Math 341 - Linear Algebra
§1.7 - 1.9

Fall 2019

There is no royal road to geometry.

- Euclid to Ptolemy

§1.7 Linear Independence

Definition (Linear Independence)

§1.7 Linear Independence

Definition (Linear Independence)

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Definition (Linear Independence)

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

Definition (Linear Independence)

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

In this case equation (1) is called a **dependence relation**.

Example: Determine if the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

are linearly independent.

Example: Determine if the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

are linearly independent.

Example: Determine if

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent.

Saying that the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$

Saying that the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$ is the same as saying that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Saying that the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$ is the same as saying that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Fact

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Fact

Fact

1. A set $\{\mathbf{v}\}$ containing a single vector is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

Fact

1. A set $\{\mathbf{v}\}$ containing a single vector is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

2. A set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent if and only if neither of $\mathbf{v}$ not $\mathbf{w}$ is a scalar multiple of the other.

Fact

1. A set $\{\mathbf{v}\}$ containing a single vector is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

2. A set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent if and only if neither of $\mathbf{v}$ not $\mathbf{w}$ is a scalar multiple of the other.

Warning! A set of 3 or more vectors may be linearly dependent even though none of them is a scalar multiple of another vector in the set.

Example: Determine if

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} 16 \\ 8 \\ -7 \end{bmatrix}$$

are linearly independent.

Theorem

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

Theorem

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Theorem

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Warning! This theorem does not say that *every* vector of a linearly dependent set can be written as a linear combination of the other vectors,

Theorem

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Warning! This theorem does not say that *every* vector of a linearly dependent set can be written as a linear combination of the other vectors, just that *some* vector can.

Example: Consider $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. What is Span $\{\mathbf{u}, \mathbf{v}\}$?

Example: Consider $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. What is Span $\{\mathbf{u}, \mathbf{v}\}$?

If $\mathbf{w}$ is another vector in $\mathbb{R}^3$, where will $\mathbf{w}$ lie if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent?

Example: Consider $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. What is Span $\{\mathbf{u}, \mathbf{v}\}$?

If $\mathbf{w}$ is another vector in $\mathbb{R}^3$, where will $\mathbf{w}$ lie if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent? Where will it lie if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent?

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

§1.7 Linear Independence

Example: Which of the following sets of vectors is linearly independent?

1. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}.$

Example: Which of the following sets of vectors is linearly independent?

1. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}.$

2. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}.$

Example: Which of the following sets of vectors is linearly independent?

1. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}.$

2. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}.$

3. $\begin{bmatrix} 2 \\ 2 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -12 \\ -6 \end{bmatrix}$

§1.8 Introduction to Linear Transformations

Given an $m \times n$ matrix A and a vector $\mathbf{x} \in \mathbb{R}^n$, we can multiply A and $\mathbf{x}$ to give a vector in $\mathbb{R}^m$.

§1.8 Introduction to Linear Transformations

Given an $m \times n$ matrix A and a vector $\mathbf{x} \in \mathbb{R}^n$, we can multiply A and $\mathbf{x}$ to give a vector in $\mathbb{R}^m$.

We can think of an $m \times n$ matrix as taking vectors in $\mathbb{R}^n$ and transforming them to vectors in $\mathbb{R}^m$.

§1.8 Introduction to Linear Transformations

Given an $m \times n$ matrix A and a vector $\mathbf{x} \in \mathbb{R}^n$, we can multiply A and $\mathbf{x}$ to give a vector in $\mathbb{R}^m$.

We can think of an $m \times n$ matrix as taking vectors in $\mathbb{R}^n$ and transforming them to vectors in $\mathbb{R}^m$.

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$. Then

$$A\mathbf{x}$$

§1.8 Introduction to Linear Transformations

Given an $m \times n$ matrix A and a vector $\mathbf{x} \in \mathbb{R}^n$, we can multiply A and $\mathbf{x}$ to give a vector in $\mathbb{R}^m$.

We can think of an $m \times n$ matrix as taking vectors in $\mathbb{R}^n$ and transforming them to vectors in $\mathbb{R}^m$.

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$$

§1.8 Introduction to Linear Transformations

Given an $m \times n$ matrix A and a vector $\mathbf{x} \in \mathbb{R}^n$, we can multiply A and $\mathbf{x}$ to give a vector in $\mathbb{R}^m$.

We can think of an $m \times n$ matrix as taking vectors in $\mathbb{R}^n$ and transforming them to vectors in $\mathbb{R}^m$.

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 35 \end{bmatrix}$$

§1.8 Introduction to Linear Transformations

Given an $m \times n$ matrix A and a vector $\mathbf{x} \in \mathbb{R}^n$, we can multiply A and $\mathbf{x}$ to give a vector in $\mathbb{R}^m$.

We can think of an $m \times n$ matrix as taking vectors in $\mathbb{R}^n$ and transforming them to vectors in $\mathbb{R}^m$.

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 35 \end{bmatrix} \in \mathbb{R}^2.$$

§1.8 Introduction to Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

§1.8 Introduction to Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T.

§1.8 Introduction to Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T.

Notation:

§1.8 Introduction to Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T.

Notation:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

§1.8 Introduction to Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T.

Notation:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{x} \mapsto T(\mathbf{x})$$

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T.

Notation:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{x} \mapsto T(\mathbf{x})$$

For a vector $\mathbf{x} \in \mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image of $\mathbf{x}$**,

§1.8 Introduction to Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

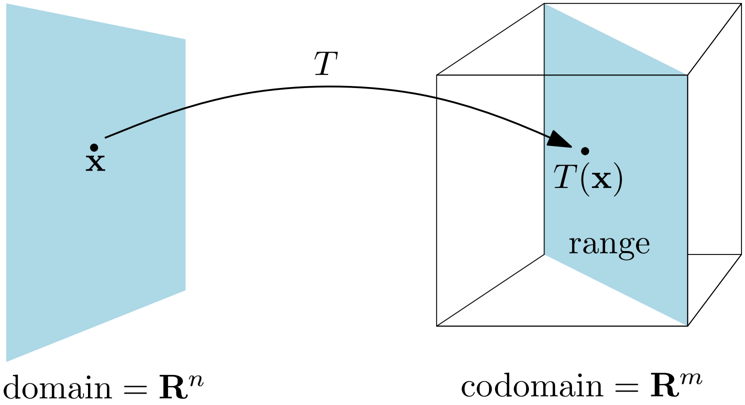
The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T.

Notation:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{x} \mapsto T(\mathbf{x})$$

For a vector $\mathbf{x} \in \mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image of $\mathbf{x}$**, and the set of all images $T(\mathbf{x})$ is called the **range of T**.



T
$\mathbf{x}$
$T(\mathbf{x})$
range
domain $= \mathbf{R}^n$
codomain $= \mathbf{R}^m$

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

What is the domain and codomain of T?

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

What is the domain and codomain of T? Let

$$\mathbf{b} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix}.$$

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

What is the domain and codomain of T? Let

$$\mathbf{b} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix}.$$

Compute $T(\mathbf{u})$.

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

What is the domain and codomain of T? Let

$$\mathbf{b} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix}.$$

Compute $T(\mathbf{u})$. Are $\mathbf{b}$ and $\mathbf{c}$ in the range of T?

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

What is the domain and codomain of T? Let

$$\mathbf{b} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix}.$$

Compute $T(\mathbf{u})$. Are $\mathbf{b}$ and $\mathbf{c}$ in the range of T? If so, find vectors $\mathbf{x}$ and $\mathbf{v}$ with $T(\mathbf{x}) = \mathbf{b}$ and $T(\mathbf{v}) = \mathbf{c}$.

Example: Let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

What is the domain and codomain of T? Let

$$\mathbf{b} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix}.$$

Compute $T(\mathbf{u})$. Are $\mathbf{b}$ and $\mathbf{c}$ in the range of T? If so, find vectors $\mathbf{x}$ and $\mathbf{v}$ with $T(\mathbf{x}) = \mathbf{b}$ and $T(\mathbf{v}) = \mathbf{c}$.

What is the range of T?

Definition

A transformation T is **linear** if for all $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars $c \in \mathbb{R}$

§1.8 Introduction to Linear Transformations

Definition

A transformation T is **linear** if for all $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars $c \in \mathbb{R}$

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

Definition

A transformation T is **linear** if for all $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars $c \in \mathbb{R}$

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

§1.8 Introduction to Linear Transformations

Definition

A transformation T is **linear** if for all $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars $c \in \mathbb{R}$

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Fact

If T is a linear transformation, then for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T, and all scalars c, d

Definition

A transformation T is **linear** if for all $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars $c \in \mathbb{R}$

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Fact

If T is a linear transformation, then for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T, and all scalars c, d

$$T(\mathbf{0}) = \mathbf{0},$$

Definition

A transformation T is **linear** if for all $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars $c \in \mathbb{R}$

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Fact

If T is a linear transformation, then for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T, and all scalars c, d

$$T(\mathbf{0}) = \mathbf{0}, \text{ and}$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

Example: Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Example: Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Described the linear transformations defined by these matrices.

Example: Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Described the linear transformations defined by these matrices.

The matrix I above is called the **identity matrix**.

§1.9 The Matrix of a Linear Transformation

Recall that for any n, we can define the following vectors in $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the jth column of the identity matrix I_n in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \ldots \quad T(\mathbf{e}_n)].$$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the jth column of the identity matrix I_n in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \ldots \quad T(\mathbf{e}_n)].$$

The matrix given above is called the **standard matrix for the linear transformation** T.

Example: Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects vectors in the line $y = -x$.

Example: Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects vectors in the line $y = -x$.

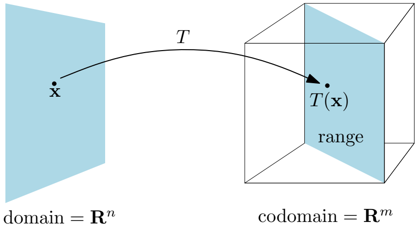
Example: Find the standard matrix of the transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ which reflects every vector through the xy-plane, and then projects to the xz-plane.

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

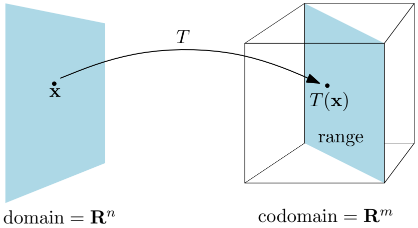
Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.



Definition

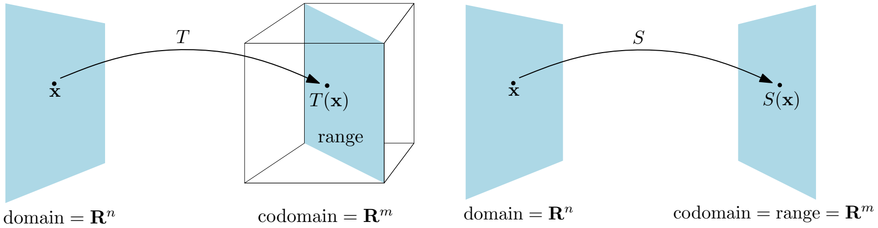
A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.



T is *not* onto.

Definition

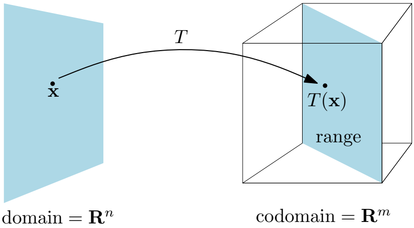
A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.



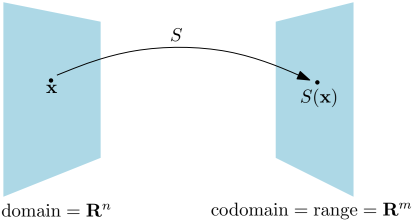
T is *not* onto.

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.



T is *not* onto.



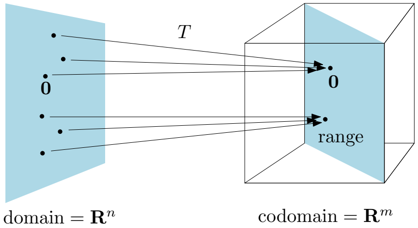
S is onto.

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **one-to-one** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$.

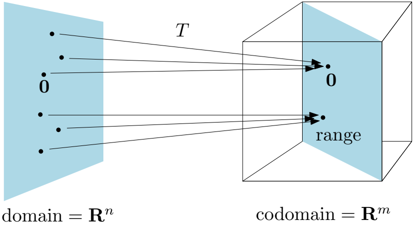
Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **one-to-one** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$.



Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **one-to-one** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$.

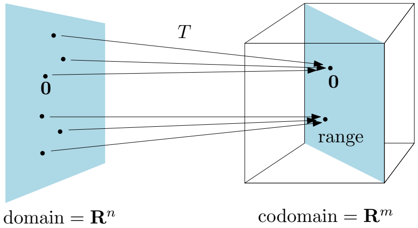


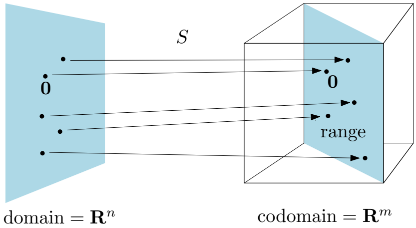
T is *not* one-to-one.

§1.9 The Matrix of a Linear Transformation

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **one-to-one** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$.



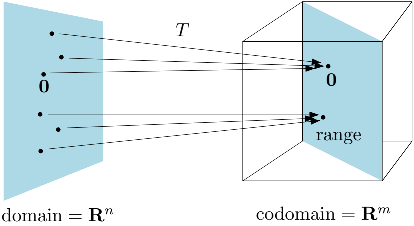


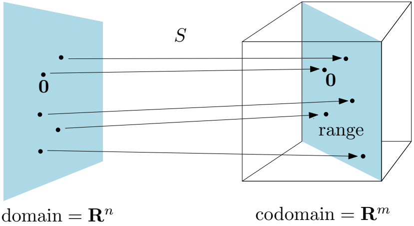
T is *not* one-to-one.

§1.9 The Matrix of a Linear Transformation

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **one-to-one** if every vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$.





T is *not* one-to-one.

S is one-to-one.

Example: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & 1 & 8 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Example: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & 1 & 8 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Is T one-to-one?

Example: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & 1 & 8 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Is T one-to-one? Is T onto?

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is onto,

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is onto,
2. the columns of A span $\mathbb{R}^m$,

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is onto,
2. the columns of A span $\mathbb{R}^m$,
3. A has a pivot in every row.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is onto,
2. the columns of A span $\mathbb{R}^m$,
3. A has a pivot in every row.

Proof.

This is essentially Theorem 4 in §1.4. $\square$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is one-to-one,

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is one-to-one,
2. $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$,

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is one-to-one,
2. $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$,
3. the columns of A are linearly independent,

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is one-to-one,
2. $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$,
3. the columns of A are linearly independent,
4. A has a pivot in every column.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A (i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$). Then the following are equivalent:

1. T is one-to-one,
2. $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$,
3. the columns of A are linearly independent,
4. A has a pivot in every column.

Proof.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. $\square$