Section 2.3

2. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$ The columns are linearly dependent. The second column is $-\frac{3}{2}$ of the first. The matrix is singular by the Invertible Matrix Theorem (IMT).

4. $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ The columns are linearly dependent. Any set of vectors containing the zero vector is linearly dependent.
The matrix is singular by IMT.

5. $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ The matrix has only 2 pivots. The matrix is singular by IMT.

11. a. True, by IMT parts b and d.
b. True, by IMT parts h and e.
c. False, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $A\vec{x} = \vec{b}$ has no solution
d. True, ~~If~~ $A\vec{x} = \vec{0}$ has a nontrivial solution, then A is not invertible by IMT, Since A is

not invertible, by IMT, A does not have n pivots. Since A is $n \times n$, A can have at most n pivots. Thus A has fewer than n pivots.

e. True A invertible implies A^T invertible by IMT. The contrapositive is logically equivalent. So, if A^T is not invertible, then A is not invertible.

13. A square upper triangular matrix is invertible if every diagonal entry is nonzero. If every diagonal entry is nonzero, then the matrix has a pivot in each column and row. Thus it is invertible by IMT.

Example: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & -5 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -5 \end{bmatrix}$

upper triangular invertible

upper triangular not invertible

15. A square matrix with two identical columns cannot be invertible. If two columns are identical, then the columns of the matrix are linearly dependent. The IMT says such a matrix is not invertible.

16. No, it is not possible for a 5×5 matrix to be invertible if its columns do not span $\mathbb{R}^5$. The IMT states that the columns of an invertible matrix span $\mathbb{R}^n$.

17. If A is invertible, then A^{-1} is invertible. (see Theorem 6a.) By the IMT, the columns of A^{-1} are linearly independent.

28. We will assume that A, B are square matrices. Since AB is invertible there exists a matrix W such that $W(AB) = I$ (by IMT part j). Thus $(WA)B = I$. Since B is square, IMT part j is satisfied and B is invertible.

31. Suppose A is an $n \times n$ matrix with the property that the equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$. By Theorem 4 in section 1.4, A has a pivot in each row. Since A is square $(n \times n)$, A has a pivot in each column. Thus there are no free variables in the system $A\vec{x} = \vec{b}$. Thus $A\vec{x} = \vec{b}$ has at most one solution, for each choice of $\vec{b}$. Therefore, $A\vec{x} = \vec{b}$ has exactly one solution for each choice of $\vec{b}$.

33. $T(x_1, x_2) = (-5x_1 + 9x_2, \; 4x_1 - 7x_2)$

By Theorem 9, T is invertible if and only the standard matrix for T is invertible. To find the standard matrix, find $T(1,0)$ and $T(0,1)$.

$T(1,0) = (-5, 4)$

$T(0,1) = (9, -7)$

$$A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$$

Since A is 2x2, we may use Theorem 4 in section 2.2.

$$A^{-1} = \frac{1}{35-36}\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$$

Thus T is invertible.

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

or

$$T^{-1}(x_1, x_2) = (7x_1 + 9x_2,\ 4x_1 + 5x_2)$$