Section 6.3

1. $\vec{x} = \underbrace{c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3}_{\text{in Span}\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}} + c_4\vec{u}_4$

Using Theorem 8,

$$\vec{x} = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{x}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3 + \frac{\vec{x}\cdot\vec{u}_4}{\vec{u}_4\cdot\vec{u}_4}\vec{u}_4$$

$$= \frac{-16}{18}\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} + \frac{-8}{36}\begin{bmatrix}3\\5\\1\\1\end{bmatrix} + \frac{12}{18}\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} + \frac{72}{36}\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}$$

$$= \frac{-8}{9}\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} + \frac{-2}{9}\begin{bmatrix}3\\5\\1\\1\end{bmatrix} + \frac{6}{9}\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} + \begin{bmatrix}10\\-6\\-2\\2\end{bmatrix}$$

$$= \begin{bmatrix}0\\-2\\4\\-2\end{bmatrix} + \begin{bmatrix}10\\-6\\-2\\2\end{bmatrix}$$

in Span$\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ in Span$\{\vec{u}_4\}$

3. To show $\{\vec{u}_1, \vec{u}_2\}$ is orthogonal set, we check their dot product.

$$\vec{u}_1 \cdot \vec{u}_2 = 1(-1) + 1(1) = 0$$

Thus $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$

$$\text{proj}_W \vec{y} = \text{proj}_{\vec{u}_1} \vec{y} + \text{proj}_{\vec{u}_2} \vec{y} \quad \text{by Theorem 8}$$

$$= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

7. $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

First we find the projection of $\vec{y}$ onto W.

$$\text{proj}_W \vec{y} = \text{proj}_{\vec{u}_1} \vec{y} + \text{proj}_{\vec{u}_2} \vec{y}$$

$$= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

since $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for W.

$$= \frac{0}{14}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$$

Next we find a vector orthogonal to W.

$$\vec{y} - \text{proj}_W \vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$$

Thus $\vec{y} = \underbrace{\begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}}_{\text{in } W} + \underbrace{\begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}}_{\text{in } W^\perp}$

11. The closest point to $\vec{y}$ in W is the projection of $\vec{y}$ onto W.

Note $\vec{v}_1 \cdot \vec{v}_2 = 0$, so $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set of nonzero vectors. Thus $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and a spanning set for W. So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for W.

By Theorem 8,

$$\begin{aligned} \text{proj}_W \vec{y} &= \text{proj}_{\vec{v}_1} \vec{y} + \text{proj}_{\vec{v}_2} \vec{y} \\ &= \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 \\ &= \frac{6}{12} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

13. To find the best approximation to $\vec{z}$ by vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$, we find the projection of z onto Span $\{\vec{v}_1, \vec{v}_2\}$.

Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Note $\vec{v}_1 \cdot \vec{v}_2 = 2 - 1 + 0 - 1 = 0$

Thus $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set. Since it is a nonzero orthogonal set, it is linearly independent. Thus it is an orthogonal basis.

By Theorem 8,

$$\text{proj}_W \vec{z} = \frac{\vec{z} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{z} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \frac{10}{15}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} + \frac{-7}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$$

15. To find the distance from $\vec{y}$ to Span $\{\vec{u}_1, \vec{u}_2\}$ we first find the proj $\vec{y}$ onto Span $\{\vec{u}_1, \vec{u}_2\}$. We subtract the projection from $\vec{y}$ to get a vector orthogonal to Span $\{\vec{u}_1, \vec{u}_2\}$. The length of this vector is the distance from

$\vec{y}$ to the plane.

Note $\vec{u}_1 \cdot \vec{u}_2 = 0$, so $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set on nonzero vectors. Thus $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

$$\text{proj}_W \vec{y} = \text{proj}_{\vec{u}_1} \vec{y} + \text{proj}_{\vec{u}_2} \vec{y}$$

$$= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

$$= \frac{35}{35} \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} + \frac{-28}{14} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

$$\vec{y} - \text{proj}_W \vec{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

The length of $\vec{y} - \text{proj}_W \vec{y}$ is

$$\sqrt{2^2 + 0^2 + 6^2} = \boxed{\sqrt{40}}$$

19. We wish to construct a nonzero vector that is orthogonal to Span $\{\vec{u}_1, \vec{u}_2\}$. First we project $\vec{u}_3$ onto Span $\{\vec{u}_1, \vec{u}_2\}$. Then we find the difference of the projection and $\vec{u}_3$. This difference will be orthogonal to Span $\{\vec{u}_1, \vec{u}_2\}$.

Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$. Note that $\vec{u}_1 \cdot \vec{u}_2 = 0$ and thus form an orthogonal set of nonzero vectors. Thus $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for W.

$$\text{proj}_W \vec{u}_3 = \text{proj}_{\vec{u}_1} \vec{u}_3 + \text{proj}_{\vec{u}_2} \vec{u}_3$$

$$= \frac{\vec{u}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{u}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

$$= \frac{-2}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{2}{30}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix}$$

$$\vec{u}_3 - \text{proj}_W \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$

(or any multiple of it)

21. a. True. Any vector in W is of the form $c_1\vec{u}_1 + c_2\vec{u}_2$.

$$(c_1\vec{u}_1 + c_2\vec{u}_2)\cdot\vec{z} = c_1(\vec{u}_1\cdot\vec{z}) + c_2(\vec{u}_2\cdot\vec{z})$$
$$= c_1(0) + c_2(0) = 0$$

Thus $\vec{z}$ is orthogonal to all vectors in W, and thus in $W^\perp$.

b. True. Theorem 8

c. False. See Theorem 8. $\hat{y}$ is determined by the vector $\vec{y}$ and the subspace W, not by the basis for W.

d. True. See colored box on page 352.

e. True. Theorem 10