Section 5.2

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ Since the rows sum to 9, 9 is an eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since the determinant is the product of the eigenvalues, $-45 = \lambda_1 \cdot \lambda_2 = 9 \cdot \lambda_2$.

The other eigenvalue is -5.

The characteristic polynomial is

$$\boxed{(\lambda-9)(\lambda+5) = \lambda^2 - 4\lambda - 45}$$

The eigenvalues are $\boxed{9, -5.}$

3. $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix} = A$ We find the characteristic polynomial by finding $\det(A - \lambda I)$.

$$\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix}$$

$$= (3-\lambda)(-1-\lambda) - (-2)$$

$$= -3 + \lambda - 3\lambda + \lambda^2 + 2$$

$$= \boxed{\lambda^2 - 2\lambda - 1}$$

So $\lambda^2 - 2\lambda - 1$ is the characteristic polynomial of A. Using the quadratic formula, we can find the roots of $\lambda^2 - 2\lambda - 1$.

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Thus the eigenvalues are $\boxed{1+\sqrt{2},\ 1-\sqrt{2}.}$

Check: $tr\ A$ = sum of eigenvalues

$$2 = 3 + (-1) = 1+\sqrt{2} + 1-\sqrt{2} = 2$$

7. $A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$$

$$= (5-\lambda)(4-\lambda) - (-12)$$

$$= \lambda^2 - 9\lambda + 20 + 12$$

$$= \boxed{\lambda^2 - 9\lambda + 32}$$

$$\lambda = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(32)}}{2(1)}$$

$$= \frac{9 \pm \sqrt{-47}}{2}$$

A has complex eigenvalues $\boxed{\frac{9+i\sqrt{47}}{2},\ \frac{9-i\sqrt{47}}{2}}$

11. $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix} = A \quad \det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{bmatrix}$

$$= (4-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} + 0 + 0$$

(cofactor expansion along 1^{st} row.)

$$= (4-\lambda)(3-\lambda)(2-\lambda)$$

or

$$= -\lambda^3 + 9\lambda^2 - 26\lambda + 24$$

15. $\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Since this is a triangular matrix its eigenvalues are 4, 3, 3, 1.

16. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$ Since this is a triangular matrix, its eigenvalues are 5, −4, 1, 1.

18.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall that the eigenspace of A corresponding to λ, is the nullspace of $A - \lambda I$.

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 1 & -\frac{h}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & -\frac{h}{2}+3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To have the eigenspace corresponding to 5 have dimension 2, we need the nullspace of $A - 5I$ to have dimension 2. Recall that the dimension of the nullspace of a matrix is equal to the number of free variables.

For $A-5I$ to have two free variables, $-\frac{h}{2}+3=0$. Thus $\boxed{h=6.}$

19. $\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\ldots(\lambda_n-\lambda)$
where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A.
If we let $\lambda=0$, then the left side is $\det A$. The right side of the equation is $(\lambda_1-0)(\lambda_2-0)\ldots(\lambda_n-0)=\lambda_1\lambda_2\ldots\lambda_n$.
This is the product of the eigenvalues.
Thus $\det A=\lambda_1\lambda_2\ldots\lambda_n$.

20. Show A and A^T have the same characteristic polynomial.

$$
\begin{aligned}
\det(A-\lambda I) &= \det\left((A-\lambda I)^T\right) && \text{since } \det B=\det B^T \\
&= \det\left(A^T-(\lambda I)^T\right) && \text{since } (A+B)^T=A^T+B^T \\
&= \det\left(A^T-\lambda I\right) && \text{since } D^T=D \text{ for any diagonal matrix.}
\end{aligned}
$$

Thus the characteristic polynomial of A, $\det(A-\lambda I)$, is equal to the characteristic polynomial of A^T, $\det(A^T-\lambda I)$.

23. Let $A = QR$ and $A_1 = RQ$ where Q is invertible.

To show A is similar to A_1, we need to show there exists an invertible matrix P such that $A = P^{-1} A_1 P$.

Since $A = QR$ and $A_1 = RQ$, we have

$$QR = P^{-1}(RQ)P.$$

Thus we want $P = Q^{-1}$. Then

$$P^{-1}A_1P = P^{-1}(RQ)P = (Q^{-1})^{-1}(RQ)Q^{-1} = QR = A.$$

Therefore A is similar to A_1

24. If A is similar to B, then there exists an investible matrix P such that $A = P^{-1}BP$.

$$\begin{aligned} \det A = \det P^{-1}BP &= \det P^{-1} \det B \det P \\ &= \det P^{-1} \det P \det B \\ &= \frac{1}{\det P} \det P \det B \\ &= \det B. \end{aligned}$$