Math 341 - Linear Algebra
§4.1 - 4.6

Fall 2019

Pure mathematics is, in its way, the poetry of logical ideas.
- Albert Einstein

§1.3 Vector Equations

Theorem (Algebraic Properties of $\mathbb{R}^n$)

For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c, d in $\mathbb{R}$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $-1 \cdot \mathbf{u}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
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Much of our intuition about $\mathbb{R}^n$ will carry over to other abstract vector spaces (i.e. we can picture general vector spaces much as we visualize $\mathbb{R}^n$).

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It shouldn't be too hard to see, that in some sense $M_{m\times n}$ is essentially the same vector space as $\mathbb{R}^{m\times n}$ (we'll make this precise later).

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Is M_n a vector space with addition defined by $\oplus$, and the usual scalar multiplication?

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$$\begin{aligned} \mathbf{p} + \mathbf{q} &= (a_0 + a_1 t + \cdots a_n t^n) + (b_0 + b_1 t + \cdots b_n t^n) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots (a_n + b_n)t^n \end{aligned}$$

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Verify the axioms.

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$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots).$$

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To tell if a subset of V is a subspace, it is enough to check the above three conditions (all three must hold, if a single property fails then W is *not* a subspace of V).

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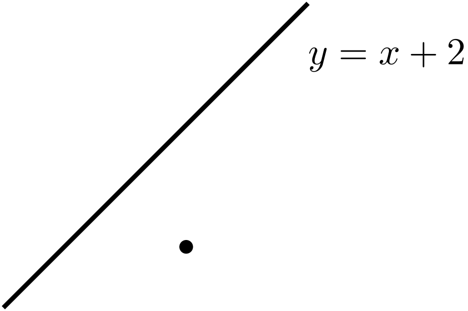

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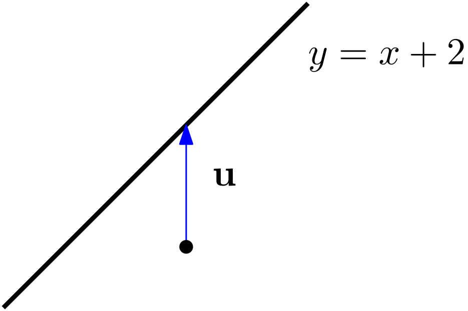

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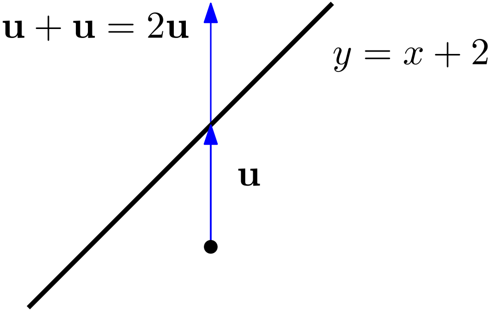

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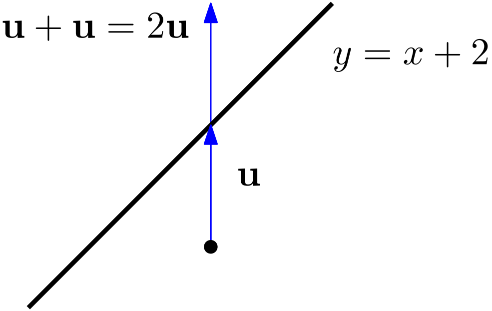

Since L doesn't contain $\mathbf{0}$, and isn't closed under scalar multiplication and vector addition it isn't a subspace of $\mathbb{R}^2$.

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Example: Let $H \subset C([0,1])$ be the subset of all differentiable functions f on $[0,1]$ with $f'(0.5) = 0$. Is H a subspace?

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$\square$

Example: Does the set H form a subspace?

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and thus by the theorem, H is a subspace.

§4.2 Null Spaces, Column Spaces & Lin. Trans.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

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Theorem

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

In other words, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Finding an explicit description of Nul A:

We can find an explicit description of Nul A by solving the homogenous system $A\mathbf{x} = \mathbf{0}$ and expressing the solution in parametric form.

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- The number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

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The **column space** of an $m \times n$ matrix A, written as Col A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$, then

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THEOREM

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Another way of describing Col A is:

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Warning! If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$, while Nul A is a subspace of $\mathbb{R}^n$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 0 \\ 2 & 4 & 9 & 1 \end{bmatrix}.$$

Find Nul A and Col A.

Example (cont.)

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$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 0 \\ 2 & 4 & 9 & 2 \end{bmatrix}$$

1. Determine if $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$ is in Nul A. Could it also be in Col A?
2. Determine if $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is in Col A. Could it also be in Nul A?

§4.2 Null Spaces, Column Spaces & Lin. Trans.

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4. No clear relation between Nul A and entries of A.	4. Obvious relation between Col A and entries of A.
5. If $\mathbf{v}$ in Nul A, then $A\mathbf{v} = \mathbf{0}$.	5. If $\mathbf{v}$ in Col A, then $A\mathbf{x} = \mathbf{v}$ is consistent.

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8. Nul $A = \{\mathbf{0}\}$ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ is onto $\mathbb{R}^m$.

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Warning! Unlike linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$, linear transformation on arbitrary vector spaces may *not* be able to be expressed in terms of a standard matrix.

Theorem

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Find the kernel (or null space) and range of D.

§4.3 Linearly Independent Sets; Bases

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

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The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there is a nontrivial solution to the equation above. (This means there must be some weights, $c_1, \ldots, c_p$ not all zero, which make the equation hold).

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In such a case, the equation above is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

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An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 0$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Example: Let $\mathbf{p}_1 = 1 - t, \mathbf{p}_2 = t + t^2$, and $\mathbf{p}_3 = 2 - t + t^2$ be vectors in $\mathbb{P}_3$ (the set of all polynomials of degree ≤ 3).

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This definition also applies to the whole space V; a basis of V is a linearly independent subset of V which spans V.

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Example: The set $\{1, t, t^2, \ldots, t^n\}$ forms a basis of $\mathbb{P}_n$, and is called the **standard basis** for $\mathbb{P}_n$.

Theorem (Spanning Set Theorem)

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and $H = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

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1. If one of the vectors in S – say, $\mathbf{v}_k$ – is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
2. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

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Example: Find a basis for Col B, where

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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The pivot columns of a matrix A from a basis for Col A.

Warning! You have to row reduce the matrix to find the pivot columns, but the basis for Col A are the original pivot columns in A, *not* the row reduced matrix.

Example: Find a basis for Col A, where

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

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Hint: A is row-equivalent to the matrix B from the previous slide.

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Theorem (Unique Representation Theorem)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

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$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

We call the weights $c_1, \ldots, c_n$ the **coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of $\mathbf{x}$**).

If $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of $\mathbf{x}$ (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of $\mathbf{x}$**.

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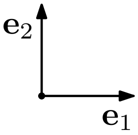
The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the **coordinate mapping (determined by $\mathcal{B}$).**

§4.4 Coordinate Systems



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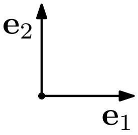

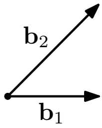

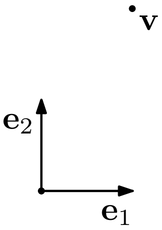
v
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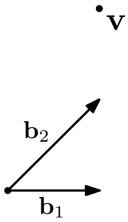
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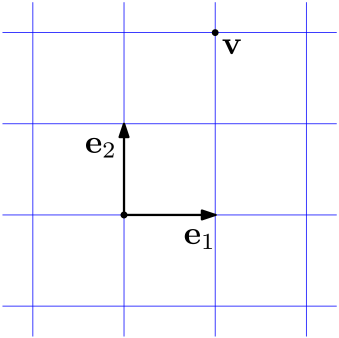

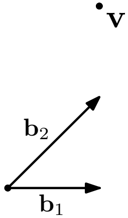

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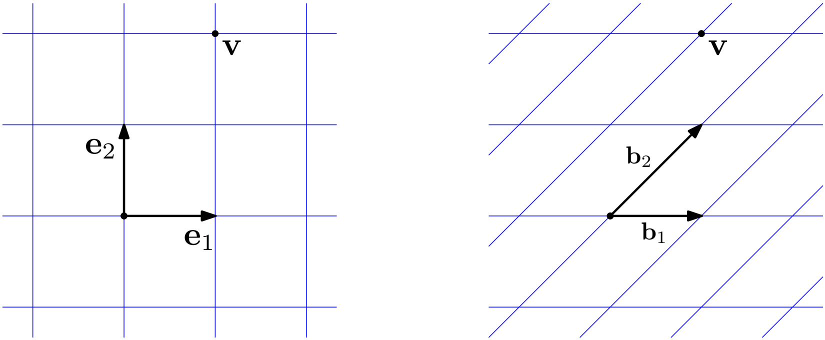

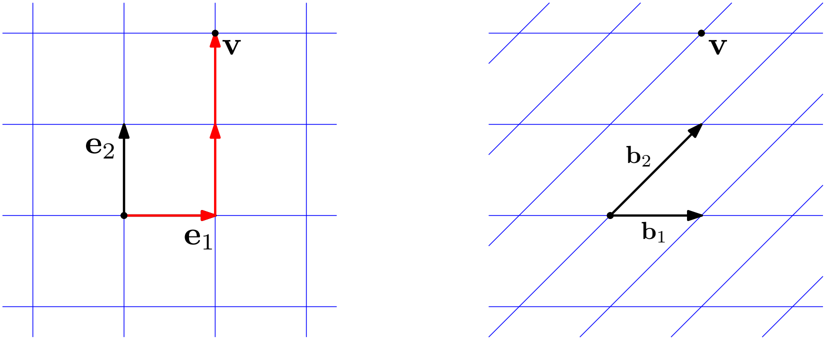
$\mathbf{v}$
$\mathbf{e}_2$
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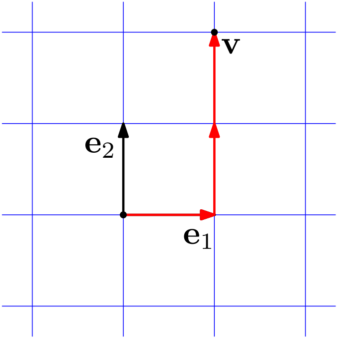
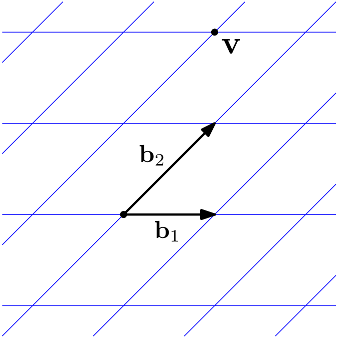
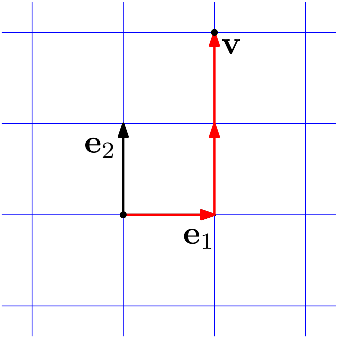
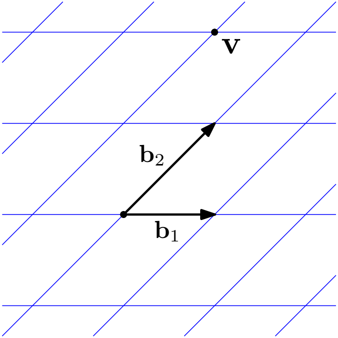




$$\mathbf{v} = 1 \cdot \mathbf{e}_1 + 2 \cdot \mathbf{e}_2$$





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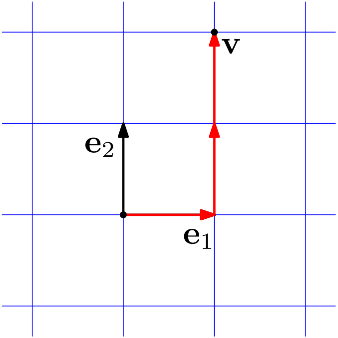
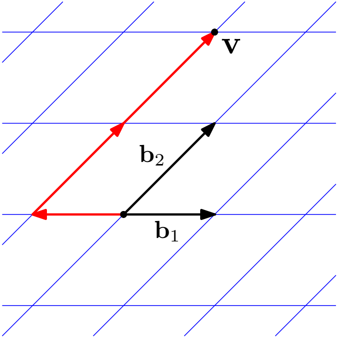
$$\Rightarrow \quad [\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$





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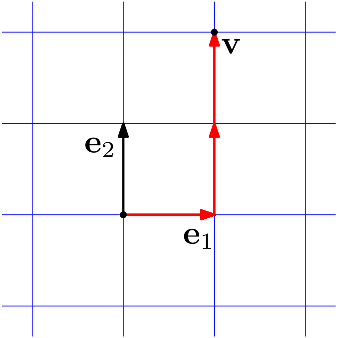

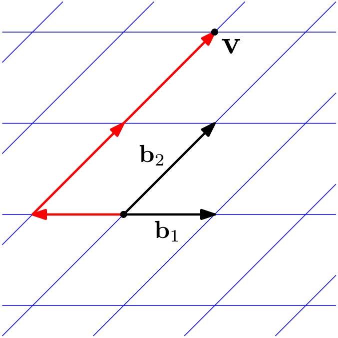

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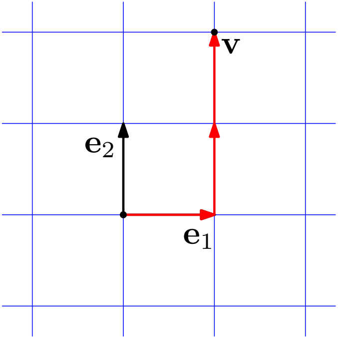

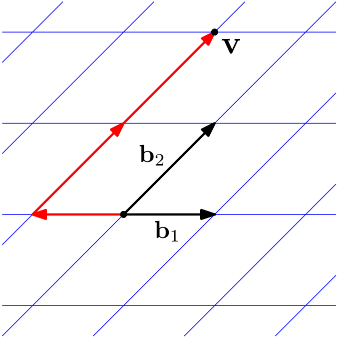

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For the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$$

find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{E}$.

Example: Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a the basis of $\mathbb{R}^3$ given by

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

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find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis, and suppose

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for some $\mathbf{x} \in \mathbb{R}^n$.

§4.4 Coordinate Systems

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis, and suppose

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We call $P_{\mathcal{B}}$ the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

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Find the change of coordinate matrix $\mathcal{P}_{\mathcal{B}}$, and use it to find the coordinate vector of $\mathbf{x}$ relative to the basis $\mathcal{B}$.

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one-to-one** linear transformation from V **onto** $\mathbb{R}^n$.

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A **one-to-one** linear transformation from V **onto** W is called an **isomorphism**.

In this case, even though the vector spaces V and W may look different, they are essentially the same.

Thus if a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ then it is essentially "the same" as $\mathbb{R}^n$.

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$$\mathbf{v} + \mathbf{w} = (v_1 + w_1)\mathbf{b}_1 + \cdots + (v_n + w_n)\mathbf{b}_n$$

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Thus $T_{\mathcal{B}}$ is linear.

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and hence $T_{\mathcal{B}}$ is onto. $\square$

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Caution: Not every vector space has a basis of the form $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. For example, the space $C(\mathbb{R})$ has no such basis. It does have a basis with infinitely many basis vectors though.

Example: Using the coordinate vectors of the polynomials $\mathbf{p}_1 = t^2 + t + 1, \mathbf{p}_2 = t^2 + 2t$, and $\mathbf{p}_3 = t^2 - 2$, determine whether they are linearly independent.

Theorem

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any subset of V containing more than n vectors must be linearly dependent.

§4.5 The Dimension of a Vector Space

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Theorem

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Definition

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V.

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The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

If V is not spanned by a finite set, then V is said to be **infinite dimensional**.

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Example: Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} \quad : \quad a, b, c, d \in \mathbb{R} \right\}.$$

Theorem

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Theorem (The Basis Theorem)

Let V be a p-dimensional vector space, for $p \geq 1$.

§4.5 The Dimension of a Vector Space

Theorem

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also H is finite-dimensional and

$$\dim H \leq \dim V.$$

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Example: Find the dimension of Nul A and Col A, where

$$A = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 3 & 3 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}.$$

§4.6 Rank

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Theorem

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Example: Find a basis for Row A, Col A, and Nul A, where

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 \\ 1 & 3 & -5 & 1 \\ 3 & 11 & -19 & 7 \end{bmatrix}.$$

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$$\text{rank } A + \dim \text{Nul } A = n.$$

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Example: A scientist has found two solutions to a homogeneous system of linear 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated non homogeneous system (with the same coefficients) has a solution?

Theorem (The Invertible Matrix Theorem (continued))

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

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