Fuzzy Reasoning

10/03/2025

Koustav Rudra

Fuzzy Reasoning

- Based on Fuzzy set theory and consequently Fuzzy Logic

- We use terms, words that are **imprecise** in nature
- Example:
  - It rains heavily
  - The door is strong
  - The color of the box is more or less red or reddish

- What is the problem?
  - These terms do not find a direct mapping to any quantification like number
  - Poses a difficulty when we try to compute with these things

- Fuzzy reasoning deals with such imprecise scenario

Types of Uncertainty and Modeling of Uncertainty

- Looks more or less like Abraham Lincoln
- How is it that we can certainly identify that this is figure of Abraham Lincoln?
- The complexity of decisions from such subjective inputs to the decision that we make in our mind is intriguing and often we do not really understand in quantified manner

Uncertainty Types

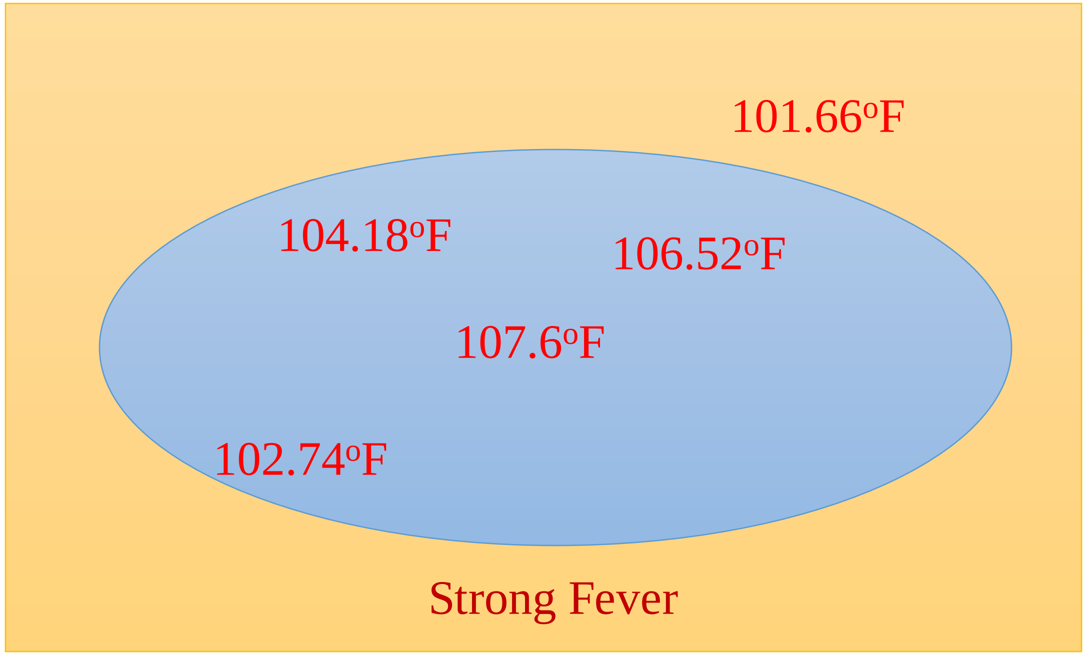
Probability and Uncertainty

- "… a person suffering from hepatitis shows in 60% of all cases a strong fever, in 45% of all cases yellowish colored skin, and in 30% of all cases suffers from nausea …"

- 60% of all cases a strong fever ➔ Probability + Fuzzy
- 45% of all cases yellowish colored skin ➔ Probability + Fuzzy
- 30% of all cases suffers from nausea ➔ Probability

Fuzzy Set Theory

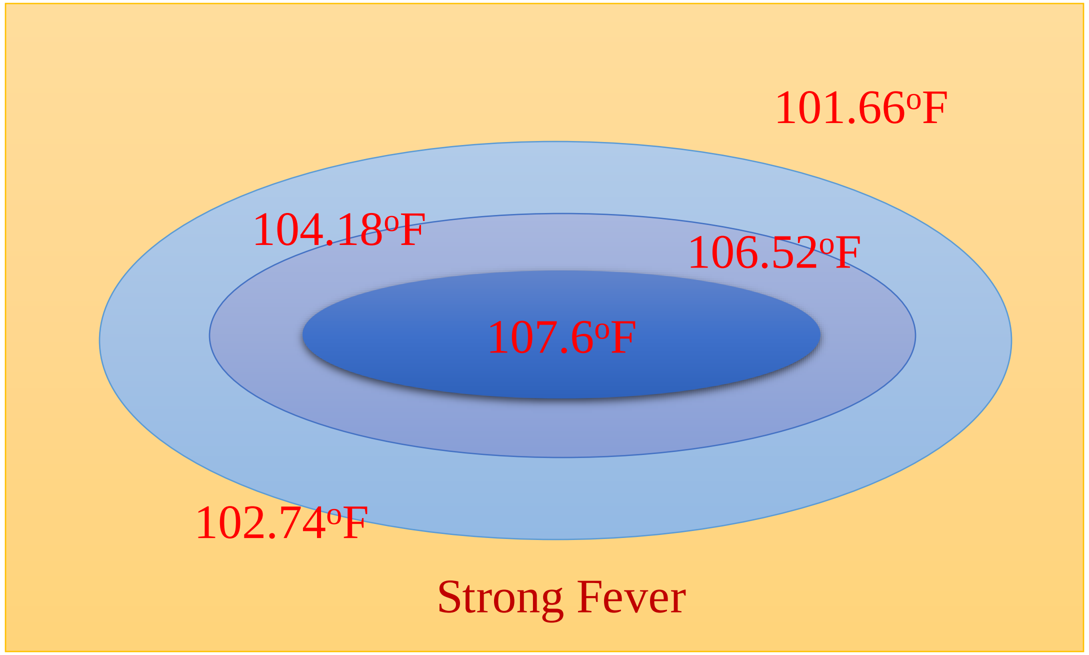
Conventional (Boolean) Set Theory



Can we make such a crisp boundary?

Either an element belong to the set or not

Fuzzy Set Theory



Boundary is gradually fading out

Chance becomes less as the point moves further

Fuzzy Logic

- Fuzzy Logic: Reasoning with qualitative information

- This is more realistic than predicate calculus
  - Because in real life we need to deal with qualitative statements

- Examples:
  - In process control: Chemical plant
  - Rule: If the temperature is moderately high & the pressure is medium then turn the knob slightly right

Fuzzy Sets vs Traditional/Crisp Sets

- Traditional set, Crisp set
  - Defined by the values that are contained within it
  - A value is either within the set, or it is not
  - e.g. a set of natural number

- Fuzzy set
  - Each value is a member of the set to some degree, or is not a member of the set to some degree
  - Example:
    - Bill is 7 feet tall
    - John is 4 feet tall
    - Jim is 5 feet tall

Crisp Set: Membership

- Membership/ Characteristics/ Discriminative Predicate
- Example:
  - S = {2,3,5,a,b,c}
- X = universe = {1,2,3,…,10,a,b,c,…,z}
- $1 \notin S$ $(does\ not\ belong)$
- $a \in S$ $(belongs)$

- U = {Set of all integers}
- X = {1,2,3,4,7,9}
- Membership of $u \in X$ is either 1 (belongs to) or 0 (does not belong)

- Fuzzy set differs from Crisp set in terms of membership

Fuzzy Set Theory: Basics

- Generalization of crisp set theory
- Fundamental observation:
  - $\mu_S(x) = no\ longer\ 0/1$
  - $\mu_S(x)$ is between [0,1], both included

- Example:
  - Crisp set, $S_1 = \{2,4,6,8,10\}$
- $\mu_{S_1}(x)$ is a predicate which denotes x to be an even number less than or equal to 10
- Given any ‘a’ which is a number, the $\mu_{S_1}(x)$ question produces 0/1

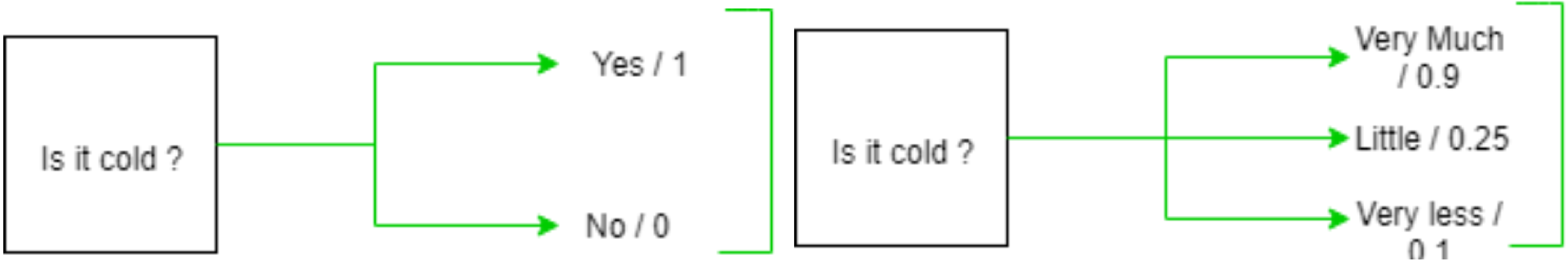
Fuzzy Set Theory: Basics

- $x \in Height$
- Fuzzy set: $Medium \subseteq Height$
- $Medium = \{\frac{0.3}{5'4''}, \frac{0.5}{5'5''}, \frac{0.8}{5'6''}, \frac{1}{5'10''}\}$
- $Medium = \{\frac{\mu_M(x)}{x}\}$

Fuzzy Set

- Fuzzy set membership function
- Fuzzy set A is defined by membership function μ_A
- Choose entirely arbitrarily, reflect a subjective view on the part of the author
- A list of pairs for representing fuzzy set in computer like
  - $A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), \ldots, (x_n, \mu_A(x_n))\}$

Basic Concept of Fuzzy Logic



Boolean Logic

Fuzzy Logic

Fuzzy Sets

- Boolean/Crisp set A is a mapping for the elements of S to the set {0,1}
  - $A: S \rightarrow \{0,1\}$
- Characteristic Function:
  - $\mu_A(x) = \begin{cases} 1 \; if \; x \in A \\ 0 \; if \; x \notin A \end{cases}$
- Fuzzy set F is a mapping for the elements of S to the interval [0,1]
  - $A: S \rightarrow \{0,1\}$
- Characteristic Function: $0 \leq \mu_F \leq 1$
- 1 means Full Membership
- 0 means No Membership
- Anything in between e.g., 0.5 is called Graded Membership

Example: Crisp Set Tall

- Crisp set Tall can be defined as
  - $\{x \mid \text{height } x > 1.8 \text{ meters}\}$

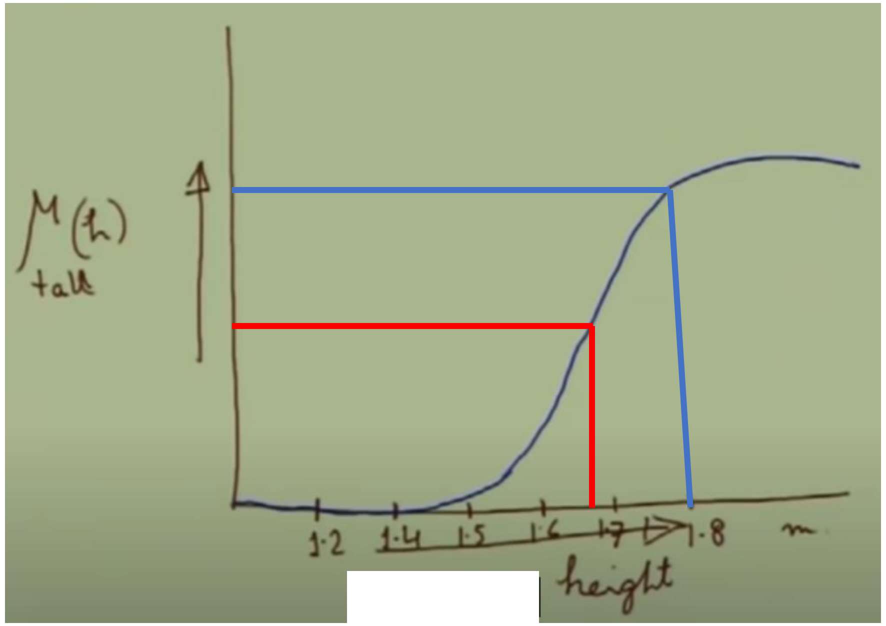
- But what about a person with height 1.79 meters
- What about 1.78 meters
- What about 1.52 meters

Example: Fuzzy Set Tall

- In a Fuzzy set a person with a height of 1.8 meters would be considered tall to a high degree

- A person with a height of 1.7 meters would be considered tall to a lesser degree

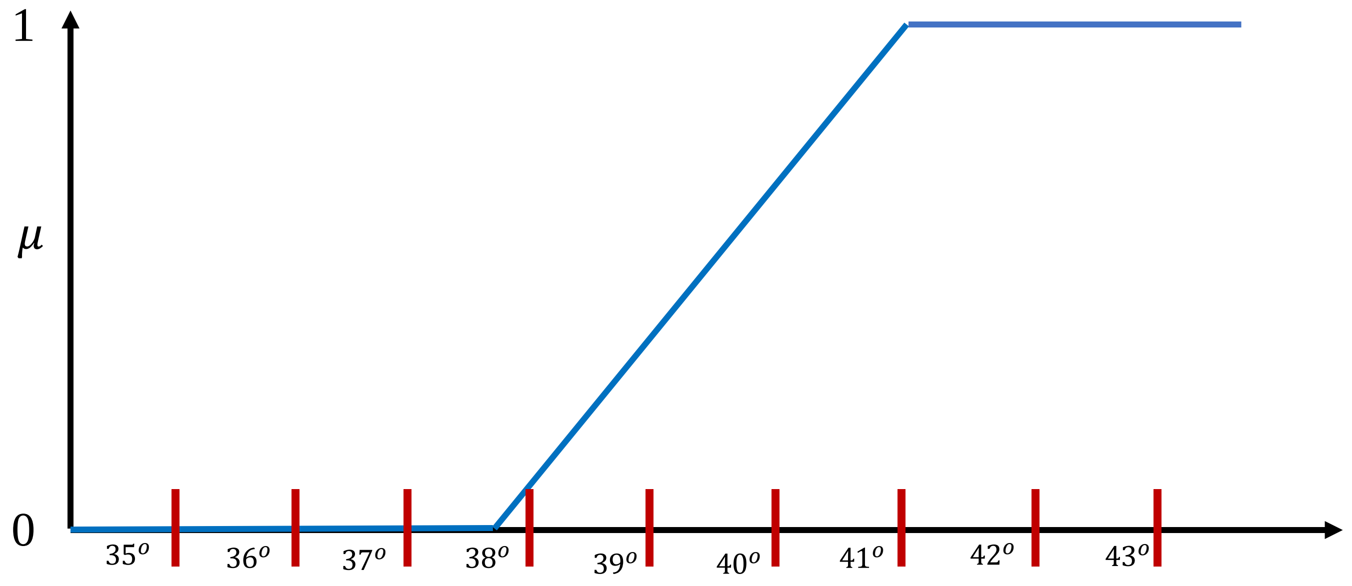
- The function can change for different domains (Basketball player, Women, …)



Fuzzy Set Definitions

- Discrete Definitions:
- $\mu_F(35^oC) = 0$ $\mu_F(38^oC) = 0.1$ $\mu_F(41^oC) = 0.9$
- $\mu_F(36^oC) = 0$ $\mu_F(39^oC) = 0.35$ $\mu_F(42^oC) = 1$
- $\mu_F(37^oC) = 0$ $\mu_F(40^oC) = 0.65$ $\mu_F(43^oC) = 1$

- Continuous Definitions:



Describing a Set

- A set is derived in one of the two ways:
  - By Extension
    - Requires listing
    - S1 is {2,4,6,8,10} --- needs finiteness
    - $X = \{6,7,8,\ldots\}$
  - By Intension
    - Needs a closed form expression related to properties
    - $X = \{x \mid x \geq 6\}$

Fuzzy Set Representation

- A finite set of elements:
  - $F = \frac{\mu_1}{x_1} + \frac{\mu_2}{x_2} + \cdots + \frac{\mu_n}{x_n}$
  - + means Boolean set union

- $Tall = \{\frac{0}{1.0}, \frac{0}{1.2}, \frac{0.1}{1.4}, \frac{0.5}{1.6}, \frac{0.8}{1.8}\}$

How do we represent a Fuzzy Set in Computer?

Membership Function

10/03/2025

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Membership Function

- A membership function for a fuzzy set A on the universe of discourse X is defined as $\mu_A: X \to [0,1]$,

- where each element of X is mapped to a value between 0 and 1

- This value, called membership value or degree of membership,
    - Quantifies the grade of membership of the element in X to the fuzzy set A

Membership Function

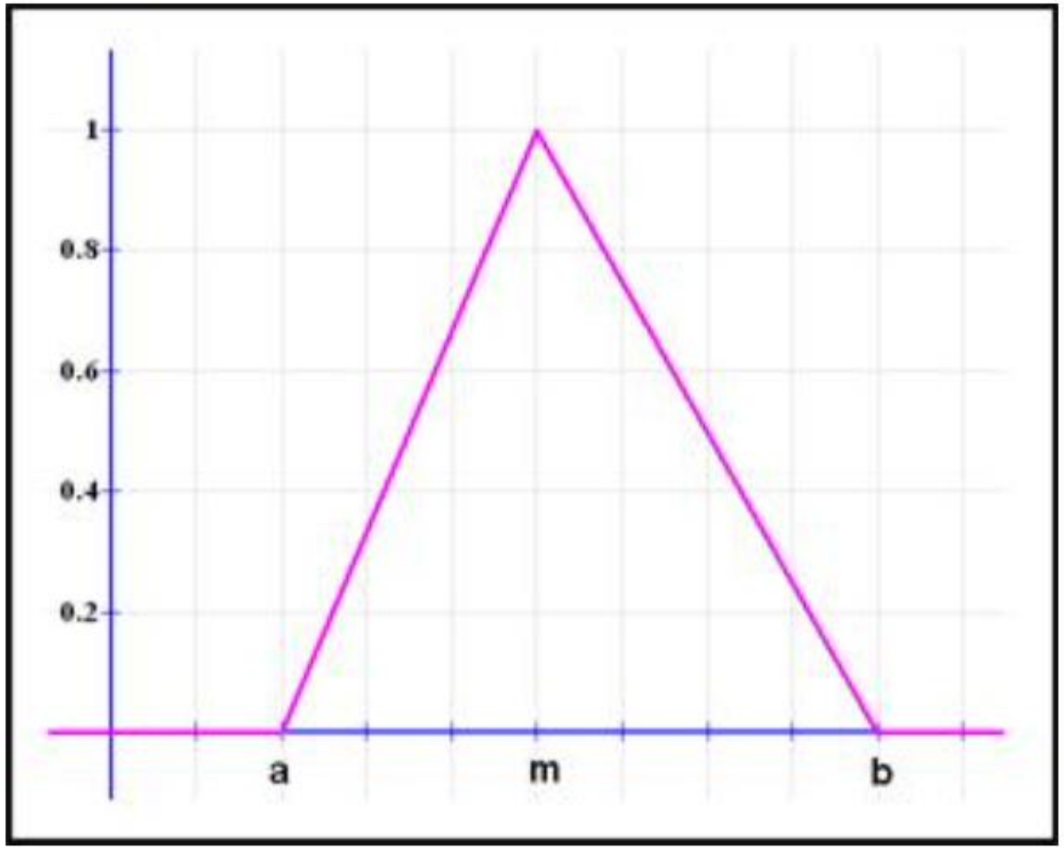
- Simple functions are used to build membership functions
  - As we are defining fuzzy concepts, using more complex functions does not add more precision

- These are some membership functions
  - Triangular function
  - Trapezoidal function
  - Gaussian function

Triangular Function

It is defined as a lower limit **a**, an upper limit **b**, and a value **m**, where **a < m < b**

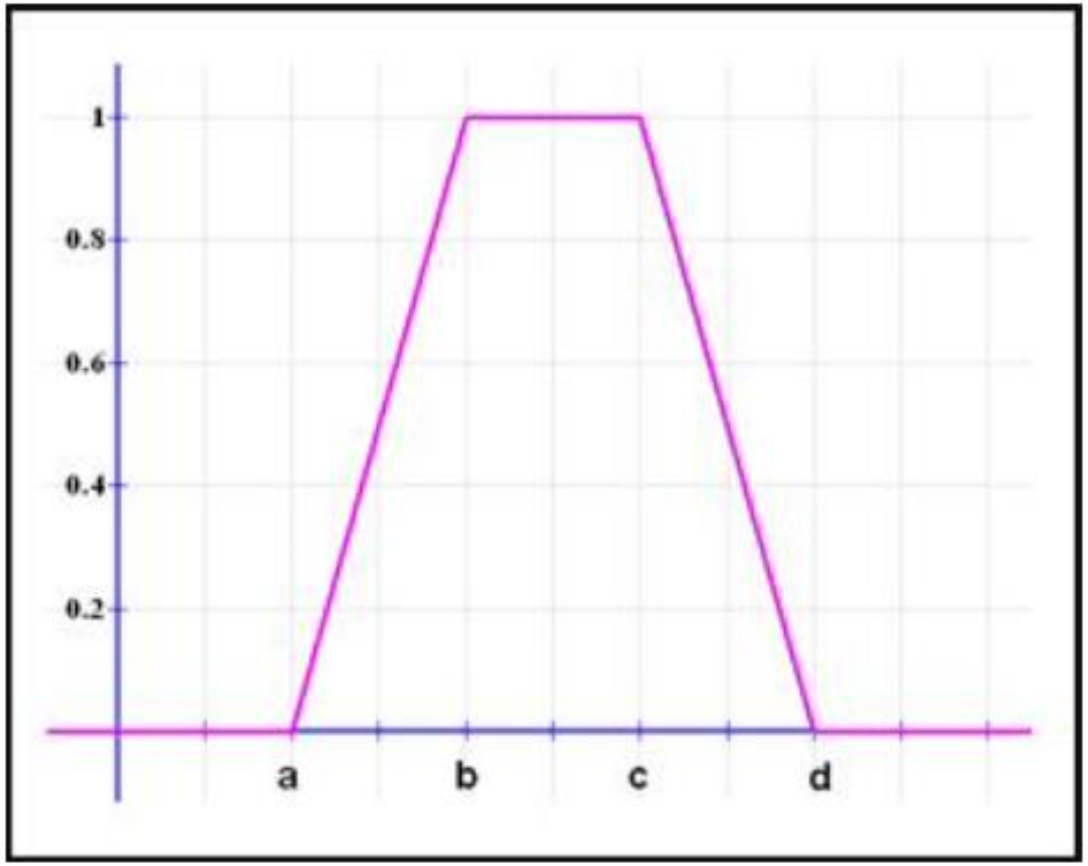
$$\mu_A(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{m-a}, & a < x \le m \\ \dfrac{b-x}{b-m}, & m < x < b \\ 0, & x \ge b \end{cases}$$



Trapezoidal Function

- It is defined by
  - a lower limit **a**,
  - an upper limit **d**,
  - a lower support limit **b**, and
  - an upper support limit **c**,
  - where **a < b < c < d**

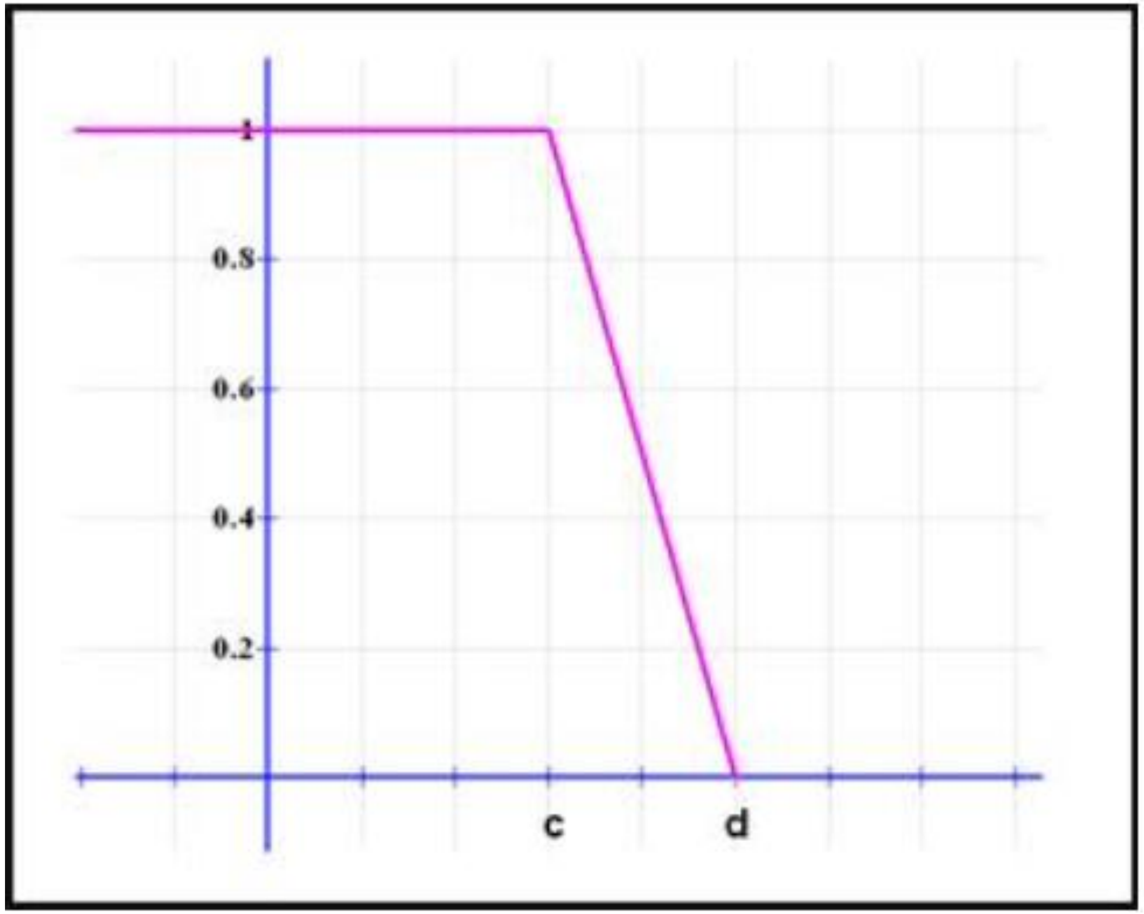
$$\mu_A(x) = \begin{cases} 0, & x < a \; or \; x > d \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c \le x \le d \end{cases}$$



Trapezoidal Function: R Function

- It is defined by
  - an upper limit **d**,
  - an upper support limit **c**,
  - where **c < d**.

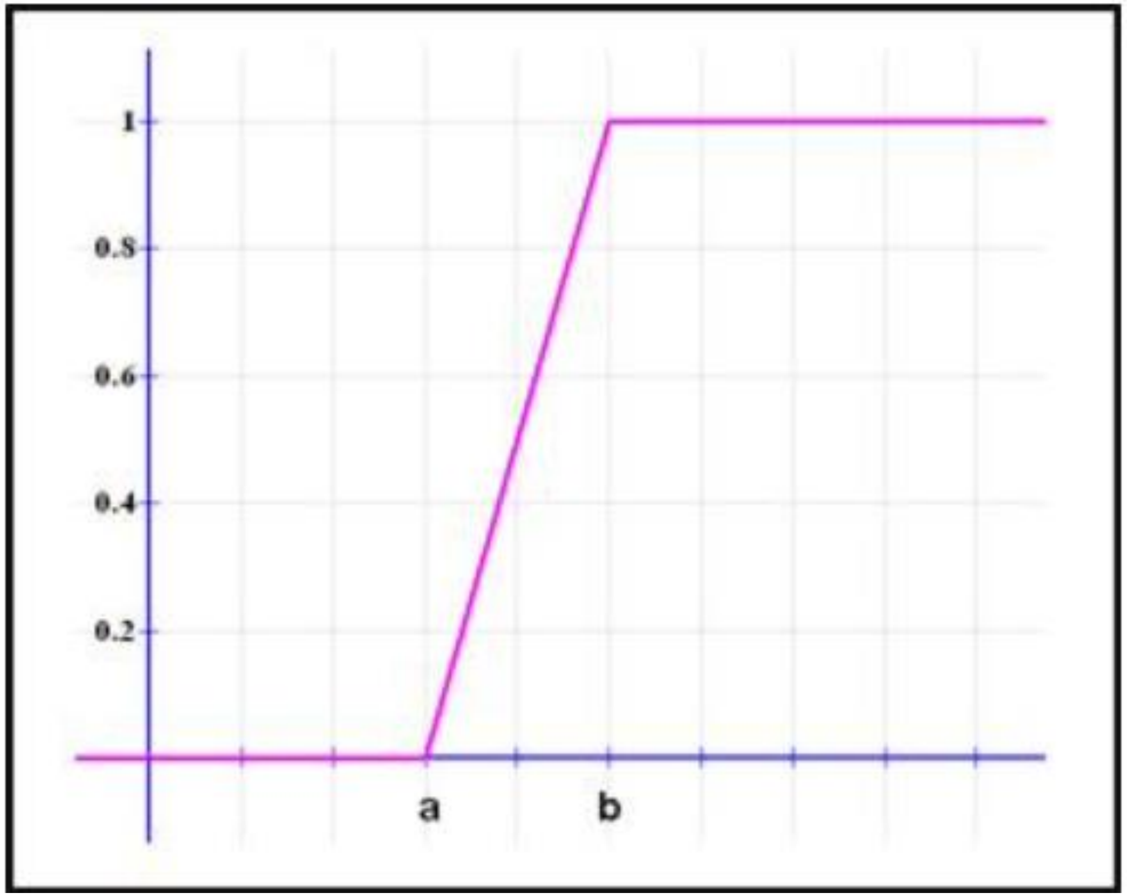
$$\mu_A(x) = \begin{cases} 0, & x > d \\ \dfrac{d-x}{d-c}, & c \le x \le d \\ 1, & x < c \end{cases}$$



Trapezoidal Function: L Function

- It is defined by
  - a lower limit **a**,
  - a lower support limit **b**, and
  - where **a < b**

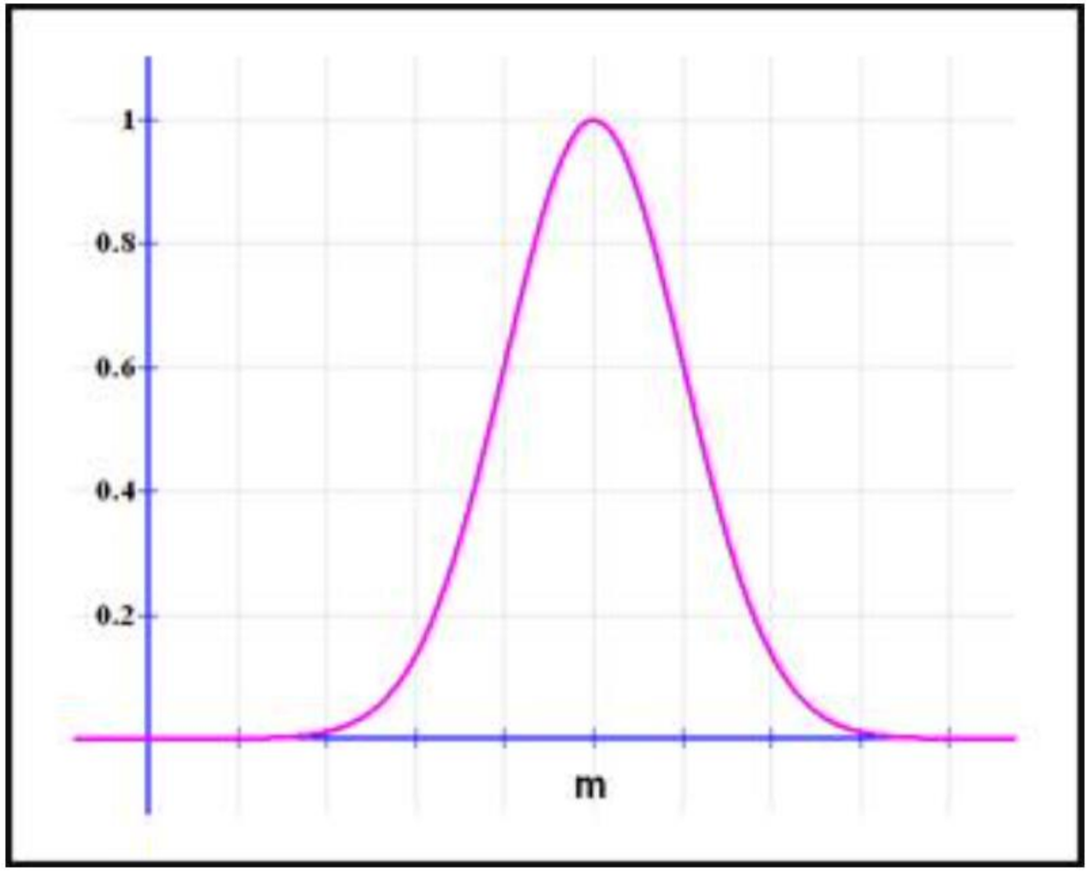
$$\mu_A(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$



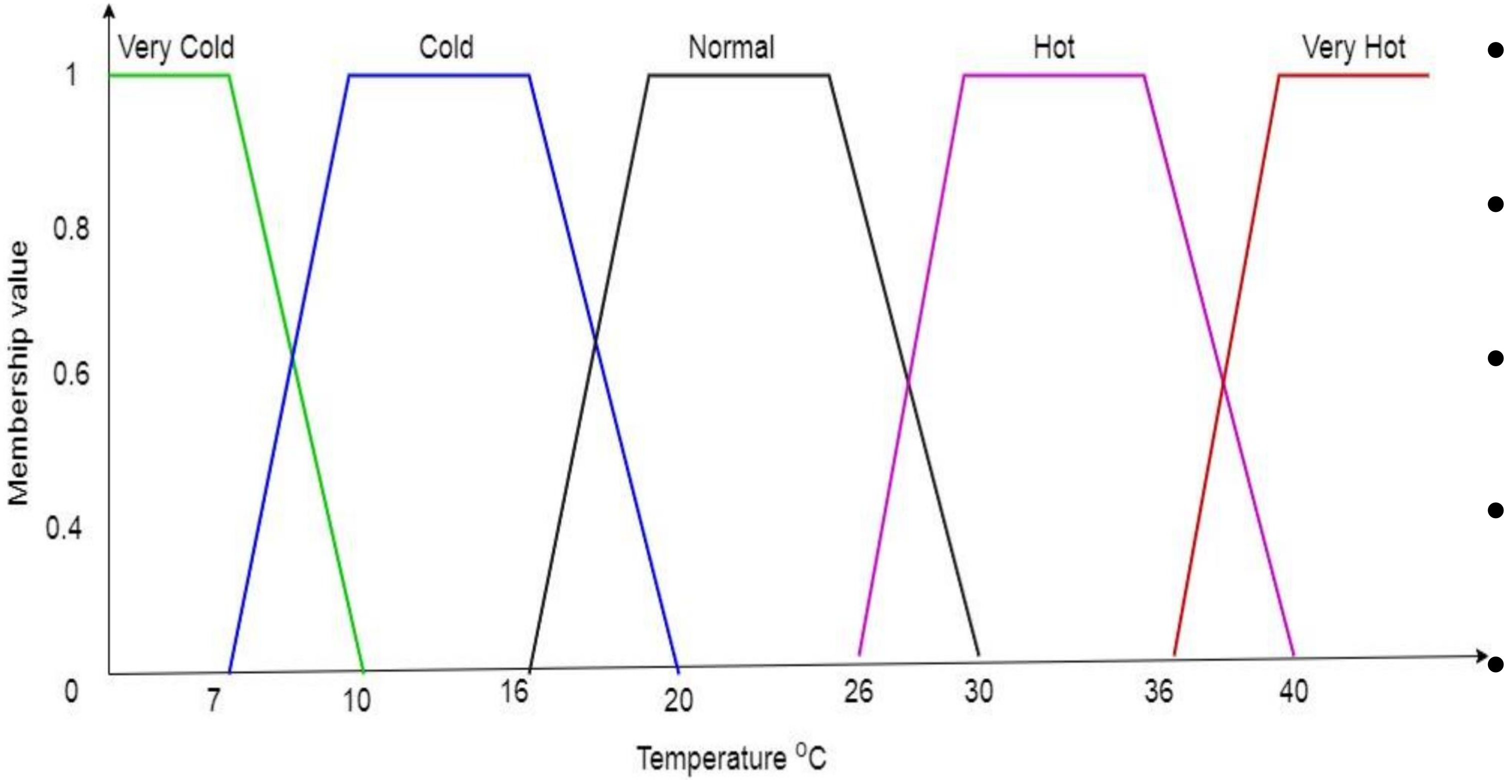
Gaussian Function

- It is defined as
  - a central value m and
  - a standard deviation k > 0
  - The smaller k is, the narrower the “bell” shape

$$\mu_A(x) = e^{-\frac{(x-m)^2}{2k^2}}$$



Temperature

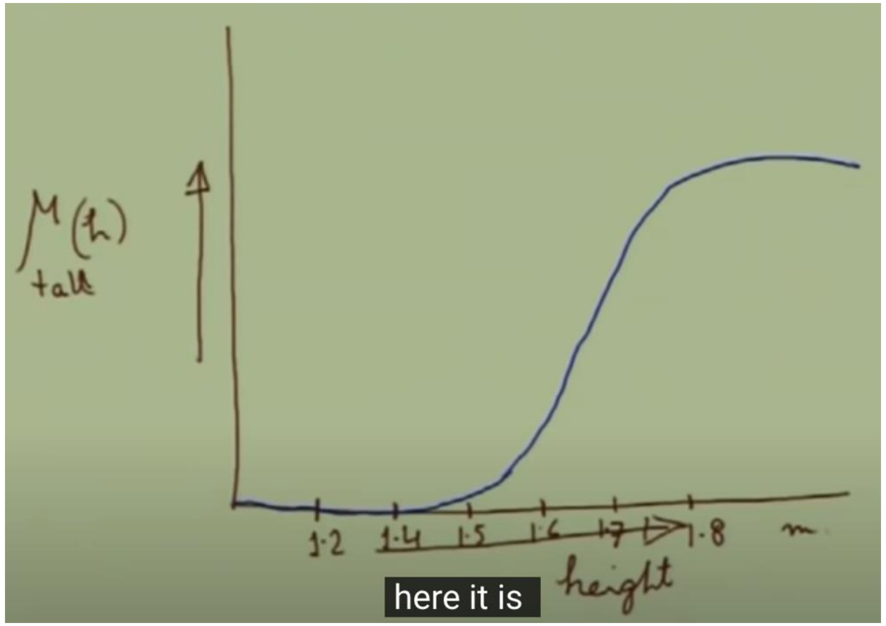


- Very cold: a<0, b<=0, c=7, d=10
- Cold: a=7, b=10, c=16, d=20
- Normal: a=16, b=20, c=26, d=30
- Hot: a=26, b=30, c=36, d=40
- Very Hot: a=36, b=40, c=46, d>46

Membership Function: S-function

- The S-function could be used to define Fuzzy sets

- $$S(x,a,b,c) = \begin{cases} 0 \; for \; x \le a \\ 2\left(\frac{x-a}{c-a}\right)^2 for \; a \le x < b \\ 1-2\left(\frac{x-c}{c-a}\right)^2 for \; b \le x < c \\ 1 \; for \; x \ge c \end{cases}$$

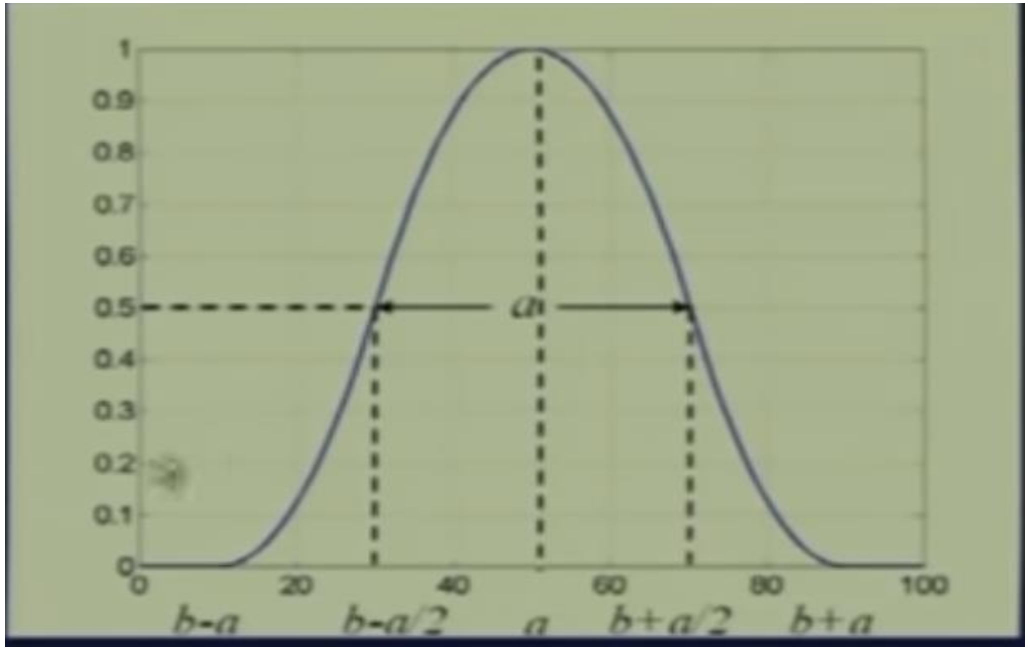


Membership Function: Close to a

$$\mu_{Close_a}(x) = \frac{1}{1 + (x - a)^2}$$

$$\mu_{Close_0}(-1) = \frac{1}{1 + (-1 - 0)^2} = \frac{1}{2} = 0.5$$

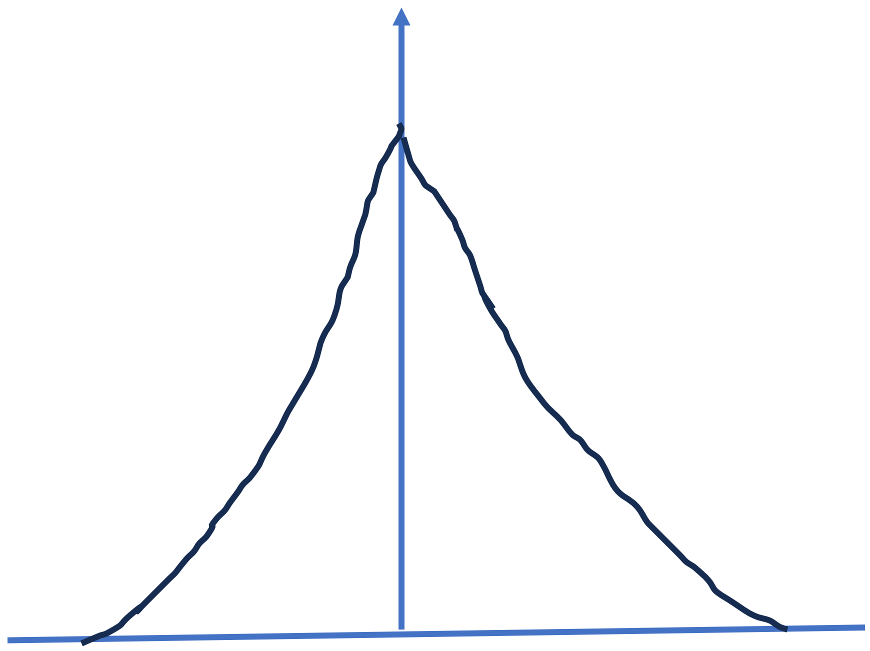
$$\mu_{Close_0}(2) = \frac{1}{1 + (2 - 0)^2} = \frac{1}{5} = 0.2$$



Membership Function: Close to a

$$\mu_{Close_a}(x) = \frac{1}{1 + |x - a|}$$

- Explicit representation of Fuzzy set: Table
- Implicit representation of Fuzzy set: Function



Linguistic Variable

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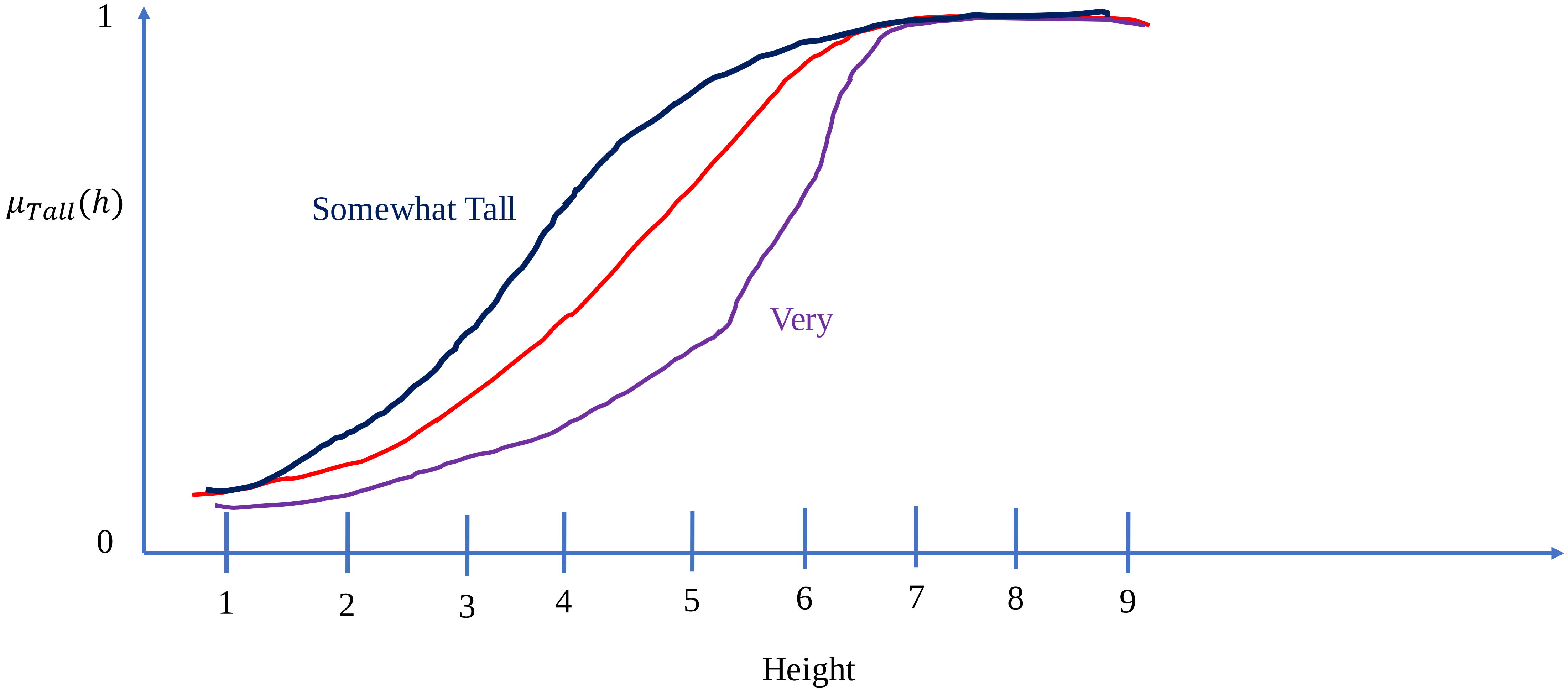
Hedges

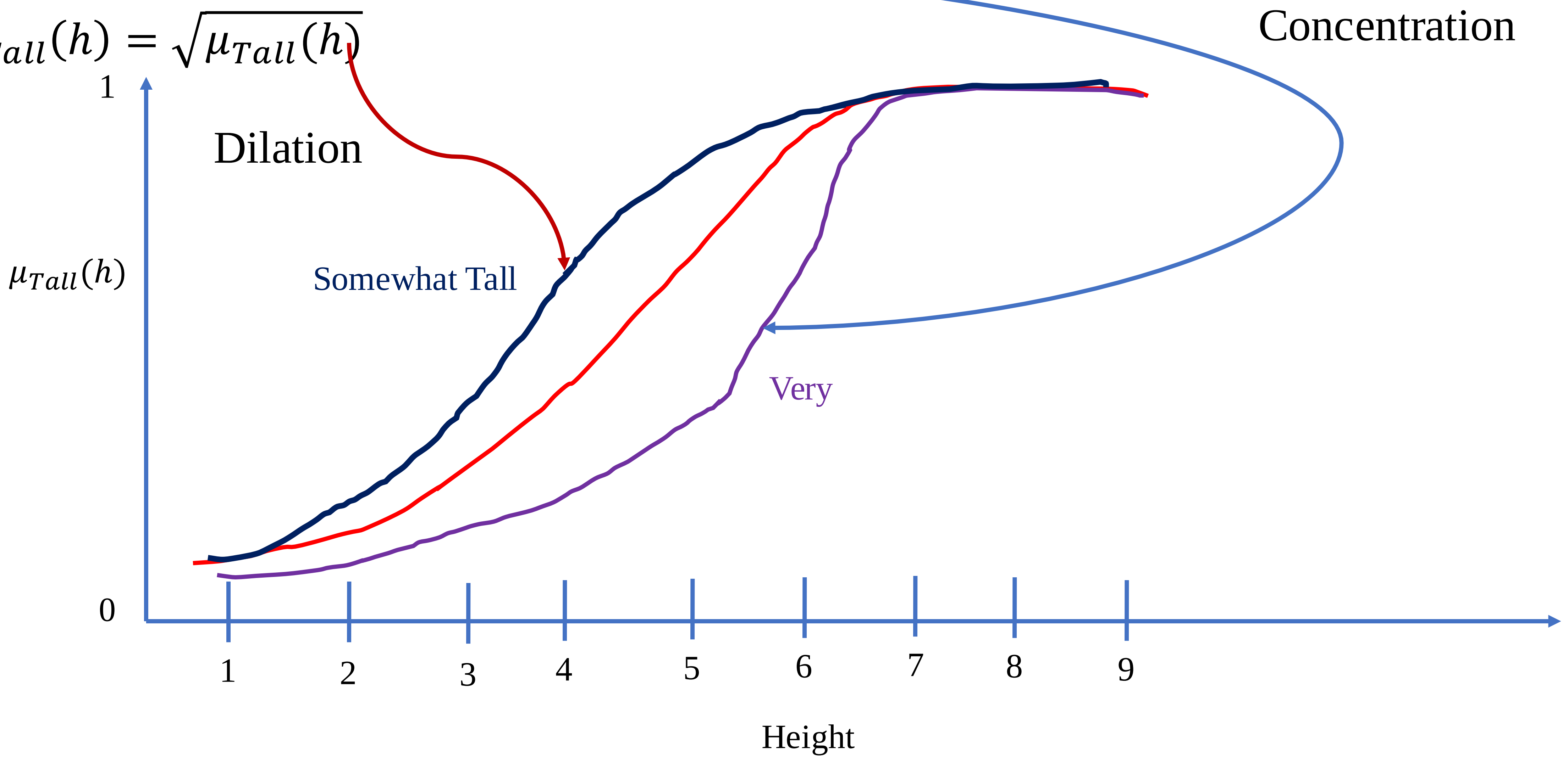
- Hedges are entities to deal with adverb
- John is tall
- Jack is very tall
- Jill is somewhat tall
- Very → squaring the μ function
- Somewhat →taking square root of μ function

Hedges



Concentration and Dilation Operator

- $\mu_{VTall}(h) = (\mu_{Tall}(h))^2$
- $\mu_{MTall}(h) = \sqrt{\mu_{Tall}(h)}$



Fuzzy Relation

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Relations

- AI = {Akash, Ashish}
- MATH = {Sundar, Sekhar}
- AI x MATH = {(Akash,Sundar), (Akash,Sekhar), (Ashish,Sundar), (Ashish, Sekhar)}

- Relation R = Close Friends
- $R \subseteq \{(Akash, Sundar), (Ashish, Sekhar)\}$

- How to go beyond absolute membership?

Fuzzy Relations

- A Fuzzy relation for N sets is defined as an extension of the crisp relation to include membership grade

- $R = \{\mu_R(x_1, \ldots, x_N)/(x_1, \ldots, x_N) | x_i \in X_i\}$

Fuzzy Relation

- Fuzzy relation describes interactions between variables

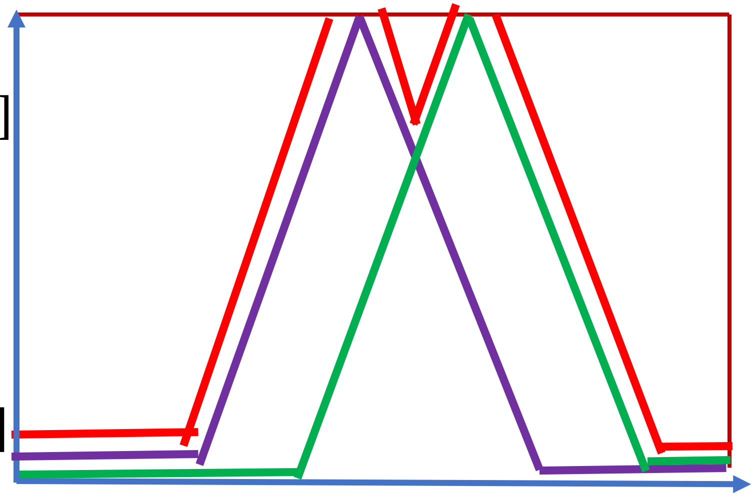
- It defines the mapping of variables from one fuzzy set to another
  - Like crisp relation, we can also define the relation over fuzzy sets

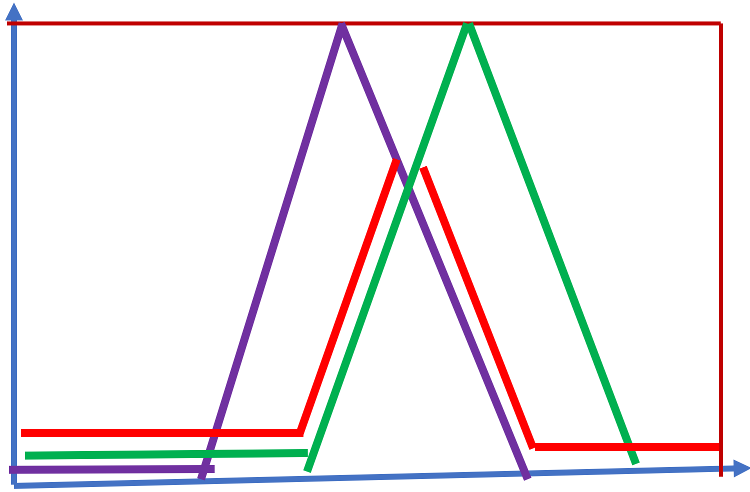
- Main fuzzy operations and compositions are followings:
  - Union
  - Intersection
  - Complement
  - Difference
  - Max-min composition
  - Max-product composition

Operations on Fuzzy Set

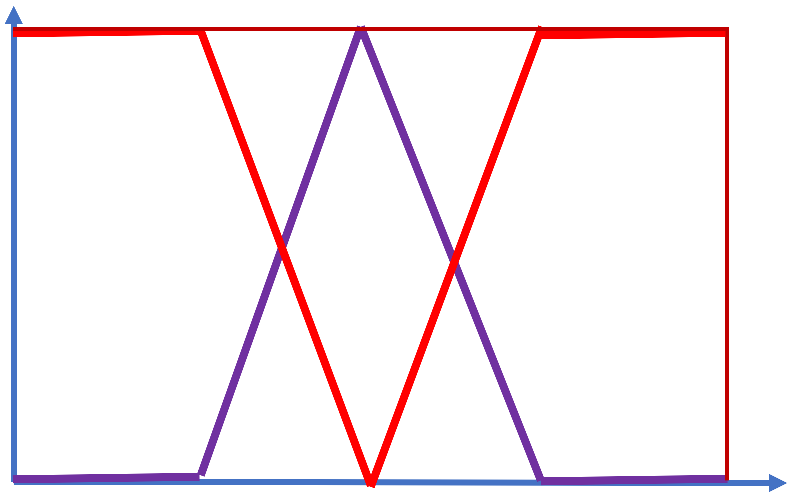
- Union:
  - $A \cup B$
  - $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X$ [Every member of A and B]

- Intersection:
  - $A \cap B$
  - $\mu_{A \cup B}(x) = min(\mu_A(x), \mu_B(x)), \forall x \in X$ [Every member of A and B]





Operations on Fuzzy Set



- Complement
  - A'
  - $\mu_{A'}(x) = 1 - \mu_A(x)$

- Difference:
  - $A - B$
  - $\mu_{A-B}(x) = \min(\mu_A(x), 1 - \mu_B(x)), \forall x \in X$ [Every member of A and B]

Operations on Fuzzy Set

- Containment
  - $A \subseteq B$
  - $\mu_A(x) \leq \mu_B(x)\ ), \forall x \in X$

- Equality:
  - $A = B$
  - $\mu_A(x) = \mu_B(x), \forall x \in X$

Compositions of two relations

- Max-Min Composition

- Given two relation matrices R and S, the max-min composition is defined as $T = R \circ S$
  - $T(x, z) = \max\{\min\{R(x, y), S(y, z) and \ \forall_y \in Y\}\}$

Max-Min Composition

- $X = \{1,3,5\}; Y = \{1,3,5\}$
- $R = \{(x,y)|y = x + 2\}; S = \{(x,y)|x < y\}$
- $R\ and\ S\ is\ on\ X \times Y$

- $R =$

	1	3	5
1	0	1	0
3	0	0	1
5	0	0	0

- $S =$

	1	3	5
1	0	1	1
3	0	0	1
5	0	0	0

- $R = \{(1,3), (3,5)\}$
- $S = \{(1,3), (1,5), (3,5)\}$

$R \circ S$

	1	3	5
1	0	0	1
3	0	0	0
5	0	0	0

- $R \circ S(1,1) = \max\{\min(R(1,1), S(1,1)), \min(R(1,3), S(3,1)), \min(R(1,5), S(5,1))\}$
- $R \circ S(1,1) = \max\{\min(0,0), \min(1,0), \min(0,0)\} = 0$

- $R \circ S(1,3) = \max\{\min(R(1,1), S(1,3)), \min(R(1,3), S(3,3)), \min(R(1,5), S(5,3))\}$
- $R \circ S(1,1) = \max\{\min(0,1), \min(1,0), \min(0,0)\} = 0$

- $R \circ S(1,5) = \max\{\min(R(1,1), S(1,5)), \min(R(1,3), S(3,5)), \min(R(1,5), S(5,5))\}$
- $R \circ S(1,1) = \max\{\min(0,1), \min(1,1), \min(0,0)\} = 1$

Fuzzy Cartesian Product

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$
- $R = A \times B \subset X \times Y$
- $\mu_R(X,Y) = \mu_{A\times B}(X,Y) = \min\{\mu_A(x), \mu_B(y)\}$
- $A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$, B $= \{(b_1, 0.5), (b_2, 0.6)\}$

$R = A \times B$

	b_1	b_2
a_1	0.2	0.2
a_2	0.5	0.6
a_3	0.4	0.4

Operations on Fuzzy Relations: Max-Min

	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.9
x_3	0.8	0.6

	z_1	z_2	z_3
y_1	0.6	0.4	0.7
y_2	0.5	0.8	0.9

- $R \circ S(x_1, z_1) = \max\{\min\big(R(x_1, y_1), S(y_1, z_1)\big), \min\big(R(x_1, y_2), S(y_2, z_1)\big)\}$
- $R \circ S(x_1, z_1) = max\{\min(0.5, 0.6), \min(0.1, 0.5)\} = 0.5$

	z_1	z_2	z_3
x_1	0.5	0.4	0.5
x_2			
x_3			

Compositions of two relations

- Max-Product Composition

- Given two relation matrices R and S, the max-product composition is defined as $T = R \circ S$
  - $T(x, z) = \max\{\{R(x, y) * S(y, z) and\ \forall_y \in Y\}\}$

Operations on Fuzzy Relations: Max-Product

	y_1	y_2
x_1	0.6	0.3
x_2	0.2	0.9

	z_1	z_2	z_3
y_1	1	0.5	0.3
y_2	0.8	0.4	0.7

	z_1	z_2	z_3
x_1	0.6	0.3	0.21
x_2			

- $R \circ S(x_1, z_1) = \max\{R(x_1, y_1) * S(y_1, z_1), R(x_1, y_2) * S(y_2, z_1)\}$
- $R \circ S(x_1, z_1) = max\{0.6 * 1, 0.3 * 0.8\} = 0.6$

Thank You