Section 4.3

1. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ Basis for $\mathbb{R}^3$

The matrix $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ has determinant equal to one and is thus invertible.

The columns of an invertible matrix span $\mathbb{R}^3$ and are linearly independent.

2. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ Not a basis for $\mathbb{R}^3$

The set is not linearly independent since it contains the zero vector.

The set does not span $\mathbb{R}^3$ since the matrix $\begin{bmatrix}1&0&0\\0&0&1\\1&0&0\end{bmatrix}$ does not have a pivot in each row.

8. $\begin{bmatrix}1\\-4\\3\end{bmatrix}, \begin{bmatrix}0\\3\\-1\end{bmatrix}, \begin{bmatrix}3\\-5\\4\end{bmatrix}, \begin{bmatrix}0\\2\\-2\end{bmatrix}$ Not a basis for $\mathbb{R}^3$

The set is not linearly independent since there are more vectors than entries in each vector.

$$\begin{bmatrix}1 & 0 & 3 & 0\\-4 & 3 & -5 & 2\\3 & -1 & 4 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0\\0 & 3 & 7 & 2\\0 & -1 & -5 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 5 & 2\\0 & 0 & -8 & -4\end{bmatrix}$$

Since there is a pivot in each row, the columns of the matrix span $\mathbb{R}^3$. Thus the set spans $\mathbb{R}^3$.

13. $$A = \begin{bmatrix}-2 & 4 & -2 & -4\\2 & -6 & -3 & 1\\-3 & 8 & 2 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 6 & 5\\0 & 2 & 5 & 3\\0 & 0 & 0 & 0\end{bmatrix} = B$$

Basis for Col A is $\left\{\begin{bmatrix}-2\\2\\3\end{bmatrix}, \begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}$

by Theorem 6.

$$\begin{bmatrix}1 & 0 & 6 & 5\\0 & 2 & 5 & 3\\0 & 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 6 & 5\\0 & 1 & 5/2 & 3/2\\0 & 0 & 0 & 0\end{bmatrix}$$

The solutions to $A\vec{x} = \vec{0}$ are given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 - 5x_4 \\ -\frac{5}{2}x_3 - \frac{3}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

x_3, x_4 in $\mathbb{R}$

Thus a basis for Nul$A$ is $\left\{ \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

(Notice that if you don't want fractions let $x_3 = x_4 = 2$.)

14.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$

by Theorem 6.

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solutions to $A\vec{x} = \vec{0}$ are written as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ 7/5\, x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$$

$x_2, x_4 \in \mathbb{R}$

Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

19. Since $H = \text{Span}\left\{ \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} \right\}$,

we have a spanning set for H.

Since $4\begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix} - 3\begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the spanning set is linearly dependent.

By Theorem 5 part a,

$$\text{Span}\left\{\begin{bmatrix}-4\\3\\7\end{bmatrix}, \begin{bmatrix}1\\9\\-2\end{bmatrix}, \begin{bmatrix}7\\11\\6\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}-4\\3\\7\end{bmatrix}, \begin{bmatrix}1\\9\\-2\end{bmatrix}\right\}.$$

(In fact any subset of 2 out of the 3 vectors would be a spanning set for H.)

Since neither of the two vectors is a scalar multiple of the other, they form a linearly independent set.

Thus $\left\{\begin{bmatrix}-4\\3\\7\end{bmatrix}, \begin{bmatrix}1\\9\\-2\end{bmatrix}\right\}$ is a basis for H.

22. a. False. $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a linearly independent set, but it is not a basis for $H = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

b. True. Theorem 5 part b.

c. True. See paragraph before example 10. A basis is a maximal linearly independent subset of the subspace.

29. $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$, k vectors in $\mathbb{R}^n$, $k < n$.

Let $A = [\vec{v}_1 \ldots \vec{v}_k]$. A is a $n \times k$ matrix with $k < n$. Since A has more rows than columns, not every row of A contains a pivot. Thus by Theorem 4 in section 1.4, the columns of A do not span $\mathbb{R}^n$.

30. $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$, k vectors in $\mathbb{R}^n$, $k > n$.

Since $k > n$, we have more vectors than entries in each vector. By Theorem 8 in section 1.7, the set is linearly dependent.

33. $\{1+t^2, 1-t^2\}$ is a linearly independent set in $\mathbb{P}_3$. Neither polynomial is a scalar multiple of the other.

34. $\{1+t, 1-t, 2\}$

$$(1+t) + (1-t) + -1(2) = 0$$

Since every polynomial is a linear combination of the other two polynomials, we may delete any of them from the spanning set. (Theorem 5 part a)

Thus $\text{Span}\{1+t, 1-t, 2\} = \text{Span}\{1+t, 2\}$

Since $1+t$ and 2 are not multiples of each other, they are a linearly independent set.

Thus $\{1+t, 2\}$ is a basis for $\text{Span}\{1+t, 1-t, 2\}$.

(In fact any two polynomials from $\{1+t, 1-t, 2\}$ will be a basis.)