Math 341 - Linear Algebra
§2.1 - 2.4

Fall 2019

Seek ye diligently and teach one another words of wisdom; yea, seek ye out of the best books words of wisdom; seek learning, even by study and also by faith.

- D&C 88:118

§2.1 Matrix Operations - Notation

If A is an $m \times n$ matrix, then we let a_{ij} denote the element in the ith row and jth column. In other words

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

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We let $\mathbf{a}_j$ denote the jth column of A, so that

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

An $n \times n$ matrix of the form

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

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is called a **diagonal matrix**, while the $m \times n$ matrix

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

is called the **zero matrix**.

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$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \end{aligned}$$

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and

$$cA = c \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}.$$

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1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + \mathbf{0} = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

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If A is an $m \times p$ matrix, and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ is a $p \times n$ matrix, then the **matrix product** of A and B, denoted AB, is the $m \times n$ matrix defined by

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i.e. the jth column of AB is A times the jth column of B.

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Example: Compute the products AB, BA, AC, and CA for the matrices

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & -3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\text{and} \qquad C = \begin{bmatrix} 5 & 1 & 1 \\ 3 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

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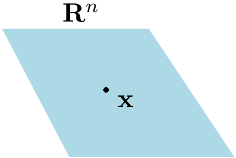

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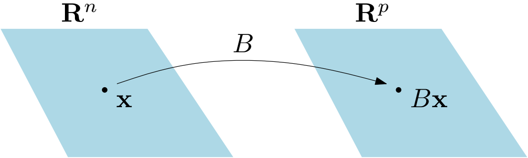

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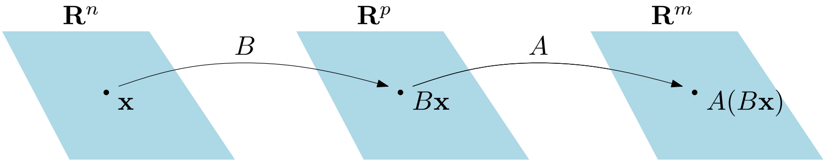

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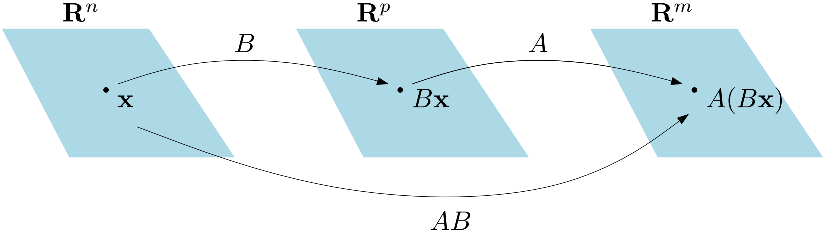

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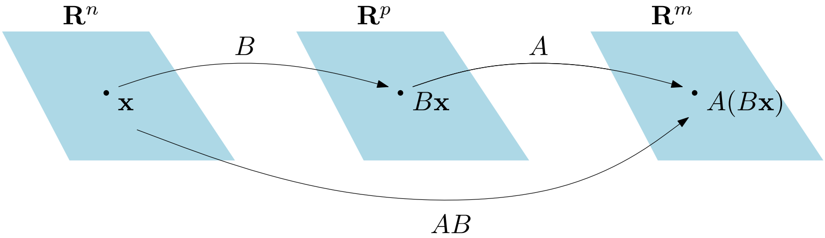

The linear transformation defined by $T(\mathbf{x}) = (AB)\mathbf{x}$ is the same as the linear transformation obtained my first mapping $\mathbf{x} \mapsto B\mathbf{x}$, followed by mapping $B\mathbf{x} \mapsto A(B\mathbf{x})$.

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$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

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Note: Only $n \times n$ (square) matrices can be invertible.

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In other words, A^{-1} is the unique matrix satisfying

$$AA^{-1} = A^{-1}A = I$$

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If $ad - bc = 0$, then A is not invertible.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

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The quantity $ad - bc$ is called the **determinant** of A, and we use the notation

$$\det A = ad - bc.$$

Example: Determine if the following matrices are invertible, and if they are, compute their inverses:

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Example: Find the solutions of

$$\begin{array}{rcrcl} x & + & 2y & = & 1 \\ -x & + & 3y & = & 3 \end{array} \quad \text{and} \quad \begin{array}{rcrcl} & & 4y & = & 4 \\ x & + & 3y & = & 2. \end{array}$$

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Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

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§2.3 Characterizations of Invertible Matrices

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§2.4 Partitioned (Block) Matrices

Consider the matrices

$$Q = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} r & s \\ p & q \\ t & v \end{bmatrix}.$$

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By dividing them up by some set of vertical and horizontal lines, we can think of them as matrices made up of smaller matrices:

$$Q = \left[\begin{array}{cc|c} a & b & c \\ d & e & f \\ \hline g & h & i \end{array}\right] \qquad \text{and} \qquad P = \left[\begin{array}{cc} r & s \\ p & q \\ \hline t & v \end{array}\right]$$

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(notice that the way we partitioned the columns of Q is the same way we partitioned the rows of P).

Now if A, B, C, D, E, and F were all real numbers, then the product of the matrices Q and P would be given by

$$QP = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}.$$

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We can also add two block matrices of the same size, though in this situation the partitions must be exactly the same.