Chapter 4-Vector Spaces

This chapter, which is central to linear algebra, unifies different and seemingly unrelated branches of mathematics. To begin with, consider the sets V of the following objects.

1. $V = \mathbb{R}^n (n \geq 1)$, the Euclidean n-space whose elements are called vectors.
2. $V = M_{m\times n}$ the set of all $m \times n$ matrices with real entries.
3. $V = \mathbb{P}_n$, the set of all polynomials of degree at most n with real coefficients.
4. $V = C[a, b]$, the set of all real valued continuous functions g on the interval [a,b].
5. $V =$ the set of all differentiable functions f on the interval [a,b].

All these sets share some (but not all) properties. Their common feature is the defining properties of a so called vector space. In all these sets V, there is a way to "add" two elements of V and to "scale" an element of V by a scalar (an element of $\mathbb{R}$). These operations which we call "addition" and "scalar multiplication" on V, have some basic properties common to all the sets listed above. A minimal set of basic properties from which other properties common to all these V follow will be our defining axioms. Before defining a vector space formally, some remarks are in order.

1. Although two continuous functions $f(x)$ and $g(x)$ are not added the same way as two matrices A and B, the sum is still denoted by $+$. To stress this point, some texts denote the operation of adding objects in V by $\oplus$. However, we will use simply $+$ and reserve the use of $\oplus$ for other less usual definitions of combining elements.
2. Some three year old children make no distinction between a dog, a cow, or a lion, and call them all a dog–the animal most familiar to them. Likewise, we shall call all objects in the set V a vector, due to our familiarity with $V = \mathbb{R}^n$. Thus one should be aware that a vector need not have a magnitude and direction. It could be for example a differentiable function.
3. Although a zero matrix is not the same thing as a zero function $f(x)$, i.e. a function which is identically zero, both are denoted by **0**.

Definition: A **vector space** (over the field of $\mathbb{R}$ of scalars a,b,c,...) is a non-empty set V of objects **u**,**v**,**w** called vectors together with two operations: addition **u**+**v** of vectors and scalar multiplication a**u** (scaling of a vector **u** by a scalar a in $\mathbb{R}$) such that the following 10 axioms are satisfied for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all $c, d \in \mathbb{R}$.

1. The sum of **u** and **v**, denoted by **u**+**v**, is in V. (closure for addition)
2. **u**+**v**=**v**+**u**. (commutative law)
3. (**u**+**v**)+**w**=**u**+(**v**+**w**) (associative law)
4. These is a **zero** vector **0** in V such that **u**+**0**=**u**.
5. For each **u** in V, there is a vector **-u** in V such that **u**+(**-u**)=**0**.
6. The scalar multiple of **u** by c, denoted by c**u**, is in V. (closure for scalar multiplication)
7. c(**u**+**v**)=c**u**+c**v**. (distributive law for scalars over vectors)
8. (c+d)**u**=c**u**+d**u**. (distributive law for vectors over scalars)
9. c(d**u**)=(cd)**u**.
10. 1**u**=**u**

Remark: Some mathematics texts use the term *linear space* for a vector space.

Examples

1. All the five sets listed at the beginning are vector spaces with the usual addition and scalar multiplication. In the next few examples, we define some unusual additions and scalar multiplications. To emphasize that they are not standard we use the symbol $\oplus$ for addition, $\odot$ for scalar multiplication, and sometimes θ for zero. Using this notation, axioms 1 through 5 look the same except each + is replaced by $\oplus$. Axioms 6 through 10 take the form:

   6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c \odot \mathbf{u}$, is in V.
   7. $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$.
   8. $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$.
   9. $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$.
   10. $1 \odot \mathbf{u} = \mathbf{u}$.

2. Fix an $n \geq 1$. On $M_{n\times n}$, define $A \oplus B = AB - BA$, where $c \odot A = cA$ is the usual scaling by c. This is not a vector space because axiom 2, $A \oplus B = B \oplus A$, is not true for all A,B.

3. On V=$\mathbb{R}$, define the scalar multiplication as multiplication in $\mathbb{R}$, but $x \oplus y = \min\{x, y\}$. Now axioms 1,2, and 3 hold, but there is no zero vector θ. In fact for a real number θ to be a zero vector, $x \oplus \theta = \min\{x, \theta\} = x$ for all x means $\theta \geq x$ for all x, which is false. So Axiom 4 does not hold.

4. For this example, we denote by [a] the largest integer $\leq a$. For example, [π]=3, but [-π]=-4. Take V=$\mathbb{Z} = 0, \pm 1, \pm 2, ...$, the set of integers. Define m $\oplus$ n to be the usual sum of integers but $a \odot m = [a]m$, e.g. $\sqrt{2} \odot 5 = 5$. Clearly $a \odot m$ is in $\mathbb{Z}$. However, if we take a=b=1/2 and m=2, then axiom 8 fails.

5. On V=$\mathbb{R}^2$, we take $\oplus$ to be the usual addition (by parallelogram law) but define $a \odot (x, y) = (ax, 0)$. Now axiom 10 fails.

   Before going to the next example, we determine that there is one and only one zero vector in V.

   Theorem: The zero vector θ in a vector space V is unique.

   Proof: If there are two, say θ_1 and θ_2, then $\theta_1 + \theta_2 = \theta_1$ (using θ_2 as a zero vector) and $\theta_1 + \theta_2 = \theta_2$ (using θ_1 as a zero vector). Therefore $\theta_1 = \theta_1 + \theta_2 = \theta_2$.

6. Again on V=$\mathbb{R}^2$, keep the usual scalar multiplication $c \odot (x, y) = (cx, cy)$ but let $(x_1, y_1) \oplus (x_2, y_2) = (x_1 \oplus x_2, 0)$. Now there are infinitely many zero vectors (0,y) with y arbitrary, so V cannot be a vector space.

7. On V=$\mathbb{P}_n$, let the scalar multiplication be the usual one: $c(a_0+a_1x+...+a_nx^n) = ca_0+ca_1x+...+ca_nx^n$, but for $f(x) = a_0 + a_1x + ... + a_nx^n, g(x) = b_0 + b_1x + ... + b_nx^n$, define $f(x) \oplus g(x)$ to be the constant polynomial $a_0b_0 + a_1b_1 + \cdots + a_nb_n$. Again V is not a vector space. Which axiom fails?

8. Finally, let V=$\mathbb{R}^+$, the set of positive real numbers. On V we define $\oplus$ and $\odot$ by $x \oplus y = xy$ (the usual product) and $c \odot x = x^c$. These two operations make V=$\mathbb{R}^+$ into a vector space. (Check that all the axioms hold.) We shall see later that as vector spaces, $\mathbb{R}^+$ and $\mathbb{R}^n$ (for n=1) are the same.

9. In the previous example (#8) let us redefine the sum by $x \oplus y = x/y$ but keep the scalar multiplication as it is. Is it a vector space?