ФЕДЕРАЛЬНОЕ ГОСУДАРСТВЕННОЕ АВТОНОМНОЕ ОБРАЗОВАТЕЛЬНОЕ УЧРЕЖДЕНИЕ ВЫСШЕГО ОБРАЗОВАНИЯ «НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ «ВЫСШАЯ ШКОЛА ЭКОНОМИКИ»

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СРАВНИТЕЛЬНЫЙ АНАЛИЗ МЕТОДОВ ОЦЕНКИ ОПЦИОНОВ

(COMPARATIVE ANALYSIS OF THE OPTION PRICING METHODS)

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Contents

1 Introduction

Economic news frequently discusses the significant losses that traders in the financial derivatives market have sustained. The financial world was shocked when major firms apparently lost hundreds of millions, if not billions, of dollars. Due to these catastrophes, several well-known institutions have failed. Often while these stories are important, fascinating, and even surprising, they have hurt the reputation of using derivatives. However, when used appropriately, they are essential tools for effective risk management, therefore a strong evaluation strategy is necessary.

the goal of this work is to explain what is option, what types of options are there, also to give mathematical reasoning to the famous methods of determination of the option's price, and to show how they could be used with modern programming language.

2 Literature and learning material review

- Breakthrough work which defined further development of derivative pricing and risk management. Black-Scholes formula provides a unique theoretical price for European options. (Black & Scholes, 1973)

- Formalized Binomial tree approach for modeling the option price. (Cox, Ross, & Rubinstein, 1979)

- Valuing American Options by estimating conditional expected payoff via Least-Squares. (Longstaff & Schwartz, 2001)

- University of London Study guide. Fundamental financial theory. (Payne, 2018)

- Book on derivatives for graduate and advanced undergraduate courses in business, economics, and financial engineering. (Hull, 2015)

- A series of internet blog posts about the Binomial Option Pricing Model with implementation in Python programming language (Thurman, 2018a, 2018b, 2018c, 2018d)

- An overview of the most common Machine Learning techniques applied to pricing European call options. (Codrut-Florin, 2021)

- Series of the lectures presented by industry professionals "on mathematical foundations of modeling in finance with those focused on important real-world problems" (Massachusetts Institute of Technology, 2013)

- Advanced undergraduate course for (Lyulko, 2021)
  Other articles: (Merton, 1973), (Merton, 1976), (Cox & Ross, 1976), (Black, 1976), (Hull & White, 1987), (Bakshi, Cao, & Chen, 1997), (Harrison & Pliska, 1981)

3 Theoretical background

Definition 1. *Financial derivatives*
Financial instruments which depend on the value of another assets.

"Financial derivatives are financial instruments that are linked to a specific financial instrument or indicator or commodity, and through which specific financial risks can be traded in financial markets in their own right." (International Monetary Fund, 2022)

Definition 2. *Derivative security (contingent claim)*
An agreement between a buyer and a seller.

As derivatives dependent upon the price of asset (usually stock), then in order to determine the present value, one has to know the payoffs schedule, as well as the stock's possible trajectories. The latter is a subject of modeling, bringing the assumptions on the type and characteristics of asset's behavior. Typically, the main division comes from assuming discrete or continuous behavior.

3.1   Options

3.1.1   Definition

Option is a contingent claim that gives one investor, holding the long position, the option (but not the obligation) to buy/sell an underlying asset

Options	Position	Long	Short
Call	option to buy	must agree to sell
Put	option to sell	must agree to buy

for a specified at the time of singing a contract price (strike price) (Thurman, 2018a)

European	at specified future time (expiration time)
American	at any date before the expiration time

One may plot the return of the option in Stock price - Profit/Loss plane:
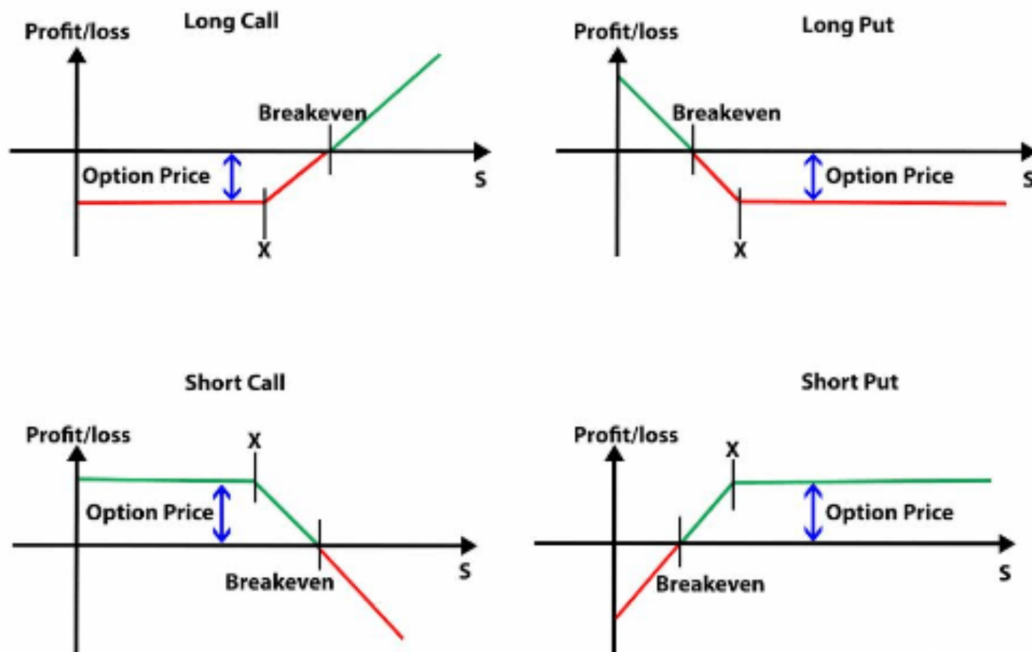

Figure 1: Payoffs for European Call and Put options (TME.net, 2021)

Why to own an option? It is cheaper than holding the asset itself, as well as decreases the downside risk of the investment. Also used as an insurance to hedge risk. For example, if investor has the stock, he may buy a put option to save the value of his asset. Note that the short position has **no limit** for the losses!

3.1.2   The Simplest Binomial Model (One period, one risky asset, two states)

Consider the following characteristics of an **European call option**:

Stock price today $(t = 0)$	$S_0 = 100$£
Stock price in 1 year $(t = 1)$	$S_L = 80$£ (Low), $S_H = 120$£ (High)
Strike price	$K = 100$£

What is fair price of the option C_0^K?

Profit at $t = 1$ (Payoff schedule):

$$C_1^K = (S_1 - K)^+ := \max(S_1 - K, 0) = \begin{cases} S_1 - K & \text{if } S_1 > K \\ 0 & \text{if } S_1 \leqslant K \end{cases}$$

Where S_T - price of 1 share, C_T^K - value of the Call option with strike price K at $t = T$.

It is reasonable to assume the **no-arbitrage condition**:

Portfolios with the same cash flows are identical in value.

From it naturally comes **pricing by replication**: create a portfolio: $\vec{H} = (H_0, H_1) = (b, a)$ - investments at $t = 0$, where a^1 amount of shares and b£ in cash.[2]

Value of the portfolio: $V_t(\vec{H}) = P(S_t) = aS_t + b$ and we know that

$$P(80£) = 0£ \text{ and } P(120£) = 20£ => a = 1/2 \text{ and } b = -40£^3$$

We replicated the option's payoff in 1 year with a portfolio by buying a half of the underlying stock and borrowing 40£. So, the price of the portfolio today is $P(100£) = 1/2 * 100£ - 40£ = 10£$. And by the 'no-arbitrage principle' the price of an option is equal to it $C_0^K = V_0(\vec{H}) = 10£$.

In general, we know: $S_L \leq S_0 \leq S_H$, $P(S_1) = C_1^K$

$$\begin{cases} P(S_L) = C_1^K{}_H = 0 \\ P(S_H) = C_1^K{}_L = S_H - S_0 \geq 0 \end{cases} \Rightarrow \begin{cases} aS_L + b = 0) \\ aS_H + b = S_H - S_0 \end{cases} \Rightarrow \begin{cases} a = \frac{S_H - S_0}{S_H - S_L} \\ b = -a * S_1 \end{cases}$$

Then, $V_0(\vec{H}) = P(S_0) = aS_0 + b = \dfrac{S_H - S_0}{S_H - S_L} * S_0 + -a * S_1 = \dfrac{(S_H - S_0)(S_0 - S_L)}{S_H - S_L} = C_0^K.$

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[1]Number of shares a is not always integer. Increasing the number of shares solves the problem.

[2]Negative values represent short selling $(a < 0)$ or borrowing $(b < 0)$.

[3]Cash b is negative. Solved by an obligation to make a payment. (ex. take a loan, write a bond)

3.1.3 Risk-Neutral Pricing:

Let us now implement probabilities into our model: $S_0 > 0$, $S_H > S_L > 0$, $p \in [0, 1]$

$$t = 1 : S_1 = \begin{cases} S_H, & \text{if } \omega_H \\ S_L, & \text{if } \omega_L \end{cases} \quad C_1^K = \begin{cases} S_H - S_0, & \text{if p} \\ S_H - S_0, & \text{if 1-p} \end{cases}$$

Two states of the world at $t = 1$: ω_H and $\omega_L \Rightarrow$

The probability space: $\Omega = \{\omega_H, \omega_L\}$, $P(\omega_H) = p$ and $P(\omega_L) = 1 - p$

The buyer pays C_0^K (non-random) at $t = 0$, and earns $C_1^K(w)$ (random) at $t = 1$

The conditions when cash is paid at each situation are fixed in the contract $\{C_1^K(\omega_H), C_1^K(\omega_L)\}$

Question: Given $C_1^K(W)$, what is fair price C_0^K?

If we ignore time value of money (interest rates), then the expected value of option in the future should be equal to the current value of it:

$$C_0^K = \mathbb{E}[C_1^K]$$

Then expected value of the option in the next year:

$$\mathbb{E}[C_1^K] = (S_H - S_0) * p + 0 * (1 - p) = \underline{(S_H - S_0)p = C_0^K} \text{ - depends on } p$$

Risk-Neutral probability

In order to price the option, we need to find the probability of exercising the option. Using risk-neutrality concept, investor ignores the riskiness of the stock price:

$$\begin{cases} S_0 = E(S_1) \\ E(S_1) = S_H * p + S_L * (1 - p) = S_L + (S_H - S_L)p \end{cases} \Rightarrow S_0 = S_L + (S_H - S_L)p$$

$$\Rightarrow p = \frac{S_0 - S_L}{S_H - S_L} \text{ - risk-neutral probability}$$

$$C_0^K = (S_H - S_0) * p = \frac{(S_H - S_0)(S_0 - S_L)}{S_H - S_L} \text{ same as in the replication approach!}$$

Now, let $q_H = \dfrac{S_0 - S_L}{S_H - S_L}$ and $q_L = \dfrac{S_H - S_L}{S_H - S_L} \Rightarrow q_H > 0, q_L > 0$ and $q_H + q_L = 1 (Zhitlukhin, 2021)$

$\Rightarrow V_0(\vec{H}) = C_1^K(\omega_H) * q_H + C_1^K(\omega_L) * q_L = \mathbb{E}^{\mathbb{Q}}[C_1^K]$ with a new **risk-neutral probability measure Q**[4]

In reality, *value today $<$ expected future value* for any stock, because the future value is uncertain and one can just keep the current asset and not take any risk.

Note that with the risk-neutral probability measure Q, the price of the option C_0^K does not depend upon the probability of the ω_H, meaning that the price is free of probability.

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[4]Q on $\Omega = \{\omega_H, \omega_L\}$:, $Q(\omega_H) = q_H$ and $(Q\omega_L) = q_L$

3.2   Option Pricing methods

There are many option pricing methods, they are different in the used assumptions such as a type of behavior of the underlying stock (ex. discrete (Random Walk), or continuous (Geometric Brownian Motion)). Most of them are based on the *risk-neutrality principle*. The most used ones are:

- Binomial/Trinomial Tree

- Black-Scholes model

- Monte Carlo simulation

- Finite difference methods

The list is not exhaustive, however it covers common techniques used both in Academia and Industry. This paper dives deep into the *Binomial tree* model, and touches the *Black-Scholes* model.

3.2.1   Binomial Tree

This method was first formalized in the article "Option Pricing: A Simplified Approach" by (Cox et al., 1979).
Idea - Random Walk: Divide the life of the option into small time intervals Δt.
Assume that at each time the price S of the underlying stock at the beginning of the interval either goes 'up' to Su with probability p, or 'down' Sd with probability $1 - p$
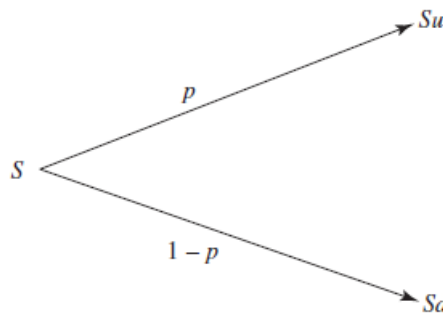

Figure 2: Paths for asset price movement in Δt under the binomial model (Hull, 2015)

As division, i.e. the number of time steps goes to infinity, Binomial tree model converges to the Black-Scholes-Merton model. Showing that discrete behaviour in the limit becomes continuous. [5]

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[5]proof why is that true

Risk-Neutral Valuation principle

The principle states that the value of a derivative can be measured under the assumption that the world is risk neutral.

- Assume $\mathbb{E}^{\mathbb{Q}}[r_i] = r_f$ for any asset A_i.

  Where r_i - return of the asset, r_f - risk-free rate, Q - risk-neutral measure.

- Value of the derivative's payoff is equal to discounted at the risk-free interest rate the expected value: $V_0 = df(t) * \mathbb{E}^{\mathbb{Q}}[X]$, where $df(t)$ discount factor after period t,

$$df(t) = \begin{cases} e^{-rt} & \text{if continuous compounding} \\ (1+r)^{-t} & \text{if discrete} \end{cases}$$

Continuous compounding will be used. [6]

Determination p, u, and d

$$\text{Parameters of the model}: S_0 > 0, u > 1, 1 > d \geqslant 0, 1 \geqslant p \geqslant 0, \Delta t > 0, n > 0, r > 0 \quad (1)$$

Let us fix the model defined above by changing some of the parameters: $S_H = S_0 * u; S_L = S_0 * d$. Where $(u-1) * 100\%$ and $(d-1) * 100\%$ are stocks percentage change in both directions. Furthermore, by adding interest rates into the model we get new risk-neutral probability:

$$S_0 = df(\Delta t) * \mathbb{E}S_1 \Rightarrow p = \frac{\mathbb{E}S_1 - S_L}{S_H - S_L} = \frac{S_0/df(\Delta t) - S_0 d}{S_0 u - S_0 d} \Rightarrow$$

$$Q(\omega_H) = q_H = \frac{e^{r\Delta t} - d}{u - d} \text{ and } Q(\omega_L) = q_L = \frac{u - e^{r\Delta t}}{u - d}$$

Knowing that $C_1^K = (S_1 - K)^+$, use the risk-neutral valuation principle to price the option

$$C_0^K = df(t) * \mathbb{E}^{\mathbb{Q}}[C_1^K] = e^{-r\Delta t}[C_1^K(\omega_H) * q_H + C_1^K(\omega_L) * q_L]$$

$$C_0^K = e^{-r\Delta t}(S_0 u - K)\frac{e^{r\Delta t} - d}{u - d} \quad (2)$$

Let's check the consistency of the model: $S_0 = 100$£, $u = 1.2$, $d = 0.8$, $r = 0\%$, $\Delta t = 1$.

$$C_0^K = e^{-0*1}(100 * 1.2 - 100)\frac{e^{0*1} - 0.8}{1.2 - 0.8} = 20 * \frac{0.2}{0.4} = 10£$$

As expected, the formula (2) gave the same result. Now add risk-free interest rate $r = 5\%$.

$$C_0^K = e^{-0.05*1}(100 * 1.2 - 100)\frac{e^{0.05*1} - 0.8}{1.2 - 0.8} = 0.951 * 20 * \frac{1.051 - 0.8}{0.4} = 19.02 * \frac{0.251}{0.4} = 11.94£$$

The price increased, meaning positive relationship between the price of the call option C_0^K and the risk-free interest rate r. The intuition is that the strike price K in the future is decreasing in its present value with increasing interest rate.

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[6]As $n \to \infty$, discrete compounding $\to$ continuous one: $\lim_{n\to\infty}(1 + \frac{r}{n})^{nt} = e^{rt}$
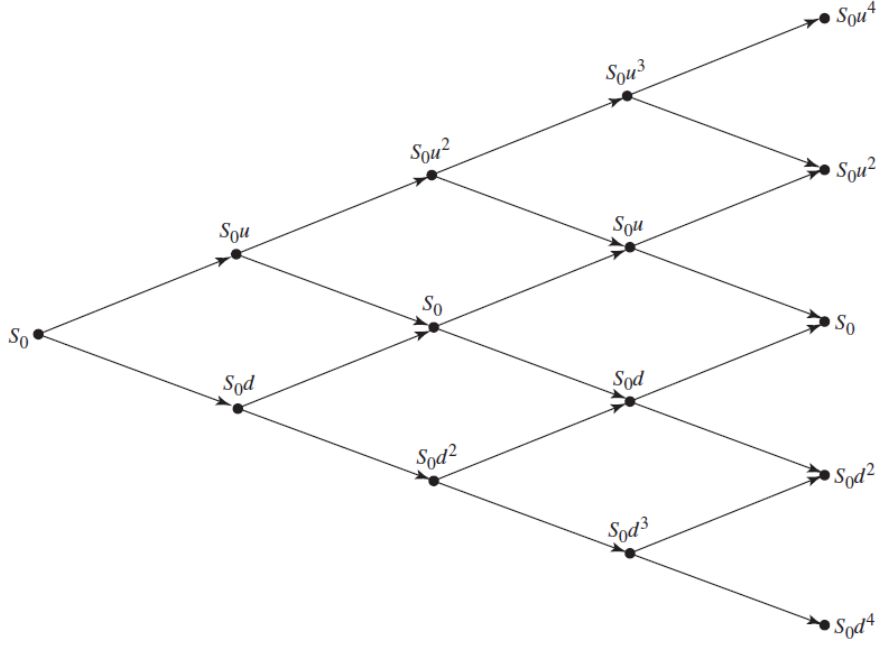
Generalize: proof by (Lyulko, 2021)

Figure 3: The tree with 4 time intervals (Hull, 2015)

The stock price S_n after n time intervals is a random variable. Denote the change after k time interval X_k. It is also a random variable.

X_k	u	d
P	p	1-p

where $p = \dfrac{e^{r\Delta t} - d}{u - d} \in [0, 1]$

Because $X_1, ..., X_n$ are independent and identically distributed (i.i.d.), then one may see that

$$S_n = S_0 X_1...X_n = S_0 * u^Y * d^{n-Y}$$

$$Y \sim Binom(n, p) \Rightarrow P(Y = k) = C_n^k p^k (1-p)^{n-k}, \ k = 0, ..., n$$

Where random variable Y is a number of "up" movements. It is distributed as *Binomial* with number of tries n and probability of success.

The option price derivation: As was already stated the current value of the option, i.e. the option price is a discounted expected payoff under the risk-neutral measure. Assumed stock price movement allow us to calculate the price analytically. $t = n * \Delta t$

$$C_0^K = df(t) * \mathbb{E}^{\mathbb{Q}}[C_1^K] = e^{-rn\Delta t} * \mathbb{E}^{\mathbb{Q}}(S_n - K)^+ = e^{-rn\Delta t} \sum_{k=1}^{n} (S_0 u^k d^{n-k} - K)^+ P(Y = k) =$$

$$= e^{-rn\Delta t} \sum_{k=1}^{n} (S_0 u^k d^{n-k} - K)^+ C_n^k p^k (1-p)^{n-k}$$

Because $(S_0 u^k d^{n-k} - K)^+ = 0$ when $S_0 u^k d^{n-k} < K$.[7]

$$(S_0 u^k d^{n-k} - K)^+ > 0 \text{ for } k \geqslant a, \text{ where } a = \left\lceil \frac{\ln\left(\frac{K}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)} \right\rceil$$

Where $\lceil x \rceil$ is the ceiling function, i.e. smallest integer greater than or equal than x.
Now continue the derivation process:

$$C_0^K = e^{-rn\Delta t} \sum_{k=1}^{n}(S_0 u^k d^{n-k} - K)^+ C_n^k p^k (1-p)^{n-k} = e^{-rn\Delta t}\sum_{k=a}^{n}(S_0 u^k d^{n-k} - K)C_n^k p^k (1-p)^{n-k} =$$

$$= \frac{S_0}{e^{n\Delta t}}\sum_{k=a}^{n}C_n^k(up)^k((1-p)d)^{n-k} - \frac{K}{e^{n\Delta t}}\sum_{k=a}^{n}C_n^k p^k(1-p)^{n-k}$$

Now denote $p' = up * e^{-n\Delta t} > 0$ to make new probability measure, then check that sum is equal to one.

$$\frac{up}{e^{n\Delta t}} + \frac{(1-p)d}{e^{n\Delta t}} = \frac{p(u-d)+d}{e^{n\Delta t}} = \frac{[(e^{r\Delta t}-d)/(u-d)]*(u-d)+d}{e^{n\Delta t}} = \frac{e^{n\Delta t}}{e^{n\Delta t}} = 1$$

So, $\frac{(1-p)d}{e^{n\Delta t}} = 1 - p'$ this allows us to define a new random variable $Y' \sim Binomial(n, p')$ and finish the formula for the price of European call option:

$$C_0^K = S_0\sum_{k=a}^{n}C_n^k(p')^k(1-p')^{n-k} - \frac{K}{e^{n\Delta t}}\sum_{k=a}^{n}C_n^k p^k(1-p)^{n-k} = S_0\sum_{k=a}^{n}P(Y'=k) - \frac{K}{e^{n\Delta t}}\sum_{k=a}^{n}P(Y=k) \Rightarrow$$

Cox-Ross-Rubinstein (CRR) formula: (Cox et al., 1979)

$$C_0^K = S_0 * P(Y \geqslant a) - \frac{K}{e^{n\Delta t}}P(Y \geqslant a) = S_0\Phi[a; n, p] - Ke^{-n\Delta t}\Phi[a; n, p'] \qquad (3)$$

Where $\Phi[a; n, p]$ - a complementary $Binomial(n, p)$ distribution function,

$$a = \left\lceil \frac{\ln\left(\frac{K}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)} \right\rceil \text{ if } a > n, C_0^K = 0$$

$$p = \frac{e^{r\Delta t} - d}{u - d}, \text{ and } p' = \frac{u}{e^{n\Delta t}}p$$

Parameters of the model: $S_0 > 0, u > 1, 1 > d \in (1, 0], p \in [0, 1], \Delta t > 0, n > 0, r > 0$ [8]

Note: As $n \to \infty$ CRR formula3 will converge to famous Black-Scholes formula for European call option.(Cox et al., 1979) Indeed, the limiting case of *Binomial* distribution by Central Limit theorem is *Normal* distribution which is present in BS formula.[9]

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[7] $S_0 u^k d^{n-k} = S_0 d^n\left(\frac{u}{d}\right)^k < K \Leftrightarrow \left(\frac{u}{d}\right)^k < \frac{K}{S_0 d^n} \Leftrightarrow k\ln\frac{u}{d} < \ln\frac{K}{S_0 d^n} \Leftrightarrow k < \frac{\ln\left(\frac{K}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)}$
[8] See the $u = 1/d$ argument at 4
[9] One may prove it both analytically and empirically

3.2.2   Black-Scholes model

As opposed to the discrete nature of the Binomial tree model 3.2.1, the Nobel Prize winning model lies in the continuous space, using full apparatus of the Stochastic Calculus.

The key feature of the model is the stochastic differential equation which defines the stock price

$$dS_t = \mu S_t dt + \sigma S_t dW_t \text{ (Black \& Scholes, 1973)}$$

The solution to the DE is random process $(S_t)_{t \geqslant 0}$ distibuted as *Geometric Brownian motion (GBM)*.(Samuelson, 1973)

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t},$$

Where μ and σ is the *drift* and *volatility* of the stock per year,

W_t is a *Brownian motion*(or *Wiener process*).

Definition 3. *(Lyulko, 2021)*

Stochastic process $W = (W_t)_{t \geqslant 0}$ is called Brownian motion if

1. *$W_0 = 0$, $W_t - W_s \sim N(0, t-s)$ for any $t > s \geqslant 0$*

2. *It is a process with independent increments:*
   $(W_{t_2} - W_{t_1})$, $(W_{t_4} - W_{t_3})$ - independent, for any $0 \leqslant t_1 < t_2$, $0 \leqslant t_3 < t_4$

Theorem 1.

The Black-Scholes formula allows us the calculate the price of European call and put options.

$$\text{C(S, t)} = \text{N(d}_1\text{)S} - \text{N(d}_2\text{)Ke}^{-rt} \tag{4}$$

$$\text{d}_1 = \frac{1}{\sigma \sqrt{t}} \left[\ln\left(\frac{S}{K}\right) + t\left(r + \frac{\sigma^2}{2}\right) \right], \text{d}_2 = \frac{1}{\sigma \sqrt{t}} \left[\ln\left(\frac{S}{K}\right) + t\left(r - \frac{\sigma^2}{2}\right) \right]$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \text{e}^{-\frac{1}{2}z^2} dz \text{ - c.d.f of standard normal distribution}$$

The **derivation** of the formula requires solving *Partial Differential Equations*. The *Taylor* expansion in *Ito Calculus* helps to solve it.(Lyulko, 2021) It comes as a surprise, but with not-trivial change of variables the underlying P.D.E. becomes the famous *Heat Equation* which has been deeply studied by physicists. [10]

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[10]

4 Empirical estimation with Binomial Tree

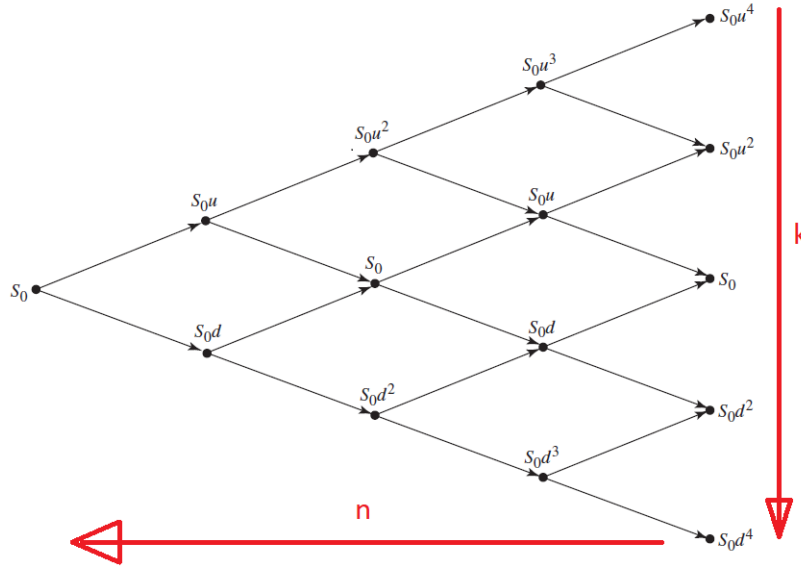Let's implement in Python Binomial tree approach to value the European Call Option.

Figure 4: Going through the tree with 4 time intervals

The algorithm stars evaluating option's value at last time interval $n = 4$. For each state of the world (node) the value of option is its payoff $C_4^K(\omega_{u^k}) = max(S_4(\omega_{u^k} - K), 0)$, $k = -4, ..., 4$. Then it goes to depth of $n - 1 = 3$ intervals and at each node it evaluates the price.

Remark: $C_n^k \neq C_n^K$, C_n^k - price of option at time interval n with k movements 'up'.

$C_{n-1}^k = df(\Delta t) * \mathbb{E}^{\mathbb{Q}}[C_n^k] =$
$e^{-r\Delta t}[C_4^{k+1}(\omega_H) * p + C_1^{k-1}(\omega_L)(1-p)],$
where p - risk-neutral probability

Repeat the process until $n = 0$:
$$C_0^K = C_0^0$$

```python
import math

def first_binomial_call(S_0, K, T, r, u, d, N):
    '''
    :param S_0: (£) current stock price
    :param K: (£) strike price
    :param T: (years) life of the option in years
    :param r: (float) risk-free interest rate in decimal
    :param u: (float > 1) increase in stock price
    :param d: (float < 1) decrease in stock price interest
    :param N: (int) number of time intervals in int
    :return C[(0,0)]: (£) price of European Call Option
    '''

    dt = T/N
    p = (math.exp(r * dt) - d)/(u - d)  # risk-neutral probability
    C = {}  # value of option at each node

    # calculating nodes at the last interval
    for k in range(N, -1, -1):  # k - number of stock's up movements
        S_k = S_0 * (u ** k) * (d ** (N-k))
        C[(N, k)] = max(S_k - K, 0)

    # moving backwards from last interval to first
    for n in range(N-1, -1, -1):  # n - depth in the tree
        for k in range(0, n + 1):
            df = math.exp(-r * dt)  # discount factor
            # expected value under risk-neutral measure
            EX = p * C[(n+1, k+1)] + (1-p) * C[(n+1, k)]
            C[(n, k)] = df * EX
    return C[(0, 0)]
```

Figure 5: Modified Python function for calculating the price of European Call option (Thurman, 2018c)

13

Examples: Let's recalculate the prices from our task in 3.2.1

```
PRMS = {"S_0":100, "K":100, "T":1, "r":0, "u":1.2, "d":0.8, "N":1}
f"r = 0.00: price = {first_binomial_call(**PRMS):.2f}£"
```

```
'r = 0.00: price = 10.00£'
```

```
PRMS = {"S_0":100, "K":100, "T":1, "r":0.05, "u":1.2, "d":0.8, "N":1}
f"r = 0.05: price = {first_binomial_call(**PRMS):.2f}£"
```

```
'r = 0.05: price = 11.95£'
```

Figure 6: Everything works as expected.

What if we try to increase number of intervals N keeping total time T constant?

```
for N in [1,10,100,200,300,400,500]:
    v = first_binomial_call(S_0=100, K=100, T=1, r=0, u=1.2, d=0.8, N=N)
    print(f"N = {N:3d}: price = {v:.2f}")

N =   1: price = 10.00
N =  10: price = 25.62
N = 100: price = 68.55
N = 200: price = 84.56
N = 300: price = 91.90
N = 400: price = 95.61
N = 500: price = 97.58
```

Figure 7: As N increases price seems to converge to the stock price. It is not wanted.

The reason behind this behavior is the constant rate of change for a stock even for any, even really small, duration of the intervals. To make it more consistent with real world u and d should be changed.

Choosing u and d: Volatility

An experienced statistician for any random variable (here S_t) would like to know its Expected value $\mathbb{E}[S_t]$ and Variance $\mathbb{V}[S_t] = \sigma^2$. Standard deviation of a variable is a measure of volatility of an asset. The bigger 'up' $(u - 1)$ or/and 'down' $(1 - d)$ movements, the more volatile the stock is.

Typically the stock price is assumed to be distributed as *Geometric Brownian motion (GBM)*.(Samuelson, 1973)

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{5}$$

Where μ and σ is the expected return and volatility (std.dev.) of the stock per year,
W_t is a *Brownian motion*3 (or *Wiener process*).

The **percentage change** of the stock is distributed approximately Normal:

$$\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma^2 \Delta t) \tag{6}$$

Hence, its mean and variance are as follows:

$$\mathbb{E}[\frac{\Delta S}{S}] = \mu \Delta t \text{ and } \mathbb{V}[\frac{\Delta S}{S}] = \sigma^2 \Delta t \tag{7}$$

By equating expected values and variances with *Random Walk*, one finds the following relationships[11]

$$u = e^{\sigma \sqrt{\Delta t}} \text{ and } d = \frac{1}{u} \tag{8}$$

```python
def binomial_call(S_0, K, T, r, std, N):
    '''
    :param S_0: (£) current stock price
    :param K: (£) strike price
    :param T: (years) life of the option in years
    :param std: (float) standard deviation of the stock
    :param r: (float) risk-free interest rate in decimal
    :param N: (int) number of time intervals in int
    :return C[(0,0)]: (£) price of European Call Option
    '''

    dt = T/N
    u = math.exp(std * math.sqrt(dt))
    d = 1/u
    p = (math.exp(r * dt) - d)/(u - d)  # risk-neutral probability
    C = {}  # value of option at each node

    # calculating nodes at the last interval
    for k in range(N, -1, -1):  # k - number of stock's up movements
            S_k = S_0 * (u ** (2*k-N))
            C[(N, k)] = max(S_k - K, 0)

    # moving backwards from last interval to first
    for n in range(N-1, -1, -1):  # n - depth in the tree
        for k in range(0, n + 1):
            df = math.exp(-r * dt)  # discount factor
            # expected value under risk-neutral measure
            EX = p * C[(n+1, k+1)] + (1-p) * C[(n+1, k)]
            C[(n, k)] = df * EX
    return C[(0, 0)]
```

Figure 8: Final function for modelling with the Binomial Tree

---

[11]Proof of the fact is in (Hull, 2015) or (Kaledin & Demidova, 2022)

```
for N in [1,10,100,200,300,400,500]:
    v = binomial_call(S_0=100, K=100, T=1, r=0, std=math.log(1.2)/math.sqrt(1), N=N)
    print(f"N = {N:3d}: price = {v:.2f}")
```

```
N =   1: price = 9.09
N =  10: price = 7.08
N = 100: price = 7.25
N = 200: price = 7.25
N = 300: price = 7.26
N = 400: price = 7.26
N = 500: price = 7.26
```

Figure 9: consistent with the real-world results

```
S_0 = float(input("Price of an asset today: "))
S_L = float(input("Lower price of an asset in future: "))
S_H = float(input("Higher price of an asset in future: "))

a = (S_H - S_0)/(S_H - S_L)
b = (-1) * a * S_L
print()
print(f"Replicating portofolio: a = {a} shares, b = {b}£ of cash")
C = a * (S_0 - S_L)
print(f"C = {C}£ - price of the option")
```

```
Price of an asset today: 100
Lower price of an asset in future: 80
Higher price of an asset in future: 120

Replicating portofolio: a = 0.5 shares, b = -40.0£ of cash
C = 10.0£ - price of the option
```

Figure 10: Python code to the Simplest Binomial Model3.1.2

5 Conclusion

Discussion of the results:

The thorough research was conducted, giving the understanding of options nature, as well as the 'state of the art' pricing methods. New branch of the financial mathematics was studied (stochastic, ito calculus) with the appropriate concepts such as (portfolio replication, risk-neutral probability measure, brownian motion, etc.). Implemented the valuation algorithms in Python programming language. Moreover, this article is written in LaTeX, giving useful tool for further personal academical development.

Possible **further research** may focus on continuing the exploration of pricing models. For instance, visualizing the Wiener processes and estimating with it the prices, the Machine Learning regression analysis (Codrut-Florin, 2021). Furthermore, the models should be tested on the real life data (MOEX, S&P 500 and others). This will give deeper understanding on the applicability of the models.

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