Section 4.9

2. a. Suppose we have foods #1, #2, #3. Let

$\vec{x}_k = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where x_i is the probability of choosing Food #i in the k^{th} trial.

As such, our stochastic matrix would be

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

b. In this situation, $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

So $\vec{x}_1 = P\vec{x}_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$.

$$\vec{x}_2 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 5/16 \\ 5/16 \end{bmatrix}$$

So the probability of choosing Food #2 on the 2^{nd} trial after the initial trial is $5/16$ or 31.25%.

3. a. Let $\vec{x}_k = \begin{bmatrix} h_k \\ s_k \end{bmatrix}$ where h_k is the percentage of healthy students and s_k is the percentage of sick students, on day k.

$$h_{k+1} = .95h_k + .45s_k$$
$$s_{k+1} = .05h_k + .55s_k$$

$$P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$$

b. $$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$

$$\left(P\vec{x}_0 = \vec{x}_1 \right)$$

So 15% of the students are likely to be sick on Tuesday.

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

$$\left(P\vec{x}_1 = \vec{x}_2 \right)$$

So 12.5% of the students are likely to be sick on Wednesday.

c. $\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$

$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$

There is a 92.5% chance that you are healthy two days after being heathy.

5. $\begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix} = A$ we want to find the steady-state vector. The steady state vector is the vector that satisfies

$A\vec{x} = \vec{x}$.

The solution to $A\vec{x} = \vec{x}$ is the solution to

$(A-I)\vec{x} = \vec{0}$.

$A-I = \begin{bmatrix} -.9 & .6 \\ .9 & -.6 \end{bmatrix}$ We row reduce to find the solutions to $(A-I)\vec{x} = \vec{0}$

$\sim \begin{bmatrix} -.9 & .6 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -9 & 6 \\ 0 & 0 \end{bmatrix}$ $\quad -9x_1 = -6x_2$

$x_1 = x_2$

A basis for the null space of $A-I$ is $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

Since we want a probability vector we multiply the vector by the reciprocal of its sum.

$$\frac{3}{5}\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

9. Is $P = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ regular stochastic?

In other words, is there a positive integer k such that P^k has positive entries?

$$P^2 = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}\begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix} = \begin{bmatrix} .84 & .2 \\ .16 & .8 \end{bmatrix}$$

Thus P is regular stochastic.

12. To answer this question, we can find a steady-state vector.

We find the nullspace of $P - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$
$x_2 = x_3$
$x_3 = x_3$

So a basis for Nul(P−I) is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Multiplying by $1/3$,

we get the probability vector $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

Thus after many trials, the animals will prefer all the foods equally.

13. a. To find the steady state vector for exercise 3, we find the nullspace of P−I. (Solutions to $P\vec{x} = \vec{x}$ are solutions to $(P-I)\vec{x} = \vec{0}$)

$$P - I = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.05 & .45 \\ .05 & -.45 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -9 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 9x_2 \\ x_2 &= x_2 \end{aligned}$$

A basis for Nul(P−I) is $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

Multiplying by $\frac{1}{10}$ we get $\begin{bmatrix} 9/10 \\ 1/10 \end{bmatrix}$

b. After many days the probability a student is ill is 10%. It doesn't matter whether the were well or sick initially.