SOLUTIONS

Fall 2009: EECS126 Final

No Collaboration Permitted. Two sheets of notes permitted. Turn in with your exam.

Be clear and precise in your answers

Write your name and student ID number on every sheet.

Come to the front if you have a question.

110 points is a very good score. There are more points than that on the exam. So look over the entire exam. Some points at the end may be easier to get than others earlier in the exam.

EE 126 Final Solutions

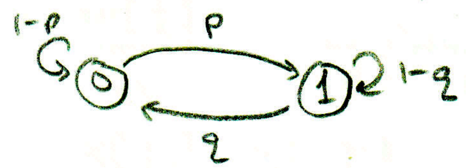
Problem 1.1 *(36pts) True or False. Prove or show a counterexample:*

a. *12pts. Let A_1, A_2, A_3 be events with $0 < P(A_3) < 1$. Suppose A_1 and A_2 are conditionally independent given A_3 and are also conditionally independent given A_3^c. Then A_1 and A_2 must be independent.*

FALSE

Counterexample:

Consider the Markov Chain



Let $X_i = \begin{cases} 1 & \text{if chain in state 1 at time } i \\ 0 & \text{o/w} \end{cases}$

with stationary initial distribution

$\left(P(X_0 = 0) = \pi_0 = \frac{q}{q+p} \right)$

Let A_1 = event that $X_0 = 1$

A_2 = event that $X_2 = 1$

A_3 = event that $X_1 = 1$

Then

$$P(A_2 | A_1 \text{ and } A_3) = P(X_2 = 1 | X_0 = 1, X_1 = 1) = P(X_2 = 1 | X_1 = 1)$$

$$= P(A_2 | A_3) \quad \text{by Markov Property}$$

Likewise

$$P(A_1 | A_2 \text{ and } A_3) = P(A_1 | A_3)$$

So A_1 and A_2 are independent given A_3. Similar arguments give A_1 and A_2 independent given A_3^c.

However

$$P(A_2 | A_1) = P(X_2 = 1 | X_0 = 1) = P(\text{state sequences } 101 \text{ or } 111 \,|\, X_0 = 1)$$

$$= pq + (1-q)^2$$

$$\neq \frac{p}{p+q}$$

$$= P(A_2)$$ since if you start in the stationary distribution, and don't know the actual realization, every subsequent step is distributed as the stationary distribution.

b. 12pts. For any random variable X and any $a > 0$ we have

$$P(|X| < a) \leq a^2 E[\frac{1}{X^2}]$$

TRUE

$$P(|X| < a) = P(\frac{1}{|X|} > \frac{1}{a})$$

$$= P(|\frac{1}{X}| > \frac{1}{a})$$

$$\leq \frac{var(\frac{1}{X})}{\frac{1}{a^2}} \quad \text{by Chebyshev inequality}$$

$$= \frac{\mathbb{E}[\frac{1}{X^2}] - (\mathbb{E}[\frac{1}{X}])^2}{\frac{1}{a^2}}$$

$$= a^2\, \mathbb{E}[\frac{1}{X^2}] - a^2 (\mathbb{E}[\frac{1}{X}])^2$$

$$\leq a^2\, \mathbb{E}[\frac{1}{X^2}] \quad \text{since } a^2(\mathbb{E}[\frac{1}{X}])^2 \geq 0.$$

c. *12pts. X and Y are two random variables with some joint distribution and each one individually has a well defined positive noninfinite variance. Then the mean-squared estimation error random variable $\tilde{X} = (X - E[X|Y])$ is independent of Y.*

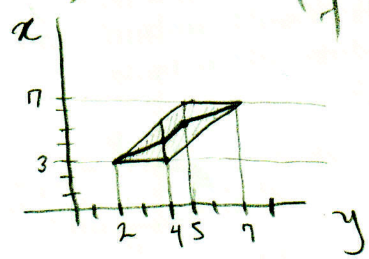
FALSE

Counterexample

Consider the uniform MMSE problem we've seen many times this semester:

$X \sim \text{unif}[3,7]$, $N \sim \text{unif}[-1,1]$, $Y = X + N$

We get a joint density of X and Y that is Unif over this space:



with the center line expressing

$g(y) = E[X \mid Y = y]$

Now define $\tilde{X} = X - E[X|Y]$ and we want to know if $\tilde{X}$ is indep of Y.

Notice that $f_{\tilde{X}|Y=8}(\tilde{x}) = \begin{cases} 1, & \tilde{x} = 0 \\ 0 & \text{o/w.} \end{cases}$

but $f_{\tilde{X}|Y=4}(\tilde{x}) = \begin{cases} \frac{1}{2}, & -1 \leq \tilde{x} \leq 1 \\ 0, & \text{o/w} \end{cases}$

$\therefore$ $\tilde{X}$ is clearly not independent of Y since knowing Y changes the distribution of $\tilde{X}$.

EE 126 Final Soln

Problem 1.2 (45pts) *Markov Fun (the parts of this problem are not connected to each other, so do them in any order)*

a. *25pts. You have a server that can be in one of two states: "busy" or "idle." Assume that the states evolve in discrete-time in a Markov fashion. Nature dictates that the probability of going from "idle" to "busy" is $\frac{9}{10}$. However, we have control of the probability q of going from "busy" to "idle."*

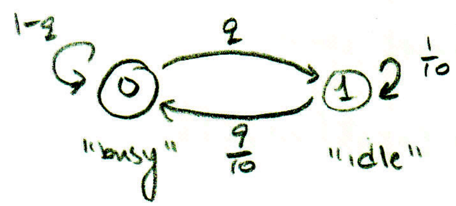
(1) – **Draw this Markov chain, labeling all the transition probabilities.**

(2) – **Calculate the stationary distribution as a function of q.**

(3) – **How low can you bring the steady state probability of the server being busy?**

(4) – **Assume that the system starts in the stationary distribution. Conditioned on the state being "busy" at time 3 what is the probability of it being "idle" at time 2?**

(1)



(2) use local balance equations:

$$\pi_0 q = \pi_1 \left(\frac{9}{10}\right)$$

$$\pi_1 = \frac{10}{9} q \pi_0$$

$$\pi_0 + \pi_1 = 1$$

$$\pi_0 + \frac{10}{9} q \pi_0 = 1$$

$$\pi_0 \left(1 + \frac{10}{9} q\right) = 1$$

$$\pi_0 = \frac{1}{1+\frac{10}{9}q} = \frac{\frac{9}{10}}{\frac{9}{10}+q}$$

stationary distribution:

$$\pi_0 = \frac{9}{9+10q}$$

$$\pi_1 = \frac{10q}{9+10q}$$

(3) we want to $\min_q \pi_0 = \min_q \frac{9}{9+10q}$ which is minimized if q is as large as possible.

Because q is a transition probability, $0 \le q \le 1$, so largest is $q = 1$.

Lowest stationary prob of being busy: $\min_q \pi_0 = \frac{9}{19}$ when $q = 1$

Extra paper in case you need it.

(1.2.a cont)

(4) System starts in stationary distribution

$\pi_0 = \frac{9}{9+10q}$, $\pi_1 = \frac{10q}{9+10q}$

Conditioned on state at time 3 being "busy" (or 0 here) what is the probability of being "idle" at time 2?

Let X_i = state at time i

Then we want

$P(X_2 = 1 \mid X_3 = 0)$

Since we start in the stationary distribution, then at time 2 we are still in the stationary distribution. Likewise at 3 if we don't condition on 2

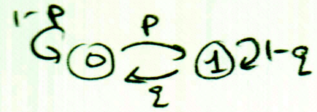
$$P(X_2=1 \mid X_3=0) = \frac{P(X_2=1, X_3=0)}{P(X_3=0)} = \frac{P(X_3=0 \mid X_2=1)\,P(X_2=1)}{P(X_3=0)}$$

$$= \frac{\frac{9}{10}\left(\frac{10q}{9+10q}\right)}{\left(\frac{9}{9+10q}\right)}$$

$$= \frac{9}{10}\left(\frac{10q}{9}\right)$$

$$= q$$

EE 126 Final Soln

Remember that w/ MC 

and transition matrix $\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = P$

eigenvalues λ_1 and λ_2:

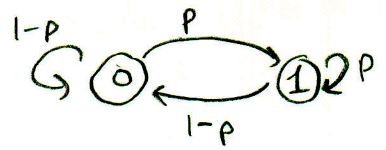
$\det(P) = \lambda_1\lambda_2$

$(1-p)(1-q) - pq = \lambda_1\lambda_2$

$1-p-q = \lambda_1\lambda_2$

Since $\lambda_1 = 1$ always, $\lambda_2 = 1-p-q$

b. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and 0.*



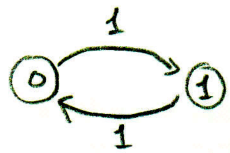
$\lambda_1 = 1$

$\lambda_2 = 1 - p - (1-p) = 0$

transition matrix

$\begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}$

c. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and -1.*



$\lambda_1 = 1$

$\lambda_2 = 1-1-1 = -1$

transition matrix

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and +1.*

$\lambda_1 = 1$

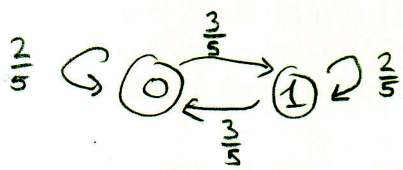
$\lambda_2 = 1-0-0 = 1.$

transition matrix

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

e. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and* $-\frac{1}{5}$.

There are lots of answers here. This is just one:



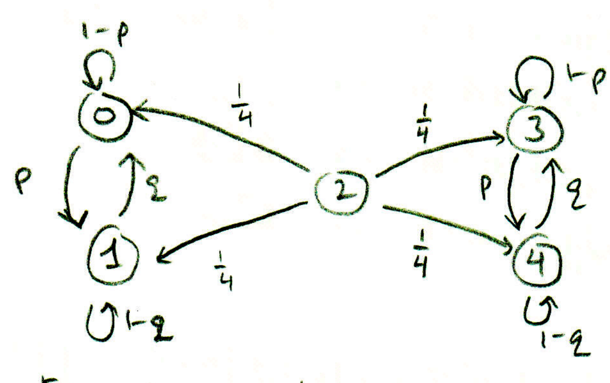
$$\lambda_1 = 1$$

$$\lambda_2 = 1 - \frac{3}{5} - \frac{3}{5} = -\frac{1}{5}$$

transition matrix

$$\begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix},$$

f. *4pts. Draw a Markov chain (and label all the transition probabilities) with two distinct recurrent classes and at least one transient state.*



recurrent class 1 | transient state | recurrent class 2

Again, lots of answers here.

EE126 Final Solutions

Estimate p to within $\pm$ 0.05, 99% of the time

Problem 1.3 (15 pts.) *There is some unknown probability p of a microchip being bad, and we know that different chips go bad independently of each other.* **How many chips should we sample in order to be able to estimate p to within ± 0.05? (Justify your answer in some detail and feel free to leave any unevaluatable integrals unevaluated)**

You can choose to use Chebyshev, Chernoff, or CLT here. Because you need to have high confidence (99%), only Chernoff and CLT solutions were given full credit. The CLT solution is shown here

Let $X_i = \begin{cases} 1 & \text{if chip } i \text{ is bad (happens w.p. } p) \\ 0 & \text{ow} \end{cases}$

By CLT:

$$\frac{\sum X_i - nE[X_i]}{\sqrt{n\, var(X_i)}} \approx_d N(0,1)$$

$$\sum X_i - nE[X_i] \approx_d N(0, n\,var(X_i))$$

$$\sum X_i \approx_d N(nE[X_i], n\,var(X_i))$$

$$\frac{1}{n}\sum X_i \approx_d N(E[X_i], \tfrac{1}{n} var(X_i))$$

$$\frac{1}{n}\sum X_i \approx_d N(p, \tfrac{1}{n} p(1-p))$$

So,

$$P\left(\left|\frac{1}{n}\sum X_i - p\right| \geq 0.05\right) \simeq P\left(\left|N(0, \tfrac{1}{n}p(1-p))\right| \geq 0.05\right)$$

$$= 2P\left(N(0, \tfrac{1}{n}p(1-p)) \geq 0.05\right)$$

$$= 2\Phi\left(\frac{-0.05\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

And we want

$$2\Phi\left(\frac{-0.05\sqrt{n}}{\sqrt{p(1-p)}}\right) \leq .01$$

$$\frac{-0.05\sqrt{n}}{\sqrt{p(1-p)}} \leq \Phi^{-1}\left(\frac{.01}{2}\right)$$

$$\sqrt{n} \geq -0.05\sqrt{p(1-p)}\,\Phi^{-1}\left(\frac{.01}{2}\right)$$

$$n \geq \left(0.05\sqrt{p(1-p)}\,\Phi^{-1}\left(\frac{.01}{2}\right)\right)^2$$

Problem 1.4 (15 pts.) *Somebody claims that it is sufficient to have just* 2 *bathrooms on an airplane that carries* 100 *people on it, each of which wants to go to the bathroom independently of the others with some probability* q. *You probe this person about what does he mean by "sufficient" and he says that he means that less than 10% of the time will someone have to wait for an empty bathroom assuming that time is slotted and the time required to finish using the bathroom is* 1 *slot long.*

What can you conclude about q**? Give your analysis and explain how you are modeling the problem using probability.**

You have lots of options on how to interpret/model this situation, depending on how you use the information "the time required to finish using the bathroom is 1 slot long." If you take this as the time for one person to finish then there is dependency between time steps. In this case, a Markov solution would work well to model the queue length, and you want the steady state probability of having 3 or more in line to be less than 10%.

For these solutions, we will assume that the "one slot" is the time required to clear the whole queue of people, so the time steps are independent slots. (Note that this assumption lowers the probability of waiting in line compared to the dependent case and so gives an upper bound to the "true" q)

So, if all slots are independent, we want the probability that only 10% of slots will see more than 2 people get up to use the bathroom. Notice that this is 10% <u>in steady state</u>, so we can assume we are looking across a very large time period and the weak law of large numbers tells us that the only thing we require is that the probability of more than 2 people getting up to use the restroom <u>at each time slot</u> must be less than 10%.

Problem 1.4 cont

Extra page in case you need more room

The number of people who need to use the bathroom at each time step, K, has a binomial distribution, with $n = 100$, $p = q$:

$$p_K(k) = \binom{100}{k} q^k (1-q)^{100-k}$$

Because $n = 100$ is large and q will likely be small, this is well approximated by a Poisson distribution with parameter $\lambda = 100q$. So:

$$p_K(k) \simeq \frac{(100q)^k}{k!} e^{-100q}$$

Now, the probability of 3 or more people:

$$P(K \geq 3) = 1 - P(K \leq 2)$$

$$= 1 - \sum_{i=0}^{2} \frac{(100q)^i}{i!} e^{-100q}$$

$$= 1 - e^{-100q} - 100q\, e^{-100q} - \frac{(100q)^2}{2} e^{-100q}$$

And we need this to be less than 0.1:

$$1 - e^{-100q} - 100q\, e^{-100q} - \frac{(100q)^2}{2} e^{-100q} \leq 0.1$$

$$e^{-100q}\left(1 + 100q + \frac{(100q)^2}{2}\right) \geq 0.9$$

This can be numerically found to give a bound on q:

$$q \lesssim 0.0112$$

Problem 1.5 (15 pts.) *Consider a networking system in which packets arrive according to a Poisson process with rate λ arrivals per second on average. I tell you that exactly* 1 *arrival has occurred between time* 0 *and some time* t. **Conditioned on this information, what is the distribution of the exact arrival time for this arrival?**

Show as complete a derivation as you can.

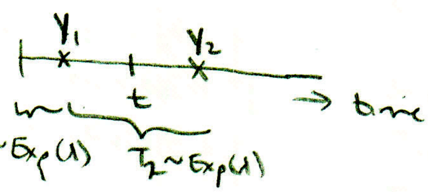
Let Y_1 and Y_2 be the arrival times of the first and 2nd packets.

Let T_1 be the time until the first packet

T_2 = interarrival between 1st + 2nd packet

So $T_1 = Y_1 \sim Exp(\lambda)$ and $T_2 = Y_2 - Y_1 \sim Exp(\lambda)$

We want this situation:



For the PDF:

$$f_{Y_1|Y_1<t, Y_2>t}(y \mid Y_1<t, Y_2>t) = \frac{f_{Y_1}(y)\, P(Y_1<t, Y_2>t \mid Y_1=y)}{P(Y_1<t, Y_2>t)}$$

$$= \frac{f_{Y_1}(y)\, P(T_2 > t-y)}{P(\text{one arrival in interval } [0,t])} \quad \text{if } 0 \le y \le t$$

$$= \frac{\lambda e^{-\lambda y}\left(1 - P(T_2 < t-y)\right)}{\lambda t e^{-\lambda t}}$$

because Poisson one arrival in $[0,t]$

$$= \frac{\lambda e^{-\lambda y}\left(1 - 1 + e^{-\lambda(t-y)}\right)}{\lambda t e^{-\lambda t}}$$

$$= \frac{1}{t}$$

$$\therefore\ f_{Y_1|Y_1<t, Y_2>t}(y) = \begin{cases} \frac{1}{t}, & 0 \le y \le t \\ 0, & \text{o.w.} \end{cases}$$

So the arrival is uniformly distributed over the interval $[0,t]$

Problem 1.6 (25 pts.) *X and Y are jointly Gaussian random variables. We know that $E[X] = E[Y] = 0$ and $E[X^2] = 1$ and $E[Y^2] = 2$. Furthermore, $E[XY] = -\frac{1}{2}$. Suppose that I tell you that X, Y are both obtainable as linear combinations of independent standard (zero mean and unit variance) Gaussian random variables W, V.* **Do you have enough information to uniquely write X and Y as linear combinations of W, V? If so, prove it and give these unique linear combinations. If not, give at least two distinct linear combinations that are both compatible with the specified information.**

We know

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}\right) \quad \text{and if} \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} W \\ V \end{bmatrix} = A\begin{bmatrix} W \\ V \end{bmatrix}$$

then $AA^T = \Sigma$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}$

so
$$a^2 + b^2 = 1$$
$$ac + bd = -\frac{1}{2}$$
$$c^2 + d^2 = 2$$

which are 3 equations and 4 unknowns

Therefore, the solution is <u>NOT UNIQUE</u>

In fact there are an infinite # of solutions

To see a couple:

let $b = 0$

then $a^2 + b^2 = 1 \Rightarrow a = 1$

$ac + bd = -\frac{1}{2} \Rightarrow c = -\frac{1}{2}$

$c^2 + d^2 = 2 \Rightarrow d^2 = \frac{7}{4}$

$d = \frac{\sqrt{7}}{2}$

So

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{7}}{2} \end{bmatrix}\begin{pmatrix} W \\ V \end{pmatrix}$$

let $d = 0$

Then

$c^2 + d^2 = 2 \Rightarrow c = \sqrt{2}$

$ac + bd = -\frac{1}{2} \Rightarrow a\sqrt{2} = -\frac{1}{2}$

$a = \frac{-1}{2\sqrt{2}}$

$a^2 + b^2 = 1 \Rightarrow \frac{1}{8} + b^2 = 1$

$b^2 = \frac{7}{8}$

$b = \sqrt{\frac{7}{8}}$

So

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & \sqrt{\frac{7}{8}} \\ \sqrt{2} & 0 \end{bmatrix}$$