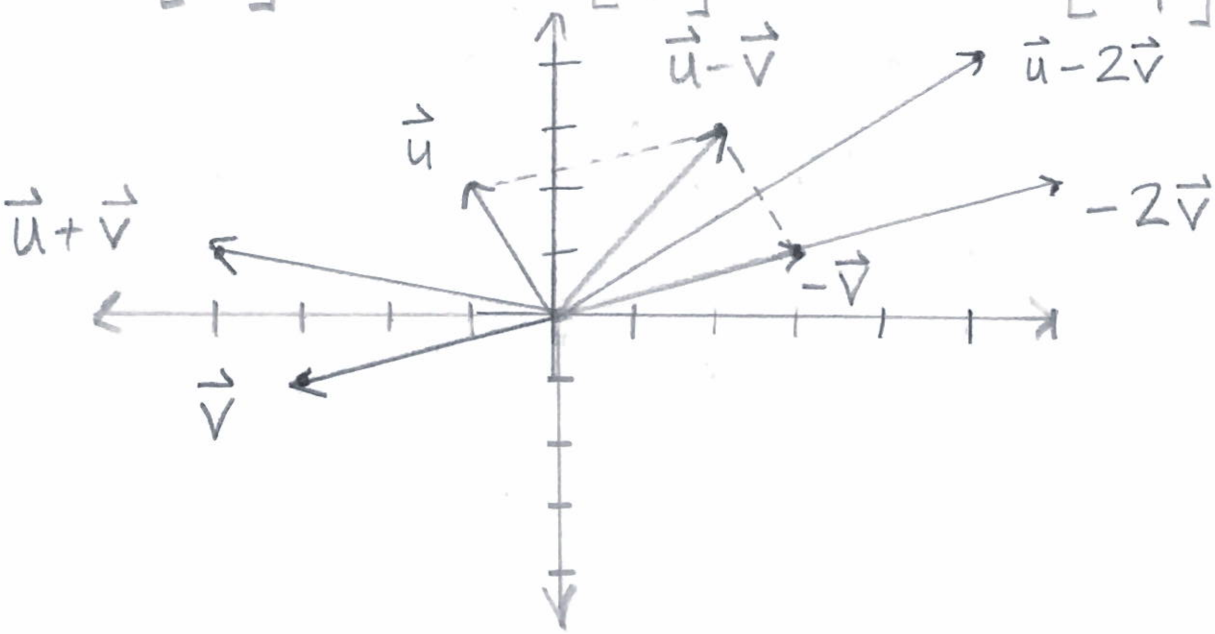
Section 1.3

3. $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, $-\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $-2\vec{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\vec{u}+\vec{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $\vec{u}-\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{u}-2\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$



5.

$$\begin{aligned} 6x_1 - 3x_2 &= 1 \\ -x_1 + 4x_2 &= -7 \\ 5x_1 &= -5 \end{aligned}$$

7.

$\vec{a} = \vec{u} - 2\vec{v}$

$\vec{b} = 2\vec{u} - 2\vec{v}$

$\vec{c} = 2\vec{u} - 3.5\vec{v}$

$\vec{d} = 3\vec{u} - 4\vec{v}$

Yes, every vector in $\mathbb{R}^2$ is a linear combination of $\vec{u}$ and $\vec{v}$.

8. $\vec{w} = 2\vec{v} - \vec{u}$

$\vec{x} = 2\vec{v} - 2\vec{u}$

$\vec{y} = 3.5\vec{v} - 2\vec{u}$

$\vec{z} = 4\vec{v} - 3\vec{u}$

9. $$x_1\begin{bmatrix}0\\4\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\6\\3\end{bmatrix} + x_3\begin{bmatrix}5\\-1\\-8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

12. If $\vec{b}$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$, then the vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ has a solution. The vector equation is a system of equations, and a system of equations can be written as an augmented matrix.

$$x_1\begin{bmatrix}1\\-2\\2\end{bmatrix} + x_2\begin{bmatrix}0\\5\\5\end{bmatrix} + x_3\begin{bmatrix}2\\0\\8\end{bmatrix} = \begin{bmatrix}-5\\11\\-7\end{bmatrix}$$

(vector equation)

$$\begin{cases} x_1 + 2x_3 = -5 \\ -2x_1 + 5x_2 = 11 \\ 2x_1 + 5x_2 + 8x_3 = -7 \end{cases}$$

(system of equations)

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + -2R_1$

$R_3 \to R_3 + -R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

This system is inconsistent by Theorem 2.

(The last column of the augmented matrix is a pivot column.) Thus the vector equation has no solution. So $\vec{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is <u>not</u> a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$.

13. Is $\vec{b}$ a linear combination of the columns of A? In other words, is there a solution to the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix} ?$$

The augmented matrix corresponding to the vector equation is

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Since the last column is a pivot column, the system is inconsistent by Theorem 2. Thus the vector equation has no solution and $\vec{b}$ is <u>not</u> a linear combination of the columns.

15. Your answers will vary.

$$\begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$$

$$\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} = 0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$$

$$\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2$$

$$\begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix} = 1 \cdot \vec{v}_1 - 1 \cdot \vec{v}_2$$

17. $\vec{b}$ is in the plane spanned by $\vec{a}_1$ and $\vec{a}_2$ if and only if $\vec{b}$ is in Span$\{\vec{a}_1, \vec{a}_2\}$

So the question is asking, is $\vec{b}$ a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

In other words, does

$$\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2$$

have a solution?

The augmented matrix associated with the above vector equation is

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & 8+h \end{array}\right]$$

$R_3 \to R_3 + 2R_1$

$R_2 \to R_2 + -4R_1$

$\frac{1}{5}R_2$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & 8+h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 17+h \end{array}\right]$$

$R_1 \to R_1 + 2R_2$
$R_3 \to R_3 + -3R_2$

By Theorem 2, the system is consistent if and only if $17+h=0$

Thus $\vec{b}$ is in the plane spamed by $\vec{a}_1$ and $\vec{a}_2$ if and only if $h=-17$.

19. $\vec{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$

Vectors in Span$\{\vec{v}_1, \vec{v}_2\}$ have the form $c_1\vec{v}_1 + c_2\vec{v}_2$ for some constants c_1 and c_2.

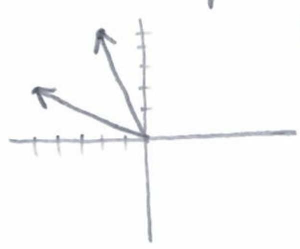
Note that $\frac{3}{2}\vec{v}_1 = \vec{v}_2$. So $\vec{v}_1$ and $\vec{v}_2$ lie on the same line through the origin. Thus

$$\text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{v}_1\} = \text{Span}\{\vec{v}_2\}$$

Since $\vec{v}_1$ is a nonzero vector, Span$\{\vec{v}_1\}$ is a line through the origin. Thus, geometrically, Span$\{\vec{v}_1, \vec{v}_2\}$ is a line through the origin.

23. a. False see example 1

b. False



$\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

c. True $\frac{1}{2}\vec{v}_1 = \frac{1}{2}\cdot\vec{v}_1 + 0\cdot\vec{v}_2$

d. True see shaded box above definition of Span.

e. False problem 19 the span of two vectors can be a line through the origin as well.

24. a. True see first paragraph of section

b. True $\vec{u}-\vec{v}+\vec{v}=\vec{u}$

c. False see definition of linear combination

d. True Since $\vec{u}$ is in Span$\{\vec{u},\vec{v}\}$ $c\vec{u}$ is in Span $\{\vec{u},\vec{v}\}$ for every choice of c. Span $\{\vec{u}\}$ is a subset of Span $\{\vec{u},\vec{v}\}$

e. True See paragraph after definition of Span.

25. a. No, $\vec{b}$ is not in $\{\vec{a}_1,\vec{a}_2,\vec{a}_3\}$
There are 3 vectors.

b. The question asks whether $\vec{b}$ is in $W=$ Span $\{\vec{a}_1,\vec{a}_2,\vec{a}_3\}$. In other words, is $\vec{b}$ a linear combination of $\vec{a}_1,\vec{a}_2,\vec{a}_3$? Or does $x_1\vec{a}_1+x_2\vec{a}_2+x_3\vec{a}_3=\vec{b}$ have a solution? The corresponding augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right] \sim$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$R_3 \to R_3 + 2R_1$

$R_3 \to R_3 + -2R_2$

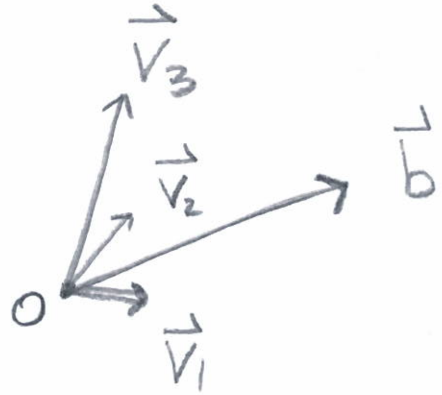
Since the last column of an echelon form of the matrix is not a pivot column, theorem 2 implies the system is consistent. Thus $\vec{b}$ is in W.

There are infinitely many vectors in W.

C. $\vec{a}_1 = 1 \cdot \vec{a}_1 + 0 \cdot \vec{a}_2 + 0 \cdot \vec{a}_3$

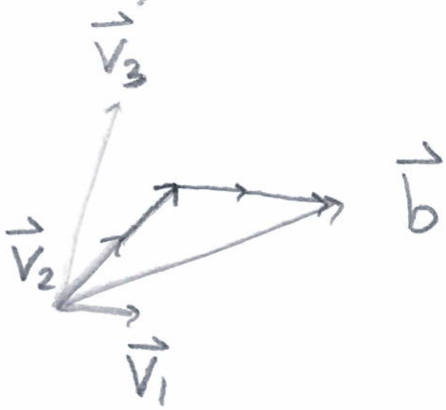
Thus $\vec{a}_1$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$. So $\vec{a}_1$ is in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

32.

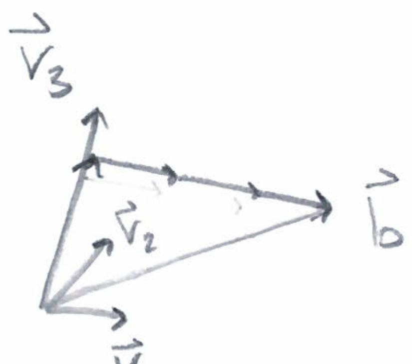


Yes, $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{b}$ has a solution.

No, the solution is not unique.



Estimating, it looks like $2\vec{v}_2 + 2\vec{v}_1 = \vec{b}$



Estimating, it looks like $3\vec{v}_1 + \frac{3}{4}\vec{v}_3 = \vec{b}$