First Order Logic

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Objective

- Formulate more types of sentences in logic?
- Write correct predicate logic formulae

Limitations of propositional logic

- All dogs are faithful
- Tommy is a dog
- Therefore, Tommy is faithful

How to represent and infer this in propositional logic?

- P: All dogs are faithful
- Q: Tommy is a dog
- Can we claim? $P \wedge Q \rightarrow$ Tommy is faithful

- Machine does not know what does “all dogs” mean?

Limitations of propositional logic

- Anil is a hardworking student
  - Hardworking_Anil

- Anil is an intelligent student
  - Intelligent_Anil

- Anil scores high marks
  - Score_High_Mark_Anil

- If Anil is hardworking and Anil is intelligent, then Anil scores high marks
  - Hardworking_Anil ∧ Intelligent_Anil → Score_High_Mark_Anil

- What about Akash and Asish?

Limitations of propositional logic

- Anil is a hardworking student
- Anil is an intelligent student

- All students who are hardworking and intelligent scores high marks

- For all x such that x is a student and x is intelligent and x is hardworking then x scores high marks
    - x is a variable
    - Need power to write such sentences

The problem of Infinite Model

- Propositional logic, we have to restrict ourselves to constants
- In general, propositional logic can deal with only a finite number of propositions

- If there are only three students Anil Akash Asish, then
  - P: Anil is intelligent
  - Q: Akash is intelligent
  - R: Asish is intelligent
- All students are intelligent: $P \wedge Q \wedge R$

- If a new student joins the class?
- How long should we go on?
- Limitation: Finiteness of statements

First Order Logic

- Generalization of propositional logic that allows us to express and infer arguments in infinite models like

  - All men are mortal
  - Some birds cannot fly
  - At least one planet has life on it

Syntax of FOL

- The syntax of First-Order Logic can be defined in terms of
  - Terms
  - Predicates
  - Quantifiers

Term

- A term denotes some object other than **true** or **false**

term

- Tommy is a dog
- All men are mortal

- Terms can be constants as well as variables
- In proposition logic, we only have constants

Term: Constants & Variables

- A constant of type W is a name that **denotes a particular object** in a set W
  - Example: 5, Anil

- A variable of type W is a name that can **denote any element** in the set W
  - Examples: $x \in N$ denotes a Natural number
  - s denotes the name of a student

Functions

- A functional term of arity n takes n objects of type W_1 to W_n as inputs and returns an object of type W
  - f(W_1, W_2, … , W_n)
  - plus(3,4) = 7
    - Two objects of type constant from the set of Natural numbers

Functional Term

Constants

Functions: Example

- Let plus be a function that takes two arguments of type Natural number and returns a Natural number

- Valid Functional Terms:
    - plus(2,3)
    - plus(5,plus(7,3))

- Invalid Functional Terms:
    - plus(0,-1)
    - Plus(1.2,3.3)

Predicates

- Predicates are like functions except that their return type is **true** or **false**

- Example:
  - gt(x,y) is true iff x>y
  - Here gt is a predicate symbol that takes two arguments of type natural number
  - gt(3,4) is a valid predicate but gt(3,-4) is not

Types of Predicates

- A predicate with no variable is a proposition
  - Anil is a student

- A predicate with one variable is called a property
  - student(x) is true iff x is student
  - mortal(y) is true iff y is mortal

Formulation of Predicates

- Let P(x,y,…) and Q(x,y,…) are two predicates

- Then so are
  - $P \wedge Q$
  - $P \vee Q$
  - ~P
  - $P \rightarrow Q$

Predicate Examples

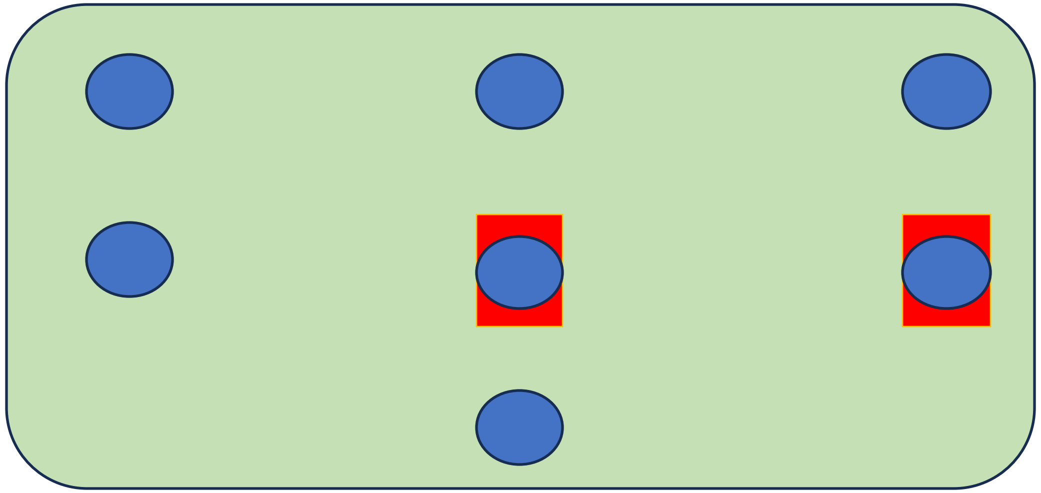
- If x is a man then x is mortal
  - $man(x) \rightarrow mortal(x)$
  - $\sim man(x) \vee mortal(x)$

- If n is a natural number then n is either even or odd
  - $natural(n) \rightarrow even(n) \vee odd(n)$

Quantifiers

- There are two basic quantifiers in FOL

- $\forall$ "for all" Universal Quantifier
- $\exists$ "there exists" Existential Quantifier



Days in a week

$$\forall_x sunrise(x)$$

$$\exists_x holiday(x)$$

Universal Quantifiers

- All dogs are faithful
  - faithful(x): x is faithful
  - dog(x): x is a dog
  - $\forall_x(\text{dog}(x) \rightarrow \text{faithful}(x))$
- All birds cannot fly
  - fly(x): x can fly
  - bird(x): x is a bird
  - $\boxed{\sim}(\forall_x(\text{bird}(x) \rightarrow \text{fly}(x)))$

Existential Quantifiers

- At least one planet has life on it
  - planet(x): x is a planet
  - haslife(x): x has life on it
  - $\exists_x$(planet(x)∧haslife(x))

- There exists a bird that can't fly
  - fly(x): x can fly
  - bird(x): x is a bird
  - $\exists_x$(bird(x)∧~fly(x))

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Duality of Quantifiers

- All men are mortal
  - No man is immortal

- $\forall_x(\text{man}(x) \rightarrow \text{mortal}(x))$
- $\boxed{\sim}(\exists_x(\text{man}(x) \wedge \sim\text{mortal}(x)))$

Universal quantifiers could be expressed in a different way with existential quantifiers

Sentences

- A predicate is a sentence

- If sen, sen' are sentences and x is a variable then following are sentences
  - (sen), ~sen, $\exists_x$sen, $\forall_x$sen
  - sen$\wedge sen'$
  - sen$\vee sen'$
  - sen$\rightarrow sen'$

- Nothing else is sentence

First-order Logic

- Sentence → AtomicSentence
    - | Sentence Connective Sentence
    - Quantifier Variable, … Sentence
    - ~ Sentence | (Sentence)

- AtomicSentence → Predicate(Term, …)
    - | Term = Term
- Term → Function(Term, …) | Constant | Variable
- Connective → $\Rightarrow | \vee | \wedge | \Leftrightarrow$
- Quantifier → $\forall | \exists$

Difference from second order logic

- In FOL we quantify only variables
- In second order logic we can quantify predicates
    - $\exists_P \forall_x \forall_y P(x, y)$

Examples

- Not all students take history and biology
  - Student(x): x is a student
  - Takes(x,y): subject x is taken by y
  - $\boxed{\sim}[\forall_x \text{Student}(x) \rightarrow \text{Takes}(\text{History},x) \land \text{Takes}(\text{Biology},x)]$
  - $\exists_x \text{Student}(x) \land [\sim\text{Takes}(\text{History},x) \lor \sim\text{Takes}(\text{Biology},x)]$

- Only one student failed biology
  - Failed(x,y): student y failed in subject x

Examples

- Only one student failed both history and biology
  - Failed(x,y): student y failed in subject x

- The best score in history is better than the best score in biology
  - Function: score(subject, student)
  - Greater(x,y): x>y

Takeaway

- Predicate logic is a more powerful version of proposition logic
- Provide support for variables and quantifiers

- Can capture sentences more naturally

How we can perform inferencing with predicates?

Thank You