Section 4.5

1. $\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\} = H$

$$\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$H = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Is the spanning set linearly independent?
Neither is a multiple of the other, so it is linearly independent.

Thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for H.

and $\dim H = 2$.

10. $H = \text{Span}\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$

Let $A = \begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix}$. Thus $H = \text{Col}\,A$.

$$\begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -3 \\ 0 & 0 & -3/2 \end{bmatrix}$$

Thus a basis for Col A is $\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$

Since $H = \text{Col } A$, the set is a basis for H as well.

Thus $\dim H = 2$. Since H is a subspace of $\mathbb{R}^2$ and $\mathbb{R}^2$ has dimension 2,

$H = \mathbb{R}^2$.

11. $H = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix} \right\}$

We have a spanning set for H. We need to determine which vectors are linear combinations of the others.

Let $A = \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix}$. Then $H = \text{Col } A$.

$$A \sim \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & -5 & -20 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$

Since Col A = H, dim H = 2, and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for H.

13. $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since there are 3 pivot columns, dim Col A = 3

Since there are 2 free variables dim Nul A = 2.

14. $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since there are 3 pivot columns dim Col A = 3.

Since there are 3 free variables dim Nul A = 3.

17. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$ Since there are 3 pivot columns, dim Col $A=3$

Since there are no freevariables dim Nul $A = 0$

21. $\{1, 2t, -2+4t^2, -12t+8t^3\} = S$

Let $\mathcal{B} = \{1, t, t^2, t^3\}$, the standard basis for $\mathbb{P}_3$.

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [2t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$[-2+4t^2] = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad [-12t+8t^3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

Using the coordinate map, the Hermite polynomials are a basis for $\mathbb{P}_3$ if and only if the four vectors in $\mathbb{R}^4$ are a basis for $\mathbb{R}^4$.

Let $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$. Since A is upper triangular, det $A = 64$.

Thus A is invertible. By the invertible matrix theorem, the columns of A span $\mathbb{R}^4$ and are linearly independent.

Therefore, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix} \right\}$ are a basis for $\mathbb{R}^4$, and the Hermite polynomials are a basis for $\mathbb{P}_3$.

28. The set of all continuous functions, $C(\mathbb{R})$, on the real line contains the vector space $\mathbb{P}$ of all polynomials. (All polynomials are continuous functions.) For each value of n, $\mathbb{P}_n$ is a subset of $\mathbb{P}$.

Since $\dim \mathbb{P}_n = n+1$, and $\dim \mathbb{P}_n \leq \dim \mathbb{P}$, the dimension of $\mathbb{P}$ is not finite. Thus $\mathbb{P}$ is infinite dimensional, and $C(\mathbb{R})$ is infinite dimensional.

29. a. True Use Theorem 5b.

b. True Use Theorem 11

c. True Since $\dim V = p$, there is a basis of size p. Thus V has a spanning set of size p. Add the zero vector to this spanning set, and we obtain a spanning set for V with $p+1$ vectors.

30. a. False $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ is a linearly dependent set of size 2, but

$$\dim \mathbb{R}^3 = 3$$

b. True Every basis is a spanning set for the vector space. If $\dim V \le p$, then there is a spanning set of size p.

c. False Let $V = \mathbb{R}^3$, so $\dim \mathbb{R}^3 = 3$.

The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ is linearly dependent.