Math 341 - Linear Algebra
§1.1 - 1.5

Fall 2019

Success isn't the absence of failure,
but going from failure to failure without any loss of enthusiasm.

Contact information

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Office Hours: TTh 1-3pm or by appt through Calendly

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§1.1 Systems of Linear Equations

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Many real world applications will involve equations with hundreds or thousands of unknowns.

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$$\begin{aligned} 2x_1 + 2x_2 - 4x_3 =& 2 \\ 2x_1 - 2x_2 + 5x_3 =& 3. \end{aligned}$$

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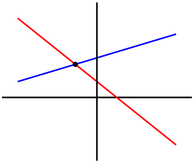

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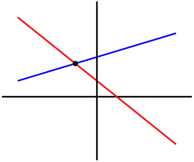

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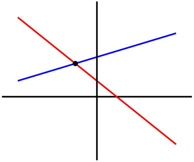

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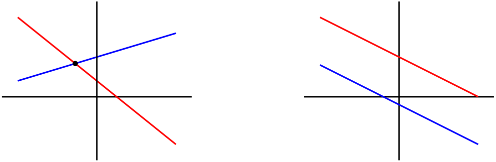

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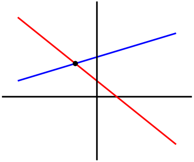

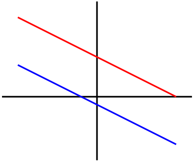

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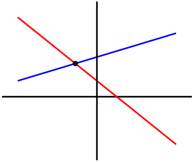

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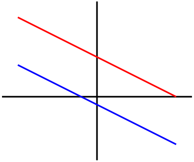

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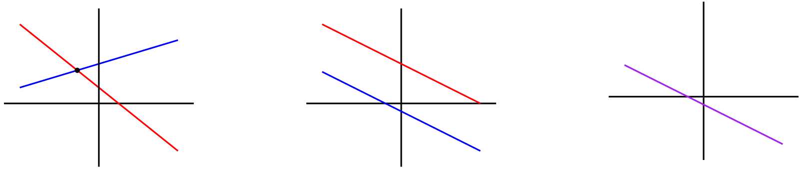

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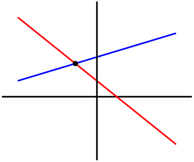

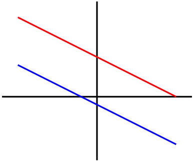

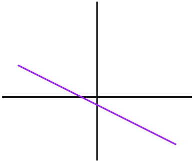

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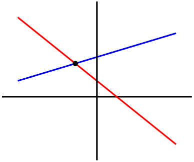

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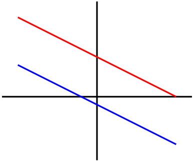

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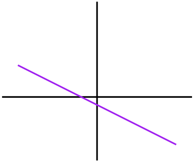

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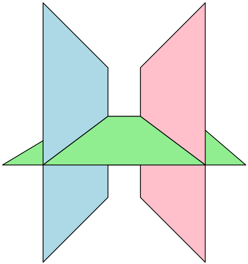

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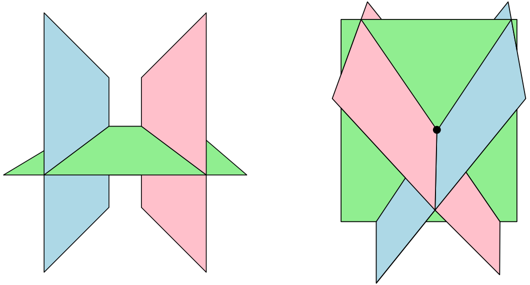

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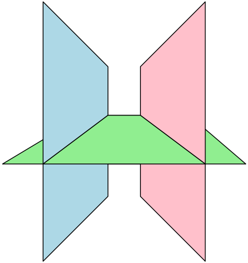
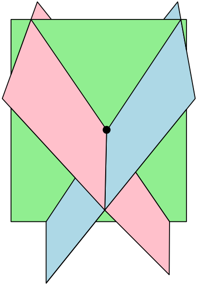
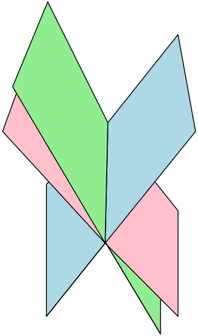

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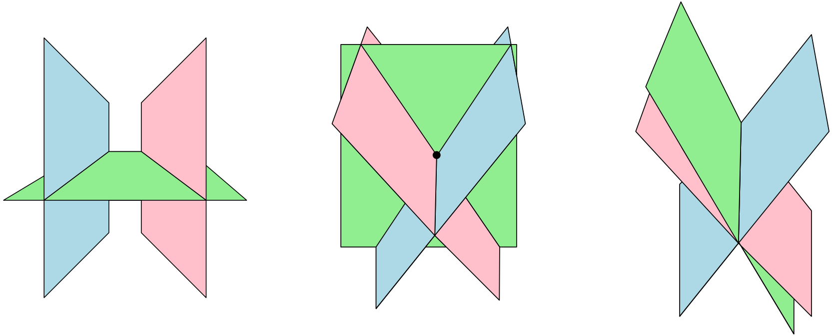

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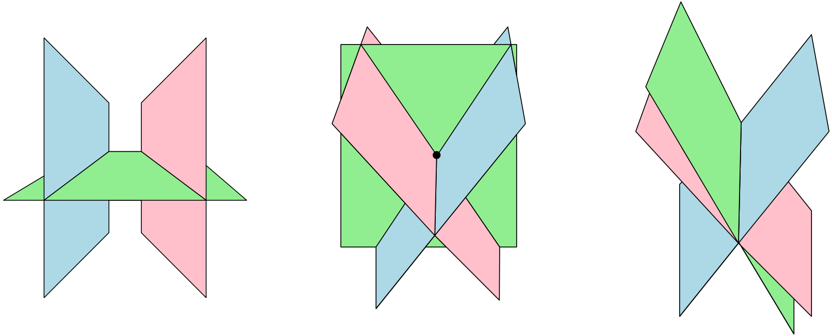

A system with at least one solution is called **consistent**. A system with no solutions is called **inconsistent**.

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1. does at least one solution exist (i.e. is the system consistent?)
2. if a solution exists, is there only one solution (i.e. is the solution unique?)

For a given system of equations

$$\begin{array}{rrrrrrrrr} & x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & -4 \\ & & & x_2 & - & 2x_3 & - & 2x_4 & = & 3 \\ - & x_1 & - & 3x_2 & + & 2x_3 & + & x_4 & = & 4 \end{array}$$

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$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 1 & -2 & -2 \\ -1 & -3 & 2 & 1 \end{bmatrix}$$

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or as an **augmented matrix**

$$\left[\begin{array}{rrrr|r} 1 & 3 & 2 & 3 & -4 \\ 0 & 1 & -2 & -2 & 3 \\ -1 & -3 & 2 & 1 & 4 \end{array}\right].$$

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A matrix with m rows and n columns is called an $\mathbf{m} \times \mathbf{n}$ **matrix**.

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Theorem

Elementary row operations do not change the solution set of the associated system of equations (row equivalent augmented matrices have the same solution sets).

Example: Is the following system consistent?

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If so, find a solution. How many solutions does it have?

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5. Each leading 1 is the only nonzero entry in its column.

§1.2 Row Reduction and Echelon Forms

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NOTE: To locate the pivot positions of a matrix A, we need to use elementary row operations to put A in RREF. The positions (columns) of the leading ones in the RREF tells us the locations of the pivots (pivot columns) in the original matrix A.

Example: Find the pivot positions and columns of the matrix

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ -6 & -4 & 3 & 1 \\ 9 & 6 & -2 & -1 \end{bmatrix}.$$

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5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Example: Use the row reduction algorithm to find the RREF matrix which is row equivalent to

$$\begin{bmatrix} 0 & 3 & -6 & 4 & -5 \\ 3 & -7 & 8 & 8 & 9 \\ 3 & -9 & 12 & 6 & 15 \end{bmatrix}.$$

Example: Find solutions to the following system

$$\begin{array}{rcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 4x_4 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & + & 8x_4 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & + & 6x_4 & = & 15 \end{array}.$$

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Hint: The augmented matrix $\left[\begin{array}{rrrr|r} 0 & 3 & -6 & 4 & -5 \\ 3 & -7 & 8 & 8 & 9 \\ 3 & -9 & 12 & 6 & 15 \end{array}\right]$ row reduces to

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & -24 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

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Different choices of values for the free variables give different solutions.

Example: Find the solutions to the following system

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & -1 \\ 2x & + & 2y & + & 6z & = & 4 \\ 3x & + & 2y & + & 1z & = & 1 \end{array} .$$

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A linear system is consistent if and only if the right-most column of the augmented matrix is *not* a pivot column i.e. if and only if an echelon form of the augmented matrix has no row of the form

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If the system is consistent, then it either has a unique solution if it has no free variables, or an infinite number of solutions if it has at least one free variable.

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Warning: We don't multiply vectors together (though we will in the future).

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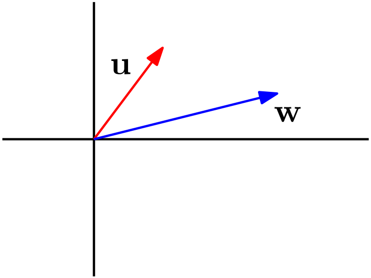
Geometric view of vector addition:



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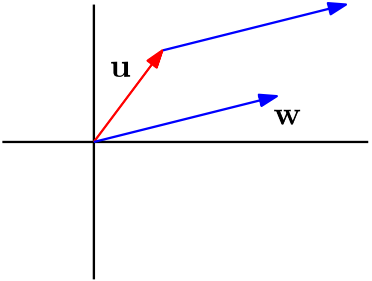

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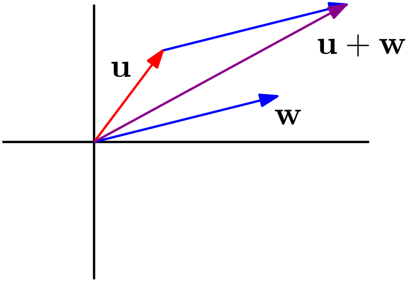

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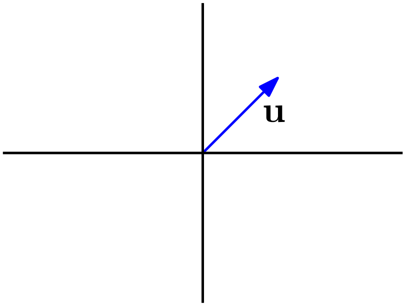

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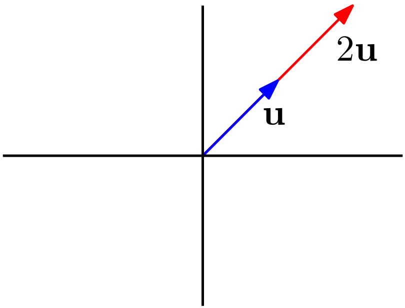

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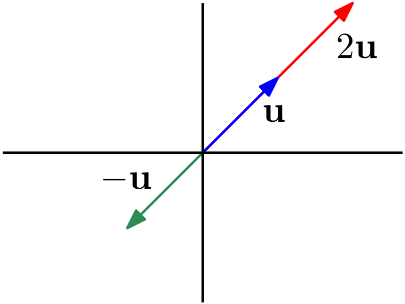

§1.3 Vector Equations

A vector in $\mathbb{R}^n$ is an ordered list of n real numbers, i.e.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Let

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

denote the zero vector in $\mathbb{R}^n$.

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§1.3 Vector Equations

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Definition (Linear Combinations)

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$, and scalars $c_1, \ldots, c_p$, a vector of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$.

Examples: The vectors

$$\sqrt{2}\cdot\mathbf{v}_1 + 3\cdot\mathbf{v}_2 - \mathbf{v}_3 \qquad \mathbf{v}_1 = 1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + 0\cdot\mathbf{v}_3 \qquad \mathbf{0} = 0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + 0\cdot\mathbf{v}_3,$$

are all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

Example: Can the vector $\mathbf{y} = \begin{bmatrix} 8 \\ -9 \\ 2 \end{bmatrix}$ be written as a linear combination of the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix}?$$

§1.3 Vector Equations

A vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has the same solution set as the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p & \mathbf{b} \end{array}\right].$$

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$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p & \mathbf{b} \end{array}\right].$$

Thus $\mathbf{b}$ can be written as a linear combination of the $\mathbf{v}_j$ if and only if the system of linear equations corresponding to the above augmented matrix is consistent.

Definition (Span of vectors)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$.

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Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. Then the **set spanned by the vectors** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, which we denote by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, is the set of all vectors which can be written as linear combinations of the $\mathbf{v}_j$.

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In other words Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the set of all vectors that can be written as

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

for some weights $c_1, \ldots, c_p$.

§1.3 Vector Equations

Example: Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in $\mathbb{R}^n$.

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Is $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix}$ in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

§1.3 Vector Equations

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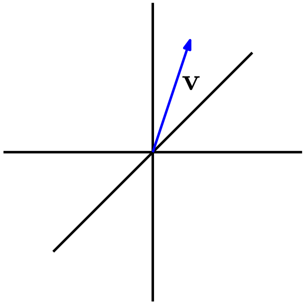

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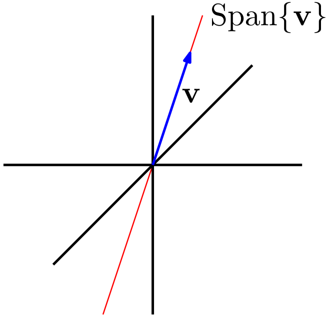

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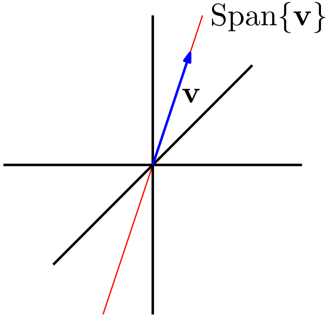

This is the line passing through the origin $\mathbf{0}$ and the vector $\mathbf{v}$.

For nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, with $\mathbf{v}$ not a multiple of $\mathbf{w}$, Span $\{\mathbf{v}, \mathbf{w}\}$ is the unique plane in $\mathbb{R}^3$ containing the vectors $\mathbf{v}, \mathbf{w}$, and $\mathbf{0}$.

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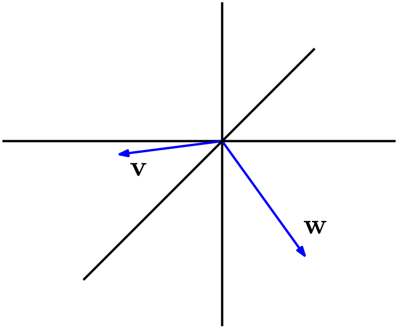

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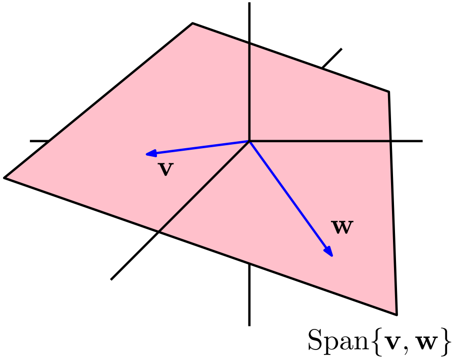

§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is a vector in $\mathbb{R}^n$,

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$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

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Note: $m \times n$ matrices can only by multiplied with vectors in $\mathbb{R}^n$.

Example: Find the product of the matrix

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 0 \\ 8 & 1 & -3 \\ 2 & -2 & 1 \end{bmatrix}$$

with the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Example: Write the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 4 \\ 3x_1 & - & 4x_2 & - & 2x_3 & = & -8 \\ x_1 & + & x_2 & + & 5x_3 & = & 0 \end{array}$$

as a vector equation, and as a matrix equation, then solve.

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Theorem

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

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If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

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$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

which has the same solution set as the system of equations whose augmented matrix is

$$\left[\begin{array}{ccc|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array}\right].$$

§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Fact: The matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

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Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

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Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

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Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

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1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

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3. The columns of A span $\mathbb{R}^m$.

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4. A has a pivot position in every row.

Warning! This theorem is about a *coefficient matrix*, not an *augmented matrix*. If an augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.

Row-Vector Rule for Computing $A\mathbf{x}$

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Example: Find the product of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix},$$

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Note: The jth entry of $A\mathbf{x}$ is just the sum of the products of the entries of the jth row of A with the corresponding entries of $\mathbf{x}$.

§1.5 Solution Sets of Linear Systems

Definition (Homogeneous System)

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

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Fact

A homogeneous system ($A\mathbf{x} = \mathbf{0}$) always has at least one solution :

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A homogeneous system ($A\mathbf{x} = \mathbf{0}$) always has at least one solution :

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This is called the **trivial solution**.

§1.5 Solution Sets of Linear Systems

Question

For a matrix A, is there a **nontrivial solution** to $A\mathbf{x} = \mathbf{0}$?

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$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

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Question

For a matrix A, is there a **nontrivial solution** to $A\mathbf{x} = \mathbf{0}$?

Fact

$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example: Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Example: Find a description of the solution set of

$$2x_1 + x_2 - x_3 = 0.$$

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Note: Any solution set (of a homogeneous equation) can be expressed as Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

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Definition

Expressing a solution set as $t\mathbf{v}_1 + s\mathbf{v}_2$, where t and s can be any real number (as above), is called **parametric vector form**.

Theorem

Suppose that $A\mathbf{x} = \mathbf{b}$ is consistent for some fixed $\mathbf{b}$, and let $\mathbf{p}$ be the solution, i.e. $A\mathbf{p} = \mathbf{b}$.

§1.5 Solution Sets of Linear Systems

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Suppose that $A\mathbf{x} = \mathbf{b}$ is consistent for some fixed $\mathbf{b}$, and let $\mathbf{p}$ be the solution, i.e. $A\mathbf{p} = \mathbf{b}$.

Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

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Warning! This theorem only applies to systems $A\mathbf{x} = \mathbf{b}$ that are consistent.

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Warning! This theorem only applies to systems $A\mathbf{x} = \mathbf{b}$ that are consistent. If the system is inconsistent then the solution set is empty.

§1.5 Solution Sets of Linear Systems

All solutions to $A\mathbf{x} = \mathbf{0}$

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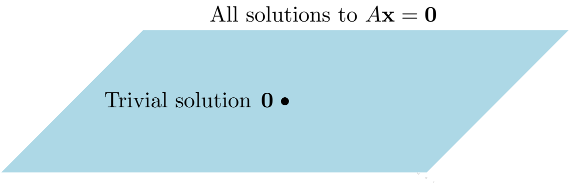

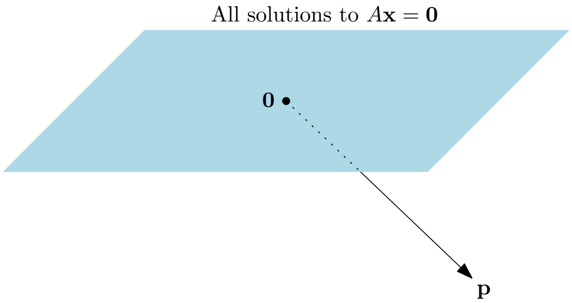
All solutions to $A\mathbf{x} = \mathbf{0}$
$\mathbf{0}$
$\mathbf{p}$

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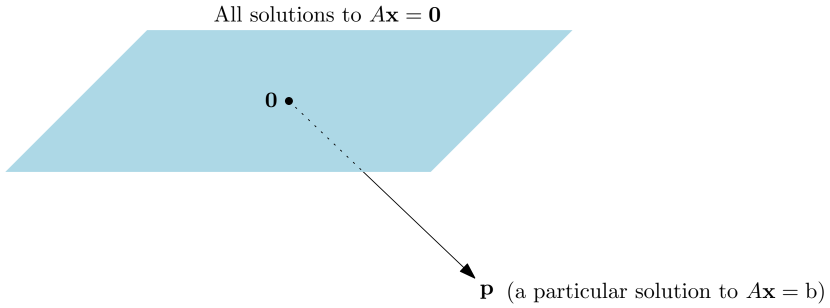

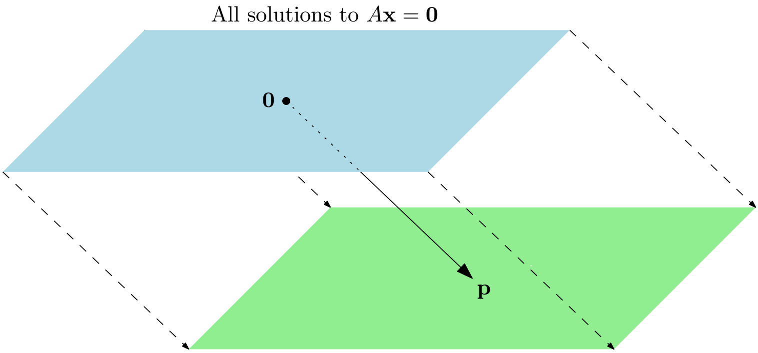
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$\mathbf{0}$
$\mathbf{p}$

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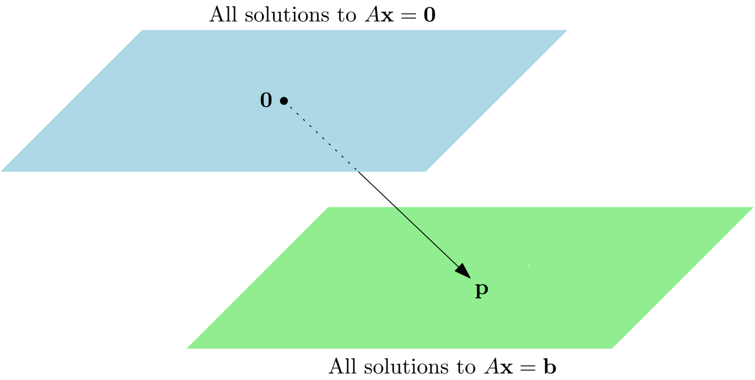

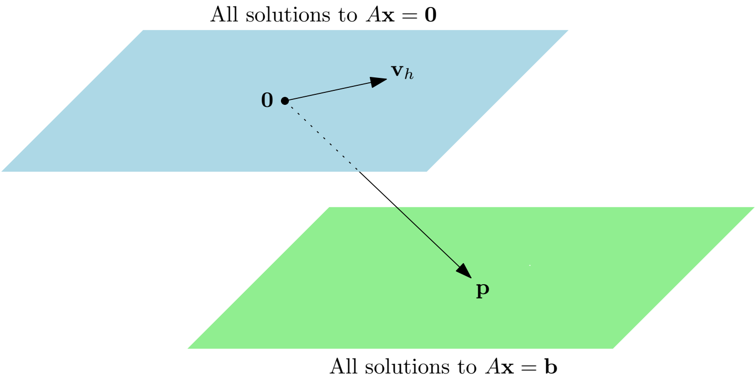
All solutions to $A\mathbf{x} = \mathbf{0}$
$\mathbf{v}_h$
$\mathbf{0}$
$\mathbf{p}$
All solutions to $A\mathbf{x} = \mathbf{b}$

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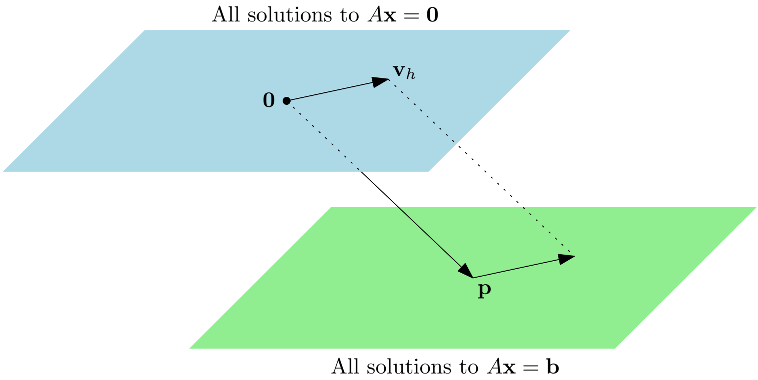
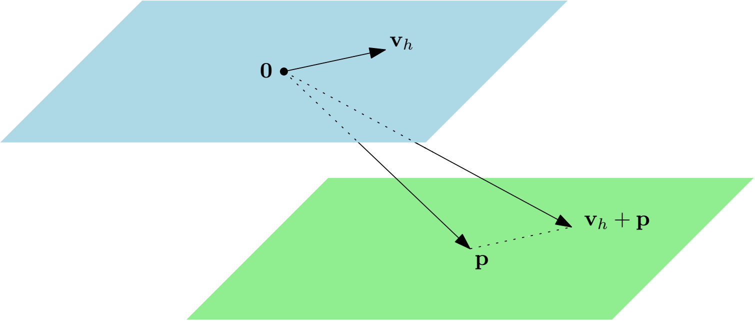


All solutions to $A\mathbf{x} = \mathbf{0}$



All solutions to $A\mathbf{x} = \mathbf{b}$

§1.5 Solution Sets of Linear Systems

Example: Find all solutions to the system

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4. \end{array}$$

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Writing solution sets in parametric vector form

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3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.