Section 4.4

1. $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \quad [\vec{x}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\vec{x} = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

3. $B = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, \quad [\vec{x}]_B = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

$$\vec{x} = 3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} + -1\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

6. $c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ Solve this vector equation.

$$\left[\begin{array}{cc|c} 1 & 5 & 4 \\ -2 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & 4 \\ 0 & 4 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 2 \end{array}\right]$$ So $c_1 = -6$, $c_2 = 2$

Thus $[\vec{x}]_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

7. $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$

Solve $c_1 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} c_1 = -1 \\ c_2 = -1 \\ c_3 = 3 \end{array}$$

$$[\vec{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

9. $\begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix} = P_B$ Remember that the columns of P_B are the basis vectors of B.

11. $P_B = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$ Recall that $P_B [\vec{x}]_B = \vec{x}$.

So $[\vec{x}_B] = P_B^{-1} \vec{x}$.

$$P_B^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix}$$

Thus $[\vec{x}]_B = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

23. A function is one-to-one if every element in the codomain has at most one preimage in the domain. A logically equivalent statement is: if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

To show that the coordinate map is one-to-one we will use the equivalent statement. Let $\vec{u}, \vec{v}$ be two vectors in a vector space V with basis $\{\vec{b}_1, \ldots, \vec{b}_n\}$.

Assume that $\vec{u}, \vec{v}$ have the same coordinate vector. In other words, $[\vec{u}]_B = [\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

Thus $\vec{u} = c_1\vec{b_1} + c_2\vec{b_2} + \ldots + c_n\vec{b_n}$.

and $\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + \ldots + c_n\vec{b_n}$.

So $\vec{u} = \vec{v}$, and since $\vec{u}, \vec{v}$ were arbitrary vectors in V, the statement is true for all vectors in V.

Therefore, the coordinate map is one-to-one.

24. To show that the coordinate map is onto $\mathbb{R}^n$ we need to show that every vector in $\mathbb{R}^n$ is the image of some vector in V.

Let $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$.

Let $\vec{v} = y_1\vec{b_1} + y_2\vec{b_2} + \ldots + y_n\vec{b_n}$ be a vector in V. Notice that $[\vec{v}]_B = \vec{y}$. Thus $\vec{y}$ has a pre image in V. Thus the coordinate map is onto $\mathbb{R}^n$.

27. $\{1, t, t^2, t^3\} = \beta$ is a basis for $\mathbb{P}_3$.

Using the coordinate map each polynomial can be sent to a vector in $\mathbb{R}^4$.

$$[1+2t^3]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \qquad [2+t-3t^2]_\beta = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}$$

$$[-t+2t^2-t^3]_\beta = \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$$

Since the coordinate map is an isomorphism $\{1+2t^3, 2+t-3t^2, -t+2t^2-t^3\}$ is a linearly independent set if and only if

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$ is a linearly independent set.

To check linear independence we will create a matrix whose columns are the vectors from the set.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since each column of the matrix is a pivot column, the columns of the matrix are linearly independent.

Thus the polynomials $1+2t^3$, $2+t-3t^2$, and $-t+2t^2-t^3$ are linearly independent.

32. a. We will use the basis $\{1, t, t^2\} = \mathcal{B}$ for $\mathbb{P}_2$.

$$[p_1(t)]_{\mathcal{B}} = [1+t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$[p_2(t)]_{\mathcal{B}} = [t-3t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$$[p_3(t)]_{\mathcal{B}} = [1+t-3t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

The polynomials $\{1+t^2, t-3t^2, 1+t-3t^2\}$ form a basis for $\mathbb{P}_2$ if and only if the set $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}, \begin{bmatrix}1\\1\\-3\end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^3$. (This is because the coordinate map is an isomorphism between $\mathbb{P}_2$ and $\mathbb{R}^3$.)

To check whether the set is a basis, we will check to see whether the matrix

$$A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -3 & -3\end{bmatrix} \text{ is invertible.}$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -3 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & -3 & -4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & -1\end{bmatrix}$$

Thus $\det A = -1$, and A is invertible.

By the invertible matrix theorem, the columns of A span $\mathbb{R}^3$ and are linearly independent.

Thus the columns are a basis for $\mathbb{R}^3$.

Therefore the polynomials $1+t^2, t-3t^2, 1+t-3t^2$

are a basis for $\mathbb{P}_2$.

b. Since $[q]_\beta = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, then

$$q = -1(1+t^2) + 1(t-3t^2) + 2(1+t-3t^2)$$

$$= -1 - t^2 + t - 3t^2 + 2 + 2t - 6t^2$$

$$= 1 + 3t - 10t^2$$