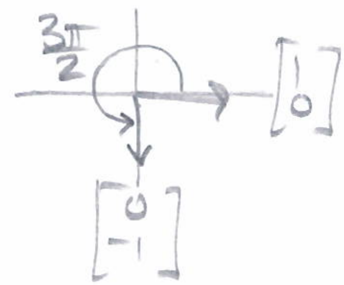
Section 1.9

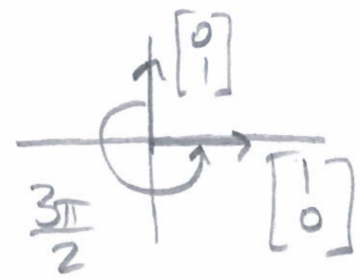
3. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotates points through $3\pi/2$ radians (counter clockwise).

The columns of the standard matrix for T are the images of the columns of identity matrix under T.

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$



$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$



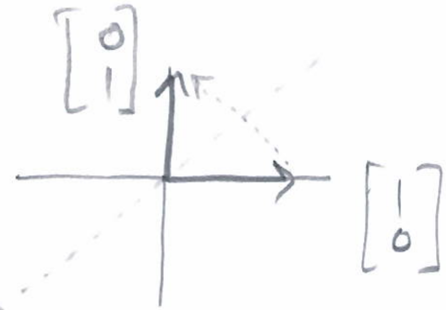
The standard matrix for T is $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$.

8. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$

Just as in problem 3, we need to find the images of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ under T.

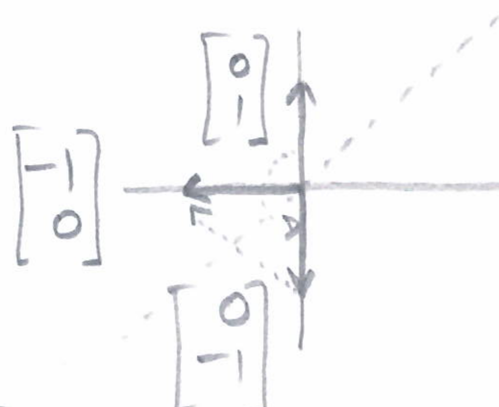
The images will be the columns of the standard matrix,

$$\boxed{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}}$$



The reflection of $\begin{bmatrix}1\\0\end{bmatrix}$ through the x_1-axis is itself. The reflection of $\begin{bmatrix}1\\0\end{bmatrix}$ through the line $x_2=x_1$ is $\begin{bmatrix}0\\1\end{bmatrix}$.

$$\boxed{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}}$$



The reflection of $\begin{bmatrix}0\\1\end{bmatrix}$ through the x_1-axis is $\begin{bmatrix}0\\-1\end{bmatrix}$. The reflection of $\begin{bmatrix}0\\-1\end{bmatrix}$ through the line $x_2=x_1$ is $\begin{bmatrix}-1\\0\end{bmatrix}$.

The standard matrix of T is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

16.

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$$

$$ax_1 + bx_2 = x_1 - x_2$$
$$cx_1 + dx_2 = -2x_1 + x_2$$
$$ex_1 + fx_2 = x_1$$

Thus the matrix is $\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}$

19. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

$T(1,0,0) = (1,0)$
$T(0,1,0) = (-5,1)$
$T(0,0,1) = (4,-6)$

These are the columns of the standard matrix

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$

27. The standard matrix for exercise 19 is $A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$

Notice there is a pivot in each row. By Theorem 4, the columns of A span $\mathbb{R}^2$. By Theorem 12, T maps $\mathbb{R}^3$ onto $\mathbb{R}^2$.

Since there are three columns, and each represents a vector in $\mathbb{R}^2$, Theorem 8 implies the columns of A form a linearly dependent set. By Theorem 12, T is not one-to-one.

29. $T: \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

Theorem 12 implies that if T is one-to-one, then the columns of the standard matrix are linearly independent.

Since T maps $\mathbb{R}^3$ to $\mathbb{R}^4$, the standard matrix is 4×3. Since the columns of the standard matrix are linearly independent, there is a pivot in each column.

The only echelon form possible is

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

31. T is one-to-one if and only if A has n pivot columns.

This follows from Theorem 12. T is one-to-one if and only if the columns of A are linearly independent. The columns of a matrix are linearly independent if and only if every column is a pivot column.

32. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has <u>m</u> pivot columns.

Theorem 12 part (a) states that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is onto if and only if the standard matrix for T has columns which span $\mathbb{R}^m$. Theorem 4 says that the columns of a matrix span $\mathbb{R}^m$ if and only if there is a pivot in each row. Since T maps $\mathbb{R}^n$ to $\mathbb{R}^m$, A is an $m \times n$ matrix. Thus A has m rows. If A has a pivot in each row, then it has m pivots, and m pivot columns.

35. By Theorem 12, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of its standard matrix span $\mathbb{R}^m$. Theorem 4 states that the columns of a matrix span $\mathbb{R}^m$ if and only if there is a pivot in each row. Thus such a matrix has at least as many columns as rows. In other words, $\underline{\underline{m \le n}}$.

By Theorem 12, T is one-to-one if and only if the columns of the standard matrix for T are linearly independent. The columns of a matrix are linearly independent if and only if each column contains a pivot. Thus such a matrix has at least as many rows as columns. In other words, $\underline{\underline{n \le m}}$.