Answers to Exercise Problems

Q5.1 [Ans: $p \approx 0.9231$]

Q5.2 [Ans: $p \approx 0.0212$]

Q5.3 [Ans $p \approx 0.9906$]

Q5.4 [Ans: $(a)\ p \approx 0.7855,\ (b)\ p \approx 0.9983$]

Q5.5 [Ans $p \approx 0.2381$]

Q5.6 [Ans: $(a)\ p \approx 0.2378, (b)\ p \approx 0.8036$]

Q5.7 [Ans $(a)\ p \approx 0.0031,\ (b)\ p \approx 0.0012$]

Q5.8 [Ans: $(a)\ p \approx 4.6 \times 10^{-5}$, (b) 250 days, (b) 111 days]

Q5.9 [Ans $(a)\ p \approx 0.0656,\ (b)\ p \approx 0.271$]

Q5.10 [Ans: (a) 20, (b) 60]

$\boxed{6}$ Continuous Distributions

Normal Distribution

The normal distribution is the most important distribution in statistical quality control.

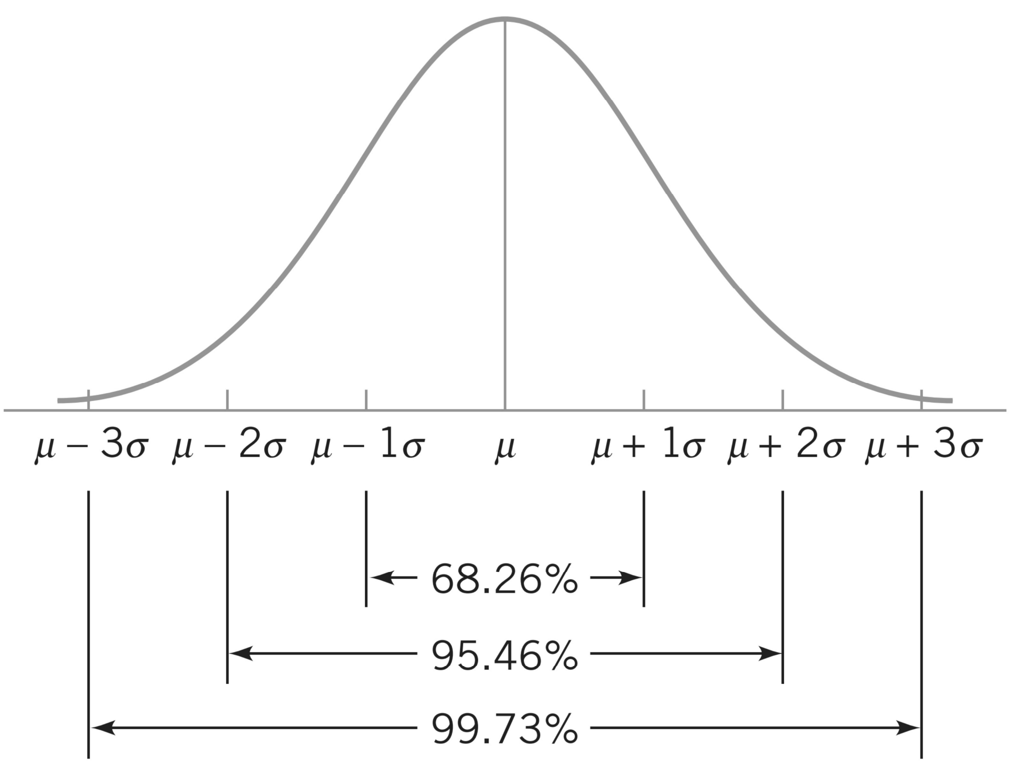
If X is a normal random variable with *location parameter* (mean) μ and *scale parameter* (standard deviation) $\sigma > 0$, then the probability density function (*PDF*) of X can be defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

The normal cumulative distribution function (*CDF*) is defined as the probability that the normal random variable is less than or equal to some value x, i.e.,

$$F(x) = \int_{-\infty}^{x} f(x)\mathrm{d}x$$

Normal Distribution (cont'd)



Standard Normal Distribution

A normal random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$ is known as the *standard normal variate.* Let us denote this variable as Z.

The *standard normal PDF* can be written as

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty$$

The *standard normal CDF* can be written as

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathrm{d}z = \Phi(z)$$

Standard Normal CDF $\Phi(z)$ Table

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

Standard Normal CDF $\Phi(z)$ Table (cont'd)

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

Normal Distribution (cont'd)

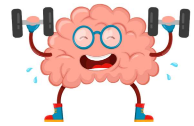
The normal distribution has many useful properties. One of these is relative to *linear combinations* of normally and independently distributed random variables:

– If $X_1, X_2, \cdots, X_n$ are normally and independently distributed random variables with means $\mu_1, \mu_2, \cdots, \mu_n$ and standard deviations $\sigma_1, \sigma_2, \cdots, \sigma_n$, then the linear combination $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$, where $a_1, \cdots, a_n$ are constants, will also follow normal distribution with mean and variance as follows:

$$\mu_y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$
$$\sigma_y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2$$

The *central limit theorem* implies that the sum of n independently distributed random variables is normal, when $n \to \infty$, regardless of the distributions of the individual variables.

Exercise Problems

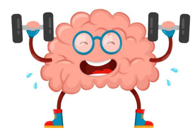


Q6.1 The time to resolve customer complaints is a critical quality characteristic for many organizations. Suppose that this time in a financial organization is a normally distributed random variable X with mean $\mu = 40$ hours and standard variation $\sigma = 2$ hours. What is the probability that a customer complaint will be resolved in less than 35 hours?

[6] Continuous Distributions • § Normal Distribution

© 2025 Indranil Hazra

Exercise Problems



Q6.1 The time to resolve customer complaints is a critical quality characteristic for many organizations. Suppose that this time in a financial organization is a normally distributed random variable X with mean $\mu = 40$ hours and standard variation $\sigma = 2$ hours. What is the probability that a customer complaint will be resolved in less than 35 hours?

Q6.2 The diameter of a metal shaft used in a disk-drive unit is normally distributed with mean 0.2508 in and standard deviation 0.0005 in. The specifications on the shaft have been established as 0.2500 ± 0.0015 in. (a) What fraction of the shafts produced conform to specifications? (b) What happens if we recenter the manufacturing process so that the process mean is exactly equal to the nominal value of 0.2500 in?

Dr. Indranil Hazra RE30003 @ IIT Kharagpur Spring 2025 103

Lognormal Distribution

Variables in a system sometimes follow an exponential relationship $X = e^Y$. If the exponent Y is a random variable, then $X = e^Y$ is also a random variable.

An important *special case* occurs when Y has a normal distribution. In that case, the distribution of X is called the lognormal distribution.

– The *lognormal PDF* with *location parameter* θ and *scale parameter* ω can be written as

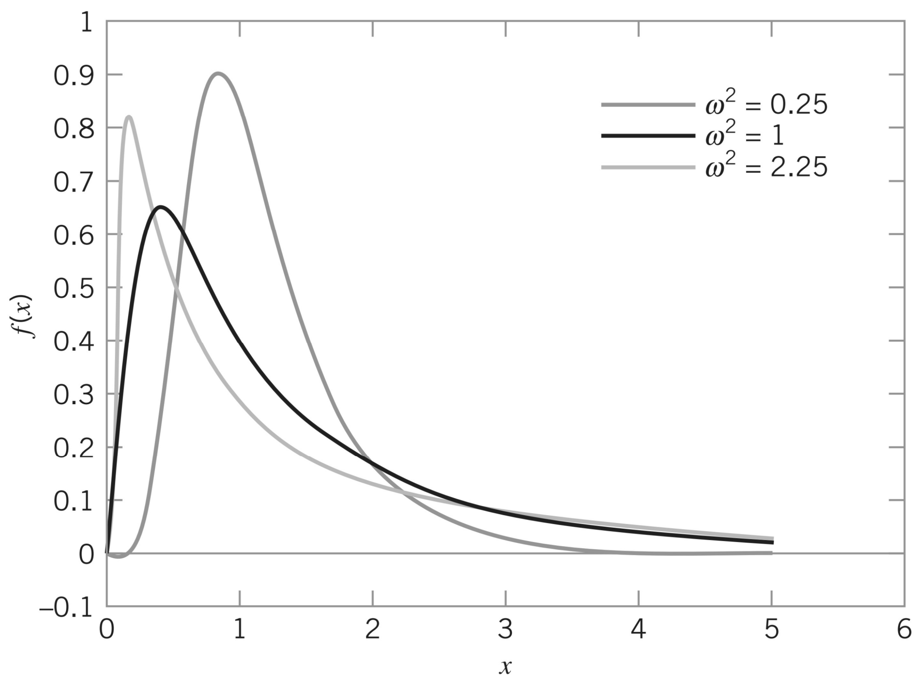
$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \theta}{\omega}\right)^2\right], \quad 0 < x < \infty$$

– The *mean* and *variance* of the distribution are

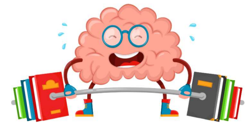
$$\mu = e^{\theta+\omega^2/2}, \qquad \sigma^2 = e^{2\theta+\omega^2}(e^{\omega^2} - 1)$$

Lognormal Distribution (cont'd)

Lognormal PDFs with $\theta = 0$ and selected values of ω:



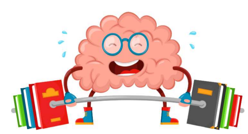
Exercise Problems



Q6.3 The lifetime of any item is an important quality characteristic. Suppose that the lifetime of a medical laser used in ophthalmic surgery has a lognormal distribution with $\theta = 6$ and $\omega = 1.2$ hours.
(a) What is the probability that the lifetime exceeds 500 hours?
(b) What lifetime is exceeded by 99% of lasers?
(c) What are the mean and standard deviation of the laser lifetime?

[6] Continuous Distributions • § Lognormal Distribution
© 2025 Indranil Hazra

Exercise Problems



Q6.3 The lifetime of any item is an important quality characteristic. Suppose that the lifetime of a medical laser used in ophthalmic surgery has a lognormal distribution with $\theta = 6$ and $\omega = 1.2$ hours.
(a) What is the probability that the lifetime exceeds 500 hours?
(b) What lifetime is exceeded by 99% of lasers?
(c) What are the mean and standard deviation of the laser lifetime?

Q6.4 A manufacturing company produces a particular type of electronic component, and the time to failure of these components follows a lognormal distribution with the location parameter $\theta = 5$ and scale parameter $\omega = 1.5$ hours.
(a) Determine the probability that a component will last more than 200 hours.
(b) Find the time by which 95% of the components are expected to fail.

Exponential Distribution

The exponential *PDF* with *rate parameter* $\lambda > 0$ can be defined as

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

- The *mean* and *variance* of the distribution are

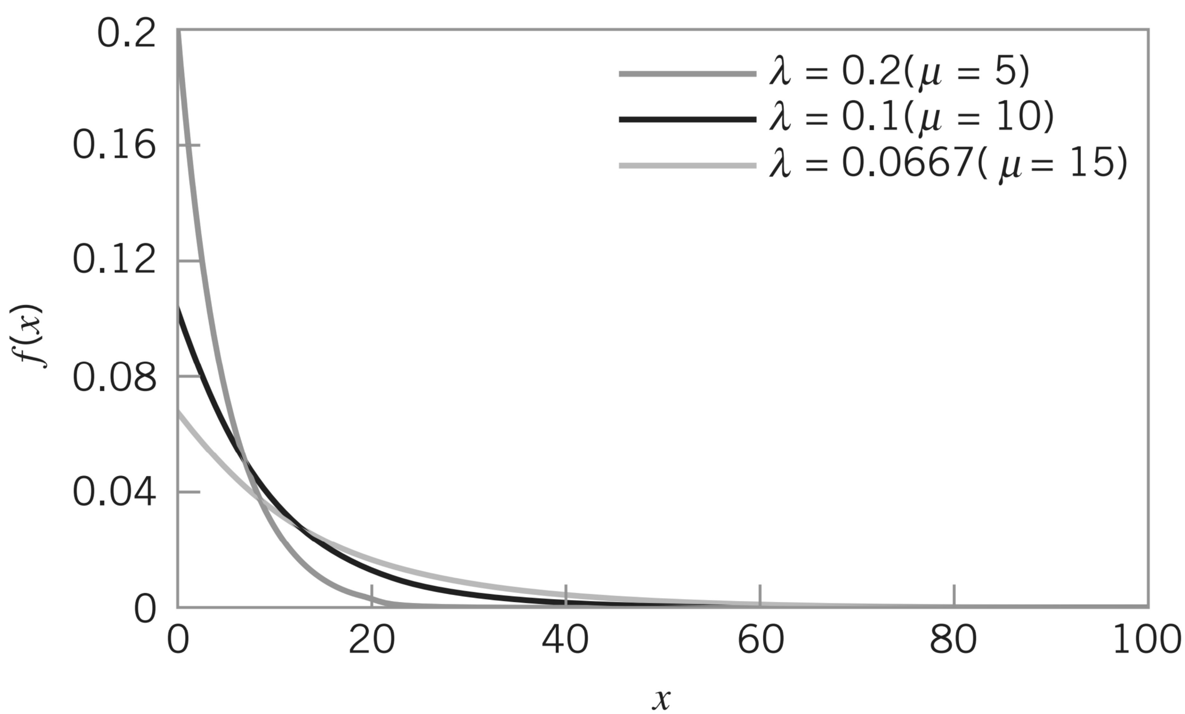
$$\mu = \frac{1}{\lambda}, \qquad \sigma^2 = \frac{1}{\lambda^2}$$

- The *CDF* for an exponential random variable can be written as

$$F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}$$

Exponential Distribution (cont'd)

Exponential PDFs with selected values of λ:



Exercise Problems

Q6.5 Suppose that an electronic component in an airborne radar system has a useful life described by an exponential distribution with failure rate $10^{-4}/\mathrm{h}$.
(a) What is the mean time to failure (MTTF) of the component ?
(b) Find the time by which 50% of the components are expected to fail.
(c) What is the probability that this component would fail before its expected life?

Exercise Problems

Q6.5 Suppose that an electronic component in an airborne radar system has a useful life described by an exponential distribution with failure rate $10^{-4}/\mathrm{h}$.
(a) What is the mean time to failure (MTTF) of the component ?
(b) Find the time by which 50% of the components are expected to fail.
(c) What is the probability that this component would fail before its expected life?

Q6.6 A manufacturing plant produces high-precision ball bearings. The time between failures of a machine used in the production process follows an exponential distribution with a mean time between failures (MTBF) of 500 hours.
(a) What is the probability that the machine will operate for at least 700 hours without a failure?
(b) If the machine has already operated for 300 hours without failure, what is the probability that it will continue to operate for an additional 700 hours without failure?
(c) Determine the time at which there is a 90% chance that the machine will have failed at least once.

Gamma Distribution

The gamma distribution with *shape parameter* $r > 0$ and *scale parameter* $\lambda > 0$ can be defined as

$$f(x) = \frac{\lambda(\lambda x)^{r-1}e^{-\lambda x}}{\Gamma(r)}, \quad x \geq 0$$

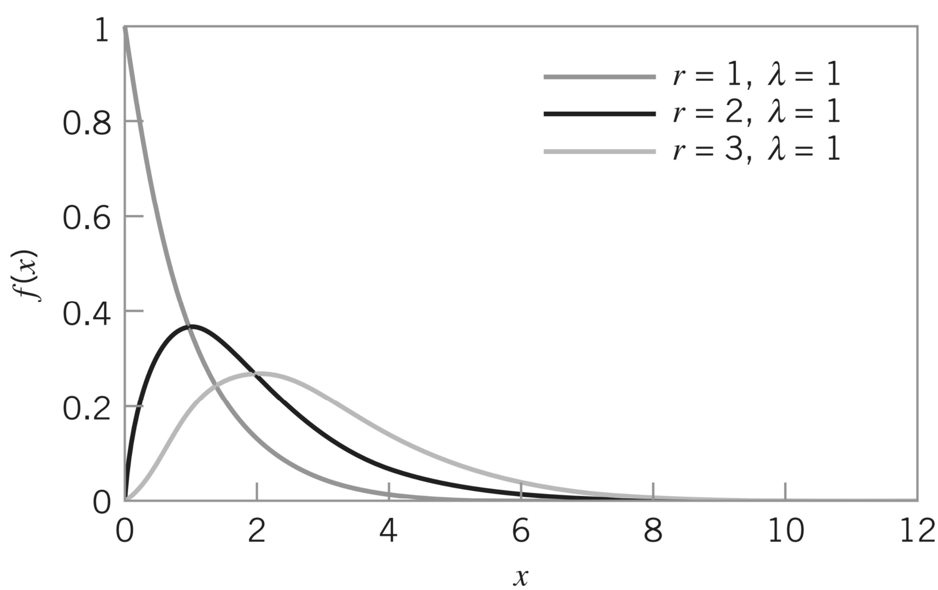
where $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$

The *mean* and *variance* of the distribution are

$$\mu = \frac{r}{\lambda}, \quad \sigma^2 = \frac{r}{\lambda^2}$$

Gamma Distribution (cont'd)

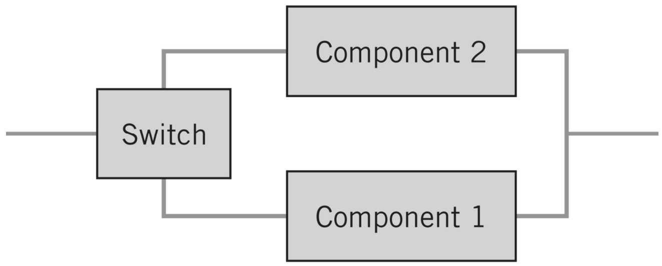
Gamma PDFs with $\lambda = 1$ and selected values of r:



Note: If $r = 1$, the gamma distribution reduces to the exponential distribution with parameter λ!

Exercise Problem

Q6.7 Consider the system shown in the figure. This is called a standby redundant system, because while component 1 is on, component 2 is off, and when component 1 fails, the switch automatically turns component 2 on. Each component has a life described by an exponential distribution with failure rate of 10^{-4}/h.



(a) Show that the system life is gamma distributed with shape parameter $r = 2$ and scale parameter $\lambda = 10^{-4}$.
(b) What is the mean time to failure of the system?
(c) What happens to the system mean time to failure if the switch is an imperfect switch that works only 99% of the time?

Weibull Distribution

The Weibull distribution with *scale parameter* $\theta > 0$ and *shape parameter* $\beta > 0$ can be defined as

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^{\beta}}, \qquad x \geq 0$$

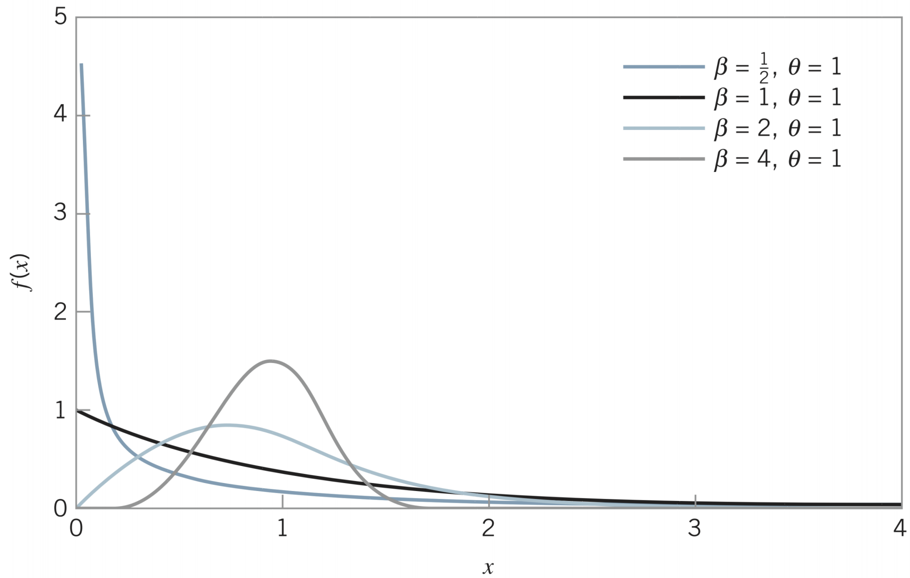
The *mean* and *variance* of the distribution are

$$\mu = \theta\Gamma\left(1 + \frac{1}{\beta}\right)$$

$$\sigma^2 = \theta^2\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \left\{\Gamma\left(1 + \frac{1}{\beta}\right)\right\}^2\right]$$

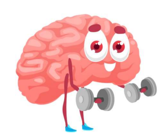
Weibull Distribution (cont'd)

Weibull PDFs with $\theta = 1$ and selected values of β:



Note: If $\beta = 1$ is fixed, the Weibull distribution turns into the exponential distribution with $\lambda = 1/\theta$.

Exercise Problems



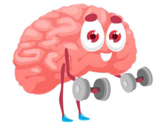
Q6.8 The time to failure for an electronic component used in a flat panel display unit is satisfactorily modeled by a Weibull distribution with $\beta = 1/2$ and $\theta = 5000$.

(a) Find the mean time to failure of the display unit.

(b) Show that the Weibull CDF is given by $1 - e^{-(x/\theta)^\beta}$

(c) What fraction of components that are expected to survive beyond $20,000$ hours?

Exercise Problems



Q6.8 The time to failure for an electronic component used in a flat panel display unit is satisfactorily modeled by a Weibull distribution with $\beta = 1/2$ and $\theta = 5000$.

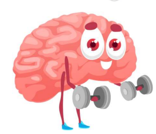
(a) Find the mean time to failure of the display unit.

(b) Show that the Weibull CDF is given by $1 - e^{-(x/\theta)^{\beta}}$

(c) What fraction of components that are expected to survive beyond $20,000$ hours?

Q6.9 In a life testing of 10000 components, it is found that 6321 components failed before 200 hours. Assuming that the lifetime distribution of these components follow the Weibull distribution, determine the scale parameter θ of the Weibull distribution. If the median life is found to be 150 hours, what is the shape parameter β of the lifetime model?

Exercise Problems (cont'd)



Q6.10 A factory produces electronic components, and each component undergoes a final quality inspection before shipment. The inspection process involves two stages:

1. Visual Inspection: The component is classified as either defective (D) or non-defective (ND). Historical data shows that 5% of the components are found to be defective (D), and 95% are non-defective (ND).

2. Performance Testing: For components that pass the visual inspection (ND), a continuous random variable X representing the time to failure (in hours) is measured. The time to failure for non-defective components follows a Weibull distribution with a shape parameter $\beta = 1.5$ and scale parameter $\theta = 600$ hours. The factory aims to ensure high performance in its products, and components that fail before 200 hours are considered under performing.

(a) Determine the overall probability that a randomly selected component is both non-defective and passes the performance testing.

(b) Calculate the expected time to failure for a randomly selected component, considering both the possibility of the component being defective and the distribution of time to failure for non-defective components.

Answers to Exercise Problems

Q6.1 [Ans $p \approx 0.0062$]

Q6.2 [Ans: $(a) p \approx 0.9192$, (b) $p \approx 0.9973$]

Q6.3 [Ans: (a) $p \approx 0.4286$; (b) 24.63 hr; (c) 828.82 hr; 1487.42 hr]

Q6.4 [Ans: (a) $p \approx 0.4213$; (b) 1743 hr]

Q6.5 [Ans: (a) 10^4 hr; (b) 6931.47 hr; (c) $p \approx 0.6321$]

Q6.6 [Ans: (a) $p \approx 0.2466$; (b) $p \approx 0.2466$; (c) 1151.3 hr]

Q6.7 [Ans: (b) 2×10^4 hr; (c) 1.99×10^4 hr]

Q6.8 [Ans: (a) 10000 hr; (c) $p \approx 0.1353$]

Q6.9 [Ans: 200 hr; 1.274]

Q6.10 [Ans: (a) $p \approx 0.7836$; (b) 514.54 hr]