Section 1.7

1. $\begin{bmatrix}5\\0\\0\end{bmatrix}, \begin{bmatrix}7\\2\\-6\end{bmatrix}, \begin{bmatrix}9\\4\\-8\end{bmatrix}$ Is their a non-trivial solution to $x_1\begin{bmatrix}5\\0\\0\end{bmatrix} + x_2\begin{bmatrix}7\\2\\-6\end{bmatrix} + x_3\begin{bmatrix}9\\4\\-8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$?

$$\begin{bmatrix}5 & 7 & 9\\0 & 2 & 4\\0 & -6 & -8\end{bmatrix} \sim \begin{bmatrix}5 & 7 & 9\\0 & 2 & 4\\0 & 0 & 4\end{bmatrix}$$

Each column of the echelon form has a pivot.

Thus $\vec{0}$ is the only solution to the above vector equation. By definition, the set of vectors in linearly independent.

3. $\begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}-3\\9\end{bmatrix}$ Note that $\begin{bmatrix}1\\-3\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}-3\\9\end{bmatrix}$.

Since $\begin{bmatrix}-3\\9\end{bmatrix}$ is a multiple of $\begin{bmatrix}1\\-3\end{bmatrix}$, the set of vectors is linearly dependent.

4. $\begin{bmatrix}-1\\4\end{bmatrix} \begin{bmatrix}-2\\-8\end{bmatrix}$ Since neither vector is a multiple of the other, the set is linearly independent.

7. $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$ The columns of the matrix form a set of 4 vectors in $\mathbb{R}^3$.

By Theorem 8, the columns of the matrix form a linearly dependent set.

16. $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$ Note that $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$

Since one vector is a multiple of the other, the set of vectors is linearly dependent.

18. $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ Since the set consists of 4 vectors in $\mathbb{R}^2$, Theorem 8 implies the set is linearly dependent.

20. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This set contains the zero vector. By Theorem 9 the set is linearly dependent.

21. a. False. $A\vec{x} = \vec{0}$ always has the trivial solution, but the columns of A aren't always linearly independent.

b. False. Theorem 7 says that <u>some</u> vector in the set will be a linear combination of the others. i.e. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ This set is linearly dependent, but $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ cannot be written as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

c. True. Theorem 8. A 4x5 matrix consists of 5 vectors in $\mathbb{R}^4$.

d. True. Theorem 7. Since $\vec{v}_1, \vec{v}_2$ is linearly independent, $\vec{v}_3$ is a linear combination of $\vec{v}_1, \vec{v}_2$. Thus $\vec{v}_3$ is in Span$\{\vec{v}_1, \vec{v}_2\}$

26. A is a 4×3 matrix $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$.

The set $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is linearly independent by Theorem 7. (Since $\vec{a}_1, \vec{a}_2$ is linearly independent, $\vec{a}_1 \neq \vec{0}$, and $\vec{a}_2$ is not a linear combination of $\vec{a}_1$. Further $\vec{a}_3$ is not a linear combination of $\vec{a}_1, \vec{a}_2$ since $\vec{a}_3$ is not in Span $\{\vec{a}_1, \vec{a}_2\}$.)

Since $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is a linearly independent set every column of A is a pivot column.

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

This is the only possible echelon form of A.

33. True. Since $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$, we have

$$2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}.$$

This may be written as

$$2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 + 0 \cdot \vec{v}_4 = \vec{0}.$$

Thus there is a dependency relation between $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and $\vec{v}_4$. In other words there is a nontrivial solution to

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}.$$

Thus the set $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ is linearly dependent.

36. False. Let $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{V}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ $\vec{V}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{V}_4 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$

$\vec{V}_3$ is not a linear combination of $\vec{V}_1, \vec{V}_2, \vec{V}_4$, but the set $\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4$ is linearly dependent.

$$-\vec{V}_1 - \vec{V}_2 + 0 \cdot \vec{V}_3 + \vec{V}_4 = \vec{0}$$