Math 341 - Linear Algebra
§5.1 - 5.3

Fall 2019

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.

- Paul Erdős

Consider the matrix

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

and the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

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What do you notice about $A\mathbf{v}$?

Notice that $A\mathbf{v} = 2\mathbf{v}$.

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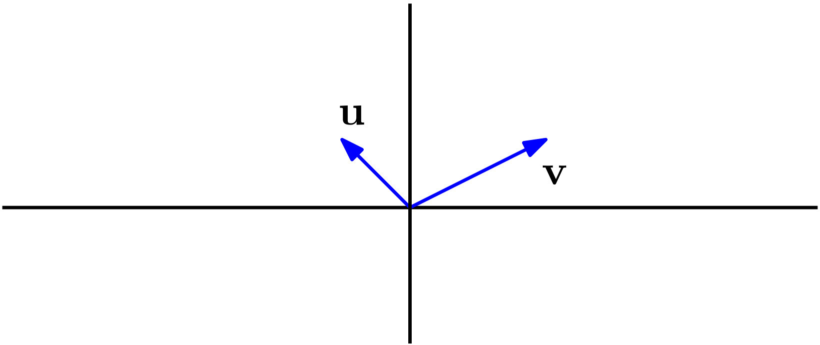

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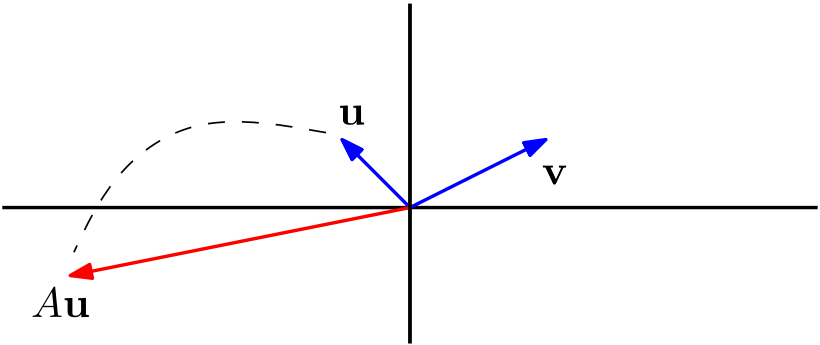

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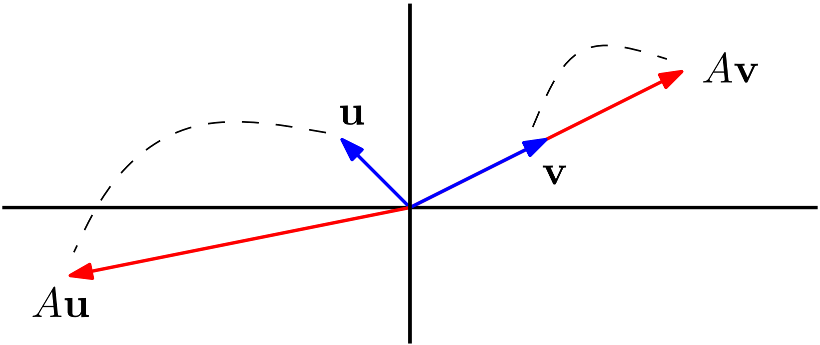

§5.1 Eigenvectors and Eigenvalues

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Note: Only $n \times n$ matrices can have eigenvectors and eigenvalues.

Example: For the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

which of the following vectors are eigenvectors? What are their associated eigenvalues?

$$\mathbf{u} = \begin{bmatrix} 3 \\ 9 \\ 8 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

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What does this tell us about the transformation defined by B?

The set of all solutions to the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

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Example: If $\lambda = 2$ is an eigenvalue for

$$C = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix},$$

find a basis for the associated eigenspace.

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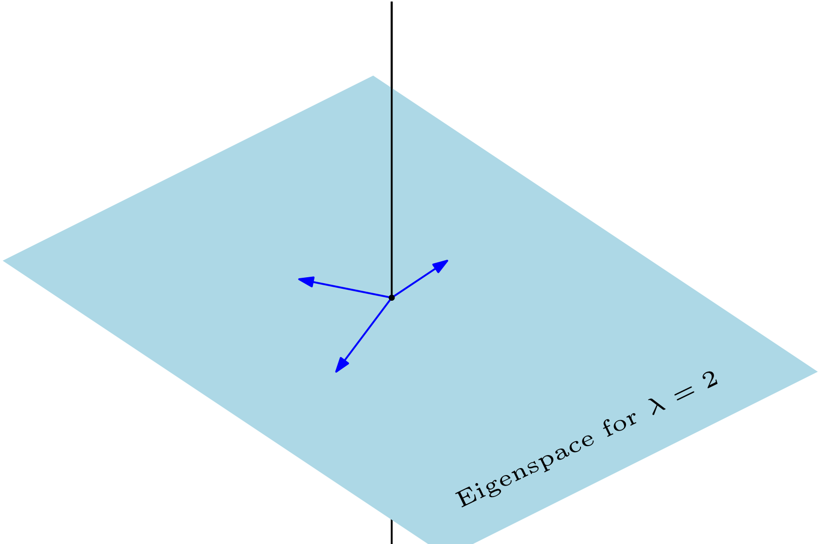

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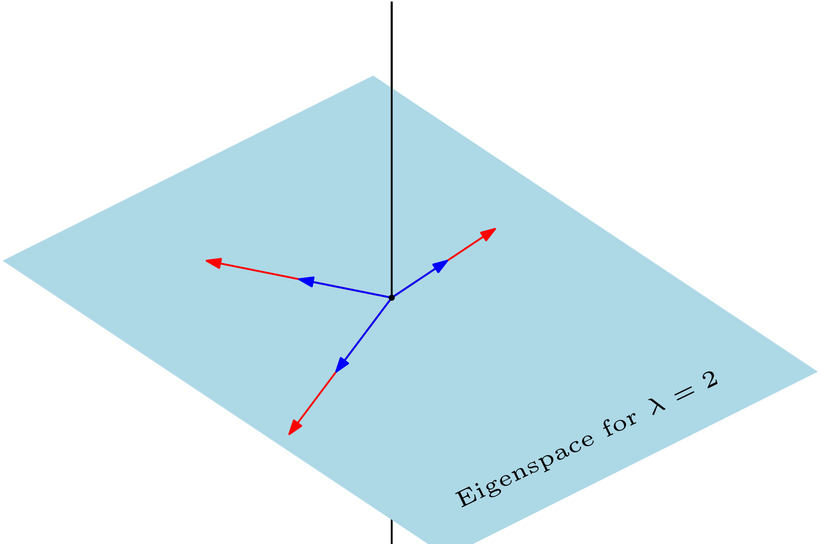

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Theorem

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

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Can we try to solve for the long term behavior explicitly?

Difference Equations and Eigenvectors

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Example: Suppose A has eigenvector $\mathbf{u}$ with corresponding eigenvalue λ. If $\mathbf{x}_0 = c \cdot \mathbf{u}$, find an explicit description of $\mathbf{x}_k$ for all $k \geq 0$.

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Example: Find the eigenvalues of

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Fact: A scalar λ will be an eigenvalue of the matrix A if and only if it satisfies the characteristic equation of A,

Equivalently, if and only if it is a root of the characteristic polynomial of A.

Example: Find the characteristic polynomial, equation, and eigenvalues of the matrix

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Example: Find the characteristic polynomial, equation, and eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

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Theorem

If A and B are similar $n \times n$ matrices, then they have the same characteristic polynomial, and hence the same eigenvalues (with multiplicities).

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First, note that A has eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .92$.

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Also, note that $\mathbf{x}_0 = .125\mathbf{v}_1 + .225\mathbf{v}_2$.

Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors, we have

$$A\mathbf{v}_1 = \mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_2 = 0.92\mathbf{v}_2.$$

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Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors, we have

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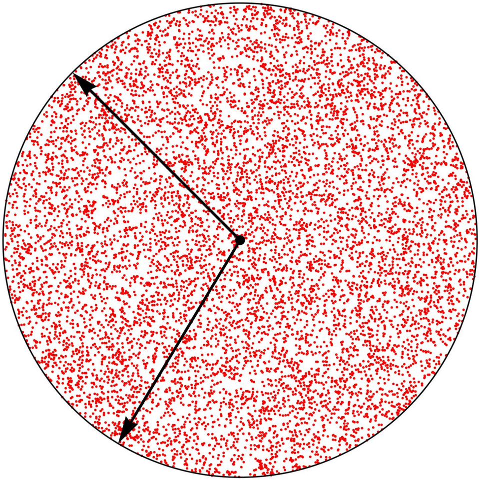
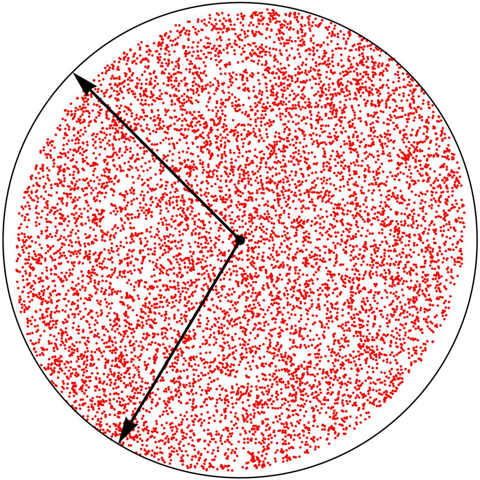
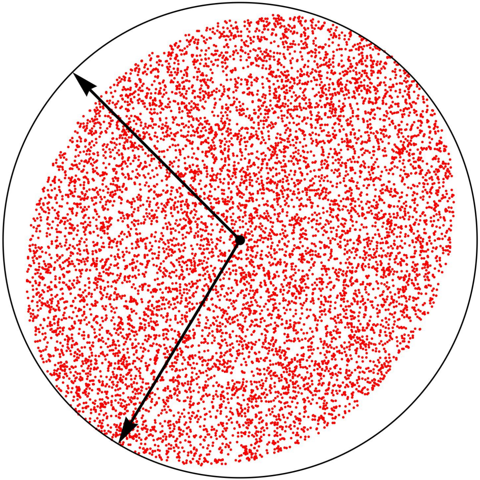


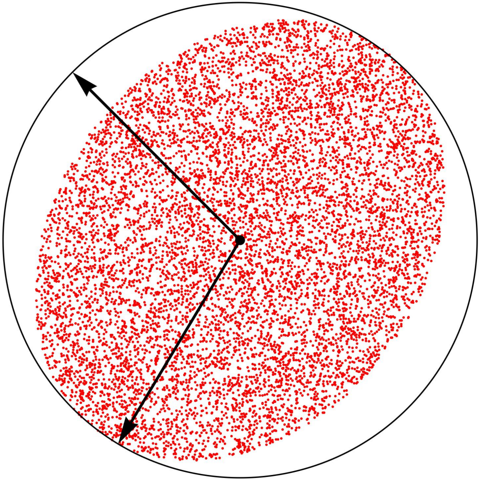
$\mathbf{x}_0$



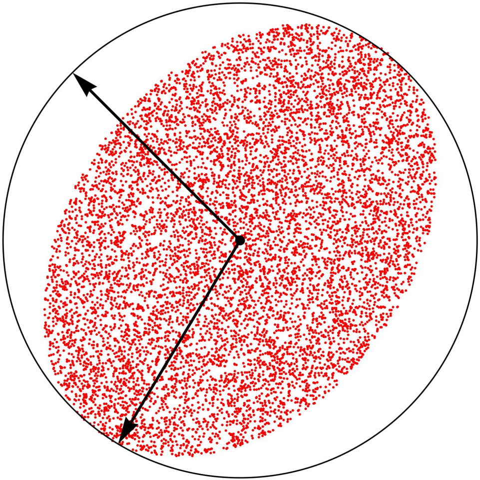
$$\mathbf{x}_1 = A\mathbf{x}_0$$



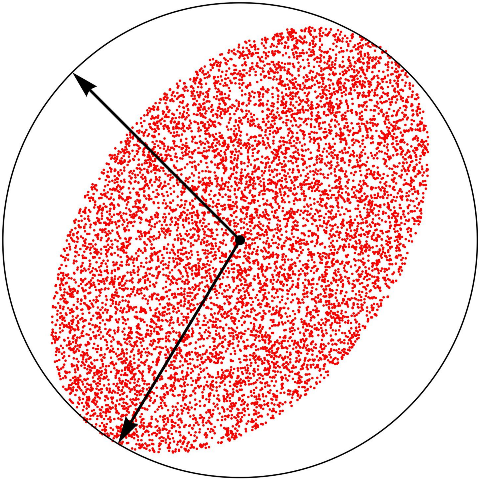
$$\mathbf{x}_2 = A^2\mathbf{x}_0$$



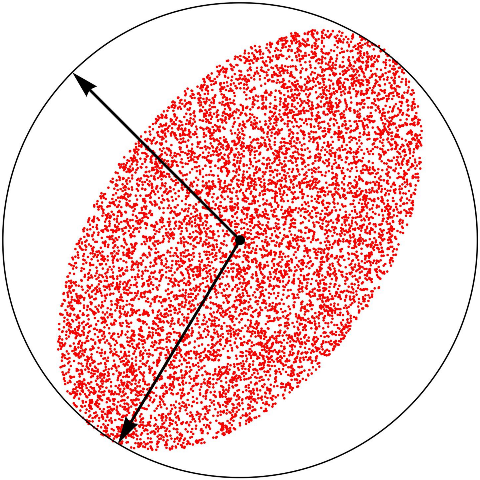
$$\mathbf{x}_3 = A^3\mathbf{x}_0$$



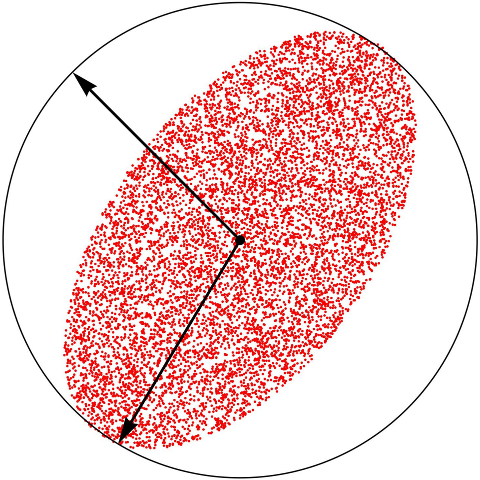
$$\mathbf{x}_4 = A^4\mathbf{x}_0$$



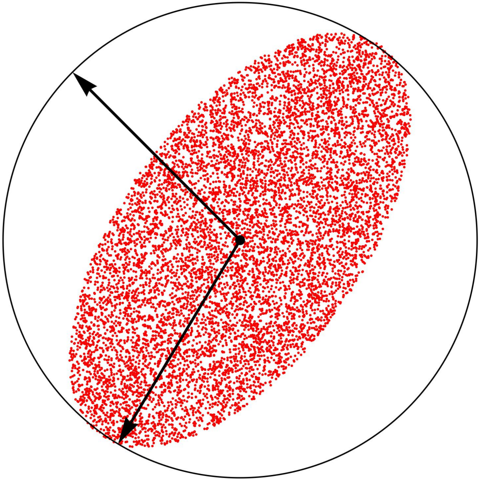
$$\mathbf{x}_5 = A^5\mathbf{x}_0$$



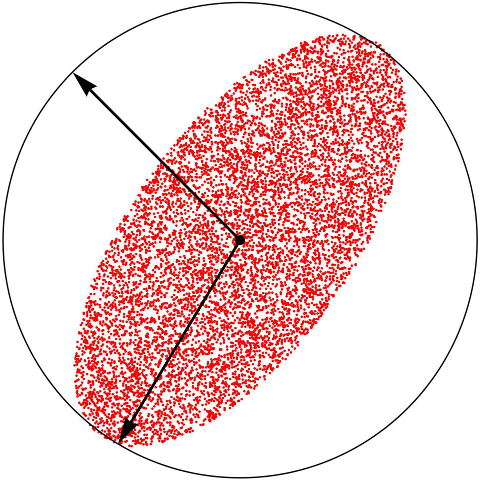
$$\mathbf{x}_6 = A^6\mathbf{x}_0$$



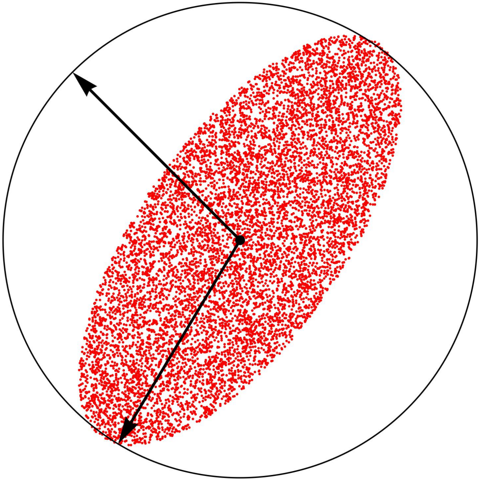
$$\mathbf{x}_7 = A^7\mathbf{x}_0$$



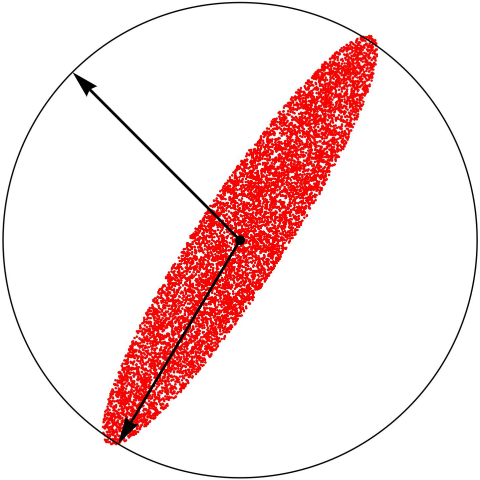
$$\mathbf{x}_8 = A^8\mathbf{x}_0$$



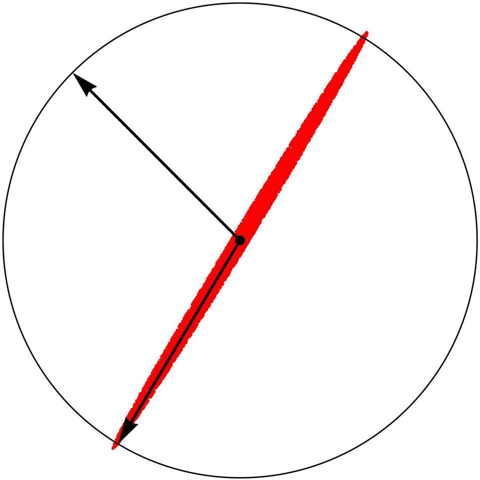
$$\mathbf{x}_9 = A^9\mathbf{x}_0$$



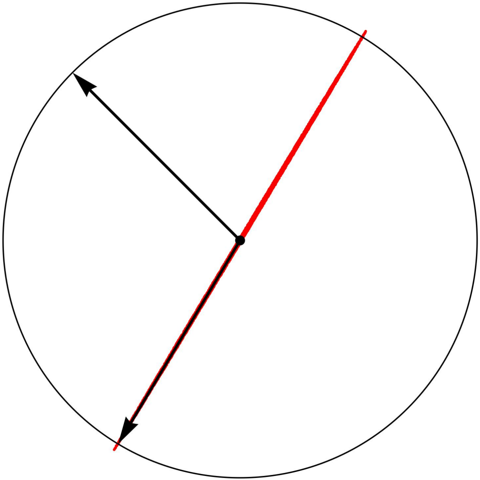
$$\mathbf{x}_{10} = A^{10}\mathbf{x}_0$$



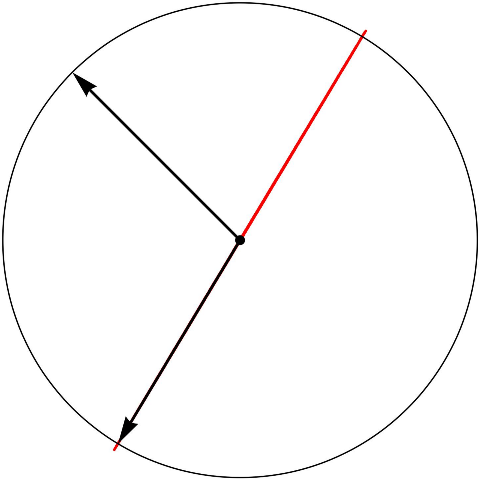
$$\mathbf{x}_{20} = A^{20}\mathbf{x}_0$$



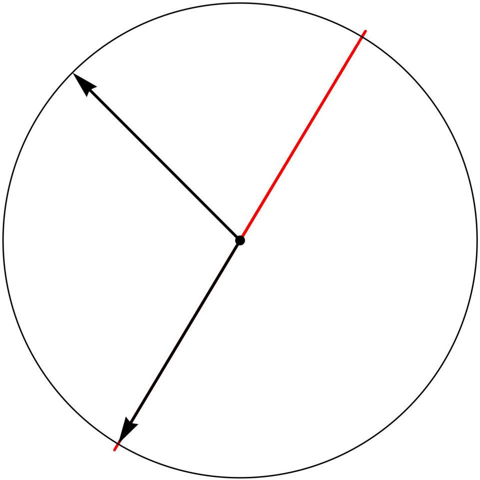
$$\mathbf{x}_{40} = A^{40}\mathbf{x}_0$$



$$\mathbf{x}_{60} = A^{60}\mathbf{x}_0$$



$$\mathbf{x}_{80} = A^{80}\mathbf{x}_0$$



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Definition

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In other words, A is diagonalizable if there is some invertible matrix P so that

$$A = PDP^{-1}$$

for some diagonal matrix D.

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for some diagonal matrix D.

Question: If A is diagonalizable, what will the entries in the diagonal matrix D be?

Example: Let

$$A = \begin{bmatrix} 6 & 2 \\ -10 & -3 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}.$$

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Find a formula for A^k.

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An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

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The diagonal entries of D will be the eigenvalues of A (in the same order as the eigenvector columns of P).

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$$A = \begin{bmatrix} 7 & -6 & 0 \\ 4 & -3 & 0 \\ -6 & 6 & 1 \end{bmatrix}.$$

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5. Check that $A = PDP^{-1}$.

§5.3 Diagonalization

Warm-up: Diagonalize the following matrix, if possible:

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}.$$

Recall the steps:

1. Find the eigenvalues of A.
2. Find linearly independent eigenvectors of A (if they exist).
3. Construct the matrix P whose columns are the above eigenvectors.
4. Construct the diagonal matrix D from the corresponding eigenvalues of D (in the same order as step 3).
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Example: Diagonalize the following matrix, if possible:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Theorem

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Remember the diagonalizable matrix

$$A = \begin{bmatrix} 7 & -6 & 0 \\ 4 & -3 & 0 \\ -6 & 6 & 1 \end{bmatrix}$$

from above had eigenvalues 1,1, and 3.

§5.3 Diagonalization

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3. If A is diagonalizable then combining bases for each eigenspace gives a basis for all of $\mathbb{R}^n$.

In other words, if E_λ is the eigenspace corresponding to λ:

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Example: A 5×5 matrix has 3 distinct eigenvalues, and one of its eigenspaces is of dimension 3. Is it diagonalizable?

§5.5 Complex Eigenvalues

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For $z = a + ib$ the *real part* of z is $\operatorname{Re} z = a$, the *imaginary part* of z is $\operatorname{Im} z = b$, and the *complex conjugate* of z is $\bar{z} = a - ib$.

Example: Find the eigenvalues and eigenvectors of the rotation matrix

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We can factor it as $\lambda^2 + 1 = (\lambda + i)(\lambda - i)$ if we allow complex numbers.

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Example: Find the eigenvalues and eigenvectors of the rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation is $\lambda^2 + 1 = 0$, which has no roots in $\mathbb{R}$ - makes sense, since A rotates everything, and doesn't scale anything, hence no eigenvectors.

We can factor it as $\lambda^2 + 1 = (\lambda + i)(\lambda - i)$ if we allow complex numbers.

Find the eigenvectors corresponding to the complex eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$.

§5.5 Complex Eigenvalues

For vectors and matrices with complex entries we can define the real and imaginary parts, as well as the complex conjugate.

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Example: A 3×3 matrix A has at least 2 eigenvalues which are real. Does it have any complex eigenvalues?

§5.5 Complex Eigenvalues

Theorem

Let A be a 2×2 matrix with a complex eigenvalue $\lambda = a - ib$ (with $b \neq 0$) and associated complex eigenvector $\mathbf{v}$. Then

$$A = PCP^{-1},$$

where

$$P = [\mathrm{Re}\,\mathbf{v} \quad \mathrm{Im}\,\mathbf{v}], \qquad \text{and} \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

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Example: Express the matrix

$$A = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$$

as $A = PCP^{-1}$ as above.

Fact: C sends $(1,0)$ to (a,b) and $(0,1)$ to $(-b,a)$.

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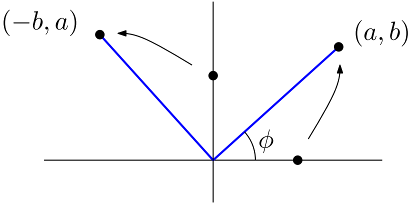

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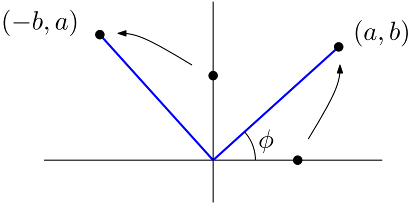

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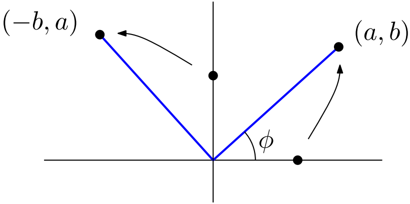

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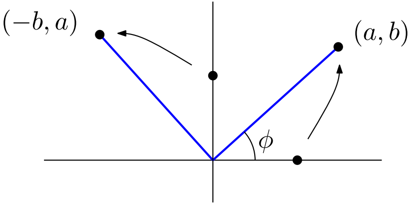

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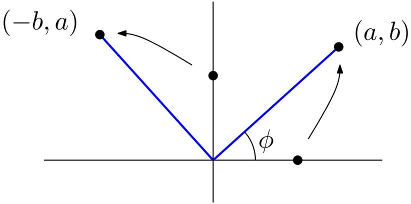

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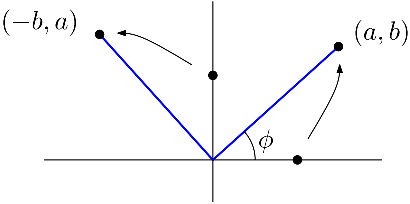

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Here, the left matrix scales vectors by $r=\sqrt{a^2+b^2}$, while the right matrix is the rotation taking $(1,0)$ to (a,b).

Thus any real 2×2 matrix A with complex eigenvalues can be written **in some basis** as a rotation followed by a scaling.

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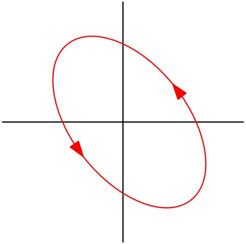

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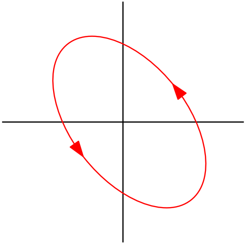

Example: Express the matrix

$$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

as the product of a scaling matrix and a rotation matrix.