Section 1.8

1. $T(\vec{u}) = A\vec{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ is the image of $\vec{u}$ under T

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

$\begin{bmatrix} 2a \\ 2b \end{bmatrix}$ is the image of $\vec{v}$ under T

5. Find a vector $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

Finding such a vector $\vec{x}$ is equivalent to solving $A\vec{x} = \vec{b}$.

$$\left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$x_1 = -3x_3 + 3$
$x_2 = -2x_3 + 1$
x_3 is free

$$\vec{x} = \begin{bmatrix} -3x_3+3 \\ -2x_3+1 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

The image of $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ under T is $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

Since there is a free variable, there are an infinite number of vectors whose image under T is $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

8. If T has domain $\mathbb{R}^4$ and codomain $\mathbb{R}^5$, the matrix A such that $T(\vec{x}) = A\vec{x}$ will have 5 rows and 4 columns.

9. The vectors that are mapped to the zero vector by T are the vectors that satisfy $T(\vec{x}) = A\vec{x} = \vec{0}$. So we are looking for solutions to the homogeneous equation $A\vec{x} = \vec{0}$.

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -8 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -9 & 7 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 9x_3 - 7x_4 \\ x_2 &= 4x_3 - 3x_4 \\ x_3, x_4 &\text{ are free} \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -7x_4 \\ -3x_4 \\ 0 \\ x_4 \end{bmatrix}$$

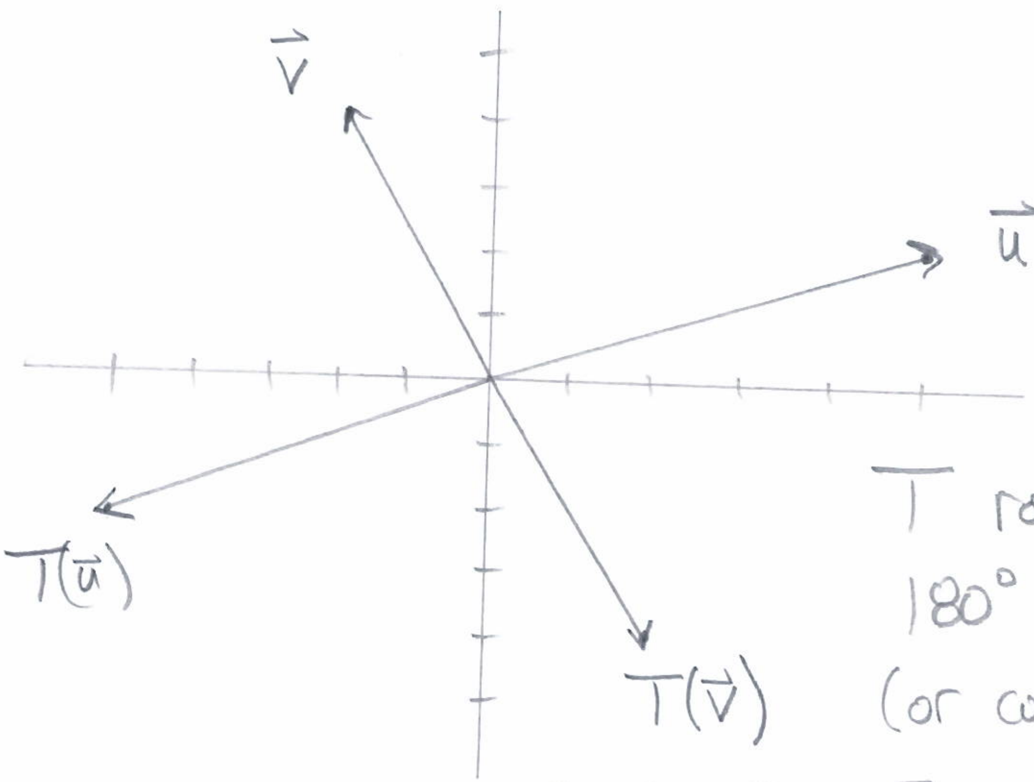
$$= x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \qquad x_3, x_4 \in \mathbb{R}$$

All the above vectors are mapped into the zero vector by the transformation $\vec{x} \longmapsto A\vec{x}$

13. $T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$

$T(\vec{u}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

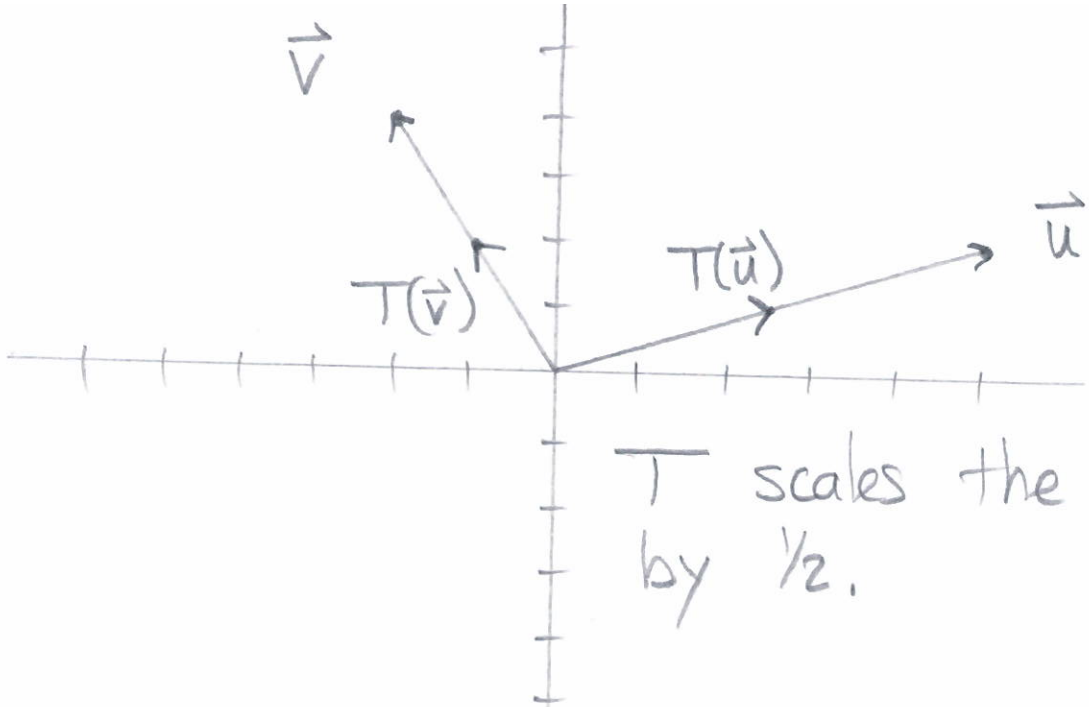


T rotates vectors 180° clockwise. (or counter clockwise)

14. $T(\vec{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .5x_1 \\ .5x_2 \end{bmatrix}$

$T(\vec{u}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



T scales the vector by ½.

17. Given $T(\vec{u}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$T(3\vec{u}) = 3\,T(\vec{u}) = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$T(2\vec{v}) = 2\,T(\vec{v}) = 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$T(3\vec{u} + 2\vec{v}) = T(3\vec{u}) + T(2\vec{v})$$

$$= \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

21. a. True. Transformation is synonymous with function. A linear transformation satisfies additional constraints.

b. False If A is 3×5, then the transformation $T(\vec{x}) = A\vec{x}$ has domain $\mathbb{R}^5$ and codomain $\mathbb{R}^3$.

c. False If A is $m\times n$, then the codomain is $\mathbb{R}^m$. It is possible that the range is a strict subset of $\mathbb{R}^m$.

d. False Differentiation is a linear transformation, but is not a matrix transformation. (see bottom of page 66)

e. True See definition of linear transformation and shaded box after the definition.

29. a. To show that f is a linear transformation we can show it satisfies the properties in the definition. Assume $f(x) = mx$

i. $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$

ii. $f(cx) = m(cx) = c(mx) = cf(x)$

Thus if $b=0$, $f(x) = mx$ is a linear

transformation.

b. Let $f(x) = mx + b$, $b \neq 0$.

$$f(x+y) = m(x+y) + b = mx + my + b$$

$$f(x) + f(y) = mx + b + my + b$$

Thus $f(x+y) \neq f(x) + f(y)$.

c. I believe functions of the form $f(x) = mx + b$ are called linear functions because their graphs are lines.

31. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Apply the linear transformation T to both sides of the vector equation.

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0})$$

Using the properties of linear transformations:

① $T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)$

$= c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + c_3 T(\vec{v}_3)$

and

② $T(\vec{0}) = \vec{0}$

Thus

$$c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + c_3 T(\vec{v}_3) = \vec{0}$$

Since c_1, c_2, c_3 are not all zero, there is a dependency relation between $T(\vec{v}_1)$, $T(\vec{v}_2)$, and $T(\vec{v}_3)$.

Thus $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$ are linearly dependent.