Math 341 - Linear Algebra
§6.1 - 6.7

Fall 2019

Seek ye diligently and teach one another words of wisdom; yea, seek ye out of the best books words of wisdom; seek learning, even by study and also by faith.

- D&C 88:118

§6.1 Inner Product, Length, and Orthogonality

Definition (Inner (Dot) Product)

Let $\mathbf{u}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^n$. Then the *inner product* (sometimes called the *dot product*) of $\mathbf{u}$ and $\mathbf{w}$ is

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{u}^T \mathbf{w}.$$

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Another way of thinking of it is

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = u_1 w_1 + u_2 w_2 + \cdots + u_n w_n.$$

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Example: If $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$, find $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{w} \cdot \mathbf{u}$.

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Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$ a scalar.

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Definition (Length of a vector)

The **length** (or **norm**) of a vector $\mathbf{v}$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2} \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

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Fact: For any $\mathbf{v}$ and any scalar c, we have

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Example: If

$$\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix},$$

compute $\|\mathbf{u}\|$ and $\|\mathbf{w}\|$, and find **unit vectors** in the same directions (i.e. vectors that point in the same direction but have length 1).

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The above procedure is called **normalizing** the vectors $\mathbf{u}$ and $\mathbf{w}$.

Definition

For $\mathbf{u}$ and $\mathbf{w}$ in $\mathbb{R}^n$, the **distance between u and w** is the length of the vector $\mathbf{u} - \mathbf{w}$, i.e.

$$\operatorname{dist}(\mathbf{u}, \mathbf{w}) = \|\mathbf{u} - \mathbf{w}\|.$$

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Example: For $\mathbf{u}, \mathbf{w}$ as above, find $\operatorname{dist}(\mathbf{u}, \mathbf{w})$.

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Theorem

Two vectors $\mathbf{u}$ and $\mathbf{w}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{w}\|^2.$$

Definition (Orthogonal complement)

Let W be a subspace of $\mathbb{R}^n$. Then a vector $\mathbf{u} \in \mathbb{R}^n$ is said to be **orthogonal to** W if it is orthogonal to every vector in W.

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$W^\perp$ is a subspace of $\mathbb{R}^n$.

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Fact: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

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Example: Find the angle between the vectors $\mathbf{u}$ and $\mathbf{w}$ (from above).

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Example: The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$$

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A set of *nonzero* orthogonal vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent, and hence forms a basis of the subspace it spans.

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Theorem

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthogonal basis for a subspace W. For each $\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m$, the weight c_j is given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}.$$

Example: The vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

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form an orthogonal basis for $\mathbb{R}^3$. Find the coordinates of

$$\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

relative to the basis $\mathcal{B}$.

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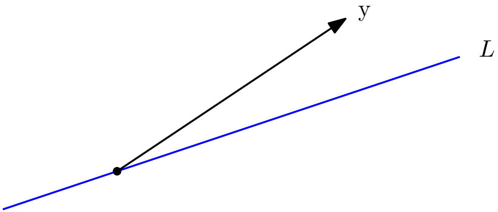
Suppose we have a line L and a vector $\mathbf{y}$, how do we find the point on L which is closest to $\mathbf{y}$?



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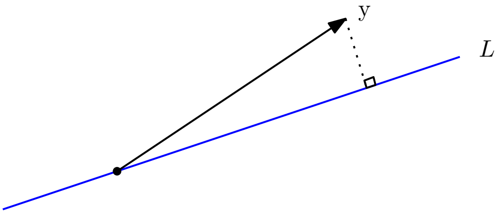

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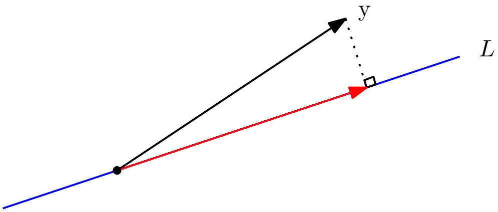

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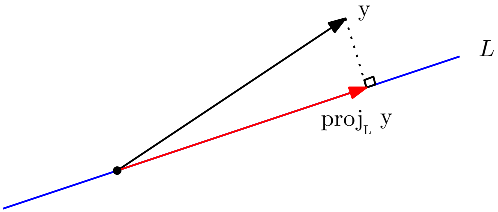

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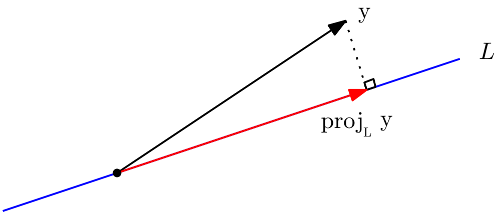

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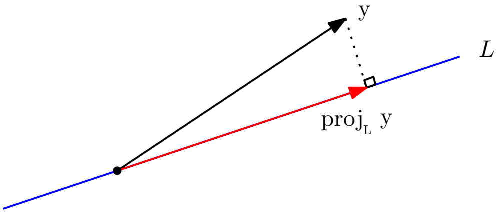

Suppose the line L is spanned by some vector $\mathbf{u}$.

Then the vector on L which is closest to $\mathbf{y}$ is called the **orthogonal projection** of $\mathbf{y}$ onto L, and is given by

$$\text{proj}_L\ \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}.$$

Notice also, that the vector

$$\mathbf{y} - \operatorname{proj}_{\mathbf{u}} \mathbf{y} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

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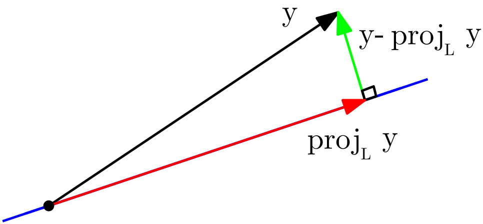

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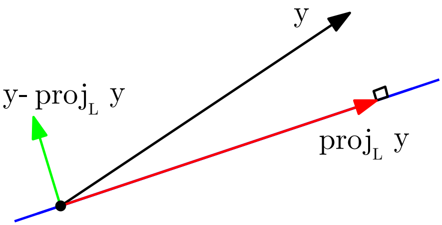

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Write the vector $\mathbf{y}$ as the sum of two vectors, one orthogonal to $\mathbf{u}$ and one in $\mathrm{Span}\{\mathbf{u}\}$.

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§6.2 Orthogonal Sets

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Example: Write the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$$

as an orthonormal set.

Let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

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(the columns of U are the orthonormal set from the previous slide).

What do you notice about the matrix products UU^T and U^TU?

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An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

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$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_5\mathbf{u}_5$$

as the sum of a vector in W and a vector in $W^\perp$.

Theorem (Orthogonal Decomposition Theorem)

Let W be a subspace of $\mathbb{R}^n$. Then any $\mathbf{y} \in \mathbb{R}^n$ can be written uniquely as

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

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§6.3 Orthogonal Projections

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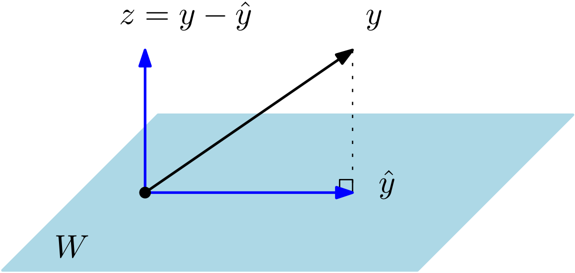
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Let $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$. Find the best approximation to $\mathbf{y}$ by an element of W, and find the distance from $\mathbf{y}$ to W.

Theorem

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal** basis for a subspace W of $\mathbb{R}^n$, then

$$\operatorname{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

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Recall: If U has orthonormal columns then $U^TU = I$, while the theorem says that UU^T gives the projection map onto Col U.

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Example: For the following orthonormal vectors

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{2} \\ \\ \frac{1}{2} \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ \\ \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

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use the theorem to compute $\text{proj}_W\ \mathbf{y}$, where $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, and

$$\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

§6.3 Orthogonal Projections

Warm up: For the following orthonormal vectors

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

compute $\mathrm{proj}_W \mathbf{y}$, where $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, and

$$\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

Question: Suppose we have a subspace W of $\mathbb{R}^n$.

§6.4 Gram-Schmidt Process

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Example: Suppose we start with $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$ which aren't orthogonal.

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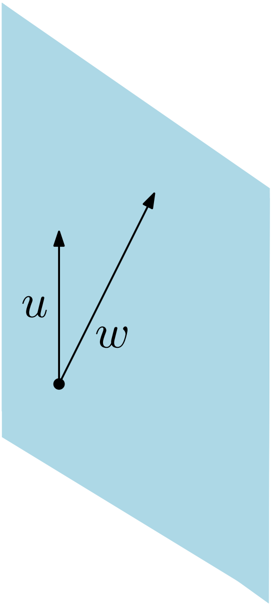
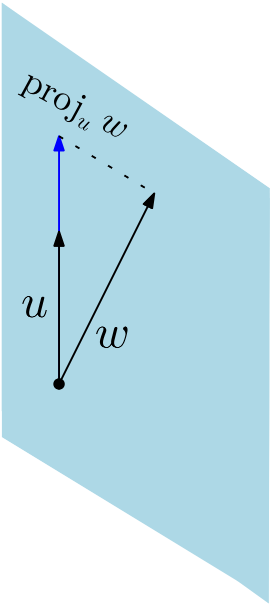
Example: Suppose we start with $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$ which aren't orthogonal.

How can we change them so that they become orthogonal and so that their span does not change?

§6.4 Gram-Schmidt Process




$\mathrm{proj}_u\, w$
u
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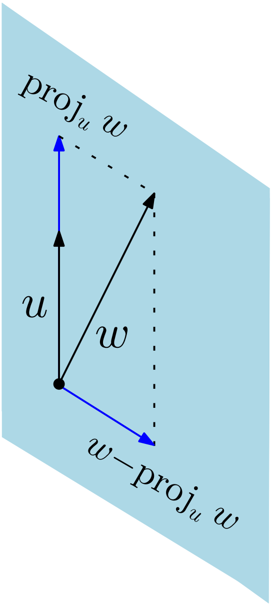

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$w - \text{proj}_u\, w$

Theorem (The Gram-Schmidt Process)

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1$$

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etc. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition,

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p.$$

Example: Let W be the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

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Example: Find an orthonormal basis of the subspace W.

Applying Gram-Schmidt gives an orthogonal basis for W:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 8/7 \\ 1/7 \\ -1/7 \\ -5/7 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 2/13 \\ 10/13 \\ 16/13 \\ 2/13 \end{bmatrix}.$$

Normalizing this orthogonal basis gives an orthonormal basis for W:

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{7} \\ 1/\sqrt{7} \\ -1/\sqrt{7} \\ 2/\sqrt{7} \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 8/\sqrt{91} \\ 1/\sqrt{91} \\ -1/\sqrt{91} \\ -5/\sqrt{91} \end{bmatrix},$$

and

$$\mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{91} \\ 5/\sqrt{91} \\ 8/\sqrt{91} \\ 1/\sqrt{91} \end{bmatrix}.$$

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A factorization of this form is called a **QR factorization of the matrix** A.

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$$Q^TA = Q^TQR = IR = R.$$

Example: Find a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix}.$$

We've already done the hard work using Gram-Schmidt to find an orthonormal basis for Col A. This was the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ we found on the previous slides. Then

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{8}{\sqrt{91}} & \frac{1}{\sqrt{91}} \\ \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{91}} & \frac{5}{\sqrt{91}} \\ -\frac{1}{\sqrt{7}} & -\frac{1}{\sqrt{91}} & \frac{8}{\sqrt{91}} \\ \frac{2}{\sqrt{7}} & -\frac{5}{\sqrt{91}} & \frac{1}{\sqrt{91}} \end{bmatrix}.$$

Finding the R matrix is now easy-peasy:

$$R = Q^T A = \begin{bmatrix} \sqrt{7} & -\frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \\ 0 & \sqrt{\frac{13}{7}} & -\frac{5}{\sqrt{91}} \\ 0 & 0 & 2\sqrt{\frac{7}{13}} \end{bmatrix}.$$

We can check that indeed $A = QR$, and that both Q and R have the properties stated in the theorem (Q has orthonormal columns with the same span as the columns of A, and R is an upper triangular invertible matrix with positive entries on the diagonal).

Example: Find a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

§6.5 Least-Squares Problems

Many real world problems involve finding solutions $\mathbf{x}$ for equations of the form

$$A\mathbf{x} = \mathbf{b},$$

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Many times no solution $\mathbf{x}$ might exist.

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If we can't find a solution $\mathbf{x}$ which satisfies $A\mathbf{x} = \mathbf{b}$ exactly, we can try to find one that makes $A\mathbf{x}$ as close as possible to $\mathbf{b}$,

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Definition (Least squares solution)

A **least-squares solution** of the equation $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that

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i.e. $\hat{\mathbf{x}}$ is the best possible approximation to a solution of $A\mathbf{x} = \mathbf{b}$.

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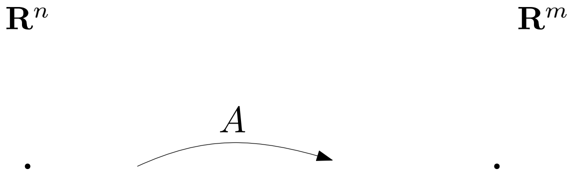

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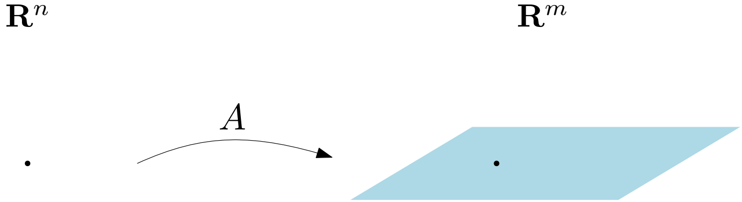

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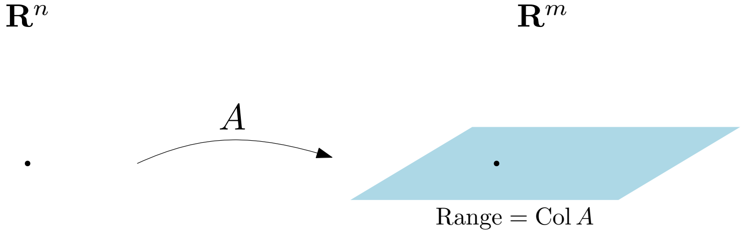

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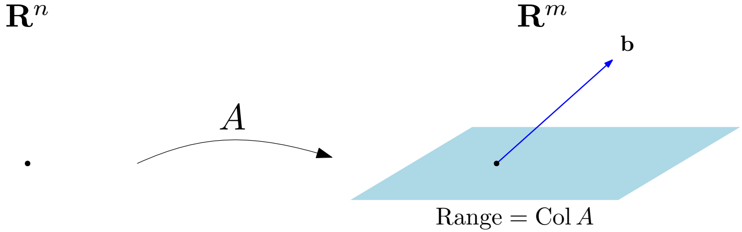

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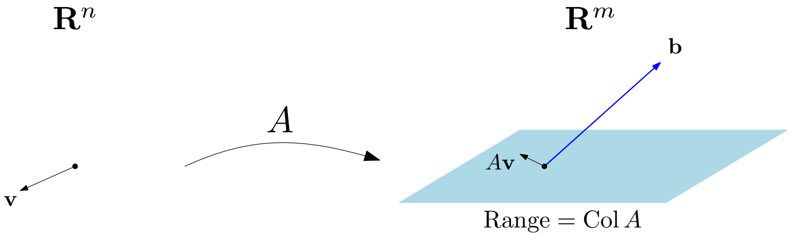

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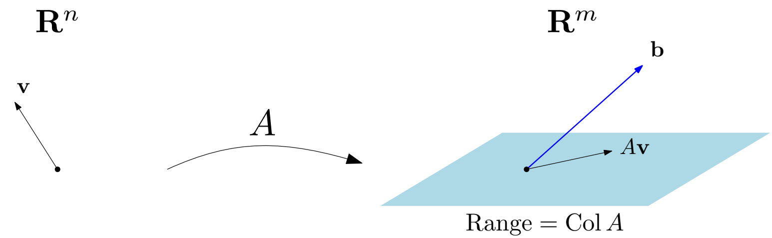

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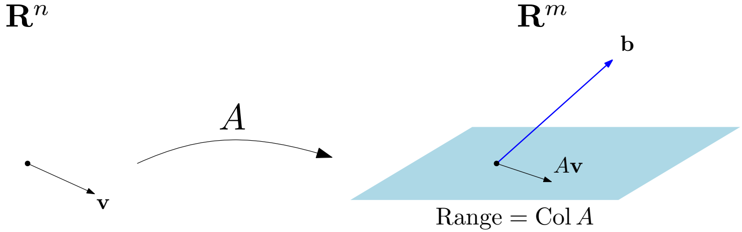

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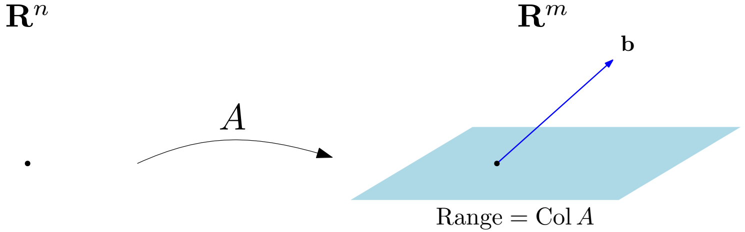

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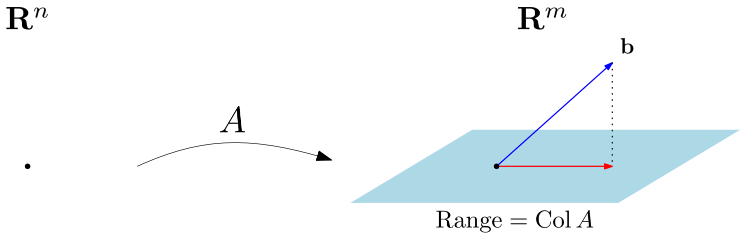

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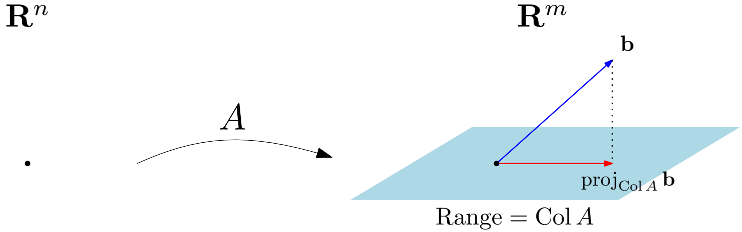

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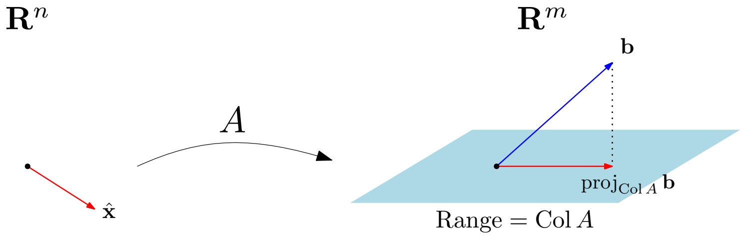

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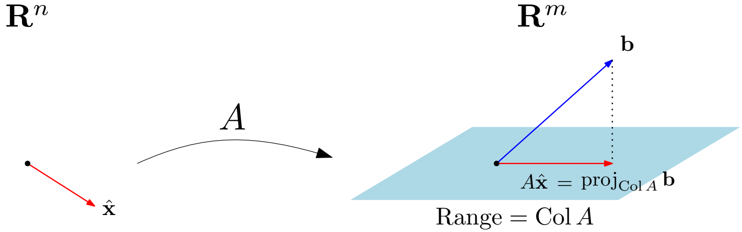

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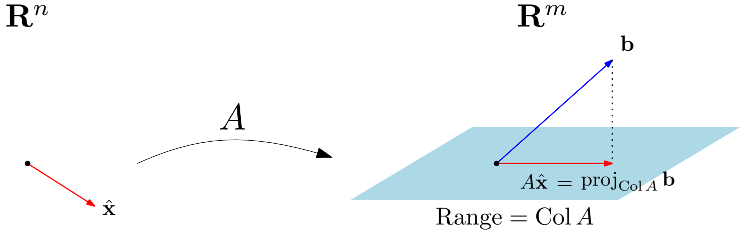

So a least-squares solution $\hat{\mathbf{x}}$ is a vector so that $A\hat{\mathbf{x}}$ is the closest vector in Col A to $\mathbf{b}$. I.e. instead of solving $A\mathbf{x} = \mathbf{b}$, we solve:

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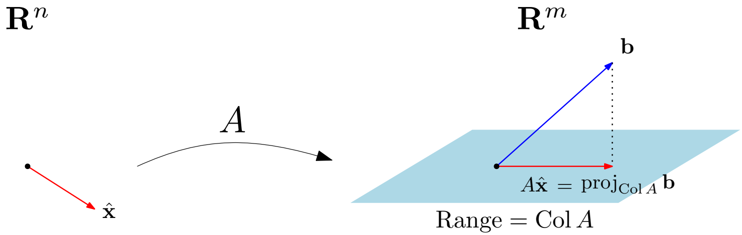

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$$A\hat{\mathbf{x}} = \text{proj}_{\text{Col } A}\ \mathbf{b}.$$

Theorem

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions to the equation

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The equations $A^T A\mathbf{x} = A^T\mathbf{b}$ are called the **normal equations** of the system $A\mathbf{x} = \mathbf{b}$.

The scalar $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ is called the **least-squares error**.

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Example: Find the least-squares approximation and error of the inconsistent system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 2 \end{bmatrix}.$$

Theorem

Let A be an $m \times n$ matrix. Then the following are equivalent:

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3. $A^T A$ is invertible.

When these statements are true, the unique least-squares solution is given by

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

Computational short-cut

Computational short-cut

Example: Find the least-squares solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}.$$

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Hint: the columns of A are orthogonal. Use this fact to compute $\text{proj}_{\text{Col } A}\, \mathbf{b}$ and then to find $\hat{\mathbf{x}}$.

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Then for each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has unique least-squares solution

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}.$$

§6.6 Applications to linear models

Problem: Suppose we have observed data points

$$(x_1, y_1), \ldots, (x_n, y_n)$$

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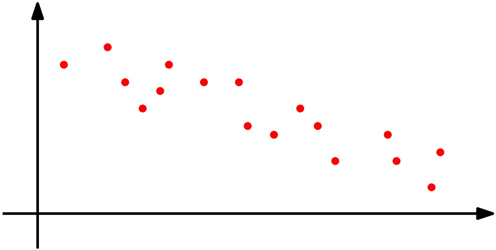

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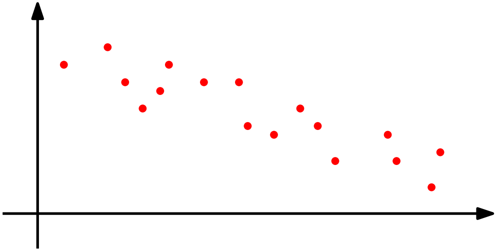

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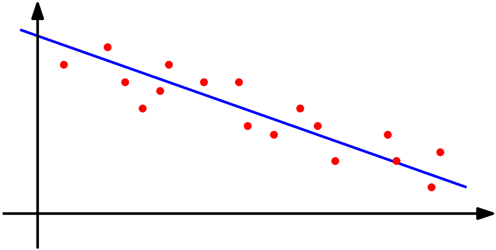

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In other words, can we find parameters β_0 and β_1 so that the line

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Let $X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

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$\Longrightarrow$ look for least-squares solution to $X\beta = \mathbf{y}$.

Example: If we have the points

$$(1, -1), \quad (2, 4), \quad (3, 12), \quad \text{and} \quad (4, 20).$$

Find the least-squares problem and the normal equations.

§6.6 Applications to linear models

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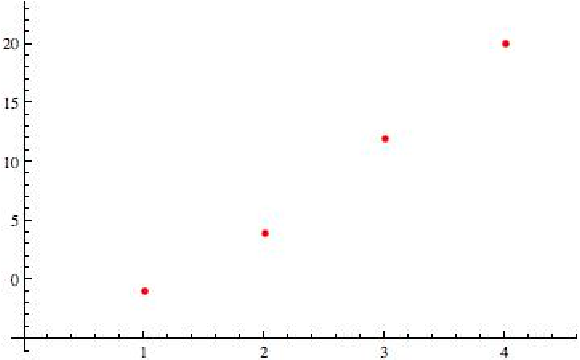
Find the least-squares problem and the normal equations.



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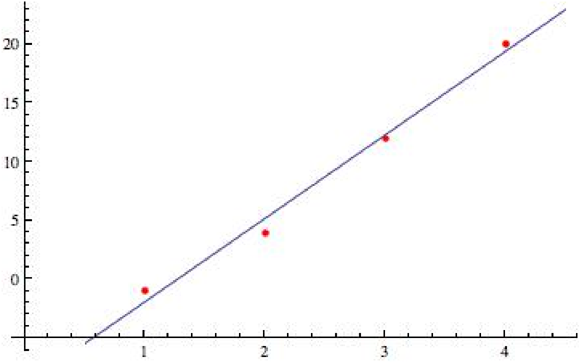

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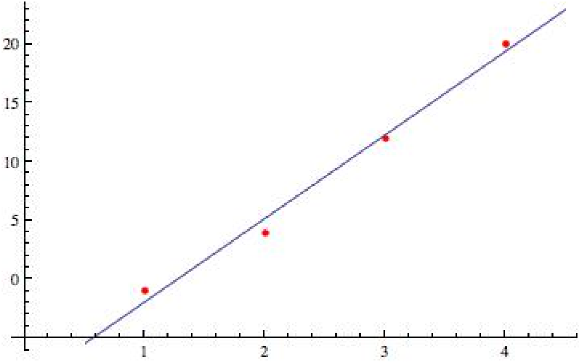

$$y = \frac{71}{10}x - 9$$

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$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 \sin x.$$

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Then we find the least-squares solution of $X\beta = \mathbf{y}$, where

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 & \sin x_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \sin x_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Definition

An **inner product** on a vector space V is a function that assigns to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$

§6.7 Inner Product Spaces

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A space with an inner product is called an **inner product space**.

Example: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$. Show that the function

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defines an inner product on $\mathbb{R}^2$.

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Show that this defines an inner product on $\mathbb{P}_n$.

Let $V = \mathbb{P}_2$, and $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$. Compute

$$\langle 1 + 2t - t^2, 2 + t^2 \rangle.$$

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Can also do orthogonal projections, Gram-Schmidt, best approximations etc. as before.

Example: Let $V = \mathbb{P}_4$, with inner product defined on previous slide (by evaluation at $-2, -1, 0, 1$ and 2).

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$$\{1, t, t^2\}$$

of $\mathbb{P}_2$ to find an orthogonal basis for $\mathbb{P}_2$.

Example: Using the above orthogonal basis of $\mathbb{P}_2$ (as a subspace of the inner product space $\mathbb{P}_4$) to find the best approximation to

$$\mathbf{p}(t) = 5 - \frac{1}{2}t^4$$

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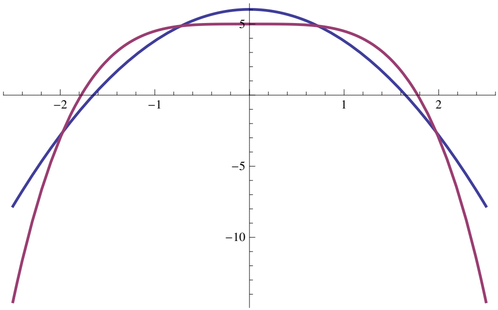

Theorem (Cauchy-Schwarz Inequality)

For all $\mathbf{u}, \mathbf{w}$ in V, we have

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Theorem (Triangle Inequality)

For all $\mathbf{u}, \mathbf{w}$ in V, we have

$$\|\mathbf{u} + \mathbf{w}\| \leq \|\mathbf{u}\| + \|\mathbf{w}\|.$$

Example: Define an inner product on $C[a,b]$ (the space of continuous functions defined on the interval $[a,b]$) by

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By Gram-Schmidt we get:

$$\mathbf{v}_1 = 1, \qquad \mathbf{v}_2 = x - \tfrac{1}{2}, \qquad \mathbf{v}_3 = e^x - (e-1) - 6(3-e)(x - \tfrac{1}{2}).$$

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