Section 3.2

1. $$\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$$ Interchange of Row 1 and Row 2

Illustrates Theorem 3b, that interchanging or swapping two rows, negates the determinant.

2. $$\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{vmatrix}$$ Replace Row 3 with the sum of Row 3 and -3 times Row 1.

Illustrates Theorem 3a, that adding a multiple of one row to another does not change the determinant.

3. $$\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}$$ Scale Row 1 by a factor of $\frac{1}{3}$

Illustrates Theorem 3c, that scaling a row by k, yields a determinant that is k times the original.

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = -3$

All row operations were replacements.

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 7 & 3 \end{vmatrix}$

$= \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & -4 \end{vmatrix} = -28$

All row operations were replacements.

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \cdot 7 = 21$

18. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$

21.

$$\begin{vmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 2 & 6 & 0 \\ 3 & 9 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & -4 \end{vmatrix} = 0$$

By Theorem 4, $\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$ is not invertible.

24.

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{vmatrix} = -\begin{vmatrix} 2 & 7 & -2 \\ 6 & 0 & -5 \\ 4 & -7 & -3 \end{vmatrix} = -\begin{vmatrix} 2 & 7 & -2 \\ 0 & -21 & 1 \\ 0 & -21 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 7 & -2 \\ 0 & -21 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

By Theorem 4, $\begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{bmatrix}$ is not invertible.

By IMT, the columns of the matrix are linearly dependent.

31. Since A is invertible, $AA^{-1} = I$.

Thus $\det(AA^{-1}) = \det I$.

- $\det I = 1$, since I is a diagonal matrix.
- $\det(AA^{-1}) = \det A \det A^{-1}$ by Theorem 6.

Thus

$$\det A \det A^{-1} = 1$$

Since A is invertible, $\det A \neq 0$ by Theorem 4.

So $$\det A^{-1} = \frac{1}{\det A}.$$

34. Let A and P be square matrices with P invertible. Show $\det(PAP^{-1}) = \det A$.

$$\begin{aligned}
\det PAP^{-1} &= \det P \det AP^{-1} && \text{by Theorem 6} \\
&= \det P \det A \det P^{-1} && \text{by Theorem 6} \\
&= \det P \det A \frac{1}{\det P} && \text{by exercise 31} \\
&= \frac{\det P}{\det P} \det A \\
&= \det A.
\end{aligned}$$

40. a. $\det AB = \det A \det B = -3 \cdot -1 = 3$

b. $\det B^5 = (\det B)^5 = (-1)^5 = -1$

c. $\det 2A = 2^4 \det A = 16 \cdot -3 = -48$

There are 4 rows in A, and 2A multiplies every row of A by 2.

d. $\det A^T BA = \det A^T B \det A = \det A^T \det B \det A$

$= \det A \det B \det A = -3 \cdot -1 \cdot -3$

$= -9$

e. $\det B^{-1}AB = \det B^{-1} \det AB = \det B^{-1} \det A \det B$

$= \frac{1}{\det B} \det A \det B = \det A = -3$

41. $\begin{vmatrix} a+e & b+f \\ c & d \end{vmatrix} = (a+e)d - c(b+f)$

$= ad + ed - cb - cf = \det A$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \det B$

$\begin{vmatrix} e & f \\ c & d \end{vmatrix} = ed - cf = \det C$

So $\det A = \det B + \det C$.