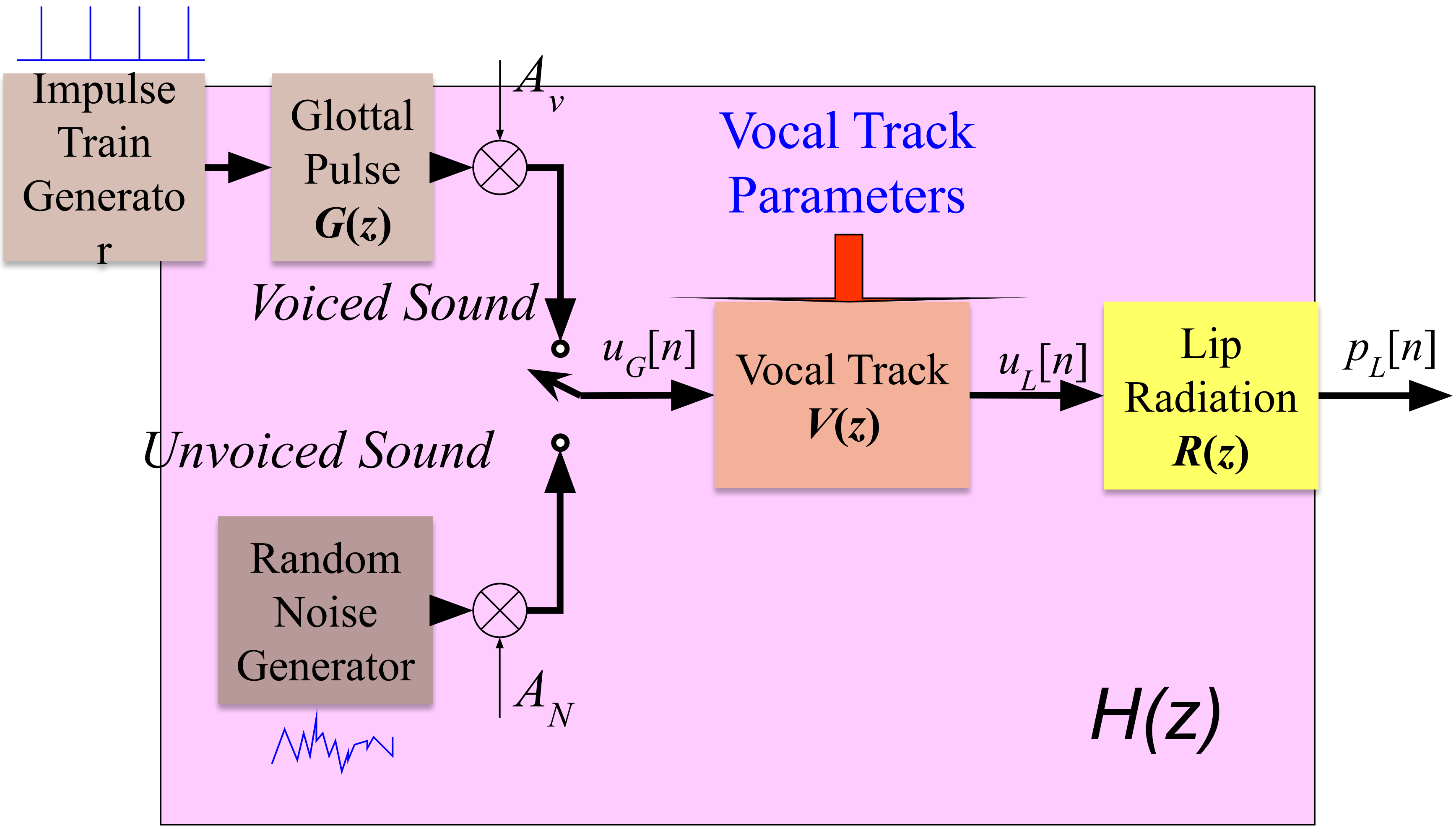
Lecture-12

Analysis and Synthesis of Pole-Zero Speech Models

Speech Production Model

$$H(z)=G(z)V(z)R(z)$$



Speech Production Model

$$H(z) = G(z)V(z)R(z)$$

$$\boxed{G(z) = \frac{1}{(1 - e^{-cT} z^{-1})^2}}$$

$$\boxed{V(z) = \frac{G}{\prod_{k=1}^{N/2} (1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2})}}$$

$$\boxed{R(z) = R_0 (1 - z^{-1})}$$

$$\boxed{H(z) = \frac{\sigma(1 - z^{-1})}{(1 - e^{-cT} z^{-1})^2 \prod_{k=1}^{N/2} (1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2})}}$$

All pole modeling

$$H(z)=G(z)V(z)R(z)$$

$$\boxed{G(z)=\frac{1}{(1-e^{-cT}z^{-1})^2}} \qquad \boxed{e^{-cT}\approx 1}$$

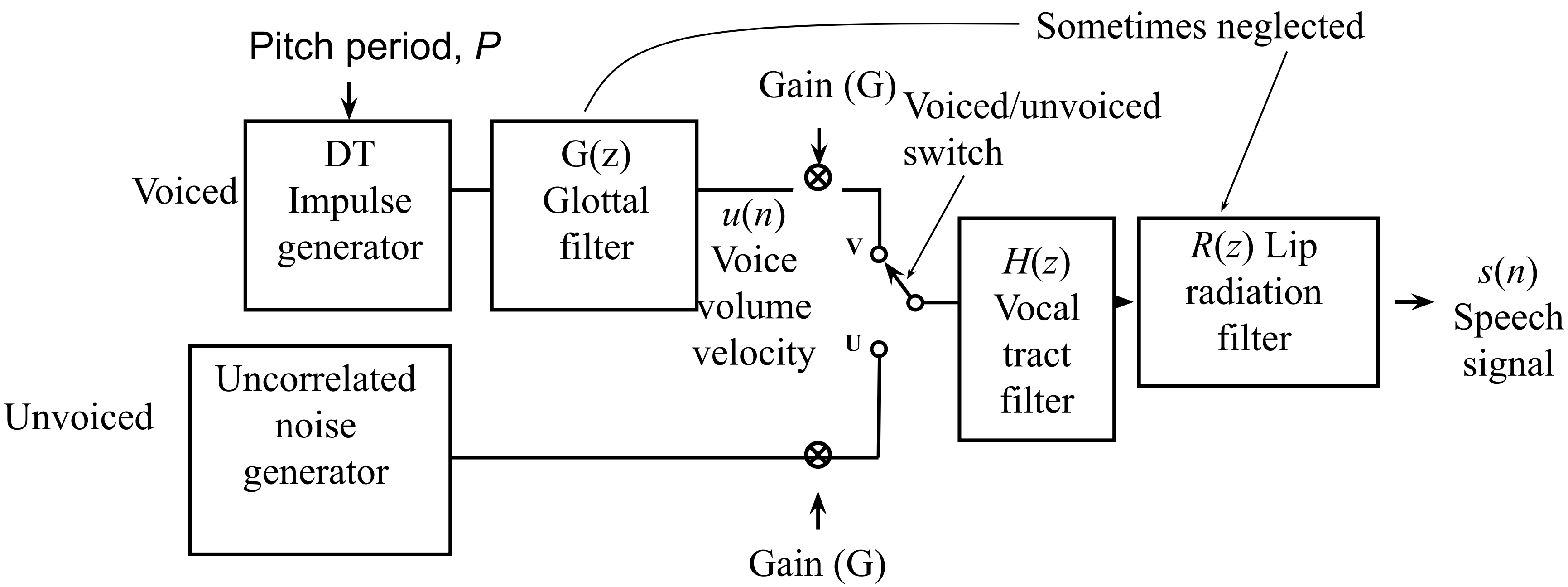
$$\boxed{V(z)=\frac{G}{\prod_{k=1}^{N/2}(1-2r_k\cos\theta_k z^{-1}+r_k^2 z^{-2})}}$$

$$\boxed{R(z)=R_0(1-z^{-1})}$$

$$\boxed{H(z)=\frac{GR_0\cancel{(1-z^{-1})}}{(1-e^{-cT}z^{-1})^{\cancel{2}}\prod_{k=1}^{N/2}(1-2r_k\cos\theta_k z^{-1}+r_k^2 z^{-2})}}$$

$$\boxed{H(z)=\frac{\sigma}{1-\sum_{k=1}^{p}a_k z^{-k}}}$$

General Speech model:



$$H(z) = \frac{G}{1 + \sum_{k=1}^{M} a_k z^{-k}}$$

The LPC Model

$$H(z) = \frac{S(z)}{U(z)} = \frac{A}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

$$S(z)[1 - \sum_{k=1}^{p} a_k z^{-k}] = AU(z)$$

$$S(z) - \sum_{k=1}^{p} a_k S(z) z^{-k} = AU_g(z)$$

$$S(z) = \sum_{k=1}^{p} a_k z^{-k} S(z) + AU_g(z)$$

$$s[n] = \sum_{k=1}^{p} a_k s[n-k] + Au_g[n],$$

$$s[n] = \sum_{k=1}^{p} a_k s[n-k] \quad when \quad u_g(n) = 0$$

$$s[n] \approx a_1 s[n-1] + a_2 s[n-2] + \ldots + a_p s[n-p]$$

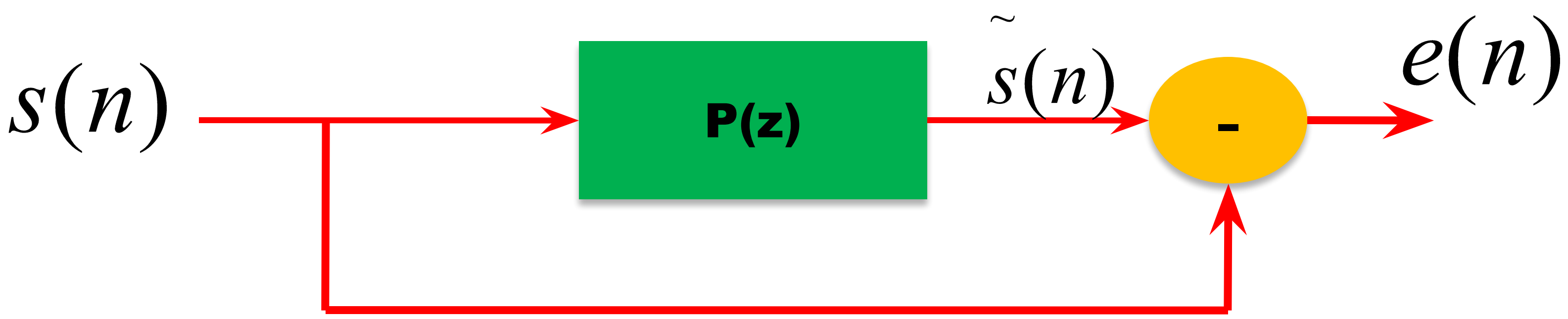
Liner predictor

$$\tilde{s}[n] = \sum_{k=1}^{p} \alpha_k s[n-k]$$

$$P(z) = \sum_{k=1}^{p} \alpha_k z^{-k}$$ FIR Filter of order P

The prediction error:

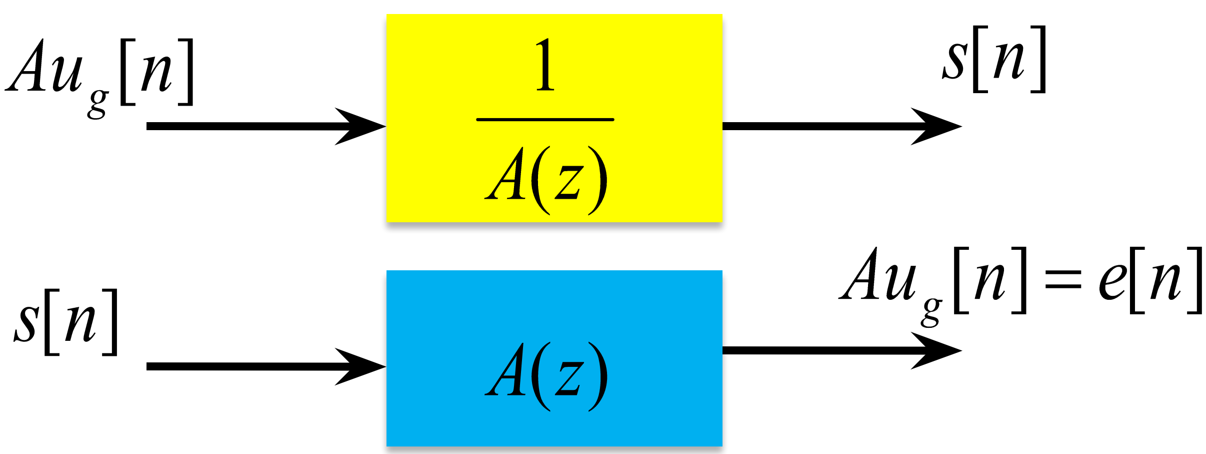
$$e[n] = s[n] - \tilde{s}[n] = s[n] - \sum_{k=1}^{p} \alpha_k s[n-k]$$



Error transfer function:

$$e[n] = s[n] - \tilde{s}[n] = s[n] - \sum_{k=1}^{p} \alpha_k s[n-k]$$

$$A(z) = \frac{E(z)}{S(z)} = 1 - \sum_{k=1}^{p} \alpha_k z^{-k} = 1 - P(z)$$



LP Estimation Issues

- Need to determine $\{\alpha_k\}$ directly from speech such that they give good estimates of the time-varying spectrum

- Need to estimate $\{\alpha_k\}$ from short segments of speech

- Need to minimize mean-squared prediction error over short segments of speech

- resulting $\{\alpha_k\}$ assumed to be the actual $\{a_k\}$ in the speech production model

all of this can be done efficiently, reliably, and accurately for speech

Solution for {α_k}

$$e_n(m) = s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k] \quad \mathit{where} \quad n-m \le m \ge n+m$$

mean squared error signal:

$$E_n = \sum_m e_n^2[m]$$

$$E_n = \sum_m \left[s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k] \right]^2$$

Can find values of α_k that minimize by setting

$$\frac{\partial E_n}{\partial \alpha_i} = 0, \qquad i = 1,2,\ldots,\ p$$

$$\frac{\partial E_n}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k])^2$$

$$= 2 \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k])(-\frac{\partial}{\partial \alpha_i} \sum_{k=1}^{p} \alpha_k s_n[m-k])$$

Wher
e

$$-s_n[m-i] = -\frac{\partial}{\partial \alpha_i} \sum_{k=1}^{p} \alpha_k s_n[m-k]$$

$$\alpha_k s_n[m-k]) \quad is \quad cons\tan t \quad with \quad respect \quad to \frac{\partial}{\partial \alpha_i} \quad for \quad k \neq i$$

$$0 = 2\sum_{-\infty}^{\infty}(s_n[m] - \sum_{k=1}^{p}\alpha_k s_n[m-k])(-s_n[m-i])$$

$$\sum_{-\infty}^{\infty} s_n[m-i]s_n[m] = \sum_{k=1}^{p}\alpha_k \sum_{-\infty}^{\infty} s_n[m-i]S_n[m-k] \qquad 1 \le i \le p \qquad (1)$$

let

$$\phi_n[i,k] = \sum_{m} S_n[m-i]S_n[m-k] \qquad 1 \le i \le p$$

then

$$\phi_n[i,0] = \sum_{k=1}^{p}\alpha_k \phi_n[i,k] \qquad i = 1,2,\ldots,p \qquad (2)$$

leading to a set of *p equations in p* unknowns that can solved in an efficient manner for the $\{\alpha_k\}$

$$
\begin{bmatrix}
\phi_n[1,1] & \phi_n[1,2] & \phi_n[1,3] & \cdots & \phi_n[1,p] \\
\phi_n[2,1] & \phi_n[2,2] & \phi_n[2,3] & \cdots & \phi_n[2,p] \\
\phi_n[3,1] & \phi_n[3,2] & \phi_n[3,3] & \cdots & \phi_n[3,p] \\
\vdots & \vdots & \vdots & & \vdots \\
\phi_n[p,1] & \phi_n[p,2] & \phi_n[p,3] & \cdots & \phi_n[p,p]
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ \alpha_p
\end{bmatrix}
=
\begin{bmatrix}
\phi_n[1,0] \\ \phi_n[2,0] \\ \phi_n[3,0] \\ \vdots \\ \phi_n[p,0]
\end{bmatrix}
$$

The resulting covariance matrix is symmetric, but not Toeplitz, and can be solved efficiently by a set of techniques called Cholesky decomposition

Minimum mean-squared prediction error can be written as

$$E_n = \sum_{-\infty}^{\infty} s_n^2[m] - \sum_{k=1}^{p} \alpha_k \sum_{-\infty}^{\infty} s_n[m] S_n[m-k]$$

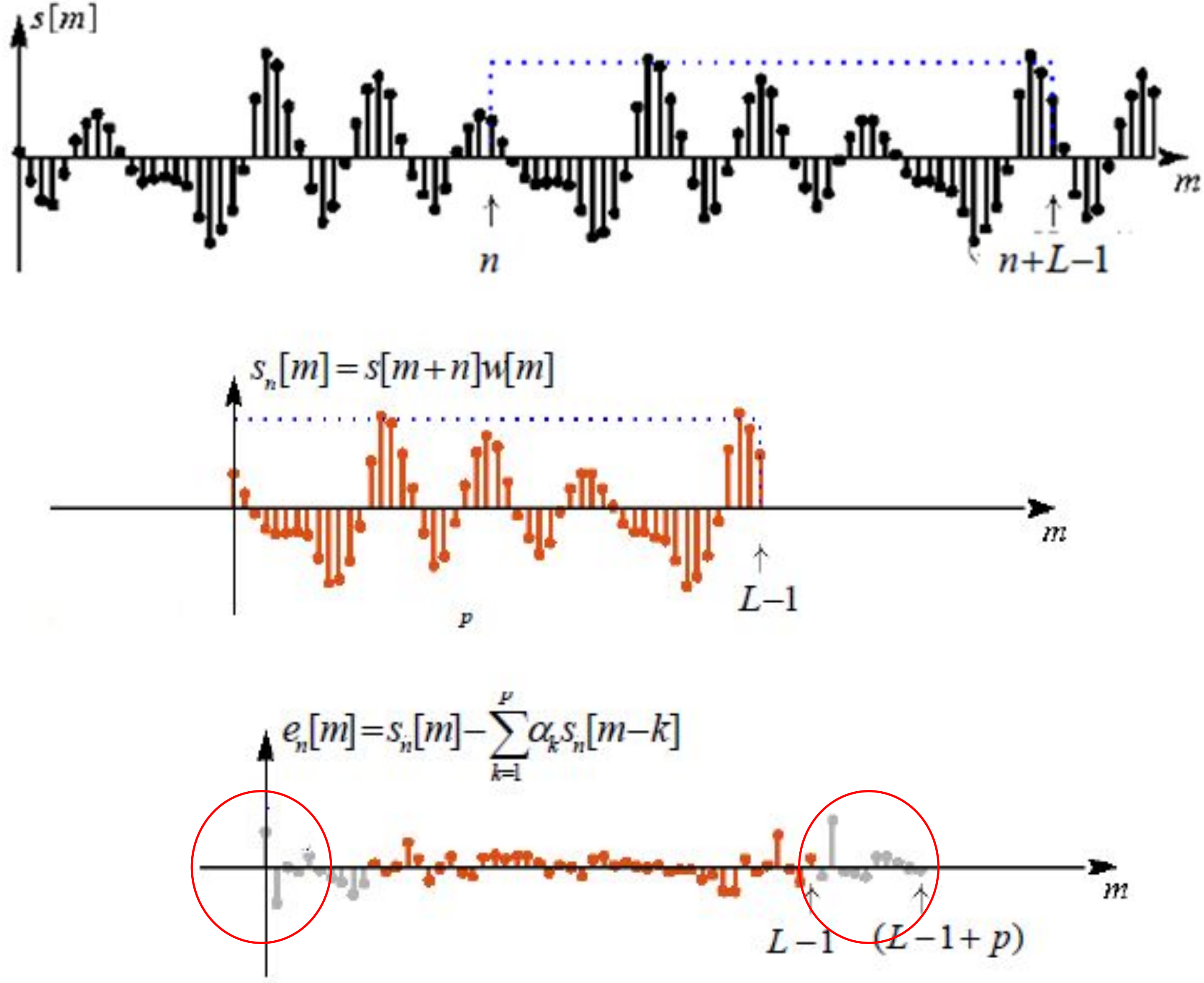
$$E_n = \phi_n[0,0] - \sum_{k=1}^{p} \alpha_k \phi_n[0,k]$$

Autocorrelation Method

Let $s_n[m]$ exists for $0 \le m \le L-1$ and is exactly zero every where else (i.e., window of length samples)

$$s_n[m] = s[m]w[m]$$

Where $w[m]$ is a finite length window of L samples


$s[m]$
m
n
$n+L-1$
$s_n[m]=s[m+n]w[m]$
m
$L-1$
p
$e_n[m]=s_n[m]-\sum_{k=1}^{p}\alpha_k s_n[m-k]$
m
$L-1$
$(L-1+p)$

$$e_n(m) = s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k]$$

Is only non zero over the interval $0 \le m \le L-1+p$

$$E_n = \sum_m e_n^2[m] = \sum_{m=0}^{L-1+p} e_n^2[m]$$

$$\phi_n[i,k] = \sum_m S_n[m-i] S_n[m-k] \quad 1 \le i \le p, \quad 1 \le k \le p$$

$\sin ce$

$$S_n[m] = 0 \quad outside \quad the \quad 0 \le m \le L-1$$

$$\phi_n[i,k] = \sum_{m=0}^{L-1+p} S_n[m-i] S_n[m-k] \quad 1 \le i \le p, \quad 0 \le k \le p$$

$$\phi_n[i,k] = \sum_{m=0}^{L-1+(i-k)} S_n[m] S_n[m+i-k] \quad 1 \le i \le p, \quad 0 \le k \le p$$

$$E_n = \sum_{m=-\infty}^{\infty} e_n^2[m]$$

$$E_n = \sum_{m=-\infty}^{\infty} [s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k]]^2$$

$$= \sum_{m=-\infty}^{\infty} s_n^2\ [m] - 2 \sum_{m=-\infty}^{\infty} s_n^2\ [m] \sum_{k=1}^{p} \alpha_k s_n[m-k]$$

$$+ \sum_{m=-\infty}^{\infty} \sum_{k=1}^{p} \alpha_k s_n[m-k] \sum_{l=1}^{p} \alpha_l s_n[m-l]$$

$$= \sum_{m=-\infty}^{\infty} s_n^2\ [m] - \sum_{k=1}^{p} \alpha_k \sum_{m=-\infty}^{\infty} s_n[m-k] s_n[m]$$

$$= \varphi_n[0,0] - \sum_{k=1}^{p} \alpha_k \varphi_n[0,k]$$

There are L-| i-k| non-zero terms in the computation of $\phi_n[i,k]$ for each value of i,k

Then $\phi_n[i,k] = R_n[i-k]$ short-time autocorrelation

$$R_n[k] = \sum_{m=0}^{L-1+k} S_n[m] S_n[m+k]$$

From equation (2)

$$\sum_{k=1}^{p} \alpha_k \phi_n[i,k] = \phi_n[i,0] \quad 1 \le i \le p$$

$$\sum_{k=1}^{p} \alpha_k R_n[i-k] = R_n[i] \quad 1 \le i \le p$$

Minimum mean-squared prediction error can be written as

$$E_n = \phi_n[0,0] - \sum_{k=1}^{p} \alpha_k \phi_n[0,k]$$

$$= R_n[0] - \sum_{k=1}^{p} \alpha_k R_n[k]$$

$$\begin{bmatrix} R_n[0) & R_n[1] & R_n[2] & \cdots & R_n[p-1] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[p-2] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[p-3] \\ \vdots & \vdots & \vdots & & \vdots \\ R_n[p-1] & R_n[p-2] & R_n[p-3] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ \alpha_p \end{bmatrix} = \begin{bmatrix} R_n[1] \\ R_n[2] \\ R_n[3] \\ \vdots \\ R_n[p] \end{bmatrix}$$

$$\Re\alpha = r$$

$$\alpha = \Re^{-1} r$$

$\Re$ is a pxp Toeplitz Matrix => symmetric with all diagonal elements equal

matrix equation solved using Levinson or Durbin method

The set of optimum predictor coefficients satisfy

$$\sum_{k=1}^{p} \alpha_k R_n[i-k] - R_n[i] = 0 \qquad 1 \le i \le p$$

with minimum mean-squared prediction error of

$$R_n[0] - \sum_{k=1}^{p} \alpha_k R_n[k] = E^{(p)}$$

$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & \cdots & \cdots & \cdots & R_n[p] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & \cdots & \cdots & \cdots & R_n[p-1] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & \cdots & \cdots & \cdots & R_n[p-2] \\ \vdots & \vdots & \vdots & & & & & \vdots \\ R_n[p-1] & R_n[p-2] & R_n[p-3] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(p)} \\ -\alpha_2^{(p)} \\ . \\ -\alpha_p^{(p)} \end{bmatrix} = \begin{bmatrix} E_n^p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Expanded matrix is still Toeplitz and It can be solved iteratively by incorporating new correlation value at each iteration and solving for next higher order predictor in terms of new correlation value and previous predictor

i^{th} order solution can be derived from $(i\text{-}1)^{st}$ order solution

given $\alpha^{(i-1)}$, the solution to $\quad R_n^{i-1}\alpha^{i-1} = E_n^{i-1}$

we derive solution to $\quad R_n^{i}\alpha^{i} = E_n^{i}$

The $(i-1)^{st}$ *solution can be expressed* as:

$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[i-1] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[i-2] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[i-3] \\ \vdots & \vdots & \vdots & & \vdots \\ R_n[i-1] & R_n[i-2] & R_n[i-3] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \end{bmatrix} = \begin{bmatrix} E_n^{i-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Appending a 0 to vector α *and multiplying by the matrix* $R_n^{(i)}$ *gives*

$$
\begin{bmatrix}
R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[i] \\
R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[i-1] \\
R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[i-2] \\
\vdots & \vdots & \vdots & & \vdots \\
R_n[i-1] & R_n[i-2] & R_n[i-3] & \cdots & R_n[1] \\
R_n[i] & R_n[i-1] & R_n[i-2] & \cdots & R_n[0]
\end{bmatrix}
\begin{bmatrix}
1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ 0
\end{bmatrix}
=
\begin{bmatrix}
E_n^{i-1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma^{(i-1)}
\end{bmatrix}
$$

$$
where \quad \gamma^{(i-1)} = R_n[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R_n[i-j]
$$

Toeplitz matrix has special symmetry we can reverse the order of the equations

$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & \cdots & \cdots & \cdots & R_n[i]] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & \cdots & \cdots & \cdots & R_n[i-1] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & \cdots & \cdots & \cdots & R_n[i-2] \\ \vdots & \vdots & \vdots & & & & & \vdots \\ R_n[i-1] & R_n[i-2] & R_n[i-3] & & & & \cdots & R_n[1] \\ R_n[i] & R_n[i-1] & R_n[i-2] & & & \cdots & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 0 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma^{(i-1)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ E_n^{i-1} \end{bmatrix}$$

Combine the two sets of matrices with a multiplicative factor k_i

$$R_n^i \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} - k_i \begin{bmatrix} 0 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ 1 \end{bmatrix} = \begin{bmatrix} E_n^{i-1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma^{(i-1)} \end{bmatrix} - k_i \begin{bmatrix} \gamma^{(i-1)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ E_n^{i-1} \end{bmatrix}$$

Choose *of* $\gamma^{(i-1)}$ so that vector on right has only a single non-zero entry, i.e

$$k_i = \frac{\gamma^{(i-1)}}{E^{i-1}} = \frac{R_n[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R_n[i-j]}{E^{i-1}} \qquad 1$$

The first element of the right hand side vector is now

$$E_n^{(i)} = E_n^{(i-1)} - k_i \gamma^{(i-1)} = E_n^{(i-1)}(1 - k_i^2) \qquad 2$$

The k_i parameters are called Partial Correlation (PARCOR) coefficients

So the vector of order i^{th} predictor coefficients is

$$\begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ -\alpha_i^{(i)} \end{bmatrix} = \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} - k_i \begin{bmatrix} 0 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ . \\ -\alpha_{i-1}^{(i-1)} \\ 1 \end{bmatrix}$$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$$ 3

$$\alpha_i^{(i)} = k_i$$ 4

A. The final solution for order *p* is:

$$\alpha_j = \alpha_j^p \qquad 1 \le j \le p$$

B. with prediction error

$$E_n^{(p)} = E_n[0] \prod_{m=1}^{p} (1 - k_m^2) = R_n[0] \prod_{m=1}^{p} (1 - k_m^2)$$

C. If we use normalized autocorrelation coefficients:

$$r_n[k] = \frac{R_n[k]}{R_n[0]}$$

D. normalized prediction error

$$v^{(i)} = \prod_{m=1}^{i} (1 - k_m^2) \quad 0 \le v^{(i)} \le 1 \quad -1 \le k_i \le 1$$

$$\mathcal{E}^{(0)} = R[0]$$

for $i = 1, 2, \ldots, p$

$$k_i = \left(R[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R[i-j] \right) / \mathcal{E}^{(i-1)}$$

$$\alpha_i^{(i)} = k_i$$

if $i > 1$ then for $j = 1, 2, \ldots, i-1$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$$

end

$$\mathcal{E}^{(i)} = (1 - k_i^2)\mathcal{E}^{(i-1)}$$

end

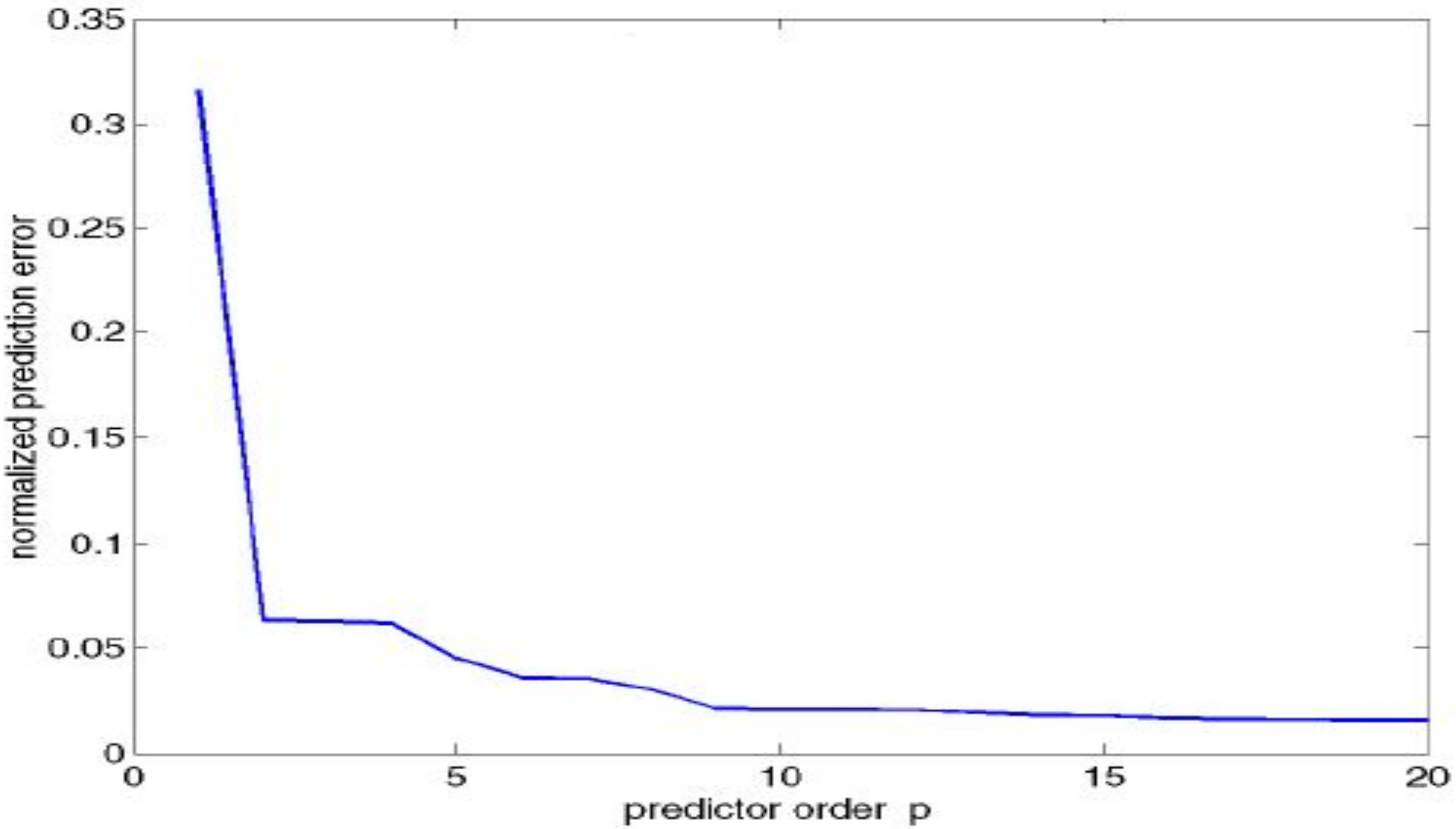
$$\alpha_j = \alpha_j^{(p)} \quad j = 1, 2, \ldots, p$$

1. Consider a simple p = *2 find the solution*

2. Consider a simple p = *3 find the solution*

3. Consider a simple p = *4 find the solution*

Prediction Error vs. Model Order



Model order is usually determined by:

- Fs/1000 poles for vocal tract
- 2-4 poles for radiation
- 2 poles for glottal pulse

End

Computation of Model Gain

$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}}$$

$$h[m] = \sum_{k=1}^{p} \alpha_k h[m-k] + G\delta[m]$$ 1

$$Multiply \quad by \quad h[m-i]$$

$$h[m]h[m-i] = \sum_{k=1}^{p} \alpha_k h[m-k]h[m-i] + G\delta[m]h[m-i]$$

$$let \quad i = 0$$

$$\sum_{m=-\infty}^{\infty} h[m]h[m] = \sum_{k=1}^{p} \alpha_k \sum_{m=-\infty}^{\infty} h[m-k]h[m] + G \sum_{m=-\infty}^{\infty} \delta[m]h[m]$$

$$r_h[0] = \sum_{k=1}^{p} \alpha_k r_h[k] + G^2 \quad \text{since} \quad h[0] = G \quad \textit{Put } m=0 \textit{ in equation (1)}$$

Energy in h[m]=Energy in short time measurement $s_n[m]$

$$r_h[0] = r_n[0]$$

$$and \quad r_h[\tau] = r_n[\tau] \quad for \quad \tau \le p$$

$$so$$

$$G^2 = r_h[0] - \sum_{k=1}^{p} \alpha_k r_h[k] = r_n[0] - \sum_{k=1}^{p} \alpha_k r_n[k] = E_n$$

Solution for Gain (Voiced)

Solution for Gain (Unvoiced)

For unvoiced speech the input is white noise with zero mean and unity variance,

Excite the system with input Au[n]

$$Y(z) = H(z)U(z)$$

$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}}$$

$$Y(z) - \sum_{k=1}^{p} \alpha_k z^{-k} Y(z) = GU(z)$$

$$Y(z) = \sum_{k=1}^{p} \alpha_k z^{-k} Y(z) + GU(z)$$

$$y[m] = \sum_{k=1}^{p} \alpha_k y[m-k] + Gu[m]$$

$$Multiply \quad by \quad y[m-i]$$

$$y[m]y[m-i] = \sum_{k=1}^{p} \alpha_k y[m-k]y[m-i] + Gu[m]y[m-i]$$

$$let \quad i = 0$$

$$\sum_{m=-\infty}^{\infty} y[m]y[m] = \sum_{k=1}^{p} \alpha_k \sum_{m=-\infty}^{\infty} y[m-k]y[m] + G \sum_{m=-\infty}^{\infty} u[m]y[m]$$

$$r_y[0] = \sum_{k=1}^{p} \alpha_k r_y[k] + G^2 \quad \sin ce \quad h[0] = G$$

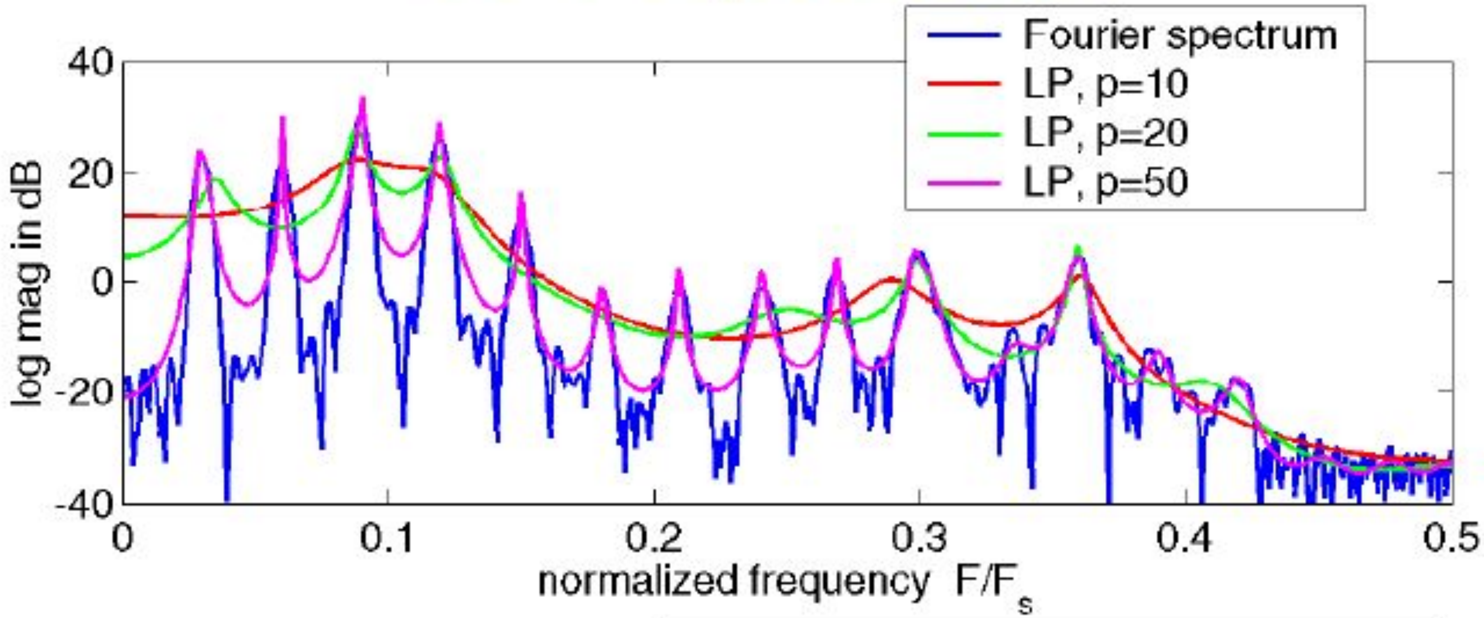
Frequency Domain Interpretations of Linear Predictive Analysis

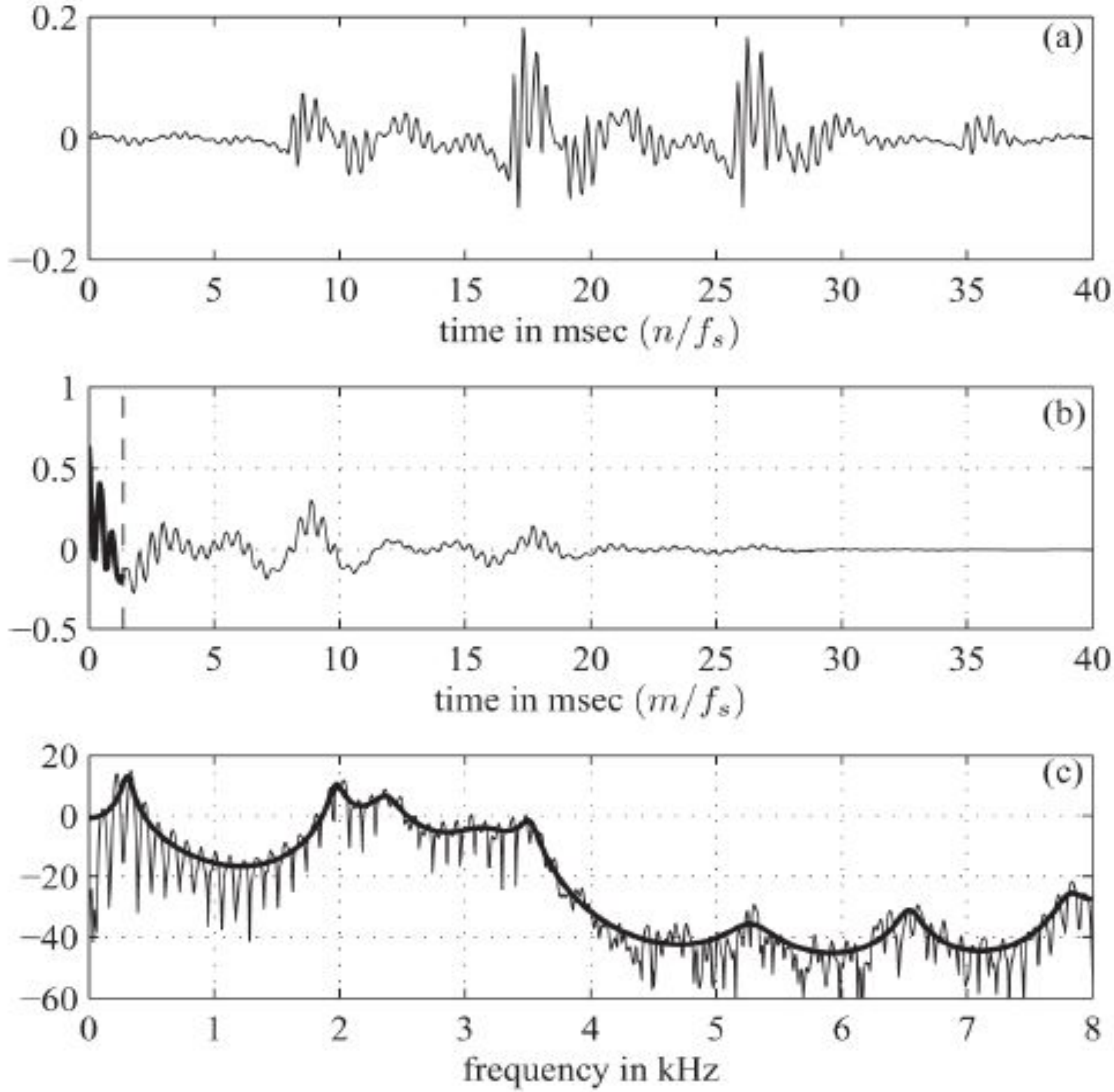
The final LPC model consists of the LPC parameters,α_k and gain A together which define the system function

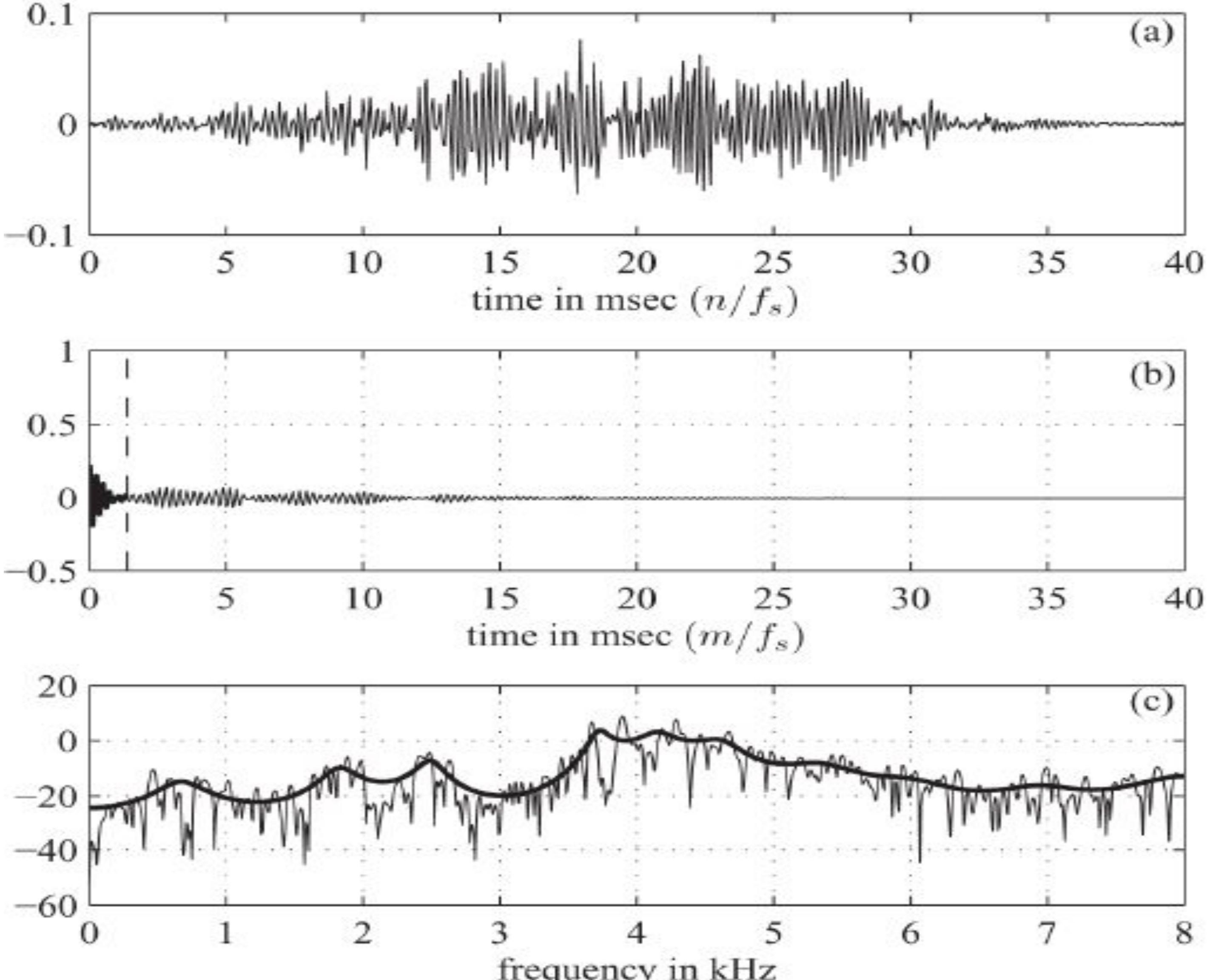
$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}}$$

$$H(e^{j\omega}) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k e^{j\omega k}}$$

LPC Spectrum




0.2
0
−0.2
(a)
0 5 10 15 20 25 30 35 40
time in msec (n/f_s)
1
0.5
0
−0.5
(b)
0 5 10 15 20 25 30 35 40
time in msec (m/f_s)
20
0
−20
−40
−60
(c)
0 1 2 3 4 5 6 7 8
frequency in kHz

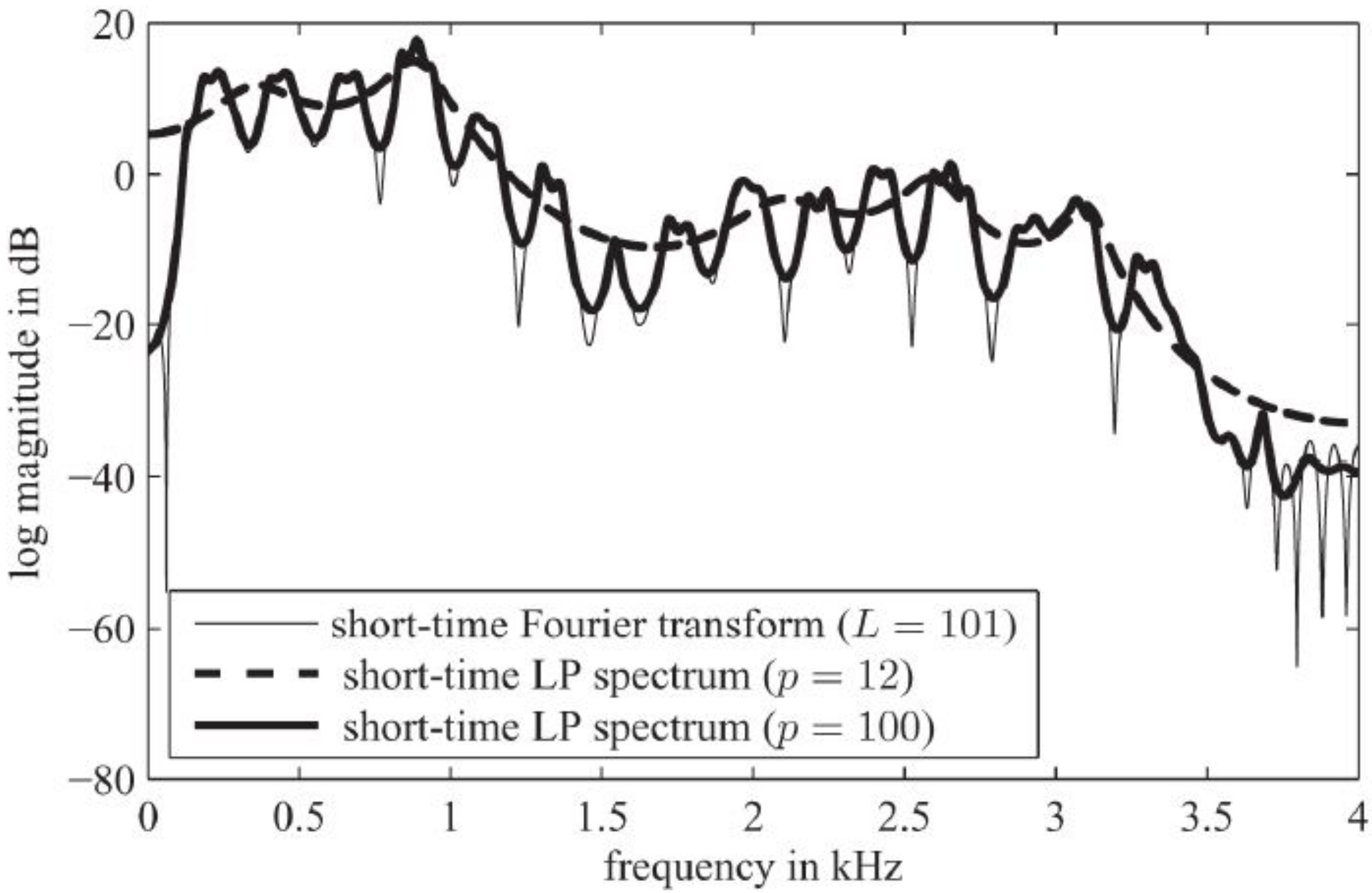


(a)
0.1
0
−0.1
0
5
10
15
20
25
30
35
40
time in msec (n/f_s)
(b)
1
0.5
0
−0.5
0
5
10
15
20
25
30
35
40
time in msec (m/f_s)
(c)
20
0
−20
−40
−60
0
1
2
3
4
5
6
7
8
frequency in kHz

Effects of Model Order

The AC $R_n[m]$ function of the speech segment $S_n[m]$, and the AC function $\hat{R}_n[m]$, of the impulse response, corresponding to the system function $h_n[m]$, are equal for the first (p +1) *values.* As p->α the AC functions are equal for all values and thus:

$$\lim_{p->\alpha} \left| \hat{H}(e^{j\omega}) \right|^2 = \left| S_n(j\omega) \right|^2$$



20
0
−20
−40
−60
−80
log magnitude in dB
0
0.5
1
1.5
2
2.5
3
3.5
4
frequency in kHz
short-time Fourier transform ($L = 101$)
short-time LP spectrum ($p = 12$)
short-time LP spectrum ($p = 100$)

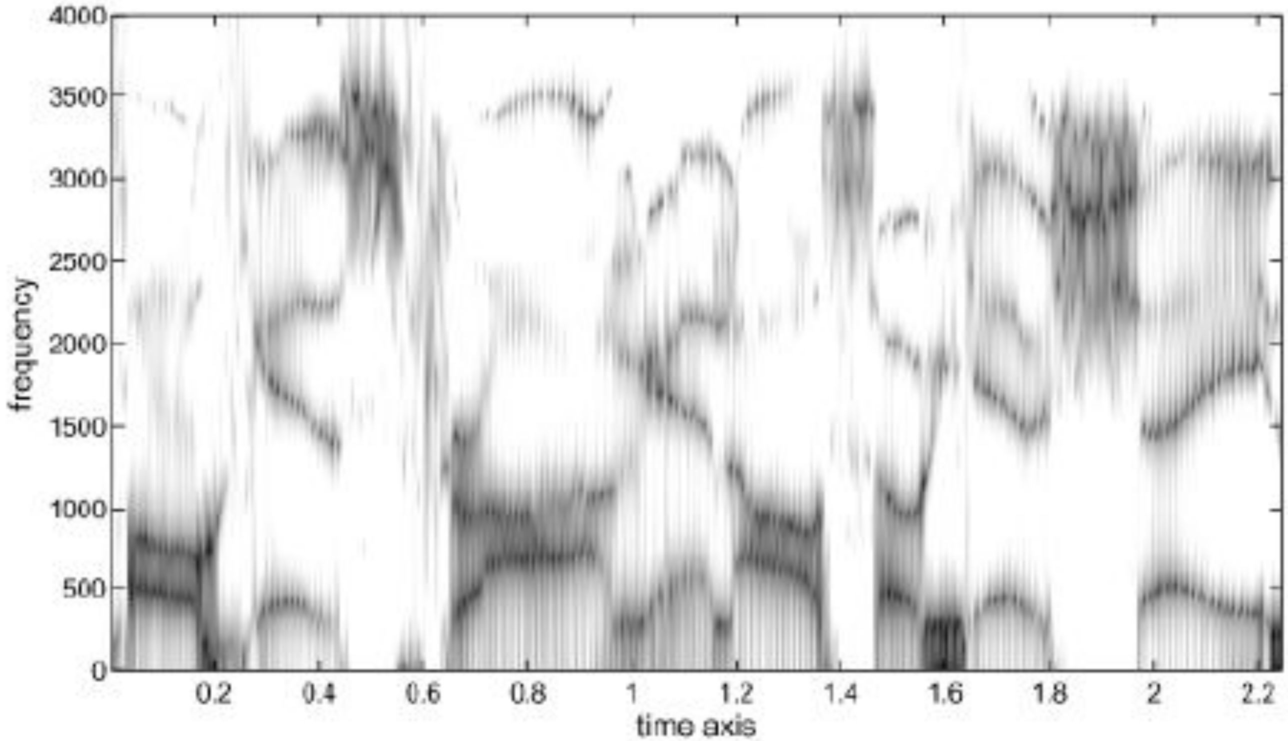
Linear Prediction Spectrogram

Speech spectrogram

$$20\log(|S_n(k)|) = 20\log\left|\sum_{m=0}^{L-1} s_n[nR+m]w[m]e^{-j(2\pi mk/N)}\right|$$

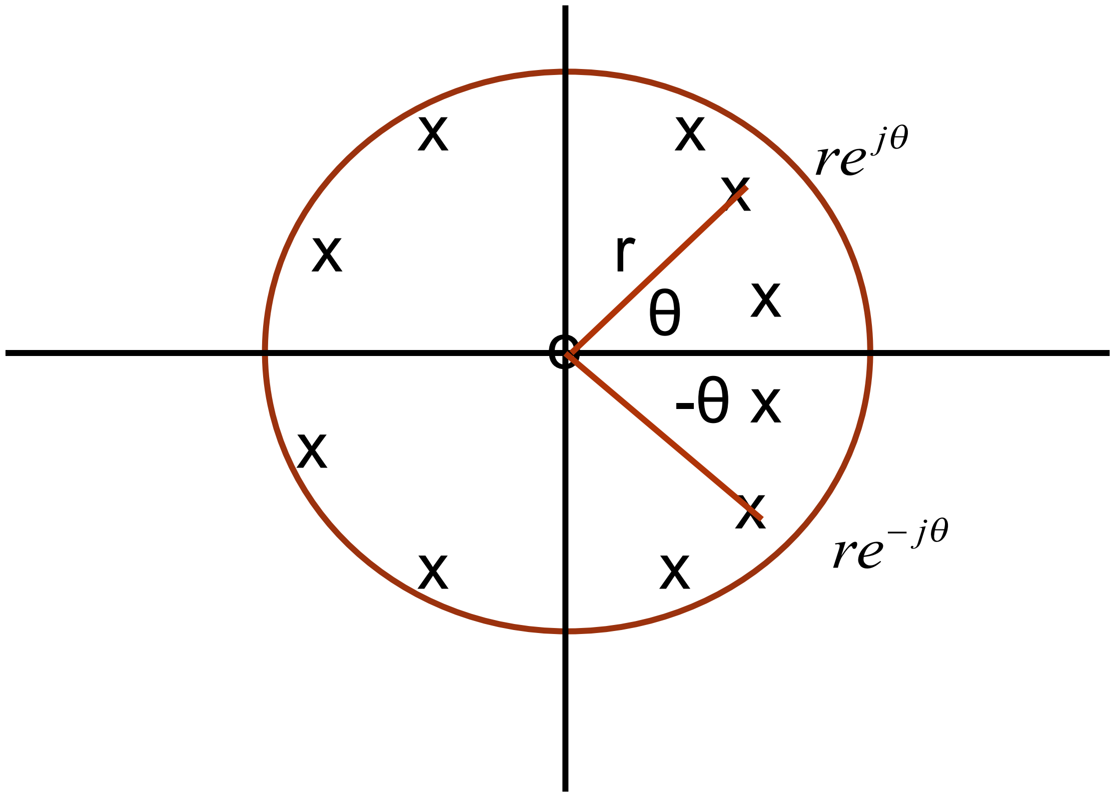
LPC spectrogram

$$20\log(|H_n(k)|) = 20\log\frac{G}{\left|\sum_{m=0}^{L-1} A_n e^{j(2\pi k/N)}\right|}$$



4000
3500
3000
2500
2000
1500
1000
500
0
frequency
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
time axis

$$H(z) = \frac{G}{A(z)} = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}} = \frac{Gz^p}{\prod_{k=1}^{p} (z - z_k)}$$



Angle corresponds to frequency, and radius corresponds to bandwidth. So we can determine the pole (or resonant) frequencies and bandwidths (converting to Hz)

$$\text{Frequency} = \frac{F_s}{2\pi}\theta$$

$$\text{Bandwidth} = \text{-log(r)}.\frac{\mathrm{F_s}}{\pi} = -\log(abs(z_k)).\frac{\mathrm{F_s}}{\pi}$$

Lecture-16

Covariance Method for Linear Prediction

Can find values of α_k that minimize by setting

$$\frac{\partial E_n}{\partial \alpha_i} = 0, \qquad i = 1,2,\dots,\ p$$

$$\frac{\partial E_n}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k])^2$$

$$= 2\sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k])(-\frac{\partial}{\partial \alpha_i} \sum_{k=1}^{p} \alpha_k s_n[m-k])$$

Wher
e

$$-s_n[m-i] = -\frac{\partial}{\partial \alpha_i} \sum_{k=1}^{p} \alpha_k s_n[m-k]$$

$$\alpha_k s_n[m-k]) \quad is \quad constant \quad with \quad respect \quad to \frac{\partial}{\partial \alpha_i} \quad for \quad k \neq i$$

$$0 = 2\sum_{-\infty}^{\infty}\left(s_n[m] - \sum_{k=1}^{p}\alpha_k s_n[m-k]\right)(-s_n[m-i])$$

$$\sum_{-\infty}^{\infty} s_n[m-i]s_n[m] = \sum_{k=1}^{p}\alpha_k \sum_{-\infty}^{\infty} s_n[m-i]S_n[m-k] \qquad 1 \le i \le p \qquad (1)$$

let

$$\boxed{\phi_n[i,k] = \sum_{m} S_n[m-i]S_n[m-k] \qquad 1 \le i \le p}$$

then

$$\phi_n[i,0] = \sum_{k=1}^{p}\alpha_k \phi_n[i,k] \qquad i = 1,2,\ldots,p \qquad (2)$$

leading to a set of *p equations in p* unknowns that can solved in an efficient manner for the *{α* $\{\alpha_k\}$

Minimum mean-squared prediction error

$$E_n = \sum_{m=-\infty}^{\infty} e_n^2[m]$$

$$E_n = \sum_{m=-\infty}^{\infty} [s_n[m] - \sum_{k=1}^{p} \alpha_k s_n[m-k]]^2$$

$$= \sum_{m=-\infty}^{\infty} s_n^2\,[m] - 2\sum_{m=-\infty}^{\infty} s_n[m] \sum_{k=1}^{p} \alpha_k s_n[m-k]$$

$$+ \sum_{m=-\infty}^{\infty} \sum_{k=1}^{p} \alpha_k s_n[m-k] \sum_{l=1}^{p} \alpha_l s_n[m-l]$$

$$\sum_{m=-\infty}^{\infty} \sum_{k=1}^{p} \alpha_k s_n[m-k] \sum_{l=1}^{p} \alpha_l s_n[m-l]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=1}^{p} \alpha_k s_n[m-k] s_n[m]$$

$$= \sum_{m=-\infty}^{\infty} s_n[m] \sum_{k=1}^{p} \alpha_k s_n[m-k]$$

$$= \sum_{m=-\infty}^{\infty} s_n^2\,[m] - \sum_{k=1}^{p} \alpha_k \sum_{m=-\infty}^{\infty} s_n[m-k] s_n[m]$$

$$= \varphi_n[0,0] - \sum_{k=1}^{p} \alpha_k \varphi_n[0,k]$$

$$\phi_n[i,k] = \sum_m S_n[m-i]S_n[m-k] \quad 1 \le i \le p$$

$$\phi_n[i,k] = \sum_{m=0}^{L-1} S_n[m-i]S_n[m-k] \quad 1 \le i \le p \quad 0 \le k \le p$$

Changing the summation index gives

$$\phi_n[i,k] = \sum_{m=-i}^{L-i-1} S_n[m]S_n[m+i-k] \quad 1 \le i \le p \quad 0 \le k \le p$$

$$\phi_n[i,k] = \sum_{m=-k}^{L-k-1} S_n[m]S_n[m+k-i] \quad 1 \le i \le p \quad 0 \le k \le p$$

key difference from Autocorrelation Method is that limits of summation include terms before *m =0. i.e* window extends *p* samples backwards

since we are extending window backwards, don't need to taper it using window function since there is ***no transition at window edges***

$$\phi_n[i,0] = \sum_{k=1}^{p} \alpha_k \phi_n[i,k] \qquad i = 1,2,\ldots,p \qquad \text{1}$$

$$E_n = \varphi_n[0,0] - \sum_{k=1}^{p} \alpha_k \varphi_n[0,k] \qquad \text{2}$$

$$\begin{bmatrix} \phi_n[1,1] & \phi_n[1,2] & \phi_n[1,3] & \cdots & \phi_n[1,p] \\ \phi_n[2,1] & \phi_n[2,2] & \phi_n[2,3] & \cdots & \phi_n[2,p] \\ \phi_n[3,1] & \phi_n[3,2] & \phi_n[3,3] & \cdots & \phi_n[3,p] \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_n[p,1] & \phi_n[p,2] & \phi_n[p,3] & \cdots & \phi_n[p,p] \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \phi_n[1,0] \\ \phi_n[2,0] \\ \phi_n[3,0] \\ \vdots \\ \phi_n[p,0] \end{bmatrix}$$

$$\varphi\alpha = \psi$$

$$\alpha = \varphi^{-1}\psi$$

the solution of the matrix equation is called the Cholesky decomposition, or square root method

φ Matrix is symmetric but not diagonal elements are same

$$\phi = ADA^t$$

Where A = lower triangular matrix with 1's on the main diagonal; D=diagonal matrix

determine elements of A and D by solving for (i, j) elements of the matrix equation

Let p=4 and matrix element $\phi_n(i,j)=\varphi_{ij}$

$$\begin{bmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi_{21} & \phi_{22} & \phi_{32} & \phi_{42} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{43} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & 1 & 0 & 0 \\ A_{31} & A_{32} & 1 & 0 \\ A_{41} & A_{42} & A_{43} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} \begin{bmatrix} 1 & A_{21} & A_{31} & A_{41} \\ 0 & 1 & A_{32} & A_{42} \\ 0 & 0 & 1 & A_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve matrix

$$d_1 = \varphi_{11}$$

$$A_{21}d_1 = \varphi_{21} --> A_{21} = \frac{\varphi_{21}}{d_1}$$

$$A_{31}d_1 = \varphi_{31} --> A_{31} = \frac{\varphi_{31}}{d_1}$$

$$A_{41}d_1 = \varphi_{41} --> A_{41} = \frac{\varphi_{41}}{d_1}$$

$$d_j = \varphi_{jj} - \sum_{k=1}^{j-1} A_{jk}^2 d_k$$

$$A_{i1} = \phi_{i1} / d_1$$

Else

$$A_{ij} = \varphi_{ij} - \sum_{k=1}^{j-1} A_{ik} d_k A_{jk} / d_j$$

$$d_p = \varphi_{pp} - \sum_{k=1}^{p-1} A_{pk}^2 d_k$$

$$d_2 = \varphi_{22} - A_{21}^2 d_1$$

$$A_{32}d_2 = \varphi_{32} - A_{31}d_1A_{21} -> A_{32} = \frac{\varphi_{32} - A_{31}d_1A_{21}}{d_2}$$

$$A_{42}d_2 = \varphi_{42} - A_{41}d_1A_{21} -> A_{42} = \frac{\varphi_{42} - A_{41}d_1A_{21}}{d_2}$$

iterate procedure to solve for d_3, A_{43}, d_4

$d_1=\varphi_{11}$
for(i=1; i<=p;i++)
{
$$A_{i1} = \phi_{i1} / d_1$$
}

for(j=2; j<p; j++)
{
$$d_j = \varphi_{jj} - \sum_{k=1}^{j-1} A_{jk}^2 d_k$$
for(i=j+1; i<=p; i++)
{
$$A_{ij} = \varphi_{ij} - \sum_{k=1}^{j-1} A_{ik} d_k A_{jk} / d_j$$
}
}

$$ADA^t\alpha = \psi$$

$$Let \quad AY = \psi$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & 1 & 0 & 0 \\ A_{31} & A_{32} & 1 & 0 \\ A_{41} & A_{42} & A_{43} & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

$$Y_1 = \psi_1$$

$$Y_2 = \psi_2 - A_{21}Y_1$$

$$Y_3 = \psi_3 - A_{31}Y_1 - A_{32}Y_2$$

$$Y_4 = \psi_4 - A_{41}Y_1 - A_{42}Y_2 - A_{43}Y_3$$

$$Y_i = \psi_i - \sum_{j=1}^{i-1} A_{ij}Y_j$$

$Y_1 = \psi_1$

for(i=1; i<=p; i++)

{

$$Y_i = \psi_i - \sum_{j=1}^{i-1} A_{ij}Y_j$$

}

$$=> DA^t\alpha = Y => A^t\alpha = D^{-1}Y$$

$$\begin{bmatrix} 1 & A_{21} & A_{31} & A_{41} \\ 0 & 1 & A_{32} & A_{42} \\ 0 & 0 & 1 & A_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1/d_1 & 0 & 0 & 0 \\ 0 & 1/d_2 & 0 & 0 \\ 0 & 0 & 1/d_3 & 0 \\ 0 & 0 & 0 & 1/d_4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$\alpha_4 = \frac{Y_4}{d_4}$$

$$\alpha_i = \frac{y_i}{d_i} - \sum_{j=i+1}^{p} A_{ji}\alpha_j$$

$$\alpha_3 = \frac{Y_3}{d_3} - A_{43}\alpha_4$$

$$\alpha_2 = \frac{Y_2}{d_2} - A_{43}\alpha_4 - A_{32}\alpha_3$$

$$\alpha_1 = \frac{Y_1}{d_1} - A_{43}\alpha_4 - A_{32}\alpha_3 - A_{21}\alpha_2$$

$\alpha_p = Y_p / d_p$

for(i=1; i<=p; i++)

{

$$\alpha_i = \frac{y_i}{d_i} - \sum_{j=i+1}^{p} A_{ji}\alpha_j$$

}

Calculation proceeds backwards from i=p-1 to 1

Lattice Formulations of Linear Prediction

Both covariance and autocorrelation methods of LP use two step solutions

Step-1: computation of a matrix of correlation values

Step-2: efficient solution of a set of linear equations

In lattice methods, two steps are combined into a recursive algorithm for determining LP parameters

It begin with Durbin algorithm--at the i^{th} *stage the set of* coefficients are coefficients of the i^{th} order optimum LP

$$H(z)=\frac{A}{1-\sum_{k=1}^{p}\alpha_k z^{-k}}$$

$$A(z)=1-\sum_{k=1}^{p}\alpha_k z^{-k}$$

i^{th} *order* prediction error filter

$$A^i(z)=1-\sum_{k=1}^{i}\alpha_k z^{-k}$$

$$e^i[m]=s[m]-\tilde{s}[m]=s[m]-\sum_{k=1}^{i}\alpha_k^i s[m-k]$$

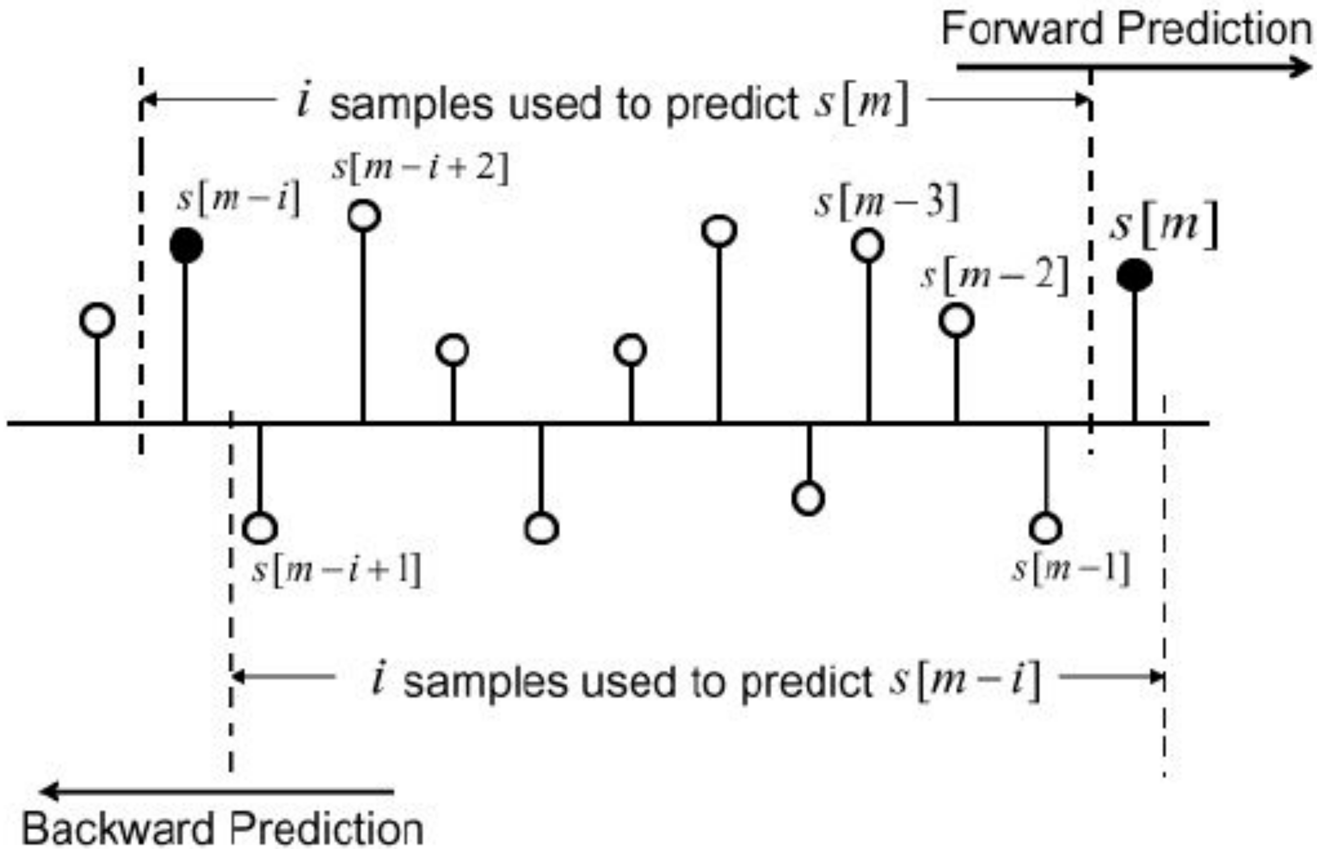
Forward prediction error $\quad E^i(z)=A^i(z)S(z)$

$$b^i[m]=s[m-i]-\sum_{k=1}^{i}\alpha_k^i s[m-i+k]$$

$$B^i(z)=z^{-i}S(z)-\sum_{k=1}^{i}\alpha_k^i z^{-i}z^{k}S(z)=z^{-i}S(z)[1-\sum_{k=1}^{i}\alpha_k^i z^{k}]$$

$$\boxed{B(z)=z^{-i}S(z)A^i(z^{-1})}$$

Lattice Formulations of LP



same set of samples is used to forward predict *s(m)* and backward predict *s(m-i)*

Levinson Recursion

Step-1

$$E^{(0)} = r(0)$$

$$\alpha_0^0 = 0$$

Step-2 *Weighting factor of i^{th}* pole model

$$k_i = \left\{ r(i) - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} r(|i-j|) \right\} \Big/ E^{(i-1)}, \qquad 1 \le i \le p$$

Step-3

$$\alpha_i^{(i)} = k_i$$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$$

Step-4 Update the mean square prediction error

$$E^{(i)} = (1 - k_i^2) E^{(i-1)}$$

$$\alpha_i^i = k_i$$
$$\alpha_j^i = \alpha_j^{i-1} - k_i \alpha_{i-j}^{i-1}$$

$$A^i(z) = 1 - \sum_{k=1}^{i} \alpha_k z^{-k}$$

$$A^i(z) = 1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k} - \alpha_i^i z^{-i} = 1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k} - k_i z^{-i}$$

$$= 1 - \sum_{k=1}^{i-1} [\alpha_k^{i-1} z^{-k} - k_i \alpha_{i-k}^{i-1} z^{-k}] - k_i z^{-i}$$

$$= [1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k}] + k_i \sum_{k=1}^{i-1} \alpha_{i-k}^{i-1} z^{-k} - k_i z^{-i}$$

$$= [1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k}] + k_i \sum_{k'=i-1}^{i} \alpha_k^{i-1} z^{-i+k} - k_i z^{-i} \qquad put \quad k' = i - k$$

$$= [1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k}] - k_i z^{-i} [1 - \sum_{k'=i-1}^{i} \alpha_k^{i-1} z^{k}]$$

$$A^{i-1}(z)$$

$$A^{i-1}(z^{-1})$$

$$A^i(z) = A^{i-1}(z) - k_i z^{-i} A^{i-1}(z^{-1})$$

$$E^i(z) = A^i(z)S(z) = A^{i-1}(z)S(z) - k_i z^{-i} A^{i-1}(z^{-1})S(z)$$

$$= E^{i-1}(z) - k_i z^{-1} B^{i-1}(z)$$

$$e^i[m] = e^{i-1}[m] - k_i b^{i-1}[m-1]$$

$$B^i(z) = z^{-i} S(z) A^i(z^{-1}) = z^{-1} B^{i-1}(z) - k_i E^{i-1}(z)$$

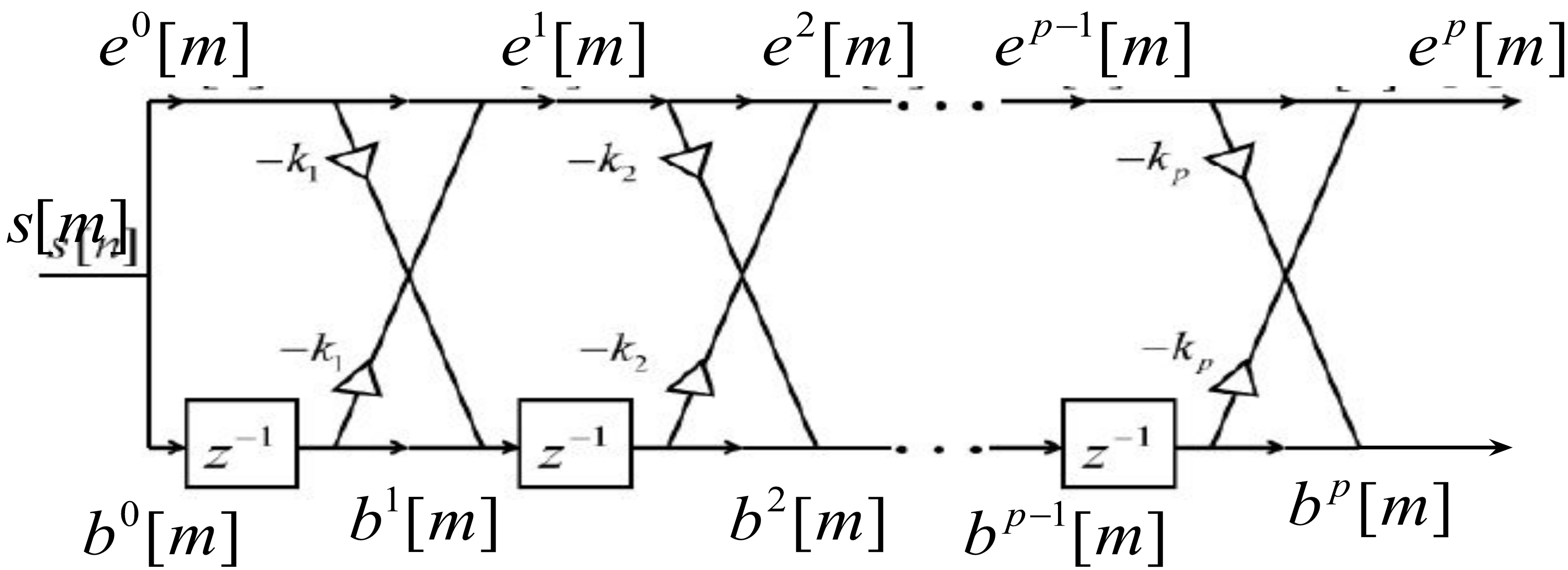
$$b^i[m] = b^{i-1}[m-1] - k_i e^{i-1}[m]$$

$$e^{i}[m] = e^{i-1}[m] - k_i b^{i-1}[m-1]$$ Equation No –(1)

$$b^{i}[m] = b^{i-1}[m-1] - k_i e^{i-1}[m]$$ Equation No –(2)

$$e^{0}[m] = b^{0}[m] = s[m]$$ Equation No –(3)

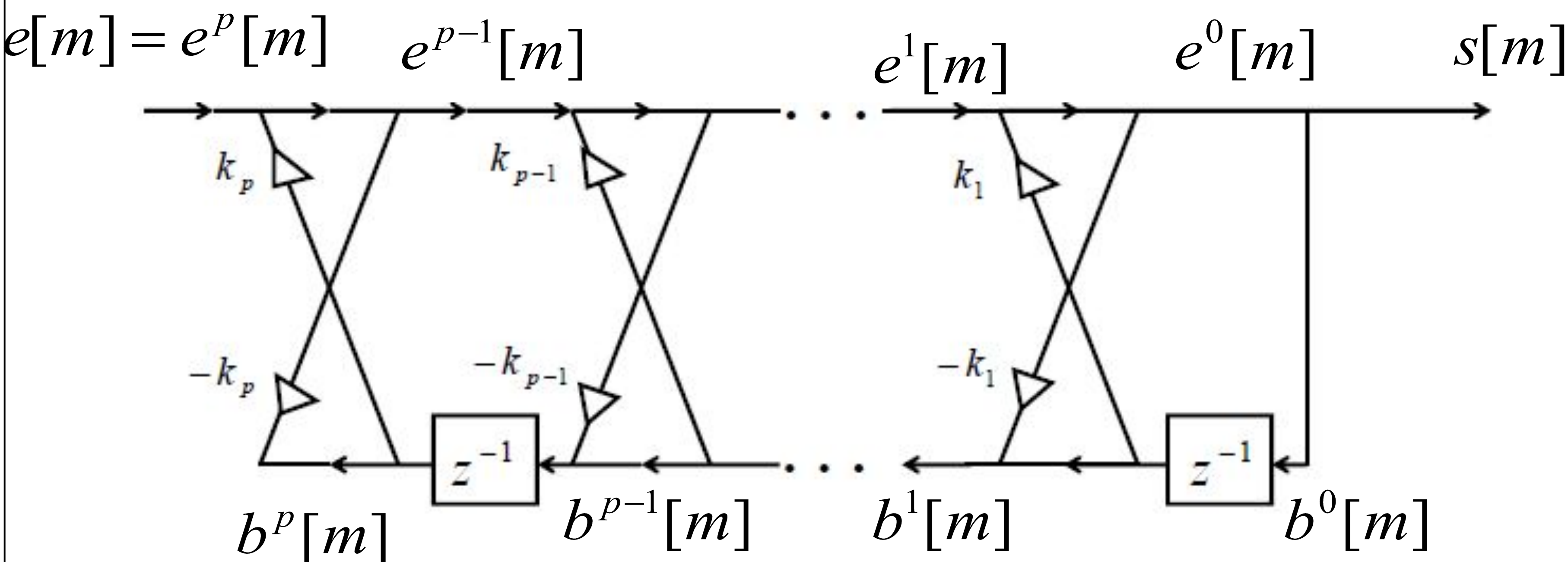
No prediction



Lattice Filter for $A(Z)$

$$e[m] = e^{p}[m]$$

All-Pole Lattice Filter for H(z)



$$e^{i-1}[m] = e^i[m] + k_i b^{i-1}[m-1]$$

$$b^{i-1}[m-1] = b^i[m] + k_i e^{i-1}[m]$$

$$H(z) = \frac{1}{A(z)}$$

$$S(z) = \frac{1}{A(z)} E(z)$$

All-Pole Lattice Filter for H(z)

1. since $b^{(i)}[-1]=0, \ \forall i,$ we can first solve for $e^{(i-1)}[0]$ for $i=p,p-1,\dots,1,$ using the relationship: $e^{(i-1)}[0]=e^{(i)}[0]$

2. since $b^{(0)}[0]=e^{(0)}[0]$ we can then solve for $b^{(i)}[0]$ for $i=1,2,\dots,p$ using the equation: $b^{(i)}[0]=-k_i\, e^{(i-1)}[0]$

3. we can now begin to solve for $e^{(i-1)}[1]$ as:

$e^{(i-1)}[1]=e^{(i)}[1]+k_i\, b^{(i-1)}[0], \ i=p,p-1,\dots,1$

4. we set $b^{(0)}[1]=e^{(0)}[1]$ and we can then solve for $b^{(i)}[1]$ for $i=1,2,\dots,p$ using the equation:

$b^{(i)}[1]=-k_i\, e^{(i-1)}[1]+b^{(i-1)}[0], \ i=1,2,\dots,p$

5. we iterate for $n=2,3,\dots,N-1$ and end up with

$s[n]=e^{(0)}[n]=b^{(0)}[n]$

Direct Computation of *k Parameters*

$$e^{i}[m]=e^{i-1}[m]-k_i b^{i-1}[m-1]$$

Minimize forward prediction error as

$$E^{i}_{forward}=\sum_{m=0}^{L-1+i}[e^{i}[m]]^2=\sum_{m=0}^{L-1+i}[e^{i-1}[m]-k_i b^{i-1}[m-1]]^2$$

$$\frac{\partial E^{i}_{forward}}{\partial k_i}=0=-2\sum_{m=0}^{L-1+i}[e^{i-1}[m]-k_i b^{i-1}[m-1]]b^{i-1}[m-1]$$

$$k_i^{forward}=\frac{\sum_{m=0}^{L-1+i}e^{i-1}[m]b^{i-1}[m-1]}{\sum_{m=0}^{L-1+i}[b^{i-1}[m-1]]^2}$$

$$b^i[m] = b^{i-1}[m-1] - k_i e^{i-1}[m]$$

Minimize backward prediction error as

$$E^i_{backward} = \sum_{m=0}^{L-1+i} [b^i[m]]^2 = \sum_{m=0}^{L-1+i} [b^{i-1}[m-1] - k_i e^{i-1}[m]]^2$$

$$\frac{\partial E^i_{backward}}{\partial k_i} = 0 = -2\sum_{m=0}^{L-1+i} [b^{i-1}[m-1] - k_i e^{i-1}[m]] e^{i-1}[m]$$

$$k_i^{backward} = \frac{\sum_{m=0}^{L-1+i} e^{i-1}[m] b^{i-1}[m-1]}{\sum_{m=0}^{L-1+i} [e^{i-1}[m]]^2}$$

$$\sum_{m=0}^{L-1+i}[e^{i-1}[m]]^2 = \sum_{m=0}^{L-1+i}[b^{i-1}[m-1]]^2$$

$$k_{\mathrm{i}}^{\mathrm{PARCOR}} = \sqrt{k_i^{forward} k_i^{backward}} = k_i^{forkward} = k_i^{backward}$$

$$= \frac{\sum_{m=0}^{L-1+i} e^{i-1}[m] b^{i-1}[m-1]}{\left(\sum_{m=0}^{L-1+i}[e^{i-1}[m]]^2 \sum_{m=0}^{L-1+i}[b^{i-1}[m-1]]^2 \right)^{1/2}}$$

Direct Computation of *k* Parameters

- minimize ***sum*** of forward and backward prediction errors over fixed interval (covariance method)

$$E_{Burg}^{(i)} = \sum_{m=0}^{L-1} \left\{ \left[e^{(i)}(m) \right]^2 + \left[b^{(i)}(m) \right]^2 \right\}$$

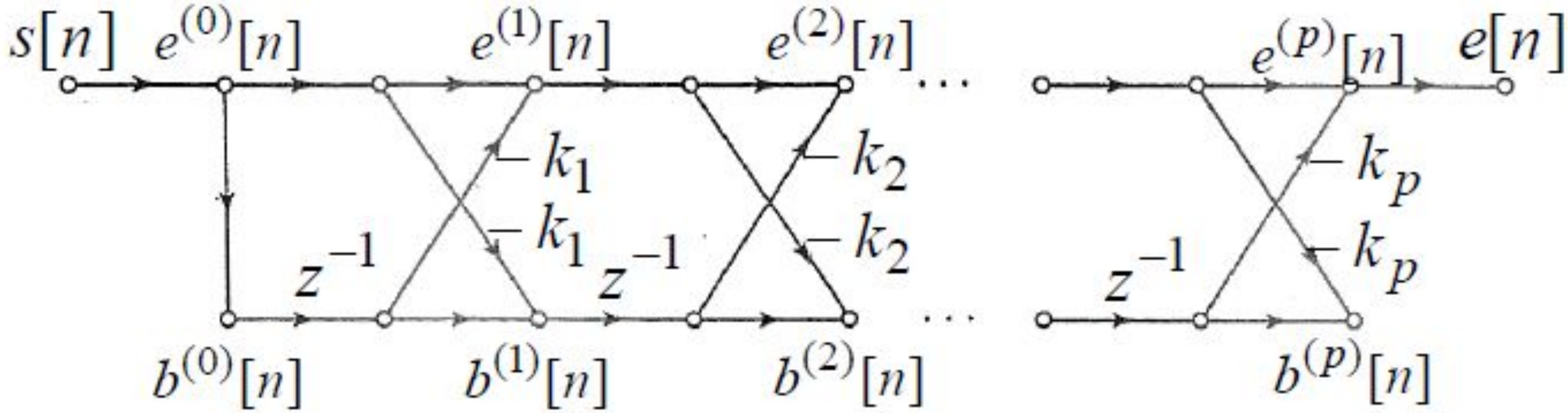
$$= \sum_{m=0}^{L-1} \left[e^{(i-1)}(m) - k_i b^{(i-1)}(m-1) \right]^2 + \sum_{m=-\infty}^{\infty} \left[-k_i e^{(i-1)}(m) + b^{(i-1)}(m-1) \right]^2$$

$$\frac{\partial E_{Burg}^{(i)}}{\partial k_i} = 0 = -2 \sum_{m=0}^{L-1} \left[e^{(i-1)}(m) - k_i b^{(i-1)}(m-1) \right] b^{(i-1)}(m-1)$$

$$-2 \sum_{m=0}^{L-1} \left[-k_i e^{(i-1)}(m) + b^{(i-1)}(m-1) \right] e^{(i-1)}(m)$$

$$k_i^{Burg} = \frac{2 \sum_{m=0}^{L-1} \left[e^{(i-1)}(m) \cdot b^{(i-1)}(m-1) \right]}{\sum_{m=0}^{L-1} \left[e^{(i-1)}(m) \right]^2 + \sum_{m=0}^{L-1} \left[b^{(i-1)}(m-1) \right]^2}$$

- $-1 \le k_i^{Burg} \le 1$ ***always***



$$\boxed{e^{(0)}(m) = b^{(0)}(m) = s(m)} \qquad \boxed{*3}$$

$$\boxed{k_i = \frac{\sum_{m=0}^{L-1+i} e^{(i-1)}(m) b^{(i-1)}(m-1)}{\left\{\left[\sum_{m=0}^{L-1+i} [e^{(i-1)}(m)]^2\right]\left[\sum_{m=0}^{L-1+i} [b^{(i-1)}(m-1)]^2\right]\right\}^{1/2}}} \qquad \boxed{*4}$$

$$\boxed{e^{(i)}(m) = e^{(i-1)}(m) - k_i b^{(i-1)}(m-1)} \qquad \boxed{*1}$$

$$\boxed{b^{(i)}(m) = b^{(i-1)}(m-1) - k_i e^{(i-1)}(m)} \qquad \boxed{*2}$$

Lattice Algorithms

$$\mathcal{E}^{(0)} = R[0] \tag{1}$$

$$e^{(0)}[n] = b^{(0)}[n] = s[n], \quad 0 \le n \le L-1 \tag{2}$$

for $i = 1, 2, \ldots, p$

compute k_i using either Eq. (9.125) or Eq. (9.128) (3)

compute $e^{(i)}[n], \quad 0 \le n \le L-1+i$ using Eq. (9.117b) (4a)

compute $b^{(i)}[n], \quad 0 \le n \le L-1+i$ using Eq. (9.117c) (4b)

$$\alpha_i^{(i)} = k_i \tag{5}$$

compute predictor coefficients

if $i > 1$ then for $j = 1, 2, \ldots, i-1$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)} \tag{6}$$

end

compute mean-squared energy

$$\mathcal{E}^{(i)} = (1 - k_i^2)\mathcal{E}^{(i-1)} \tag{7}$$

end

$$\alpha_j = \alpha_j^{(p)} \quad j = 1, 2, \ldots, p \tag{8}$$

$$e[n] = e^{(p)}[n], \quad 0 \le n \le L-1+p \tag{9}$$