Math 341 - Linear Algebra
§3.1 - 3.3

Fall 2019

Everyone should learn something new everyday.
- Joseph F. Smith

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A.

§3.1 Introduction to Determinants

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A.

For instance if

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A.

For instance if

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Then A_{32} is obtained by crossing out row 3 and column 2

§3.1 Introduction to Determinants

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A.

For instance if

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Then A_{32} is obtained by crossing out row 3 and column 2

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Then A_{32} is obtained by crossing out row 3 and column 2

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Then A_{32} is obtained by crossing out row 3 and column 2

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

so that

$$A_{32} =$$

Then A_{32} is obtained by crossing out row 3 and column 2

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & -2 & 4 & 5 \\ 3 & 8 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

so that

$$A_{32} = \begin{bmatrix} 1 & 7 & 9 \\ 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

Definition (Determinant of A)

Definition (Determinant of A)

For a 1×1 matrix $A = [a]$, the determinant of A, denoted by $\det A$, is defined to be

$$\det A = a.$$

Definition (Determinant of A)

For a 1×1 matrix $A = [a]$, the determinant of A, denoted by $\det A$, is defined to be

$$\det A = a.$$

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum

§3.1 Introduction to Determinants

Definition (Determinant of A)

For a 1×1 matrix $A = [a]$, the determinant of A, denoted by $\det A$, is defined to be

$$\det A = a.$$

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{n+1} a_{1n} \det A_{1n}$$

Definition (Determinant of A)

For a 1×1 matrix $A = [a]$, the determinant of A, denoted by $\det A$, is defined to be

$$\det A = a.$$

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{n+1} a_{1n} \det A_{1n} \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det A_{1i}. \end{aligned}$$

Example: Find the determinant of the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and verify that it matches the definition from §2.2.

Example: Find the determinant of the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and verify that it matches the definition from §2.2.

Example: Find the determinants of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix}.$$

Definition ((i, j)-cofactor)

Given $A = [a_{ij}]$, the (i, j)-**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Definition ((i,j)-cofactor)

Given $A = [a_{ij}]$, the (i,j)-**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example: Find C_{11}, C_{12}, and C_{13} for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

Definition ((i, j)-cofactor)

Given $A = [a_{ij}]$, the (i, j)-**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example: Find C_{11}, C_{12}, and C_{13} for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

$$C_{11} = 2,$$

Definition ((i, j)-cofactor)

Given $A = [a_{ij}]$, the (i, j)-**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example: Find C_{11}, C_{12}, and C_{13} for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

$$C_{11} = 2, \qquad C_{12} = 2,$$

Definition ((i, j)-cofactor)

Given $A = [a_{ij}]$, the (i, j)-**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example: Find C_{11}, C_{12}, and C_{13} for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

$$C_{11} = 2, \qquad C_{12} = 2, \qquad \text{and} \qquad C_{13} = -3.$$

So, we can state the determinant of A (from the previous example) in terms of its cofactors:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

So, we can state the determinant of A (from the previous example) in terms of its cofactors:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This is called a **cofactor expansion of** $\det A$ **across the first row** of A.

So, we can state the determinant of A (from the previous example) in terms of its cofactors:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This is called a **cofactor expansion of** $\det A$ **across the first row** of A.

The sign we multiply each A_{ij} to get C_{ij} is given by the following checkerboard pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Theorem (Cofactor Expansions)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

Theorem (Cofactor Expansions)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The expansion across the ith row is

Theorem (Cofactor Expansions)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

Theorem (Cofactor Expansions)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

Theorem (Cofactor Expansions)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Theorem (Cofactor Expansions)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Example: Find the determinant of

$$A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}.$$

Example: Find the determinant of

$$B = \begin{bmatrix} 6 & -1 & -3 & 2 \\ 0 & 3 & 9 & -11 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Example: Find the determinant of

$$B = \begin{bmatrix} 6 & -1 & -3 & 2 \\ 0 & 3 & 9 & -11 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Theorem

Let A be a square matrix.

Theorem

Let A be a square matrix.

1. If a multiple of one row is added to another row to produce a matrix B, then

$$\det B = \det A.$$

§3.2 Properties of Determinants

Theorem

Let A be a square matrix.

1. If a multiple of one row is added to another row to produce a matrix B, then

$$\det B = \det A.$$

2. If two rows of A are interchanged to produce B then

$$\det B = -\det A.$$

§3.2 Properties of Determinants

Theorem

Let A be a square matrix.

1. If a multiple of one row is added to another row to produce a matrix B, then

$$\det B = \det A.$$

2. If two rows of A are interchanged to produce B then

$$\det B = -\det A.$$

3. If one row of A is multiplied by k to produce B, then

$$\det B = k \cdot \det A.$$

Example: Find the determinant of

$$\begin{bmatrix} 0 & 2 & 1 & -2 \\ 3 & 6 & -3 & 0 \\ -1 & -2 & 2 & 11 \\ 1 & -2 & -5 & -2 \end{bmatrix}$$

using row reduction operations.

Fact: Suppose that a square matrix A has been reduced to an echelon form U by row replacements (adding multiples of one row to another) and row interchanges. If there are r interchanges, then

Fact: Suppose that a square matrix A has been reduced to an echelon form U by row replacements (adding multiples of one row to another) and row interchanges. If there are r interchanges, then $\det A = (-1)^r \det U$.

Fact: Suppose that a square matrix A has been reduced to an echelon form U by row replacements (adding multiples of one row to another) and row interchanges. If there are r interchanges, then $\det A = (-1)^r \det U$.

This also means that

$\det A =$

Fact: Suppose that a square matrix A has been reduced to an echelon form U by row replacements (adding multiples of one row to another) and row interchanges. If there are r interchanges, then $\det A = (-1)^r \det U$.

This also means that

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in U} \end{pmatrix} & \text{when } A \text{ is invertible} \\ & \end{cases}$$

§3.2 Properties of Determinants

Fact: Suppose that a square matrix A has been reduced to an echelon form U by row replacements (adding multiples of one row to another) and row interchanges. If there are r interchanges, then $\det A = (-1)^r \det U$.

This also means that

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in U} \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

§3.2 Properties of Determinants

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

By Invertible Matrix Theorem, this also tells us that

1. $\det A = 0 \Leftrightarrow$ the columns of A are linearly dependent.

§3.2 Properties of Determinants

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

By Invertible Matrix Theorem, this also tells us that

1. $\det A = 0 \Leftrightarrow$ the columns of A are linearly dependent.
2. $\det A = 0 \Leftrightarrow$ the rows of A are linearly dependent.

§3.2 Properties of Determinants

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

By Invertible Matrix Theorem, this also tells us that

1. $\det A = 0 \Leftrightarrow$ the columns of A are linearly dependent.
2. $\det A = 0 \Leftrightarrow$ the rows of A are linearly dependent.
3. $\det A \neq 0 \Leftrightarrow$ the columns of A span $\mathbb{R}^n$.

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

By Invertible Matrix Theorem, this also tells us that

1. $\det A = 0 \Leftrightarrow$ the columns of A are linearly dependent.
2. $\det A = 0 \Leftrightarrow$ the rows of A are linearly dependent.
3. $\det A \neq 0 \Leftrightarrow$ the columns of A span $\mathbb{R}^n$.
4. $\det A \neq 0 \Leftrightarrow$ the rows of A span $\mathbb{R}^n$.

§3.2 Properties of Determinants

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

By Invertible Matrix Theorem, this also tells us that

1. $\det A = 0 \Leftrightarrow$ the columns of A are linearly dependent.
2. $\det A = 0 \Leftrightarrow$ the rows of A are linearly dependent.
3. $\det A \neq 0 \Leftrightarrow$ the columns of A span $\mathbb{R}^n$.
4. $\det A \neq 0 \Leftrightarrow$ the rows of A span $\mathbb{R}^n$.

Example: Compute

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 5 & 1 \end{vmatrix}$$

§3.2 Properties of Determinants

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

By Invertible Matrix Theorem, this also tells us that

1. $\det A = 0 \Leftrightarrow$ the columns of A are linearly dependent.
2. $\det A = 0 \Leftrightarrow$ the rows of A are linearly dependent.
3. $\det A \neq 0 \Leftrightarrow$ the columns of A span $\mathbb{R}^n$.
4. $\det A \neq 0 \Leftrightarrow$ the rows of A span $\mathbb{R}^n$.

Example: Compute

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 5 & 1 \end{vmatrix} = 0$$

because $R3 = R1 + R2$, and hence the rows are not linearly independent (and A is not invertible).

§3.2 Properties of Determinants

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

§3.2 Properties of Determinants

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

$$\det(A^T) = \det A.$$

§3.2 Properties of Determinants

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

$$\det(A^T) = \det A.$$

Warning! In general

$$\det(A + B) \neq \det A + \det B.$$

You have been warned.

§3.2 Properties of Determinants

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

$$\det(A^T) = \det A.$$

Warning! In general

$$\det(A + B) \neq \det A + \det B.$$

You have been warned.

Example:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 1+4$$

§3.2 Properties of Determinants

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

$$\det(A^T) = \det A.$$

Warning! In general

$$\det(A + B) \neq \det A + \det B.$$

You have been warned.

Example:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 1 + 4 \neq 9$$

§3.2 Properties of Determinants

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

$$\det(A^T) = \det A.$$

Warning! In general

$$\det(A + B) \neq \det A + \det B.$$

You have been warned.

Example:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 1+4 \neq 9 = \det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right).$$

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i with the vector $\mathbf{b}$.

§3.3 Cramer's Rule, Volume & Linear Transf.

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i with the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = \begin{array}{ccccccc} \Big[& \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n & \Big] \\ & & & \uparrow & & & \\ & & & \text{column } i & & & \end{array}$$

§3.3 Cramer's Rule, Volume & Linear Transf.

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i with the vector $\mathbf{b}$.

$$\begin{array}{rccccccl} A_i(\mathbf{b}) = & \Big[& \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n & \Big] \\ & & & & \uparrow & & & \\ & & & & \text{column } i & & & \end{array}$$

Theorem (Cramer's rule)

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i with the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = \begin{array}{c} \left[\begin{array}{ccccc} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{array}\right] \\ \uparrow \\ \text{column } i \end{array}$$

Theorem (Cramer's rule)

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n.$$

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$,

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$, and $\det I_j(\mathbf{x}) = x_j$.

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$, and $\det I_j(\mathbf{x}) = x_j$.

$$\begin{aligned} A \cdot I_j(\mathbf{x}) &= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} \\ &= A_j(\mathbf{b}). \end{aligned}$$

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$, and $\det I_j(\mathbf{x}) = x_j$.

$$\begin{aligned} A \cdot I_j(\mathbf{x}) &= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} \\ &= A_j(\mathbf{b}). \end{aligned}$$

Thus

$$\det A_j(\mathbf{b}) = \det(A \cdot I_j(\mathbf{x}))$$

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$, and $\det I_j(\mathbf{x}) = x_j$.

$$\begin{aligned} A \cdot I_j(\mathbf{x}) &= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} \\ &= A_j(\mathbf{b}). \end{aligned}$$

Thus

$$\det A_j(\mathbf{b}) = \det(A \cdot I_j(\mathbf{x})) = \det A \cdot \det I_j(\mathbf{x})$$

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$, and $\det I_j(\mathbf{x}) = x_j$.

$$\begin{aligned} A \cdot I_j(\mathbf{x}) &= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} \\ &= A_j(\mathbf{b}). \end{aligned}$$

Thus

$$\det A_j(\mathbf{b}) = \det(A \cdot I_j(\mathbf{x})) = \det A \cdot \det I_j(\mathbf{x}) = \det A \cdot x_j,$$

Proof.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, and suppose $A\mathbf{x} = \mathbf{b}$.

Then $I_j(\mathbf{x}) = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$, and $\det I_j(\mathbf{x}) = x_j$.

$$\begin{aligned} A \cdot I_j(\mathbf{x}) &= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} \\ &= A_j(\mathbf{b}). \end{aligned}$$

Thus

$$\det A_j(\mathbf{b}) = \det(A \cdot I_j(\mathbf{x})) = \det A \cdot \det I_j(\mathbf{x}) = \det A \cdot x_j,$$

and $x_j = \dfrac{\det A_j(\mathbf{b})}{\det A}$. $\square$

Example: Find the solutions x and y to the following system:

$$\begin{aligned} x + 3y &= 6 \\ 2x + y &= 4. \end{aligned}$$

Example: Find the solutions x and y to the following system:

$$\begin{aligned} x + 3y &= 6 \\ 2x + y &= 4. \end{aligned}$$

$$x = \frac{\begin{vmatrix} 6 & 3 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}$$

Example: Find the solutions x and y to the following system:

$$\begin{aligned} x + 3y &= 6 \\ 2x + y &= 4. \end{aligned}$$

$$x = \frac{\begin{vmatrix} 6 & 3 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{6}{5}$$

Example: Find the solutions x and y to the following system:

$$\begin{aligned} x + 3y &= 6 \\ 2x + y &= 4. \end{aligned}$$

$$x = \frac{\begin{vmatrix} 6 & 3 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{6}{5} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}$$

Example: Find the solutions x and y to the following system:

$$\begin{aligned} x + 3y &= 6 \\ 2x + y &= 4. \end{aligned}$$

$$x = \frac{\begin{vmatrix} 6 & 3 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{6}{5} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{8}{5}.$$

Example: Find the solutions x, y and z to the following system:

$$\begin{aligned} x + 3y - 2z &= 0 \\ 4x + y + 3z &= 1 \\ 6x + 7y - 2z &= 2. \end{aligned}$$

If A is an $n \times n$ matrix, then the **adjugate** of A is the following matrix of cofactors:

§3.3 Cramer's Rule, Volume & Linear Transf.

If A is an $n \times n$ matrix, then the **adjugate** of A is the following matrix of cofactors:

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

§3.3 Cramer's Rule, Volume & Linear Transf.

If A is an $n \times n$ matrix, then the **adjugate** of A is the following matrix of cofactors:

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

i.e. the adjugate is the transpose of the **cofactor matrix** (the matrix whose (ij)th entry is the (ij)th cofactor C_{ij}).

§3.3 Cramer's Rule, Volume & Linear Transf.

If A is an $n \times n$ matrix, then the **adjugate** of A is the following matrix of cofactors:

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

i.e. the adjugate is the transpose of the **cofactor matrix** (the matrix whose (ij)th entry is the (ij)th cofactor C_{ij}).

Theorem

Let A be an invertible $n \times n$ matrix.

If A is an $n \times n$ matrix, then the **adjugate** of A is the following matrix of cofactors:

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

i.e. the adjugate is the transpose of the **cofactor matrix** (the matrix whose (ij)th entry is the (ij)th cofactor C_{ij}).

Theorem

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$

Example: Find the adjugate of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix},$$

and use it to find A^{-1}.

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, with standard matrix A, and let $S \subset \mathbb{R}^2$ be a region with finite area.

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, with standard matrix A, and let $S \subset \mathbb{R}^2$ be a region with finite area. Then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, with standard matrix A, and let $S \subset \mathbb{R}^2$ be a region with finite area. Then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

If instead $T : \mathbb{R}^3 \to \mathbb{R}^3$, with standard matrix A, and $S \subset \mathbb{R}^3$ is a region with finite volume, then

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, with standard matrix A, and let $S \subset \mathbb{R}^2$ be a region with finite area. Then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

If instead $T : \mathbb{R}^3 \to \mathbb{R}^3$, with standard matrix A, and $S \subset \mathbb{R}^3$ is a region with finite volume, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, with standard matrix A, and let $S \subset \mathbb{R}^2$ be a region with finite area. Then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

If instead $T : \mathbb{R}^3 \to \mathbb{R}^3$, with standard matrix A, and $S \subset \mathbb{R}^3$ is a region with finite volume, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

i.e. the size of the determinant tells us how much area/volume are scaled under a linear transformation.

Example: Find the area of a parallelogram P with corners at $(0,0),(2,3),(1,-2)$, and $(3,1)$.

Example: Find the area of a parallelogram P with corners at $(0,0),(2,3),(1,-2)$, and $(3,1)$.

Example: Find the volume of the ellipsoid E given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1.$$