Section 4.1

1. a. Let $\vec{u}, \vec{v}$ be vectors in V.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Since $\vec{u}$ and $\vec{v}$ are in V, $u_1 \geq 0$, $u_2 \geq 0$, $v_1 \geq 0$, $v_2 \geq 0$.

Thus $u_1 + v_1 \geq 0$ and $u_2 + v_2 \geq 0$.

So $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ is in V.

b. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = -1$.

Then $c\vec{u} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ which is not in V.

3. $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$

Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Both $\vec{u}$ and $\vec{v}$ are in H.

$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in H.

(your example may vary from mine.)

6. The zero polynomial is not of the form $a+t^2$ for some a in $\mathbb{R}$. Thus the set of all such polynomials is not a subspace of $\mathbb{P}_n$.

OR

The set of all polynomials of the form $a+t^2$ for some a in $\mathbb{R}$ is not closed under vector addition. For example.

$(1+t^2)+(2+t^2) = 3+2t^2$ which is not of the form $a+t^2$ for some a in $\mathbb{R}$.

7. The set of all polynomials of degree at most 3, with integer coefficients is not closed under scalar multiplication. For example,

$$\frac{1}{2}\left(t^3-t+1\right) = \frac{1}{2}t^3-\frac{1}{2}t+\frac{1}{2}$$

would not have integer coefficients.

8. Let $S = \{ p(t) \text{ in } \mathbb{P}_n : p(0) = 0 \}$.

(1) The zero polynomial in $\mathbb{P}_n$ is
$0t^n + 0t^{n-1} + \dots + 0t + 0$ or just 0.
S contains the zero polynomial, since
$p(t) = 0$ for all t.

(2) Let $p(t)$ and $q(t)$ be polynomials in S.

$(p+q)(t) = p(t) + q(t)$ and so

$(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$.

Thus $(p+q)(t)$ is in S.

(This can also be explained as the sum of polynomials with zero constant term, is a polynomial with zero constant term.)

(3) Let $p(t)$ be a polynomial in S, and c a constant in $\mathbb{R}$.

$(cp)(t) = c \cdot p(t)$ and so

$(cp)(0) = c \cdot p(0) = c \cdot 0 = 0$.

Thus $(cp)(t)$ is in S.

(This can also be explained as the product of a polynomial with zero constant term and a constant, is a polynomial with zero constant term.)

By the subspace theorem, S is a subspace of $\mathbb{P}_n$.

9. $H = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right\}$, This means H is a subspace by theorem 1.

11. If $\vec{x}$ is a vector in W then

$$\vec{x} = \begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 2c \\ 0 \\ c \end{bmatrix} = b\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

for some constants b, c.

Thus $W = \text{span}\left\{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

By Theorem 1, W is a subspace of $\mathbb{R}^3$.

13. a. No, $\vec{w}$ is not in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

There are three vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

b. There are an infinite number of vectors in Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

c. In other words, is $\vec{w}$ a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 1$

$x_2 = -2x_3 + 1$

x_3 is free

There is an infinite number of solutions, so $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$

Note $\vec{v}_1 + \vec{v}_2 = \vec{w}$.

Thus $\vec{w}$ is in the subspace spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

15. $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ Notice that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the set. Thus the set of vectors of the form $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ is not a subspace of $\mathbb{R}^3$.

17. $$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ -c \\ c \\ 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$ for some constants a, b, c

$$W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Thus W is a subspace of $\mathbb{R}^4$ by Theorem 1.

21. $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$

(1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in H (let a, b, d be all zero.)

(2) Let A, B be matrices in H.

$$A + B = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ 0 & a_3 + b_3 \end{bmatrix}$$

Thus A+B is in H.

(3) Let A be a matrix in H and c a real number.

$$cA = c \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} = \begin{bmatrix} ca_1 & ca_2 \\ 0 & ca_3 \end{bmatrix}$$

Thus cA is in H.

By the Subspace Theorem, H is a subspace of $M_{2\times 2}$.

26. a) Axiom 3 (vector addition is associative)

b) Axiom 5 ($\vec{u} + (-\vec{u}) = \vec{0}$)

c) Axiom 4 ($\vec{u} + \vec{0} = \vec{u}$)