Section 1.2

1. a) RREF
   b) RREF
   c) Neither
   d) REF

3. $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

$R_2 \to R_2 + -4R_1$
$R_3 \to R_3 + -6R_1$

$-\frac{1}{3} R_2$
$-\frac{1}{5} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + -R_2$
$R_1 \to R_1 + -2R_2$

columns 1 and 2 are pivot columns

4. $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$

$R_2 \to R_2 + -3R_1$

$R_3 \to R_3 + -5R_1$

$-\frac{1}{4}R_2$

$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$

$R_1 \to R_1 + -3R_2$

$R_3 \to R_3 + 8R_2$

$-\frac{1}{10}R_3$

$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 \to R_1 + 2R_3$

$R_2 \to R_2 + -3R_3$

columns 1, 2, and 4 are pivot columns

5. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$

7. $$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$R_2 \to R_2 + -3R_1$ $\qquad$ $-\frac{1}{5}R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 + -4R_2$

The system corresponding to the RREF is

$$x_1 + 3x_2 = -5$$
$$x_3 = 3$$

Solve for basic variables.

$$\boxed{\begin{aligned} x_1 &= -3x_2 - 5 \\ x_2 &\text{ is free} \\ x_3 &= 3 \end{aligned}}$$

8.

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix}$$

$R_2 \to R_2 + -2R_1$ $\qquad$ $-R_2$

$$\sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

$R_1 \to R_1 + -4R_2$

$$\boxed{\begin{aligned} &x_1 = -9 \\ &x_2 = 4 \\ &x_3 \text{ is free} \end{aligned}}$$

12.

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix}$$

$R_3 \to R_3 + R_1$ $\qquad$ $R_3 \to R_3 + 4R_2$

$$\sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

New system:

$$x_1 - 7x_2 + 6x_4 = 5$$
$$x_3 - 2x_4 = -3$$

Solution: $\boxed{\begin{array}{l} x_1 = 7x_2 - 6x_4 + 5 \\ x_2 \text{ is free} \\ x_3 = 2x_4 - 3 \\ x_4 \text{ is free} \end{array}}$

14. $= \begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 + -2R_2$

$\sim \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

New system: $x_1 + 7x_3 = -9$

$x_2 - 6x_3 - 3x_4 = 2$

$x_5 = 0$

Solution: $\boxed{\begin{array}{l} x_1 = -7x_3 - 9 \\ x_2 = 6x_3 + 3x_4 + 2 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{array}}$

16. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$ No row of form $\begin{bmatrix} 0 & 0 & \blacksquare \end{bmatrix}$ so by Theorem 2 the system is consistent. No free variables, so solution is unique.

b. $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$ No row of form $\begin{bmatrix} 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ so by Theorem 2 the system is consistent.

Column 2 corresponds to a free variable, so there are an infinite number of solutions.

17. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$

$R_2 \rightarrow R_2 + -2R_1$

By Theorem 2, the system is consistent if and only if $7-2h=0$.

Thus the system is consistent only when $h = 7/2$.

21. a. False - Theorem 1 states that the RREF of a matrix is unique.

b. False - See second paragraph of this section

c. True - See definition of basic variable

d. True - See section "Parametric Descriptions of Solution Sets"

e. False - Theorem 2 states that for a system to be inconsistent the augmented matrix has a row echelon form with a row [0 0 0 0 b] with b nonzero.

22. a. False - The RREF is unique, but not the REF.

b. False - See first paragraph in section "Pivot Positions".

c. True - See definition of forward phase

d. False - See last paragraph of section "Parametric Descriptions of Solution Sets"

e. True – See paragraph after definition of basic and free variables.

24. If a 3×5 augmented matrix has its 5th column as a pivot column, the corresponding system is inconsistent. The matrix would have its 3rd row be of the form $\begin{bmatrix} 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ and so by Theorem 2, the system would be inconsistent.

25. If the coefficient matrix has a pivot in every row, then the last column of the augmented matrix would not be a pivot column. By Theorem 2, the system would be consistent.

29. If a system has fewer equations than variables, then the coefficient matrix for the system has fewer rows than columns. Thus not every column

of the coefficient matrix can be a pivot column. Thus there is at least one free variable. The problem states that the system is consistent. Thus the system has an infinite number of solutions.