Section 1.5

3. $-3x_1 + 5x_2 - 7x_3 = 0$
$-6x_1 + 7x_2 + x_3 = 0$

$$\left[\begin{array}{ccc|c} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{array}\right]$$

Since there are more columns than rows in the coefficient matrix, there is at least one free variable. Since it is homogeneous, the system is consistent. Thus there are an infinite number of solutions.

5. $x_1 + 3x_2 + x_3 = 0$
$-4x_1 - 9x_2 + 2x_3 = 0$
$-3x_2 - 6x_3 = 0$

$$\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: This is the coefficient matrix.

$x_1 - 5x_3 = 0$
$x_2 + 2x_3 = 0$ $\longrightarrow$

$x_1 = 5x_3$
$x_2 = -2x_3$
$x_3 = x_3$
(x_3 is free)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

7. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

$$\begin{aligned} x_1 + 9x_3 - 8x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \end{aligned} \longrightarrow \begin{aligned} x_1 &= -9x_3 + 8x_4 \\ x_2 &= 4x_3 - 5x_4 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

11. $$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 + 5x_6 = 0$$
$$x_3 - x_6 = 0$$
$$x_5 - 4x_6 = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_6 \\ 0 \\ x_6 \\ 0 \\ 4x_6 \\ x_6 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \quad x_2, x_4, x_6 \in \mathbb{R}$$

15. $$x_1 + 3x_2 + x_3 = 1$$
$$-4x_1 - 9x_2 + 2x_3 = -1$$
$$-3x_2 - 6x_3 = -3$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 5x_3 = -2$

$x_2 + 2x_3 = 1$

x_3 is free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 - 2 \\ -2x_3 + 1 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

The solution set is a line parallel to the solution set for exercise 5.

17. $x_1 + 9x_2 - 4x_3 = 0$

$$x_1 = -9x_2 + 4x_3$$

x_2 is free

x_3 is free

$$\vec{x} = \begin{bmatrix} -9x_2 + 4x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$x_2, x_3 \in \mathbb{R}$

$x_1 + 9x_2 - 4x_3 = -2$

$$x_1 = -9x_2 + 4x_3 - 2$$

x_2 is free

x_3 is free

$$\vec{x} = \begin{bmatrix} -9x_2 + 4x_3 - 2 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$x_2, x_3 \in \mathbb{R}$

The solutions sets of the two equations differ only by the vector $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$. This is meant to highlight the differences between homogeneous systems + non-homogeneous.

The solution set of the homogeneous system is a plane through the origin. The solution set of the non homogeneous system is a plane that has been translated by the vector $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$. The two planes are parallel.

20. $\vec{a} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$

$$\vec{x} = t\begin{bmatrix} -7 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad t \in \mathbb{R}$$

Think of $\begin{bmatrix} -7 \\ 8 \end{bmatrix}$ as the "slope" of the line.

It is often times called a direction vector.

Think of $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ as an "intercept". When $t = 0$, the output is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

23. a. True, see paragraph with definition of homogeneous.

b. False, see paragraph with definition of parametric vector equation.

c. False, the trivial solution is always a solution for the homogeneous equation.

d. False, $\vec{x} = \vec{p} + t\vec{v}$ describes a line parallel to $\vec{v}$ through $\vec{p}$.

e. False, it is close to being true. Look at Theorem 6. The statement doesn't say whether $A\vec{x} = \vec{b}$ has a solution, and it doesn't say what the vector $\vec{p}$ represents.

25. Let $\vec{w}$ be a solution to $A\vec{x} = \vec{b}$, and define $\vec{v}_h = \vec{w} - \vec{p}$. We want to show $\vec{v}_h$ is a solution to $A\vec{x} = \vec{0}$.

$$A(\vec{v}_h) = A(\vec{w} - \vec{p}) \underset{\substack{\uparrow \\ \text{Theorem 5}}}{=} A\vec{w} - A\vec{p} \underset{\substack{\uparrow \\ \text{Since } \vec{w} \text{ and } \vec{p} \\ \text{are solutions to} \\ A\vec{x} = \vec{b}}}{=} \vec{b} - \vec{b} = \vec{0}$$

Since $A\vec{v}_h = \vec{0}$, $\vec{v}_h$ is a solution to $A\vec{x} = \vec{0}$

29. A is a 3×3 matrix with 3 pivot positions.

a) Since A has 3 pivot positions, each column is a pivot column, and there are no free variables. Thus $A\vec{x} = \vec{0}$ has only the trivial solution.

b) Since A has 3 pivot positions, each row of A has a pivot. By Theorem 4, $A\vec{x} = \vec{b}$ has a solution for every choice of $\vec{b}$.

30. A is a 3×3 matrix with two pivot positions

a) Since A has 2 pivot positions, there is a column without a pivot position. Thus A has a free variable and nontrivial solutions to $A\vec{x} = \vec{0}$.

b) Since A has two pivot positions, there is a row of A without a pivot. By Theorem 4, there exists a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution.

35. Your solution may vary.

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$