Section 5.1

3. $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix}$ since $\begin{bmatrix} 1 \\ 29 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is not an eigenvector.

6. $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector for the matrix corresponding to the eigenvalue -2.

9. $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1, 5$.

To find a basis for the eigenspace corresponding to $\lambda = 1$, we find a basis for the null space of $A - I$.

$$A - I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $x_1 = 0$, and x_2 is free.

$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2 \in \mathbb{R}$, are the solutions

to $(A - I)\vec{x} = \vec{0}$.

Since $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the nullspace of $A - I$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to $\lambda = 1$.

To find a basis for the eigenspace of A corresponding to $\lambda = 5$, we find a basis for the nullspace of $A - 5I$.

$$A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_2 \\ x_2 &= x_2 \end{aligned} \qquad \vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ x_2 \in \mathbb{R}.$$

Thus $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to $\lambda = 5$.

14.

$$A-(-2I)=\begin{bmatrix}3&0&-1\\1&-1&0\\4&-13&3\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\3&0&-1\\4&-13&3\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-1&0\\0&3&-1\\0&-9&3\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\0&3&-1\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-\frac{1}{3}\\0&1&-\frac{1}{3}\\0&0&0\end{bmatrix}$$

$$\begin{aligned}x_1&=\tfrac{1}{3}x_3\\x_2&=\tfrac{1}{3}x_3\\x_3&=x_3\end{aligned}\qquad \vec{x}=x_3\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\1\end{bmatrix},\ x_3\in\mathbb{R}.$$

So $\begin{bmatrix}1\\1\\3\end{bmatrix}$ is a basis for the eigenspace of A corresponding to $\lambda=-2$.

17. Since the matrix is upper triangular, Theorem 1 states the eigenvalues are the diagonal entries. Thus the eigenvalues are $0, 2, -1$.

19.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The sum of the rows are all 6, so 6 is an eigenvalue.

The columns are all multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so the matrix is not invertible.

Thus 0 is an eigenvalue.

(You just need to identify one of the eigenvalues.)

23. An $n \times n$ matrix can have at most n distinct eigenvalues because of Theorem 2. The eigenvectors corresponding to distinct eigenvalues are linearly independent. Since the dimension of $\mathbb{R}^n$ is n, we can have at most n vectors, in $\mathbb{R}^n$, in a linearly independent set. Thus an $n \times n$ matrix can have at most n distinct eigenvalues.

25. Since λ is an eigenvalue of A, there exists an eigenvector $\vec{x}$ satisfying, $A\vec{x} = \lambda\vec{x}$.

Since A is invertible, we may multiply A^{-1} on the left of each side of the equation.

$$A^{-1}(A\vec{x}) = A^{-1}(\lambda\vec{x})$$

$$I\vec{x} = A^{-1}(\lambda\vec{x})$$

$$\vec{x} = A^{-1}\lambda\vec{x}$$

Since A is invertible, $\lambda \neq 0$.

$$\frac{1}{\lambda}\vec{x} = \frac{1}{\lambda}A^{-1}\lambda\vec{x} = A^{-1}\vec{x},$$

Thus $A^{-1}\vec{x} = \lambda^{-1}\vec{x}$.

Since $\vec{x}$ is an eigenvector, $\vec{x} \neq \vec{0}$. Thus λ^{-1} is an eigenvalue of A^{-1}.

26. Let λ be an eigenvalue of A, and suppose $A^2 = 0$.

Since λ is an eigenvalue of A, there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}.$$

Multiply A on the left of each side of the equation.

$$A(A\vec{x}) = A(\lambda\vec{x})$$

On one hand,

$$A(A\vec{x}) = A^2\vec{x} = 0\vec{x} = \vec{0}.$$

On the other hand,

$$A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}.$$

Thus $\lambda^2\vec{x} = \vec{0}$. Since $\vec{x}$ is an eigenvector, $\vec{x} \neq \vec{0}$. Thus $\lambda^2 = 0$, and $\lambda = 0$.

Since λ was an arbitrary eigenvalue of A, all eigenvalues of A are equal to 0.

27. Recall that λ is an eigenvalue of A if and only if $A-\lambda I$ has a non-trivial nullspace. $A-\lambda I$ has a non-trivial nullspace if and only if $A-\lambda I$ is not invertible. (Invertible Matrix Theorem). $A-\lambda I$ is not invertible if and only if $(A-\lambda I)^T$ is not invertible. Note $(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$.

Thus $A-\lambda I$ is not invertible if and only if $A^T-\lambda I$ is not invertible. $A^T-\lambda I$ is not invertible if and only if $A^T-\lambda I$ has a non-trivial null space. $A^T-\lambda I$ has a nontrivial null space if and only if λ is an eigenvalue of A^T.

29. Let A be a matrix whose row sums are all equal to s. Let $\vec{x}$ be the all-ones vector. Then $A\vec{x}=s\vec{x}$. Thus s is an eigenvalue of A.