Section 1.4

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$ The product is not defined. The vector $\vec{x}$ should be in $\mathbb{R}^2$ since the matrix A has 2 columns.

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} + (-3)\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$

Alternatively,

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 + 5\cdot(-3) \\ -4\cdot 2 + (-3)\cdot(-3) \\ 7\cdot 2 + 6\cdot(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

5. $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$ matrix equation

$$5\begin{bmatrix} 5 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3\begin{bmatrix} -8 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

vector equation

7.

$$x_1\begin{bmatrix}4\\-1\\7\\4\end{bmatrix}+x_2\begin{bmatrix}-5\\3\\-5\\1\end{bmatrix}+x_3\begin{bmatrix}7\\-8\\0\\2\end{bmatrix}=\begin{bmatrix}6\\-8\\0\\-7\end{bmatrix}$$ vector equation

$$\begin{bmatrix}4&-5&7&6\\-1&3&-8&-8\\7&-5&0&0\\4&1&2&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}6\\-8\\0\\7\end{bmatrix}$$ matrix vector equation

10.

$$x_1\begin{bmatrix}8\\5\\1\end{bmatrix}+x_2\begin{bmatrix}-1\\4\\-3\end{bmatrix}=\begin{bmatrix}4\\1\\2\end{bmatrix}$$ vector equation

$$\begin{bmatrix}8&-1\\5&4\\1&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4\\1\\2\end{bmatrix}$$ matrix equation

12.

$$\left[\begin{array}{ccc|c}1&2&1&0\\-3&-1&2&1\\0&5&3&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&1&0\\0&5&5&1\\0&5&3&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&1&0\\0&5&5&1\\0&0&-2&-2\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&2&1&0\\0&1&1&1/5\\0&0&1&1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-4/5\\0&0&1&1\end{array}\right]\sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} x_1 = 3/5 \\ x_2 = -4/5 \\ x_3 = 1 \end{array} \quad \vec{x} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$$

13. If $\vec{u}$ is in the plane spanned by

$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix}$, then $\vec{u}$ can be written as

a linear combination of the two vectors. So, is there a solution to the vector equation

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} ?$$

The augmented matrix corresponding to this vector equation is:

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array}\right]$$

So $\frac{5}{2}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$.

Thus $\vec{u}$ is a linear combination of the columns of A, and is in the plane spanned by the columns of A.

15. $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

Since A does not have a pivot position in each row, theorem 4 states that there exists a $\vec{b}$ in $\mathbb{R}^m$ for which $A\vec{x} = \vec{b}$ does not have a solution.

$$\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{array}\right]$$

$A\vec{x}=\vec{b}$ will have a solution as long as $b_2+3b_1=0$. This is the set of all points of the form $(b_1, {}^-3b_1)$. This can be written as the line $y={}^-3x$, or $x_2=-3x_1$.

17. $\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim$

$R_2 \to R_2 + R_1$
$R_4 \to R_4 - 2R_1$

$R_3 \to R_3 + 2R_2$
$R_4 \to R_4 + 3R_2$

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \leftrightarrow R_4$

3 rows of A contain a pivot. The equation $A\vec{x}=\vec{b}$ does not have a solution for each choice of $\vec{b}$, since it does not have a pivot

position in each row. (see theorem 4 parts (a) and (d).)

19. From problem 17, we know that A does not have a pivot position in each row. Thus by Theorem 4, part (b), there is some $\vec{b}$ in $\mathbb{R}^4$ which is not a linear combination of the columns of A. Similarly, part (c) says the columns of A do not span $\mathbb{R}^4$.

22. By Theorem 4, if $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ has a pivot in each row, then the columns of A span $\mathbb{R}^3$.

$$A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus A has a pivot in each row, and the columns of A span $\mathbb{R}^3$.

23. a. False, it is a matrix equation.
b. True, see shaded box on page 37.
c. False, the coefficient matrix needs a pivot in each row. (See theorem 4)
d. True, see Row-Vector Rule for computing Ax
e. True, see parts (a) and (c) of Theorem 4.
f. True, see parts (a) and (d) of Theorem 4.

24. a. True, see Theorem 3.
b. True, see definition of matrix-vector multiplication.
c. True, see Theorem 3.
d. True, see shaded box on page 37.
e. False, $A\vec{x} = \vec{b}$ being inconsistent depends on whether the last column of the augmented matrix is a pivot column. (see theorem 2)

f. True, see parts (c) and (a) of Theorem 4.

25. If $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

then $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$.

Using the definition of matrix-vector multiplication, c_1, c_2, and c_3 are $-3, -1$, and 2 respectively.

26. Since $3\vec{u} - 5\vec{v} - \vec{w} = \vec{0}$, then

$$3\vec{u} - 5\vec{v} = \vec{w}.$$

So $3\begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} - 5\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$

This vector equation can be written as a matrix equation.

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

Thus
$x_1 = 3$
$x_2 = -5$.

27. $x_1 \vec{q}_1 + x_2 \vec{q}_2 + x_3 \vec{q}_3 = \vec{v}$

$Q = [\vec{q}_1 \ \vec{q}_2 \ \vec{q}_3]$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The vector equation can be written as

$$\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{v}$$

which is the same as

$Q\vec{x} = \vec{v}$ (matrix equation).

32. A set of 3 vectors in $\mathbb{R}^4$ could not span $\mathbb{R}^4$. The 3 vectors can be the columns of a 4×3 matrix. Since there are 3 columns and 4 rows the matrix has at most 3 pivot positions. Thus the matrix cannot have a pivot position in each row.

By Theorem 4, the columns of the matrix would not span $\mathbb{R}^4$.
The same reasoning would apply to n vectors in $\mathbb{R}^m$ where $n < m$.
There is not enough columns to have a pivot in each row.