Section 4.2

2. $\vec{w}$ is in Nul A if $A\vec{w} = \vec{0}$.

$$A\vec{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\vec{w}$ is in Nul A.

3. $$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 7x_3 - 6x_4 \\ x_2 &= -4x_3 + 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \text{ in } \mathbb{R}$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

5.

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A is already in RREF.

$$\begin{aligned} x_1 &= 2x_2 - 4x_4 \\ x_2 &= x_2 \\ x_3 &= 9x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}, \quad \text{for } x_2, x_4 \text{ in } \mathbb{R}$$

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$$

7.

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c = 2 \right\}$$

Assuming the usual vector addition and scalar multiplication, for W to be a subspace of $\mathbb{R}^3$, it would need to contain the zero vector. But $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy the condition $a+b+c=2$.

You could also give examples to show that W is not closed under vector addition nor under scalar multiplication.

8. $W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$

This is the same as problem 7. The zero vector does not satisfy $5r-1 = s+2t$. Thus the zerovector is not in W, and W is not a subspace of $\mathbb{R}^3$.

18. $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$

A is a 4×3 matrix.
Nul A is a subspace of $\mathbb{R}^3$
Col A is a subspace of $\mathbb{R}^4$.

21. $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$ is a nonzero vector in Col A.

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a nonzero vector in Nul A.

23. $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{w}$ is in Col A if it is a linear combination of the columns of A. In other words, $\vec{w}$ is in Col A if $A\vec{x} = \vec{w}$ is consistent.

Notice that $\begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If you didn't see that you could see if $A\vec{x} = \vec{w}$ is consistent by row reducing

$$\left[\begin{array}{cc|c} -6 & 12 & 2 \\ -3 & 6 & 1 \end{array}\right].$$

Thus $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in Col A.

$\vec{w}$ is in Nul A if $A\vec{w} = \vec{0}$.

$$\begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in Nul A as well.

26. a. T see theorem 2

b. T see theorem 3

c. F Col A is the set of all $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent. (blue box pg 203)

d. T If $\vec{x} \mapsto A\vec{x}$, then $T(\vec{x}) = A\vec{x}$. The set of vectors $\vec{x}$ such that $T(\vec{x}) = \vec{0}$ is the kernel of T. If $T(\vec{x}) = \vec{0}$, then $A\vec{x} = \vec{0}$. Thus the kernel of T is the nullspace of A.

e. T See paragraphs below definition of kernel and range.

f. T See example 9.

33. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

a. Show T is a linear transformation. (see definition on page 206.)

i) Let A, B be matrices in $M_{2\times2}$.

$T(A+B) = (A+B) + (A+B)^T$ by definition of T

$= A+B + A^T + B^T$ property of transpose

$= A + A^T + B + B^T$ matrix addition is commutative

$= T(A) + T(B)$ definition of T

ii) Let c be a scalar.

$T(cA) = cA + (cA)^T$ definition of T

$= cA + cA^T$ property of transpose

$= c(A + A^T)$ scalar distribution

$= c\,T(A)$ definition of T

Thus T satisfies both properties necessary to be a linear transformation.

b. Let $A = \frac{1}{2}B$.

$T(A) = \frac{1}{2}B + (\frac{1}{2}B)^T = \frac{1}{2}B + \frac{1}{2}B^T$

$= \frac{1}{2}B + \frac{1}{2}B$ (since $B = B^T$)

$= B$

Thus $T(A) = B$ when $A = \frac{1}{2}B$ and $B = B^T$.

c. Show range $T = \{ B \text{ in } M_{2\times2} : B^T = B \}$

Part b) shows that

$\{ B \text{ in } M_{2\times2} : B^T = B \}$ is a subset of the

range of T.

We need to show that the range of T

is a subset of $\{ B \text{ in } M_{2\times2} : B^T = B \}$

- Let A be a matrix in range of T.

Then $A = C + C^T$ for some matrix C.

So $A^T = (C + C^T)^T = C^T + (C^T)^T = C^T + C$

$$= C + C^T = A.$$

Thus $A^T = A$ and A is in the set

$\{ B \text{ in } M_{2\times2} : B^T = B \}$.

d. The kernel of T would be the set

of matrices in $M_{2\times2}$ that are mapped to

the zero "vector" in $M_{2\times2}$. The zero

"vector" in $M_{2\times2}$ is the zero matrix,

We want $T(A) = 0$, in other words

$$T(A) = A + A^T = 0$$

$$A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

If $A + A^T$ is the zero matrix, then

$2a = 0$, $2d = 0$, and $b + c = 0$.

Thus $a = 0$, $d = 0$, and $b = -c$

The matrices in the kernel of T are the matrices of the form

$\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ where b is any real number.

OR

$$\text{Kernel } T = \left\{ \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} : b \text{ is any real number} \right\}$$