Section 2.4

1. $\begin{bmatrix} I & 0 \\ E & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} IA+0C & IB+0D \\ EA+IC & EB+ID \end{bmatrix}$

I is an identity matrix and

0 is the zero matrix.

$$= \begin{bmatrix} A & B \\ EA+C & EB+D \end{bmatrix}$$

7. $\begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} XA & XZ \\ YA+B & YZ+I \end{bmatrix}$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

① $XA = I$

② $YA + B = 0$

③ $XZ = 0$

④ $YZ + I = I$

If A is square, then equation 1 implies that X and A are invertible.

Thus $XAA^{-1} = IA^{-1}$ and $X = A^{-1}$.

(See Invertible Matrix Theorem parts j, k.)

Since X is invertible and equation 3 states $XZ=0$, $X^{-1}XZ=X^{-1}0$. Simplifying $Z=0$, and Z is a zero matrix.

Using equation 2, $YA+B=0$, we see $YA=-B$. We assumed A was square and thus equation 1 implied A is invertible. Thus $YAA^{-1}=-BA^{-1}$, and $Y=-BA^{-1}$.

Therefore, $X=A^{-1}$, $Y=-BA^{-1}$, and $Z=0$.

10.

$$\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ C+Z & I & 0 \\ A+BZ+X & B+Y & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

So $C+Z=0$ ①

$A+BZ+X=0$ ②

$B+Y=0$ ③

Using equation 1, $Z = -C$

Using equation 3, $Y = -B$

Using equation 2, $X = -A - BZ$

$$= -A - B(-C)$$

$$= -A + BC$$

13. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ where B, C are square.

① Assume A is invertible. Show B, C are invertible.

Since A is invertible, it has an inverse A^{-1}.

Let $A^{-1} = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$ where A^{-1} is partitioned as A is partitioned.

$$AA^{-1} = I$$

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} BD & BE \\ CF & CG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Thus $BD = I$ and $CG = I$. Since A^{-1} was partitioned similarly to A, D and G are square. By IMT part k, B and C are invertible.

(2) Assume B, C are invertible. Show A is invertible

Let $K = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$

$$AK = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} BB^{-1} & 0 \\ 0 & CC^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$ which is the identity matrix.

Since B and C are square, K is a square matrix. By IMT part k, A is invertible.

16. Note that $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ and $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$

are invertible by IMT part c.

Thus $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1}$

The problem says to assume $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is invertible. Thus $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$ is a product of invertible matrices. By Theorem 6, part b, $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$ is invertible. Since A_{11} is invertible, it is a square matrix. Similarly A is a square matrix. Thus S is a square matrix. Using problem 13, S is invertible.

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17. $X_k = [\vec{x}_1 \ldots \vec{x}_k]$ $\quad$ $X_{k+1} = [\vec{x}_1 \ldots \vec{x}_k \, \vec{x}_{k+1}]$

$$G_k = X_k X_k^T = \vec{x}_1 \vec{x}_1^T + \vec{x}_2 \vec{x}_2^T + \ldots + \vec{x}_k \vec{x}_k^T$$

using column-row expansion

$$G_{k+1} = X_{k+1} X_{k+1}^T = \vec{x}_1 \vec{x}_1^T + \vec{x}_2 \vec{x}_2^T + \ldots + \vec{x}_k \vec{x}_k^T + \vec{x}_{k+1} \vec{x}_{k+1}^T$$

using column-row expansion

To update G_k to G_{k+1}, the matrix $\vec{X}_{k+1}\vec{X}_{k+1}^T$ must be added to G_k.

i.e. $G_{k+1} = G_k + \vec{X}_{k+1}\vec{X}_{k+1}^T$

21. a. $\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

so $A^2 = I$

b. $M = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$M^2 = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}\begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}$$

$$= \begin{bmatrix} A^2 & 0 \\ A-A & A^2 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_4$$