussian processes $\{\mathbf{\Lambda}_t\} \in \mathrm{GP}(k)$,*
$$\frac{\mathrm{FAIR}^{\pi^{re}}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0)}{\mathrm{FAIR}^{*}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0)} \quad \geq \quad \frac{1.3}{k} \ln\left(\frac{k}{2} + 1\right).$$
3. *For all reflected Gaussian processes such that $\sigma_{i,t} = 0$ for all $i \in \{1, \cdots, I\}$ and $t \in [0, T]$,*
$$\frac{\mathrm{FAIR}^{\pi^{re}}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0)}{\mathrm{FAIR}^{*}(\boldsymbol{B}_0, T, \boldsymbol{\lambda}_0)} \quad = \quad 1.$$

We pause to emphasize some of the features of this result:

1. *Rate volatility robustness*: The upper bounds from Parts 1 and 2 of the theorem are obtained without imposing any condition on the absolute scale of volatility of the goods arrival rate process; this indicates that our re-optimization policy is robust across the entire family of processes we examine, no matter how volatile they are. Moreover, for the natural rGP family, the bound is uniform. Lastly, as per Part 3, the algorithm will precisely match the offline optimal fairness as the volatility of $\{\mathbf{\Lambda}_t\}$ tends to 0.
2. *Correlation structure robustness*: The uniform upper bound is obtained without imposing any condition on the correlation among different good types.
3. *Almost sure market clearance*: Our policy is simultaneously guaranteed to clear the market even in the case that the size of market is uncertain, and does not need to assume that $\{\mathbf{\Lambda}_t\}$ is from the rGP family.
4. *Practicality*: Recall that implementing our re-optimization policy simply requires the platform to frequently implement myopic resolves for the static optimal fairness. The myopic nature of this algorithm makes it implementable for any type of rate process and places no forecasting burden on the platform.

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