Section 6.2

1. $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = -3 + -16 + 21 = 2$

The set of vectors is not orthogonal.

5. $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} = -3 + -6 + -3 + 12 = 0$

$\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = 9 + -16 + 7 + 0 = 0$

$\begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = -3 + 24 + -21 + 0 = 0$

Since every pair of vectors is orthogonal, the set is orthogonal.

7. $\vec{u}_1 \cdot \vec{u}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 12 + -12 = 0$

Thus $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set. By theorem 4, $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent. Since $\mathbb{R}^2$ is 2-dimensional, any linearly independent set of size two is a basis for $\mathbb{R}^2$. Thus $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$.

We wish to write $\vec{x}$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$. i.e. $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2$.

Theorem 5 states that $c_1 = \frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$ and $c_2 = \frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}$.

$$\text{Thus } \vec{x} = \frac{39}{13}\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \frac{26}{52}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -9 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

11. $\vec{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ The projection of $\vec{y}$ onto $\vec{u}$ is

$$\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} = \frac{10}{20}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

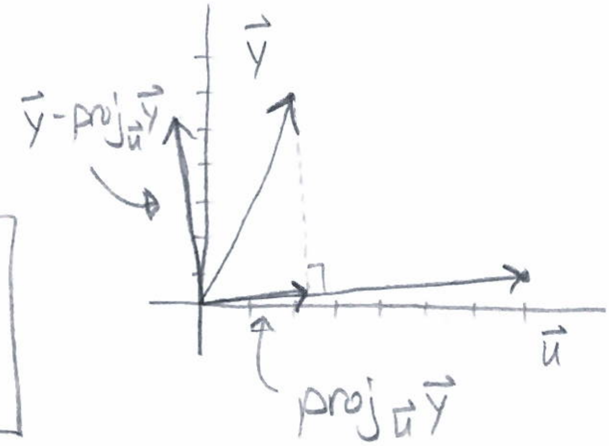
14. The projection of $\vec{y}$ onto $\vec{u}$ will be in Span $\{\vec{u}\}$.

$$\text{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{20}{50} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

$\vec{y} - \text{proj}_{\vec{u}} \vec{y}$ will be orthogonal to $\vec{u}$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$




17. $$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = 0$$

$$\sqrt{(1/3)^2 + (1/3)^2 + (1/3)^2} = \sqrt{1/3}$$

$$\sqrt{(-1/2)^2 + 0^2 + (1/2)^2} = \sqrt{1/2}$$

Both vectors need to be normalized.

$$\sqrt{3} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad \sqrt{2} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

So $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ is an orthonormal set.

23. a. True. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a linearly independent set which is not orthogonal.

b. True. This is a consequence of Theorem 5.

c. False. The process of normalization only scales the vectors to have length 1. This will not affect orthogonality. See top of page 345.

25. We prove part b. first.

$$(U\vec{x})\cdot(U\vec{y}) = (U\vec{x})^T(U\vec{y}) = (\vec{x}^T U^T)(U\vec{y})$$

$$= \vec{x}^T (U^T U)\vec{y}$$

$$= \vec{x}^T I \vec{y}$$ since $U$ has orthonormal columns (Theorem 6)

$$= \vec{x}^T \vec{y}$$

$$= \vec{x} \cdot \vec{y}$$

Part c. is true since in part b. is true.

Part a. follows from part b. as follows:

$$\|U\vec{x}\|^2 = U\vec{x} \cdot U\vec{x} = \vec{x} \cdot \vec{x} \quad \text{by part b.}$$
$$= \|\vec{x}\|^2$$

Since $\|\vec{x}\| \geq 0$, we have that

$\|U\vec{x}\|^2 = \|\vec{x}\|^2$ implies $\|U\vec{x}\| = \|\vec{x}\|$.

27. Let $U$ be a square matrix with orthonormal columns.

By Theorem 6, $U^T U = I$.

By Invertible Matrix Theorem, part j, $U$ is invertible.