Section 4.6

1. $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$

rank A = # pivot columns = 2

dim Nul A = # of free variables = 2

Basis for Col A $= \left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$

Basis for Row A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \\ -6 \end{bmatrix} \right\}$

Solving $A\vec{x} = \vec{0}$:

$x_1 = x_3 - 5x_4$

$x_2 = \frac{5}{2}x_3 - 3x_4$

$x_3 = x_3$

$x_4 = x_4$

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

Basis for Nul A $= \left\{ \begin{bmatrix} 2 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. $A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$

rank A = # of pivot columns = 3

dim Nul A = # of free variables = 2

Basis for Col A = $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$

Basis for Row A = $\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \right\}$

Solving $A\vec{x} = \vec{0}$:

$x_1 = 3x_2 - 5x_4$

$x_2 = x_2$

$x_3 = \frac{3}{2}x_4$

$x_4 = x_4$

$x_5 = 0$

$\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$, x_2, x_4 in $\mathbb{R}$

Basis for Nul A = $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} \right\}$

5. 3×8 matrix A, $\operatorname{rank} A = 3$

Using Rank-Nullity Theorem

$$\operatorname{rank} A + \operatorname{nullity} A = \# \text{ of columns}$$

$$3 + \operatorname{nullity} A = 8$$

$$\boxed{\dim \operatorname{Nul} A = 5}$$

Using Rank-Nullity Theorem

$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A$$

$$\boxed{\dim \operatorname{Row} A = 3}$$

$$\operatorname{rank} A^T = \dim \operatorname{Col} A^T = \dim \operatorname{Row} A = \operatorname{rank} A$$

So $\boxed{\operatorname{rank} A^T = 3}$

6. 6×3 matrix A, $\operatorname{rank} A = 3$

Using Rank-Nullity Theorem

$$\operatorname{rank} A + \operatorname{nullity} A = \# \text{ of columns}$$

$$3 + \operatorname{nullity} A = 3$$

$$\operatorname{nullity} A = \boxed{\dim \operatorname{Nul} A = 0.}$$

$\underline{\dim \operatorname{Row} A} = \dim \operatorname{Col} A = \operatorname{rank} A = \underline{3}$

$$\operatorname{rank} A^T = \dim \operatorname{Col} A^T = \dim \operatorname{Row} A = \operatorname{rank} A$$

so $\boxed{\operatorname{rank} A^T = 3.}$

7. 4×7 matrix A has 4 pivots.

$\rightarrow$ $\dim \operatorname{Col} A = 4 = \operatorname{rank} A$.

So $\operatorname{Col} A$ is a 4 dimensional subspace of $\mathbb{R}^4$

The only 4 dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$.

Thus $\operatorname{Col} A = \mathbb{R}^4$.

$\rightarrow$ $\dim \operatorname{Nul} A = 3$, by the Rank-Nullity Theorem.

$\operatorname{Nul} A$ is a 3-dimensional subspace of $\mathbb{R}^7$.

Thus $\operatorname{Nul} A$ is isomorphic to $\mathbb{R}^3$, but

$\operatorname{Nul} A \neq \mathbb{R}^3$.

8. A is 5×6 and has 4 pivot columns.

Since A has 4 pivot columns, $\operatorname{rank} A = 4$.

By Rank-Nullity Theorem, $\dim \operatorname{Nul} A = 2$.

$\operatorname{Col} A \neq \mathbb{R}^4$. $\dim \operatorname{Col} A = 4$, and $\operatorname{Col} A$ is a subspace of $\mathbb{R}^5$, not $\mathbb{R}^4$. You could say that $\operatorname{Col} A$ is isomorphic to $\mathbb{R}^4$.

15. A is 6×8 matrix. Since A is 6×8, it can have at most 6 pivots. Thus $\operatorname{rank} A \leq 6$. We have

$$\dim \operatorname{Nul} A = 8 - \operatorname{rank} A.$$

So $\dim \operatorname{Nul} A \geq 2$.

19. Let A be the coefficient matrix of the homogeneous system. A is a 5×6 matrix. Since the solutions to the homogeneous system are solutions to the matrix equation $A\vec{x}=\vec{0}$, the fact that all solutions are multiples of a nonzero solution implies Nul A has a basis consisting of one vector. Thus dim Nul $A=1$. By the rank-nullity theorem, rank $A=5$. Since Col A is a subspace of $\mathbb{R}^5$ and rank $A=5$, Col $A=\mathbb{R}^5$. Thus $A\vec{x}=\vec{b}$ will have a solution for every choice of $\vec{b}$. So the answer is yes.

26. Let A be an $m\times n$ matrix with full rank and $m>n$. Since A has full rank and rank $A\le \min\{m,n\}$, rank $A=n$. In other words every column of A is a pivot column. Since every column of A is a pivot column, $A\vec{x}=\vec{0}$ has only the trivial solution. Thus the columns of A are linearly independent.

On the other hand, if A has linearly independent columns, every column of A is a pivot column. Since every column is a pivot column, A cannot have more pivots than the number of columns of A. Thus A has full rank.

31. $\vec{u}\vec{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$

Every column of $\vec{u}\vec{v}^T$ is a multiple of $\vec{u}$.

Thus, if $\vec{v} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $\text{Col } A = \text{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \right\}$

So rank $\vec{u}\vec{v}^T = 1$.

If $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $\vec{u}\vec{v}^T$ is the zero matrix

and $\text{Col } A = \{\vec{0}\}$. In this case rank $\vec{u}\vec{v}^T = 0$.

Therefore rank $\vec{u}\vec{v}^T \leq 1$.