Section 5.3

1. $A^4 = (PDP^{-1})^4 = (PD\underline{P^{-1})(P}D\underline{P^{-1})(P}D\underline{P^{-1})(P}DP^{-1})$

$= PD^4P^{-1}$ since $P^{-1}P = I$

$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ so $P^{-1} = \frac{1}{\det P}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$P^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

Thus $A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^4\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$

6. The diagonal entries of D are the eigenvalues of A. Thus the eigenvalues of A are $5, 5, 4$.

The columns of P are the eigenvectors of A, corresponding to the eigenvalues of A in the order of appearance.

A basis for the eigenspace corresponding to 5 is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

A basis for the eigenspace corresponding to 4 is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

7. $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ Since the matrix is lower triangular, the eigenvalues are $1, -1$.

To find the eigenspace corresponding to $\lambda = 1$, we find the nullspace of $A - I$.

$$A - I = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= \tfrac{1}{3} x_2 \\ x_2 &= x_2 \end{aligned} \qquad \vec{x} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, \quad x_2 \in \mathbb{R}$$

A basis for the eigenspace corr. to $\lambda = 1$

is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Next, we find the eigenspace corresponding to $\lambda = -1$.

$$A + I = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = 0 \\ x_2 = x_2 \end{matrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; x_2 \text{ in } \mathbb{R}$$

A basis for the eigenspace corr. to $\lambda = -1$

is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Finding P, D is sufficient. Remember, there are many correct answers.

9. $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ First let's find the characteristic polynomial

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda)-(-1)$$

$$= \lambda^2 - 8\lambda + 15 + 1$$

$$= \lambda^2 - 8\lambda + 16$$

$$= (\lambda-4)^2$$

So the eigenvalues are 4 with multiplicity 2.

To find eigenvectors, we find the eigenspace corresponding to each eigenvalue.

$$A - 4I = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \qquad \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x_2 \in \mathbb{R}.$$

So a basis for the $\lambda=4$ eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus the dimension of the $\lambda=4$ eigenspace is 1. By Theorem 7 part b, A is not diagonalizable.

12. $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ We are given that $\lambda = 2, 8$.

$\lambda = 2$ eigenspace:

$$A - 2I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned} \qquad \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\lambda = 2$ eigenspace.

$\lambda = 8$ eigenspace:

$$A - 8I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$
$x_2 = x_3$ $\quad \vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ x_3 \in \mathbb{R}$
$x_3 = x_3$

So a basis for the $\lambda = 8$ eigenspace is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (Note that we could have seen this without calculation since the row sums are equal to 8.)

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Remember, there are many solutions. As long as each column of P is an eigenvector corresponding to the diagonal entry in the corresponding column of D, it should be correct.

21. a. False: It doesn't specify that D is diagonal.

b. True: Theorem 5. A basis is a linearly independent set.

c. False: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has 2 eigenvalues, counting multiplicities, but it is not diagonalizable

d. False: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a diagonal matrix and thus diagonalizable. (It is similar to itself.) It is not invertible since its determinant is 0.

25. Since A is 4x4 with 3 eigenvalues, there are 3 eigenspaces. One eigenspace has dimension 2, one has dimension 1, and the remaining dimension is not given. Theorem 7, part a) states that the dimension of an eigenspace is at least 1. So the sum of the dimensions of the eigenspaces is at least 4. Since the dimension of an eigenspace is no more than the alg. multiplicity, the sum of the dimensions of the eigenspaces is at most 4. Thus by Theorem 7

part b, A is diagonalizable.

27. Since A is diagonalizable, there exists a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Since A is invertible, no eigenvalue of A is zero. Since the diagonal entries of D are the eigenvalues of A, the diagonal entries of D are nonzero. Thus $\det D \neq 0$, and D is invertible. Further, the inverse of a diagonal matrix is a diagonal matrix with the reciprocals of the diagonal entries of D.

i.e. if $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ then $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$

So $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$

$= P D^{-1} P^{-1}$

Thus A^{-1} is similar to a diagonal matrix, and thus diagonalizable.

31. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible since $\det A = 1$.

The dimension of the eigenspace corresponding to $\lambda = 1$ is 1, but the algebraic multiplicity of 1 is 2. By theorem 7 part b, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

$A - \lambda I$ for $\lambda = 1$

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = x_1 \\ x_2 = 0 \end{matrix} \quad \vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ x_1 \text{ in } \mathbb{R}$$

Thus a basis for the eigenspace is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the dimension is 1.