Section 2.1

1. $-2A = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}$ $B-2A = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$

$$AC = \underset{2\times 3}{\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}$$

: Matrix multiplication not defined. A has more columns than B has rows.

$$CD = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot -1 & 1\cdot 5 + 2\cdot 4 \\ -2\cdot 3 + 1\cdot -1 & -2\cdot 5 + 1\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$$

3. $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$ $3I_2 - A = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$$

$$(3I_2)A = 3(I_2A) = 3A = 3\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}$$

6. $$\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = AB$$

$$A\vec{b}_1 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$$

So $AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

Using row-column rule:

$$\begin{bmatrix} 4\cdot 1 + -2\cdot 2 & 4\cdot 3 + -2\cdot -1 \\ -3\cdot 1 + 0\cdot 2 & -3\cdot 3 + 0\cdot -1 \\ 3\cdot 1 + 5\cdot 2 & 3\cdot 3 + 5\cdot -1 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$$

7. $\underset{5\times 3}{A}\ \underset{3\times 7}{B} = \underset{5\times 7}{AB}$ B must have the same # of rows as the columns of A.

B must have the same # of columns as the columns of AB.

Thus B is 3×7 matrix

10. $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

So $AB = AC$, but $B \neq C$.

11. $AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$

$$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$$

When A is multiplied on the right of D, the diagonal entries of D scale the columns of A. When A is multiplied on the left of D, the diagonal entries of D scale the rows of A.

Let $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $B \neq I_3$
B is not the zero matrix.

$$AB = A(2I_3) = 2(AI_3) = 2A = 2(I_3 A)$$
$$= (2I_3)A = BA.$$

Thus $AB = BA$.

(In fact any matrix cI commutes with every matrix.)

12. We want to construct B so that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Remember from the definition of matrix multiplication, that $AB = A[\vec{b}_1\ \vec{b}_2] = [A\vec{b}_1\ A\vec{b}_2]$

Since $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we want to find $\vec{b}_1, \vec{b}_2$ so that $A\vec{b}_1 = \vec{0}$ and $A\vec{b}_2 = \vec{0}$. Thus any vectors that are solutions to the homogeneous system $A\vec{x} = \vec{0}$ will work.

$$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x_2 \in \mathbb{R}$

Since we want B to have distinct columns, choose distinct values for x_2. Let $x_2 = 1$ and $x_2 = -1$, then

$$B = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$ (Your answers will vary)

16. a. False $AB = [A\vec{b}_1 \ A\vec{b}_2 \ A\vec{b}_3]$

b. True The second row of AB is equal to the second column of $(AB)^T$. $(AB)^T = B^TA^T$. The second column of B^TA^T is equal to B^T times the second column of A^T. The second column of A^T is the second row of A.

c. False $BC \neq CB$ in general.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$(AB)C = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$

$(AC)B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

d. False $(AB)^T = B^TA^T$ by Theorem 3 d

e. True $(A+B)^T = A^T + B^T$ by Theorem 3 b.

22. Let B have linearly dependent columns. Since the columns of B are linearly dependent there exists a nontrivial solution to $B\vec{x} = \vec{0}$. (i.e. there exists a vector $\vec{u}$, $\vec{u} \neq \vec{0}$ such that $B\vec{u} = \vec{0}$)

Then $$A(B\vec{u}) = A(\vec{0}) = \vec{0}.$$

So $$(AB)\vec{u} = \vec{0}.$$

Thus $\vec{u}$ is a non-trivial solution to $(AB)\vec{x} = \vec{0}$. Therefore, the columns of AB are linearly dependent.

23. Assume $CA = I_n$. Then

$$(CA)\vec{x} = I_n\vec{x} = \vec{x} \text{ for every } \vec{x}.$$

Suppose $A\vec{x} = \vec{0}$ has a nontrivial solution $\vec{v}$. In other words, $A\vec{v} = \vec{0}$ and $\vec{v} \neq \vec{0}$.

Then $(CA)\vec{v} = C(A\vec{v}) = C\vec{0} = \vec{0} \neq \vec{v}$

This contradicts that $(CA)\vec{x} = \vec{x}$ for every vector $\vec{x}$. Thus $A\vec{x} = \vec{0}$ has only the trivial solution.

A cannot have more columns than rows because if it did, the columns of A would be linearly dependent; and if the columns of A are linearly dependent, then $A\vec{x} = \vec{0}$ has a non-trivial solution.

27. $\vec{u}^T\vec{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a + 3b - 4c$

$\vec{v}^T\vec{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = -2a + 3b - 4c$

$$\vec{u}\vec{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$\vec{v}\vec{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$